UNIVERSIDAD DE CASTILLA~LA MANCHA

Ph.D. THESIS

ADVANCES IN PARALLEL AND SEQUENTIAL DYNAMICAL SYSTEMS OVER GRAPHS

Submitted for the degree of Doctor of Philosophy at the University of Castilla-La Mancha

by

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January 2020

To my family.

To my friends.

Contents

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Preface

In accordance with the requirements of the procedure established by the University of Castilla-La Mancha (UCLM) for the elaboration, defense and examination of doctoral theses, once I have received the positive examiners' reports and the corresponding authorizations of the supervisors of the doctoral thesis, Silvia Martínez Sanahuja and José Carlos Valverde Fajardo, and the Coordinator of the Doctoral Programme FISYMAT, Gabriel Fernández Calvo, I submit this work as a dissertation for the degree of Doctor of Philosophy at the University of Castilla-La Mancha.

This dissertation, entitled "Advances in Parallel and Sequential Dynamical Systems over Graphs", constitutes an own original research work related to the fields *Dynamical Systems* and *Computational Algebra*, included in the research lines *Mathematical Models and Methods in Science* and *Algebraic Models* of the Doctoral Programme FISYMAT.

The scientific contributions of the doctoral thesis have been published or are in process of publication by international journals of the JCR (Web of Science) Index:

- Paper [6], *On the Periods of Parallel Dynamical Systems*, Complexity Volume 2017 (2017) Article ID 7209762, 6 pages, corresponds to Subsections 3.1.2 and 3.1.3 of Section 3.1 in Chapter 3.

- Paper [7], *On periods and equilibria of computational sequential systems*, Info. Sci. 409–410 (2017) 27–34, corresponds to Subsections 4.1.1, 4.1.2 and 4.1.3 of Section 4.1 in Chapter 4.

- Paper [8], *Maximum number of periodic orbits in parallel dynamical systems*, Inf. Sci. 468 (2018) 63–71, corresponds to Subsections 3.1.4, 3.1.5 and 3.1.6 of Section 3.1 in Chapter 3.

- Paper [9], *Predecessors and Garden-of-Eden configurations in parallel dynamical systems on maxterm and minterm Boolean functions*, J. Comput. Appl.

Math. 348 (2019) 26–33, corresponds to Subsection 3.2.1 of Section 3.2 in Chapter 3.

- Paper [10], *Solution to the predecessors and Gardens-of-Eden problems for synchronous systems over directed graphs*, Appl. Math. Comput. 347 (2019) 22–28, corresponds to Subsection 5.2.1 of Section 5.2 in Chapter 5.
- Paper [11], *Predecessors Existence Problems and Gardens of Eden in Sequential Dynamical Systems*, Complexity Volume 2019 (2019) Article ID 6280960, 10 pages, corresponds to Subsection 4.2.1 of Section 4.2 in Chapter 4.
- Paper [12], *Predecessors and Gardens of Eden in sequential dynamical systems over directed graphs*, Appl. Math. Nonlinear Sci. 3(2) (2018) 593–602, corresponds to Subsection 5.2.2 of Section 5.2 in Chapter 5.
- Paper [13], *Dynamical attraction in parallel network models*, Appl. Math. Comput. 361 (2019) 874–888, corresponds to Subsection 3.2.2 of Section 3.2 in Chapter 3.
- Paper [14], *Attractors and transient in sequential dynamical systems*, Int. J. Comput. Math. (2019), corresponds to Subsection 4.2.2 of Section 4.2 in Chapter 4.
- Paper [15], *Enumerating periodic orbits in sequential dynamical systems*, Under review, corresponds to Subsections 4.1.4, 4.1.5 and 4.1.6 of Section 4.1 in Chapter 4.
- Paper [16], *Coexistence of periods in parallel and sequential dynamical systems over directed graphs*, Under review, corresponds to Subsections 5.1.1 and 5.1.2 of Section 5.1 in Chapter 5.

The quality indicators of the papers according to the data in Web of Science are:
Sum of times cited: 31.
Average citations per item: 3.88.
H-index: 4.
Q1-publications: 6.

Albacete, January 2020

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Abstract

In this dissertation, the dynamics of homogeneous parallel and sequential dynamical systems on maxterm and minterm Boolean functions are analyzed.

In particular, for parallel and sequential dynamical systems over undirected graphs, the dynamics are described completely, while some advances are provided for such systems over directed graphs.

Specifically, for the case of homogeneous parallel dynamical systems on maxterm or minterm Boolean functions over undirected graphs, it is proved that they can present only two kinds of periodic orbits: fixed points and 2-periodic orbits. Furthermore, it is demonstrated that fixed points and 2-periodic orbits cannot coexist. In addition, uniqueness results of such periodic orbits are provided. Finally, the study of the periodic structure of such systems is completed by showing optimal upper bounds for the number of fixed points and 2-periodic orbits, and examples where these bounds are attained.

The dynamics of non-periodic orbits are also studied for this kind of systems, by solving the classical predecessor problems (existence, uniqueness, coexistence and number of predecessors), obtaining a characterization of the Garden-of-Eden configurations and an optimal bound for the number of them. Additionally, it is provided a characterization of attractors and a method to obtain their basins of attraction. Finally, optimal upper bounds for the transient in such systems are shown.

In the case of homogeneous sequential dynamical systems on maxterm or minterm Boolean functions over undirected graphs, it is demonstrated that they can present periodic orbits of any period. Besides, it is proved that periodic orbits with different periods greater than or equal to 2 can coexist, but when these systems have fixed points, periodic orbits of other periods cannot appear. Finally, as in the parallel update case, the study of the periodic structure of such systems is completed by showing optimal upper bounds for the number of fixed points and periodic orbits of period greater than 1, and examples where these bounds are attained.

In this case, the dynamics of non-periodic orbits are also studied, by solving the same problems as in the case of parallel dynamical systems on maxterm or minterm Boolean functions over undirected graphs. Indeed, the classical predecessor problems (existence, uniqueness, coexistence and number of predecessors) are solved, providing a characterization of the Garden-of-Eden configurations and an optimal bound for the number of them. A characterization of attractors and a method to obtain their basins of attraction are shown, also providing optimal upper bounds for the transient in such systems.

Finally, for homogeneous parallel and sequential dynamical systems on maxterm or minterm Boolean functions over directed graphs, it is proved that periodic orbits of any periods can appear and coexist, even fixed points and periodic orbits with greater periods. Also, a solution to the predecessor problems is provided, so extending the results given for systems over undirected graphs. Consequently, a characterization of the Garden-of-Eden states is achieved, providing the best bound for the number of them.

Chapter 1

Introduction

A (mathematical) model is the mathematical formalization of a real phenomenon. Therefore, in its formulation, it is necessary to include, on one hand, the elements involved in the phenomenon and, on the other hand, the relationships among them that determine their evolution. Thus, the elements (or more precisely their states) are represented in the model using variables, and the relationships among them that determine their evolution are expressed through equations, functions, logical operators, etc., trying to formalize the laws or principles that govern the phenomenon in the reality (technological principles, laws of physics, biological principles, etc.). The evolution of the states of the elements can be influenced by certain (quantifiable/estimable) conditions that are incorporated into the mathematical model as parameters.

Mathematical models are useful in sciences and engineering, especially in the study of the dynamics or evolution of phenomena from the real world in which it is very costly or it is not possible to experiment with real elements. In such cases, the (mathematical) analysis of the model allows to know the asymptotic behavior of the phenomenon under consideration, providing a very useful tool for evaluating decisions in relation to the phenomenon as well as their possible consequences.

Studying the dynamics of a model has as main objectives: the knowledge of their periodic and non-periodic states, as well as attractiveness and repellent relations among them. These are the objectives that we achieve with this dissertation for different models coming from electronics or computation, which have also turned out useful for other sciences as physics, biology, chemistry, mathematics, etc.

Network models are the natural way to formalize any phenomenon involving cells, gens, entities or any kind of distinct elements which interact among them. In particular, network models where the states of the elements have Boolean states ([39]) and the functions determining their evolution are Boolean functions, are called *Boolean network models* (BN).

The emergence of the concepts of *Cellular Automata* (CA) [94] and *Kauffman Networks* (KN) [78], in the late 1960s, for the formalization of computational processes and genetic regulation respectively, supposed a first step in the modeling of evolutionary phenomena through Boolean networks (BN). This type of models has proved to be useful not only for computational processes or genetic regulation, but also to solve several problems coming from other sciences, such as chemistry (see [83, 84, 99]), mathematics (see [40, 43, 46, 47, 51, 57, 76, 93]), physics (see [41, 42, 44, 48]), biology (see [1, 5, 53, 78, 79, 61, 86, 90, 105, 106, 113]), ecology (see [54, 72, 73]), or even social sciences such as psychology or sociology (see [2, 70, 82]), among others. Moreover, this new paradigm has served as a basis to establish other new concepts, as Graph Turing Machines, which can be seen as the infinite version of such models (see [3]).

The first CA consisted of a grid of cells, where each cell could have a state belonging to the set $\{0,1\}$, which evolved synchronously at (discrete) intervals of time, giving rise to (discrete) iterations [38]. The update of the state of each cell was carried out according to a Boolean (local) function, the same for all the cells, depending only on the cell to be updated and the cells surrounding it. That is, in a one-dimensional CA, if x_i^t was the state value of the cell i at time t, its state value at the instant $t+1$, x_i^{t+1}, was obtained by applying the local (common) Boolean function to the state values x_{i-1}^t, x_i^t and x_{i+1}^t. Thus, CA were the first mathematical model to capture the essential characteristics of digital systems: synchronicity, regular distribution and locally dependent iterations.

Although CA were firstly introduced in the works of Ulam and von Neumann [94], they did not arouse a great interest until 1970, when Martin Gardner [59, 60] published in Scientific American an explanation of the so-called "Game of Life" by John Conway [49]. However, it was not until 1983, when Wolfram [117] established the first results on its complexity, despite its simple construction. In fact, in 1984, as a result of further investigation [118], he suggested that most CA can be classified, according to their dynamic behavior, into four types: the first three ones, exhibiting a behavior similar to fixed-point systems, and the fourth one, with *unpredictable* asymptotic properties. In later years, Wolfram continued his research (see [119, 120, 121]), which was finally collected in [122]. During this same period, other works helped to develop the theory of CA (see [110, 111, 112]) which was well presented later, for instance, in [74, 77, 100].

On the other hand, the first KN of size n and connectivity k, (n, k)-networks, consisted of n interconnected vertices so that each one was connected to another k vertices. In this case, the dependency graph was considered directed, the states of the vertices belonged to the set $\{0, 1\}$ and the local update Boolean functions could be different for each vertex. It follows that the first CA could be considered as a particular case of $(n, 2)$-networks.

The KN as said above, appeared for the first time in the work carried out by Kauffman [78], as models applied to the simulation of genetic networks in which genes had an on-off behavior that could be formalized, respectively, as 1 and 0. Kauffman's results suggested that, if each gene is affected or influenced by only 2 or 3 genes, then the system seems to behave stably. The most of the results obtained on this model during the last quarter of the 20th century were published by Kauffman in his book [79].

These first BN models have evolved in recent years, giving rise to the paradigm of *(discrete) dynamical systems on graphs and Boolean functions* (GDS), firstly coined in [28]. Nevertheless, its origins can be situated in a series of papers by Barret, Mortveit and Reydis entitled *Elements of a theory of computer simulation I, II, III, IV* at the very beginning of this century [33, 34, 35, 36]. This new concept, whose denomination emphasizes its deterministic character, generalizes the previous ones in the sense that it contemplates the possibility that the relations among the elements of the system can be arbitrary. Alternatively, this kind of models was named *Boolean finite dynamical systems* (BFDS) in the recent works [80, 81], what evoques their principal features of deterministic Boolean models involving a finite number of elements. In this dissertation, we focus on the study of the dynamics of this kind of models.

For these new models, the relationships among the elements of the system can be represented by a graph called *dependency graph* that could be arbitrary. In this sense, the smallest aggregation units of the phenomenon are now called *nodes* (or *vertices*), in relation to their belonging to the dependency graph, relieving the term of *cells* in CA and *entities* in KN, although we will use all of them along this document. Thus, the relationship between two elements is represented by an edge between them or by an arc if the relationship is not bidirectional. In this last case, the dependency graph is directed.

Likewise, each vertex i of the graph has assigned a variable, x_i, representing its state which is called *state variable* of the vertex i. In this kind of systems, these variables usually take values in the (Boolean) set $\{0, 1\}$ to indicate the deactivated or activated state of the corresponding vertex. However, it is possible that they take values in a more general *Boolean algebra* (see [22, 23]).

The evolution of the system is implemented through *local (Boolean) functions*, which generally come from the restriction of a (global) Boolean function to each vertex and its adjacent ones in the graph. In this case, the system is said to be *homogeneous*. Nevertheless, there exists the possibility of considering independent local functions to update the state of each vertex (see, for example, [19, 25, 115]). Observe that, once the local functions are defined, they (automatically) determine the dependency graph of the system and this is why the word graph is usually avoided in their denomination. However, in models emerging from experimental phenomena, it is natural to determine firstly the relationships among the entities and, subsequently, how the relations interfere in the evolution of their states.

The evolution of the state of all the nodes of the system can occur synchronously or asynchronously. In the first case, the models are called *parallel dynamical systems* (PDS) or, alternatively, *synchronous dynamical systems* (SyDS) ([6, 8, 17, 18, 19, 20, 21, 22, 23, 28, 123]). In the second case, they are called *sequential dynamical systems* (SDS) or, alternatively, *asynchronous dynamical systems* (AsyDS) ([7, 29, 30, 33, 34, 35, 36, 50, 51, 91, 92, 95]). In view of them, a mixed situation could be contemplated by considering that some of the nodes update at the same time in a asynchronous scheme of updating. This last models are known as *semi-synchronous* or *mixed dynamical systems* (MDS) [62, 65].[1]

In addition, since the beginning of the present century, another generalization of BN models is being developed by considering the possibility of non-determinism or stochasticity. This generalization has lead to the concept of *stochastic or probabilistic Boolean networks* [69, 103, 104]. The stochastic character arises when any of the fundamental elements (dependency graph, Boolean state set, local functions, updating scheme) is chosen randomly, usually from a finite set of them, iteration by iteration. They are often called *random Boolean networks* (RBN), although this term is also used for deterministic ones to indicate that the dependency graph is arbitrary [62, 65].

Regarding the possible generalizations of BN, it is worth mentioning the (pioneer) works by Gershenson [62, 65], where classifications of the most important generalizations of this kind of models are established. In this line, other interesting works by the same author on RBN are ([63, 64, 66, 67]).

[1] The abbreviations GDS, PDS, SyDS, SDS, AsyDS, MDS, CA, BN and GOE will be written along this document for the singular and plural forms of the corresponding terms, since it seems better from an aesthetic point of view.

Both deterministic an non-deterministic BN can be classified depending on the type of fundamental elements which constitute them: the dependency graph, the (Boolean) set of states of the entities, the (Boolean) evolution operator and the update scheme. That is, they can be classified depending on whether: the dependency graph is undirected or directed; the set of states is the basic Boolean algebra $\{0, 1\}$ or a more general one; the local (Boolean) functions are all inferred from a general one or they are independent; the updating of all the entities is synchronous, semi-synchronous (mixed) or asynchronous. In this sense, different classifications of systems, which are transversal among them, can be established.

In this dissertation, we focus on deterministic BN over an (arbitrary) undirected or directed dependency graph (which is not loop-free, although for convenience the loops are not drawn); with the basic Boolean algebra $\{0, 1\}$ as set of states of the entities; whose evolution operator is homogeneous, induced by a Boolean function of the type maxterm or minterm; and with parallel or sequential update scheme according to a permutation.

The main objective in the study of the dynamics is to get to know the (asymptotic) behavior of all the orbits of the system [85]. The graphical representation of the dynamics of the system is called its *phase portrait*. In our context, it is also named *phase diagram* or *transition diagram*, since it is a directed graph representing the transit from each state to its corresponding successor, according to the evolution of the system.

In the methodology to study the dynamics of a system, a first step is to study the structure of its periodic orbits. Specifically, as pointed out in [52], it means to determine the length and number of coexisting periodic orbits. Thus, the methodology for the study of the periodic structure of these systems consists in solving the following problems:

- Periodic orbits existence (POE): It consists in determining what periods can exist.
- Periodic orbits coexistence (POC): It consists in determining which periods can coexist and, in this case, if a period determines the existence of others.
- Periodic orbits uniqueness (POU): It consists in determining if a certain periodic orbit is the unique one in the system.
- Maximum number of orbits of a certain period ($\#n$-PO): It consists in determining the number of periodic orbits of a certain period.

In this sense, in [28], the POE and POC problems were solved for homogeneous PDS and SDS when the evolution operator is one of the simplest Boolean functions, that is, for OR (resp. AND) and NAND (resp. NOR). In [34], the authors demonstrated that the fixed points of a SDS induced by symmetric Boolean functions as PAR, MAJ, MIN and XOR are independent of the order of updating. Later, the POE problem was solved, in a more general context, for homogeneous PDS on maxterm and minterm Boolean functions over undirected dependency graphs [17] and over directed ones [18]. Likewise, the POE problem was also solved for non-homogeneous PDS when the independent local functions belong to {AND, OR, NAND, NOR} in [19]. Finally, the results were generalized, solving the POE problem for homogeneous PDS on maxterm an minterm Boolean functions where the state set is any Boolean algebra [22]. The POE and POC problems for SDS on bi-threshold functions over non-uniform networks (where the threshold parameters depend on each vertex) have been recently studied in [123], proving that the presence of fixed points excludes the existence of other periodic orbits.

Regarding the last problem, some recent related works prove that it is computationally intractable even in the case of fixed points for a certain class of non-homogeneous systems (see [25, 115]). Actually, only upper bounds for the number of fixed points are provided. In the case of SDS, in [29], it was proved that this problem is NP-complete even for some simple cases, although, in the case of SDS on linear or monotone local updating functions, the problem can be solved efficiently. Later, in [114] , it was confirmed that the problem of counting fixed points is computationally intractable even when the local updating functions are symmetric Boolean functions or when every node has a number of adjacent vertices bounded by a small constant. Nevertheless, in [50], the problem of enumerating periodic points is solved for certain SDS, namely $[C_n, parity_3, id]$ and $[C_n, (1 + parity)_3, id]$.

Once the periodic structure of a system is known, the next step in the study of the dynamics is to analyze the asymptotic behavior of non-periodic orbits. In particular, in this context, it means to determine which different non-periodic states arrive in the same periodic orbit. In the case of GDS, each non-periodic orbit converges to a periodic one. In such a case, it is said that the non-periodic orbit is in the *basin of attraction* of the periodic orbit. The number of iterations needed by the non-periodic point to reach the corresponding periodic orbit is known as its *transient*. The maximum of the transients is called the *transient* or *width* of the system. The methodology for the analysis of the behavior of non-periodic orbits is carried out through the study of predecessors, which allow us to determine towards which periodic orbits they converge and, at the same time, to infer their basins of attraction, as well as the width or transient of the system. In particular, the non-existence of predecessors makes possible to identify the (non-periodic) GOE states of the system (which are the heads of the branches constituting the basins of attraction), the width of the system and the attractive periodic orbits can be established. Specifically, the methodology for the study of predecessors consists in solving several problems similar to the previous ones:

- Predecessors existence (PRE): It consists in determining which states have at least a predecessor.
- Predecessors coexistence (PREC): It consists in determining when there exists more than one predecessor for a state.
- Predecessors uniqueness (PREU): It consists in determining when the predecessor of a state is unique.
- Maximum number of predecessors of a certain state (#PRE): It consists in determining the number of predecessors of a certain state.

The study of predecessors, which leads to find out the corresponding GOE at the same time, has been treated by several researchers in this field in different environments related to Boolean network models [29, 30, 31, 32]. In [33], results for GOE states existence are given in relation to the invertibility of the systems. Specifically, the invertibility of a system characterize the non-existence of GOE.

So far, the resolution of these problems has been mainly faced from the point of view of its computational complexity [29, 30, 31, 32], as previously done in [109] and [68] for CA. In particular, it was shown that the PRE problem is NP-complete for some restricted classes of SDS. On the other hand, polynomial time algorithms are given for the PRE problem for SDS on the simplest maxterm OR (resp. minterm AND) and symmetric Boolean operators, which can be extended to the corresponding PDS. In the recent works [80, 81], the authors study the computational complexity of generalized t-predecessor problems an t-GOE for some particular cases of PDS corresponding to a variety of sets of local functions.

The study can be performed computationally, by "brute force", when the number of entities of the system is not excessively large, using computer algorithms such as those in [20] or [45], or specific software such as [65]: DDLab, `http://www.ddlab.com`, which allows to simulate the dynamics for synchronous RBN and CA; RBNLab, `http://rbn.sourceforge.net`, which is able to simulate RBN with different update schemes; or BN/PBN Toolbox for Matlab, `https://code.google.com/archive/p/pbn-matlab-toolbox/downloads`, which allows to simulate both deterministic and probabilistic Boolean networks.

In this dissertation, in contrast with this numerical studies, we have applied algebraic methods and techniques coming from graph theory, combinatorics and discrete mathematics to give analytical proofs of our results. Actually, for an arbitrarily large number of entities, the solution needs to involve these methods and techniques. In this sense, the results are obtained through the combinatorial analysis of the possibilities of evolution of the states. Likewise, these methods and techniques are different from those usually employed in other kinds of dynamical systems, such as those ones defined over (infinite) metric spaces through continuous functions or differential equations.

As a result of this dissertation, we complete the analysis of the dynamics of PDS and SDS on maxterm and minterm Boolean functions over undirected dependency graphs, and provide some important advances in the case of directed dependency graphs, denoted by PDDS and SDDS respectively. Specifically, regarding the study of periodic orbits, we prove that in this kind of PDS only fixed points and 2-periodic orbits can appear, while SDS can present periodic orbits of any period. In both cases, we demonstrate that fixed points and periodic orbits of greater periods cannot coexist. However, periodic orbits of any period greater than 1 can coexist in SDS. Furthermore, we prove that PDDS and SDDS can present periodic orbits of any period which can coexist in any case, even fixed points and periodic orbits of greater periods.

Additionally, in the case of PDS and SDS, we give results on the uniqueness and maximum number of periodic orbits, distinguishing the cases of fixed points and periodic orbits of period greater than 1.

Concerning the study of non-periodic orbits, we deal with the classical predecessor problems (existence, uniqueness, coexistence and maximum number or predecessors) in PDS, SDS, PDDS and SDDS, providing a characterization and upper bounds for the GOE states of the systems. Moreover, in the case of PDS and SDS, we give a characterization of attractors and a method to obtain their basins of attraction, showing optimal upper bounds for the transient of these systems.

This thesis provides not only several new results in the field but also some new ideas and tools to extend them. Thus, from this approach, a further progress in some future research directions can be expected. The main future research directions correspond to extension of these results to other kinds of models, according to the classifications shown before. Specifically, generalizations on the set of states of the entities, the dependency graph, the local update functions and the evolution schedule of deterministic BN models can be considered as a natural continuation of this research work, as well as the analysis of non-deterministic models. Additionally, from this complete theoretical study, a direct application of these results to models coming from sciences, engineering or real-word situations seems to be feasible.

This dissertation is organized as follows. In Chapter 2, we include the most important notation and basic concepts to understand the results in the rest of the chapters.

In Chapter 3, the study of the dynamics in homogeneous PDS on maxterm and minterm Boolean functions over undirected dependency graphs is performed. The first section of this chapter is dedicated to the analysis of the dynamics of periodic orbits, showing that these systems can present, as periodic orbits, only fixed point and 2-periodic orbits. Besides, we prove that periodic orbits with different periods cannot coexist, which implies that a kind of Sharkovsky's order is not valid for this class of dynamical systems. Additionally, we provide conditions to obtain a Fixed-Point Theorem and a 2-Periodic-Orbit Theorem in this context, based on the uniqueness of these periodic orbits. Finally, the study of the periodic structure of such systems is completed by showing upper bounds for the number of fixed points and 2-periodic orbits. Actually, we provide examples where these bounds are attained, demonstrating that they are the best possible ones. This section provides a relevant advance in the knowledge of the dynamics of such systems. Moreover, the ideas developed here help to obtain similar results for other related systems. In the second section of this chapter, we study the dynamics of non-periodic orbits. Firstly, we solve the classical predecessor problems for PDS on maxterm and minterm Boolean functions. Actually, we solve analytically the predecessor existence problem by giving a characterization to have a predecessor for any given configuration. As a consequence, we also get a characterization of the Garden-of-Eden (GOE) configurations of these systems and an optimal bound for the number of them. Moreover, the structure of the predecessors found out allows us to give a solution to the unique predecessor problem, the coexistence of predecessors problem and the number of predecessors problem. Later in this section, we give a characterization of attractors in PDS, which evolve by means of maxterm or minterm Boolean functions, and provide a method to obtain their basins of attraction. This makes possible to obtain a detailed description of their phase diagrams. Furthermore, we state necessary and sufficient conditions to know when a fixed point or a 2-periodic orbit is globally attractive. Besides, we provide optimal upper bounds for the transient in such systems, i.e., for the maximum number of iterations required to reach a periodic orbit. In order to do that, we distinguish the two possible cases: attractive fixed points and attractive 2-periodic orbits. Moreover, we establish patterns that allow us to obtain a PDS on a maxterm or minterm Boolean function for which any given optimal upper bound for the transient is reached.

In Chapter 4, the study of the dynamics in homogeneous SDS on maxterm and minterm Boolean functions over undirected dependency graphs is performed. According with the methodology established, as in the previous chapter, the first section is dedicated to the study of periodic orbits in this kind of systems. We show that sequential systems with (Boolean) maxterms and minterms as global evolution operators can present orbits of any period, so breaking the pattern found in the parallel case. Besides, we prove that periodic orbits with different periods greater than or equal to 2 can coexist. Nevertheless, when an SDS has fixed points, we demonstrate that periodic orbits of other periods cannot appear. Additionally, we provide conditions to obtain a Fixed-Point Theorem and an m-Periodic-Orbit Theorem ($m > 1$) in this context, based on the uniqueness of these periodic orbits. Finally, we give an upper bound for the number of fixed points and periodic orbits of period greater than 1, so completing the study of the periodic structure of such systems. We also demonstrate that these bounds are the best possible ones, providing examples where they are attained. In the second section of this chapter, we analyze the dynamics of non-periodic orbits. To do that, we deal with the predecessor problems. In particular, we solve algebraically such problems in SDS on maxterm and minterm Boolean functions. We also provide a description of the GOE configurations of any system, giving the best upper bound for the number of GOE points. On the other hand, we show how to determine attractors in SDS on maxterm and minterm Boolean functions and their corresponding basins of attraction, making possible to obtain a detailed description of the phase diagrams. Furthermore, we study when an attractor is globally attractive. As another interesting result, upper bounds for the transient in such systems are provided. In order to do that, we distinguish two possible scenarios again: fixed points and periodic orbits with period greater than 1.

In Chapter 5, the study of the dynamics in homogeneous PDDS and SDDS on maxterm and minterm Boolean functions over directed dependency graphs is introduced, showing some results obtained directly from the previous ones related to PDS and SDS. As in the previous chapters, the first section is dedicated to the analysis of periodic orbits. In this case, we study the existence and coexistence of periodic orbits in PDDS and SDDS. We prove that periodic orbits of any periods can appear and coexist in such systems, even fixed points and periodic orbits with greater periods, so breaking the patterns shown in the case of undirected dependency graphs. Thus, in this case of dynamical systems over directed dependency graphs, no order, as the one provided by Sharkovsky, applies. Also, the simplest maxterm and minterm Boolean functions are analyzed, showing that PDDS and SDDS on the maxterm OR (resp. minterm AND) only present fixed points as periodic orbits, while PDDS and SDDS on the maxterm NAND (resp. minterm NOR) can present periodic orbits of any period, except fixed points. In the first case of fixed points as periodic orbits, the configurations with all the entities activated and all the entities deactivated are always some of them, but with the possibility of other alternatives, so breaking again the pattern shown in PDS and SDS. In the second section of this chapter, we study the dynamics of non-periodic orbits in this kind of systems. We solve all the predecessor problems in PDDS and SDDS, so extending the results given for systems over undirected graphs. In this same sense, we provide a characterization to have at least one predecessor for any given state and, consequently, a characterization of the GOE states. Furthermore, we solve the unique predecessor problem, the coexistence of predecessors problem and the number of predecessors problem, providing the best bounds for such a number and for the number of GOE configurations.

Finally, we summarize the most important conclusions of this thesis and show the future research directions which appear as a natural continuation of this work.

Chapter 3 corresponds to our articles [6, 8, 9, 13]; Chapter 4 corresponds to our papers [7, 11, 14, 15]; and Chapter 5 corresponds to our works [10, 12, 16]. In particular, six of these eleven articles were published in Q1-JCR journals.

Chapter 2

Preliminaries

In our context, a system consists of several entities and each entity has a (Boolean) state at any given time (see [33, 34, 35, 36]). Entities are related and they get information from their associated ones. As done in this dissertation, every entity is usually represented by a vertex of an undirected graph and two vertices are adjacent if their states influence each other in the update of the system. Such a graph is said to be the (undirected) *dependency graph* of the system (see [28]). However, in the last chapter at the end of the thesis, we present some generalizations of the previous results in which the entities are vertices of a directed dependency graph, representing asymmetrical influence of the states of the entities.

In the case of symmetrical influence, we denote the dependency graph by $G = (V, E)$ while, in the asymmetrical situation, by $D = (V, A)$, being $V = \{1, \ldots, n\}$ the vertex set, E the edge set and A the arc set. Along this dissertation, we will assume that G (resp. D) is connected (resp. weekly connected) because, otherwise, the results can be generalized simply by working on each connected component.

For every entity $i \in V$, it is defined its *state value*, $x_i \in \{0, 1\}$, to indicate if the entity i is deactivated, $x_i = 0$, or activated, $x_i = 1$. Nevertheless, it is possible to have state values in a more general *Boolean algebra* (see [22, 23]). A vector constituted by the state values of the entities, $x = (x_1, \ldots, x_n)$, is denominated a (global) *state* or *configuration* of the system. There are two special configurations, the one with all the entities activated and the one with all of them deactivated, which will be denoted by $\mathcal{I}$ and $\mathcal{O}$ respectively.

Since the general situation in this dissertation is the study of systems with symmetrical influence of the states of the entities, hereinafter in this chapter, the concepts will be defined for undirected dependency graphs. Anyway, all the definitions here can be easily translated to the case of digraphs. They will be shown explicitly where appropriate in the last chapter at the end of the thesis.

For every entity $i \in V$, subset $U \subseteq V$ or subgraph $G_0 = (V_0, E_0)$ of G, in the case of symmetrical influence, we consider all the vertices that interfere with them in their updating process:

$$\overline{A_G(i)} = \{j \in V : \{i, j\} \in E\} \cup \{i\},$$

$$\overline{A_G(U)} = \bigcup_{i \in U} \overline{A_G(i)}.$$

For our purposes, later on this dissertation, we will also need to consider these other sets:

$$A_G(i) = \{j \in V : \{i, j\} \in E\},$$

$$A_G(U) = \bigcup_{i \in U} A_G(i),$$

$$A_G(G_0) = A_G(V_0),$$

$$A_G^*(U) = A_G(U) \setminus U.$$

We will denote the complementary set of any of them as usual. That is, for instance, $\overline{A_G(U)}^c$ will denote the subset of entities in V which are not in $\overline{A_G(U)}$, or equivalently the subset of entities which are neither in U nor in $A_G^*(U)$.

The update or evolution of the system is performed by local functions. Thereby, to update the state of an entity i, the corresponding local function acts only on $\overline{A_G(i)}$. As the states of the entities are Boolean, the local function are Boolean too.

The state of the entities can be updated in a synchronous way or in an asynchronous or sequential manner. In this last case, a permutation on V, $\pi = \pi_1 | \ldots | \pi_n$, which settles the order of updating is usually considered, being π_1 the first entity whose state updates, π_2 the second one, and so on. In each step, the updated states of the entities are involved in the following evolution of the state of the other vertices.

Definition 2.1. Let $G = (V, E)$ be an undirected graph on $V = \{1, \ldots, n\}$ and a map

$$F : \{0,1\}^n \to \{0,1\}^n, \qquad F(x_1, \ldots, x_i, \ldots, x_n) = (y_1, \ldots, y_i, \ldots, y_n),$$

where y_i is the updated state value of the entity i by applying a local function f_i over the state values of the entities in $\overline{A_G\left(i\right)}$. They constitute a discrete dynamical system called *parallel dynamical system* over G, which will be denoted by $[G, F]-\mathrm{PDS}$ or $F-\mathrm{PDS}$ when specifying the dependency graph is not necessary. □

Accordingly with this definition, in this dissertation, generical PDS with a maxterm MAX (resp. minterm MIN) as evolution operator will be denoted by $\mathrm{MAX}-\mathrm{PDS}$ (resp. $\mathrm{MIN}-\mathrm{PDS}$).

Let us illustrate this definition with the following example:

Example 2.1. Let $G = (V, E)$ be the graph defined by $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$ (see Figure 2.1).

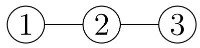

Figure 2.1: Graph $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$.

The adjacency set associated to each vertex $i \in V$, $\overline{A_G\left(i\right)}$, is: $\overline{A_G\left(1\right)} = \{1, 2\}$, $\overline{A_G\left(2\right)} = \{1, 2, 3\}$ and $\overline{A_G\left(3\right)} = \{2, 3\}$.

Let us consider the following local functions $\{f_i\}_{i=1}^{3}$ defined over the states of the entities belonging to each set $\overline{A_G\left(i\right)}$:

- $f_1\left(x_1, x_2\right) = x_1' \vee x_2'$,
- $f_2\left(x_1, x_2, x_3\right) = x_1' \vee x_2' \vee x_3$,
- $f_3\left(x_2, x_3\right) = x_2' \vee x_3$.

They constitute the evolution operator $F : \{0, 1\}^n \to \{0, 1\}^n$,

$$F\left(x_1, x_2, x_3\right) = \left(f_1\left(x_1, x_2\right), f_2\left(x_1, x_2, x_3\right), f_3\left(x_2, x_3\right)\right) = \left(y_1, y_2, y_3\right).$$

These elements define a PDS which is denoted by $[G, F]-\mathrm{PDS}$.

In this case, a state $x = (x_1, x_2, x_3) \in \{0, 1\}^3$ evolves to other state $y = (y_1, y_2, y_3) \in \{0, 1\}^3$ if $y = F(x)$. For instance, the state $x = (0, 0, 0)$ evolves to $y = F(0, 0, 0) = (1, 1, 1)$. In Figure 2.2, a phase diagram with the successor of each configuration can be seen.

□

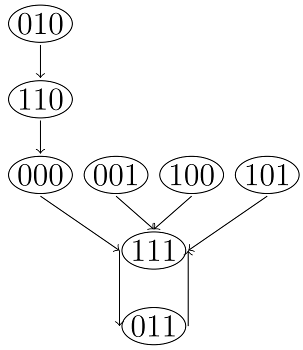

Figure 2.2: Phase portrait of the system $[(\{1,2,3\},\{\{1,2\},\{2,3\}\}),(x_1' \vee x_2', x_1' \vee x_2' \vee x_3, x_2' \vee x_3)] -$ PDS.

Definition 2.2. Let $G = (V, E)$ be an undirected graph on $V = \{1, \ldots, n\}$, $\pi = \pi_1 | \ldots | \pi_n$ a permutation on V and a map

$$[F, \pi] = F_{\pi_n} \circ \cdots \circ F_{\pi_1} : \{0,1\}^n \to \{0,1\}^n,$$

$$[F, \pi](x_1, \ldots, x_i, \ldots, x_n) = (y_1, \ldots, y_i, \ldots, y_n),$$

where $F_{\pi_i} : \{0,1\}^n \to \{0,1\}^n$ updates the state value of the entity $\pi_i \in V$ from x_{π_i} to y_{π_i} considering the state values of the entities belonging to $\overline{A_G(\pi_i)}$ and keeping the other states unaltered, i.e., $F_{\pi_i} = (\mathrm{id}_1, \ldots, f_{\pi_i}, \ldots, \mathrm{id}_n)$, being id_j the identity function over the entity j and $f_{\pi_i} : \{0,1\}^n \to \{0,1\}$ the local function which performs the update for the entity π_i. They constitute a discrete dynamical system called *sequential dynamical system* over G, which will be denoted by $[G, F, \pi] -$ SDS or $F -$ SDS when specifying the dependency graph is not necessary and the updating order can be implicit in this context of sequential evolution. $\square$

As in the case of PDS, in this dissertation, generical SDS with a maxterm MAX (resp. minterm MIN) as evolution operator will be denoted by MAX $-$ SDS (resp. MIN $-$ SDS).

Again, let us illustrate this definition with the following example:

Example 2.2. Consider G and $\{f_i\}_{i=1}^3$ as in Example 2.1 and let $\pi = 1|2|3$ be a permutation on V. In terms of Definition 2.2, let us consider:

- $F_1(x_1, x_2, x_3) = (x_1' \vee x_2', x_2, x_3)$,
- $F_2(y_1, x_2, x_3) = (y_1, y_1' \vee x_2' \vee x_3, x_3)$,

- $F_3(y_1, y_2, x_3) = (y_1, y_2, y_2' \vee x_3)$.

They constitute the evolution operator $[F, \pi] : \{0,1\}^n \to \{0,1\}^n$,

$$[F, \pi](x_1, x_2, x_3) = F_3 \circ F_2 \circ F_1 (x_1, x_2, x_3) = (y_1, y_2, y_3).$$

These elements define an SDS which is denoted by $[G, F, \pi] -$ SDS.

In this case, a state $x = (x_1, x_2, x_3) \in \{0,1\}^3$ evolves to other state $y = (y_1, y_2, y_3) \in \{0,1\}^3$ if $y = [F, \pi](x)$. For instance, the state $x = (0,0,0)$ evolves to $y = [F, \pi](0,0,0) = (1,1,0)$. In Figure 2.3, a phase diagram with the successor of each configuration can be seen.

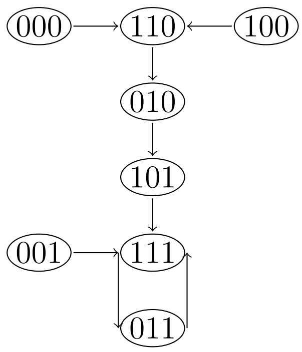

Figure 2.3: Phase portrait of the system $[(\{1,2,3\}, \{\{1,2\},\{2,3\}\}), F_3 \circ F_2 \circ F_1 (x_1, x_2, x_3), 1|2|3] -$ SDS.

□

Every f_i is usually the restriction of a global Boolean function[1] $f : \{0,1\}^n \to \{0,1\}$ acting only over the states of the entities in $\overline{A_G(i)}$, as we are going to suppose in this dissertation. In this case, since the Boolean function f originates F, along this thesis we will identify f and F. If all the local functions are the restriction of a global one, then the system is called *homogeneous*. Nevertheless, such local functions could be independent (see [19]).

In Examples 2.1 and 2.2, each local function f_i, $i = 1, 2, 3$, is originated as the restriction of the global function $f : \{0,1\}^n \to \{0,1\}$, $f(x_1, x_2, x_3) = x_1' \vee x_2' \vee x_3$, over the states of the entities in $\overline{A_G(i)}$.

[1] A Boolean function on n (Boolean) variables can be understood as a function $f : \{0,1\}^n \to \{0,1\}$, where the evaluation $f(x_1, \ldots, x_n)$ is computed from the values $x_1, \ldots, x_n \in \{0,1\}$ using the logical operators AND ($\wedge$), OR ($\vee$), NOT ($'$) as a propositional formula. For further information about general Boolean functions and their properties, [89] can be consulted.

As said before, throughout this dissertation, the local functions of the systems under study will be given by Boolean functions, which describe how to determine a Boolean output from some Boolean inputs. Special cases of Boolean functions are the *maxterms* and the *minterms* [17]. Recall that a maxterm (resp. minterm) of n variables $x_1, \ldots, x_n$ is a Boolean function f such that

$$f(x_1, \ldots, x_n) = z_1 \vee \ldots \vee z_n \qquad (\text{resp. } f(x_1, \ldots, x_n) = z_1 \wedge \ldots \wedge z_n),$$

where $z_i = x_i$ or $z_i = x_i'$. Minterm is the dual concept of maxterm, changing the disjunction operator for the conjunction one.

The simplest maxterm is the one where each variable appears once in its direct form:

$$\mathrm{OR}(x_1, \ldots, x_n) = x_1 \vee \cdots \vee x_n.$$

Similarly, we have the maxterm NAND, considering all the variables in their complemented form:

$$\mathrm{NAND}(x_1, \ldots, x_n) = x_1' \vee \cdots \vee x_n'.$$

On the other hand, the simplest minterm is the one where each variable appears once in its direct form:

$$\mathrm{AND}(x_1, \ldots, x_n) = x_1 \wedge \cdots \wedge x_n.$$

Similarly, we have the maxterm NOR, considering all the variables in their complemented form:

$$\mathrm{NOR}(x_1, \ldots, x_n) = x_1' \wedge \cdots \wedge x_n'.$$

The great importance of this special class of Boolean functions is that (see [24, 37, 98]) any Boolean function, except $F \equiv 1$ (resp. $F \equiv 0$) can be expressed in a canonical form as a conjunction (resp. disjunction) of maxterms (resp. minterms). Thus, it is natural to start with the study of the dynamics for this kind of Boolean functions, as we do in this dissertation. For this reason, all the dynamical systems in this thesis, PDS or SDS (PDDS or SDDS in the context or directed dependency graphs), will be named dynamical systems over graphs (GDS) on maxterm (resp. minterm) Boolean functions.

Since in this dissertation the evolution operator of each system will be given by a unique maxterm (resp. minterm), each variable of the system will be named *direct* or *complemented variable*, as it appears in the associated maxterm (resp. minterm), and respectively its associated entity/vertex will be called *direct* or *complemented entity/vertex*.

According to [85, 116], understanding a dynamical system means knowing its orbit structure and, consequently, its phase portrait. That is, the partitioning of the state space into its orbits, which provides a complete visual idea of the asymptotic behavior of the whole system. This is the main aim of this thesis, which gives a complete study of the dynamics in PDS and SDS.

Definition 2.3. The *orbit* of an $F-\mathrm{PDS}$ or $F-\mathrm{SDS}$ starting at $x=(x_1,\ldots,x_n)\in\{0,1\}^n$ is the subset of the state space $\{0,1\}^n$ given by

$$\mathrm{Orb} = (x)\left\{y=(y_1,\ldots,y_n)\in\{0,1\}^n : F^t(x)=y,\ \forall t\in\mathbb{N}\cup\{0\}\right\}\subseteq\{0,1\}^n,$$

being $F^t = F\circ\overset{t)}{\cdots}\circ F$. □

An orbit is said to be a *periodic orbit*, if each configuration x belonging to it satisfies that there exists $t\geq 1$ such that $F^t(x)=x$. A state belonging to a periodic orbit is named *periodic point*. When a periodic orbit consists of a unique state, it is named *fixed point*, while in other case, the orbit is named *cycle*. In contrast, an orbit is said to be *non-periodic*, if there is a configuration x of the orbit such that $F^t(x)\neq x,\ \forall t\in\mathbb{N}$. A non-periodic orbit that reaches a periodic point (resp. fixed point) after a finite number of iterations is said to be an *eventually periodic orbit* (resp. *eventually fixed point*), and the states of such an orbit are named *eventually periodic points* (resp. *eventually fixed points*). Since the set of states is finite, in our context, every non-periodic orbit is eventually periodic (or eventually fixed).

Example 2.3. In Example 2.1, the phase portrait of the system is described in Figure 2.2. Thus, $\mathrm{Orb}(1,1,1)=\mathrm{Orb}(0,1,1)=\{(1,1,1),(0,1,1)\}$ is the only periodic orbit of the system, while, for instance, $\mathrm{Orb}(0,0,0)=\{(0,0,0),(1,1,1),(0,1,1)\}$ and $\mathrm{Orb}(0,1,0)=\{(0,1,0),(1,1,0),(0,0,0),(1,1,1),(0,1,1)\}$. □

The orbits of PDS, SDS, PDDS and SDDS are ordered sequences of configurations that can be enumerated by increasing integers. In this context, it is worthwhile to recall that, given a discrete dynamical system with evolution operator F and an initial state x, $y=F(x)$ is said to be the *successor* of the state x. Since a dynamical system is deterministic, there exists exactly one successor for each state, but a configuration could be the successor of more than one state, which are called its *predecessors*. If there is not a state x such that $F(x)=y$, then y is said to be a *Garden-of-Eden* (GOE) point of the system. That is, y is a state without predecessors. The set of GOE states of the system is denoted by $\mathrm{GOE}[F]$.

In our particular case, as the state space of the system is finite, every orbit is either periodic or eventually periodic. This means that the evolution from any initial state always reaches a periodic orbit, named an *attractor* [71]. If the attractor consists of one state, this periodic orbit is called an *attractor point*, whereas if it consists of two or more states, it is said to be an *attractor cycle*. Each state belonging to an attractor is a periodic point.

A periodic point x belonging to a cycle satisfies that there exists an integer $p > 1$ such that $F^p(x) = x$. Also, if for any integer $0 < t < p$, $F^t(x) \neq x$, then p is called the period of the orbit of x.

At this point, the following concepts related to orbits arise naturally:

Definition 2.4. In a $F -$ PDS or $F -$ SDS, an *attractive periodic orbit* is an orbit such that there exists a state of the system which is not part of the periodic orbit, but reaches it. Contrarily, a repulsive periodic orbit is an isolated periodic orbit, non-reachable from any external state. $\square$

Definition 2.5. In a $F-$PDS or $F-$SDS, we call *basin of attraction* of an attractive periodic orbit to the set of states that reach such a periodic orbit. $\square$

Definition 2.6. The time (number of iterations) that a state needs to reach an attractor is denominated as its *transient*. $\square$

Given a generic $[G, F] -$ PDS or $[G, F, \pi] -$ SDS and a configuration y, we define the following subsets of V and subgraphs of G, which will be useful throughout this thesis:

Let $W \subseteq V$ (resp. $W' \subseteq V$) be the set of entities such that the corresponding variables appear in the maxterm or minterm generating the evolution operator F in direct (resp. complemented) form. These sets W and W' are such that $W' = W^c$.

