Scuola Normale Superiore
Classe di Lettere e Filosofia
Corso di Perfezionamento in Filosofia

The Principle of Analyticity of Logic
A Philosophical and Formal Perspective

Thesis Advisors:
Prof. Marcello D'Agostino
Prof. Massimo Mugnai

Candidate:
Costanza Larese

July 2019

Contents

Introduction

The subject of the present work is the principle of analyticity of logic. In order for the question 'Is logic analytic?' to make sense and before trying to find an answer to this problem, it is obviously necessary to specify two preliminary issues, namely, the meaning of the term 'analytic' and the meaning of the term 'logic'. The former issue is somehow justified and expected: after all, analyticity represents one of the philosophical concepts *par excellence* and, as such, it has been at the core of a lively debate throughout the history of the discipline. But, despite possible appearances to the contrary, the second issue is probably more decisive than the former in determining the answer to the initial question: both the contents and the philosophical conceptions of logic play a fundamental role in the study of the epistemological status of this discipline. We could even say that the clarification of the concepts of analyticity and of logic *constitutes in itself* the decision on the analyticity of logic.

This thesis studies the principle of analyticity of logic through two different, but related, methodologies, which individuate the two main parts of the work: the former offers a historical and philosophical reconstruction of the problem; the latter proposes two formal characterizations of the analytic-synthetic distinction. The reconstruction of the first part does not presume to be exhaustive and is restricted to the theories of the following philosophers: Kant, Bolzano, Frege and Hintikka. The material has been chosen according to the following criteria. First, this work aims at showing the 'historical' nature of the principle of analyticity of logic, which has a certain genealogy and a precise starting point. Although after the Vienna Circle this tenet has been taken for granted, there are many and significant conceptions that criticize it. Theories holding that logic is either not analytic or synthetic are the main characters of our reconstruction. This explains, for example, why we have dedicated great attention to Bolzano, while leaving little margin to the logical empiricist movement, despite the fact that analyticity is probably more fundamental for the latter's thought than for the former's philosophical construction. As a result of this choice, theories of meaning and their connection to analyticity are completely overlooked, since they belong to the logical empiricists' interpretation of the analytic-synthetic distinction. In other words, the principle of analyticity of logic and the philosophers arguing for it are taken as a critical target, but the true focus is on the varieties of reactions against them.

Second, each author that has been selected might be seen as a symbol for a peculiar answer to the question of the analyticity of logic: as we will see, each of them represent a definite approach to the problem. This observation clarifies why, for example, we have chosen only one author between Bolzano and Mill: although their philosophical presuppositions are completely different, the conclusion they reach on the status of logic goes surprisingly in the same direction. Last, but not least, we have chosen our protagonists among the most excellent figures of the modern and contemporary philosophical panorama for learning purposes. Despite the indisputable advantages, this choice has exposed our research to the risk of being too general and of privileging the horizontal rather than the vertical direction.

We hope we have avoided this danger through an adequate methodological equipment. First of all, we have offered hopefully accurate textual readings of the relevant passages of our primary sources. This is the case for the excerpts of Kant's *Critique of Pure Reason* A 6-7/B 10-11 and A 151-152/B 190-191; Bolzano's *Theory of Science* §148; Frege's *Foundations of Arithmetic* §3 and Hintikka's articles in *Logic, Language-Games and Information*. At the same time, we have tried to take into account a wide selection of secondary literature: we have considered both the most recent productions, such as, to make just one example, de Jong's useful article *The Analytic-Synthetic Distinction and the Classical Model of Science: Kant, Bolzano and Frege*, and the traditional interpretations, such as Coffa's *The Semantic Tradition from Kant to Carnap. To the Vienna Station* and Proust's *Questions of Form. Logic and the Analytic Proposition from Kant to Carnap.*

Consistently with the reflections expressed above, we have examined, for each author, first, his characterization of the analytic-synthetic distinction; second, his conception of logic; and, third, his standpoint on the status of logic with respect to analyticity. This should be evident through a quick look at the index of the work, where Hintikka constitutes a partial exception, due to his contemporary understanding of logic. As a result, the historical and philosophical reconstruction of the first part of this thesis might be read as the weaving of three micro-stories, in which the same protagonists are involved. Despite the continuous cross-references and the unity of purpose, the chapters of the philosophical reconstruction are ideally autonomous. This means that they both contribute to the overall picture and are self-contained. Each chapter, while trying to argue for a specific micro-thesis, is marked by different peculiarities and demands.

The crucial challenge of the first chapter was to provide an accurate reading of the texts in order to make Kant say what he really said and the target was to avoid anachronistic interpretations. To this end, the secondary literature has proved to be a precious instrument and a valid support. The picture was different for Bolzano's case. In the second chapter, space has been left to the exposition of key concepts necessary to understand the specificity of this author and attention has been paid in trying to find the right position of Bolzano in the historical and philosophical panorama. The third chapter intended to distinguish Frege's position from the standpoint defended by the logical empiricists' movement. For this reason, the question of the analyticity of logic is clearly separated from the problems posed by its alleged tautologicity. The aim of the fourth chapter was to offer a comprehensive interpretation of Hintikka's work on the Kantian themes. The objective was to keep the interpretative and philosophical side together with the technical and formal part of the Finnish philosopher's proposal and to understand the influence of Hintikka's reading of Kant on his logical achievements.