Additionally, let us consider the following two subsets of W':

$$W'_D = \{i \in W' : A_G(i) \cap W \neq \emptyset\},$$

$$W'_C = \{i \in W' : A_G(i) \cap W = \emptyset\}.$$

Each element of W'_D must be subclassified according to the following condition: if all its adjacent complemented vertices are adjacent to a direct vertex or not. Specifically, we consider the following sets:

$$W'^{\alpha}_D = \{i \in W'_D : A_G(i) \cap W' \subseteq W'_D\},$$

$$W'^{\beta}_D = \{i \in W'_D : A_G(i) \cap W'_C \neq \emptyset\}.$$

Let $G_1, \ldots, G_p$ be the connected components which result from G when we remove all the vertices in W' and the edges adjacent to those vertices. On the other hand, let $C_1, \ldots, C_q$ be the connected components which result from G when we remove all the vertices in $W \cup W'_D$ and the edges adjacent to those vertices. Additionally, let G^* be the graph which results from G when we remove all the vertices in W'_C and the edges which are incident to those vertices.

For our purposes, sometimes, we will need to consider the subsystem restricted to one of these subgraphs or restricted to the union of some of them. Let S be a subgraph of G. Then, the subsystem restricted to S will be denoted as $\left[S, F_{|S}\right]-$PDS in the case of parallel update and $\left[S, F_{|S}, \pi_{|S}\right]-$SDS in the case of sequential update, where $F_{|S}$ is the restriction of F to the vertices in S and $\pi_{|S}$ is the restriction of π to the vertices in S.

It should be pointed out that this subgraph S may not be connected (as is, in general, G^*). Therefore, $\left[S, F_{|S}\right]$ – PDS and $\left[S, F_{|S}, \pi_{|S}\right]$ – SDS may be understood as a set of independent PDS and SDS, respectively, in the sense that the evolution in each one only depends on the restriction of F (and π in the case of sequential update) to the connected component of S over which it is defined.

We will say that a vertex $i \in W'$ and a connected component G_j are adjacent if i is adjacent to any vertex of G_j. In this context, we define the subsets W'_1 and W'_2 of W':

$$W'_1 = \{i \in W' : \text{there exists a unique } j,\ 1 \leq j \leq p,\ \text{such that } i \text{ is adjacent to } G_j\},$$

$$W'_2 = W' \setminus W'_1.$$

For a configuration $y = (y_1, \ldots, y_n)$, we will consider the split of V into the following sets:

$$V_0 = \{i \in V : y_i = 0\},$$
$$V_1 = \{i \in V : y_i = 1\}.$$

Namely, V_0 (resp. V_1) is the set of deactivated (resp. activated) entities of y. Although, such sets are associated with the configuration y and could be denoted by $V_0(y)$ and $V_1(y)$, for simplicity, we avoid to include y in the notation.

Additionally, only in the context of SDS, let us consider the following sets contained in $\overline{A_G\,(V_0)}$:

$$P_0 = \{i \in V : \exists j \in V_0 \text{ such that } \{i, j\} \in E, i = \pi_r, j = \pi_s \text{ and } s < r\},$$

$$Q_0 = \{i \in V : \exists j \in V_0 \text{ such that } \{i, j\} \in E, i = \pi_r, j = \pi_s \text{ and } s > r\}.$$

In other words, each element i belonging to P_0 (resp. Q_0) is adjacent to a vertex $j \in V_0$ which is updated, according the order expressed in π, before (resp. after) i. As in the case of V_0 (resp. V_1, we will avoid to include y in the notation of P_0 (resp. Q_0).

Similarly for V_1, we consider the sets P_1 and Q_1 contained in $\overline{A_G(V_1)}$:

$$P_1 = \{i \in V : \exists j \in V_1 \text{ such that } \{i, j\} \in E, i = \pi_r, j = \pi_s \text{ and } s < r\},$$

$$Q_1 = \{i \in V : \exists j \in V_1 \text{ such that } \{i, j\} \in E, i = \pi_r, j = \pi_s \text{ and } s > r\}.$$

Once all these subsets of V and subgraphs of G have been defined, for the sake of clarity, let us illustrate all these definitions with an example:

Example 2.4. Let $G = (V, E)$ be the graph defined by $V = \{1, \ldots, 8\}$ and $E = \{\{1, 2\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 6\}, \{4, 7\}, \{7, 8\}\}$ (see Figure 2.4).

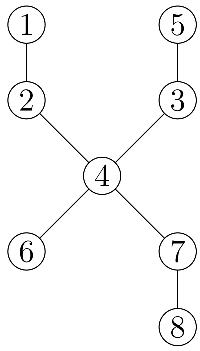

Figure 2.4: Graph $G = (\{1, \ldots, 8\}, \{\{1, 2\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 6\}, \{4, 7\}, \{7, 8\}\})$.

Let us consider the identity permutation over 8 elements, $\pi = \text{id}$, as updating order, and the maxterm:

$$\text{MAX} = x_1 \vee x_2 \vee x_3 \vee x'_4 \vee x'_5 \vee x'_6 \vee x'_7 \vee x'_8.$$

Let $[G, \text{MAX}, \pi] - \text{SDS}$ be the sequential dynamical system over G associated with the maxterm Boolean function MAX.

From MAX, we obtain the sets of entities whose associated variables are direct (W) or complemented (W'): $W = \{1, 2, 3\}$ and $W' = \{4, 5, 6, 7, 8\}$. Inside W', the set of entities which are adjacent to a vertex associated to a direct variable is $W'_D = \{4, 5\}$, while $W'_C = \{6, 7, 8\}$ is the set of entities not satisfying this condition. Finally, inside W'_D, 4 is adjacent to 6, which is adjacent only to complemented vertices, and 5 do not have adjacent vertices satisfying this condition, so $W_D'^{\alpha} = \{5\}$ and $W_D'^{\beta} = \{4\}$.

In this case, there are $p = 2$ connected components, subgraphs of G, which result when we remove all the vertices in W' and the edges adjacent to them: $G_1 = (\{1,2\}, \{\{1,2\}\})$ and $G_2 = (\{3\}, \emptyset)$; and $q = 2$ connected components, subgraphs of G, which result when we remove all the vertices in $W \cup W_D'$ and the edges adjacent to them: $C_1 = (\{6\}, \emptyset)$ and $C_2 = (\{7,8\}, \{\{7,8\}\})$.

The subsystems restricted to these subgraphs are, respectively:

$$[(\{1,2\}, \{\{1,2\}\}), x_1 \vee x_2, \mathrm{id}] - \mathrm{SDS},$$
$$[(\{3\}, \emptyset), x_3, \mathrm{id}] - \mathrm{SDS},$$
$$[(\{6\}, \emptyset), x_6', \mathrm{id}] - \mathrm{SDS},$$
$$[(\{7,8\}, \{\{7,8\}\}), x_7' \vee x_8', \mathrm{id}] - \mathrm{SDS}.$$

Additionally, the graph $G^* = (\{1,2,3,4,5\}, \{\{1,2\}, \{2,4\}, \{3,4\}, \{3,5\}\})$, being $\left[G^*, \mathrm{MAX}_{|G^*}, \pi_{|G^*}\right] - \mathrm{SDS}$ the system defined over this graph G^* on a maxterm $\mathrm{MAX}_{|G^*} = x_1 \vee x_2 \vee x_3 \vee x_4' \vee x_5'$ and order relationship $\pi_{|G^*} = 1|2|3|4|5$.

Knowing W', G_1 and G_2, we have that the only vertex belonging to W' adjacent to a unique G_j generates $W_1' = \{5\}$, and the others in W' form the set $W_2' = \{4,6,7,8\}$.

Lastly, let us consider the configuration $y = (0,1,1,0,0,1,1,1)$. The set of entities activated is $V_1 = \{2,3,6,7,8\}$ and deactivated is $V_0 = \{1,4,5\}$. Thus, $1 \in Q_1$ because $\{1,2\} \in E$, $2 \in V_1$ and 2 updates after 1; $2 \in P_0$ because of 1, and $2 \in Q_0$ because of 4; $3 \in Q_0$ because of 4 (and 5); $4 \in P_1$ because of 2 (and 3), and $4 \in Q_1$ because of 6 (and 7); $5 \in P_1$ because of 3; $6 \in P_0$ because of 4; $7 \in P_0$ because of 4, and $7 \in Q_1$ because of 8; and $8 \in P_1$ because of 7. So, $P_0 = \{2,6,7\}$, $Q_0 = \{2,3\}$, $P_1 = \{4,5,8\}$ and $Q_1 = \{1,4,7\}$. $\square$

As last appreciations in terms of notation, when a max or $\sum$ expression appears along this dissertation acting over an empty set, we will consider 0 as default value in this situation.

Finally, in this thesis, the results will be expressed (and proved) in terms of evolution operators generated from general maxterms. That way, the dual results in terms of minterms (obtained by interchanging OR ($\vee$) by AND ($\wedge$) and the elements 0's (resp. 1's) by 1's (resp. 0's)) become automatically proved by the duality principle in Boolean algebras (see [24, 98]).

Chapter 3

Advances in Parallel Dynamical Systems

In this chapter, we perform a complete analysis of the dynamics in parallel dynamical systems on maxterm and minterm Boolean functions over undirected dependency graphs. The study is divided into two sections according to the methodology of research: dynamics of periodic orbits and dynamics of non-periodic orbits.

3.1 Dynamics of periodic orbits

In this section, the dynamics of periodic orbits are analyzed, solving the problems of existence and coexistence of them in $\mathrm{MAX}-\mathrm{PDS}$ and $\mathrm{MIN}-\mathrm{PDS}$, and obtaining an upper bound for their number. We specially focus on that cases in which there is a unique periodic orbit, which acts as the unique (global) attractor for the rest of orbits of the system.

3.1.1 Existence of periodic orbits

As a starting point in the study of the dynamics of periodic orbits in PDS, we begin by analyzing the orbital structure of a PDS on a general maxterm or minterm as evolution operator. Specifically, we study the type of periodic orbits that such a system can present.

Some previous works (see, for example, [28]) deal with this topic in the case of the simplest maxterm or minterm functions. Also, a generalization of these previous studies is given in [17], where it is shown that the only periodic orbits of a PDS on a general maxterm or minterm Boolean function are fixed points and periodic orbits of period 2.

Theorem 3.1 (Periodic structure of MAX$-$PDS)**.** *Let* $[G, \mathrm{MAX}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, all the periodic orbits of this system are fixed points or* 2*-periodic orbits, while the rest of the orbits are eventually fixed points or eventually periodic orbits.*

Proof. See [17]. □

Remark 3.1. The proof of Theorem 3.1 provides information about asymptotic behavior of the entities in a MAX $-$ PDS, which will be essential throughout this dissertation:

- Each $i \in W$ fixes its state value after a certain number of iterations.
- When a periodic orbit is reached, each $i \in W'_D$ has state value 1.
- The period comes from the evolution of the vertices belonging to W'_C.

□

Dually, we have the following theorem also proved in [17].

Theorem 3.2 (Periodic structure of MIN$-$PDS)**.** *Let* $[G, \mathrm{MIN}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, all the periodic orbits of this system are fixed points or* 2*-periodic orbits, while the rest of the orbits are eventually fixed points or eventually periodic orbits.*

As a direct consequence of these theorems, the following results for some special classes of maxterm and minterm Boolean functions can be established [17].

Corollary 3.1. *Let* $[G, \mathrm{OR}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* OR*. Then, all the periodic orbits of this system are fixed points. In fact, there are exactly two fixed points, namely,* $\mathcal{I}$ *and* $\mathcal{O}$*.*

Corollary 3.2. *Let* $[G, \mathrm{NAND}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND*. Then, all the periodic orbits of this system are* 2*-periodic orbits.*

Dually, we have the following results.

Corollary 3.3. *Let* $[G, \mathrm{AND}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* AND*. Then, all the periodic orbits of this system are fixed points. In fact, there are exactly two fixed points, namely,* $\mathcal{O}$ *and* $\mathcal{I}$*.*

Corollary 3.4. *Let* $[G, \mathrm{NOR}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR*. Then, all the periodic orbits of this system are* 2*-periodic orbits.*

3.1.2 Coexistence of periodic orbits

In Theorem 3.1 (resp. Theorem 3.2) of Subsection 3.1.1, it has been shown that the periodic orbits of a MAX − PDS (resp. MIN − PDS) are fixed points or 2-periodic orbits. Moreover, when the maxterm MAX (resp. minterm MIN) has all the variables in its direct form, then only (eventually) fixed points can appear, while if it has all the variables in its complemented form, then only (eventually) 2-periodic orbits are possible.

However, some important questions remained open. One of them consists in studying the coexistence of periodic orbits with different periods in the same sense of Sharkovsky's Theorem [102, 108] for PDS with general maxterm (resp. minterm) functions as evolution operators. In other words, it consists in guessing whether the existence of certain periods implies the appearance of other ones in a similar way to Sharkovsky's order. Concerning this question, next, we demonstrate that periodic orbits with different periods cannot coexist.

In order to do that, we describe the structure that a PDS must have in order to admit (eventually) fixed points. We will outline the reasonings for the case of a MAX − PDS, although all of them can be dually rewritten for a MIN − PDS.

Theorem 3.3. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, all the periodic orbits of this system are fixed points if, and only if,* $W'_C = \emptyset$*.*

Proof. In the case $W' = \emptyset$, the system has only two fixed points: $\mathcal{I}$ and $\mathcal{O}$ (see Corollary 3.1 in Subsection 3.1.1 and [28]). Let us analyze now the general case when $W' \neq \emptyset$:

First, assume that all the periodic orbits of this system are fixed points. Take $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ a fixed point, where $\hat{x}_i$ represents the (fixed) value of the vertex $i \in V$. Note that, for all $i \in W'$, it must be $\hat{x}_i = 1$. Otherwise (i.e., if $\hat{x}_i = 0$), it would change to 1 after the following iteration.

Suppose that there exists $i \in W'_C$. In such a case $A_G(i) \subseteq W'$ and so, for every $j \in A_G(i)$, $\hat{x}_j = \hat{x}_i = 1$. But this is not possible, since in the following iteration the value of i would change to 0, which is a contradiction.

To prove the converse implication, let us suppose that for all $i \in W'$, it is $W \cap A_G(i) \neq \emptyset$. We will write x_i^k to indicate the state value of the entity i after k iterations of the evolution operator MAX. Thus, let us consider an arbitrary initial value for the variables $(x_1^0, \ldots, x_n^0)$. Since the dependency graph is finite (and so is the state space), note that after a certain number of iterations, let us say $r \in \mathbb{N}$, the states of all the vertices belonging to W become fixed (see Remark 3.1 in Subsection 3.1.1). Let us take $i \in W'$ and let us prove that $x_i^{r+1} = 1$. In fact, let us suppose that $x_i^{r+1} = 0$ and take $j \in W \cap A_G(i)$. Then, it would be $x_j^{r+2} = 1 = x_j^r$ (since we are assuming that the value of j is fixed from the iteration r). But then, since $x_j^r = 1$, it must be $x_i^{r+1} = 1$, which is a contradiction. Thus, $x_i^{r+1} = 1$ for all $i \in W'$ and these state values do not change, as can be easily inferred from the reasoning above.

Therefore, all the variables of the system become fixed after $r+1$ iterations and the proof finishes. □

Dually, we have the following theorem.

Theorem 3.4. *Let* $[G, \mathrm{MIN}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, all the periodic orbits of this system are fixed points if, and only if,* $W'_C = \emptyset$.

In view of Theorems 3.3 and 3.4, we will call *fixed-point PDS* to a PDS which only presents fixed points as periodic orbits.

Since all the periodic orbits of the system are fixed points or 2-periodic orbits (see Theorem 3.1 of Subsection 3.1.1), as a consequence of Theorem 3.3 we have the following result.

Theorem 3.5. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, all the periodic orbits of this system are* 2*-periodic orbits if, and only if,* $W'_C \neq \emptyset$.

Proof. First, assume that all the periodic orbits of this system are 2-periodic orbits. If the theses were not true, that is, if for all $i \in W'$ it is $W \cap A_G(i) \neq \emptyset$, then from Theorem 3.3 we have that all the periodic orbits of this system are fixed points, which is a contradiction.

Conversely, let us suppose that there exists $i \in W'$ such that $W \cap A_G(i) = \emptyset$. If the system has a fixed point, reasoning as in the proof of Theorem 3.3, we get that every complemented entity is adjacent to a direct one, which is a contradiction. Hence, all the periodic orbits of the system must be 2-periodic orbits. □

Dually, we have the following theorem.

Theorem 3.6. *Let* $[G, \mathrm{MIN}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, all the periodic orbits of this system are* 2*-periodic orbits if, and only if,* $W'_C \neq \emptyset$.

In view of Theorems 3.5 and 3.6, we will call 2-*Periodic PDS* to a PDS which only presents periodic orbits of period 2.

Remark 3.2. Note that the cases when the evolution operator of the system is AND, OR, NAND and NOR, studied in [28], can be immediately obtained as particular cases of these theorems. □

As a direct consequence of Theorems 3.3 and 3.5, we get the main result of this subsection.

Corollary 3.5 (Coexistence of periods in $\mathrm{MAX}-\mathrm{PDS}$)**.** *Let* $[G, \mathrm{MAX}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, (eventually) fixed points and (eventually)* 2*-periodic orbits cannot coexist.*

And its dual version.

Corollary 3.6 (Coexistence of periods in $\mathrm{MIN}-\mathrm{PDS}$)**.** *Let* $[G, \mathrm{MIN}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, (eventually) fixed points and (eventually)* 2*-periodic orbits cannot coexist.*

3.1.3 Uniqueness of fixed points

Our main objective in this subsection is to obtain a Fixed-Point Theorem for PDS on maxterm and minterm Boolean functions. Observe that, although fixed points and 2-periodic orbits cannot coexist, there are PDS whose state spaces contain more than one fixed point, as shown in the following example.

Example 3.1. Let us consider the graph $G = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$, and let us take the evolution operator given by the maxterm

$$\mathrm{MAX} = x_1 \vee x_2' \vee x_3.$$

The fixed points of this PDS are $(1, 1, 0)$, $(0, 1, 1)$, and $(1, 1, 1)$. □

Thus, it would be desirable to find conditions to assure that the system has a unique fixed point. In order to do that, assume that the MAX − PDS has at least a fixed point, that is, $W_C' = \emptyset$ (see Theorems 3.3, 3.5 and Corollary 3.5 in Subsection 3.1.2).

Let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ be a fixed point of the system. As we have commented in Remark 3.1 in Subsection 3.1.1, for all $i \in W'$, $\hat{x}_i = 1$.

Furthermore, if j, $1 \leq j \leq p$, is such that $A_G(G_j) \cap W_1' \neq \emptyset$, then for all $i \in W$ in G_j, $\hat{x}_i = 1$. In order to see that, note that in the fixed point $\hat{x}$ all the vertices in G_j must be either activated or deactivated. Let us take $k \in A_G(G_j) \cap W_1'$. Then, if $\hat{x}_i = 0$ for all $i \in W$ in G_j, and since $\hat{x}_i = 1$ for all $i \in A_G(k) \cap W'$, then the state of k will change to 0 after the next iteration, which is a contradiction.

Therefore, if for every j, $1 \leq j \leq p$, $A_G(G_j) \cap W_1' \neq \emptyset$, then the system has a unique fixed point: $\mathcal{I}$.

On the other hand, if for a j, $1 \leq j \leq p$, $A_G(G_j) \cap W_1' = \emptyset$, then in a fixed point two situations are possible: either all the vertices in G_j are activated or all of them are deactivated. Regarding this last comment, we must point out that given $i \in W_2'$, if $G_{i_1}, \ldots, G_{i_{l(i)}}$, $2 \leq l(i) \leq p$, are the connected components adjacent to i, then not all these components can be deactivated simultaneously in the fixed point; otherwise, the value of x_i would change from 1 to 0 in the following iteration.

In view of the explanations above, we have proved the following result.

Theorem 3.7 (Fixed-Point Theorem for MAX − PDS)**.** *Let* $[G, \mathrm{MAX}]$ − PDS *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Assume that* $W_C' = \emptyset$. *Then, this* PDS *has a unique fixed point if, and only if, for every* j, $1 \leq j \leq p$, $A_G(G_j) \cap W_1' \neq \emptyset$. *In this situation, the unique fixed point is* $\mathcal{I}$, *and all the orbits converge to this fixed point.*

Dually, we have the following theorem.

Theorem 3.8 (Fixed-Point Theorem for MIN − PDS)**.** *Let* $[G, \mathrm{MIN}]$ − PDS *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Assume that* $W_C' = \emptyset$. *Then, this* PDS *has a unique fixed point if, and only if, for every* j, $1 \leq j \leq p$, $A_G(G_j) \cap W_1' \neq \emptyset$. *In this situation, the unique fixed point is* $\mathcal{O}$, *and all the orbits converge to this fixed point.*

3.1.4 Uniqueness of 2-periodic orbits

In Theorems 3.3, 3.5 and Corollary 3.5 of Subsection 3.1.2, it has been shown that all the periodic orbits of a MAX − PDS are 2-periodic orbits if, and only if, $W'_C \neq \emptyset$, i.e., there exists $i \in W'$ such that $W \cap A_G(i) = \emptyset$.

Next, we look for conditions to assure the uniqueness of a 2-periodic orbit in a MAX−PDS. In order to do that, we will use the following result where the particular case MAX = NAND is analyzed. As usual, we denote by K_n the complete graph of n vertices.

Proposition 3.1 (2-Periodic-Orbit Theorem for NAND − PDS). *Let* $[G, \mathrm{NAND}] -$ PDS *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND. *Then, there is a unique* 2*-periodic orbit if, and only if,* G *is a complete graph. In this situation, the unique* 2*-periodic orbit is* $\{\mathcal{O}, \mathcal{I}\}$*, and all the orbits of the system converge to this* 2*-periodic orbit.*

Proof. Firstly, note that according to the results quoted above (and independently of the graph G), the only periodic orbits of a NAND − PDS are 2-periodic orbits (see Corollary 3.2 in Subsection 3.1.1). Also, observe that the alternation of the configurations $\mathcal{I}/\mathcal{O}$ is always a 2-periodic orbit of the system. We will call this orbit the activated/deactivated 2-period. Thus, the proof consists in proving that the activated/deactivated 2-period is the unique 2-periodic orbit if, and only if, G is complete.

First, let us assume that $G = K_n$. We will denote by x_i^k the state value of the vertex i after k iterations of the evolution operator. Let us consider an arbitrary initial value for the variables $(x_1^0, \ldots, x_n^0)$.

- If $x_i^0 = 1$ for all $i \in V$, then $x_i^1 = 0$ for all $i \in V$ and we get the activated/deactivated 2-period.
- Otherwise, if $x_j^0 = 0$ for any $j \in V$, then $x_i^1 = 1$ for all $i \in V$ and, again, we get the activated/deactivated 2-period.

To prove the converse implication, we will show that if $G \neq K_n$, then there exists a 2-periodic orbit different from the activated/deactivated 2-period. Thus, assume that $G \neq K_n$ (which in particular implies that $n \geq 3$) and choose $i \in V$ such that $\overline{A_G(i)} \neq V$.

Let us consider the following initial values for the variables:

$x_j^0 = 1$ for all $j \in \overline{A_G(i)}$, and

$x_j^0 = 0$ for all $j \notin \overline{A_G(i)}$.

Then, we have

$x_j^1 = 0$ for all $j \in V$ such that $\overline{A_G(j)} \subseteq \overline{A_G(i)}$, and

$x_j^1 = 1$ for all $j \in V$ such that $\overline{A_G(j)} \nsubseteq \overline{A_G(i)}$.

And finally,

$x_j^2 = x_j^0$ for all $j \in V$.

Thus, we have found a 2-periodic orbit different from the activated/deactivated 2-period, which ends the proof. □

Dually, we have the following result.

Proposition 3.2 (2-Periodic-Orbit Theorem for NOR−PDS). *Let* $[G, \mathrm{NOR}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR. *Then, there is a unique* 2*-periodic orbit if, and only if,* G *is a complete graph. In this situation, the unique* 2*-periodic orbit is* $\{\mathcal{I}, \mathcal{O}\}$*, and all the orbits of the system converge to this* 2*-periodic orbit.*

Let us take $[G, \mathrm{MAX}] - \mathrm{PDS}$ a PDS over a dependency graph $G = (V, E)$ associated with the maxterm MAX, and let us assume $W_C' \neq \emptyset$, which means that all the periodic orbits of the system are 2-periodic orbits. For a 2-periodic orbit reached after r_0 iterations (see Remark 3.1 in Subsection 3.1.1):

- If $i \in W$, then $x_i^r = x_i^{r_0}$ for all $r \geq r_0$. In other words, the direct vertices do not change their state (activated or deactivated) from the r_0-th iteration on. Moreover, in each connected component G_j, $1 \leq j \leq p$, either all the variables associated with the vertices in G_j have state value 1 or all of them have state value 0.
- If $i \in W_D'$ (note that if $\mathrm{MAX} \neq \mathrm{NAND}$ and $W_C' \neq \emptyset$, then $W_D' \neq \emptyset$), it follows that $x_i^r = 1$ for all $r \geq r_0$. In fact, suppose that there exists $i \in W_D'$ such that (after the r_0-th iteration) x_i alternates the values 1 and 0, and assume (without loss of generality) that $x_i^{r_0} = 0$. Since $i \in W_D'$, there exists a direct

vertex j adjacent to i. Observe that it must be $x_j^{r_0} = 1$. Otherwise (i.e., if $x_j^{r_0} = 0$), it would be $x_j^{r_0+1} = x_i^{r_0+1} = 1$, which is not possible (we are assuming that the variables of all the direct vertices do not change their value after the r_0-th iteration). Then, being $x_j^{r_0} = 1$, we get that $x_i^r = 1$ for all $r > r_0$, which contradicts our initial assumption.

- The restriction of the system to each connected component C_j performs as a $\mathrm{NAND} - \mathrm{PDS}$.

In view of this, it should be pointed out that, as commented in Chapter 2 about G^*, this graph is not, in general, connected. Therefore, $\mathrm{MAX} - \mathrm{PDS}_{|G^*}$ may be understood as a set of independent PDS, in the sense that the evolution in each one only depends on the restriction of MAX to the connected component of G^* over which it is defined. In this setting, by saying that $\mathrm{MAX} - \mathrm{PDS}_{|G^*}$ has a unique fixed point, we mean that all of these independent PDS over the connected components of G^* (if more than one) have a unique fixed point.

Then, we have the following theorem.

Theorem 3.9 (2-Periodic-Orbit Theorem for $\mathrm{MAX} - \mathrm{PDS}$)**.** *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, this system has a unique* 2*-periodic orbit if, and only if, the following conditions are simultaneously satisfied:*

i) $W'_C \neq \emptyset$.

ii) The subgraph of G *generated by* W'_C *is complete.*

iii) Either $\mathrm{MAX} = \mathrm{NAND}$ *or* $\mathrm{MAX} - \mathrm{PDS}_{|G^*}$ *has a unique fixed point.*

Proof. In Proposition 3.1, we have already demonstrated the result when $\mathrm{MAX} = \mathrm{NAND}$. Thus, let us assume that $\mathrm{MAX} \neq \mathrm{NAND}$.

Under this assumption, let us suppose that the $\mathrm{MAX} - \mathrm{PDS}$ has a unique 2-periodic orbit. First, note that, from Theorems 3.3, 3.5 and Corollary 3.5 in Subsection 3.1.2, there exists $i \in W'$ such that $W \cap A_G(i) = \emptyset$, i.e., $W'_C \neq \emptyset$.

Moreover, we know that, once the 2-period has been reached, the restriction of the system to each connected component C_j, $1 \leq j \leq q$, performs as a $\mathrm{NAND} - \mathrm{PDS}$. Then, as the system has a unique 2-period, it must be $q = 1$. Consequently, from Proposition 3.1, C_1 (which coincides with the subgraph of G generated by W'_C) must be a complete graph.

Finally, let us see that $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ has a unique fixed point. Note that for the system $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ over G^*, the subset of direct vertices coincides with the one of $\mathrm{MAX}-\mathrm{PDS}$, W. On the other hand, the subset of complemented vertices in $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ is W'_D. In particular, since in G^* every complemented vertex is adjacent to a direct vertex, all the periodic orbits of $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ are fixed points.

Observe that, given a fixed point of $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$, we can construct a 2-periodic orbit of $\mathrm{MAX}-\mathrm{PDS}$ by fixing the values of the variables associated with the vertices in $W \cup W'_D$ as in such fixed point, and considering the activated/deactivated 2-period for the variables associated with the vertices in W'_C. Hence, since $\mathrm{MAX}-\mathrm{PDS}$ has a unique 2-periodic orbit, it follows that $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ has a unique fixed point.

Conversely, assume that $W'_C \neq \emptyset$, the subgraph of G generated by W'_C is complete, and $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ has a unique fixed point.

Firstly, since $W'_C \neq \emptyset$, the only periodic orbits of $\mathrm{MAX}-\mathrm{PDS}$ are 2-periodic orbits (see Theorems 3.3, 3.5 and Corollary 3.5 of Subsection 3.1.2). On the other hand, we know that the restriction of MAX to the subgraph of G generated by W'_C performs as a $\mathrm{NAND}-\mathrm{PDS}$ once the 2-periodic orbit has been reached. Then, since such a subgraph is complete, from Proposition 3.1, we get that the restriction of a 2-periodic orbit of the system $\mathrm{MAX}-\mathrm{PDS}$ to this subgraph is the activated/deactivated 2-period. We also know that in a 2-periodic orbit of $\mathrm{MAX}-\mathrm{PDS}$, all the variables associated with the vertices in W'_D fix their values to 1. To finish the proof, we will see that if $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ has a unique fixed point, then in a 2-period of the system $\mathrm{MAX}-\mathrm{PDS}$ all the variables associated to the vertices in W fix their values to 1.

Observe that the connected components which result in G^* when we remove all its complemented vertices (i.e., the vertices in W'_D) and the edges which are incident to those vertices coincide with the ones for G, i.e., they are $G_1, \ldots, G_p$.

Since $\mathrm{MAX}-\mathrm{PDS}_{|G^*}$ has a unique fixed point, from Theorem 3.7 in Subsection 3.1.3, we have that, for every j, $1 \leq j \leq p$, there exists $i_j \in A_{G^*}(G_j) \cap W'_D$ such that $i_j \notin A_{G^*}(G_k)$ for $k \neq j$.

Reasoning by reduction to the absurd, suppose that for a 2-periodic orbit of $\mathrm{MAX}-\mathrm{PDS}$ there exists a vertex $i \in W$ whose variable fixes its value to 0. Let G_j be the connected component containing i, and take $i_j \in A_{G^*}(G_j) \cap W'_D$ such that $i_j \notin A_{G^*}(G_k)$ for $k \neq j$. In particular, since $i_j \in W'_D$, its associated variable fixes its value to 1 in the 2-periodic orbit. Regarding the variables associated with the vertices in G_j, all of them fix their values to 0 in the 2-periodic orbit. Recall also that the variables associated with the vertices in W'_C alternates the states: all of them activated/all of them deactivated. In particular, for an iteration in which all the variables associated with the vertices in W'_C are activated, we have that the variables associated with direct vertices adjacent to i_j are deactivated, and the variables associated with complemented vertices adjacent to i_j are activated. But it implies that the value of the variable x_{i_j} would change from 1 to 0 in the next iteration, which is a contradiction. Therefore, all the variables associated with the vertices in W fix their values to 1 in the 2-periodic orbit and the proof finishes. □

Dually, we have the following result.

Theorem 3.10 (2-Periodic-Orbit Theorem for MIN − PDS)**.** *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, this system has a unique* 2*-periodic orbit if, and only if, the following conditions are simultaneously satisfied:*

i) $W'_C \neq \emptyset$.

ii) The subgraph of G *generated by* W'_C *is complete.*

iii) Either $\mathrm{MIN} = \mathrm{NOR}$ *or* $\mathrm{MIN} - \mathrm{PDS}_{|G^*}$ *has a unique fixed point.*

3.1.5 Maximum number of fixed points

In Theorems 3.3, 3.5 and Corollary 3.5 of Subsection 3.1.2, it has been proved that all the periodic orbits of a $\mathrm{MAX} - \mathrm{PDS}$ are fixed points if, and only if, $W'_C = \emptyset$, i.e., $W \cap A_G(i) \neq \emptyset$ for all $i \in W'$. Moreover, from Theorem 3.7 in Subsection 3.1.3 we know that there is a unique fixed point if, and only if, $A_G(G_j) \cap W'_1 \neq \emptyset$ for all j, $1 \leq j \leq p$.

Throughout this subsection, we will deal with a $\mathrm{MAX} - \mathrm{PDS}$ over a dependency graph G such that $W \cap A_G(i) \neq \emptyset$ for all $i \in W'$, i.e., such that its periodic orbits are only fixed points. Our aim is to find an upper bound for the number of fixed points, taking into account the adjacency structure of G and the maxterm MAX.

In the simplest case, when $W' = \emptyset$ (i.e., $\mathrm{MAX} = \mathrm{OR}$), there are two fixed points: $\mathcal{I}$ and $\mathcal{O}$ (see Corollary 3.1 in Subsection 3.1.1 and [28]).

Hence, let us assume that $W' \neq \emptyset$. Recall that, if $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ is a fixed point, where $\hat{x}_i$ represents the (fixed) state value of the vertex $i \in V$, then $\hat{x}_i = 1$ for all $i \in W'$ (see Remark 3.1 in Subsection 3.1.1 for the details). Moreover, for each j, $1 \leq j \leq p$, two situations are possible: either all the vertices in G_j are activated or all of them are deactivated in the fixed point. Regarding this last comment, note that not all the vertices in W can be deactivated simultaneously; otherwise, the values of the variables associated to the vertices in W' would change from 1 to 0 in the following iteration.

At this point, we have the following theorem.

Theorem 3.11. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then:*

- *If* $W' = \emptyset$ *(i.e.,* $\mathrm{MAX} = \mathrm{OR}$*), there are exactly two fixed points:* $\mathcal{I}$ *and* $\mathcal{O}$.
- *If* $W' \neq \emptyset$ *and* $W \cap A_G(i) \neq \emptyset$ *holds for all* $i \in W'$ *(i.e., the periodic orbits are only fixed points), then there are at most* $2^p - 1$ *fixed points, being* p *the number of connected components which result from* G *when we remove all the vertices in* W' *and the edges which are incident to those vertices.*

Dually, we have the following result.

Theorem 3.12. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then:*

- *If* $W' = \emptyset$ *(i.e.,* $\mathrm{MIN} = \mathrm{AND}$*), there are exactly two fixed points:* $\mathcal{O}$ *and* $\mathcal{I}$.
- *If* $W' \neq \emptyset$ *and* $W \cap A_G(i) \neq \emptyset$ *holds for all* $i \in W'$ *(i.e., the periodic orbits are only fixed points), then there are at most* $2^p - 1$ *fixed points, being* p *the number of connected components which result from* G *when we remove all the vertices in* W' *and the edges which are incident to those vertices.*

Next, we construct a PDS associated with a particular maxterm MAX where the upper bound obtained in Theorem 3.11 is attained. A dual example of a $\mathrm{MIN} - \mathrm{PDS}$ can be similarly constructed where the upper bound obtained in Theorem 3.12 is attained.

Example 3.2. Let us consider the star graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, $n = p + 1 \geq 2$, and $E = \{\{1, i\} : i = 2, \ldots, n\}$, and take the PDS over G whose evolution operator is the maxterm:

$$\mathrm{MAX} = x_1' \vee x_2 \vee \cdots \vee x_n.$$

For this system, $W' = \{1\}$ and there are $n - 1 = p$ connected components $G_j = \{\{j + 1\}, \emptyset\}$, $1 \leq j \leq p$. It can be easily checked that this system has $2^p - 1$ fixed points which result by fixing $x_1 = 1$ and taking whichever combination for the values of the variables x_i, $2 \leq i \leq n$, except the one with $x_2 = \cdots = x_n = 0$. □

3.1.6 Maximum number of 2-periodic orbits

In this subsection, we establish an upper bound for the number of 2-periodic orbits of a PDS with a Boolean maxterm (resp. minterm) as evolution operator.

To begin with, we analyze the particular case when MAX = NAND (resp. MIN = NOR).

Proposition 3.3. *Let* $[G, \mathrm{NAND}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND. *Let* $\mathcal{P}(V)$ *be the power set of* V *and*

$$\Theta = \{\overline{A_G(Q)} : Q \in \mathcal{P}(V)\}.$$

Then, the number of 2*-periodic orbits of the system is* $|\Theta|/2$.

Proof. Let $x^0 = (x_1^0, \ldots, x_n^0)$ be a configuration of state values for the variables associated with the vertices in $V = \{1, \ldots, n\}$, and consider the sets V_0 and V_1 associated to x^0:

$$V_0 = \{i \in V : x_i^0 = 0\},$$
$$V_1 = \{i \in V : x_i^0 = 1\}.$$

We will see that x^0 is a 2-periodic point of $\mathrm{NAND} - \mathrm{PDS}$ if, and only if, $V_1 \in \Theta$ (i.e., if, and only if, there exists $Q \in \mathcal{P}(V)$ such that $\overline{A_G(Q)} = V_1$).

Firstly, assume that x^0 is a 2-periodic point of $\mathrm{NAND} - \mathrm{PDS}$. Let x^1 be the configuration after an iteration and take $Q = \{i \in V : x_i^1 = 0\}$. Then, taking into account that after another iteration $x^2 = x^0$, we have that $\overline{A_G(Q)} = V_1$.

Conversely, assume that for the configuration x^0 there exists $Q \in \mathcal{P}(V)$ such that $\overline{A_G(Q)} = V_1$.

If $Q = \emptyset$, then $V_1 = \overline{A_G(Q)} = \emptyset$, i.e., all the entities are deactivated in x^0 and so it is a 2-periodic point (part of the activated/deactivated 2-period). If $Q \neq \emptyset$, after one iteration:

- If $i \in V_1 = \overline{A_G(Q)}$, we distinguish two cases:
  - If $i \in Q$, then $x_i^1 = 0$ since $x_j^0 = 1$ for all $j \in \overline{A_G(i)}$.
  - If $i \in \overline{A_G(Q)} \setminus Q$, then
    - either $x_i^1 = 1$, provided that i is adjacent to any vertex $j \in V_0 = V \setminus \overline{A_G(Q)}$,

◇ or $x_i^1 = 0$, provided that $\overline{A_G(i)} \cap V_0 = \emptyset$.

- If $i \in V_0$, then $x_i^1 = 1$.

Now, it is straightforward to check that $x_i^2 = x_i^0$ after another iteration for all $i \in V$, namely, x^0 is a 2-periodic point.

The proof finishes by observing that every 2-periodic orbit consists of two (distinct) 2-periodic points. □

Dually, we have the following result.

Proposition 3.4. *Let* $[G, \mathrm{NOR}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR. *Let* $\mathcal{P}(V)$ *be the power set of* V *and*

$$\Theta = \{\overline{A_G(Q)} : Q \in \mathcal{P}(V)\}.$$

Then, the number of 2*-periodic orbits of the system is* $|\Theta|/2$.

Thanks to Proposition 3.3 (resp. Proposition 3.4), we can obtain an upper bound for the number of 2-periodic orbits of a NAND − PDS (resp. NOR − PDS) over a dependency graph $G = (V, E)$, which only depends on the number n of vertices of V.

Proposition 3.5. *Let* $[G, \mathrm{NAND}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND. *Then, the number of* 2*-periodic orbits of the system is, at most,* $\max\{1, 2^{n-2}\}$.

Proof. The cases $n = 1$ and $n = 2$ follow directly by observing that G is complete (see Proposition 3.1 in Subsection 3.1.4) and therefore there is a unique 2-periodic orbit. Thus, let us assume that $n \geq 3$.

Let us take $T \in \Theta$, where Θ is the subset of $\mathcal{P}(V)$ defined in Proposition 3.3. Then, there exists $Q \in \mathcal{P}(V)$ such that $\overline{A_G(Q)} = T$.

Let us define $P_i = V \setminus \overline{A_G(i)}$. Namely, a vertex $j \in V$ belongs to P_i if $j \neq i$ and it is not adjacent to i. Then, it can be easily checked that $V \setminus T = \cap_{j \in Q} P_j$ if $T \neq \emptyset$. For our purposes, we will take $\cap_{j \in \emptyset} P_j = V \setminus \emptyset = V$.

Consequently, if T belongs to Θ, we define

$$\Omega = \{\cap_{j \in Q} P_j : Q \in \mathcal{P}(V)\} \ni V \setminus T.$$

Note that $|\Theta| = |\Omega|$. Let us see that, for $n \geq 3$, $|\Omega| \leq 2^{n-1}$.

Since G is connected, there exist $i_1, i_2 \in V$ which are adjacent, and so $i_1 \notin P_{i_2}$ and $i_2 \notin P_{i_1}$.

Let us take $\Gamma = \{Q \in \mathcal{P}(V) : i_1 \in Q \text{ or } i_2 \in Q\}$. Then, $|\Gamma| = (3/4)\, 2^n$. Moreover, for $Q \in \Gamma$, we have that $i_1, i_2 \notin \cap_{j\in Q} P_j$, which means that

$$|\{\cap_{j\in Q} P_j : Q \in \Gamma\}| \leq \frac{1}{4} 2^n.$$

Therefore, considering that $|\mathcal{P}(V) \setminus \Gamma| = (1/4)2^n$,

$$|\Theta| = |\Omega| = |\{\cap_{j\in Q} P_j : Q \in \Gamma\}| + |\{\cap_{j\in Q} P_j : Q \in \mathcal{P}(V) \setminus \Gamma\}| \leq \frac{1}{4}2^n + \frac{1}{4}2^n = 2^{n-1}.$$

This inequality, jointly with Proposition 3.3, allow us to get our thesis. □

Dually, we have the following result.

Proposition 3.6. *Let* $[G, \mathrm{NOR}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR. *Then, the number of* 2*-periodic orbits of the system is, at most,* $\max\{1, 2^{n-2}\}$.

Next, we construct a $\mathrm{NAND} - \mathrm{PDS}$ where the upper bound obtained in Proposition 3.5 is attained. A dual example for a $\mathrm{NOR} - \mathrm{PDS}$ can be similarly constructed where the upper bound obtained in Proposition 3.6 is attained.

Example 3.3. Let us consider the start graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, $n \geq 1$, $E = \{\{1, i\} : i = 2, \ldots, n\}$, and take the $\mathrm{NAND} - \mathrm{PDS}$ over V.

This system has $\max\{2, 2^{n-1}\}$ 2-periodic points:

- If $n = 1$, the configurations $\mathcal{I}$ and $\mathcal{O}$.
- If $n \geq 2$:
  - Whichever configuration of values with $x_1^0 = 1$, except the one with $x_j^0 = 0$ for all $j \neq 1$. That is, $2^{n-1} - 1$ configurations.
  - The configuration with $x_j^0 = 0$ for all $j \in V$.

Therefore, the system has 2^{n-2} 2-periodic orbits. □

In view of this, we are able to obtain an upper bound for the number of 2-periodic orbits of a PDS over a dependency graph $G = (V, E)$ associated with an arbitrary maxterm MAX.

Let n_j be the number of vertices of C_j, $1 \leq j \leq q$. In particular, if $p = 0$ then $\mathrm{MAX} = \mathrm{NAND}$, $q = 1$, and we are under the assumptions of Proposition 3.5. Thus, the number of 2-periodic orbits of the system is, at most, $\max\{1, 2^{n_1-2}\}$.

We also know that, if x^0 is a 2-periodic point of the system, then for all $i \in W'_D$, $x_i^0 = 1$ (and so $x_i^r = 1$ for all $r \in \mathbb{N}$). Moreover, the restriction of the system to each connected component G_j (resp. C_j) performs as an $\mathrm{OR} - \mathrm{PDS}$ (resp. a $\mathrm{NAND} - \mathrm{PDS}$) once a 2-periodic orbit has been reached. In particular, in each connected component G_j, $1 \leq j \leq p$, either all the vertices in G_j are activated or all of them are deactivated in the 2-periodic point, which leads to 2^p combinations.

Finally, observe that the configuration where all the direct vertices are deactivated and all the complemented vertices are activated is not a 2-periodic point, since under these assumptions the variables associated with vertices in W'_D would change their values to 0 after an iteration, which is not possible as remarked above.

Therefore, using also Proposition 3.5, we have the following theorem proved.

Theorem 3.13. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX, $\mathrm{MAX} \neq \mathrm{NAND}$, *such that* $W'_C \neq \emptyset$. *Then, all the periodic orbits of the system are* 2*-periods and its number is, at most,*

$$\left(2^p \frac{\prod_{j=1}^{q} \max\{2, 2^{n_j-1}\}}{2} \right) - 1.$$

Observe that, in Theorem 3.13, we need that the maxterm is different from NAND, since otherwise $p = 0$, $q = 1$ and this upper bound is not available for the number of 2-periodic orbits of the system (see Proposition 3.5 and Example 3.3).

Dually, we have the following result.

Theorem 3.14. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN, $\mathrm{MIN} \neq \mathrm{NOR}$, *such that* $W'_C \neq \emptyset$. *Then, all the periodic orbits of the system are* 2*-periods and its number is, at most,*

$$\left(2^p \frac{\prod_{j=1}^{q} \max\{2, 2^{n_j-1}\}}{2} \right) - 1.$$

To finish, in the following example, we construct a $\mathrm{MAX}-\mathrm{PDS}$ where the upper bound obtained in Theorem 3.13 is attained. A dual example of a $\mathrm{MIN}-\mathrm{PDS}$ can be similarly constructed where the upper bound obtained in Theorem 3.14 is attained.

Example 3.4. Let us fix $p > 0$, $q > 0$, $n_1, \ldots, n_q > 0$, and consider the following sets of vertices:

- $W = \{d_1, \ldots, d_p\}$.
- $W'_D = \{c_1\}$.
- $W'_{C_j} = \{c_{j1}, \ldots, c_{jn_j}\}$ for $j = 1, \ldots, q$.

Then, we take as vertex set of the dependency graph $V = W \cup W'_D \cup \left(\cup_{j=1}^{q} W'_{C_j}\right)$.

Regarding the adjacencies among these vertices, we take the edges:

- $\{d_i, c_1\}$ for all $1 \leq i \leq p$.
- $\{c_{jk}, c_1\}$ for all $1 \leq j \leq q$ and $1 \leq k \leq n_j$.
- $\{c_{j1}, c_{jk}\}$ for all $1 \leq j \leq q$ and $1 \leq k \leq n_j$.

All these edges form the edge set E.

Over this dependency graph $G = (V, E)$, we consider the maxterm MAX whose directed variables are the ones associated with the vertices in W and whose complemented variables are those associated with the vertices in $V \setminus W$.

In this PDS, we have p connected components $G_1, \ldots, G_p$, given by $G_i = \{\{d_i\}, \emptyset\}$ for each $i \in \{1, \ldots, p\}$, and q connected components $C_1, \ldots, C_q$, given by $C_j = \left(W'_{C_j}, \{\{c_{j1}, c_{jk}\} : 1 \leq k \leq n_j\}\right)$ for each $j \in \{1, \ldots, q\}$.

The system $\mathrm{MAX}-\mathrm{PDS}$ over G constructed above has exactly

$$\left(2^p \frac{\prod_{j=1}^{q} \max\{2, 2^{n_j-1}\}}{2}\right) - 1.$$

2-periodic orbits, as can be reasoned taking into account that every connected component C_j is a star graph like the one described in Example 3.3, and so the restriction of the PDS to C_j has $\max\{1, 2^{n_j-2}\}$ 2-periodic orbits. $\square$

3.2 Dynamics of non-periodic orbits

In this last section of the chapter, we study the dynamics of non-periodic orbits until a periodic orbit is reached.

Firstly, we study the existence and uniqueness of predecessor, what naturally leads us to explore the field of the Garden-of-Eden configurations of the system.

After that, we expose results about attractiveness of periodic orbits, basins of attraction, and we analyze the maximum number of iterations needed to ensure that any non-periodic point reaches a periodic one.

3.2.1 Predecessors and GOE configurations

The study of predecessors in network models is usually divided into four more specific problems [31, 32]:

- Predecessor existence problem (PRE): Determining whether a predecessor exists for a given state.
- Unique predecessor problem (UPRE): Determining whether a predecessor is the unique one for a given state.
- Coexistence of predecessors problem (APRE): Determining whether a predecessor is not unique for a given state.
- Number of predecessors problem (#PRE): Counting the number of predecessors of a given state, in case of non-uniqueness.

In this subsection, we solve the first one in the context of PDS on maxterm and minterm Boolean functions. This allows us to get also a characterization of the GOE of such systems. These results lead us to describe the structure of the potential predecessors of a given state, what allows us to give results to solve the rest of the problems in the mentioned context.

In order to solve the PRE problem, in the next theorem, we provide sufficient and necessary conditions to know when a certain configuration y is the successor of another configuration x, i.e., when y has at least a predecessor. These conditions are expressed in simple and direct terms, becoming an agile and fast procedure to determine whether a specific state of the entities has predecessors in a particular PDS.

Theorem 3.15. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, a configuration has a predecessor if, and only if, every activated entity adjacent to a deactivated one in such a configuration is also adjacent to an activated entity which is not adjacent to any deactivated one.*

In other words, a configuration y *has a predecessor if, and only if,* $\overline{A_G(A^c)} = V_1$, *being* $A = \overline{A_G(V_0)}$.

Proof. This constructive proof generates a predecessor state x of a current state y, whenever possible, highlighting those conditions of y under which the existence of a predecessor x is impossible.

In this case, we can split the set V_1 associated to y into two subsets, corresponding to the vertices adjacent to some vertices in V_0, $A_G^*(V_0)$, and the vertices which are not adjacent to any vertex in V_0, $\overline{A_G(V_0)}^c$, that is,

$$V_1 = A_G^*(V_0) \cup \overline{A_G(V_0)}^c.$$

Suppose, by reduction to the absurd, that there exists a configuration y which has a predecessor x, but one of the (activated) entities k such that $y_k = 1$, which is adjacent to one of V_0, i.e., $k \in A_G^*(V_0)$ is not adjacent to any entity in $\overline{A_G(V_0)}^c$.

Observe that if $i \in V_0$, then for every entity $j \in \overline{A_G(i)}$ it must be:

- $x_j = 0$ when $j \in W$, and
- $x_j = 1$ when $j \in W'$,

since otherwise, $y_i = 1$ and $i \notin V_0$. In particular, this occurs for every $j \in \overline{A_G(V_0)}$.

In such a context, since $k \in A_G^*(V_0) \subset V_1$ is not adjacent to any entity in $\overline{A_G(V_0)}^c$, it would be $y_k = 0$, what is a contradiction.

Reciprocally, if, in a given configuration y, every activated entity adjacent to a deactivated one, is also adjacent to an activated entity which is not adjacent to any deactivated one, then, to get a predecessor, x, of the given configuration y, it should be sufficient to take x as follows:

- For every entity $j \in \overline{A_G(V_0)}$
  - $x_j = 0$ when $j \in W$, and

    - $x_j = 1$ when $j \in W'$.
- For every entity $j \in \overline{A_G(V_0)}^c$
    - $x_j = 1$ when $j \in W$, and
    - $x_j = 0$ when $j \in W'$.

□

Remark 3.3. Observe that, in the conditions of existence of predecessors in Theorem 3.15, each activated entity adjacent to a deactivated one acts as an articulation node between deactivated entities and activated entities which are not adjacent to any deactivated one. That is, in terms of the proof, the vertices in $A^*_G(V_0)$ act as connectors between vertices in V_0 and vertices in $\overline{A_G(V_0)}^c$. □

Dually, we have the following theorem.

Theorem 3.16. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, a configuration has a predecessor if, and only if, every deactivated entity adjacent to an activated one in such a configuration is also adjacent to a deactivated entity which is not adjacent to any activated one.*

In other words, a configuration y *has a predecessor if, and only if,* $\overline{A_G(A^c)} = V_0$, *being* $A = \overline{A_G(V_1)}$.

Remark 3.4. Dually to the case in Theorem 3.15, in the conditions of Theorem 3.16, each deactivated entity adjacent to an activated one acts as an articulation node between activated entities and deactivated entities which are not adjacent to any activated one. That is, in terms of the proof, the vertices in $A^*_G(V_1)$ act as connectors between vertices in V_1 and vertices in $\overline{A_G(V_1)}^c$. □

Theorems 3.15 and 3.16 solve the PRE problem for PDS on maxterm and minterm Boolean functions and allow us to establish a characterization of GOE states of such systems.

Corollary 3.7 (Characterization of GOE in MAX − PDS). *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, a configuration is a* GOE *if, and only if, there exists an activated entity adjacent to a deactivated one in such a configuration, but not adjacent to an activated entity which is not adjacent to any deactivated one.*

In other words, a configuration y *is a* GOE *if, and only if,* $\overline{A_G(A^c)} \neq V_1$, *being* $A = \overline{A_G(V_0)}$.

Corollary 3.8 (Characterization of GOE in MIN $-$ PDS)**.** *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, a configuration is a* GOE *if, and only if, there exists a deactivated entity adjacent to an activated one in such a configuration, but not adjacent to a deactivated entity which is not adjacent to any activated one.*

In other words, a configuration y *is a* GOE *if, and only if,* $\overline{A_G\left(A^c\right)} \neq V_0$*, being* $A = \overline{A_G\left(V_1\right)}$*.*

In particular, taking this characterization into account, one can discover particular cases of GOE, and bounds for the number of them.

Corollary 3.9. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$*, with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$*, associated with the maxterm* MAX. *A state of the system with only one activated entity has no predecessors and, consequently, this configuration is a* GOE. *Also, the configurations* $\mathcal{O}$ *and* $\mathcal{I}$ *are never* GOE *so, the number of* GOE *points of the system,* #GOE*, is such that*

$$n \leq \#\mathrm{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

Proof. The classification as GOE or not GOE of these configurations is directly obtained from Theorem 3.15 and Corollary 3.7. Thus, we must only prove that these bounds are reachable, as it happens in the following example:

Let us consider the only PDS with $n = 2$ entities over the maxterm NAND. In this case, there are 4 possible configurations: $\mathcal{O}$ and $\mathcal{I}$, which are not GOE points (in fact, they belong to a 2-cycle), and, on the other hand, $(0, 1)$ and $(1, 0)$, which are GOE points of the system. □

And now its dual version.

Corollary 3.10. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$*, with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$*, associated with the minterm* MIN. *A state of the system with only one deactivated entity has no predecessors and, consequently, this configuration is a* GOE. *Also, the configurations* $\mathcal{I}$ *and* $\mathcal{O}$ *are never* GOE *so, the number of* GOE *points of the system,* #GOE*, is such that*

$$n \leq \#\mathrm{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

Remark 3.5. In Corollary 3.9 (resp. Corollary 3.10), $n \geq 2$ has been imposed. This is necessary because a $[G, \mathrm{MAX}] - \mathrm{PDS}$ (resp. $[G, \mathrm{MIN}] - \mathrm{PDS}$) with $n = 1$ has 2 fixed points, if $W' = \emptyset$, or one 2-cycle, if $W = \emptyset$. That is, all the orbits are periodic and, consequently, it has not GOE points in any case. □

The proof of Theorem 3.15 is constructive and provides information about the structure of a predecessor of a given state of the system, when it exists. This information is collected in the following two results.

Corollary 3.11. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *If a configuration* y *has a predecessor state* x*, such a predecessor has the following structure:*

- *If* $y_i = 0$*, then, for every entity* $j \in \overline{A_G(i)}$*:*
    - $x_j = 0$ *when* $j \in W$*, and*
    - $x_j = 1$ *when* $j \in W'$*.*
- *If* $y_i = 1$*, then there exists an entity* $j \in \overline{A_G(i)}$ *such that it accomplishes one of the following conditions:*
    - $x_j = 1$ *with* $j \in W$*, or*
    - $x_j = 0$ *with* $j \in W'$*.*

Proof. We can see it in the constructive process shown to prove Theorem 3.15. □

Remark 3.6. In terms of Corollary 3.11, in the case of existence of predecessor for a state y, there is always a configuration x corresponding to a predecessor, that we will call *fundamental predecessor* of y. This configuration, which is proposed in the (second part of the) proof of Theorem 3.15, is as follows:

- If $i \in \overline{A_G(V_0)}$, then:
    - $x_i = 0$ when $i \in W$, and
    - $x_i = 1$ when $i \in W'$.
- If $i \in \overline{A_G(V_0)}^c$, then:
    - $x_i = 1$ with $i \in W$, and
    - $x_i = 0$ with $i \in W'$.

Thus, to know if a configuration y has a predecessor, we only need to verify if the corresponding configuration x, candidate to be its fundamental predecessor, is such that $\mathrm{MAX}(x) = y$. $\square$

Dually, we have the following corollary.

Corollary 3.12. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *If a configuration* y *has a predecessor state* x*, such a predecessor has the following structure:*

- *If* $y_i = 1$*, then, for every entity* $j \in \overline{A_G(i)}$*:*
    - $x_j = 1$ *when* $j \in W$*, and*
    - $x_j = 0$ *when* $j \in W'$*.*
- *If* $y_i = 0$*, then there exists an entity* $j \in \overline{A_G(i)}$ *such that it accomplishes one of the following conditions:*
    - $x_j = 0$ *with* $j \in W$*, or*
    - $x_j = 1$ *with* $j \in W'$*.*

Remark 3.7. As before, in terms of Corollary 3.12, in the case of existence of predecessors for a state y, there is always a configuration x corresponding to a predecessor, that we will call *fundamental predecessor* of y. This configuration is as follows:

- If $i \in \overline{A_G(V_1)}$, then:
    - $x_i = 1$ when $i \in W$, and
    - $x_i = 0$ when $i \in W'$.
- If $i \in \overline{A_G(V_1)}^c$, then:
    - $x_i = 0$ with $i \in W$, and
    - $x_i = 1$ with $i \in W'$.

Thus, to know if a configuration y has a predecessor, as before, we only need to verify if the corresponding configuration x, candidate to be its fundamental predecessor, is such that $\mathrm{MIN}(x) = y$. $\square$

Observe that the entities whose state values are 0 (resp. 1) in the configuration y determine univocally their state values and the state values of their adjacent ones in any predecessor x with respect to $\mathrm{MAX}-\mathrm{PDS}$ (resp. $\mathrm{MIN}-\mathrm{PDS}$), when such a predecessor exits. Nevertheless, for any entity whose state value is 1 (resp. 0) in the configuration y, it is only necessary the existence of an appropriate adjacent one which provides such a value with the $\mathrm{MAX}-\mathrm{PDS}$ (resp. $\mathrm{MIN}-\mathrm{PDS}$) updating. This points out how to look for the solution to the UPRE, APRE and #PRE problems in our context.

The following results are concerned with the determination of the possible predecessors of a given configuration y, once we know that at least one predecessor x exists. As a consequence, we solve the UPRE and APRE problems for PDS on maxterm and minterm Boolean functions.

Theorem 3.17. *Let $[G, \mathrm{MAX}]-\mathrm{PDS}$ be a parallel dynamical system over a dependency graph $G=(V,E)$ associated with the maxterm MAX. Let y be a configuration and suppose that it has a predecessor. Then, the predecessor of y is not unique if, and only if, there exists an activated entity $i \in \overline{A_G(V_0)}^c$ such that one of its adjacent entities also belong to $\overline{A_G(V_0)}^c$ and the rest of its adjacent ones, if any, are also adjacent to other entities in $\overline{A_G(V_0)}^c$.*

In other words, the predecessor of y is not unique if, and only if, there exists and entity $i \in \overline{A_G(V_0)}^c$ such that $\overline{A_G(A^c \setminus \{i\})}=V_1$, being $A=\overline{A_G(V_0)}$.

Proof. First of all, suppose that there exists such an activated entity $i \in \overline{A_G(V_0)}^c$ in the configuration y which is not adjacent to any deactivated one, such that one of its adjacent entities also belong to $\overline{A_G(V_0)}^c$ and the rest of its adjacent ones, if any, are also adjacent to other entities in $\overline{A_G(V_0)}^c$. Remember that, since y has a predecessor, the entities in $\overline{A_G(V_0)}$ at y determine univocally their state values in any predecessor with respect to $\mathrm{MAX}-\mathrm{PDS}$. On the other hand, we can act similarly as in the (second part of the) proof of Theorem 3.15 and for every entity $j \in \overline{A_G(V_0)}^c$, $j \neq i$, to construct a predecessor configuration, we can take as follows:

- $x_j = 1$ when $j \in W$, and
- $x_j = 0$ when $j \in W'$.

Now, taking into account that some of the entities which are adjacent to i also belong to $\overline{A_G(V_0)}^c$ and the rest of its adjacent ones, if any, are also adjacent to other entities in $\overline{A_G(V_0)}^c$, i and its adjacent vertices become activated, independently of the state value of i. That is, in such a predecessor construction, we can choose either $x_i = 0$ or $x_i = 1$, so obtaining two different configurations which are predecessors of y.

Reciprocally, suppose that the configuration y has more than one predecessor. Again, the entities in $\overline{A_G(V_0)}$ in the configuration y determine univocally their state values in any predecessor with respect to MAX $-$ PDS. Thus, the discrepancies should be in the state values of entities belonging to $\overline{A_G(V_0)}^c$, that is, activated and not adjacent to any deactivated one. Suppose that there is a discrepancy of two predecessors in the state values of an entity $i \in \overline{A_G(V_0)}^c$. This means that the entity i and its adjacent ones become activated in y independently of the state value of i in such predecessors. Therefore, there should exist activated entities belonging to $\overline{A_G(V_0)}^c$ which are adjacent to i and its adjacent ones in $A^*_G(V_0)$ to provide that all of them have state value equal to 1 in the configuration y. □

Remark 3.8. In terms of Theorem 3.17, given a state y, if the configuration x defined as in Remark 3.6 is its (fundamental) predecessor, to know if this is its unique predecessor, we must only verify if y has a predecessor belonging to the following set:

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\ i \in \overline{A_G(V_0)}^c \textit{ such that } \hat{x}_i \neq x_i \textit{ and } \hat{x}_j = x_j\ \forall j \in V \setminus \{i\}\}.$$

This result reduces an initial exponentially-sized problem, that is, the search of a particular configuration among the 2^n possible states of the system, into another one in which, at most, n cases must be analyzed. In this case, a short list of possible candidates is provided and the evaluation of the evolution operator only over the elements of this set provides the answer to the global problem of existence of a unique predecessor for the state y. □

Dually, we have the following theorem.

Theorem 3.18. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Let* y *be a configuration and suppose that it has a predecessor. Then, the predecessor of* y *is not unique if, and only if, there exists a deactivated entity* $i \in \overline{A_G(V_1)}^c$ *such that one of its adjacent entities also belong to* $\overline{A_G(V_1)}^c$ *and the rest of its adjacent ones, if any, are also adjacent to other entities in* $\overline{A_G(V_1)}^c$.

In other words, the predecessor of y *is not unique if, and only if, there exists and entity* $i \in \overline{A_G(V_1)}^c$ *such that* $\overline{A_G(A^c \setminus \{i\})} = V_0$*, being* $A = \overline{A_G(V_1)}$.

Remark 3.9. As for the MAX $-$ PDS case, in terms of Theorem 3.18 for MIN $-$ PDS, given a state y, if the configuration x defined as in Remark 3.7 is its (fundamental) predecessor, to know if this is its unique predecessor, we must only verify if y has a predecessor belonging to the following set:

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\, i \in \overline{A_G(V_1)}^c \textit{ such that } \hat{x}_i \neq x_i \textit{ and } \hat{x}_j = x_j\ \forall j \in V \setminus \{i\}\}.$$

As before, this result reduces the search of a particular configuration among the 2^n possible states of the system to, at most, n cases. □

Once the existence of more than one predecessor is known, the following step is to try to obtain the number of them for any given state. In the next two corollaries, we explain how to obtain theoretically the set of all of them and, consequently, its number, in order to solve the classical predecessor problem #PRE.

Corollary 3.13. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Let* y *a configuration and*

$$P_i = \{x \textit{ state} : x \textit{ satisfies the conditions in Corollary } 3.11 \textit{ for } y_i\}.$$

Then, $P = \bigcap_{i \in V} P_i$ *is the set of all the predecessor states of* y.

Dually, we have the following corollary.

Corollary 3.14. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Let* y *be a configuration and*

$$P_i = \{x \textit{ state} : x \textit{ satisfies the conditions in Corollary } 3.12 \textit{ for } y_i\}.$$

Then, $P = \bigcap_{i \in V} P_i$ *is the set of all the predecessor states of* y.

In the case of $\mathrm{MAX} - \mathrm{PDS}$ (resp. $\mathrm{MIN} - \mathrm{PDS}$), the configuration $\mathcal{O}$ (resp. $\mathcal{I}$) has always a unique predecessor, by Theorems 3.15 and 3.17 (resp. Theorems 3.16 and 3.18). However, the calculus of the number of predecessors for a general state of the entities different from these ones depends on the connections among the entities in the particular system. As traditionally done in other contexts, we have been able to get a bound for the number of predecessors of a general configuration, which is given in the following theorem.

Theorem 3.19. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, the number of predecessors of a given configuration* y *different from* $\mathcal{O}$ *is upper bounded by* $2^{\#\overline{A_G(V_0)}^c} - 1$. *In fact, such a bound is reachable.*

Proof. From Theorems 3.15 and 3.17, it is clear that the possible discrepancies between predecessors correspond to differences in state values of entities in $\overline{A_G\left(V_0\right)}^c$. Since the state values of any of these entities are either 0 or 1, a first bound for the number of predecessors is $2^{\#\overline{A_G(V_0)}^c}$.

However, if $\overline{A_G\left(V_0\right)}^c = \emptyset$, as we are assuming that $y \neq \mathcal{O}$, at least one of the entities has state value 1 and, by Theorem 3.15, y has not predecessors, while if $\overline{A_G\left(V_0\right)}^c \neq \emptyset$, a configuration such that:

- $x_i = 0$ when $i \in W \cap \overline{A_G\left(V_0\right)}^c$, and
- $x_i = 1$ when $i \in W' \cap \overline{A_G\left(V_0\right)}^c$,

cannot be a predecessor of y. Thus, in any case, one configuration must be discarded from the previous bound.

In fact, such a bound $2^{\#\overline{A_G(V_0)}^c} - 1$ is reachable. It is sufficient to consider a $[G, \mathrm{MAX}] - \mathrm{PDS}$ such that the subgraph corresponding to $\overline{A_G\left(V_0\right)}^c$ is complete and with the condition that each entity in such a set is adjacent to all the entities in $A^*_G\left(V_0\right)$. $\square$

Dually, we have the following theorem.

Theorem 3.20. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, the number of predecessors of a given configuration* y *different from* $\mathcal{I}$ *is upper bounded by* $2^{\#\overline{A_G(V_1)}^c} - 1$*. In fact, such a bound is reachable.*

3.2.2 Convergence to periodic orbits: attractors, global attractors, basins of attraction and transient

The study performed above helps us to give some results concerning the attractive character of fixed points and 2-periodic orbits of any PDS with a maxterm or a minterm Boolean function as global evolution operator, their basins of attraction and the *transient* (or *width*) of the system.

Attractive and repulsive periodic orbits

The concepts of attractive and repulsive periodic orbit have already been introduced in this thesis in Chapter 2, in Definition 2.4. In this context, for the determination of attractors, one can find some precedents in the literature, where the problem is set out from the point of view of constructing a (numerical) algorithm to find these attractors (see [4, 55, 75]).

It should be noted that, for a periodic orbit, the concept of attractiveness is equivalent to have at least one of the states of the periodic orbit with a predecessor different from the one that it has in such a periodic orbit. That is, a periodic orbit is attractive if one of the states of the orbit has at least two predecessors. In particular, a fixed point with an additional predecessor different from itself is an attractive fixed point, while a fixed point without more predecessors than itself is a repulsive one. Note that, in this last case, the repulsive fixed point cannot be considered a GOE, although no other different state converges to it. Thus, we can state the following results that characterize the attractive or repulsive character.

Theorem 3.21. *Let* $[G, \mathrm{MAX}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G=(V,E)$ *associated with the maxterm* MAX. *Then, a periodic orbit is attractive if, and only if, for a state* y *of such a periodic orbit, there exists an activated entity* $i \in \overline{A_G(V_0)}^c$, *such that one of its adjacent entities also belong to* $\overline{A_G(V_0)}^c$ *and the rest of its adjacent ones, if any, are also adjacent to other entities in* $\overline{A_G(V_0)}^c$.

In other words, a periodic orbit is attractive if, and only if, for a state y *of such a periodic orbit, there exists and entity* $i \in \overline{A_G(V_0)}^c$ *such that* $\overline{A_G(A^c \setminus \{i\})} = V_1$, *being* $A = \overline{A_G(V_0)}$.

Proof. Since a periodic orbit is attractive if one of the states in such an orbit has at least two predecessors, the result is a consequence of Theorem 3.17 in Subsection 3.2.1, where the existence of non-unique predecessors is characterized for this type of PDS. □

Dually, we have the following result.

Theorem 3.22. *Let* $[G, \mathrm{MIN}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G=(V,E)$ *associated with the minterm* MIN. *Then, a periodic orbit is attractive if, and only if, for a state* y *of such a periodic orbit, there exists a deactivated entity* $i \in \overline{A_G(V_1)}^c$, *such that one of its adjacent entities also belong to* $\overline{A_G(V_1)}^c$ *and the rest of its adjacent ones, if any, are also adjacent to other entities in* $\overline{A_G(V_1)}^c$.

In other words, a periodic orbit is attractive if, and only if, for a state y *of such a periodic orbit, there exists and entity* $i \in \overline{A_G(V_1)}^c$ *such that* $\overline{A_G(A^c \setminus \{i\})} = V_0$, *being* $A = \overline{A_G(V_1)}$.

Observe that, when a PDS presents a unique fixed point, this is globally attractive. That is, the rest of the orbits of the system converges to such a fixed point. Thus, we can state the conditions that characterize globally attractive fixed points for this class of PDS.

Theorem 3.23. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, this* PDS *has a globally attractive fixed point if, and only if,* $W'_C = \emptyset$ *and* $A_G(G_j) \cap W'_1 \neq \emptyset$ *for every* j, $1 \leq j \leq p$. *In this situation, the globally attractive fixed point is* $\mathcal{I}$.

Proof. Since a fixed point of a $\mathrm{MAX} - \mathrm{PDS}$ is globally attractive if, and only if, it is the unique fixed point of such a system, the result is a consequence of Theorem 3.7 in Subsection 3.1.3, which is, indeed, a Fixed-Point Theorem for this kind of PDS. □

Dually, we have the following theorem for $\mathrm{MIN} - \mathrm{PDS}$.

Theorem 3.24. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, this* PDS *has a globally attractive fixed point if, and only if,* $W'_C = \emptyset$ *and* $A_G(G_j) \cap W'_1 \neq \emptyset$ *for every* j, $1 \leq j \leq p$. *In this situation, the globally attractive fixed point is* $\mathcal{O}$.

A similar situation occurs when a PDS presents a unique 2-periodic orbit, being such an orbit globally attractive. Thus, we can state the conditions that characterize globally attractive 2-periodic orbits for this class of PDS.

Theorem 3.25. *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, this system has a globally attractive* 2*-periodic orbit if, and only if, the following conditions are simultaneously satisfied:*

i) $W'_C \neq \emptyset$.

ii) The subgraph of G *generated by* W'_C *is complete.*

iii) Either $\mathrm{MAX} = \mathrm{NAND}$ *or* $\mathrm{MAX} - \mathrm{PDS}_{|G^*}$ *has a unique fixed point.*

Proof. Since a 2-periodic orbit of a $\mathrm{MAX} - \mathrm{PDS}$ is globally attractive if, and only if, it is the unique 2-periodic orbit of such a system, the result is a consequence of Theorem 3.9 in Subsection 3.1.4, where it is proved that the conditions *i*), *ii*) and *iii*) allow us to assure this uniqueness. □

Theorem 3.26. *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, this system has a globally attractive* 2*-periodic orbit if, and only if, the following conditions are simultaneously satisfied:*

i) $W'_C \neq \emptyset$.

ii) The subgraph of G *generated by* W'_C *is complete.*

iii) Either $\mathrm{MIN} = \mathrm{NOR}$ *or* $\mathrm{MIN} - \mathrm{PDS}_{|G^*}$ *has a unique fixed point.*

Other interesting dynamical situation occurs when a PDS presents more than one periodic orbit, but only one of such periodic orbits is attractive. In such a case, every non-periodic orbit converges to the unique attractive periodic one, which could be said to be *quasi-globally attractive.* Quasi-globally attractive equilibria often appear in experimental models as, for example, epidemiological dynamical models, where the endemic equilibrium attracts all the states of the systems except the disease free equilibrium (see, for instance, [26]). In our setting, this situation also occurs. For instance, in the case of a PDS on the maxterm OR (resp. minterm AND), there is always two fixed points, namely, $\mathcal{I}$ and $\mathcal{O}$. As is well known ([28]), $\mathcal{I}$ (resp. $\mathcal{O}$) is a quasi-globally attractive fixed point for such $\mathrm{OR} - \mathrm{PDS}$ (resp. $\mathrm{AND} - \mathrm{PDS}$), since every non-periodic orbit converges to it.

Basin of attraction for attractive periodic orbits

Observe that periodic orbits act as organizational kernels of the dynamics of a PDS, since every state finally reaches one of such periodic orbits. In this sense, their basins of attraction (see Definition 2.5 in Chapter 2) allow us to describe the phase diagram as much as possible, fractionating it into the different trees that reach the corresponding periodic orbits.

In particular, the basin of attraction of a repulsive periodic orbit is the empty set. A mechanism for obtaining the basin of attraction of any attractive periodic orbit is to get all the predecessors of such a periodic orbit, proceeding as explained in Corollaries 3.13 and 3.14 of Subsection 3.2.1.

Dynamical concepts as attractiveness and basins of attraction of periodic orbits highlight the importance of determining the set of GOE states, whose characterization can be seen in Corollaries 3.7 and 3.8 of Subsection 3.2.1. These states are the beginning of a branch in the tree constituting a basin of attraction associated with an attractive periodic orbit. For this reason, they are crucial in order to establish the different basins of attraction. Actually, all the orbits are periodic (and, consequently, repulsive), if the PDS does not present GOE states. But, in view of Corollaries 3.9, 3.10 and Remark 3.5 of Subsection 3.2.1, this is not possible except in the trivial case corresponding to only one entity. Thus, we can state the following corollary.

Corollary 3.15. *Every homogeneous* PDS *on a maxterm or minterm Boolean function with more than one entity has attractors.*

Transient to a fixed point in PDS

When studying basins of attraction in general, one of their interesting dynamic features is the width of them, i.e., the maximum number of iterations needed by an eventually periodic orbit to reach its corresponding periodic orbit, as we do hereafter.

As is well known, all the periodic orbits in a PDS on a maxterm or a minterm Boolean function are either fixed points or 2-periodic orbits, while the rest of orbits are eventually fixed points or eventually 2-periodic orbits (see Theorems 3.1 and 3.2 of Subsection 3.1.1). In particular, the simplest maxterm OR and minterm AND, which present only fixed points, are studied in [28], showing that the maximum number of iterations needed by an eventually fixed point to reach the corresponding fixed point is, at most, the diameter of the dependency graph.

Regarding the more general context of PDS on general maxterm and minterm Boolean functions, recall that fixed points and 2-periodic orbits cannot coexists (see Corollaries 3.5 and 3.6 of Subsection 3.1.2). Due to that, we have to distinguish between these two cases. Actually, we study now the transient of the non-periodic orbits to fixed points.

Lemma 3.1. *Let* $[G, \mathrm{MAX}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, a vertex* $i \in W_D'^{\alpha}$ *takes permanent state value* 1 *at a maximum of* $m_i + 2$ *iterations, being* $m_i = \mathrm{card}\left(A_G(i) \cap W'\right)$.

Proof. Consider a vertex $i \in W'$ such that $A_G(i) \cap W \neq \emptyset$ and $A_G(i) \cap W' \subseteq A_G(W)$. Then, we can have only one of the following two possibilities:

- $\forall t \geq 0$, $x_i^t = 1$. Thus, i takes permanent state value 1 at 0 iterations. Since $0 < 2 \leq 2 + m_i$, the result is proved in this case.

- $\exists T \geq 0$ such that $x_i^T = 0$, being the iteration T the first time that the variable x_i takes the value 0. In this situation, after the next iteration $T+1$, we have $x_i^{T+1} = 1$ and $x_j^{T+1} = 1$ $\forall j \in A_G(i) \cap W \neq \emptyset$.

  Due to that, any $j \in A_G(i) \cap W \neq \emptyset$ makes i take permanent state value 1 from this iteration on.

  In particular, if $T \leq 1$, then i takes permanent state value 1 after, at most, 2 iterations. Since $2 \leq 2 + m_i$, the result is proved also in this case.

  Thus, suppose that $T > 1$. At this point, it must be $\forall t < T$, $x_i^t = 1$ and $x_j^t = 0$ $\forall j \in A_G(i) \cap W$.

  But, in such a situation, to make i have state value 1 for every $0 < t < T$, $\exists k \in A_G(i) \cap W' \subseteq A_G(W)$ such that $x_k^{t-1} = 0$ to provide that $x_i^t = 1$. After iteration t, the vertex k takes permanent state value 1.

  Since there are m_i vertices in $A_G(i) \cap W'$, the maximum number of iterations that x_i can maintain its value equal to 1, before changing to 0, is m_i. That is, after at most m_i iterations, every $k \in A_G(i) \cap W'$ takes permanent state value 1. This makes i have state value 0 after the next iteration, what means that $T \leq m_i + 1$. Consequently, $T + 1 \leq m_i + 2$, what proves the result in this last case.

$\square$

Theorem 3.27 (Transient in Fixed-Point $\mathrm{MAX} - \mathrm{PDS}$). *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*, where the structure of* G *only allows fixed points as periodic orbits. Then, every state of the system reaches a fixed point after a maximum of*

$$\max_{k \in \{1,\dots,p\}} \{\mathrm{diam}\,(G_k)\} + \max_{i \in W'} \{m_i + 2\}$$

iterations, being $m_i = \mathrm{card}\,(A_G(i) \cap W')$*,* $\forall i \in W'$*.*

Proof. First of all, observe that $p > 0$, because $p = 0$ implies that $\mathrm{MAX} = \mathrm{NAND}$ and this system does not allow fixed points, as it can be seen in Corollary 3.2 of Subsection 3.1.1.

In particular, if $W' = \emptyset$, then $\mathrm{MAX} = \mathrm{OR}$ and G is the only connected subgraph resulting from the elimination in G of the vertices belonging to W' and their incident edges. Then, as proved in [28], the system converges to a fixed point after, at most, $\mathrm{diam}\,(G_k)$ iterations, what fits with the expression of the upper bound.

Thus, suppose that $W' \neq \emptyset$ (and $p > 0$). Recall that, as proved in Theorems 3.3, 3.5 and Corollary 3.5 of Subsection 3.1.2, a MAX − PDS only presents fixed points as periodic orbits if, and only if, every complemented vertex is adjacent to a direct vertex. For this reason, Lemma 3.1 can be applied to every $i \in W'$.

Taking this into account, after $\max_{i\in W'}\{m_i + 2\}$ iterations, every $i \in W'$ has permanent state value 1. Therefore, after such a number of iterations, the complemented vertices neither change their state value nor affect the state of other vertices. Thus, the study of the evolution of the system can be reduced to analyze what happens in the restriction to the subgraph induced by $V \setminus W'$.

That is, at this point, the behavior of the entire system can be obtained from the study of the evolution in each connected subgraph $G_1, \ldots, G_p$. Since they only have vertices associated with direct variables, we know that each local system restricted to each G_k, $k \in \{1, \ldots, p\}$, converges to a fixed point, and it takes, at most, $\mathrm{diam}\,(G_k)$ iterations to reach it (see [28]). In view of this, in at most $\max_{k\in\{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\}$ iterations, after the $\max_{i\in W'}\{m_i + 2\}$ iterations needed for ensuring that the state values of the vertices in W' are fixed, every vertex in $V \setminus W'$ reaches a state value which will not change anymore.

Therefore, after at most

$$\max_{k\in\{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\} + \max_{i\in W'}\{m_i + 2\}$$

iterations, the system reaches a fixed point of the MAX − PDS. □

Theorem 3.28 (Transient in Fixed-Point MIN − PDS)**.** *Let* $[G, \mathrm{MIN}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*, where the structure of* G *only allows fixed points as periodic orbits. Then, every state of the system reaches a fixed point after a maximum of*

$$\max_{k\in\{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\} + \max_{i\in W'}\{m_i + 2\}$$

iterations, being $m_i = \mathrm{card}\,(A_G\,(i) \cap W')$, $\forall i \in W'$*.*

Observe that, depending on the conditions of each parallel dynamical system, some max expressions in Theorems 3.27 and 3.28 can be taken over an empty set. Along this dissertation, we consider 0 as default value in these situations.

Theorem 3.27 gives an upper bound for the transient or width of a fixed-point MAX − PDS. Actually, this upper bound is the best possible, since it is reachable, as we show in the example below. A dual example in the case of a minterm MIN as evolution operator can be similarly constructed, where the upper bound obtained in Theorem 3.28 is reached.

Example 3.5. Let us consider the fixed-point MAX $-$ PDS over the line graph $G = (V, E)$, with $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$ (see Figure 3.1) on the maxterm Boolean function given by $MAX = x_1' \vee x_2 \vee x_3$.

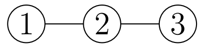

Figure 3.1: Graph $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$.

In this case, according to the notation in Theorem 3.27:

- $W = \{2, 3\}$ and $W' = \{1\}$ are the sets of vertices whose corresponding variables appear in MAX in direct and complemented form, respectively.
- There is $p = 1$ connected component which results from G when the only complemented vertex and the only edge which incides to it are removed,$G_1 = (\{2, 3\}, \{\{2, 3\}\})$. In this case, $\operatorname{diam}(G_1) = 1$.
- There are $m_1 = 0$ vertices in W' adjacent to the only element in W'.

Associated to this PDS, consider the initial (global) configuration: $(1, 0, 0)$. Then, its orbit consists of the states:

$$(1, 0, 0) \rightarrow (0, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1).$$

That is, the orbit starting at $(1, 0, 0)$ reaches the fixed point $(1, 1, 1)$ after $3 = \operatorname{diam}(G_1) + m_1 + 2$ iterations, and therefore the upper bound in Theorem 3.27 is reached.

In Figure 3.2, we show the phase diagram of this PDS, where it can be easily checked that 3 is its transient.

$\square$

In addition, we are able to provide a pattern that can be considered to obtain a PDS on a maxterm Boolean function for which any given (optimal upper bound) transient is reached. A dual construction can be considered to achieve a pattern in the case of PDS on minterm Boolean functions.

Example 3.6. Let us consider the following collection of parallel dynamical systems $\mathfrak{F} = \{\mathrm{PDS}_k : k \in \mathbb{N}\}$ where each PDS_k is defined as follows:

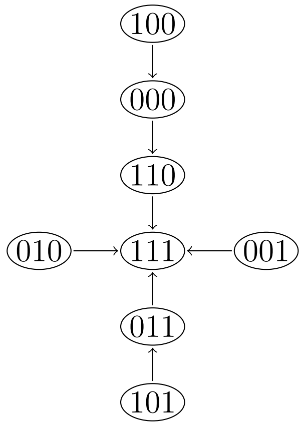

Figure 3.2: Phase portrait of the system $[(\{1,2,3\},\{\{1,2\},\{2,3\}\}), x_1' \vee x_2 \vee x_3] - \mathrm{PDS}$.

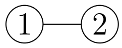

Figure 3.3: Graph $G_1 = (\{1,2\},\{\{1,2\}\})$.

- $k = 1$: PDS_1 is defined as the parallel dynamical system over $G_1 = (V_1, E_1)$ with $V_1 = \{1,2\}$, $E_1 = \{\{1,2\}\}$ (see Figure 3.3),

  and whose evolution operator is the maxterm $\mathrm{MAX}_1 = x_1' \vee x_2$.

  In this case, according to the notation in Theorem 3.27:

  - $W_1 = \{2\}$ and $W_1' = \{1\}$ are the sets of vertices whose corresponding variables appear in MAX_1, respectively, in direct and complemented form.
  - There is $p_1 = 1$ connected component which results from G_1 when the only complemented vertex and the only edge which are incident to it are removed, $G_{1,1} = (\{2\}, \emptyset)$. In this case, $\operatorname{diam}(G_{1,1}) = 0$.
  - There are $m_{1,1} = 0$ vertices in W_1' adjacent to the only element in W_1'.

- $k = 2$: PDS_2 is defined as the parallel dynamical system over $G_2 = (V_2, E_2)$ with $V_2 = \{1,2,3,4\}$, $E_2 = \{\{1,2\},\{3,4\},\{1,3\}\}$ (see Figure 3.4),

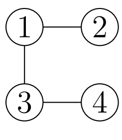

Figure 3.4: Graph $G_2 = (\{1,2,3,4\},\{\{1,2\},\{3,4\},\{1,3\}\})$.

and whose evolution operator is the maxterm $\mathrm{MAX}_2 = x_1' \vee x_2 \vee x_3' \vee x_4$. According to the notation in Theorem 3.27:

- $W_2 = \{2,4\}$ and $W_2' = \{1,3\}$ are the sets of vertices whose corresponding variables appear in MAX_2, respectively, in direct and complemented form.
- There are $p_2 = 2$ connected components which result from G_2 when all the vertices in W_2' and the edges which are incident to them are removed, $G_{2,1} = (\{2\}, \emptyset)$ and $G_{2,2} = (\{4\}, \emptyset)$. In this case, $\operatorname{diam}(G_{2,i}) = 0$ for $i = 1, 2$.
- For all $i \in W_2'$, $m_{2,i} = \operatorname{card}(A_{G_2}(i) \cap W_2') = 1$.

- $k \geq 3$: PDS_k is recursively defined as the parallel dynamical system over $G_k = (V_k, E_k)$ with $V_k = V_{k-1} \cup \{2k-1, 2k\}$, $E_k = E_{k-1} \cup \{\{2k-1, 2k\}\} \cup \{\{i, 2k-1\} : i \in W_{k-2}'\}$ and whose evolution operator is the maxterm $\mathrm{MAX}_k = \mathrm{MAX}_{k-1} \vee x_{2k-1}' \vee x_{2k}$. For example, some particular cases are:

  For $k = 3$, see Figure 3.5,

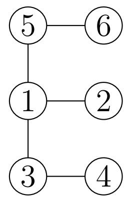

Figure 3.5: Graph $G_3 = (V_3, E_3) = (\{1,2,3,4,5,6\}, \{\{1,2\}, \{3,4\}, \{1,3\}, \{5,6\}, \{1,5\}\})$.

$$\mathrm{MAX}_3 = x_1' \vee x_2 \vee x_3' \vee x_4 \vee x_5' \vee x_6.$$

For $k = 4$, see Figure 3.6,

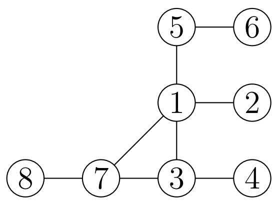

Figure 3.6: Graph $G_4 = (V_4, E_4) = (\{1, \ldots, 8\}, E_3 \cup \{\{7,8\}, \{1,7\}, \{3,7\}\})$.

$$\mathrm{MAX}_4 = x_1' \vee x_2 \vee x_3' \vee x_4 \vee x_5' \vee x_6 \vee x_7' \vee x_8.$$

And for $k = 5$, see Figure 3.7,

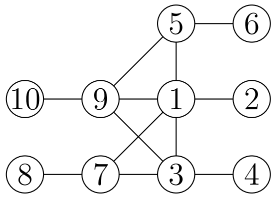

Figure 3.7: Graph $G_5 = (\{1, \ldots, 10\}, E_4 \cup \{\{i, 9\} : i \in \{1, 3, 5, 10\}\})$.

$$\mathrm{MAX}_5 = x_1' \vee x_2 \vee x_3' \vee x_4 \vee x_5' \vee x_6 \vee x_7' \vee x_8 \vee x_9' \vee x_{10}.$$

According to the notation in Theorem 3.27:

- $W_k = W_{k-1} \cup \{2k\}$ and $W_k' = W_{k-1}' \cup \{2k-1\}$ are the sets of vertices whose corresponding variables appear in MAX_k, respectively, in direct and complemented form.
- There are $p_k = k$ connected components which result from G_k when all the vertices in W_k' and the edges which are incident to them are removed, $G_{k,i} = (\{2i\}, \emptyset)$ for $i = 1, \ldots, k$. In this case, $\operatorname{diam}(G_{k,i}) = 0$ for $i = 1, \ldots, k$.
- Finally, $m_{k,i} = m_{k-1,i} + 1$ for each $i \in W_{k-2}'$, $m_{k,2(k-1)-1} = m_{k-1,2(k-1)-1}$ and $m_{k,2k-1} = k - 2$.

Associated to these parallel dynamical systems, let us consider the initial state values of the variables:

- $x_1^0 = \left(x_{1,1}^0, x_{1,2}^0\right) = (1, 0)$.
- $x_k^0 = \left(x_{k,1}^0, \ldots, x_{k,2k}^0\right)$ for $k \geq 2$, with $x_{k,i}^0 = 1$, if $i \in W_{k-1}'$, and $x_{k,i}^0 = 0$ in other case. For example, some particular cases:

  $x_2^0 = (1, 0, 0, 0)$.

  $x_3^0 = (1, 0, 1, 0, 0, 0)$.

  $x_4^0 = (1, 0, 1, 0, 1, 0, 0, 0)$.

  $x_5^0 = (1, 0, 1, 0, 1, 0, 1, 0, 0, 0)$.

Conditions above ensure, for all $k \in \mathbb{N}$:

- $m_{k,2k-1} \leq k-1$ (initialization condition).
- $m_{k+1,i} \leq m_{k,i}+1$ for all $i \in W'_k$ (propagation condition).

which implies that $m_{k,i} \leq k-1$ for all $i \in W'_k$.

On the other hand,

- $m_{1,1} = 0$.
- $m_{k+1,1} = m_{k,1}+1$.

Therefore, $m_{k,1} = k-1$ and then, $\max_{i \in W'_k}\{m_{k,i}\} = m_{k,1} = k-1$.

Now, let us see that the valoration x_k^0 in PDS_k reaches a fixed point in, exactly, $k+1$ iterations, value of the upper bound in Theorem 3.27 for this PDS. In fact, note that:

- x_1^0 reaches a fixed point in PDS_1 after 2 iterations.
- x_{k+1}^0 reaches a fixed point in PDS_{k+1} after one iteration more than x_k^0 in PDS_k, for all $k \in \mathbb{N}$.

The simplest case ($k=1$) is direct, since the evolution of x_1^0 in PDS_1 is:

$$(1,0) \to (0,0) \to (1,1)\,.$$

Regarding the general case, it follows directly by observing that:

- $x_{k+1,2(k+1)-1}^1 = x_{k+1,2(k+1)}^1 = 1$, and they continue activated onwards.
- $x_{k+1,i}^1 = x_{k,i}^0$ for all $i \in V_k$, and the restriction of PDS_{k+1} to G_k evolves as PDS_k onwards.

When the component $\max_{k \in \{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\}$ in Theorem 3.27 is greater than 0, the upper bound can be also reached. To see that, let us take the parallel dynamical system PDS_k and refurbish it as follows:

- We join a line graph of length l adjacent to vertex 2 in G_k. Thus, $\mathrm{diam}\,(G_{k,1}) = l$.

- The new evolution operator is such that its restriction to G_k is MAX_k and its restriction to the added line graph is the maxterm OR.

Then, if we fix the initial state values as above for the vertices in G_k and as 0 for the vertices in the added line graph, the system reaches a fixed point (the one with all the vertices activated) in $l + k + 1$ iterations. $\square$

Remark 3.10. Observe that the upper bound for a fixed-point PDS on the simplest maxterm OR (resp. on the simplest minterm AND), which presents only fixed points, is given by the diameter of the dependency graph (see [28]). This previous result is not valid for general maxterm and minterm Boolean functions, as can be seen in Example 3.5 where $\text{diam}\,(G) = 2 < 3$, being 3 the transient of the system. This reveals the relevance of our extended result, given by Theorems 3.27 and 3.28, due to the breakdown found in the upper bound of the transient for general PDS. $\square$

Transient to 2-periodic orbits in PDS

By Theorems 3.3, 3.5 and Corollary 3.5 of Subsection 3.1.2, we know that a $\text{MAX} - \text{PDS}$ only presents 2-periodic orbits if, and only if, there exists a complemented vertex which is not adjacent to a direct vertex. In [28], it is shown that the transient for PDS on NAND or NOR Boolean functions is 1. Here, we extend these results to the case of PDS on a general maxterm or minterm Boolean function.

Lemma 3.2. *Let* $[G, \text{MAX}] - \text{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*, where the structure of* G *only allows* 2*-periodic orbits. Then, a vertex* $i \in W_D'^{\beta}$ *takes permanent state value* 1 *at a maximum of* $2s_i + 3$ *iterations, being* $s_i = \text{card}\,(A_G\,(i) \cap W_D')$*.*

Proof. Consider a vertex $i \in W_D'^{\beta}$. As in the proof of Lemma 3.1, we can have only one of the following two possibilities:

- $\forall t \geq 0$, $x_i^t = 1$, i.e., i takes permanenet state value 1 after 0 iterations. Since $0 < 3 \leq 3 + 2s_i$, the result is proved in this case.

- $\exists T \geq 0$ such that $x_i^T = 0$, being the iteration T the first time that the variable x_i takes the value 0. In this situation, after the next iteration $T + 1$, we have $x_i^{T+1} = 1$ and $x_j^{T+1} = 1\ \forall j \in A_G\,(i) \cap W \neq \emptyset$.

  Due to that, any $j \in A_G\,(i) \cap W \neq \emptyset$ makes i take permanent state value 1 from this iteration on.

In particular, if $T \leq 2$, then i takes permanent state value 1 after, at most, 3 iterations. Since $3 \leq 3 + 2s_i$, the result is proved also in this case.

Finally, suppose that $T > 2$. At this point, it must be $\forall t < T$, $x_i^t = 1$ and $x_j^t = 0\ \forall j \in A_G(i) \cap W$.

But, in this situation, to make i have state value 1 after every iteration t, $1 \leq t < T$, $\exists k \in A_G(i) \cap W'$ such that $x_k^{t-1} = 0$, what provides that $x_i^t = 1$. Now, we have also two possibilities:

- If $k \in W'_D$, after the iteration t, $x_k^t = 1$ and $x_j^t = 1\ \forall j \in A_G(k) \cap W$. Due to that, any $j \in A_G(k) \cap W \neq \emptyset$ makes k take permanent state value 1 from this iteration on. That is, the vertex k does not influence the state value of the vertex i anymore.
- If $k \in W'_C$, after the iteration t, $x_k^t = 1$ and $x_j^t = 1\ \forall j \in A_G(k) \subseteq W'$. Consequently, $x_k^{t+1} = 0 = x_k^{t-1}$, repeating this alternation of state values after every two iterations. In particular, observe that, in such a case, the vertex k does not influence the state of the vertex i for the intermediate iteration $t+1$.

In view of this, once a vertex $k \in A_G(i) \cap W'_D$ provides the state value 1 for the vertex i, it cannot provide it anymore. On the other hand, once a vertex $k \in A_G(i) \cap W'_C$ provides state value 1 for the vertex i after the iteration t, in the next iteration $t+1$, another vertex $l \in A_G(i) \cap W'$ with state value 0 is needed in order to keep it. That is, in the following iteration, without any other vertex in $A_G(i) \cap W'_D$ with state value equals 0, the state value of x_i becomes 0, or 1 definitively if there exists $l \in A_G(i) \cap W'_C$ such that $x_l^t = 0$ while $x_k^{t-1} = 0$.

Therefore, the maximum number of iterations that x_i can maintain its value equal to 1, before changing to 0 for the first time, can be obtained intercalating the influences of vertices of both sets $A_G(i) \cap W'_D$ and $A_G(i) \cap W'_C$, that is, vertices with state values equal to 0.

In particular if $s_i = \mathrm{card}(A_G(i) \cap W'_D) = 0$, and we assume (without loos of generality) that $x_i^0 = 1$.

- If every vertex in $A_G(i) \cap W'_C = A_G(i) \cap W'$ has state value equals 1, then $x_i^1 = 0$, being $1 + 1 \leq 3 + 2 \cdot 0$.
- If there is a vertex $k \in A_G(i) \cap W'_C$ such that $x_k^0 = 0$, then, after the first iteration, we can have:

- ◇ The state values of the rest of the vertices $l \in A_G(i) \cap W'_C$ become 1 and, consequently, $x_i^2 = 0$, being $2 + 1 \leq 3 + 2 \cdot 0$.
- ◇ There exists a vertex $l \in A_G(i) \cap W'_C$ different from k, such that $x_l^1 = 0$, what means that $x_i^1 = x_i^2 = 1$ and this state does not change anymore.

For the general case $s_i > 0$, observe that, at most, $T - 1 \leq 2s_i + 1$ and, therefore, after $T + 1 \leq 2s_i + 3$ iterations the vertex i has permanent state value 1.

□

Theorem 3.29 (Transient in 2-Periodic $\mathrm{MAX} - \mathrm{PDS}$)**.** *Let* $[G, \mathrm{MAX}] - \mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*, where the structure of* G *only allows* 2*-periodic orbits. Then, every state of the system reaches a* 2*-periodic orbit after a maximum of*

$$\max\left\{\max_{k\in\{1,\dots,p\}}\{\mathrm{diam}\,(G_k)\}, 1\right\} + \max\left\{\max_{i\in W_D'^{\alpha}}\{m_i + 2\}, \max_{i\in W_D'^{\beta}}\{2s_i + 3\}\right\}$$

iterations, being $m_i = \mathrm{card}\,(A_G(i) \cap W'_D)$, $\forall i \in W_D'^{\alpha}$ *and* $s_i = \mathrm{card}\,(A_G(i) \cap W'_D)$, $\forall i \in W_D'^{\beta}$.

Proof. In this case, as said above, $W'_C \neq \emptyset$.

Firstly, if $W'_D = \emptyset$, then $p = 0$, $\mathrm{MAX} = \mathrm{NAND}$ and the system reaches a 2-periodic orbit, at most, after 1 iteration (see [28]). It fits with the expression of the upper bound for this case.

Let us see now the general case when $W'_D \neq \emptyset$, which also implies $p > 0$:

Consider $i \in W'_D$. If $i \in W_D'^{\alpha}$, it satisfies the hypotheses of Lemma 3.1, being $m_i = \mathrm{card}\,(A_G(i) \cap W')$. Thus, at a maximum of $m_i + 2$ iterations, x_i will have permanent state value 1. On the other hand, if $i \in W_D'^{\beta}$, it satisfies the hypotheses of Lemma 3.2. Thus, at a maximum of $2s_i + 3$ iterations, x_i will have permanent state value 1.

Taking this into account, in $\max\{\max_{i\in W_D'^{\alpha}}\{m_i + 2\}, \max_{i\in W_D'^{\beta}}\{2s_i + 3\}\}$ iterations, for all $i \in W'_D$, x_i reaches a permanent state value 1. It must be noted that this calculation is also valid even if $W_D'^{\alpha} = \emptyset$ or $W_D'^{\beta} = \emptyset$, because of the value 0 considered by default when a max expression is taken over an empty set.

After that, for the future evolution of the system, the vertices belonging to W'_D do not change their state value, and the study of the evolution of the system can be reduced to analyze what happens in the restriction of G to $W \cup W'_C$. Given that a vertex in W is not adjacent to a vertex in W'_C, there cannot be interference between these sets and the behavior of the entire system can be obtained from the study of the evolution in each connected subgraph $G_1, \ldots, G_p$ and the connected components which result from G when we remove all the vertices in $W \cup W'_D$ and the edges which are incident to them, $C_1, \ldots, C_q$.

Regarding $G_1, \ldots, G_p$, since they only have vertices associated with direct variables, we know that the restriction of the PDS to each G_k performs as an $\mathrm{OR}-\mathrm{PDS}$ which converges to a fixed point in, at most, $\mathrm{diam}\,(G_k)$ iterations (see [28]). On the other hand, the restriction of the PDS to each component C_k performs as a $\mathrm{NAND}-\mathrm{PDS}$, and so a 2-periodic orbit is reached in, at most, 1 iteration (see [28]). In view of this, after $\max\{\max_{k\in\{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\}, 1\}$ iterations (after the $\max\{\max_{i\in W'^{\alpha}_D}\{m_i+2\}, \max_{i\in W'^{\beta}_D}\{2s_i+3\}\}$ iterations needed for ensuring that the state values of the vertices in W'_D are fixed) all the vertices in W will reach a state value that they will permanently preserve, and all the vertices in W'_C will repeat their state value every 2 iterations, alternating values 0 and 1 or preserving state value 1 onwards.

Therefore, after at most

$$\max\{\max_{k\in\{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\}, 1\} + \max\{\max_{i\in W'^{\alpha}_D}\{m_i+2\}, \max_{i\in W'^{\beta}_D}\{2s_i+3\}\}$$

iterations, any (initial) state reaches a 2-periodic orbit. □

Dually, we have the following result.

Theorem 3.30 (Transient in 2-Periodic $\mathrm{MIN}-\mathrm{PDS}$)**.** *Let* $[G, \mathrm{MIN}]-\mathrm{PDS}$ *be a parallel dynamical system over a dependency graph* $G=(V,E)$ *associated with the minterm* MIN, *where the structure of* G *only allows* 2*-periodic orbits. Then, every state of the system reaches a* 2*-periodic orbit after a maximum of*

$$\max\left\{\max_{k\in\{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\}, 1\right\} + \max\left\{\max_{i\in W'^{\alpha}_D}\{m_i+2\}, \max_{i\in W'^{\beta}_D}\{2s_i+3\}\right\}$$

iterations, being $m_i = \mathrm{card}\,(A_G\,(i)\cap W'_D)$, $\forall i\in W'^{\alpha}_D$ *and* $s_i = \mathrm{card}\,(A_G\,(i)\cap W'_D)$, $\forall i \in W'^{\beta}_D$.

This new upper bound obtained in Theorem 3.29 is also the best possible one for this kind of PDS, since it is reachable, as we show in the example below. A dual example in the case of a minterm MIN as evolution operator can be similarly constructed, where the upper bound obtained in Theorem 3.30 is reached.

Example 3.7. Let us consider the PDS over the line graph $G = (V, E)$, with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{1, 4\}\}$ (see Figure 3.8) on the maxterm Boolean function given by $MAX = x_1' \vee x_2 \vee x_3 \vee x_4'$.

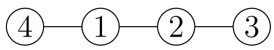

Figure 3.8: Graph $G = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{1, 4\}\})$.

In this case, according to the notation in Theorem 3.29:

- $W = \{2, 3\}$ and $W' = \{1, 4\}$ are the sets of vertices whose corresponding variables appear in MAX in direct and complemented form, respectively. Moreover, $W_D' = \{1\}$ and $W_C' = \{4\}$. Finally, $W_D'^{\alpha} = \emptyset$ and $W_D'^{\beta} = \{1\}$.
- There is $p = 1$ connected component which results from G when all the vertices in W' and the edges which are incident to them are removed, $G_1 = (\{2, 3\}, \{\{2, 3\}\})$. In this case, $\operatorname{diam}(G_1) = 1$.
- There are $s_1 = 0$ vertices in W_D' adjacent to the only element in W_D'.

The upper bound of Theorem 3.29 for this PDS is 4 iterations. If we consider the initial configuration $x^0 = (1, 0, 0, 0)$, the evolution of x^0 until reaching the periodic orbit is:

$$(1, 0, 0, 0) \to (1, 0, 0, 1) \to (0, 0, 0, 0) \to (1, 1, 0, 1) \to (1, 1, 1, 0)\,.$$

In Figure 3.9, we show the phase diagram of this PDS, where it can be easily checked that 4 is its transient.

□

As in the case of fixed-point PDS, additionally, we are able to provide a pattern that can be considered to obtain a 2-Periodic MAX − PDS for which any given (optimal upper bound) transient is reached. A dual construction can be considered to achieve a pattern in the case of PDS on minterm Boolean functions.

Example 3.8. Let us consider the following collection of parallel dynamical systems $\mathfrak{F}_2 = \{\overline{\mathrm{PDS}}_k : k \in \mathbb{N}\}$, where each $\overline{\mathrm{PDS}}_k$ is defined as follows:

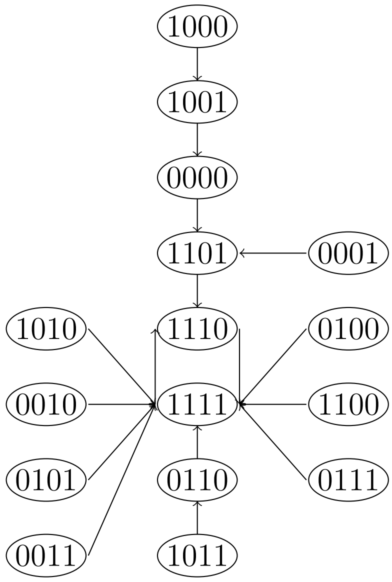

Figure 3.9: Phase portrait of the system $[(\{1,2,3,4\},\{\{1,2\},\{2,3\},\{1,4\}\}), x_1' \vee x_2 \vee x_3 \vee x_4'] -$ PDS.

- $k = 1$: $\overline{\text{PDS}}_1$ is defined as the parallel dynamical system over $\overline{G}_1 = \left(\overline{V}_1, \overline{E}_1\right)$ with $\overline{V}_1 = \{1,2,3,4\}$, $\overline{E}_1 = \{\{1,2\},\{2,3\},\{1,4\}\}$ (the same as in Example 3.7, which is exposed in Figure 3.8).

  The evolution operator is the maxterm $\overline{\text{MAX}}_1 = x_1' \vee x_2 \vee x_3 \vee x_4'$.

  In this case, the relevant elements in Theorem 3.29 related to this PDS are:

  - $\overline{W}_1 = \{2,3\}$ and $\overline{W}'_1 = \{1,4\}$ are the sets of vertices whose corresponding variables appear in $\overline{\text{MAX}}_1$, respectively, in direct and complemented form. Inside $\overline{W}'_1$, $\overline{W}'_{D,1} = \{1\}$ and $\overline{W}'_{C,1} = \{4\}$. Finally, $\overline{W}'^{\alpha}_{D,1} = \emptyset$ and $\overline{W}'^{\beta}_{D,1} = \{1\}$.
  - There is $\overline{p}_1 = 1$ connected component which results from $\overline{G}_1$ when all the vertices in $\overline{W}'_1$ and the edges which are incident to them are removed, $\overline{G}_{1,1} = (\{2,3\},\{\{2,3\}\})$. In this case, $\operatorname{diam}\left(\overline{G}_{1,1}\right) = 1$.
  - There are $\overline{s}_{1,1} = 0$ vertices in $\overline{W}'_{D,1}$ adjacent to the only element in $\overline{W}'^{\beta}_{D,1}$.

- $k \geq 2$: $\overline{\text{PDS}}_k$ is defined from the elements of $\mathfrak{F}$ shown in Example 3.6. To avoid duplication in the name of the vertices of each $\text{PDS}_t \in \mathfrak{F}$, we will denote as i_t to the vertex i in PDS_t:

This way, $\overline{\mathrm{PDS}}_k$ is recursively defined as the parallel dynamical system over $\overline{G}_k = \left(\overline{V}_k, \overline{E}_k\right)$ with $\overline{V}_k = \overline{V}_{k-1} \cup V_{2k-3}$, $\overline{E}_k = \overline{E}_{k-1} \cup E_{2k-3} \cup \{\{1, 1_{2k-3}\}\}$ and whose evolution operator is the maxterm $\overline{\mathrm{MAX}}_k = \overline{\mathrm{MAX}}_{k-1} \vee \mathrm{MAX}_{2k-3}$.

The relevant elements in Theorem 3.29 related to this PDS are:

- $\overline{W}_k = \overline{W}_{k-1} \cup W_{2k-3} = \left(\bigcup_{t=1}^{k-1} W_{2t-1}\right) \cup \{2,3\}$ and $\overline{W}'_k = \overline{W}'_{k-1} \cup W'_{2k-3} = \left(\bigcup_{t=1}^{k-1} W'_{2t-1}\right) \cup \{1,4\}$ are the sets of vertices whose corresponding variables appear in $\overline{\mathrm{MAX}}_k$, respectively, in direct and complemented form. Also, inside $\overline{W}'_k$:
  - $\overline{W}'_{D,k} = \overline{W}'_{D,k-1} \cup W'_{2k-3} = \overline{W}'_k \setminus \{4\}$.
  - $\overline{W}'_{C,k} = \overline{W}'_{C,k-1} = \{4\}$.
  - Finally, inside $\overline{W}'_{D,k}$, $\overline{W}'^{\alpha}_{D,k} = \overline{W}'^{\alpha}_{D,k-1} \cup W'_{2k-3} = \overline{W}'_{D,k} \setminus \{1\}$ and $\overline{W}'^{\beta}_{D,k} = \overline{W}'^{\beta}_{D,k-1} = \{1\}$.
- There are $\overline{p}_k = \left(\sum_{t=1}^{k-1} 2t-1\right) + 1 = (k-1)^2 + 1$ connected components which result from $\overline{G}_k$ when all the vertices in $\overline{W}'_k$ and the edges which are incident to them are removed, those ones of PDS_1, PDS_3,...,PDS_{2k-3} and $\overline{G}_{1,1}$ defined below. In this case, all these connected components have diameter 0 except $\overline{G}_{1,1}$, with $\operatorname{diam}\left(\overline{G}_{1,1}\right) = 1$.
- Finally, for each $i \in \overline{W}'_{D,k}$, $\overline{m}_{k,i}$ if $i \in \overline{W}'^{\alpha}_{D,k}$ or $\overline{s}_{k,i}$ if $i \in \overline{W}'^{\beta}_{D,k}$ can be obtained as:
  - If $i \in W'_{2t-1} \setminus \{1_{2t-1}\}$ for $t \leq k-1$, $\overline{m}_{k,i} = m_{2t-1,i} \leq (2t-1) - 1 \leq 2k-4$.
  - If $i = 1_{2t-1}$ for $t \leq k-1$, because of the new adjacency $\{1, 1_{2t-1}\} \in \overline{E}_k$, $\overline{m}_{k,i} = m_{2t-1,i} + 1 = (2t-1) - 1 + 1 \leq 2k-3$.
  - $\overline{s}_{k,1} = k-1$.

Therefore, the upper bound in Theorem 3.29 for each $\overline{\mathrm{PDS}}_k \in \mathfrak{F}_2$ is $1 + 2(k-1) + 3 = 2k+2$ iterations.

Associated to these parallel dynamical systems, let us consider the initial state values of the variables:

- $\overline{x}_1^0 = \left(\overline{x}_{1,1}^0, \overline{x}_{1,2}^0, \overline{x}_{1,3}^0, \overline{x}_{1,4}^0\right) = (1,0,0,0)$.

- $\overline{x}_k^0 = \left(\overline{x}_{k,i}^0\right)_{i\in \overline{V}_k}$ for $k \geq 2$, with $\overline{x}_{k,i}^0 = \overline{x}_{1,i}^0$ for $i = 1, 2, 3, 4$ and $\overline{x}_{k,i}^0 = x_{2t-1,i}^0$ as defined in Example 3.6 if $i \in V_{2t-1}$.

Let us see that the valoration $\overline{x}_k^0$ in $\overline{\mathrm{PDS}}_k$ reaches a 2-periodic orbit in, exactly, $2k + 2$ iterations. In fact, note that:

- $\overline{x}_1^0$ reaches a 2-periodic orbit in $\overline{\mathrm{PDS}}_1$ after 4 iterations.
- $\overline{x}_{k+1}^0$ reaches a 2-periodic orbit in $\overline{\mathrm{PDS}}_{k+1}$ after two iterations more than $\overline{x}_k^0$ in $\overline{\mathrm{PDS}}_k$, for all $k \in \mathbb{N}$.

The simplest case ($k = 1$) is direct, since the evolution of $\overline{x}_1^0$ in $\overline{\mathrm{PDS}}_1$ is:

$$(1,0,0,0) \to (1,0,0,1) \to (0,0,0,0) \to (1,1,0,1) \to (1,1,1,0) \leftrightarrow (1,1,1,1)\,.$$

Regarding the general case, it follows directly by observing that:

$\overline{x}_{k+1,1}^0 = 1$ and, since $\overline{x}_{k+1,4}^0 = 0$, then $\overline{x}_{k+1,1}^1 = 1$. Furthermore, the evolution of the whole system $\overline{\mathrm{PDS}}_{k+1}$ during the first two iterations is the same as the restriction of $\overline{\mathrm{PDS}}_{k+1}$ to each connected component which results from $\overline{G}_{k+1}$ when the vertex 1 and the edges which are incident to it are removed. Thus:

- $\overline{x}_{k+1,2}^2 = 0 = \overline{x}_{k,2}^0$.
- $\overline{x}_{k+1,3}^2 = 0 = \overline{x}_{k,3}^0$.
- $\overline{x}_{k+1,4}^2 = 0 = \overline{x}_{k,4}^0$.
- $\overline{x}_{k+1,1_1}^1 = 0$, so $\overline{x}_{k+1,1}^2 = 1 = \overline{x}_{k,1}^0$.
- $\overline{x}_{k+1,i}^2 = 1$ for all $i \in V_1$, and they continue activated onwards.
- If $V_{2t-1} = \{1_{2t-1}, \ldots, 2\,(2t-1)_{2t-1}\}$ and $V_{2t-3} = \{1_{2t-3}, \ldots, 2\,(2t-3)_{2t-3}\}$, with $t \in \{2, \ldots, k\}$, then:
  - $\overline{x}_{k+1,j_{2t-1}}^2 = 1$ for all $j \geq 2\,(2t-1) - 3$, and they continue activated onwards.
  - $\overline{x}_{k+1,j_{2t-1}}^2 = \overline{x}_{k,j_{2t-3}}^0$ for $j \leq 2\,(2t-1) - 4 = 2\,(2t-3)$, and the restriction of $\overline{\mathrm{PDS}}_{k+1}$ to these G_{2t-3} evolves as $\overline{\mathrm{PDS}}_k$ onwards.

$\square$

Remark 3.11. Observe that the upper bound for a 2-Periodic PDS on the simplest maxterm NAND (resp. on the simplest minterm NOR), which presents only periodic orbits of period 2, is 1 (see [28]). This previous result is not valid for general maxterm and minterm Boolean functions, as can be seen in Example 3.7 where the transient of the system is 4. This reveals the relevance of our extended results, given in Theorems 3.29 and 3.30, due to the breakdown found in the upper bound of the transient for general PDS. □

Chapter 4

Advances in Sequential Dynamical Systems

In this chapter, we perform a complete analysis of the dynamics in sequential dynamical systems on maxterm and minterm Boolean functions over undirected dependency graphs. Following a similar scheme as in the parallel case, the study is divided into two sections according to the methodology of research: dynamics of periodic orbits and dynamics of non-periodic orbits.

4.1 Dynamics of periodic orbits

In this section, the dynamics of periodic orbits are analyzed, identifying which of them can exist and coexist, and obtaining an upper bound for their number. In particular, as in the parallel case, we are able to give a Fixed-Point Theorem and a Periodic-Orbit Theorem, so characterizing the uniqueness of them.

4.1.1 Existence of periodic orbits

In Theorem 3.1 (resp. Theorem 3.2) of Subsection 3.1.1, the orbital structure of PDS with general maxterm (resp. minterm) functions as evolution operators has been analyzed, proving that the only periodic orbits of such systems are fixed points and periodic orbits of period 2.

In contrast to these results for PDS, in the case of SDS with general maxterm (resp. minterm) functions as evolution operators, we will see that they can present orbits of whichever period.

Theorem 4.1 (Periodic structure of $\mathrm{MAX} - \mathrm{SDS}$)**.** *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, it can present periodic orbits of any period.*

Proof. Given $n \in \mathbb{N}$, $n \geq 2$, we will give a pattern to construct an SDS with an orbit of period $n-1$. Let us take $V = \{1, \ldots, n-1, n\}$ and consider the following adjacency structure among the vertices:

- Each $i \in \{1, \ldots, n-1\}$ is adjacent to all the vertices in $\{1, \ldots, n-1\} \setminus \{i\}$. In other words, we take the complete graph K_{n-1} of $n-1$ vertices.
- The vertex n is adjacent to the vertex $n-1$.

As permutation on V we take $\pi = \mathrm{id}$, the identity permutation.

Finally, we choose the updating operator

$$\mathrm{MAX} = x_1' \vee \cdots \vee x_{n-1}' \vee x_n.$$

Let us write x_i^k to indicate the state value of the entity i after k iterations of the evolution operator MAX. Then, let us consider the initial value for the variables $x_i^0 = 1$ for all $i \in V$. It is a straightforward computation to check that the system evolves in the following way:

- After k iterations, $1 \leq k \leq n-2$: $x_k^k = 0$, $x_i^k = 1$ for all $i \in V \setminus \{k\}$.
- After $n-1$ iterations, all the state values coincide with the initial ones, i.e., $x_i^{n-1} = x_i^0 = 1$ for all $i \in V$.

Namely, the SDS so constructed presents a periodic orbit of period $n-1$.

To illustrate the designed patterns, in Figure 4.1 the cases for $n = 2$, $n = 3$, $n = 4$ and $n = 5$ can be seen.

$\square$

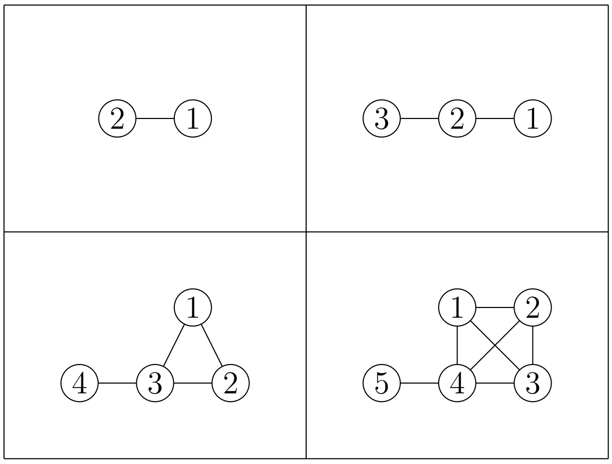

Figure 4.1: Patterns for $n = 2$, $n = 3$, $n = 4$ and $n = 5$.

Remark 4.1. It is direct to perform a similar argument as in the proof of Theorem 3.1 of Subsection 3.1.1, but in the context of SDS, to achieve analogous information to Remark 3.1 about asymptotic behavior of the entities in SDS, essential in the study of these systems throughout this thesis:

- Each $i \in W$ fixes its state value after a certain number of iterations.
- When a periodic orbit is reached, each $i \in W'_D$ has state value 1.
- The period comes from the evolution of the vertices belonging to W'_C.

□

Dually we have the following theorem.

Theorem 4.2 (Periodic structure of $\mathrm{MIN} - \mathrm{SDS}$)**.** *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, it can present periodic orbits of any period.*

It is worth to analyze the particular relevant cases when the evolution operator is the maxterm OR or NAND (resp. minterm AND or NOR). Recall that the only periodic orbits of PDS over undirected dependency graphs with OR (resp. AND) as updating operator are fixed points and with NAND (resp. NOR), 2-periodic orbits (see Corollaries 3.1, 3.2, 3.3 and 3.4 of Subsection 3.1.1). In this case, when the updating of the state values is asynchronous and the evolution operator is given by

OR (resp. AND), the only periodic orbits of the system are fixed points, as in PDS. In contrast, when the evolution operator is NAND (resp. NOR), the corresponding SDS could present periodic orbits of any period greater than or equal to 2.

To see that, let us turn to the following well-known lemma (see [35, 91]).

Lemma 4.1. *Let* $[G, F, \pi]-\mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm or minterm* F*. Then, the systems* $[G, F, \pi] - \mathrm{SDS}$ *and* $[G, F] - \mathrm{PDS}$ *have the same fixed points.*

Firstly, in the case of OR as evolution operator, we have the following result (which is already proved in [28], although an alternative proof is shown here).

Theorem 4.3. *Let* $[G, \mathrm{OR}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* OR*. Then, all the periodic orbits of this system are fixed points. In fact, there are exactly two fixed points, namely,* $\mathcal{I}$ *and* $\mathcal{O}$*.*

Proof. By Lemma 4.1 and Corollary 3.1 in Subsection 3.1.1, $\mathcal{I}$ and $\mathcal{O}$ are the only fixed points of this SDS.

It only remains to show that there cannot be periodic orbits of greater period: since all the entities belong to W, we can have only one of the following two possibilities for each $i \in V$:

- $\forall t \geq 0$, $x_i^t = 0$. In this case, the state value 0 is permanent for this entity from the initial configuration.
- $\exists T \geq 0$ such that $x_i^T = 1$, being the iteration T the first time that the variable x_i takes the value 1. In this situation, the state value 1 is permanent from this iteration on.

Thus, after a certain number of iterations, all the entities reach a fixed value that they preserve onwards. □

In the case of NAND as evolution operator, we have the following result.

Theorem 4.4. *Let* $[G, \mathrm{NAND}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND*. Then, this system can present periodic orbits of any period, except fixed points.*

Proof. First, notice that such an SDS cannot present fixed points. In fact, we know that PDS with NAND as evolution operator cannot present fixed points. Then, this first assertion follows from Lemma 4.1.

Given $n \in \mathbb{N}$, we will provide a pattern to construct an SDS with NAND as evolution operator which has an orbit of period $n+1$.

Let us take $V = \{1, \ldots, n\}$. Assume that the adjacency structure among the vertices is the one of a complete graph K_n of n vertices.

As permutation on V, we take $\pi = \mathrm{id}$, the identity permutation.

Let us consider the initial state values $x_i^0 = 1$ for all $i \in V$. Then, the system evolves as follows:

- After k iterations, $1 \leq k \leq n$: $x_k^k = 0$, $x_i^k = 1$ for all $i \in V \setminus \{k\}$.
- After $n+1$ iterations, all the state values coincide with the initial ones, i.e., $x_i^{n+1} = x_i^0 = 1$ for all $i \in V$.

Namely, the SDS so constructed presents a periodic orbit of period $n+1$. $\square$

Dually, in the case of AND and NOR, we have the following results (the first one of them is already proved in [28]).

Theorem 4.5. *Let* $[G, \mathrm{AND}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* AND*. Then, all the periodic orbits of this system are fixed points. In fact, there are exactly two fixed points, namely,* $\mathcal{O}$ *and* $\mathcal{I}$*.*

Theorem 4.6. *Let* $[G, \mathrm{NOR}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR*. Then, this system can present periodic orbits of any period, except fixed points.*

4.1.2 Coexistence of periodic orbits

In Corollaries 3.5 and 3.6 of Subsection 3.1.2, the orbital structure of PDS with general maxterm (resp. minterm) functions as evolution operators has been analyzed, proving that the coexistence of periodic orbits with different periods is not possible.

In contrast to these results for PDS, in the case of SDS with general maxterm (resp. minterm) functions as evolution operators, we will see that periodic orbits with different periods greater than or equal to 2 can coexist in an SDS. Nevertheless, when an SDS has fixed points, no other periods can appear.

It is well known that, in a PDS over a dependency graph $G = (V, E)$ associated with the maxterm MAX (resp. minterm MIN), if there is a fixed point, then there are no periodic orbits of other periods (see Theorems 3.3, 3.5 and Corollary 3.5 in Subsection 3.1.2). This situation remains true for SDS, as we see next. That is, for both PDS and SDS, the existence of fixed points excludes the presence of other periodic orbits.

Theorem 4.7. *Let* $[G, \mathrm{MAX}] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, all the periodic orbits of this system are fixed points if, and only if,* $W'_C = \emptyset$*.*

Proof. In the case $W' = \emptyset$, the system has only two fixed points: $\mathcal{I}$ and $\mathcal{O}$ (see Theorem 4.3 of Subsection 4.1.1 and [28]). Let us analyze now the general case when $W' \neq \emptyset$:

First, assume that all the periodic orbits of this system are fixed points and let us see that $W'_C = \emptyset$.

Take $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ a fixed point, where $\hat{x}_i$ represents the (fixed) value of the vertex $i \in V$. Note that, for all $i \in W'$, it must be $\hat{x}_i = 1$. Otherwise (i.e., if $\hat{x}_i = 0$), it would change to 1 after the following iteration.

Suppose that there exists $i \in W'$ such that $W \cap A_G(i) = \emptyset$. In such a case, $A_G(i) \subseteq W'$ and so, for every $j \in A_G(i)$, it is $\hat{x}_j = \hat{x}_i = 1$. Since we are assuming that $\hat{x}$ is a fixed point, independently of the order in which the state values are updated (i.e. independently of π), these are the state values of the vertices $j \in A_G(i)$ when the state value of i is updated. Then, the state value of i would change to 0 in the following iteration, which is a contradiction.

To prove the converse implication, let us suppose that for all $i \in W'$, $W \cap A_G(i) \neq \emptyset$. Let us consider an arbitrary initial value for the variables $(x_1^0, \ldots, x_n^0)$. Since the dependency graph is finite (and so is the state space), note that after a certain number of iterations, let us say $r \in \mathbb{N}$, the states of all the vertices belonging to W become fixed (see Remark 4.1 in Subsection 4.1.1). Let us take $i \in W'$ and let us prove that $x_i^{r+1} = 1$. In fact, let us suppose that $x_i^{r+1} = 0$ and take $j \in W \cap A_G(i)$. Then, it would be $x_j^{r+2} = 1 = x_j^r$ (since we are assuming that the state value of j is fixed from the iteration r). Then, since $x_j^r = 1$, it must be $x_i^{r+1} = 1$, which is a contradiction. Thus, $x_i^{r+1} = 1$ for all $i \in W'$ and these state values do not change.

Therefore, all the variables of the system become fixed after $r+1$ iterations and the proof finishes. □

Theorem 4.8. *Let* $[G, \mathrm{MAX}] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, all the periodic orbits of this system have period greater than* 1 *if, and only if,* $W'_C \neq \emptyset$.

Proof. First, assume that all the periodic orbits of this system have period greater than 1. If the theses were not true, that is, if for all $i \in W'$ it is $W \cap A_G(i) \neq \emptyset$, then from Theorem 4.7 we have that all the periodic orbits of this system are fixed points, which is a contradiction.

Conversely, let us suppose that there exists $i \in W'$ such that $W \cap A_G(i) = \emptyset$. If there is a fixed point, reasoning as in the proof of Theorem 4.7, we have $W'_C = \emptyset$, which is a contradiction. Hence, all the periodic orbits of the system must have period greater than 1. □

Corollary 4.1 (Coexistence of periods in $\mathrm{MAX} - \mathrm{SDS}$). *Let* $[G, \mathrm{MAX}] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, (eventually) fixed points and (eventually) periodic orbits of other periods* $m \geq 2$ *cannot coexist.*

Proof. It is a direct consequence of Theorems 4.7 and 4.8. □

Remark 4.2. It is important to note that all, the impossibility of coexistence of fixed points and periodic orbits of other periods $m \geq 2$, the condition which implies that all the periodic orbits of an SDS are fixed points and the complementary condition which implied that all the periodic orbits of an SDS have period greater than or equal to 2, are equal to the respective ones in the case of a PDS (see Theorems 3.3, 3.5 and Corollary 3.5 in Subsection 3.1.2). □

Dually, we have the following results.

Theorem 4.9. *Let* $[G, \mathrm{MIN}] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, all the periodic orbits of this system are fixed points if, and only if,* $W'_C = \emptyset$.

Theorem 4.10. *Let* $[G, \mathrm{MIN}] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, all the periodic orbits of this system have period greater than* 1 *if, and only if,* $W'_C \neq \emptyset$.

Corollary 4.2 (Coexistence of periods in MIN − SDS). *Let* $[G, \mathrm{MIN}] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, (eventually) fixed points and (eventually) periodic orbits of other periods* $m \geq 2$ *cannot coexist.*

In the light of Theorems 4.1 and 4.2 in Subsection 4.1.1, it naturally arises the problem of determining the possible coexistence of periodic orbits of different periods for SDS with maxterm or minterm Boolean functions as updating operators. In this sense, in the following theorems, we show that periodic orbits of whichever periods greater than or equal to 2 can coexist in such discrete dynamical systems.

Theorem 4.11. *Given* $\{n_1, \ldots, n_r\} \subset \mathbb{N}$ *with* $n_i \geq 2$ *for every* $i = 1, \ldots, r$, $r \geq 2$, *there exists an* SDS *with a maxterm as evolution operator which presents periodic orbits of periods* $n_1, \ldots, n_r$ *simultaneously.*

Proof. Let us construct an SDS with orbits of periods $n_1, \ldots, n_r$.

For each n_i, let us take two complete graphs K_{n_i-1} and $\overline{K}_{n_i-1}$ of $n_i - 1$ vertices. We will denote by $\{v_{i,1}, \ldots, v_{i,n_i-1}\}$ and $\{\overline{v}_{i,1}, \ldots, \overline{v}_{i,n_i-1}\}$ the vertices of K_{n_i-1} and $\overline{K}_{n_i-1}$, respectively, for every $i \in \{1, \ldots, r\}$. Thus, the dependency graph has $2\,(n_1 + \cdots + n_r - r)$ vertices.

Apart from the internal adjacencies in each complete graph K_{n_i-1} and $\overline{K}_{n_i-1}$, we will consider the following adjacency structure among the vertices:

- $v_{i,j}$ is adjacent to $v_{l,m}$ for all j, m and $i \neq l$. In other words, the vertices in K_{n_i-1} are adjacent to all the vertices in K_{n_l-1} for $i \neq l$.
- $\overline{v}_{i,j}$ is adjacent to $\overline{v}_{l,m}$ for all j, m and $i \neq l$. In other words, the vertices in $\overline{K}_{n_i-1}$ are adjacent to all the vertices in $\overline{K}_{n_l-1}$ for $i \neq l$.
- $v_{i,j}$ is adjacent to $\overline{v}_{l,m}$ for all j, m and $i \neq l$. In other words, the vertices in K_{n_i-1} are adjacent to all the vertices in $\overline{K}_{n_l-1}$ for $i \neq l$ (but they are not adjacent to the vertices in $\overline{K}_{n_i-1}$).

As permutation on the set of vertices, we will consider:

$$\pi = v_{1,1}|\cdots|v_{1,n_1-1}|\overline{v}_{1,1}|\cdots|\overline{v}_{1,n_1-1}|\cdots|v_{r,1}|\cdots|v_{r,n_r-1}|\overline{v}_{r,1}|\cdots|\overline{v}_{r,n_r-1}.$$

Namely, first the state values of the vertices in K_{n_1-1} are updated, then the ones in $\overline{K}_{n_1-1}$, next the vertices in K_{n_2-1} followed by the ones in $\overline{K}_{n_2-1}$, and so on until all the vertices are updated.

As evolution operator, we take MAX = NAND.

As initial state values for the variables, we consider that the vertex $v_{i,1}$ is deactivated and the rest of vertices are activated.

This system evolves as follows:

- After k iterations, $1 \leq k \leq n_i - 2$, $v_{i,k+1}$ and $\overline{v}_{i,k}$ are deactivated and the rest of vertices are activated.
- After $n_i - 1$ iterations, $\overline{v}_{i,n_i-1}$ is deactivated and the rest of vertices are activated.
- After n_i iterations, all the state values coincide with the initial ones.

Namely, the $[G, \mathrm{MAX}, \pi]-$SDS so constructed presents a periodic orbit of period n_i.

Thus, by considering the different r initial state values obtained by varying i in $\{1, \ldots, r\}$, r periodic orbits with periods $n_1, \ldots, n_r$ result. $\square$

Let us illustrate this result with the following example to clarify the notation.

Example 4.1. The SDS proposed by Theorem 4.11 in which a 2-periodic orbit and a 3-periodic orbit coexist, is defined by

- $G = (\{1, 2, 3, 4, 5, 6\}, \{\{3, 4\}, \{5, 6\}\} \cup \{\{i, j\} : 1 \leq i \leq 2, 3 \leq j \leq 6\})$ (see Figure 4.2),
- MAX = NAND,
- $\pi = \mathrm{id}$, the identity permutation.

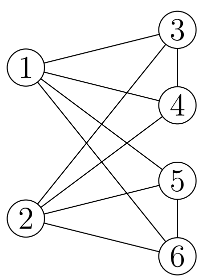

Figure 4.2: Graph $G = (\{1, 2, 3, 4, 5, 6\}, \{\{3, 4\}, \{5, 6\}\} \cup \{\{i, j\} : 1 \leq i \leq 2, 3 \leq j \leq 6\})$.

According to the notation in Theorem 4.11, 1 is the only vertex in K_1, 2 is the only vertex in $\overline{K}_1$, 3, 4 are the vertices in K_2, and 5, 6 are the vertices in $\overline{K}_2$.

In this case, the 2-periodic orbit proposed by Theorem 4.11 can be seen in Figure 4.3.

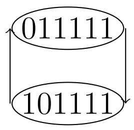

Figure 4.3: 2-periodic orbit of the system proposed by Theorem 4.11.

On the other hand, the 3-periodic orbit proposed by Theorem 4.11 can be seen in Figure 4.4.

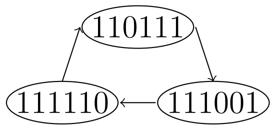

Figure 4.4: 3-periodic orbit of the system proposed by Theorem 4.11.

□

Finally, we can state the dual result.

Theorem 4.12. *Given* $\{n_1, \ldots, n_r\} \subset \mathbb{N}$ *with* $n_i \geq 2$ *for every* $i = 1, \ldots, r$, $r \geq 2$, *there exists an* SDS *with a minterm as evolution operator which presents periodic orbits of periods* $n_1, \ldots, n_r$ *simultaneously.*

4.1.3 Uniqueness of fixed points

Our main objective in this subsection is to obtain a Fixed-Point Theorem for SDS. Observe that, although fixed points and periodic orbits of greater period cannot coexist, there are SDS whose state spaces contain more than one fixed point. Actually, there is a total resemblance regarding the fixed points of SDS and PDS with whichever maxterm or minterm Boolean function as evolution operator, as said in Lemma 4.1 in Subsection 4.1.1.

In Theorem 3.7 (resp. Theorem 3.8) of Subsection 3.1.3, a Fixed-Point Theorem was obtained for PDS on maxterm (resp. minterm) Boolean functions as evolution operators. Thus, as a direct consequence of this result for PDS, we have the following theorem in the context of SDS.

Theorem 4.13 (Fixed-Point Theorem for $\mathrm{MAX}-\mathrm{SDS}$)**.** *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Assume that,* $W'_C = \emptyset$. *Then, this* SDS *has a unique fixed point if, and only if, for every* j, $1 \leq j \leq p$, $A_G(G_j) \cap W'_1 \neq \emptyset$. *In this situation, the unique fixed point is* $\mathcal{I}$, *and all the orbits converge to this fixed point.*

Proof. It is a direct consequence of Lemma 4.1 in Subsection 4.1.1, Theorems 4.7, 4.8 and Corollary 4.1 in Subsection 4.1.2 and Theorem 3.7 in Subsection 3.1.3. □

Dually, we have the following theorem.

Theorem 4.14 (Fixed-Point Theorem for $\mathrm{MIN} - \mathrm{SDS}$)**.** *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Assume that,* $W'_C = \emptyset$. *Then, this* SDS *has a unique fixed point if, and only if, for every* j, $1 \leq j \leq p$, $A_G(G_j) \cap W'_1 \neq \emptyset$. *In this situation, the unique fixed point is* $\mathcal{O}$, *and all the orbits converge to this fixed point.*

4.1.4 Uniqueness of periodic orbits

By Theorems 4.7, 4.8, 4.9, 4.10 and Corollaries 4.1, 4.2 of Subsection 4.1.2, it is well known that all the periodic orbits of an SDS have period greater than 1 if, and only if, $W'_C \neq \emptyset$. Our aim now is to characterize the uniqueness of periodic orbits of period greater than 1 in a $\mathrm{MAX} - \mathrm{SDS}$. To achieve this objective, first, we analyze the particular case $\mathrm{MAX} = \mathrm{NAND}$.

In Proposition 3.1 of Subsection 3.1.4, it is established that, in the case of parallel update associated with a NAND Boolean operator, the periodic orbit (order equal to 2 in such a case) is unique if, and only if, the associated dependency graph is complete. In this case of sequential update, this pattern is broken, as we show below.

When a sequential update schedule with a maxterm NAND as evolution operator is considered, as will be seen in Proposition 4.3 of Subsection 4.2.1, we have that the GOE points of the system are the configurations such that there are two adjacent vertices $i, j \in V$ with $x_i = x_j = 0$. Furthermore, the rest of the configurations belong to periodic orbits.

As said above, in a $\mathrm{NAND} - \mathrm{SDS}$, the completeness of the dependency graph is not a necessary condition for the uniqueness of a periodic orbit, breaking the pattern found for the parallel update, shown in Proposition 3.1 of Subsection 3.1.4. We can see this break down in the following example.

Example 4.2. Let us consider the $\mathrm{NAND} - \mathrm{SDS}$ defined by

- $G = \{\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}\}$ (see Figure 4.5),
- $\text{MAX} = x_1' \vee x_2' \vee x_3' \vee x_4'$,
- $\pi = 1|2|3|4$.

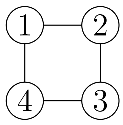

Figure 4.5: Graph $\{\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}\}$.

The configurations without adjacent complemented entities with state values equal to 0 are:

$$(1,1,1,1)\,,(0,1,1,1)\,,(1,0,1,1)\,,(1,1,0,1)\,,(1,1,1,0)\,,(0,1,0,1)\,,(1,0,1,0)\,.$$

In Figure 4.6, it can be seen that all of them belong to the same periodic orbit.

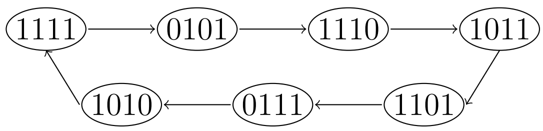

Figure 4.6: Unique periodic orbit of the $[\{\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}\}, \text{NAND}, \text{id}]-\text{SDS}$.

The rest of the configurations are GOE states from which this unique periodic orbit is reached after 1 iteration. □

Thus, we can enunciate the following uniqueness result for a periodic orbit in a $[G, \text{NAND}, \pi] - \text{SDS}$.

Lemma 4.2 (m-Periodic-Orbit Theorem for NAND$-$SDS)**.** *Let* $[G, \text{NAND}, \pi]-\text{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND*. Then, there is a unique* m*-periodic orbit (*$m > 1$*) if, and only if, any configuration reaches the state* $\mathcal{I}$*, being* m *the number of configurations without adjacent (complemented) vertices with state values equal to* 0*. In this situation, all the orbits of the system converge to this* m*-periodic orbit.*

Proof. From Theorem 4.4 in Subsection 4.1.1, this kind of SDS only presents periodic orbits of period greater than 1. On the other hand, as will be seen in Proposition 4.3 of Subsection 4.2.1, the periodic points of the system are those configurations without adjacent (complemented) vertices with state values equal to 0. In particular, the state $\mathcal{I}$ always belongs to a periodic orbit. Thus, the result follows from the fact that in an SDS all the configurations belong to or reach a periodic orbit. □

Dually, we have the following lemma.

Lemma 4.3 (m-Periodic-Orbit Theorem for NOR$-$SDS)**.** *Let* $[G, \text{NOR}, \pi]-$SDS *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR. *Then, there is a unique m-periodic orbit ($m > 1$) if, and only if, any configuration reaches the state $\mathcal{O}$, being m the number of configurations without adjacent (complemented) vertices with state values equal to* 1. *In this situation, all the orbits of the system converge to this m-periodic orbit.*

Once these results have been established, we provide the following system to show how they can determine the uniqueness of a periodic orbit in this kind of SDS:

Let us consider a $[G, \text{NAND}, \pi]-$SDS such that $\pi = 1|\dots|n$, the identity permutation, and take G the complete graph K_n. Then, the system has a unique periodic orbit because each configuration reaches the state $\mathcal{I}$. In fact, if $x \in \{0, 1\}^n$ and there exists $i \in V$ such that $x_i = 0$, let $j = \max\{k \in V : x_k = 0\}$. If $j = n$, after one iteration the configuration $\mathcal{I}$ is reached. Otherwise, after one iteration the configuration reached is the one with all the entities activated except $j + 1$. Similarly, in the next iteration, the configuration reached is that one with all the entities activated except $j + 2$, and so on. Thus, after $n - j + 1$ iterations, the configuration $\mathcal{I}$ is reached.

As a consequence of Example 4.2 and this last comment, we can enunciate the following corollaries.

Corollary 4.3. *In* NAND $-$ SDS*, the completeness of the dependency graph is a sufficient (but not necessary) condition for the existence of a unique periodic orbit in the system.*

And now the dual result.

Corollary 4.4. *In* NOR $-$ SDS*, the completeness of the dependency graph is a sufficient (but not necessary) condition for the existence of a unique periodic orbit in the system.*

Finally, once all the results above have been established for the particular case in which the maxterm is NAND (and their dual versions for the minterm NOR), now let us work with a general maxterm MAX as evolution operator.

According to what has been seen for the parallel update case, when a periodic orbit of period greater than 1 is reached after r_0 iterations, the system performs as follows:

- If $i \in W$, then $x_i^r = x_i^{r_0}$ for all $r \geq r_0$. In other words, the direct vertices do not change their state (activated or deactivated) from the r_0-th iteration on. Moreover, in each connected component G_j, $1 \leq j \leq p$, either every variable associated with the vertices in G_j has state value 1 or every variable has state value 0.

- If $i \in W'_D$ (note that if MAX $\neq$ NAND and $W'_C \neq \emptyset$, then $W'_D \neq \emptyset$), it follows that $x_i^r = 1$ for all $r \geq r_0$. In fact, suppose that there exists $i \in W'_D$ such that (after the r_0-th iteration) x_i takes the value 0 in an infinite number of future updates of the system, and assume (without loss of generality) that $r_0 \geq 1$ and $x_i^{r_0} = 0$. Since $i \in W'_D$, there exists a direct vertex j adjacent to i. Observe that it must be $x_j^{r_0+1} = 1$: if j updates before i, $x_i^{r_0} = 0$ makes $x_j^{r_0+1} = 1$; while if j updates after i, it is already $x_j^{r_0} = 1$ due to $x_i^{r_0} = 0$. Then, being $x_j^{r_0+1} = 1$, we get that $x_i^r = 1$ for all $r > r_0 + 1$, which contradicts our initial assumption.

- The restriction of the system to each connected component C_j performs as a NAND $-$ SDS. The permutation which determines the updating order in the system restricted to C_j is the restriction of π to the vertices in C_j.

In view of this, it should be pointed out that, as commented in Chapter 2 about G^*, this graph is not, in general, connected. Thus, the system $\left[G^*, \mathrm{MAX}_{|G^*}, \pi_{|G^*}\right] -$ SDS may be understood as a set of independent SDS, in the sense that the evolution in each one only depends on the restriction of MAX and π to the connected component of G^* over which it is defined. In this setting, by saying that $\left[G^*, \mathrm{MAX}_{|G^*}, \pi_{|G^*}\right] -$ SDS has a unique fixed point, we mean that all these independent SDS over the connected components of G^* (if more than one) have a unique fixed point.

Additionally, for every $j \in \{1, \ldots, q\}$, let $\left[C_j, \mathrm{MAX}_{|C_j}, \pi_{|C_j}\right]-$SDS be the sequential dynamical system over the dependency graph C_j associated with the maxterm $\mathrm{MAX}_{|C_j}$, i.e., the restriction of MAX to the vertices in C_j, and $\pi_{|C_j}$, the restriction of π to the vertices in C_j.

As said before, the restriction of the system to each connected component C_j performs as an independent SDS with the maxterm NAND as evolution operator. Thus, we must analyze the periodic orbits of the restricted systems $\left[C_j, \mathrm{MAX}_{|C_j}, \pi_{|C_j}\right] -$ SDS, $j \in \{1, \ldots, q\}$, and the relationships among them to study the periodic orbits of the global system.

Let us consider the systems restricted to C_{j_1} and C_{j_2}, with $j_1 \neq j_2$, and respective periodic orbits Orb_{j_1} and Orb_{j_2}. The combination of these orbits in the system $\left[C_{j_1} \cup C_{j_2}, \mathrm{MAX}_{|C_{j_1} \cup C_{j_2}}, \pi_{|C_{j_1} \cup C_{j_2}}\right] -$ SDS is also a periodic orbit.

Observe that, depending on the synchronization of the different state vectors of Orb_{j_1} and Orb_{j_2} in the evolution of the system, different periodic orbits can appear in the system restricted to $C_{j_1} \cup C_{j_2}$, as shown in the following example.

Example 4.3. Let us consider the following SDS:

$$C_1 = [(\{1\}, \emptyset), \mathrm{NAND}, \mathrm{id}] - \mathrm{SDS},$$

$$C_2 = [(\{2\}, \emptyset), \mathrm{NAND}, \mathrm{id}] - \mathrm{SDS},$$

being id the identity permutation. For each SDS, let us consider the following set of states belonging to a periodic orbit:

$$\mathrm{Orb}_1 = \{0, 1\},$$

$$\mathrm{Orb}_2 = \{0, 1\}.$$

From the union of these systems, we obtain the SDS

$$[(\{1, 2\}, \emptyset), \mathrm{NAND}, 1|2] - \mathrm{SDS},$$

in which two periodic orbits are generated from Orb_1 and Orb_2:

- When $x_1 = 0$ and $x_2 = 0$ update simultaneously, $\mathrm{Orb}_{u_1} = \{(0,0), (1,1)\}$.
- When $x_1 = 0$ and $x_2 = 1$ update simultaneously, $\mathrm{Orb}_{u_2} = \{(0,1), (1,0)\}$.

Note that in the union system $C_1 \cup C_2$ also the order $2|1$ may be chosen, since C_1 and C_2 update independently. □

Thereby, we define the product of the periodic orbits Orb_{j_1} and Orb_{j_2} as the set of all the periodic orbits of the restricted system to $C_{j_1} \cup C_{j_2}$ generated by the states belonging to Orb_{j_1} and Orb_{j_2}.

Finally, let us consider the following lemmas in which, as usual, the functions lcm and gcd return, respectively, the least common multiple and the great common divisor of a set of integer numbers.

Lemma 4.4. *Let* $\left[C_{j_1}, \mathrm{NAND}_{|C_{j_1}}, \pi_{|C_{j_1}}\right] - \mathrm{SDS}$ *and* $\left[C_{j_2}, \mathrm{NAND}_{|C_{j_2}}, \pi_{|C_{j_2}}\right] - \mathrm{SDS}$ *be two independent sequential dynamical systems over dependency graphs* C_1 *and* C_2 *respectively, and associated with the maxterm* NAND. *Given a periodic orbit of the system restricted to* C_{j_1}*, with period* $m_{j_1} > 1$*, and a periodic orbit of the system restricted to* C_{j_2}*, with period* $m_{j_2} > 1$*, then the system over* $C_{j_1} \cup C_{j_2}$ *has* $\gcd(m_{j_1}, m_{j_2})$ *different periodic orbits of period* $\mathrm{lcm}(m_{j_1}, m_{j_2})$ *associated with the product of such orbits.*

Proof. Firstly, by Theorem 4.4 in Subsection 4.1.1, all the periodic orbits of these systems have period greater than 1.

In this case, we can give an algebraical structure to the periodic orbits Orb_{j_1} and Orb_{j_2} by considering them as cyclic groups generated by a state of each orbit, y and z respectively, as follows:

$$\mathrm{Orb}_{j_1} = \langle y \rangle = \{x \in \{0,1\}^{n_{j_1}} : x = \mathrm{NAND}^t_{|C_{j_1}}(y), \text{ for } t \in \mathbb{N}\},$$

$$\mathrm{Orb}_{j_2} = \langle z \rangle = \{x \in \{0,1\}^{n_{j_2}} : x = \mathrm{NAND}^t_{|C_{j_2}}(z), \text{ for } t \in \mathbb{N}\}.$$

Thus, results about direct product of cyclic groups can be applied in this case. Since $\mathrm{card}(\mathrm{Orb}_{j_1}) = m_{j_1}$ and $\mathrm{card}(\mathrm{Orb}_{j_2}) = m_{j_2}$, it is known (see [58]) that the direct product of these cyclic groups is a group with $m_{j_1} \cdot m_{j_2}$ elements of order $\mathrm{lcm}(m_{j_1}, m_{j_2})$. That is, from the point of view of periodic orbits, this product generates

$$\frac{m_{j_1} \cdot m_{j_2}}{\mathrm{lcm}(m_{j_1}, m_{j_2})} = \gcd(m_{j_1}, m_{j_2})$$

different periodic orbits of period $\mathrm{lcm}(m_{j_1}, m_{j_2})$. □

This definition of product of periodic orbits can be easily extended to a product of a finite number of orbits.

Lemma 4.5. *Let* $\left\{\left[C_j, \mathrm{NAND}_{|C_j}, \pi_{|C_j}\right] - \mathrm{SDS}\right\}_{j=1}^{q}$ *be a family of independent sequential dynamical systems over dependency graphs* $C_1, \ldots, C_q$ *respectively, all of* *them associated with the maxterm* NAND*. Given a periodic orbit of each restricted system, with period $m_j > 1$ respectively, then the system over $\cup_{j=1}^{q} C_j$ has*

$$\frac{\prod_{j=1}^{q} m_j}{\operatorname{lcm}(m_1, \ldots, m_q)}$$

different periodic orbits of period $\operatorname{lcm}(m_1, \ldots, m_q)$ *associated with the product of such orbits.*

Proof. This result is direct for $q = 1$ and coincides with Lemma 4.4 when $q = 2$. For $q > 2$, it follows from the following recurrence rule: set a fixed value t, $2 \leq t < q$, if the product of the orbits $\mathrm{Orb}_1, \ldots, \mathrm{Orb}_t$ when considering the t systems $\left\{ \left[C_j, \mathrm{NAND}_{|C_j}, \pi_{|C_j} \right] - \mathrm{SDS} \right\}_{j=1}^{t}$ generates

$$\frac{\prod_{j=1}^{t} m_j}{\operatorname{lcm}(m_1, \ldots, m_t)}$$

periodic orbits of period $\operatorname{lcm}(m_1, \ldots, m_t)$ each one, then, by Lemma 4.4, the product of only one of these periodic orbits with the periodic orbit of the following system $\left[C_{t+1}, \mathrm{NAND}_{|C_{t+1}}, \pi_{|C_{t+1}} \right] - \mathrm{SDS}$ generates

$$\gcd(\operatorname{lcm}(m_1, \ldots, m_t), m_{t+1}) = \frac{\operatorname{lcm}(m_1, \ldots, m_t) \cdot m_{t+1}}{\operatorname{lcm}(\operatorname{lcm}(m_1, \ldots, m_t), m_{t+1})}$$

periodic orbits. Therefore, the number of periodic orbits when considering the product of the $t+1$ orbits $\mathrm{Orb}_1, \ldots, \mathrm{Orb}_{t+1}$ is

$$\frac{\prod_{j=1}^{t} m_j}{\operatorname{lcm}(m_1, \ldots, m_t)} \frac{\operatorname{lcm}(m_1, \ldots, m_t) \cdot m_{t+1}}{\operatorname{lcm}(\operatorname{lcm}(m_1, \ldots, m_t), m_{t+1})} = \frac{\prod_{j=1}^{t+1} m_j}{\operatorname{lcm}(m_1, \ldots, m_{t+1})},$$

and all of them with period $\operatorname{lcm}(\operatorname{lcm}(m_1, \ldots, m_t), m_{t+1}) = \operatorname{lcm}(m_1, \ldots, m_{t+1})$, by Lemma 4.4. □

Additionally, we establish the following lemma.

Lemma 4.6. *Let* $\left[C_j, \mathrm{NAND}_{|C_j}, \pi_{|C_j} \right] - \mathrm{SDS}$ *be a sequential dynamical system and assume that this system has a unique periodic orbit of period m_j. Then, for any initial state $x^0 \in \{0,1\}^{n_j}$ and a vertex i belonging to the graph C_j, exists $r_i > 1$ such that in the iteration r_i, after the update of the state of the vertex i, all the entities are activated. Furthermore, this pattern is periodically repeated onwards every m_j iterations.*

Proof. By Theorem 4.4 in Subsection 4.1.1, all the periodic orbits of this system have period greater than 1. Since it has a unique periodic orbit of period $m_j > 1$, as will be seen in Proposition 4.3 of Subsection 4.2.1, there are exactly m_j states, $x^1, \ldots, x^{m_j}$, without adjacent complemented vertices with state values equal to 0, being $x^k = \mathrm{NAND}_{|C_j}\left(x^{k-1}\right)$, $1 \leq k \leq m_j$. Note that, one of these configurations is $\mathcal{I}$.

On the other hand, the update of the state of a vertex t from a configuration without adjacent vertices with state values equal to 0 originates other configuration without adjacent vertices with state values equal to 0, because if t has an adjacent vertex with state value 0, the evolution operator NAND updates the state value of t to 1. Thus, from x^1 each update of the state of a unique vertex generates a configuration belonging to $\{x^1, \ldots, x^{m_j}\}$.

Let us analyze the configurations reached after the update of the state of the vertex i in the iterations $2, \ldots, m_j + 1$, respectively $x^{i,2}, \ldots, x^{i,m_j+1}$. Note that $x^{i,k} \neq x^{i,l}$ for all $k, l \in \{2, \ldots, m_j + 1\}$, $k \neq l$, since otherwise we had a periodic orbit with period smaller than m_j. Thereby, $\{x^1, \ldots, x^{m_j}\} = \{x^{i,2}, \ldots, x^{i,m_j+1}\}$ and there is $k \in \{2, \ldots, m_j + 1\}$ such that, in $x^{i,k}$, every variable has state value 1. $\square$

As a consequence of the previous results, we have the following theorem.

Theorem 4.15 (m-Periodic-Orbit Theorem for MAX − SDS)**.** *Let* $[G, \mathrm{MAX}, \pi] -$ SDS *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, it has a unique periodic orbit of period* m*, with* $m > 1$*, if, and only if, the following conditions are satisfied simultaneously:*

i) $W'_C \neq \emptyset$.

ii) Either $\mathrm{MAX} = \mathrm{NAND}$ *or* $\left[G^*, \mathrm{MAX}_{|G^*}, \pi_{|G^*}\right] -$ SDS *has a unique fixed point* ($\mathcal{I}$)*.*

iii) For every $j \in \{1, \ldots, q\}$, $\left[C_j, \mathrm{MAX}_{|C_j}, \pi_{|C_j}\right] -$ SDS *has a unique periodic orbit with period* $m_j > 1$ *(i.e., in the restricted system every state reaches the subconfiguration with all the entities activated).*

iv) For all $i, j \in \{1, \ldots, q\}$, $i \neq j$, m_i *and* m_j *are coprime integers, being* $m = \prod_{j=1}^{q} m_j$.

Proof. In Lemma 4.2, we have proved the result for the particular case $\mathrm{MAX} = \mathrm{NAND}$. In this case, $W'_C = V$, $q = 1$, $C_1 = G$ and the graph G^* is not defined since the set of direct vertices is empty. Thus, let us assume that $\mathrm{MAX} \neq \mathrm{NAND}$.

Under this assumption, let us suppose that $[G, \text{MAX}, \pi] - \text{SDS}$ has a unique periodic orbit of period m, with $m > 1$. First, note that from Theorems 4.7, 4.8 and Corollary 4.1 in Subsection 4.1.2, there exists $i \in W'$ such that $W \cap A_G(i) = \emptyset$, i.e., $W'_C \neq \emptyset$.

Moreover, we know that, once the m-period has been reached, the restriction of the system to each connected component C_j, $1 \leq j \leq q$, performs as an SDS with NAND as evolution operator. Then, since the system has a unique m-period, by Lemma 4.5, each $\left[C_j, \text{MAX}_{|C_j}, \pi_{|C_j}\right] - \text{SDS}$, $1 \leq j \leq q$, must have a unique m_j-periodic orbit, and these periods, $m_1, \ldots, m_q$, must be pairwise coprime, being $m = \prod_{j=1}^{q} m_j$.

Finally, let us see that $\left[G^*, \text{MAX}_{|G^*}, \pi_{|G^*}\right] - \text{SDS}$ has a unique fixed point. Note that for this system, the subset of direct vertices coincides with the one in the global system $[G, \text{MAX}, \pi] - \text{SDS}$, i.e., W. On the other hand, the subset of complemented vertices in $\left[G^*, \text{MAX}_{|G^*}, \pi_{|G^*}\right] - \text{SDS}$ is W'_D. In particular, since in G^* every complemented vertex is adjacent to a direct vertex, all the periodic orbits of $\left[G^*, \text{MAX}_{|G^*}, \pi_{|G^*}\right] - \text{SDS}$ are fixed points (see Theorems 4.7, 4.8 and Corollary 4.1 of Subsection 4.1.2).

Observe that given a fixed point of $\left[G^*, \text{MAX}_{|G^*}, \pi_{|G^*}\right] - \text{SDS}$, we can construct a periodic orbit for the system $[G, \text{MAX}, \pi] - \text{SDS}$ by fixing the values of the variables associated with the vertices in $W \cup W'_D$ as in such fixed point, and considering the periodic orbit starting at the configuration with all the vertices in W'_C activated. Hence, since the initial system $[G, \text{MAX}, \pi] - \text{SDS}$ has a unique periodic orbit of period m, it follows that $\left[G^*, \text{MAX}_{|G^*}, \pi_{|G^*}\right] - \text{SDS}$ has a unique fixed point.

Conversely, let us assume that i), ii), iii) and iv) hold, being $\text{MAX} \neq \text{NAND}$.

Firstly, since $W'_C \neq \emptyset$, the only periodic orbits of $[G, \text{MAX}, \pi] - \text{SDS}$ are periodic orbits with period greater than 1 (see Theorems 4.7, 4.8 and Corollary 4.1 of Subsection 4.1.2).

On the other hand, we know that the restriction of the system to each C_j, $1 \leq j \leq q$, performs as a $\text{NAND} - \text{SDS}$ once a periodic orbit has been reached. Then, since $\left[C_j, \text{MAX}_{|C_j}, \pi_{|C_j}\right] - \text{SDS}$ has a unique periodic orbit with period $m_j > 1$ for each $j \in \{1, \ldots, q\}$, and m_i, m_j are coprime integers for $i, j \in \{1, \ldots, q\}$ with $i \neq j$, by Lemma 4.5, the product of the sub-cycles of the subsystems $\left[C_j, \text{MAX}_{|C_j}, \pi_{|C_j}\right] - \text{SDS}$ generates a unique periodic orbit of period $m = \prod_{j=1}^{q} m_j$.

We also know that, in an m-periodic orbit of the system, all the variables associated with vertices in W'_D fix their values to 1. To finish the proof, we will see that if $\left[G^*, \text{MAX}_{|G^*}, \pi_{|G^*}\right] - \text{SDS}$ has a unique fixed point, then in a periodic orbit of the system $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ all the variables associated with the direct vertices in $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ (i.e. the vertices in W) fix their values to 1.

Observe that the connected components which result in G^* when we remove all its complemented vertices (i.e. the vertices in W'_D) and the edges which are incident to those vertices coincide with the ones for G, i.e., they are $G_1, \dots, G_p$.

Since $\left[G^*, \mathrm{MAX}_{|_{G^*}}, \pi_{|_{G^*}}\right] - \mathrm{SDS}$ has a unique fixed point, from Theorem 4.13 in Subsection 4.1.3, we have that for every j, $1 \leq j \leq p$, there exists $i_j \in A_{G^*}(G_j) \cap W'_D$ such that $i_j \notin A_{G^*}(G_k)$ for $k \neq j$.

Reasoning by reduction to the absurd, let us suppose that for a periodic orbit of $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ there exists a vertex $i \in W$ whose variable fixes its value to 0. Let G_j be the connected component containing i, and take $i_j \in A_{G^*}(G_j) \cap W'_D$ such that $i_j \notin A_{G^*}(G_k)$ for $k \neq j$. In particular, since $i_j \in W'_D$, its associated variable fixes its value to 1 in the periodic orbit. Regarding the variables associated with the vertices in G_j, all of them fix their values to 0 in the periodic orbit.

On the other hand, once the periodic orbit has been reached, for each k, $1 \leq k \leq q$, there exists $r_k > 1$ such that every vertex in C_k has state value 1 when i_j updates in the r_k-th iteration. In fact, this r_k can be chosen as

- the one obtained by applying Lemma 4.6 over the last vertex in C_k updating before i_j according the order established in π, if any, or, in other case,
- the following iteration to the one obtained by applying Lemma 4.6 over the last vertex in C_k to update according the order in π.

Also, since this pattern is repeated after m_k iterations (see Lemma 4.6) and the integers $m_1, \dots, m_q$ are pairwise coprime, by the Chinese Remainder Theorem, there exists $r > 1$ such that every vertex in C_k, for all $k \in \{1, \dots, q\}$, has state value 1 when the entity i_j updates in the r-th iteration, once the periodic orbit has been reached.

In particular, for this iteration, when the state of the entity i_j updates, all the variables associated with direct vertices adjacent to i_j are deactivated, and the variables associated with complemented vertices adjacent to i_j are activated. But it implies that the state of the vertex i_j would change from 1 to 0 in the iteration r, once the periodic orbit has been reached, which is a contradiction. Therefore, all the variables associated with the vertices in W fix their values to 1 in the periodic orbit and the proof finishes. □

Dually, we have the following result.

Theorem 4.16 (m-Periodic-Orbit Theorem for MIN − SDS)**.** *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, it has a unique periodic orbit of period* m*, with* $m > 1$*, if, and only if, the following conditions are satisfied simultaneously:*

i) $W'_C \neq \emptyset$.

ii) Either $\mathrm{MIN} = \mathrm{NOR}$ *or* $\left[G^*, \mathrm{MIN}_{|G^*}, \pi_{|G^*}\right] - \mathrm{SDS}$ *has a unique fixed point (*$\mathcal{O}$*).*

iii) For every $j \in \{1, \ldots, q\}$, $\left[C_j, \mathrm{MIN}_{|C_j}, \pi_{|C_j}\right] - \mathrm{SDS}$ *has a unique periodic orbit with period* $m_j > 1$ *(i.e., in the restricted system every state reaches the sub-configuration with all the entities deactivated).*

iv) For all $i, j \in \{1, \ldots, q\}$, $i \neq j$, m_i *and* m_j *are coprime integers, being* $m = \prod_{j=1}^{q} m_j$.

4.1.5 Maximum number of fixed points

By Theorems 4.7, 4.8, 4.9, 4.10 and Corollaries 4.1, 4.2 of Subsection 4.1.2, it is known that all of the periodic orbits of an SDS are fixed points if, and only if, $W'_C = \emptyset$. Also, by Theorem 4.13 in Subsection 4.1.3, there is a unique fixed point if, and only if, $A_G(G_j) \cap W'_1 \neq \emptyset$ for all j, $1 \leq j \leq p$.

In this subsection, we consider a $\mathrm{MAX} - \mathrm{SDS}$ over a dependency graph G such that its periodic orbits are only fixed points. Our purpose is to find an upper bound for the number of fixed points, according to the maxterm MAX and the structure of G.

In this case, we have the following theorem.

Theorem 4.17. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX *such that* $W'_C = \emptyset$*. Then:*

- *If* $W' = \emptyset$ *(i.e.,* $\mathrm{MAX} = \mathrm{OR}$*), there are exactly two fixed points:* $\mathcal{I}$ *and* $\mathcal{O}$*.*
- *If* $W' \neq \emptyset$*, then there are at most* $2^p - 1$ *fixed points, being* p *the number of connected components,* $G_1, \ldots, G_p$*, which result from* G *when we remove all the vertices in* W' *and the edges which are incident to those vertices.*

Proof. This result follows directly from the upper bound obtained in Theorem 3.11 of Subsection 3.1.5 for PDS, since the fixed points in a $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ coincide with the fixed points of the associated $[G, \mathrm{MAX}] - \mathrm{PDS}$ (see Lemma 4.1 in Subsection 4.1.1). □

Dually, for an SDS associated with a minterm MIN, we have the following result.

Theorem 4.18. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN *such that* $W'_C = \emptyset$. *Then:*

- *If* $W' = \emptyset$ *(i.e.,* $\mathrm{MIN} = \mathrm{AND}$*), there are exactly two fixed points:* $\mathcal{O}$ *and* $\mathcal{I}$.
- *If* $W' \neq \emptyset$*, then there are at most* $2^p - 1$ *fixed points, being* p *the number of connected components,* $G_1, \ldots, G_p$*, which result from* G *when we remove all the vertices in* W' *and the edges which are incident to those vertices.*

Moreover, the upper bound in Theorem 4.17 is attained in the SDS generated from the PDS shown in Example 3.2 of Subsection 3.1.5 with the identity as updating permutation. A dual example of an SDS on a minterm can be constructed where the upper bound in Theorem 4.18 is also attained.

4.1.6 Maximum number of periodic orbits

In this subsection, we establish an upper bound for the number of periodic orbits of period greater than 1 in a $\mathrm{MAX} - \mathrm{SDS}$ (resp. $\mathrm{MIN} - \mathrm{SDS}$).

Firstly, we analyze the particular case when $\mathrm{MAX} = \mathrm{NAND}$ (resp. $\mathrm{MIN} = \mathrm{NOR}$).

Thanks to Proposition 4.3 (resp. Proposition 4.4) in Subsection 4.2.1, we obtain an upper bound for the number of periodic orbits of a $[G, \mathrm{NAND}, \pi] - \mathrm{SDS}$ (resp. $[G, \mathrm{NOR}, \pi] - \mathrm{SDS}$) over a dependency graph $G = (V, E)$, which only depends on the number n of vertices of V.

Proposition 4.1. *Let* $[G, \mathrm{NAND}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND*. Then, the number of periodic orbits is upper bounded by* $\max\{1, 2^{n-2}\}$.

Proof. From Proposition 4.3 in Subsection 4.2.1, the periodic points are the configurations such that no pair of adjacent vertices have state values equal to 0. Let us see that the number of such configurations in a connected graph is, at most, $2^{n-1}+1$. Then, the result is completed considering that a periodic orbit must have at least 2 states.

Reasoning by reduction to the absurd, let us suppose that there exists a graph G with $V = \{1, \dots, n\}$ and more than $2^{n-1} + 1$ periodic points. We can assume, without loss of generality, that n is the minimum with the property that there is a connected graph with n vertices and more than $2^{n-1} + 1$ periodic points. Since, clearly, this is not the case for $n = 1$ and $n = 2$ (i.e., $G = K_2$), it must be $n \geq 3$.

Since G is a connected finite graph, there is a vertex i such that it is not an articulation point[1]. Removing i from G and the edges incident to it, we have a connected graph $\overline{G}$ with $n-1 \geq 2$ vertices in which the upper bound is valid because of the way in which n in taken. Thus, $\overline{G}$ allows, at most, $2^{n-2} + 1$ configurations without adjacent vertices with state value 0.

Therefore, returning to G, there are a maximum of $2^{n-2} + 1$ configurations with $x_i = 1$ and without adjacent vertices with state value 0 (those ones obtained from $\overline{G}$). On the other hand, there are, at most, 2^{n-2} configurations with $x_i = 0$ and without adjacent vertices with state value 0. Observe that, if $x_i = 0$, it must be $x_j = 1$ for all $j \in A_G(i)$, and $\mathrm{card}(A_G(i)) \geq 1$ since G is connected.

This means that G allows a maximum of $2^{n-2}+1+2^{n-2} = 2^{n-1}+1$ configurations without adjacent vertices with state value 0, contradicting the initial assumption, and so the proof finishes. □

Dually, we have the following proposition.

Proposition 4.2. *Let* $[G, \mathrm{NOR}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR. *Then, the number of periodic orbits is upper bounded by* $\max\{1, 2^{n-2}\}$.

Next, we construct a $[G, \mathrm{NAND}, \pi] - \mathrm{SDS}$ where the upper bound obtained in Proposition 4.1 is attained. A dual example for a $[G, \mathrm{NOR}, \pi]-\mathrm{SDS}$ can be similarly constructed where the upper bound obtained in Proposition 4.2 is attained.

Example 4.4. Let us consider the star graph $G = (V, E)$ with $V = \{1, \dots, n\}$, $n > 1$, $E = \{\{1, i\} : i = 2, \dots, n\}$, $\pi = 1|\dots|n$, the identity permutation, and take the system $[G, \mathrm{NAND}, \pi] - \mathrm{SDS}$.

[1] Recall that an articulation point or cut vertex is a vertex that, if removed, disconnects the graph.

This system has $2^{n-1}+1$ states without adjacent vertices with state values equal to 0, i.e., from Proposition 4.3 in Subsection 4.2.1, $2^{n-1}+1$ periodic points:

- Whichever configuration of values with $x_1 = 1$.
- The one with $x_1 = 0$ and $x_j = 1$ for all $j \in V \setminus \{1\}$.

Specifically, there is the 3-periodic orbit:

$$(0,1,\ldots,1) \to (1,0,\ldots,0) \to (1,1,\ldots,1) \to (0,1,\ldots,1)\,,$$

and the other periodic points, if any, belong to 2-periodic orbits. In such 2-periodic orbits, x_1 always has state value 1 and the other vertices alternate state values 0 and 1 after each iteration.

Therefore, the system has 2^{n-2} periodic orbits. $\square$

In view of this, we are able to obtain an upper bound for the number of periodic orbits of a MAX $-$ SDS taking into account Lemma 4.5 of Subsection 4.1.4.

Let n_j be the number of vertices of C_j. In particular, if $p = 0$, then $\mathrm{MAX} = \mathrm{NAND}$ and $q = 1$. That is, we are under the assumptions of Proposition 4.1, i.e., the number of periodic orbits of the system is, at most, $\max\{1, 2^{n_1-2}\}$.

With all of this, in the case of a general maxterm MAX, we have the following theorem.

Theorem 4.19. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*, such that* $W'_C \neq \emptyset$ *and* $\mathrm{MAX} \neq \mathrm{NAND}$*. Then, all the periodic orbits of the system have period greater than* 1*, and the number of them is, at most,*

$$\left(2^p \frac{\prod_{j=1}^{q}\left(2^{n_j-1}+1\right)}{2}\right) - 1.$$

Proof. Since $W'_C \neq \emptyset$, from Theorems 4.7, 4.8 and Corollary 4.1 in Subsection 4.1.2, every periodic orbit of this system has period greater than 1. Also, since $\mathrm{MAX} \neq \mathrm{NAND}$, then $W'_D \neq \emptyset$.

We also know that, if x^0 is a periodic point of the system, then for all $i \in W'_D$, $x_i^0 = 1$ (and so, $x_i^m = 1$ for all $m \in \mathbb{N}$). Moreover, the restriction of the system to each connected component G_j (resp. C_j) performs as an $\mathrm{OR}-\mathrm{SDS}$ (resp. $\mathrm{NAND}-\mathrm{SDS}$), once a periodic orbit has been reached.

Thus, in each connected component G_j, $1 \leq j \leq p$, either all the entities associated with the vertices in G_j are activated or all of them are deactivated in the periodic point, which leads to 2^p combinations.

On the other hand, if in a periodic orbit of the global system the restriction to each C_j performs as a $\mathrm{NAND}-\mathrm{SDS}$, the periodic orbits in the global system are originated from the product of independent periodic orbits of these restricted systems.

Let us assume that, for $1 \leq j \leq q$, the system restricted to C_j has m_j periodic points, distributed into s_j periodic orbits of respective periods $m_{j,k}$, for $1 \leq k \leq s_j$. By Lemma 4.5 in Subsection 4.1.4, the number of periodic orbits resulting from the product considering in each system restricted to C_j, $1 \leq j \leq q$, a periodic orbit Orb_{j,k_j} of period m_{j,k_j} is

$$\frac{\prod_{j=1}^{q} m_{j,k_j}}{\mathrm{lcm}\left(\{m_{j,k_j}\}_{j=1}^{q}\right)} \leq \frac{\prod_{j=1}^{q} m_{j,k_j}}{2}$$

since $\mathrm{lcm}\left(\{m_{j,k_j}\}_{j=1}^{q}\right) \geq 2$ because each period $m_{j,k_j} \geq 2$. Therefore, the total number of periodic orbits is upper bounded by

$$\sum_{k_1=1}^{s_1} \cdots \sum_{k_q=1}^{s_q} \frac{\prod_{j=1}^{q} m_{j,k_j}}{2} = \frac{\sum_{k_1=1}^{s_1} \cdots \sum_{k_q=1}^{s_q} \prod_{j=1}^{q} m_{j,k_j}}{2}$$

and, by applying the distributive property,

$$\frac{\sum_{k_1=1}^{s_1} \cdots \sum_{k_q=1}^{s_q} \prod_{j=1}^{q} m_{j,k_j}}{2} = \frac{\left(\sum_{k_1=1}^{s_1} m_{1,k_1}\right) \cdots \left(\sum_{k_q=1}^{s_q} m_{q,k_q}\right)}{2} = \frac{\prod_{j=1}^{q} m_j}{2}.$$

Since in a periodic orbit each system restricted to C_j acts as a $\mathrm{NAND}-\mathrm{SDS}$, from Proposition 4.1, this total number of periodic orbits is upper bounded by

$$\frac{\prod_{j=1}^{q} \left(2^{n_j-1}+1\right)}{2}.$$

Finally, not all these combinations correspond to periodic orbits. Indeed, if i is the entity in W'_D which updates in the first place according to the order π, let us consider the following configuration:

- $x_k = 0$ for all $k \in W$.
- $x_k = 1$ for all $k \in W'_D$.
- For each $1 \leq j \leq q$, the following states of the vertices of the graph C_j:
  - $x_k = 1$, if k updates after i.
  - If $k_1, \ldots, k_s$ are the entities which update before i sorted in descending update order (k_s updates before k_{s-1}, which updates before k_{s-2}, and so on), let us consider:
    - $x_{k_1} = 0$.
    - For $2 \leq t \leq s$, if k_t is adjacent to a deactivated vertex of C_j, k_{t_0} with $t_0 < t$, then $x_{k_t} = 1$. Otherwise, $x_{k_t} = 0$.

Since this state is such that there are no adjacent entities with state values 0 in each C_j, $1 \leq j \leq q$, by Proposition 4.3 in Subsection 4.2.1, this configuration generates a periodic point of the system restricted to each C_j, $1 \leq j \leq q$.

Under these assumptions, in the evolution of this configuration, when i updates, all the direct vertices are deactivated and all the complemented vertices are activated. Thus, the state value of $i \in W'_D$ would become 0 after one iteration, which is not possible in a periodic point, as remarked above. □

Observe that, in Theorem 4.19, we need a maxterm different from NAND since, otherwise, $p = 0$, $q = 1$ and the upper bound obtained for the number of periodic orbits of the system is not valid (see Proposition 4.1 and Example 4.4).

Dually, we have the following result.

Theorem 4.20. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN, *such that* $W'_C \neq \emptyset$ *and* MIN $\neq$ NOR. *Then, all the periodic orbits of the system have period greater than* 1, *and the number of them is, at most,*

$$\left(2^p \frac{\prod_{j=1}^{q} (2^{n_j - 1} + 1)}{2}\right) - 1.$$

To finish, in the following example, we construct a $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ where the upper bound obtained in Theorem 4.19 is attained. A dual example of a $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ can be similarly constructed where the upper bound obtained in Theorem 4.20 is attained.

Example 4.5. Let us fix $p > 0$, $q > 0$, and consider the following sets of vertices:

- $W = \{d_1, \ldots, d_p\}$.
- $W'_D = \{c_1\}$.
- $W'_{C_j} = \{\bar{c}_j\}$ for $j = 1, \ldots, q$.

Then, we take as vertex set of the dependency graph $V = W \cup W'_D \cup \left(\cup_{j=1}^{q} W'_{C_j}\right)$.

Regarding the adjacencies among these vertices, we take the edges:

- $\{d_i, c_1\}$ for all $1 \leq i \leq p$.
- $\{c_1, \bar{c}_j\}$ for all $1 \leq j \leq q$.

All these edges constitute the edge set E.

Over this dependency graph $G = (V, E)$, we consider the maxterm MAX whose directed variables are the ones associated with the vertices in W and whose complemented variables are those associated with the vertices in $V \setminus W$.

Also, we consider the updating order given by $\pi = d_1 | \cdots | d_p | c_1 | \bar{c}_1 | \cdots | \bar{c}_q$.

For this $[G, \mathrm{MAX}, \pi]-$SDS over G, we have p connected components, $G_1, \ldots, G_p$, given by $G_i = \{\{d_i\}, \emptyset\}$, and q connected components, $C_1, \ldots, C_q$, with $n_j = 1$ vertex each one, given by $C_j = \left(W'_{C_j}, \emptyset\right)$.

The system constructed above has exactly

$$\left(2^p \frac{\prod_{j=1}^{q} (2^{n_j - 1} + 1)}{2}\right) - 1$$

periodic orbits. Certainly, every connected component C_j is a graph with 1 vertex and so, the restriction of the SDS to C_j has one 2-periodic orbit; and the configuration with $d_i = 0$, $1 \leq i \leq p$, $c_1 = 1$ and $\bar{c}_j = 1$, $1 \leq j \leq q$, does not belong to a periodic orbit, since after one iteration $c_1 = 0$. □

4.2 Dynamics of non-periodic orbits

Following the same scheme as in the parallel case, in this section, we study the dynamics of non-periodic orbits of SDS.

First, we study the existence and uniqueness of predecessor, what naturally leads us to explore the field of the Garden-of-Eden configurations of the system.

Finally, we expose results about attractiveness of periodic orbits, basins of attraction and, specially, we analyze the maximum number of iterations needed to ensure that any configuration reaches a periodic orbit.

4.2.1 Predecessors and GOE configurations

As has been already said in Subsection 3.2.1, in [30, 31, 32], the study of predecessors is divided into four specific problems:

- Predecessor existence (PRE).
- Predecessor uniqueness (UPRE).
- Predecessors coexistence (APRE).
- Number of predecessors of a given state (#PRE).

We will start solving the first of these problems in the context of SDS on maxterm and minterm Boolean functions and we will also provide a characterization of GOE points. Therefrom, some results will be also reached which will allow us to solve the rest of the problems in the previous list.

The next theorem provides a characterization of the existence of predecessors, finding a particular predecessor, named *fundamental predecessor*, of a specific state of the system in the context of an SDS, when it exists.

Theorem 4.21. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, a configuration* y *has a predecessor if, and only if, the state* x *defined as follows is such that* $\mathrm{MAX}(x) = y$*:*

- *For every entity* $i \in V_0 \cup P_0$*,*
  - $x_i = 0$*, if* $i \in W$*,*

  - $x_i = 1$, *if* $i \in W'$.

- *For every entity* $i \in (V_0 \cup P_0)^c$,
  - $x_i = 1$, *if* $i \in W$,
  - $x_i = 0$, *if* $i \in W'$.

Proof. It must only be shown that this condition is necessary for the existence of a predecessor. For this purpose, let us see that if there is a predecessor of y, $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$, then x defined as in this theorem is also a predecessor of y.

Thus, if $i \in V_0 \cup P_0$, it must be

- $\hat{x}_i = 0 = x_i$, if $i \in W$,
- $\hat{x}_i = 1 = x_i$, if $i \in W'$,

since, otherwise, $y_i = 1$, if $i \in V_0$, or $y_j = 1$ for some $j \in A_G(i) \cap V_0$ (those ones that update before i) if $i \in P_0$.

Suppose, by reduction to the absurd, that x is not a predecessor of y. Let $i \in V$ be the first entity, according to the order established by π, such that x_i does not update to y_i. It must be $i \in V_0 \cup P_0$, because the entities in $(V_0 \cup P_0)^c \subseteq V_1$ update to the activated state because of their own state values in x.

If $i \in P_0 \setminus V_0 \subseteq V_1$, let us analyze the possible states of the entities belonging to $\overline{A_G(i)}$:

- Since i is the first entity not updating to the state given by $y_i = 1$, then $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\hat{x}$.
- $x_i = \hat{x}_i$.
- $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$:
  - If $j \in P_0 \cup V_0$, then $x_j = \hat{x}_j$.
  - If $j \in (P_0 \cup V_0)^c$, then $x_j = 1$, if $j \in W$, or $x_j = 0$, if $j \in W'$.

Since $\hat{x}_i$ updates to $y_i = 1$, x_i must also do it, but this is a contradiction and, consequently, $i \notin P_0 \setminus V_0$.

Therefore $i \in V_0$. In this situation:

- Since i is the first entity not updating to the state given by $y_i = 0$, then $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\hat{x}$.
- $x_i = \hat{x}_i$.
- $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$, the entity $j \in P_0$, so $x_j = 0$, if $j \in W$, or $x_j = 1$, if $j \in W'$.

Since $\hat{x}_i$ updates to $y_i = 0$, x_i must also do it, which is a contradiction and, consequently, $i \notin V_0$.

Therefore, there cannot exist $i \in V$ like that and x updates to y. □

Dually, we have the following result.

Theorem 4.22. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, a configuration* y *has a predecessor if, and only if, the state* x *defined as follows is such that* $\mathrm{MIN}(x) = y$*:*

- *For every entity* $i \in V_1 \cup P_1$,
  - $x_i = 1$, *if* $i \in W$,
  - $x_i = 0$, *if* $i \in W'$.
- *For every entity* $i \in (V_1 \cup P_1)^c$,
  - $x_i = 0$, *if* $i \in W$,
  - $x_i = 1$, *if* $i \in W'$.

Theorems 4.21 and 4.22 solve the PRE problem for SDS on maxterm and minterm Boolean functions, respectively, and allow us to establish the following characterization of the GOE points of these systems.

Corollary 4.5 (Characterization of GOE in MAX − SDS)**.** *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, a configuration* y *is a* GOE *point of the system if, and only if, the state* x *defined as in Theorem 4.21 is such that* $\mathrm{MAX}(x) \neq y$.

Corollary 4.6 (Characterization of GOE in MIN − SDS)**.** *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, a configuration* y *is a* GOE *point of the system if, and only if, the state* x *defined as in Theorem 4.22 is such that* $\mathrm{MIN}(x) \neq y$.

Next, we provide sufficient conditions to determine GOE points.

Corollary 4.7. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. If a state* y *is such that* $(Q_0 \cap V_0) \cap W' \neq \emptyset$*, then* y *is a* GOE *point.*

Proof. If $(Q_0 \cap V_0) \cap W' \neq \emptyset$, there is an entity $i \in V_0$ whose corresponding variable in MAX appears in complemented form and with an adjacent entity $j \in V_0$ updating after it. In this situation, the configuration y cannot be obtained as the update of another state x because the evolution of the entity i to the deactivated state makes it impossible the posterior update of the entity j to this state. □

Corollary 4.8. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. If a state* y *is such that* $(Q_0 \cap V_0^c) \cap W \neq \emptyset$*, then* y *is a* GOE *point.*

Proof. If $(Q_0 \cap V_0^c) \cap W \neq \emptyset$, there is an entity $i \in V_1$ whose corresponding variable in MAX appears in direct form and with an adjacent entity $j \in V_0$ updating after it. The proof finishes reasoning as in Corollary 4.7. □

Dually, we have the following results.

Corollary 4.9. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. If a state* y *is such that* $(Q_1 \cap V_1) \cap W' \neq \emptyset$*, then* y *is a* GOE *point.*

Corollary 4.10. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. If a state* y *is such that* $(Q_1 \cap V_1^c) \cap W \neq \emptyset$*, then* y *is a* GOE *point.*

From Corollary 4.7 (resp. Corollary 4.9), it can be deduced that a configuration with two adjacent complemented vertices with state value 0 (resp. 1) is a GOE point of a $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ (resp. $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$). Actually, when MAX = NAND (resp. MIN = NOR), they are the only GOE points of the system.

Indeed, if a configuration y of $[G, \mathrm{NAND}, \pi] - \mathrm{SDS}$ is such that $y_i = y_j = 0$ implies that $\{i, j\} \notin E$, then y has a predecessor given by

- $x_i = 1$, if $i \in V_0 \cup P_0$,
- $x_i = 0$, if $i \in (V_0 \cup P_0)^c$.

Additionally, since in such kinds of systems any configuration reaches a periodic orbit at a maximum of one iteration (see [28]), we have the following result.

Proposition 4.3. *Let* $[G, \mathrm{NAND}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* NAND. *Then, the* GOE *points of the system are the configurations such that there are two adjacent vertices* $i, j \in V$ *with* $x_i = x_j = 0$. *Furthermore, the other configurations belong to periodic orbits.*

Dually, we have the following result.

Proposition 4.4. *Let* $[G, \mathrm{NOR}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* NOR. *Then, the* GOE *points of the system are the configurations such that there are two adjacent vertices* $i, j \in V$ *with* $x_i = x_j = 1$. *Furthermore, the other configurations belong to periodic orbits.*

From Proposition 4.3 (and, dually, from Proposition 4.4), we can see that the pattern of the parallel update case for a $\mathrm{MAX} - \mathrm{PDS}$ over a dependency graph $G = (V, E)$, with $V = \{1, \ldots, n\}$ and $n \geq 2$, whereby a configuration with only one activated entity has no predecessors, is broken in the case of the sequential update, as shown in the following example.

Example 4.6. In the case of the SDS defined by

- $G = (\{1, 2\}, \{\{1, 2\}\})$,
- $\mathrm{MAX} = x_1' \vee x_2'$,
- $\pi = 1|2$,

the configuration $y = (0, 1)$ is not a GOE point because $x = (1, 1)$ is its predecessor. □

In view of these results, we can state the following corollaries about the number of GOE points in an SDS.

Corollary 4.11. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX, *with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$. *Then, the number of* GOE *points,* #GOE, *is such that*

$$1 \leq \#\mathrm{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

Proof. First, we will prove that any SDS with $n \geq 2$ has GOE points. Observe that, as $n \geq 2$, there exist two adjacent entities i and j with i updating before j. If $i \in W$, then a configuration with $y_i = 1$ and $y_j = 0$ has no predecessor; otherwise $i \in W'$ and the same occurs for a configuration with $y_i = y_j = 0$.

In fact, the lower bound is reached, as shown in the example below. Let us consider the $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ defined by

- $G = (\{1,2\}, \{\{1,2\}\})$,
- $\mathrm{MAX} = x_1' \vee x_2'$,
- $\pi = 1|2$.

In this case, $(0,0)$ is a GOE of the system and the rest of states belong to a 3-cycle, as can be checked in Figure 4.7.

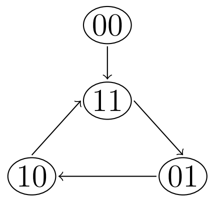

Figure 4.7: Phase portrait of the system $[(\{1,2\}, \{\{1,2\}\}), x_1' \vee x_2', 1|2] - \mathrm{SDS}$.

On the other hand, $\mathcal{I}$ is never a GOE point of the system, because the state x defined as follows is its predecessor:

- $x_i = 1$, if $i \in W$,
- $x_i = 0$, if $i \in W'$.

Also, there is always another configuration with a predecessor, because if $\overline{x}$ is defined as

- $\overline{x}_i = 0$, if $i \in W$,
- $\overline{x}_i = 1$, if $i \in W'$,

then $\overline{x}$ updates to a state $\overline{y}$ such that $\overline{y}_1 = 0$.

As shown in the example below, this upper bound is also reached. Let us consider the following $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$, determined by

- $G = (\{1, 2\}, \{\{1, 2\}\})$,
- $\text{MAX} = x_1 \vee x_2'$,
- $\pi = 1|2$.

The phase portrait of this system is shown in Figure 4.8.

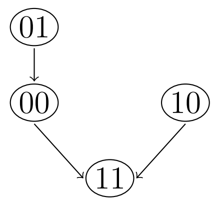

Figure 4.8: Phase portrait of the system $[(\{1, 2\}, \{\{1, 2\}\}), x_1 \vee x_2', 1|2] - \text{SDS}$.

□

Remark 4.3. In Corollary 4.11, $n \geq 2$ has been imposed. This is necessary because a $[G, \text{MAX}, \pi] - \text{SDS}$ with $n = 1$ has 2 fixed points, if $W' = \emptyset$, or one 2-cycle, if $W = \emptyset$. That is, it has not GOE points in any case. □

Dually, we have the following result.

Corollary 4.12. *Let* $[G, \text{MIN}, \pi] - \text{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN, *with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$. *Then, the number of* GOE *points,* #GOE, *is such that*

$$1 \leq \#\text{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

In Theorem 4.21, a constructive proof about the existence of a fundamental predecessor is shown. The conditions regarding a predecessor exposed in that reasoning inspires the following result.

Corollary 4.13. *Let* $[G, \text{MAX}, \pi] - \text{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *If a configuration* y *has a predecessor* x, *the following conditions are verified:*

- *If* $y_i = 0$, *for every entity* $j \in \overline{A_G(i)}$, *with* $i = \pi_r$ *and* $j = \pi_s$*:*

- *If* $i = j$ *or* $r < s$*:*
  - $\diamond$ $x_j = 0$*, if* $j \in W$*, or*
  - $\diamond$ $x_j = 1$*, if* $j \in W'$*.*
- *If* $r > s$*:*
  - $\diamond$ $y_j = 0$*, if* $j \in W$*, or*
  - $\diamond$ $y_j = 1$*, if* $j \in W'$*.*

- *If* $y_i = 1$*, there exists an entity* $j \in \overline{A_G(i)}$ *such that if* $i = \pi_r$ *and* $j = \pi_s$*, at least one of the following conditions is accomplished:*
  - $i = j$ *or* $r < s$*, and:*
    - $\diamond$ $x_j = 1$*, if* $j \in W$*, or*
    - $\diamond$ $x_j = 0$*, if* $j \in W'$*.*
  - $r > s$*, and:*
    - $\diamond$ $y_j = 1$*, if* $j \in W$*, or*
    - $\diamond$ $y_j = 0$*, if* $j \in W'$*.*

Proof. On one hand, if $y_i = 0$ and there is $j \in \overline{A_G(i)}$ such that the conditions shown are not satisfied in this case, the entity i will update to the activated state due to this adjacent entity j, which is a contradiction. On the other hand, if $y_i = 1$ and $\forall j \in \overline{A_G(i)}$ these conditions are not satisfied, the entity i will update to the deactivated state, which is also a contradiction. $\square$

Dually, we have the following result.

Corollary 4.14. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. If a configuration* y *has a predecessor* x*, the following conditions are verified:*

- *If* $y_i = 1$*, for every entity* $j \in \overline{A_G(i)}$*, with* $i = \pi_r$ *and* $j = \pi_s$*:*
  - *If* $i = j$ *or* $r < s$*:*
    - $\diamond$ $x_j = 1$*, if* $j \in W$*, or*
    - $\diamond$ $x_j = 0$*, if* $j \in W'$*.*
  - *If* $r > s$*:*
    - $\diamond$ $y_j = 1$*, if* $j \in W$*, or*

◇ $y_j = 0$, *if* $j \in W'$.

- *If* $y_i = 0$, *there exists an entity* $j \in \overline{A_G(i)}$ *such that if* $i = \pi_r$ *and* $j = \pi_s$, *at least one of the following conditions is accomplished:*
  - $i = j$ *or* $r < s$, *and:*
    - ◇ $x_j = 0$, *if* $j \in W$, *or*
    - ◇ $x_j = 1$, *if* $j \in W'$.
  - $r > s$, *and:*
    - ◇ $y_j = 0$, *if* $j \in W$, *or*
    - ◇ $y_j = 1$, *if* $j \in W'$.

In a $\mathrm{MAX}-\mathrm{SDS}$ (resp. $\mathrm{MIN}-\mathrm{SDS}$), the entities whose state is deactivated (resp. activated) in y determine univocally their state and the state of their adjacent entities in P_0 (resp. P_1) in any predecessor x, if such a predecessor exists. However, for any entity whose state value is 1 (resp. 0) in y, it is only necessary the intervention of a timely adjacent entity, or itself, with the appropriate state in the moment of its update. This point is the key to solve the UPRE, APRE and #PRE problems hereafter.

The following theorems allow us to determine if, given a state y with a predecessor x, there are other configurations different from x such that they are also predecessors of y. Thus, the UPRE and APRE problems in the context of an SDS on maxterm and minterm Bolean functions are solved.

Theorem 4.23. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Let* y *be a configuration of the system such that it has a predecessor. Then, this predecessor of* y *is not unique if, and only if, there is a predecessor of* y *belonging to the following set:*

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\, i \in (V_0 \cup P_0)^c \text{ such that } \hat{x}_i \neq x_i \text{ and } \hat{x}_j = x_j\ \forall j \in V \setminus \{i\}\},$$

being x *the fundamental predecessor of* y *described in Theorem 4.21.*

Proof. Since the fundamental predecessor x defined as in Theorem 4.21 is such that $x \notin \mathfrak{P}$, it must only be shown that this condition is necessary for the existence of a predecessor different from x. For this purpose, let us see that, if there is a predecessor of y different from x, $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$, then there exists a state $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathfrak{P}$ such that $\hat{x}$ is also a predecessor of y.

Given that $\overline{x} \neq x$, being $\overline{x}$ a predecessor of y, by Corollary 4.13, there is an entity $i_0 \in (V_0 \cup P_0)^c$ such that $\overline{x}_{i_0} \neq x_{i_0}$. Let us take $\hat{x}$ as the only element of $\mathfrak{P}$ such that $\hat{x}_{i_0} \neq x_{i_0}$ (consequently, $\hat{x}_{i_0} = \overline{x}_{i_0}$), and let us see that this state is a predecessor of y.

Suppose, by reduction to the absurd, that $\hat{x}$ is not a predecessor of y. Let $i \in V$ be the first entity, according to the order established by π, such that $\hat{x}_i$ does not update to y_i. It must be $i \in V_0 \cup P_0 \cup \{i_0\}$, because the entities in $(V_0 \cup P_0 \cup \{i_0\})^c \subseteq V_1$ update to the activated state because of their own state values in $\hat{x}$.

If $i \in P_0 \setminus (V_0 \cup \{i_0\}) = P_0 \setminus V_0 \subseteq V_1$, let us analyze the possible states of the entities belonging to $\underline{A_G(i)}$:

- Since i is the first entity not updating to the state given by $y_i = 1$, then $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\overline{x}$.
- $\hat{x}_i = \overline{x}_i$.
- $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$:
  - If $j \in P_0 \cup V_0 \cup \{i_0\}$, then $\hat{x}_j = \overline{x}_j$.
  - If $j \in (P_0 \cup V_0 \cup \{i_0\})^c$, then either $\hat{x}_j = 1$, if $j \in W$, or $\hat{x}_j = 0$, if $j \in W'$.

Since $\overline{x}_i$ updates to $y_i = 1$, $\hat{x}_i$ must also do it, but this is a contradiction and, consequently, $i \notin P_0 \setminus V_0$.

If $i \in V_0 \setminus \{i_0\} = V_0$, we have the following:

- Since i is the first entity not updating to the state given by $y_i = 0$, then $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\overline{x}$.
- $\hat{x}_i = \overline{x}_i$.
- $\forall j \in A_G(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$, the entity $j \in P_0 = P_0 \setminus \{i_0\}$, so either $x_j = 0$, if $j \in W$, or $x_j = 1$, if $j \in W'$.

Since $\overline{x}_i$ updates to $y_i = 0$, $\hat{x}_i$ must also do it, which is a contradiction and, consequently, $i \notin V_0$.

Therefore $i = i_0$. In this situation, we have the following:

- Since i is the first entity not updating to the state given by $y_i = 1$ ($i_0 \in (V_0 \cup P_0)^c \subseteq V_1$), then $\forall j \in A_G(i_0)$ with $i_0 = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\overline{x}$.

- $\hat{x}_{i_0} = \overline{x}_{i_0}$.

- $\forall j \in A_G(i_0)$ with $i_0 = \pi_r$, $j = \pi_s$ and $s > r$:

    - If $j \in P_0 \cup V_0$, then $\hat{x}_j = \overline{x}_j$.
    - If $j \in (P_0 \cup V_0)^c$, then either $\hat{x}_j = 1$, if $j \in W$, or $\hat{x}_j = 0$, if $j \in W'$.

Since $\overline{x}_i$ updates to $y_i = 1$, $\hat{x}_i$ must also do it, but this is also a contradiction and, consequently, $i \neq i_0$.

Therefore, there cannot exist $i \in V$ like that and $\hat{x}$ updates to y. □

Remark 4.4. The previous result reduces an initial exponentially-sized problem, the search of a particular configuration among the 2^n possible states of the system, into another one in which, at most, n cases must be analyzed. In this case, a short list of possible candidates is given and the evaluation of the evolution operator only over the elements of this set provides the answer to the global problem of existence of a unique predecessor for a state y. □

Dually, we have the following result.

Theorem 4.24. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Let* y *be a configuration of the system such that it has a predecessor. Then, this predecessor of* y *is not unique if, and only if, there is a predecessor of* y *belonging to the following set:*

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\, i \in (V_1 \cup P_1)^c \text{ such that } \hat{x}_i \neq x_i \text{ and } \hat{x}_j = x_j \ \forall j \in V \setminus \{i\}\},$$

being x *the fundamental predecessor of* y *described in Theorem 4.22.*

These results respond to the question of the existence of more than one predecessor for a state y. The next step is to go deeper into this topic, getting the number of them. In the following results, we explain a method to obtain all the predecessors of y and, consequently, this number in order to solve the predecessor problem #PRE.

Corollary 4.15. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Let* y *be a configuration and let us consider the following iterative process:*

- $\mathfrak{P}_{n+1} = \{y\}$.
- *If* $i \in V$, *then*

$$\mathfrak{P}_i = \{x \in \{0,1\}^n : \exists \overline{y} \in \mathfrak{P}_{i+1} \ / \ x_j = \overline{y}_j \ if \ j \neq \pi_i \ and \ \mathrm{MAX}_{|\overline{A_G(\pi_i)}}(x) = y_{\pi_i}\},$$

being $\mathrm{MAX}_{|\overline{A_G(\pi_i)}}$ *the restriction of* MAX *over* $\overline{A_G(\pi_i)}$.

Then, $\mathfrak{P}_1$ *is the set of all the predecessors of* y.

Example 4.7. Let us illustrate this procedure with a particular example, in order to clarify the notation. We consider the $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ defined by

- $G = (\{1,2,3\}, \{\{1,3\}, \{2,3\}\})$,
- $\mathrm{MAX} = x_1 \vee x_2 \vee x_3$,
- $\pi = 1|2|3$.

The phase portrait of this system is shown in Figure 4.9.

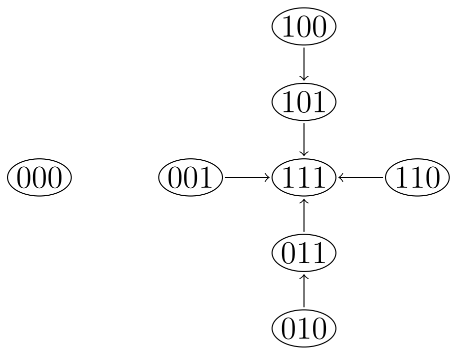

Figure 4.9: Phase portrait of the system $[(\{1,2,3\}, \{\{1,3\}, \{2,3\}\}), x_1 \vee x_2 \vee x_3, 1|2|3] - \mathrm{SDS}$.

The set of predecessors of the configuration $y = (1,1,1)$ is

$$\{(1,1,1), (1,1,0), (0,0,1), (0,1,1), (1,0,1)\}.$$

Let us see that this set is obtained as $\mathfrak{P}_1$ at the end of the iterative process starting with $\mathfrak{P}_4 = \{(1,1,1)\}$.

Firstly, $\mathfrak{P}_3$ is obtained: since $\pi_3 = 3$, the only configurations that can belong to $\mathfrak{P}_3$ are $(1,1,0)$ and $(1,1,1)$. Besides, since $\mathrm{MAX}_{|\overline{A_G(3)}}(1,1,0) = \mathrm{MAX}_{|\overline{A_G(3)}}(1,1,1) = 1 = y_3$, then $\mathfrak{P}_3 = \{(1,1,0),(1,1,1)\}$.

Then, to obtain $\mathfrak{P}_2$, since $\pi_2 = 2$ and considering the elements in $\mathfrak{P}_3$, the only configurations that can belong to this set $\mathfrak{P}_2$ are $(1,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(1,1,1)$. Now, $\mathrm{MAX}_{|\overline{A_G(2)}}(1,0,0)$ is equal to $0 \neq y_2$, while $\mathrm{MAX}_{|\overline{A_G(2)}}(1,1,0)$, $\mathrm{MAX}_{|\overline{A_G(2)}}(1,0,1)$ and $\mathrm{MAX}_{|\overline{A_G(2)}}(1,1,1)$ are all equal to $1 = y_2$. Hence, the set $\mathfrak{P}_2$ is as follows: $\mathfrak{P}_2 = \{(1,1,0),(1,0,1),(1,1,1)\}$.

Finally, since $\pi_1 = 1$ and knowing the set $\mathfrak{P}_2$, the only configurations that can belong to $\mathfrak{P}_1$ are $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$. Now, $\mathrm{MAX}_{|\overline{A_G(1)}}(0,1,0)$ is equal to $0 \neq y_1$, while $\mathrm{MAX}_{|\overline{A_G(1)}}(1,1,0)$, $\mathrm{MAX}_{|\overline{A_G(1)}}(0,0,1)$, $\mathrm{MAX}_{|\overline{A_G(1)}}(1,0,1)$, $\mathrm{MAX}_{|\overline{A_G(1)}}(0,1,1)$ and $\mathrm{MAX}_{|\overline{A_G(1)}}(1,1,1)$ are all equal to $1 = y_1$. Thus, $\mathfrak{P}_1 = \{(1,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}$. $\square$

Dually, we have the following result.

Corollary 4.16. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Let* y *be a configuration and let us consider the following iterative process:*

- $\mathfrak{P}_{n+1} = \{y\}$.
- *If* $i \in V$, *then*

  $$\mathfrak{P}_i = \{x \in \{0,1\}^n : \exists \overline{y} \in \mathfrak{P}_{i+1} \ / \ x_j = \overline{y}_j \ if \ j \neq \pi_i \ and \ \mathrm{MIN}_{|\overline{A_G(\pi_i)}}(x) = y_{\pi_i}\},$$

  being $\mathrm{MIN}_{|\overline{A_G(\pi_i)}}$ *the restriction of* MIN *over* $\overline{A_G(\pi_i)}$.

Then, $\mathfrak{P}_1$ *is the set of all the predecessors of* y.

These last procedures allow us to know all the predecessors of a state y in an SDS and, consequently, the number of them. However, the calculus of the number of predecessors for a state of the entities depends on the connections among entities in the particular graph. In this case, as traditionally done in other contexts, we have been able to get a bound for the number of predecessors of a configuration, which is given in the following theorems.

Theorem 4.25. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX. *Then, the number of predecessors of a given state* y *is upper bounded by* $2^{\#(V_0 \cup P_0)^c}$. *Moreover, this bound is the best possible because it is reachable.*

Proof. From Theorem 4.21 and Corollary 4.13, the states of the entities belonging to $V_0 \cup P_0$ in a possible predecessor of y are fixed. Since the state values of the rest of entities are either 0 or 1, a first upper bound for the number of predecessors is $2^{\#(V_0 \cup P_0)^c}$.

This upper bound is the best possible because it is reached in the following example. Let us consider the $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ defined by

- $G = (V, E)$, with $V = \{1, \ldots, n\}$, $n \geq 2$, and $E = \{\{2, i\} : i \in V \setminus \{2\}\}$,
- $\mathrm{MAX} = x_1' \vee x_2 \vee \cdots \vee x_n$,
- $\pi = 1|\ldots|n$.

In this context, if $y = (0, 1, \ldots, 1)$, then $V_0 = \{1\}$, $P_0 = \{2\}$, $Q_0 = \emptyset$ and $V_1 = \{2, \ldots, n\}$.

For any predecessor, x, it must be $x_1 = 1$ and $x_2 = 0$ but, in this case, all the other choices for the states of the rest of entities generate predecessors of y. $\square$

Dually, we have the following result.

Theorem 4.26. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN. *Then, the number of predecessors of a given state* y *is upper bounded by* $2^{\#(V_1 \cup P_1)^c}$. *Moreover, this bound is the best possible because it is reachable.*

4.2.2 Convergence to periodic orbits: attractors, global attractors, basins of attraction and transient

The study performed above helps us to give in this subsection some results concerning the attractive character of fixed points and periodic orbits of greater period of any SDS with a maxterm or a minterm Boolean function as global evolution operator, their basins of attraction and the transient (or width) of the system.

Attractive and repulsive periodic orbits

The concept of attractive or repulsive periodic orbit has already been introduced in this thesis in Chapter 2, in Definition 2.4.

As said before, for a periodic orbit, the concept of attractiveness in our context is equivalent to have at least one of the states of the periodic orbit with a predecessor different from the one that it has in its periodic orbit. Thus, we can state the following results that characterize the attractive or repulsive character.

Theorem 4.27. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, a periodic orbit is attractive if, and only if, for a state* y *of such a periodic orbit, there is a predecessor of it belonging to the following set:*

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\ i \in (V_0 \cup P_0)^c \ \textit{such that}\ \hat{x}_i \neq x_i \ \textit{and}\ \hat{x}_j = x_j\ \forall j \in V \setminus \{i\}\},$$

being x *the fundamental predecessor of* y *defined as:*

- *For every entity* $i \in V_0 \cup P_0$,
  - $x_i = 0$, *if* $i \in W$,
  - $x_i = 1$, *if* $i \in W'$.
- *For every entity* $i \in (V_0 \cup P_0)^c$,
  - $x_i = 1$, *if* $i \in W$,
  - $x_i = 0$, *if* $i \in W'$.

Proof. Since a periodic orbit is attractive if one of the states in such an orbit has at least two predecessors, the result is a consequence of Theorem 4.23 in Subsection 4.2.1, where the existence of non-unique predecessors is characterized for this type of SDS. □

Dually, we have the following result.

Theorem 4.28. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, a periodic orbit is attractive if, and only if, for a state* y *of such a periodic orbit, there is a predecessor of it belonging to the following set:*

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\ i \in (V_1 \cup P_1)^c \ \textit{such that}\ \hat{x}_i \neq x_i \ \textit{and}\ \hat{x}_j = x_j\ \forall j \in V \setminus \{i\}\},$$

being x *the fundamental predecessor of* y *defined as:*

- *For every entity* $i \in V_1 \cup P_1$,
  - $x_i = 1$, *if* $i \in W$,

- $x_i = 0$, *if* $i \in W'$.

- *For every entity* $i \in (V_1 \cup P_1)^c$,
  - $x_i = 0$, *if* $i \in W$,
  - $x_i = 1$, *if* $i \in W'$.

Observe that, when an SDS presents a unique fixed point, this is globally attractive. That is, the rest of the orbits of the system converges to such a fixed point. Thus, we can state the conditions that characterize globally attractive fixed points for this class of SDS.

Theorem 4.29. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, this* SDS *has a globally attractive fixed point if, and only if,* $W_C' = \emptyset$*, and* $A_G(G_j) \cap W_1' \neq \emptyset$ *for every* j*,* $1 \leq j \leq p$*. In this situation, the globally attractive fixed point is* $\mathcal{I}$*.*

Proof. Since a fixed point of a $\mathrm{MAX} - \mathrm{SDS}$ is globally attractive if, and only if, it is the unique fixed point of such a system, the result is a consequence of Theorem 4.13 in Subsection 4.1.3, which is, indeed, a Fixed-Point Theorem for this kind of SDS. □

Theorem 4.30. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, this* SDS *has a globally attractive fixed point if, and only if,* $W_C' = \emptyset$*, and* $A_G(G_j) \cap W_1' \neq \emptyset$ *for every* j*,* $1 \leq j \leq p$*. In this situation, the globally attractive fixed point is* $\mathcal{O}$*.*

A similar situation occurs when an SDS presents a unique periodic orbit of period m, being such an orbit globally attractive. Thus, we can state the conditions that characterize globally attractive m-periodic orbits with $m > 1$ for this class of SDS.

Theorem 4.31. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, this system has a globally attractive* m*-periodic orbit with* $m > 1$ *if, and only if, the following conditions are simultaneously satisfied:*

i) $W_C' \neq \emptyset$.

ii) *Either* $\mathrm{MAX} = \mathrm{NAND}$ *or* $\left[G^*, \mathrm{MAX}_{|G^*}, \pi_{|G^*}\right] - \mathrm{SDS}$ *has a unique fixed point* $(\mathcal{I})$*.*

iii) *For every* $j \in \{1, \ldots, q\}$, $\left[C_j, \mathrm{MAX}_{|C_j}, \pi_{|C_j}\right] - \mathrm{SDS}$ *has a unique periodic orbit with period* $m_j > 1$ *(i.e., in the restricted system every state reaches the sub-configuration with all the entities activated).*

iv) *For all* $i, j \in \{1, \ldots, q\}$, $i \neq j$, m_i *and* m_j *are coprime integers, being* $m = \prod_{j=1}^{q} m_j$.

Proof. Since an m-periodic orbit, $m > 1$, of a MAX $-$ SDS is globally attractive if, and only if, it is the unique periodic orbit of period greater than 1 of such a system, the result is a consequence of Theorem 4.15 in Subsection 4.1.4, where it is proved that the conditions *i*), *ii*), *iii*) and *iv*) allow to assure this uniqueness. □

Theorem 4.32. *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*. Then, this system has a globally attractive* m*-periodic orbit with* $m > 1$ *if, and only if, the following conditions are simultaneously satisfied:*

i) $W'_C \neq \emptyset$.

ii) *Either* $\mathrm{MIN} = \mathrm{NOR}$ *or* $\left[G^*, \mathrm{MAX}_{|G^*}, \pi_{|G^*}\right] - \mathrm{SDS}$ *has a unique fixed point* $(\mathcal{O})$.

iii) *For every* $j \in \{1, \ldots, q\}$, $\left[C_j, \mathrm{MAX}_{|C_j}, \pi_{|C_j}\right] - \mathrm{SDS}$ *has a unique periodic orbit with period* $m_j > 1$ *(i.e., in the restricted system every state reaches the sub-configuration with all the entities deactivated).*

iv) *For all* $i, j \in \{1, \ldots, q\}$, $i \neq j$, m_i *and* m_j *are coprime integers, being* $m = \prod_{j=1}^{q} m_j$.

Basin of attraction for attractive periodic orbits

As in the case of PDS, periodic orbits act as organizational kernels of the dynamics of an SDS, since every state finally reaches one of such periodic orbits. Their basins of attraction (see Definition 2.5 in Chapter 2) allow us to describe the phase diagram as much as possible, fractionating it into the different trees that reach the corresponding periodic orbits.

A mechanism to obtain the basin of attraction of any attractive periodic orbit is to get all the predecessors of such a periodic orbit, proceeding as explained in Corollaries 4.15 and 4.16 of Subsection 4.2.1.

As in the parallel case, the importance of determining the set of GOE states (whose characterization can be seen in Corollaries 4.5 and 4.6 of Subsection 4.2.1) is clear in the light of dynamical concepts as attractiveness and basins of attraction of periodic orbits. These states are the beginning of a branch in the tree constituting a basin of attraction associated with an attractive periodic orbit. Thus, they are crucial in order to establish the different basins of attraction. Actually, all the orbits are periodic (and, consequently, repulsive), if the SDS does not present GOE states. But, in view of Corollaries 4.11, 4.12 and Remark 4.3 of Subsection 4.2.1, this is not possible except in the trivial case corresponding to only one entity. Thus, we can state the following corollary.

Corollary 4.17. *Every homogeneous* SDS *on a maxterm or minterm Boolean function with more than one entity has attractors.*

Transient to a fixed point in SDS

One of the most interesting dynamic features of the basins of attraction is the maximum number of iterations needed, by an eventually periodic orbit, to reach its corresponding periodic orbit, which is the width or transient of the system.

As commented in Theorems 4.1 and 4.2 of Subsection 4.1.1, in an SDS on a maxterm or minterm Boolean function there can be orbits of any period. In particular, the simplest maxterm OR and minterm AND, which present only fixed points, are studied in [28], showing that the maximum number of iterations needed by an eventually fixed point to reach the corresponding fixed point is, at most, the diameter of the dependency graph.

Regarding the more general context of SDS on arbitrary maxterm and minterm Boolean functions, recall that fixed points cannot coexist with periodic orbits of greater period (see Corollaries 4.1 and 4.2 in Subsection 4.1.2). Due to that, we have to distinguish between these two cases. In particular, in this subsection, we study the transient of non-periodic orbits to fixed points.

In Subsection 3.2.2, which concerns dynamical attraction in the parallel case, Lemma 3.1 shows an upper bound for the number of iterations needed to ensure that a vertex $i \in W'$ such that $A_G(i) \cap W \neq \emptyset$ and $A_G(i) \cap W' \subseteq A_G(W)$ (i.e., $i \in W_D'^{\alpha}$) takes permanent state value 1 for a general maxterm.

This proof shows that a vertex i can preserve a non-permanent state value 1 for, at most, as many iterations as the number of elements of $A_G(i) \cap W'$. In every update, this non-permanent state 1 can be preserved by the influence of one of these adjacent elements in $A_G(i) \cap W'$, each one just for a unique iteration in the optimal case. After that, i takes state value 0 and, one iteration later, permanent state value 1. For more details, see the proof in Lemma 3.1 of Subsection 3.2.2.

In the case of sequential update, the state value associated to a vertex $i \in V$ at time t is involved, in the same iteration t, in the update of the states of the vertices belonging to $A_G(i)$ which are in the updating order after i and, in the iteration $t+1$, in the update of the states of the vertices belonging to $A_G(i)$ which are in the updating order before i. Thus, the initial state value of the vertex i, x_i^0, does not affect to the update of the state of the vertices belonging to $A_G(i)$ which are in the updating order after i, but x_i^1 does.

For this reason, a vertex $i \in W'_D$ with initial state deactivated can recover this state some iterations later if the entities in $A_G(i) \cap W$ update after i, according to π. Hereafter, i gets permanent state value 1 because all the entities in $A_G(i) \cap W$ become activated.

Thus, in the case of SDS, the same reasoning that in the proof of Lemma 3.1 in Subsection 3.2.2 can be considered to achieve an upper bound for the number of iterations needed to ensure that such a vertex $i \in W_D'^{\alpha}$ takes permanent state value 1, but starting from the configuration reached after the first iteration. This gives an additional iteration with respect to the upper bound shown in Lemma 3.1 of Subsection 3.2.2.

Taking all of this into account, in this case of sequential update, we have the following result:

Lemma 4.7. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*. Then, a vertex* $i \in W_D'^{\alpha}$ *takes permanent state value* 1 *after a maximum of* $m_i + 3$ *iterations, being* $m_i = \mathrm{card}\,(A_G(i) \cap W')$*.*

From Lemma 4.7, we can obtain the following theorem.

Theorem 4.33 (Transient in Fixed-Point $\mathrm{MAX} - \mathrm{SDS}$)**.** *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*, where the structure of* G *only allows fixed points as periodic orbits. Then, every orbit of the system reaches a fixed point after a maximum of*

$$\max_{k \in \{1,\ldots,p\}} \{\mathrm{diam}\,(G_k)\} + \max_{i \in W'} \{m_i + 3\}$$

iterations, being $m_i = \mathrm{card}\,(A_G(i) \cap W')$ *for each* $i \in W'$*.*

Proof. First of all, observe that it must be $p > 0$, because $p = 0$ implies that MAX $=$ NAND and the system does not allow fixed points, as shown in Theorem 4.4 of Subsection 4.1.1.

In particular, if $W' = \emptyset$, then MAX $=$ OR, $p = 1$, $G_1 = G$ and $m_1 = 0$. Then, as proved in [28], the system converges to a fixed point after, at most, $\mathrm{diam}\,(G)$ iterations, what fits with the expression of the upper bound.

Thus, suppose that $W' \neq \emptyset$ (and so $p > 0$). Recall that, as proved in Theorems 4.7, 4.8 and Corollary 4.1 of Subsection 4.1.2, a $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ only presents fixed points as periodic orbits if, and only if, every complemented vertex is adjacent to a direct vertex, i.e., $W'_C = \emptyset$. For this reason, Lemma 4.7 can be applied to every $i \in W'$.

Taking this into account, after $\max_{i \in W'}\{m_i + 3\}$ iterations, every $i \in W'$ has permanent state value 1. Therefore, after such a number of iterations, the complemented vertices neither change their state value nor affect the state of other vertices. Thus, the study of the evolution of the system can be reduced to analyze what happens in the restriction to the subgraph induced by $V \setminus W' = W$.

That is, at this point, the behavior of the entire system can be obtained from the study of the evolution in each connected subgraph $G_1, \ldots, G_p$. Since they only have vertices associated with direct variables, we know that each local system restricted to G_k, $k \in \{1, \ldots, p\}$, converges to a fixed point, and it takes at most $\mathrm{diam}\,(G_k)$ iterations to reach it (see [28]). In view of this, in at most $\max_{k \in \{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\}$ iterations, after the $\max_{i \in W'}\{m_i + 3\}$ iterations needed for ensuring that the state values of the vertices in W' are fixed, every vertex in W reaches a permanent state value.

Therefore, after at most

$$\max_{k \in \{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\} + \max_{i \in W'}\{m_i + 3\}$$

iterations, the system reaches a fixed point. □

Dually, we have the following result.

Theorem 4.34 (Transient in Fixed-Point MIN $-$ SDS)**.** *Let* $[G, \mathrm{MIN}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the minterm* MIN*, where the structure of* G *only allows fixed points as periodic orbits. Then, every orbit of the system reaches a fixed point after a maximum of*

$$\max_{k \in \{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\} + \max_{i \in W'}\{m_i + 3\}$$

iterations, being $m_i = \mathrm{card}\,(A_G\,(i) \cap W')$ *for each* $i \in W'$.

Observe that some max expressions in Theorems 4.33 and 4.34 could be considered over empty sets. In this context, we consider 0 as default value in these cases and also when summations appear over empty sets.

The upper bound provided in Theorem 4.33 is the best possible, since it is reachable as we show in the example below. A dual example can be obtained, in which the upper bound in Theorem 4.34 is also reached.

Example 4.8. Let us consider the graph $G = (V, E)$ of Figure 4.10, where

- $V = \{1, 2, 3, 4, 5, 6, 7\}$, and
- $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 6\}, \{5, 7\}\}$.

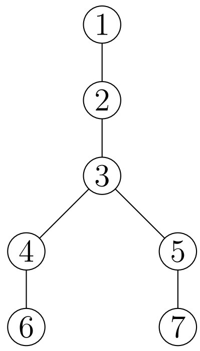

Figure 4.10: Graph $G = (\{1, 2, 3, 4, 5, 6, 7\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 6\}, \{5, 7\}\})$.

Now, let us consider the SDS over G on the maxterm Boolean function given by

$$\mathrm{MAX} = x_1 \vee x_2 \vee x_3' \vee x_4' \vee x_5' \vee x_6 \vee x_7$$

and $\pi = \mathrm{id}$, the identity permutation.

In this case, according to the notation in Theorem 4.33:

- $W = \{1, 2, 6, 7\}$ and $W' = \{3, 4, 5\}$ are the sets of vertices whose corresponding variables appear in MAX in direct and complemented form, respectively.
- There are $p = 3$ connected components which result from G when the complemented vertices and the edges which are incident to them are removed,

$$G_1 = (\{1, 2\}, \{\{1, 2\}\}), \ G_2 = (\{6\}, \emptyset) \text{ and } G_3 = (\{7\}, \emptyset).$$

In this case, $\mathrm{diam}\,(G_1) = 1$ and $\mathrm{diam}\,(G_2) = \mathrm{diam}\,(G_3) = 0$.

- $m_3 = 2$ and $m_4 = m_5 = 1$.

Associated with this SDS, let us consider the initial configuration

$$x^0 = (0, 0, 1, 1, 0, 0, 0).$$

The transit starting at x^0 can be seen in Figure 4.11, being $(1, 1, 1, 1, 1, 1, 1)$ an attractive fixed point of the system.

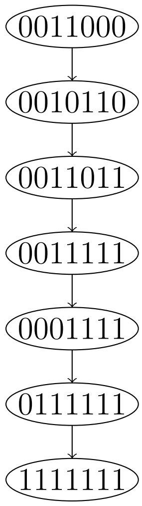

Figure 4.11: Orbit of $x^0 = (0, 0, 1, 1, 0, 0, 0)$.

That is, x^0 reaches a fixed point after $6 = \operatorname{diam}(G_1) + m_3 + 3$ iterations and, therefore, the upper bound in Theorem 4.33 is reached. $\square$

Remark 4.5. Observe that the upper bound for the simplest maxterm OR and minterm AND, given by the diameter of the dependency graph in [28], is not valid for general maxterm and minterm Boolean functions when the periodic orbits of the system are fixed points. This can be seen in Example 4.8, where $\operatorname{diam}(G) = 4 < 6$, being 6 the transient of the system. This reveals the relevance of our more general results, given by Theorems 4.33 and 4.34, due to the breakdown found in the upper bound of the transient for general SDS. $\square$

Transient to periodic orbits in SDS

From Theorems 4.7, 4.8 and Corollary 4.1 in Subsection 4.1.2, we know that a $\mathrm{MAX} - \mathrm{SDS}$ only presents periodic orbits of period greater than 1 if, and only if, there exists a complemented vertex which is not adjacent to a direct vertex. In [28], it is shown that the transient for SDS on NAND or NOR Boolean functions is 1 (except for the simplest system when there is only 1 vertex, in which case all the configurations are periodic points). Here, we extend these results to the case of SDS on a general maxterm or minterm Boolean function.

Let $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ be a sequential dynamical system such that it only allows periodic orbits of period greater than 1. In this context, W'_C is not empty (see Theorems 4.7, 4.8 and Corollary 4.1 of Subsection 4.1.2).

Note that, in such an SDS, W'_D can be empty (see Theorems 4.7, 4.8 and Corollary 4.1 of Subsection 4.1.2) and, therefore, also W'^{α}_D and W'^{β}_D would be empty. This case corresponds to the maxterm NAND, where a periodic orbit is always reached after a maximum of 1 iteration ([28]).

Once this particular case has been exposed, we can suppose that $W'_D \neq \emptyset$ and, thus, $W'^{\beta}_D \neq \emptyset$. Lemma 4.7 shows a stability result, which can be also applied in this case for the vertices belonging to W'^{α}_D, if any. In this context, it is crucial to analyze the performance of the elements of W'^{β}_D.

As in the case of the elements in W'^{α}_D (see Lemma 4.7), if $i \in W'^{\beta}_D$, there exists an iteration such that i has permanent state value 1 after such an iteration. Observe that, if i becomes deactivated after the iteration $t \geq 1$, and we take $j \in W \cap A_G(i)$ (which is not an empty set since $i \in W'^{\beta}_D \subseteq W'_D$), then:

- If j updates after i, then j updates to a permanent activated state after the iteration t.
- If j updates before i, then j updates to a permanent activated state after the iteration $t+1$.

As a consequence, i has permanent state value 1 after the iteration $t+1$ in both cases.

Thus, when a periodic orbit has been reached, all the vertices belonging to W'_D have permanent state value 1, and the period comes from the evolution of the vertices in W'_C.

To analyze the number of iterations needed to ensure that a vertex $i \in W'^{\beta}_D$ reaches a permanent state value 1, we must study its environment within the global SDS: its adjacent direct vertices; the subsystems restricted to the graphs C_j adjacent to i (i.e., with some vertices adjacent to i); and the adjacent entities belonging to W'_D. Formally, for each $i \in W'^{\beta}_D$, the following subsystem arise in a natural way for the study of its asymptotic stability.

Let us consider the set of vertices

$$\begin{aligned} V_i \;=\;& \{i\} \cup (A_G(i) \cap W) \cup \\ & (A_G(i) \cap W'_D) \cup (A_G(A_G(i) \cap W'_D) \cap W) \cup \\ & \{k \in V : \exists j \in \{1, \ldots, q\} / i \in A_G(C_j) \text{ and } k \text{ is a vertex of the graph } C_j\} \end{aligned}$$

and consider the SDS defined by:

- $S_i = (V_i, E_i)$, the subgraph of G where V_i is the vertex set and the edges in E_i are these ones in E between two vertices belonging to V_i.
- $\mathrm{MAX}_{|S_i}$, the restriction of MAX to the vertices in the graph S_i.
- $\pi_{|S_i}$, the restriction of π to the vertices in the graph S_i.

This SDS, $\left[S_i, \mathrm{MAX}_{|S_i}, \pi_{|S_i}\right] - \mathrm{SDS}$, will be named onwards as the *local subsystem* associated with i, or SDS_i, and i will be called the *central vertex* of the local subsystem.

Following with the notation in Lemma 4.7, consider $m_i = \mathrm{card}\,(A_G(i) \cap W'_D)$.

Since $i \in W'^{\beta}_D$, by Theorems 4.7, 4.8 and Corollary 4.1 in Subsection 4.1.2, the local subsystem only presents periodic orbits of period greater than 1. Also, let s_i be the maximum period of this system, which can be obtained from the periods of the SDS generated by the graphs C_j contained in S_i, as studied in Lemma 4.6 of Subsection 4.1.4.

With this, we have the following lemma.

Lemma 4.8. *Let* $[G, \mathrm{MAX}, \pi] - \mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G = (V, E)$ *associated with the maxterm* MAX*, where the structure of* G *only allows periodic orbits of period greater than* 1*. Then, any vertex* $i \in W'^{\beta}_D$ *takes permanent state value* 1 *after a maximum of*

$$1 + \sum_{j \in W'^{\beta}_D} [1 + s_j (m_j + 1)]$$

iterations, being $m_j = \mathrm{card}\,(A_G(j) \cap W'_D)$ *and* s_j *the maximum period of the local subsystem associated with* j.

Proof. First, the following analysis applies to valuations obtained after an evolution of the system. Hence, an extra iteration must be added to the upper bound which will be obtained below. Observe that the initial state can be a configuration of the states of the entities impossible to be repeated again after an update of the system. Consequently, these initial configurations will be excluded from the reasoning. These situations are:

- In the initial configuration, the state of an entity can influence only to some of its adjacent vertices, those updating before it (as said above in the contextualization of Lemma 4.7).
- The initial configuration in the subsystem restricted to a graph C_j may correspond to an eventually periodic orbit of this subsystem. However, after an update of the global system, the reached configuration will be always a periodic point of this subsystem since it will not have two adjacent complemented vertices with state value 0 (see Proposition 4.3 of Subsection 4.2.1). Observe that, after an update of the global system, there cannot be two adjacent complemented vertices with state value 0, since the first of them updating to state value 0 would provoke the update of the other one to state value 1.

If we analyze each local subsystem SDS_j independently, $j \in W_D'^{\beta}$, a state value 1 of the central vertex j can be preserved after an update of the system by one of these ways:

- If, at the moment of the update of j, there is an adjacent entity $k \in W$ which is activated. Also, this state will be permanently preserved onwards.
- If, at the moment of the update of j, for all $k \in W \cap A_G(i)$, k is deactivated and j has an adjacent entity $l \in W'$ which is deactivated. In this case, when j updates again in the following iteration, l is activated, not being able to perform this effect over j.

Since the entities belonging to W_C' which are adjacent to j can show a periodical evolution, indeed this second option can be permanent even without an adjacent activated entity.

Thus, if we consider each local subsystem independent from the others, its upper bound is reached by the maximum number of iterations in which the second options does not become permanent, updating after that the entity j to the deactivated state and then, to the permanent state activated given by the first option with an activated adjacent vertex.

In this case, for this entity j, let us suppose that it takes the maximum number of iterations, $s_j - 1$, to appear states of the entities belonging to W'_C which does not preserve the state value 1 of the entity j in the following iteration. In this case, an adjacent vertex belonging to W'_D can delay the change of state of this entity, having to wait again $s_j - 1$ iterations for a valuation of the states of the entities belonging to W'_C which does not preserve the state value 1 of the entity j in the following iteration. Again, other adjacent vertex belonging to W'_D can delay the change of state of this entity, and so on until the last adjacent vertex of j belonging to W'_D gets permanent state value 1 by means of an adjacent direct vertex with state value 1.

When the local subsystem completes s_j-1 iterations again, after the next update, j evolves to state value 0, which has taken $s_j\,(m_j+1)$ iterations overall. Next, j evolves to a permanent state value 1 by means of an adjacent direct vertex with state value 1.

However, since the local subsystems are connected through the graph, the extreme case is satisfied when this situation spreads stepwise from a unique local subsystem to another one, which means to add the individual upper bounds to achieve a global upper bound. □

Remark 4.6. Note that in Lemma 4.7 a different upper bound could be obtained for each vertex verifying the hypothesis, considering the particular conditions of each one. In this case, the upper bound is the same for all the vertices in W'^{β}_D, showing the number of iterations needed to ensure that all the elements of this set have permanent state value 1 simultaneously. The order in which each element fixes its state value to 1 depends on the dependency graph and the order of update. □

And, finally, we can state the following theorem.

Theorem 4.35 (Transient in m-Periodic $\mathrm{MAX}-\mathrm{SDS}$)**.** *Let* $[G,\mathrm{MAX},\pi]-\mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G=(V,E)$ *associated with the maxterm* MAX*, where the structure of* G *only allows periodic orbits of period greater than* 1*. Then, every orbit of the system reaches a periodic orbit after a maximum of*

$$\max_{k\in\{1,\ldots,p\}}\{\mathrm{diam}\,(G_k)\}+\max\{\max_{i\in W'^{\alpha}_D}\{m_i+3\},(1+\sum_{j\in W'^{\beta}_D}[1+s_j\,(m_j+1)])\delta_{W'^{\beta}_D},1\}$$

iterations, being $m_i=\mathrm{card}\,(A_G\,(i)\cap W'_D)$*,* s_i *the maximum period of the local subsystem associated with* i *for each* $i\in W'^{\beta}_D$ *and* $\delta_{W'^{\beta}_D}$ *a dummy variable taking value* 0*, if* $W'^{\beta}_D=\emptyset$*, and value* 1 *in other case.*

Proof. In this case, as is well known (see Theorems 4.7, 4.8 and Corollary 4.1 in Subsection 4.1.2), $W'_C \neq \emptyset$.

First, if $W'_D = \emptyset$, then $p = 0$, MAX = NAND and the system reaches a periodic orbit, at most, after 1 iteration (see [28]). It fits with the upper bound for this case.

Now, let us see the general case when $W'_D \neq \emptyset$, which also implies $p > 0$:

Consider $i \in W'_D$.

- If $i \in W'^{\alpha}_D$, the hypotheses of Lemma 4.7 are satisfied considering $m_i = \mathrm{card}\left(A_G\left(i\right) \cap W'\right)$. So, at a maximum of $m_i + 3$ iterations, x_i will have permanent state value 1.
- On the other hand, if $i \in W'^{\beta}_D$, then it satisfies the hypotheses of Lemma 4.8 and so, at a maximum of $1 + \sum_{j \in W'^{\beta}_D} \left[1 + s_j\left(m_j + 1\right)\right]$ iterations, x_i will have permanent state value 1.

Taking all of this into account, in

$$\max\{\max_{i \in W'^{\alpha}_D}\{m_i + 3\}, (1 + \sum_{j \in W'^{\beta}_D}\left[1 + s_j\left(m_j + 1\right)\right])\delta_{W'^{\beta}_D}\}$$

iterations, for all $i \in W'_D$, x_i reaches a permanent state value 1.

It must be noted that this calculation is also valid even if $W'^{\alpha}_D = \emptyset$ or $W'^{\beta}_D = \emptyset$, because of the value 0 considered by default when a max or a $\sum$ expression is taken over an empty set, besides the use of the dummy variable $\delta_{W'^{\beta}_D}$.

Also, in this case, this number of iterations is always greater than or equal to 1. Thus, it is the same as $\max\{\max_{i \in W'^{\alpha}_D}\{m_i+3\}, (1+\sum_{j \in W'^{\beta}_D}\left[1 + s_j\left(m_j + 1\right)\right])\delta_{W'^{\beta}_D}, 1\}$, to include in the same formula the case $W'_D = \emptyset$.

After that, for the future evolution of the system, vertices belonging to W'_D do not change their state values, and the study of the evolution of the system can be reduced to analyze what happens in the restriction of G to $W \cup W'_C$. Since a vertex in W is not adjacent to a vertex in W'_C, there cannot be interference between these sets. Hence, the behavior of the entire system can be obtained from the study of the evolution in each connected subgraph $G_1, \ldots, G_p$ and $C_1, \ldots, C_q$.

Regarding to $G_1, \ldots, G_p$, since they only have vertices associated with direct variables, we know that the restriction of the SDS to each G_k performs as an SDS on a maxterm OR as evolution operator. Therefore, an attractive fixed point is attained in, at most, $\mathrm{diam}\left(G_k\right)$ iterations (see [28]).

On the other hand, the restriction of the SDS to each component C_k performs as an SDS on a maxterm NAND as evolution operator. In this case, let us consider the following facts:

- The periodic points of the subsystem restricted to a subgraph C_k are those configurations without adjacent complemented vertices with state value 0 (see Proposition 4.3 in Subsection 4.2.1). Also, after an update of the global system, there cannot be two adjacent complemented vertices with state value 0, since the first of them updating to state value 0 would provoke the update of the other one to state value 1. Thus:
    - ◦ After one iteration, the system restricted to C_k reaches a periodic point of this subsystem.
    - ◦ A periodic point of this subsystem over C_k cannot evolve to an eventually periodic orbit of the subsystem due to the interference of the rest of the entities in the global system.
- The number of iterations needed to ensure that all the elements in W'_D have permanent state value 1 is greater than or equal to 1.

Thus, after the iterations needed to ensure that all the elements in W'_D have permanent state value 1, all the elements in W'_C already have a state belonging to a periodic orbit in each subsystem restricted to each subgraph C_k. Finally, the composition of these local periodic orbits generates a periodic orbit in the global system, as can be seen in Lemma 4.6 of Subsection 4.1.4.

In view of this, after $\max_{k\in\{1,\ldots,p\}}\{\operatorname{diam}(G_k)\}$ iterations (following the previous $\max\{\max_{i\in W'^{\alpha}_D}\{m_i+3\},(1+\sum_{j\in W'^{\beta}_D}[1+s_j\,(m_j+1)])\delta_{W'^{\beta}_D},1\}$ iterations needed for ensuring that the state value of the vertices in W'_D is fixed) all the vertices in W will reach a state that they will preserve permanently, and all the vertices in W'_C will have a state that they will repeat simultaneously in a future update.

Therefore, after at most

$$\max_{k\in\{1,\ldots,p\}}\{\operatorname{diam}(G_k)\}+\max\{\max_{i\in W'^{\alpha}_D}\{m_i+3\},(1+\sum_{j\in W'^{\beta}_D}[1+s_j\,(m_j+1)])\delta_{W'^{\beta}_D},1\}$$

iterations, the system reaches a periodic orbit. □

Dually, we have the following result.

Theorem 4.36 (Transient in m-Periodic $\mathrm{MIN}-\mathrm{SDS}$)**.** *Let* $[G, \mathrm{MIN}, \pi]-\mathrm{SDS}$ *be a sequential dynamical system over a dependency graph* $G=(V, E)$ *associated with the minterm* MIN*, where the structure of* G *only allows periodic orbits of period greater than* 1*. Then, every orbit of the system reaches a periodic orbit after a maximum of*

$$\max_{k\in\{1,\ldots,p\}}\{\operatorname{diam}(G_k)\}+\max\{\max_{i\in W_D'^{\alpha}}\{m_i+3\},(1+\sum_{j\in W_D'^{\beta}}[1+s_j\,(m_j+1)])\delta_{W_D'^{\beta}},1\}$$

iterations, being $m_i=\operatorname{card}(A_G(i)\cap W_D')$*,* s_i *the maximum period of the local subsystem associated with* i *for each* $i\in W_D'^{\beta}$ *and* $\delta_{W_D'^{\beta}}$ *a dummy variable taking value* 0*, if* $W_D'^{\beta}=\emptyset$*, and value* 1 *in other case.*

This new upper bound shown in Theorem 4.35 is also the best possible one for this kind of SDS, since it is reachable, as shown in the example below. A dual example can be obtained, in which the upper bound in Theorem 4.36 is also reached.

Example 4.9. Let us consider the graph $G=(V, E)$ (see Figure 4.12), with

- $V=\{1,2,3,4,5,6,7,8,9\}$, and
- $E=\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{3,6\},\{6,7\},\{3,8\},\{8,9\}\}$.

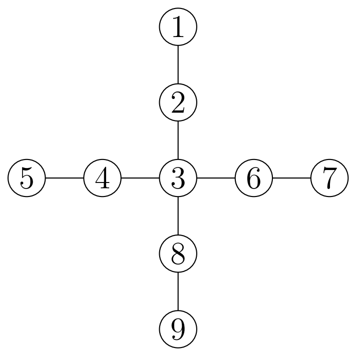

Figure 4.12: Graph $G=(\{1,\ldots,9\},\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{3,6\},\{6,7\},\{3,8\},\{8,9\}\})$.

Now, let us consider the SDS over G on the maxterm Boolean function given by

$$\mathrm{MAX}=x_1\vee x_2\vee x_3'\vee x_4'\vee x_5'\vee x_6'\vee x_7'\vee x_8'\vee x_9'$$

and $\pi=\mathrm{id}$, the identity permutation.

In this case, according to the notation in Theorem 4.35:

- $W = \{1,2\}$ and $W' = \{3,4,5,6,7,8,9\}$ are the sets of vertices whose corresponding variables appear in MAX in direct and complemented form, respectively. Moreover, $W'_D = \{3\}$ and $W'_C = \{4,5,6,7,8,9\}$. Finally, $W'^{\alpha}_D = \emptyset$ and $W'^{\beta}_D = \{3\}$.

- There is $p = 1$ connected component resulting from G when the complemented vertices and the edges which are incident to them are removed, $G_1 = (\{1,2\},\{\{1,2\}\})$. In this case, $\operatorname{diam}(G_1) = 1$.

- $m_3 = 0$.

- The local subsystem associated with the only element in W'^{β}_D, SDS_3, is the global system in which the vertex 1 and the edge adjacent to it have been removed. In this case, $s_3 = 3$ (see Lemma 4.6 in Subsection 4.1.4).

Associated with this SDS, let us consider the initial configuration

$$x^0 = (0,0,1,1,1,0,0,1,0)\,.$$

The transit starting at x^0 can be seen in Figure 4.13, being $(1,1,1,0,1,1,0,1,0)$ a periodic point of the system.

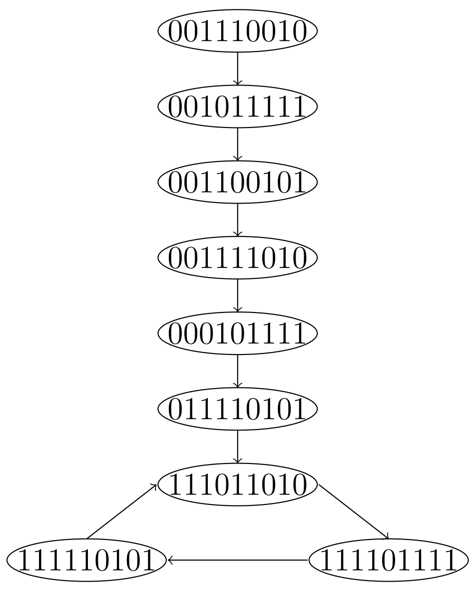

Figure 4.13: Orbit of $x^0 = (0,0,1,1,1,0,0,1,0)$.

That is, the orbit starting at x^0 reaches its attractor after $6 = \operatorname{diam}(G_1) + (1 + [1 + s_3(m_3 + 1)])$ iterations. Therefore, the upper bound in Theorem 4.35 is reached. □

Remark 4.7. Observe that the upper bound for the simplest maxterm NAND and minterm NOR, which present periodic orbits of period greater than 1, is 1 (see [28]). The theorems above proves that this upper bound is not valid for general maxterm and minterm Boolean functions when the periodic orbits of the system have period greater than 1. This can be seen in Example 4.9, where the transient of the system is 6. Again, this reveals the relevance of our more general results, given in Theorems 4.35 and 4.36, due to the breakdown found in the upper bound of the transient for general SDS. □

Chapter 5

Advances in Parallel and Sequential Directed Dynamical Systems

In Chapters 3 and 4, a complete study of the dynamics in parallel and sequential dynamical systems on maxterm and minterm Boolean functions over undirected dependency graphs has been exposed. Some of these results can be immediately generalized for the case of directed dependency graphs. This is the case of the results on the existence and coexistence of periodic orbits and predecessors and Garden-of-Eden configurations. As a closure of the results exposed in this dissertation and as a sign of continuity in the study of the dynamics of these systems, they will be exposed along this chapter.

As in the last two chapters, the study here is divided into two sections: dynamics of periodic orbits and dynamics of non-periodic orbits.

The preliminaries in Chapter 2 can be considered also in this chapter, when appropriate. Anyway, although the generalization of the concepts from undirected graphs to directed graphs is immediate, it is worth to devote here some lines to delimit the notation when there are changes regarding what was said in Chapter 2.

First of all, in the case of directed dependency graphs, we consider all the vertices that influence $i \in V$ or the entities belonging to $U \subseteq V$ in their evolution:

$$\overline{I_D\left(i\right)} = \{j \in V : (j, i) \in A\} \cup \{i\},$$

$$\overline{I_D\left(U\right)} = \bigcup_{i \in U} \overline{I_D\left(i\right)}.$$

And also, for our purposes, we will need to consider these other sets:

$$I_D\left(i\right) = \{j \in V : (j, i) \in A\},$$

$$I_D\left(U\right) = \bigcup_{i \in U} I_D\left(i\right),$$

$$I_D^*\left(U\right) = I_D\left(U\right) \setminus U.$$

Thus, in this case of directed dependency graphs, the formal definition of the a discrete dynamical system in which all the entities update their states in a synchronous way is as follows.

Definition 5.1. Let $D = (V, A)$ be a directed graph on $V = \{1, \ldots, n\}$ and a map

$$F : \{0,1\}^n \to \{0,1\}^n, \qquad F\left(x_1, \ldots, x_i, \ldots, x_n\right) = \left(y_1, \ldots, y_i, \ldots, y_n\right),$$

where y_i is the updated state value of the entity i by applying a local function f_i over the state values of the entities in $\overline{I_D\left(i\right)}$. They constitute a discrete dynamical system called *parallel directed dynamical system* over D, which will be denoted by $[D, F] -$ PDDS or $F -$ PDDS when specifying the dependency digraph is not necessary. $\square$

Accordingly with this definition, in this dissertation, generical PDDS with a maxterm MAX (resp. minterm MIN) as evolution operator will be denoted by MAX $-$ PDDS (resp. MIN $-$ PDDS).

On the other hand, in the case of sequential update, the definition is as follows.

Definition 5.2. Let $D = (V, A)$ be a directed graph on $V = \{1, \ldots, n\}$, $\pi = \pi_1 | \ldots | \pi_n$ a permutation on V and a map

$$[F, \pi] = F_{\pi_n} \circ \cdots \circ F_{\pi_1} : \{0,1\}^n \to \{0,1\}^n,$$

$$[F, \pi]\left(x_1, \ldots, x_i, \ldots, x_n\right) = \left(y_1, \ldots, y_i, \ldots, y_n\right),$$

where $F_{\pi_i} : \{0,1\}^n \to \{0,1\}^n$ updates the state value of the entity $\pi_i \in V$ from x_{π_i} to y_{π_i} considering the state values of the entities belonging to $\overline{I_D\left(\pi_i\right)}$ and keeping the other states unaltered, i.e., $F_{\pi_i} = (\mathrm{id}_1, \ldots, f_{\pi_i}, \ldots, \mathrm{id}_n)$, being id_j the identity function over the entity j and $f_{\pi_i} : \{0,1\}^n \to \{0,1\}$ the local function which performs the update for the entity π_i. They constitute a discrete dynamical system called *sequential directed dynamical system* over D, which will be denoted by $[D, F, \pi] -$ SDDS or $F -$ SDDS when specifying the dependency graph is not necessary and the updating order is implicit in this context of sequential evolution. $\square$

As in the case of PDDS, in this dissertation, generical SDDS with a maxterm MAX (resp. minterm MIN) as evolution operator will be denoted by $\mathrm{MAX}-\mathrm{SDDS}$ (resp. $\mathrm{MIN}-\mathrm{SDDS}$).

Additionally, we need to redefine the sets P_0, Q_0, P_1 and Q_1 introduced previously for SDS. Let us consider now, only in the context of SDDS, the following sets contained in $\overline{I_D(V_0)}$:

$$P_0 = \{i \in V : \exists j \in V_0 \text{ such that } (i,j) \in A, i = \pi_r, j = \pi_s \text{ and } s < r\},$$

$$Q_0 = \{i \in V : \exists j \in V_0 \text{ such that } (i,j) \in A, i = \pi_r, j = \pi_s \text{ and } s > r\}.$$

In other words, each element i belonging to P_0 (resp. Q_0) is influencing to a vertex $j \in V_0$ which is updated, according the order expressed in π, before (resp. after) i.

Similarly for V_1, we consider the sets P_1 and Q_1 contained in $\overline{I_D(V_1)}$:

$$P_1 = \{i \in V : \exists j \in V_1 \text{ such that } (i,j) \in A, i = \pi_r, j = \pi_s \text{ and } s < r\},$$

$$Q_1 = \{i \in V : \exists j \in V_1 \text{ such that } (i,j) \in A, i = \pi_r, j = \pi_s \text{ and } s > r\}.$$

5.1 Dynamics of periodic orbits

In this section, we solve the problems of existence and coexistence of periodic orbits in $\mathrm{MAX}-\mathrm{PDDS}$, $\mathrm{MIN}-\mathrm{PDDS}$, $\mathrm{MAX}-\mathrm{SDDS}$ and $\mathrm{MIN}-\mathrm{SDDS}$, in the context of the analysis of the dynamics of periodic orbits in this kind of systems.

5.1.1 Existence and coexistence of periodic orbits in PDDS

As a starting point in the study of the dynamics of PDDS, we begin by analyzing the orbital structure of a PDDS on a general maxterm or minterm as evolution operator. Specifically, we study the type of periodic orbits that such a system can present.

In Sections 3.1 and 4.1, we deal with this topic in the case of general maxterm or minterm functions over undirected graphs, performing a complete analysis of the dynamics in these cases. In this occasion, we propose a generalization of some of these previous results by studying the dynamics in the case of dynamical systems on maxterm and minterm Boolean functions over directed graphs, which also is of great interest from the point of view of circuit theory or computer science.

In the following results, we show that a PDDS on a general maxterm or minterm Boolean function can present periodic orbits of any period, so breaking the pattern shown in the case of parallel dynamical systems over undirected dependency graphs, in which only fixed points and 2-periodic orbits are available.

Theorem 5.1 (Periodic structure of $\mathrm{MAX}-\mathrm{PDDS}$)**.** *Let* $[D, \mathrm{MAX}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Then, it can present periodic orbits of any period.*

Proof. The proof can be seen in [18]. Nevertheless, due to its constructive character and its utility in the following, we include it here.

Firstly, the case of period 1 (fixed point) is direct since the simplest system with $n = 1$ and the maxterm OR has two fixed points: $\mathcal{I}$ and $\mathcal{O}$.

Orbits of period 2 can also appear in this kind of systems. For example, let us consider the PDDS defined by: $D = (\{1,2,3\}, \{(1,3), (2,3)\})$ (see Figure 5.1) and $\mathrm{MAX} = \mathrm{NAND}$. In this case, the configuration $x^0 = (0,1,1)$ belongs to a periodic orbit whose elements are: $\mathrm{Orb}(x^0) = \{(0,1,1), (1,0,1)\}$.

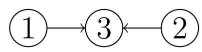

Figure 5.1: Graph $D = \{\{1,2,3\}, \{(1,3), (2,3)\}\}$.

Finally, given $n \in \mathbb{N}$, $n \geq 3$, we will give a pattern to construct a PDDS with an orbit of period n. Let us consider the complete digraph $K_n = (V_{K_n}, E_{K_n})$ and, from it, construct the following directed graph $D = (V, A)$:

- $V = V_{K_n} = \{1, \ldots, n\}$.
- $A = E_{K_n} \setminus (\{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(n,1)\})$.

Finally, we choose the updating operator $\mathrm{MAX} = \mathrm{NAND}$.

Let us write x_i^k to indicate the state value of the entity i after k iterations of the evolution operator MAX. Then, let us consider the initial value for the variables $x_1^0 = 0$ and $x_i^0 = 1$ for all $i \in V \setminus \{1\}$. It is a straightforward computation to check that the system evolves in the following way:

- After k iterations, $1 \leq k \leq n-1$: $x_{k+1}^k = 0$, $x_i^k = 1$ for all $i \in V \setminus \{k+1\}$.
- After n iterations, all the state values coincide with the initial ones, i.e., $x_i^n = x_i^0$ for all $i \in V$.

Namely, the PDDS so constructed presents a periodic orbit of period n.

To illustrate the designed patterns, in Figure 5.2 the cases for $n = 3$, $n = 4$, $n = 5$ and $n = 6$ can be seen.

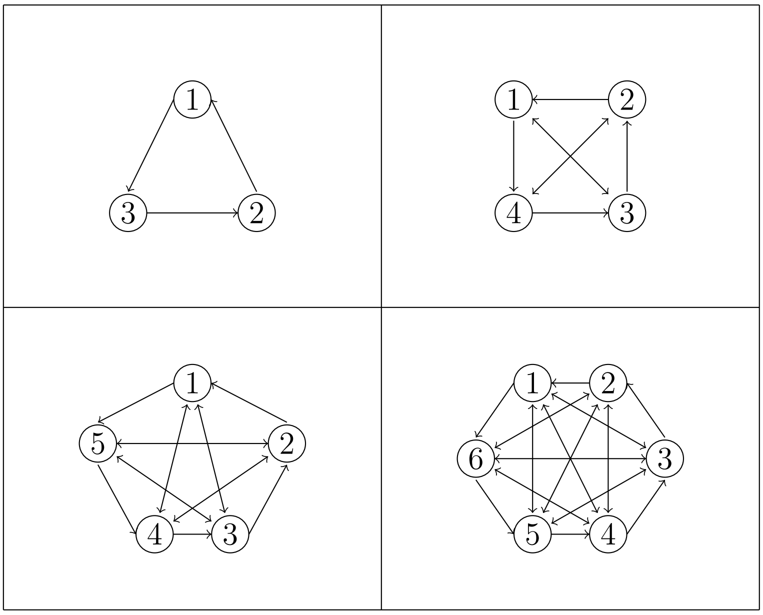

Figure 5.2: Patterns for $n = 3$, $n = 4$, $n = 5$ and $n = 6$.

$\square$

Dually, we have the following theorem [18].

Theorem 5.2 (Periodic structure of $\mathrm{MIN}-\mathrm{PDDS}$)**.** *Let* $[D, \mathrm{MIN}]-\mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*. Then, it can present periodic orbits of any period.*

As a direct consequence of these theorems, we can establish the following results for some special classes of maxterm and minterm Boolean functions [18].

Corollary 5.1. *Let* $[D, \mathrm{OR}]-\mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* OR*. Then, all the* *periodic orbits of this system are fixed points. In fact, $\mathcal{I}$ and $\mathcal{O}$ are always fixed points of the system, but other fixed points can appear, so breaking the pattern for* $\mathrm{OR}-\mathrm{PDS}$ *in the case of undirected dependency graphs.*

Corollary 5.2. *Let* $[D, \mathrm{NAND}]-\mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph $D=(V,A)$ associated with the maxterm* NAND*. Then, this system can present periodic orbits of any period, except fixed points.*

Dually, we have the following results.

Corollary 5.3. *Let* $[D, \mathrm{AND}]-\mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph $D=(V,A)$ associated with the minterm* AND*. Then, all the periodic orbits of this system are fixed points. In fact, $\mathcal{O}$ and $\mathcal{I}$ are always fixed points of the system, but other fixed points can appear, so breaking the pattern for* $\mathrm{AND}-\mathrm{PDS}$ *in the case of undirected dependency graphs.*

Corollary 5.4. *Let* $[D, \mathrm{NOR}]-\mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph $D=(V,A)$ associated with the minterm* NOR*. Then, this system can present periodic orbits of any period, except fixed points.*

So far in this subsection, we have recall the results on the existence of periodic orbits in PDDS. However, some important questions still remain open. One of them consists in analyzing the coexistence of periodic orbits with different periods.

In Subsections 3.1.2 and 4.1.2, a complete analysis of the coexistence of periodic orbits in PDS and SDS, respectively, has been performed. This analysis shows that fixed points and periodic orbits of greater period cannot coexist, although all the possible periodic orbits with periods greater than 1 (only 2-periodic orbits in the case of PDS and any period greater than 1 in the case of SDS) can coexist.

For PDDS with general maxterm (resp. minterm) functions as evolution operators, we will show that periodic orbits with different periods can coexist. Even fixed points and periodic orbits of period greater than 1 can now coexist, so breaking the pattern observed for PDS and SDS over undirected dependency graphs.

Theorem 5.3 (Coexistence of periods in $\mathrm{MAX}-\mathrm{PDDS}$)**.** *Given $\{n_1,\ldots,n_r\}\subset\mathbb{N}$, $r\geq 2$, there exists a* PDDS *with a maxterm as evolution operator which presents periodic orbits of periods $n_1,\ldots,n_r$ simultaneously.*

Proof. Let us construct a PDDS with orbits of periods $n_1,\ldots,n_r$.

Firstly, let us consider the case in which $n_i\geq 2$ for all $i\in\{1,\ldots,r\}$.

For each n_i, let us take the PDDS in the proof of Theorem 5.1 for which the n_i-periodic orbit is achieved. To avoid duplication in the name of the vertices of each PDDS, we will denote by $v_{i,j}$ to the vertex j in the digraph of the PDDS associated to the n_i-periodic orbit.

Let us consider the digraphs associated to these PDDS. Apart from the internal influences in each case, we will consider the following influence structure among the vertices: $v_{i,j}$ is adjacent (reciprocal influence) to $v_{l,m}$ for all j, m and $i \neq l$. In other words, the vertices in two digraphs from different PDDS have reciprocal influence among them. This way, we reach a connected digraph D.

As evolutionary operator we take MAX = NAND.

As initial state values for the variables, we consider that the vertex $v_{i,1}$ is deactivated and the rest of vertices are activated.

This system evolves as follows:

- After k iterations, $1 \leq k \leq n_i - 1$, $v_{i,k+1}$ is deactivated and the rest of vertices are activated.
- After n_i iterations, all the state values coincide with the initial ones.

Namely, the $[D, \mathrm{MAX}] - \mathrm{PDDS}$ so constructed presents a periodic orbit of period n_i.

Thus, by considering the different r initial state values obtained by varying i in $\{1, \ldots, r\}$, r periodic orbits with periods $n_1, \ldots, n_r$ result.

On the other hand, when there exists $i \in \{1, \ldots, r\}$ such that $n_i = 1$, let $[D, \mathrm{MAX}] - \mathrm{PDDS}$ be the PDDS constructed above (or in the proof of Theorem 5.1 in the case $r = 2$) in which the other periodic orbits $n_j \geq 2$, $j \neq i$, coexist, being $D = (V, A)$. Based on this systems, let us consider the PDDS defined from these elements:

- $\overline{D} = (V \cup \{d\}, E \cup \{(d, k) : k \in V\})$, the dependency digraph of the PDDS.
- $\overline{\mathrm{MAX}} = \mathrm{MAX} \vee d$, the maxterm Boolean function of the PDDS.

$\left[\overline{D}, \overline{\mathrm{MAX}}\right] - \mathrm{PDDS}$ is as $[D, \mathrm{MAX}] - \mathrm{PDDS}$, but with an additional direct entity influencing all the other ones but influenced by none (apart from itself).

In this case, an initial state $x_d^0 = 0$ will be preserved permanently and the system will evolve as $[D, \mathrm{MAX}] - \mathrm{PDDS}$, appearing all the periodic orbits of period greater than 1; while an initial state $x_d^0 = 1$ reaches the fixed point $\mathcal{I}$ after, at most, 1 iteration. □

Let us illustrate this result with the following example to clarify the notation.

Example 5.1. The PDDS proposed by Theorem 5.3 in which a fixed point, a 2-periodic orbit and a 3-periodic orbit coexist, is defined by:

- $D = (V, A)$ (see Figure 5.3), with
  - $V = \{1, 2, 3, 4, 5, 6, 7\}$, and
  - $A = \{(2,4), (3,4)\} \cup \{(5,7), (6,5), (7,6)\} \cup \{\{i,j\} : 2 \leq i \leq 4, 5 \leq j \leq 7\} \cup \{(1,i) : 2 \leq i \leq 7\}$.

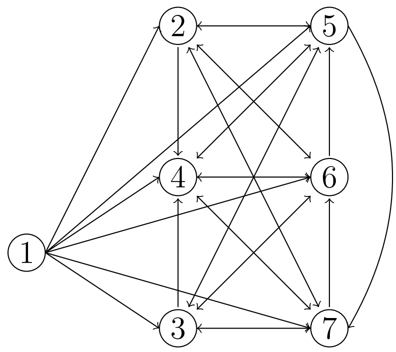

Figure 5.3: Graph $D = (V, A)$.

- $\mathrm{MAX} = x_1 \vee x_2' \vee x_3' \vee x_4' \vee x_5' \vee x_6' \vee x_7'$.

According to the notation in Theorem 5.3: 1 is the vertex d in the PDDS of the proof of Theorem 5.1 generating the fixed point; 2, 3, 4 are the vertices in the PDDS of the proof of Theorem 5.1 generating the 2-periodic orbit; and 5, 6, 7 are the vertices in the PDDS of the proof of Theorem 5.1 generating the 3-periodic orbit.

In this case, the fixed point proposed by Theorem 5.3 is $\mathcal{I}$. The 2-periodic orbit proposed by Theorem 5.3 can be seen in Figure 5.4.

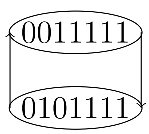

Figure 5.4: 2-periodic orbit of the system proposed by Theorem 5.3.

And finally, the 3-periodic orbit proposed by Theorem 5.3 can be seen in Figure 5.5.

□

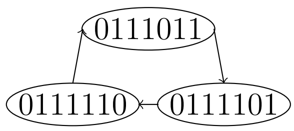

Figure 5.5: 3-periodic orbit of the system proposed by Theorem 5.3.

Finally, we can state the dual result.

Theorem 5.4 (Coexistence of periods in MIN − PDDS)**.** *Given* $\{n_1, \ldots, n_r\} \subset \mathbb{N}$, $r \geq 2$, *there exists a* PDDS *with a minterm as evolution operator which presents periodic orbits of periods* $n_1, \ldots, n_r$ *simultaneously.*

5.1.2 Existence and coexistence of periodic orbits in SDDS

In Theorem 5.1 (resp. Theorem 5.2) of Subsection 5.1.1, some results about the orbital structure of PDDS with general maxterm (resp. minterm) functions as evolution operators have been shown, proving that periodic orbits of any period can appear in such systems. In this subsection, we analyze the case of SDDS with general maxterm (resp. minterm) functions as evolution operators, showing that this situation remains when considering sequential update in SDDS.

Theorem 5.5 (Periodic structure of MAX − SDDS)**.** *Let* $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX*. Then, it can present periodic orbits of any period.*

Proof. It is a direct consequence of Theorem 4.1 in Subsection 4.1.1, since an SDS is a particular case of SDDS. □

Dually, we have the following theorem.

Theorem 5.6 (Periodic structure of MIN−SDDS)**.** *Let* $[D, \mathrm{MIN}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*. Then, it can present periodic orbits of any period.*

It worths analyzing the particular relevant cases when the evolution operator of a discrete dynamical system is the maxterm OR or NAND (resp. minterm AND or NOR). Recall that the only periodic orbits of PDDS over directed dependency graphs with OR (resp. AND) as updating operator are fixed points while with NAND (resp. NOR), periodic orbits of period greater than 1 (see Corollaries 5.1, 5.2, 5.3 and 5.4 in Subsection 5.1.1). In this case, this situation remains when considering sequential update in SDDS.

To see that, let us establish first the following lemma.

Lemma 5.1. *Let* $[D, F, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm or minterm* F*. Then, the systems* $[D, F, \pi] - \mathrm{SDDS}$ *and* $[D, F] - \mathrm{PDDS}$ *have the same fixed points.*

Proof. Let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ be a fixed point of the $[G, F] - \mathrm{PDDS}$, where $\hat{x}_i$ represents the (fixed) state value of the vertex $i \in V$. Since the updating of the state value of each i only depends on the state values of the entities in $\overline{I_D(i)}$ and the restriction of F to that set, it is straightforward to check that, independently of the election of π, $\hat{x}$ is also a fixed point of $[G, F, \pi] - \mathrm{SDDS}$.

Conversely, let $\hat{x}$ be a fixed point for $[G, F, \pi] - \mathrm{SDDS}$ for a certain updating permutation π. Since the state values of all the vertices remain equal in the successive updating steps determined by π, it becomes clear that $\hat{x}$ is also a fixed point of $[G, F] - \mathrm{PDDS}$. □

Firstly, in the case of OR as evolution operator, we have the following result.

Corollary 5.5. *Let* $[D, \mathrm{OR}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* OR. *Then, all the periodic orbits of this system are fixed points. In fact,* $\mathcal{I}$ *and* $\mathcal{O}$ *are always fixed points of the system, but other fixed points can appear, so breaking the pattern for* $\mathrm{OR} - \mathrm{SDS}$ *in the case of undirected dependency graphs.*

Proof. This proof is very similar to the one for the analogous result in PDDS.

By Lemma 5.1 and Corollary 5.1 in Subsection 5.1.1, $\mathcal{I}$ and $\mathcal{O}$ are always fixed points of this SDDS, but there can appear another ones different from them.

It only remains to show that there cannot be periodic orbits of greater period: since all the entities belong to W, we can have only one of the following two possibilities for each $i \in V$:

- $\forall t \geq 0$, $x_i^t = 0$. In this case, the state value 0 is permanent for this entity from the initial configuration.
- $\exists T \geq 0$ such that $x_i^T = 1$, being the iteration T the first time that the variable x_i takes the value 1. In this situation, the state value 1 is permanent from this iteration on.

Thus, after a certain number of iterations, all the entities reach a fixed value that they preserve onwards. $\square$

In the case of NAND as evolution operator, we have the following result.

Corollary 5.6. *Let* $[D, \mathrm{NAND}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* NAND*. Then, this system can present periodic orbits of any period, except fixed points.*

Proof. First, notice that such an SDDS cannot present fixed points. In fact, we know that PDDS with NAND as evolution operator cannot present fixed points. Then, this first assertion follows from Lemma 5.1.

On the other hand, this system can present periodic orbits of any period greater than 1 is a direct consequence of Theorem 4.4 in Subsection 4.1.1, since an SDS is a particular case of SDDS. $\square$

Dually, we have the following results.

Corollary 5.7. *Let* $[D, \mathrm{AND}] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* AND*. Then, all the periodic orbits of this system are fixed points. In fact,* $\mathcal{O}$ *and* $\mathcal{I}$ *are always fixed points of the system, but other fixed points can appear, so breaking the pattern for* $\mathrm{AND} - \mathrm{SDS}$ *in the case of undirected dependency graphs.*

Corollary 5.8. *Let* $[D, \mathrm{NOR}] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* NOR*. Then, this system can present periodic orbits of any period, except fixed points.*

Once studied the existence of periodic orbits in SDDS, we continue analyzing the possible coexistence of them in this kind of systems. In Theorem 5.1 (resp. Theorem 5.2) of Subsection 5.1.1, some results about the orbital structure of PDDS with general maxterm (resp. minterm) functions as evolution operators have been shown, proving that the coexistence of any periodic orbits with different periods is possible, even with fixed points.

In line with the results for PDDS, in the context of SDDS with general maxterm (resp. minterm) functions as evolution operators, we will show that periodic orbits with different periods can coexist in an SDDS, even fixed points and periodic orbits of period greater than 1, so breaking the pattern observed for systems over undirected dependency graphs.

Theorem 5.7 (Coexistence of periods in $\mathrm{MAX} - \mathrm{SDDS}$)**.** *Given* $\{n_1, \ldots, n_r\} \subset \mathbb{N}$, $r \geq 2$, *there exists an* SDDS *with a maxterm as evolution operator which presents periodic orbits of periods* $n_1, \ldots, n_r$ *simultaneously.*

Proof. If $n_i \geq 2$ for all $i \in \{1, \ldots, r\}$, the result is a direct consequence of Theorem 4.11 in Subsection 4.1.2, since an SDS is a particular case of SDDS.

On the other hand, if there exists $i \in \{1, \ldots, r\}$ with $n_i = 1$, let $[G, \mathrm{MAX}, \pi] -$ SDS be the SDS constructed in Theorem 4.11 in Subsection 4.1.2 (or in Theorem 4.4 in Subsection 4.1.1 in the case $r = 2$) in which the other periodic orbits $n_j \geq 2$, $j \neq i$, coexist, being $G = (V, E)$. Based on this SDS, let us consider the SDDS defined from these elements:

- $D = (V \cup \{d\}, E \cup \{(d, k) : k \in V\})$, the dependency digraph of the SDDS.
- $\overline{\mathrm{MAX}} = \mathrm{MAX} \vee d$, the maxterm Boolean function of the SDDS.
- $\overline{\pi} = d|\pi$, the order permutation of the SDDS.

$\left[D, \overline{\mathrm{MAX}}, \overline{\pi}\right] -$ SDDS is as $[G, \mathrm{MAX}, \pi] -$ SDS, but with an additional direct entity influencing all the other ones but influenced by none (apart from itself).

In this case, an initial state $x_d^0 = 0$ will be preserved permanently and the system will evolve as $[G, \mathrm{MAX}, \pi] -$ SDS, appearing all the periodic orbits of period greater than 1; while an initial state $x_d^0 = 1$ reaches the fixed point $\mathcal{I}$ after, at most, 1 iteration. □

Dually, we have the following result.

Theorem 5.8 (Coexistence of periods in MIN − SDDS). *Given $\{n_1, \ldots, n_r\} \subset \mathbb{N}$, $r \geq 2$, there exists an* SDDS *with a minterm as evolution operator which presents periodic orbits of periods $n_1, \ldots, n_r$ simultaneously.*

5.2 Dynamics of non-periodic orbits

In this last section, we study the existence and uniqueness of predecessor, what naturally leads us to explore the Garden-of-Eden configurations of the system, in the context of the study of the dynamics of non-periodic orbits in PDDS and SDDS.

5.2.1 Predecessor and GOE configurations in PDDS

As said in Subsection 3.2.1 in the context of PDS, the study of predecessors in network models is usually divided into four more specific problems [31, 32]:

- Predecessor existence problem (PRE): Determining whether a predecessor exists for a given state.
- Unique predecessor problem (UPRE): Determining whether a predecessor is the unique one for a given state.
- Coexistence of predecessors problem (APRE): Determining whether a predecessor is not unique for a given state.
- Number of predecessors problem (#PRE): Counting the number of predecessors of a given state, in case of non-uniqueness.

In this subsection, to begin with, we solve the first one in the context of PDDS on maxterm and minterm Boolean functions. This allows us to get also a characterization of the GOE of such systems. These results lead us to describe the structure of the potential predecessors of a given state, what allows us to give results to solve the rest of the problems in the mentioned context.

In order to solve the PRE problem, in the next theorem, we provide sufficient and necessary conditions to know when a certain configuration y is the successor of another configuration x, i.e., when y has at least a predecessor.

Theorem 5.9. *Let* $[D, \mathrm{MAX}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Then, a configuration has a predecessor if, and only if, every activated entity influencing to a deactivated one in such a configuration is also influenced by an activated entity which is not influencing to any deactivated one.*

In other words, a configuration has a predecessor if, and only if, $\overline{I_D\left(I^c\right)} = V_1$, *being* $I = \overline{I_D\left(V_0\right)}$.

Proof. This constructive proof generates a predecessor state x of a current state y, whenever possible, highlighting those conditions of y under which the existence of a predecessor x is impossible.

In this case, we can split the set V_1 associated to y into two subsets, corresponding to the vertices influencing to some vertices in V_0, $I_D^*\left(V_0\right)$, and the vertices which are not influencing to any vertex in V_0, $\overline{I_D\left(V_0\right)}^c$, that is,

$$V_1 = I_D^*\left(V_0\right) \cup \overline{I_D\left(V_0\right)}^c$$

Suppose, by reduction to the absurd, that there exists a configuration y which has a predecessor x, but one of the (activated) entities k such that $y_k = 1$, which is influencing to someone of V_0, i.e., $k \in I_D^*(V_0)$, is not influenced by anyone in $\overline{I_D(V_0)}^c$.

Observe that if $i \in V_0$, then, for every entity $j \in \overline{I_D(i)}$, it must be:

- $x_j = 0$ when $j \in W$, and
- $x_j = 1$ when $j \in W'$,

since otherwise, $y_i = 1$ and $i \notin V_0$. In particular, this occurs for every $j \in \overline{I_D(V_0)}$.

In such a context, since $k \in I_D^*(V_0) \subset V_1$ is not influenced by any entity in $\overline{I_D(V_0)}^c$, it would be $y_k = 0$, what is a contradiction.

Reciprocally, if for a given configuration y every activated entity influencing to a deactivated one, is also influenced by an activated entity which is not influencing to any deactivated one, to get a predecessor, x, of the given configuration y, it should be sufficient to take x as follows:

- For every entity $j \in \overline{I_D(V_0)}$,
  - $x_j = 0$, if $j \in W$, and
  - $x_j = 1$, if $j \in W'$.
- For every entity $j \in \overline{I_D(V_0)}^c$,
  - $x_j = 1$, if $j \in W$, and
  - $x_j = 0$, if $j \in W'$.

□

Remark 5.1. Observe that, in the conditions of existence of predecessors in Theorem 5.9, each activated entity influencing to a deactivated one acts as an articulation node between deactivated entities and activated entities which are not influencing to any deactivated one. That is, in terms of the proof, the vertices in $I_D^*(V_0)$ act as connectors between vertices in V_0 and vertices in $\overline{I_D(V_0)}^c$. □

Dually, we have the following theorem.

Theorem 5.10. *Let* $[D, \mathrm{MIN}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *Then, a configuration has a predecessor if, and only if, every deactivated entity influencing to an* *activated one in such a configuration is also influenced by a deactivated entity which is not influencing to any activated one.*

In other words, a configuration has a predecessor if, and only if, $\overline{I_D(I^c)} = V_0$, *being* $I = \overline{I_D(V_1)}$.

Remark 5.2. Dually to the case in Theorem 5.9, in the conditions of Theorem 5.10, each deactivated entity influencing to an activated one acts as an articulation node between activated entities and deactivated entities which are not influencing to any activated one. That is, in terms of the proof, the vertices in $I_D^*(V_1)$ act as connectors between vertices in V_1 and vertices in $\overline{I_D(V_1)}^c$. □

Theorems 5.9 and 5.10 solve the PRE problem for PDDS on maxterm and minterm Boolean functions and allow us to establish a characterization of GOE states of such systems.

Corollary 5.9 (Characterization of GOE in MAX−PDDS)**.** *Let* $[D, \mathrm{MAX}]-\mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX*. Then, a configuration is a* GOE *if, and only if, there exists an activated entity influencing to a deactivated one in such a configuration, but not influenced by an activated entity which is not influencing to any deactivated one.*

In other words, a configuration is a GOE *if, and only if,* $\overline{I_D(I^c)} \neq V_1$, *being* $I = \overline{I_D(V_0)}$.

Corollary 5.10 (Characterization of GOE in MIN−PDDS)**.** *Let* $[D, \mathrm{MIN}]-\mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*. Then, a configuration is a* GOE *if, and only if, there exists a deactivated entity influencing to an activated one in such a configuration, but not influenced by a deactivated entity which is not influencing to any activated one.*

In other words, a configuration is a GOE *if, and only if,* $\overline{I_D(I^c)} \neq V_0$, *being* $I = \overline{I_D(V_1)}$.

Remark 5.3. Contrarily to the case of MAX − PDS (Corollary 3.9 in Subsection 3.2.1), in the case of MAX − PDDS, a state of the system with only one activated entity could have a predecessor. Specifically, if the activated entity has out-degree 0, such a state has a predecessor. Dually, in the case of MIN − PDDS, a state of the system with only one deactivated entity could have a predecessor. Specifically, if the deactivated entity has out-degree 0, such a state has a predecessor. □

Actually, we have obtained bounds for the number of GOE points of such systems, as shown below.

Corollary 5.11. *Let* $[D, \mathrm{MAX}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$*, with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$*, associated with the maxterm* MAX*. Then, the number of* GOE *points,* #GOE*, is such that*

$$1 \leq \#\mathrm{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

Proof. Firstly, observe that, as $n \geq 2$, there exist a vertex $i \in V$ influencing to other vertex $j \in V$. Then, a configuration such that $y_i = 1$ and $y_k = 0$, if $k \neq i$, has no predecessors by Theorem 5.9. This bound is reached in the following example:

Let $D = (V, A)$ be the digraph defined by $V = \{1, 2\}$ and $A = \{(1, 2)\}$. Additionally, let us consider the maxterm $\mathrm{MAX} = x_1 \vee x_2'$.

Let $[D, \mathrm{MAX}] - \mathrm{PDDS}$ be the parallel directed dynamical system over D associated with the maxterm MAX. This system has a unique GOE configuration, as can be seen in its phase portrait in Figure 5.6.

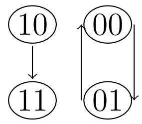

Figure 5.6: Phase portrait of the system $[\{1, 2\}, \{(1, 2)\}, x_1 \vee x_2'] - \mathrm{PDDS}$.

On the other hand, in view of the characterization of GOE given in Corollary 5.9, it is clear that the states $\mathcal{I}$ and $\mathcal{O}$ are not GOE points. Moreover, this upper bound is reached for the complete digraph K_2, whichever the maxterm MAX, since the other two states, $(0, 1)$ and $(1, 0)$, are GOE. □

And now its dual version.

Corollary 5.12. *Let* $[D, \mathrm{MIN}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$*, with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$*, associated with the minterm* MIN*. Then, the number of* GOE *points,* #GOE*, is such that*

$$1 \leq \#\mathrm{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

Remark 5.4. In Corollary 5.11 (resp. Corollary 5.12), $n \geq 2$ has been imposed. This is necessary because a $[D, \mathrm{MAX}] - \mathrm{PDDS}$ (resp. $[D, \mathrm{MIN}] - \mathrm{PDDS}$) with $n = 1$ has 2 fixed points, if $W' = \emptyset$, or one 2-cycle, if $W = \emptyset$. That is, it has not GOE points in any case. □

The proof of Theorem 5.9 is constructive and provides information about the structure of a predecessor of a given state of the system, when it exists. This information is collected in the following two results.

Corollary 5.13. *Let* $[D, \mathrm{MAX}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX*. If a configuration* y *has a predecessor state* x*, such a predecessor verifies:*

- *If* $y_i = 0$*, for every entity* $j \in \overline{I_D(i)}$*, then*
  - $x_j = 0$*, if* $j \in W$*, and*
  - $x_j = 1$*, if* $j \in W'$*.*
- *If* $y_i = 1$*, then there exists an entity* $j \in \overline{I_D(i)}$ *such that it accomplishes one of the following conditions:*
  - $x_j = 1$ *and* $j \in W$*, or*
  - $x_j = 0$ *and* $j \in W'$*.*

Proof. We can see it in the constructive process shown to prove Theorem 5.9. □

Remark 5.5. In terms of Corollary 5.13, in the case of existence of predecessor for a state y, there is always a structure x corresponding to a predecessor, named *fundamental predecessor* of y, as in other contexts. This configuration, which is proposed in the (second part of the) proof of Theorem 5.9, is as follows:

- If $i \in \overline{I_D(V_0)}$, then
  - $x_i = 0$, if $i \in W$, and
  - $x_i = 1$, if $i \in W'$.
- If $i \in \overline{I_D(V_0)}^c$, then
  - $x_i = 1$, if $i \in W$, and
  - $x_i = 0$, if $i \in W'$.

Thus, to know if a configuration y has a predecessor in a MAX$-$PDDS, one needs only to verify if the corresponding configuration x, candidate to be its fundamental predecessor, is such that $\mathrm{MAX}(x) = y$. □

Dually, we have the following corollary.

Corollary 5.14. *Let* $[D, \mathrm{MIN}]-$PDDS *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*. If a configuration* y *has a predecessor state* x*, such a predecessor verifies:*

- *If* $y_i = 1$*, for every entity* $j \in \overline{I_D(i)}$*, then:*
  - $x_j = 1$*, if* $j \in W$*, and*
  - $x_j = 0$*, if* $j \in W'$*.*
- *If* $y_i = 0$*, then there exists an entity* $j \in \overline{I_D(i)}$ *such that it accomplishes one of the following conditions:*
  - $x_j = 0$ *and* $j \in W$*, or*
  - $x_j = 1$ *and* $j \in W'$*.*

Remark 5.6. As for Corollary 5.13, in terms of Corollary 5.14, in the case of existence of predecessor for a state y, there is always a structure x corresponding to a predecessor, named *fundamental predecessor* of y. This configuration is as follows:

- If $i \in \overline{I_D(V_1)}$, then
  - $x_i = 1$, if $i \in W$, and
  - $x_i = 0$, if $i \in W'$.
- If $i \in \overline{I_D(V_1)}^c$, then
  - $x_i = 0$, if $i \in W$, and
  - $x_i = 1$, if $i \in W'$.

Thus, as before, to know if a configuration y has a predecessor in a MIN$-$PDDS, one needs only to verify if the corresponding configuration x, candidate to be its fundamental predecessor, is such that $\mathrm{MIN}(x) = y$. □

As in the case of undirected dependency graphs, observe that the entities whose state values are 0 (resp. 1) in the configuration y determine univocally their state values and the state values of their influencing ones in any predecessor x with respect to MAX − PDDS (resp. MIN − PDDS), when such a predecessor exits. Nevertheless, for any entity whose state value is 1 (resp. 0) in the configuration y, it is only necessary the existence of an appropriate influencing one which provides such a value with the MAX − PDDS (resp. MIN − PDDS) updating. This points out how to look for the solution to the UPRE, APRE and #PRE problems in our context.

The following results are concerned with the determination of the possible predecessors of a given configuration y, once we know that at least one predecessor x exists. As a consequence, we solve the UPRE and APRE problems for PDDS on maxterm and minterm Boolean functions.

Theorem 5.11. *Let* $[D, \mathrm{MAX}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Let* y *be a configuration and suppose that it has a predecessor. Then, the predecessor of* y *is not unique if, and only if, there exists an activated entity* $i \in \overline{I_D(V_0)}^c$ *such that one of its influencing entities also belong to* $\overline{I_D(V_0)}^c$ *and its influenced ones in* $I_D^*(V_0)$*, if any, are also influenced by other entities in* $\overline{I_D(V_0)}^c$*.*

In other words, the predecessor of y *is not unique if, and only if, there exists and entity* $i \in \overline{I_D(V_0)}^c$ *such that* $\overline{I_D(I^c \setminus \{i\})} = V_1$*, being* $I = \overline{I_D(V_0)}$*.*

Proof. First of all, suppose that there exists such an activated entity $i \in \overline{I_D(V_0)}^c$ in the configuration y which is not influencing to any deactivated one, such that one of its influencing entities also belong to $\overline{I_D(V_0)}^c$ and its influenced ones in $I_D^*(V_0)$, if any, are also influenced by other entities in $\overline{I_D(V_0)}^c$. Remember that, since y has a predecessor, the entities in $\overline{I_D(V_0)}$ at y determine univocally their state values in any predecessor with respect to MAX − PDDS. On the other hand, we can act similarly as in the (second part of the) proof of Theorem 5.9 and for every entity $j \in \overline{I_D(V_0)}^c$, $j \neq i$, to construct a predecessor configuration, we can take as follows:

- $x_j = 1$ when $j \in W$, and
- $x_j = 0$ when $j \in W'$.

Now, taking into account that one of the entities which are influencing to i also belong to $\overline{I_D(V_0)}^c$ and the rest of its influenced ones in $I_D^*(V_0)$, if any, are also influenced by other entities in $\overline{I_D(V_0)}^c$, i and its influenced vertices in $I_D^*(V_0)$ become activated, independently of the state value of i. That is, in such a predecessor construction, we can choose either $x_i = 0$ or $x_i = 1$, so obtaining two different configurations which are predecessors of y.

Reciprocally, suppose that the configuration y has more than one predecessor. Again, the entities in $\overline{I_D(V_0)}$ in the configuration y determine univocally their state values in any predecessor with respect to MAX $-$ PDDS. Thus, the discrepancies should be in the state values of entities belonging to $\overline{I_D(V_0)}^c$, that is, activated and not influencing to any deactivated one. Suppose that there is a discrepancy of two predecessors in the state values of an entity $i \in \overline{I_D(V_0)}^c$. This means that the entity i and its influenced ones in $I_D^*(V_0)$ become activated in y independently of the state value of i in such predecessors. Therefore, there should exist activated entities belonging to $\overline{I_D(V_0)}^c$ which are influencing to i and its influenced ones in $I_D^*(V_0)$ to provide that all of them have state value equal to 1 in the configuration y. □

Remark 5.7. In terms of Theorem 5.11, given a state y, if the configuration x defined as in Remark 5.5 is its (fundamental) predecessor, to know if this is its unique predecessor, we must only verify if y has a predecessor belonging to the following set:

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\, i \in \overline{I_D(V_0)}^c \textit{ such that } \hat{x}_i \neq x_i \textit{ and } \hat{x}_j = x_j\ \forall j \in V \setminus \{i\}\}.$$

This result reduces an initial exponentially-sized problem, i.e., the search of a particular configuration among the 2^n possible states of the system, into another one in which, at most, n cases must be analyzed. □

Dually, we have the following theorem.

Theorem 5.12. *Let* $[D, \mathrm{MIN}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *Let* y *be a configuration and suppose that it has a predecessor. Then, the predecessor of* y *is not unique if, and only if, there exists a deactivated entity* $i \in \overline{I_D(V_1)}^c$ *such that one of its influencing entities also belong to* $\overline{I_D(V_1)}^c$ *and its influenced ones in* $I_D^*(V_1)$*, if any, are also influenced by other entities in* $\overline{I_D(V_1)}^c$.

In other words, the predecessor of y *is not unique if, and only if, there exists and entity* $i \in \overline{I_D(V_1)}^c$ *such that* $\overline{I_D(I^c \setminus \{i\})} = V_0$*, being* $I = \overline{I_D(V_1)}$.

Remark 5.8. Similarly as in Remark 5.7, in terms of Theorem 5.12, given a state y, if the configuration x defined as in Remark 5.6 is its (fundamental) predecessor, to know if this is its unique predecessor, we must only verify if y has a predecessor belonging to the following set:

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\, i \in \overline{I_D(V_1)}^c \textit{ such that } \hat{x}_i \neq x_i \textit{ and } \hat{x}_j = x_j\ \forall j \in V \setminus \{i\}\}.$$

As in the case of MAX − PDDS, this result reduces for MIN − PDDS the initial exponentially-sized problem, into another one in which, at most, n cases must be analyzed. □

Once the existence of more than one predecessor is known, the following step is to try to obtain the number of them for any given state. In the next two corollaries, we explain how to obtain theoretically the set of all of them and, consequently, its number, in order to solve the classical predecessor problem #PRE.

Corollary 5.15. *Let* $[D, \mathrm{MAX}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX*. Let* y *a configuration and*

$$P_i = \{x\ state : x\ satisfies\ the\ conditions\ in\ Corollary\ 5.13\ for\ y_i\}.$$

Then, $P = \bigcap_{i \in V} P_i$ *is the set of all the predecessor states of* y*.*

Dually, we have the following corollary.

Corollary 5.16. *Let* $[D, \mathrm{MIN}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*. Let* y *be a configuration and*

$$P_i = \{x\ state : x\ satisfies\ the\ conditions\ in\ Corollary\ 5.14\ for\ y_i\}.$$

Then, $P = \bigcap_{i \in V} P_i$ *is the set of all the predecessor states of* y*.*

In the case of MAX − PDDS (resp. MIN − PDDS), the configuration $\mathcal{O}$ (resp. $\mathcal{I}$) has always a unique predecessor, by Theorems 5.9 and 5.11 (resp. Theorems 5.10 and 5.12). However, the calculus of the number of predecessors for a general state of the entities different from these ones depends on the connections among the entities in each particular system. Nevertheless, as traditionally done, we have been able to get a bound for the number of predecessors of a general configuration, which is given in the following theorem.

Theorem 5.13. *Let* $[D, \mathrm{MAX}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX*. Then, the number of predecessors of a given configuration* y *different from* $\mathcal{O}$ *is upper bounded by* $2^{\#\overline{I_D(V_0)}^c} - 1$*. In fact, such a bound is reachable.*

Proof. From Theorems 5.9 and 5.11, it is clear that the possible discrepancies between predecessors correspond to differences in state values of entities in $\overline{I_D(V_0)}^c$. Since the state values of any of these entities are either 0 or 1, a first bound for the number of predecessors is $2^{\#\overline{I_D(V_0)}^c}$.

However, if $\overline{I_D(V_0)}^c = \emptyset$, by Theorem 5.9, y has not predecessors, while if $\overline{I_D(V_0)}^c \neq \emptyset$, a configuration such that:

- $x_i = 0$ when $i \in W \cap \overline{I_D(V_0)}^c$, and
- $x_i = 1$ when $i \in W' \cap \overline{I_D(V_0)}^c$,

cannot be a predecessor of y. Thus, in any case, one configuration must be discarded from the previous bound.

In fact, such a bound $2^{\#\overline{I_D(V_0)}^c} - 1$ is reachable. It is sufficient to consider a $[D, \mathrm{MAX}] - \mathrm{PDDS}$ such that the subdigraph corresponding to $\overline{I_D(V_0)}^c$ is complete and with the condition that each entity in such a set is influencing to all the entities in $I_D^*(V_0)$. □

Dually, we have the following theorem.

Theorem 5.14. *Let* $[D, \mathrm{MIN}] - \mathrm{PDDS}$ *be a parallel directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *Then, the number of predecessors of a given configuration y different from* $\mathcal{I}$ *is upper bounded by* $2^{\#\overline{I_D(V_1)}^c} - 1$. *In fact, such a bound is reachable.*

5.2.2 Predecessor and GOE configurations in SDDS

As said in Subsection 4.2.1 within the ambit of SDS, in [30, 31, 32], the study of predecessors is divided into four specific problems (PRE, UPRE, APRE and #PRE).

Proceeding as in the case of SDS, the next theorem provides us with a characterization of existence of predecessors in terms of sufficient and necessary conditions, finding a particular predecessor, named fundamental predecessor, of a specific state of the entities in the context of an SDDS, when it exists.

Theorem 5.15. *Let* $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Then, a configuration y has a predecessor if, and only if, the state x defined as follows is such that* $\mathrm{MAX}(x) = y$*:*

- *For every entity* $i \in V_0 \cup P_0$,
  - $x_i = 0$, *if* $i \in W$,
  - $x_i = 1$, *if* $i \in W'$.
- *For every entity* $i \in (V_0 \cup P_0)^c$,
  - $x_i = 1$, *if* $i \in W$,
  - $x_i = 0$, *if* $i \in W'$.

Proof. The arguments in this proof are similar to those in the case of SDS. However, for the sake of completeness, we include them here.

It must only be shown that this condition is necessary for the existence of a predecessor. For this purpose, let us see that if there is a predecessor of y, $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$, then x defined as in this theorem is also a predecessor of y.

Thus, if $i \in V_0 \cup P_0$, it must be,

- $\hat{x}_i = 0 = x_i$, if $i \in W$,
- $\hat{x}_i = 1 = x_i$, if $i \in W'$,

since otherwise, $y_i = 1$, if $i \in V_0$, or $y_j = 1$ for some $j \in V_0$ influenced by i (those ones that update before i) if $i \in P_0$.

Suppose, by reduction to the absurd, that x is not a predecessor of y. Let $i \in V$ be the first entity, according to the order established by π, such that x_i does not update to y_i. It must be $i \in V_0 \cup P_0$, because the entities in $(V_0 \cup P_0)^c \subseteq V_1$ update to the activated state because of their own state values in x.

If $i \in P_0 \setminus V_0 \subseteq V_1$, let us analyze the possible state of the entities belonging to $\overline{I_D(i)}$:

- Since i is the first entity not updating to the state given by $y_i = 1$, then $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\hat{x}$.
- $x_i = \hat{x}_i$.
- $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$:
  - If $j \in P_0 \cup V_0$, then $x_j = \hat{x}_j$.

- If $j \in (P_0 \cup V_0)^c$, then $x_j = 1$, if $j \in W$, or $x_j = 0$, if $j \in W'$.

Since $\hat{x}_i$ updates to $y_i = 1$, x_i must also do it, but this is a contradiction and, consequently, $i \notin P_0 \setminus V_0$.

Therefore $i \in V_0$. In this situation:

- Since i is the first entity not updating to the state given by $y_i = 0$, then $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\hat{x}$.
- $x_i = \hat{x}_i$.
- $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$, the entity $j \in P_0$, so $x_j = 0$, if $j \in W$, or $x_j = 1$, if $j \in W'$.

Since $\hat{x}_i$ updates to $y_i = 0$, x_i must also do it, which is a contradiction and, consequently, $i \notin V_0$.

Therefore, there cannot exist $i \in V$ like that and x updates to y. □

Dually, we have the following result.

Theorem 5.16. *Let* $[D, \mathrm{MIN}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*. Then, a configuration* y *has a predecessor if, and only if, the state* x *defined as follows is such that* $\mathrm{MIN}(x) = y$*:*

- *For every entity* $i \in V_1 \cup P_1$*,*
  - $x_i = 1$*, if* $i \in W$*,*
  - $x_i = 0$*, if* $i \in W'$*.*
- *For every entity* $i \in (V_1 \cup P_1)^c$*,*
  - $x_i = 0$*, if* $i \in W$*,*
  - $x_i = 1$*, if* $i \in W'$*.*

Theorems 5.15 and 5.16 solve the PRE problem for SDDS on maxterm and minterm Boolean functions, respectively, and allow us to establish the following characterization of the GOE points of these systems.

Corollary 5.17 (Characterization of GOE in MAX $-$ SDDS)**.** *Let* $[D, \mathrm{MAX}, \pi]-$ SDDS *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Then, a configuration* y *is a* GOE *point of the system if, and only if, the state* x *defined as in Theorem 5.15 is such that* $\mathrm{MAX}\,(x) \neq y$.

Corollary 5.18 (Characterization of GOE in MIN$-$SDDS)**.** *Let* $[D, \mathrm{MIN}, \pi]-$SDDS *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *Then, a configuration* y *is a* GOE *point of the system if, and only if, the state* x *defined as in Theorem 5.16 is such that* $\mathrm{MIN}\,(x) \neq y$.

Next, we provide sufficient conditions to determine GOE points.

Corollary 5.19. *Let* $[D, \mathrm{MAX}, \pi]-$SDDS *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *If a state* y *is such that* $(Q_0 \cap V_0) \cap W' \neq \emptyset$, *then* y *is a* GOE *point.*

Proof. If $(Q_0 \cap V_0) \cap W' \neq \emptyset$, there is an entity $i \in V_0$ whose corresponding variable in MAX appears in complemented form and influencing an entity $j \in V_0$ which updates after it. In this situation, the configuration y cannot be obtained as the update of another state x because the evolution of the entity i to the deactivated state makes it impossible the posterior update of the entity j to this state. □

Corollary 5.20. *Let* $[D, \mathrm{MAX}, \pi]-$SDDS *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *If a state* y *is such that* $(Q_0 \cap V_0^c) \cap W \neq \emptyset$, *then* y *is a* GOE *point.*

Proof. If $(Q_0 \cap V_0^c) \cap W \neq \emptyset$, there is an entity $i \in V_1$ whose corresponding variable in MAX appears in direct form and influencing an entity $j \in V_0$ which updates after it. The proof finishes reasoning as in Corollary 5.19. □

Dually, we have the following results.

Corollary 5.21. *Let* $[D, \mathrm{MIN}, \pi]-$SDDS *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *If a state* y *is such that* $(Q_1 \cap V_1) \cap W' \neq \emptyset$, *then* y *is a* GOE *point.*

Corollary 5.22. *Let* $[D, \mathrm{MIN}, \pi]-$SDDS *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *If a state* y *is such that* $(Q_1 \cap V_1^c) \cap W \neq \emptyset$, *then* y *is a* GOE *point.*

Corollary 5.19 means that a configuration y with a deactivated entity associated to a complemented variable and influencing other deactivated one which updates after it, is a GOE point. In Subsection 4.2.1, an homologous result is shown for the case of SDS (see Corollary 4.7). Indeed, when MAX = NAND, they are the only GOE points and the other configurations belong to periodic orbits (see Proposition 4.3). In NAND − SDDS, these configurations are not the only GOE points of the system and there are eventually periodic points different from the GOE, as shown in the following example.

Example 5.2. In the case of the NAND − SDDS defined by

- $D = (\{1, 2\}, \{(2, 1)\})$,
- $\mathrm{MAX} = x_1' \vee x_2'$,
- $\pi = 1|2$,

the configuration $(0, 1)$ is a GOE point and $(1, 0)$ is neither a GOE nor a periodic point, as can be seen in the phase portrait of the system in Figure 5.7.

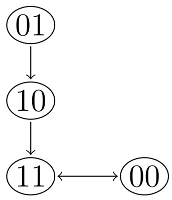

Figure 5.7: Phase portrait of the system $[(\{1, 2\}, \{(2, 1)\}), x_1' \vee x_2', 1|2] - \mathrm{SDDS}$.

□

Additionally, the pattern of the synchronous update over undirected graphs shown in Corollary 3.9 of Subsection 3.2.1 for a MAX − PDS over a dependency graph $G = (V, E)$, with $V = \{1, \ldots, n\}$ and $n \geq 2$, whereby a configuration with only one activated entity has no predecessors, is broken in this case of SDDS, since it is already broken in the case of SDS, as said in Example 4.6 of Subsection 4.2.1, and SDS is a particular case of SDDS.

In view of these results, we can state the following corollaries about the number of GOE points in an SDDS.

Corollary 5.23. *Let* $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX*, with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$*. Then, the number of* GOE *points,* $\#\mathrm{GOE}$*, is such that*

$$1 \leq \#\mathrm{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

Proof. First, we will prove that any SDDS with $n \geq 2$ has GOE points.

If there is an arc from an entity i towards an entity j with i updating before j, if $i \in W$ then $y_i = 1$ and $y_j = 0$ cannot be obtained after the update of a configuration, and the same if $i \in W'$ for the values $y_i = y_j = 0$.

Otherwise, each arc of D is such that its initial vertex updates after its final vertex, according to the order π. Thus, the vertex π_n is not influenced by another one in its update and, in any predecessor of a state with $y_{\pi_n} = 1$, π_n must be activated if $\pi_n \in W$ or deactivated if $\pi_n \in W'$. Since D in weakly connected, there exists $k \in V$ such that $(\pi_n, k) \in A$ and, therefore, $y_k = 0$ and $y_{\pi_n} = 1$ cannot be obtained after the update of a configuration.

In fact, the lower bound is reached, as shown in the example below. Let us consider the $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ defined by

- $D = (\{1, 2\}, \{(1, 2)\})$,
- $\mathrm{MAX} = x'_1 \vee x'_2$,
- $\pi = 1|2$.

In this case, $(0, 0)$ is a GOE point, being the phase portrait of the system as can be checked in Figure 5.8.

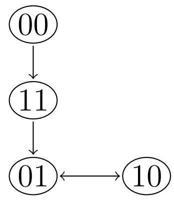

Figure 5.8: Phase portrait of the system $[(\{1, 2\}, \{(1, 2)\}), x'_1 \vee x'_2, 1|2] - \mathrm{SDDS}$.

On the other hand, $\mathcal{I}$ is never a GOE point of the system, because the state x defined as follows is its predecessor:

- $x_i = 1$, if $i \in W$,
- $x_i = 0$, if $i \in W'$.

Also, there is always another configuration with a predecessor, because if $\overline{x}$ is defined as

- $\overline{x}_i = 0$, if $i \in W$,
- $\overline{x}_i = 1$, if $i \in W'$,

then $\overline{x}$ updates to a state $\overline{y}$ such that $\overline{y}_1 = 0$.

As shown in the example below, this upper bound is also reached. Let us consider the following $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$, determined by

- $D = (\{1,2\}, \{(1,2), (2,1)\})$,
- $\mathrm{MAX} = x_1 \vee x_2'$,
- $\pi = 1|2$.

The phase portrait of this system is shown in Figure 5.9.

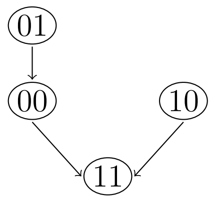

Figure 5.9: Phase portrait of the system $[(\{1,2\}, \{(1,2), (2,1)\}), x_1 \vee x_2', 1|2] - \mathrm{SDDS}$.

$\square$

Remark 5.9. In Corollary 5.23, $n \geq 2$ has been imposed. This is necessary because a $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ with $n = 1$ has 2 fixed points, if $W' = \emptyset$, or one 2-cycle, if $W = \emptyset$. That is, it has not GOE points in any case. $\square$

Dually, we have the following result.

Corollary 5.24. *Let* $[D, \mathrm{MIN}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*, with* $V = \{1, \ldots, n\}$ *and* $n \geq 2$*. Then, the number of* GOE *points,* #GOE*, is such that*

$$1 \leq \#\mathrm{GOE} \leq 2^n - 2.$$

Moreover, these bounds are the best possible because they are reachable.

In Theorem 5.15, a constructive proof about the existence of a fundamental predecessor is shown. The structure of such a predecessor exposed in that reasoning inspires the following result.

Corollary 5.25. *Let* $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX*. If a configuration* y *has a predecessor* x*, the following conditions are verified:*

- *If* $y_i = 0$*, for every entity* $j \in \overline{I_D(i)}$*, with* $i = \pi_r$ *and* $j = \pi_s$*:*
  - *If* $i = j$ *or* $r < s$*:*
    - ◇ $x_j = 0$*, if* $j \in W$*, or*
    - ◇ $x_j = 1$*, if* $j \in W'$*.*
  - *If* $r > s$*:*
    - ◇ $y_j = 0$*, if* $j \in W$*, or*
    - ◇ $y_j = 1$*, if* $j \in W'$*.*
- *If* $y_i = 1$*, there exists an entity* $j \in \overline{I_D(i)}$ *such that if* $i = \pi_r$ *and* $j = \pi_s$*, at least one of the following conditions is accomplished:*
  - $i = j$ *or* $r < s$*, and:*
    - ◇ $x_j = 1$*, if* $j \in W$*, or*
    - ◇ $x_j = 0$*, if* $j \in W'$*.*
  - $r > s$*, and:*
    - ◇ $y_j = 1$*, if* $j \in W$*, or*
    - ◇ $y_j = 0$*, if* $j \in W'$*.*

Proof. On the one hand, if $y_i = 0$ and there is $j \in \overline{I_D(i)}$ such that the conditions shown are not satisfied in this case, the entity i will update to the activated state due to this influencing entity j, which is a contradiction. On the other hand, if $y_i = 1$ and $\forall j \in \overline{I_D(i)}$ these conditions are not satisfied, the entity i will update to the deactivated state, which is also a contradiction. □

Dually, we have the following result.

Corollary 5.26. *Let $[D, \mathrm{MIN}, \pi] - \mathrm{SDDS}$ be a sequential directed dynamical system over a dependency digraph $D = (V, A)$ associated with the minterm MIN. If a configuration y has a predecessor x, the following conditions are verified:*

- *If $y_i = 1$, for every entity $j \in \overline{I_D(i)}$, with $i = \pi_r$ and $j = \pi_s$:*
  - *If $i = j$ or $r < s$:*
    - ◇ *$x_j = 1$, if $j \in W$, or*
    - ◇ *$x_j = 0$, if $j \in W'$.*
  - *If $r > s$:*
    - ◇ *$y_j = 1$, if $j \in W$, or*
    - ◇ *$y_j = 0$, if $j \in W'$.*
- *If $y_i = 0$, there exists an entity $j \in \overline{I_D(i)}$ such that if $i = \pi_r$ and $j = \pi_s$, at least one of the following conditions is accomplished:*
  - *$i = j$ or $r < s$, and:*
    - ◇ *$x_j = 0$, if $j \in W$, or*
    - ◇ *$x_j = 1$, if $j \in W'$.*
  - *$r > s$, and:*
    - ◇ *$y_j = 0$, if $j \in W$, or*
    - ◇ *$y_j = 1$, if $j \in W'$.*

In a MAX − SDDS (resp. MIN − SDDS), the entities whose state is deactivated (resp. activated) in y determine univocally their state and the state of their influencing entities in P_0 (resp. P_1) in any predecessor x, if such a predecessor exists. However, for any entity whose state value is 1 (resp. 0) in y, it is only necessary the intervention of a timely influencing entity, or itself, with the appropriate state in the moment of its update. This point is the key to solve the UPRE, APRE and #PRE problems hereafter.

The following theorems allow us to determine if, given a state y with a predecessor x, there are other configurations different from x such that they are also predecessors of y. Thus, the UPRE and APRE problems in the context of an SDDS on maxterm and minterm Bolean functions are solved.

Theorem 5.17. *Let* $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Let* y *be a configuration of the system such that it has a predecessor. Then, this predecessor of* y *is not unique if, and only if, there is a predecessor of* y *belonging to the following set:*

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\, i \in (V_0 \cup P_0)^c \text{ such that } \hat{x}_i \neq x_i \text{ and } \hat{x}_j = x_j \ \forall j \in V \setminus \{i\}\},$$

being x *the fundamental predecessor of* y *described in Theorem 5.15.*

Proof. The arguments in this proof are similar to those in the case of SDS. However, to the sake of completeness, we include them here.

Since the fundamental predecessor x defined as in Theorem 5.15 is such that $x \notin \mathfrak{P}$, it must only be shown that this condition is necessary for the existence of a predecessor different from x. For this purpose, let us see that if there is a predecessor of y different from x, $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$, then there exists a state $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathfrak{P}$ such that $\hat{x}$ is also a predecessor of y.

Given that $\overline{x} \neq x$ and $\overline{x}$ is a predecessor of y, by Corollary 5.25, there is an entity $i_0 \in (V_0 \cup P_0)^c$ such that $\overline{x}_{i_0} \neq x_{i_0}$. Let us take $\hat{x}$ as the only element of $\mathfrak{P}$ such that $\hat{x}_{i_0} \neq x_{i_0}$ (consequently, $\hat{x}_{i_0} = \overline{x}_{i_0}$), and let us see that this state is a predecessor of y.

Suppose, by reduction to the absurd, that $\hat{x}$ is not a predecessor of y. Let $i \in V$ be the first entity, according to the order established by π, such that $\hat{x}_i$ does not update to y_i. It must be $i \in V_0 \cup P_0 \cup \{i_0\}$, because the entities in $(V_0 \cup P_0 \cup \{i_0\})^c \subseteq V_1$ update to the activated state because of their own state values in $\hat{x}$.

If $i \in P_0 \setminus (V_0 \cup \{i_0\}) = P_0 \setminus V_0 \subseteq V_1$, let us analyze the possible state of the entities belonging to $\overline{I_D(i)}$:

- Since i is the first entity not updating to the state given by $y_i = 1$, then $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\overline{x}$.

- $\hat{x}_i = \overline{x}_i$.

- $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$:
    - If $j \in P_0 \cup V_0 \cup \{i_0\}$, then $\hat{x}_j = \overline{x}_j$.
    - If $j \in (P_0 \cup V_0 \cup \{i_0\})^c$, then $\hat{x}_j = 1$, if $j \in W$, or $\hat{x}_j = 0$, if $j \in W'$.

Since $\overline{x}_i$ updates to $y_i = 1$, $\hat{x}_i$ must also do it, but this is a contradiction and, consequently, $i \notin P_0 \setminus V_0$.

If $i \in V_0 \setminus \{i_0\} = V_0$, we have the following:

- Since i is the first entity not updating to the state given by $y_i = 0$, then $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\overline{x}$.
- $\hat{x}_i = \overline{x}_i$.
- $\forall j \in I_D(i)$ with $i = \pi_r$, $j = \pi_s$ and $s > r$, the entity $j \in P_0 = P_0 \setminus \{i_0\}$, so $x_j = 0$, if $j \in W$, or $x_j = 1$, if $j \in W'$.

Since $\overline{x}_i$ updates to $y_i = 0$, $\hat{x}_i$ must also do it, which is a contradiction and, consequently, $i \notin V_0$.

Therefore $i = i_0$. In this situation, we have the following:

- Since i is the first entity not updating to the state given by $y_i = 1$ ($i_0 \in (V_0 \cup P_0)^c \subseteq V_1$), then $\forall j \in I_D(i_0)$ with $i_0 = \pi_r$, $j = \pi_s$ and $s < r$, the entity j has updated to the state given by y_j, the same as for $\overline{x}$.
- $\hat{x}_{i_0} = \overline{x}_{i_0}$.
- $\forall j \in I_D(i_0)$ with $i_0 = \pi_r$, $j = \pi_s$ and $s > r$:
    - If $j \in P_0 \cup V_0$, then $\hat{x}_j = \overline{x}_j$.
    - If $j \in (P_0 \cup V_0)^c$, then $\hat{x}_j = 1$, if $j \in W$, or $\hat{x}_j = 0$, if $j \in W'$.

Since $\overline{x}_i$ updates to $y_i = 1$, $\hat{x}_i$ must also do it, but this is also a contradiction and, consequently, $i \neq i_0$.

Therefore, there cannot exist $i \in V$ like that and $\hat{x}$ updates to y. $\square$

Remark 5.10. As in the case of PDDS, this result reduces an initial exponentially-sized problem, into another one in which, at most, n cases must be analyzed. $\square$

Dually, we have the following result.

Theorem 5.18. *Let* $[D, \mathrm{MIN}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *Let* y *be a configuration of the system such that it has a predecessor. Then, this predecessor of* y *is not unique if, and only if, there is a predecessor of* y *belonging to the following set:*

$$\mathfrak{P} = \{\hat{x} \in \{0,1\}^n : \exists\ i \in (V_1 \cup P_1)^c \ \text{ such that } \hat{x}_i \neq x_i \text{ and } \hat{x}_j = x_j \ \forall j \in V \setminus \{i\}\},$$

being x *the fundamental predecessor of* y *described in Theorem 5.16.*

These results respond to the question of the existence of more than one predecessor for a state y. The next step is to go deeper into this topic, getting the number of them. In the following results we explain a method to obtain all the predecessors of y and, consequently, this number in order to solve the predecessor problem #PRE.

Corollary 5.27. *Let* $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Let* y *be a configuration and let us consider the following iterative process:*

- $\mathfrak{P}_{n+1} = \{y\}$.
- *If* $i \in V$, *then*

  $$\mathfrak{P}_i = \{x \in \{0,1\}^n : \exists \overline{y} \in \mathfrak{P}_{i+1} \ / \ x_j = \overline{y}_j \ \text{if } j \neq \pi_i \text{ and } \mathrm{MAX}_{|\overline{I_D(\pi_i)}}(x) = y_{\pi_i}\},$$

  being $\mathrm{MAX}_{|\overline{I_D(\pi_i)}}$ *the restriction of* MAX *over* $\overline{I_D(\pi_i)}$.

Then, $\mathfrak{P}_1$ *is the set of all the predecessors of* y.

Example 5.3. Let us illustrate this procedure with a particular example in order to clarify the notation. We consider the $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ defined by

- $D = (\{1,2,3\}, \{(1,2), (2,3), (3,1)\})$,
- $\mathrm{MAX} = x_1 \vee x_2 \vee x_3$,
- $\pi = 1|2|3$.

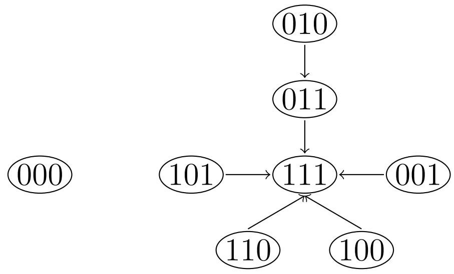

Figure 5.10: Phase portrait of the system $[(\{1,2,3\},\{(1,2),(2,3),(3,1)\}), x_1 \vee x_2 \vee x_3, 1|2|3] -$ SDDS.

The phase portrait of this system is shown in Figure 5.10.

The set of predecessors of the configuration $y = (1,1,1)$ is

$$\{(1,1,1),(0,1,1),(1,0,1),(1,1,0),(1,0,0),(0,0,1)\}.$$

Let us see that this set is obtained as $\mathfrak{P}_1$ at the end of the iterative process starting with $\mathfrak{P}_4 = \{(1,1,1)\}$.

Firstly, $\mathfrak{P}_3$ is obtained: since $\pi_3 = 3$, the only configurations that can belong to $\mathfrak{P}_3$ are $(1,1,0)$ and $(1,1,1)$. Besides, since $\mathrm{MAX}_{|\overline{I_D(3)}}(1,1,0) = \mathrm{MAX}_{|\overline{I_D(3)}}(1,1,1) = 1 = y_3$, then $\mathfrak{P}_3 = \{(1,1,0),(1,1,1)\}$.

Then, to obtain $\mathfrak{P}_2$, since $\pi_2 = 2$ and considering the elements in $\mathfrak{P}_3$, the only configurations that can belong to this set $\mathfrak{P}_2$ are $(1,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(1,1,1)$. Now, $\mathrm{MAX}_{|\overline{I_D(2)}}(1,0,0)$, $\mathrm{MAX}_{|\overline{I_D(2)}}(1,1,0)$, $\mathrm{MAX}_{|\overline{I_D(2)}}(1,0,1)$ and $\mathrm{MAX}_{|\overline{I_D(2)}}(1,1,1)$ are all equal to $1 = y_2$. Thus, the set $\mathfrak{P}_2$ is as follows: $\mathfrak{P}_2 = \{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\}$.

Finally, since $\pi_1 = 1$ and knowing the set $\mathfrak{P}_2$, the only configurations that can belong to $\mathfrak{P}_1$ are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$. Now, $\mathrm{MAX}_{|\overline{I_D(1)}}(0,0,0)$ and $\mathrm{MAX}_{|\overline{I_D(1)}}(0,1,0)$ are equal to $0 \neq y_1$, while $\mathrm{MAX}_{|\overline{I_D(1)}}(1,0,0)$, $\mathrm{MAX}_{|\overline{I_D(1)}}(1,1,0)$, $\mathrm{MAX}_{|\overline{I_D(1)}}(0,0,1)$, $\mathrm{MAX}_{|\overline{I_D(1)}}(1,0,1)$, $\mathrm{MAX}_{|\overline{I_D(1)}}(0,1,1)$ and $\mathrm{MAX}_{|\overline{I_D(1)}}(1,1,1)$ are all equal to $1 = y_1$. Thus, $\mathfrak{P}_1 = \{(1,0,0),(1,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}$. $\square$

Dually, we have the following result.

Corollary 5.28. *Let* $[D, \mathrm{MIN}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN. *Let* y *be a configuration and let us consider the following iterative process:*

- $\mathfrak{P}_{n+1} = \{y\}$.
- *If* $i \in V$, *then*

  $$\mathfrak{P}_i = \{x \in \{0,1\}^n : \exists \overline{y} \in \mathfrak{P}_{i+1} \ / \ x_j = \overline{y}_j \ if \ j \neq \pi_i \ and \ \mathrm{MIN}_{|\overline{I_D(\pi_i)}}(x) = y_{\pi_i}\},$$

  being $\mathrm{MIN}_{|\overline{I_D(\pi_i)}}$ *the restriction of* MIN *over* $\overline{I_D(\pi_i)}$.

Then, $\mathfrak{P}_1$ *is the set of all the predecessors of* y.

As in the case of SDS, these last procedures allow us to know all the predecessors of a state y in an SDDS and, consequently, the number of them. However, the calculus of the number of predecessors for a state of the entities depends on the connections among entities in the particular digraph. In this case, as traditionally done in other contexts, we have been able to get a bound for the number of predecessors of a configuration, which is given in the following theorems.

Theorem 5.19. *Let* $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the maxterm* MAX. *Then, the number of predecessors of a given state* y *is upper bounded by* $2^{\#(V_0 \cup P_0)^c}$. *Moreover, this bound is the best possible because it is reachable.*

Proof. From Theorem 5.15 and Corollary 5.25, the states of the entities belonging to $V_0 \cup P_0$ in a possible predecessor of y are fixed. Since the state values of the rest of entities are either 0 or 1, a first upper bound for the number of predecessors is $2^{\#(V_0 \cup P_0)^c}$.

This upper bound is the best possible because it is reached in the following example. Let us consider the $[D, \mathrm{MAX}, \pi] - \mathrm{SDDS}$ defined by

- $D = (V, A)$, with $V = \{1, \ldots, n\}$, $n \geq 2$, and $A = \{(2, i) : i \in V \setminus \{2\}\} \cup \{(1, 2)\}$,
- $\mathrm{MAX} = x_1' \vee x_2 \vee \cdots \vee x_n$,
- $\pi = 1|\ldots|n$.

In this context, if $y = (0, 1, \ldots, 1)$, then $V_0 = \{1\}$, $P_0 = \{2\}$, $Q_0 = \emptyset$ and $V_1 = \{2, \ldots, n\}$.

In any predecessor, x, it must be $x_1 = 1$ and $x_2 = 0$ but, in this case, all the other choices for the states of the rest of entities generate predecessors of y. $\square$

Dually, we have the following result.

Theorem 5.20. *Let* $[D, \mathrm{MIN}, \pi] - \mathrm{SDDS}$ *be a sequential directed dynamical system over a dependency digraph* $D = (V, A)$ *associated with the minterm* MIN*. Then, the number of predecessors of a given state* y *is upper bounded by* $2^{\#(V_1 \cup P_1)^c}$*. Moreover, this bound is the best possible because it is reachable.*

Remark 5.11*.* As can be checked, these upper bounds coincide with the ones obtained for SDS over undirected dependency graphs. □

Conclusions and future research directions

This dissertation supposes a complete description of the dynamics of homogeneous PDS and SDS on maxterm and minterm Boolean functions over undirected graphs.

The results obtained in PDS and SDS have turned out fundamental to identify different features of the dynamics in PDDS and SDDS. On the other hand, the methods and techniques developed in this work provide novel ingenious ideas to research this theory in the future.

In this sense, this work gives us hope for further progress in some future research directions which arise from it. The main future research directions correspond to extensions of the results to other kinds of models. Specifically, the study of models coming from the generalizations of the basic elements involved in the definitions (i.e., the set of states of the entities, the dependency graph, the local update functions and the updating schedule) of deterministic BN models can be considered as a natural continuation of this research work, as well as the analysis of non-deterministic models. Additionally, from this complete theoretical study, a direct application of these results to models coming from sciences, engineering or real-word situations seems to be feasible.

More specifically, we have:

1. Related to the set of states of the entities:

    - Generalization to a (general) Boolean algebra: In [22], a generalization of PDS over graphs is introduced by considering that the states of the entities can take values in an arbitrary Boolean algebra $(B, \curlyvee, \curlywedge, \prime, \mathcal{O}, \mathcal{I})$ with 2^p elements, $p \in \mathbb{N}$, $p \geq 1$. Note that the results in this dissertation correspond to the particular case where $B = \{0, 1\}$, that is, $p = 1$. In such a paper, a generalization of some results related to PDS is performed by using the Stone Representation Theorem for Boolean Algebras in a suitable way to decompose this general dynamical system into p PDS, where the entities take values in $\{0, 1\}$. Thus, a direct research line consists in generalizing the analysis shown in this dissertation for PDS and SDS to the case of general Boolean algebras just following this idea.

On the other hand, the states of all the variables usually belong to the same Boolean algebra, but there exits also the possibility of belonging to different Boolean algebras (with different number of elements) since, even so, the interaction between two variables with different states can be modeled through a Boolean function.

- Generalization to a (general) finite set of states: The original CA ([117]) were defined assuming that each cell can have, not only binary states activated and deactivated, but any state value belonging to a finite set. The study in this case is usually performed by considering the binary situation, although some papers ([88, 107]) explore the general situation, under some physical assumptions of uniformity. Nevertheless, observe that these models are not Boolean network models.
- Generalization to an infinite set of entities or states: Although, by computational reasons, the study of this kind of dynamical systems is usually performed considering a finite set of entities, already in the original conception of CA ([117]), a system can have an infinite number of them. The possibility of analyzing the dynamics in systems with an infinite number of entities, which can take an infinite number of states, can be consider as an interesting future line of research.
- Generalization to fuzzy sets: A direct generalization of our results could be obtained by using fuzzy sets and following the ideas in [124].

2. Related to dependency graph:

    - Generalization to directed graphs and to special kinds of them: In Chapter 5 of this thesis, and in some previous works (see [18] for example), the importance of analyzing the behavior of the system when the entities are not influenced in a symmetrical form is shown. In this Chapter 5, some of the results obtained in the previous chapters of this dissertation are generalized. As a future research direction, we propose a global generalization of all the other previous results, which will lead to a complete understanding of the dynamics in PDDS and SDDS.

      On the other hand, in this thesis, overall results have been obtained for PDS and SDS over general undirected dependency graphs. However, sometimes, it can be important to know the behavior of a system restricted to a particular kind of graph, looking for a specific performance. Some previous works have already dealt with this topic (see, for example, [21]), being line graphs, star graphs, arborescence structures, acyclic graphs and circle graphs some special graph classes which are worthwhile to be considered.

- Generalization to free-loop graphs: Structures different from the usual are becoming popular because of their appearance in disciplines such as genetic or biology. This is the case, for example, of the study of the dynamics in systems in which the update of the state of each entity does not depend on the state of the own entity (see [97]). This change in the fundamental principles implies a very promising research direction, with multiple proposals for use in the field of applied sciences.
- Generalization to mixed-loop graphs: Inspired by the previous proposal, the study can be extended to the most general case of systems where loops are not in all the nodes, but in some of them, which is the real situation in some biological systems (see [97]).

3. Related to the local functions:

- Generalization to general Boolean functions: As said in Chapter 2, the great importance of the particular class of Boolean functions formed by maxterms and minterms is that (see [24, 37, 98]) any Boolean function, except $F \equiv 1$ (resp. $F \equiv 0$) can be expressed in a canonical form as a conjunction (resp. disjunction) of maxterms (resp. minterms). Thus, it is natural to start with the study of the dynamics for this kind of Boolean functions, as done in this thesis. Once this analysis has been done, the following step is to study how the results can be used in order to achieve a complete study of the dynamics in parallel and sequential dynamical systems over graphs on general Boolean functions as evolution operators.
- Generalization to independent local (Boolean) functions: Along this dissertation, we have studied the dynamics of systems in which, the evolution of the states of the entities is performed by local functions acting over the adjacency or influence sets associated with them. All the results in this thesis are related to the case of homogeneous systems, i.e., those ones in which the evolution of the states of the entities is performed by local functions which are the restriction of a global maxterm or minterm Boolean function.

When generic structures are modeled, the situation could be more complex, being able to have independent local functions to update the state of the entities of the system (see [78]). Some works have already studied this situation (see [19, 25, 115]) and, as a line of continuity of the results shown in this dissertation, we propose the generalization of them in this context.

4. Related to the updating schedule:

    - Generalization to mixed schedule (MDS): Since a complete study of the dynamics has been performed in this thesis in the case of dynamical systems with parallel and sequential update of the state of the entities, a direct generalization consists in consider a mixture of both kind of evolution (see [62, 65]). This schedule is present in some computational processes when the updating task is not performed neither in parallel nor a sequential manner, but in a mixed way.

      In this case, the vertex set V is divided into some subsets, $V_1, \ldots, V_k \subseteq V$, being $V_i \cap V_j = \emptyset$ when $i \neq j$, such that the update of the states is performed in a parallel way inside every set V_i, but sequentially among different sets, being the entities belonging to V_1 the first evolving their states, after that the entities in V_2, and so on.

    - Generalization to updating order given by "words": In the sequential update case, in this thesis, we have used a permutation among the vertices to specify the sequence in which the states of the entities evolve. The update order is usually formalized this way, but also through finite words over the vertex set of the corresponding wiring graph (see [91, 92]). That is, in an iteration a node could be updated more than one time. This possibility can be considered as a future research line.

In addition, other interesting future research direction could be to use these results in the analysis of non-deterministic models. Some recent works (see [91, 92]) show the importance of the research in non-deterministic models, which allow more flexibility, adjust to the reality and significance of the results than in deterministic models. The first steps towards this objective, which would extend greatly the results, could be the study of the dynamics in systems in which:

- The adjacency relationships change after each iteration.
- The local functions change after each iteration.

- The updating order (in the case of sequential update) changes after each iteration.

Along this dissertation, we have mentioned the versatility of CA and KN in applications to several branches of sciences as biology (see [53, 78, 79, 113]), ecology (see [54, 72, 73]), psychology (see [2, 70]), mathematics (see [41, 43, 47, 57, 76]), physics (see [42, 44, 48]) and chemistry (see [83]), among others. Thus, other future research challenge could be to find applications to models coming from sciences and engineering (as, for instance, those in [27, 56], where some of our works are taken as a theoretical basis) or to study ad hoc real-word models related with the ones treated here that other researchers can propose us.

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