The second part of this thesis is marked by the use of formal logical tools and by the proposals of a new characterization of the analytic-synthetic distinction. The reason why we have chosen this method for advancing our contribution must be searched in the clarity and lucidity of the logical equipment. Two are the crucial premises for this part of the work: on the one hand, Hintikka's theory of distributive normal forms; on the other, D'Agostino and Floridi's formulation of Depth Bounded Boolean Logics.

As for the ones that constitute the first part of this thesis, also the last two chapters might be read independently of one another, although they represent essential elements for the understanding of this work. The fifth chapter enunciates the basic definitions of Depth Bounded First-Order Logics. This result is obtained through the extension of Depth Bounded Boolean Logics to the quantified case following Hintikka's suggestions. In this context, great attention has been paid to the discussion of the possibilities through which this aim could be achieved. While in chapter five we have exposed this family of logics through a proof-system, the sixth chapter employs semantic means. Depth Bounded Epistemic Logics are the result of a shift of focus to epistemic considerations and to the strictly related problem of logical omniscience. The aim of this chapter is to show that theoretical reasoning on the principle of analyticity of logic might have practical applications. This motivates the choice of the muddy children puzzle as a case study. Last, the appendix provides the formal proofs of the propositions and theorems expressed in chapter six.

The natural worry that might arise at this point is whether the two parts of this thesis are connected. We argue that they are strictly interwoven. The last stage of the reconstruction of the first part has underlined a key reason why the logic of quantification cannot be said to be analytic. This reason, according to Hintikka, must be found in the theory of computability. The formal proposals of the second part of this work start from this point and, following D'Agostino and Floridi, move to the field of theoretical computer science to cover both the propositional and the epistemic contexts. In other words, the historical and philosophical researches are not only interesting *in se*; rather, they have offered the indispensable bases on which our own answer to the question whether logic is analytic has been constructed through formal tools.

Summary

The subject of this thesis is the principle of analyticity of logic. **Part I** offers a historical and philosophical reconstruction of this principle and takes into account the conceptions of four authors: Kant, Bolzano, Frege and Hintikka. **Part II** proposes two logical systems that provide formal characterizations of the analytic-synthetic distinction: Depth Bounded First-Order Logics and Depth Bounded Epistemic Logics.

Chapter 1 is devoted to Kant and it is divided into three main sections. **Section 1.1** explores Kant's analytic-synthetic distinction. We provide an overview of how the relationship between the four criteria of analyticity of the *Critique* is explained in the literature (**1.1.1**). We show that the containment criterion enjoys a definitional priority over the other versions of the distinctions; we examine the restrictions on the applicability of this criterion and we deal with the charges of narrowness, psychologism and obscurity moved against it (**1.1.2**). We then consider the remaining criteria of analyticity: clarification, identity and contradiction, with special attention to the role played by the latter in Kant's theory (**1.1.3**). Then, we focus on Kant's notions of synthetic judgments and of intuition (**1.1.4**). **Section 1.2** studies Kant's conception of pure general logic. We provide an overview of both Kant's logical notions and the topics that constituted the discipline of logic at the time, clarifying which truths were considered as 'logical' truths in Kant's view (**1.2.1**). We investigate Kant's defining features of logic: pureness, generality and formality (**1.2.2**). **Section 1.3** analyzes the application and the applicability of the analytic-synthetic distinction to the discipline of logic. First, we examine Kant's perspective on the role of logic as an instrument for both defining and applying the analytic-synthetic distinction (**1.3.1**). Second, we analyze the epistemological status of logic in order to provide an answer to the question of whether logical truths are, following Kant's definitions, analytic or synthetic a priori (or neither of them) (**1.3.2**).

Chapter 2 is dedicated to Bolzano and is organized into three main parts. **Section 2.1** deals with Bolzano's analytic-synthetic distinction. We introduce some preliminary concepts of Bolzano's logic, such as the method of substitution and the notions of validity and derivability (**2.1.1**). We examine Bolzano's dis-

tinction and we provide a commentary of §148 of the *Wissenschaftslehre*. First, we concentrate on the relation between, on the one hand, analyticity and, on the other hand, truth-values, syntheticity and conceptuality (**2.1.2**). Second, we discuss Bolzano's definition of logical analyticity and its connection with logical truths (**2.1.3**). Third, we make some observations on the link between analyticity and language (**2.1.4**). We provide an interpretation of Bolzano's criticism about the Kantian conception of analysis and of the analytic-synthetic distinction as understood in the *Critique* (**2.1.5**). **Section 2.2** focuses on Bolzano's logic, which is conceived as a kind of deductive science. We show the role that the grounding relation plays in axiomatic structures and point out the differences between Bolzano's notion of synthetic *a priori* with respect to Kant's (**2.2.1**). Then, we concentrate on distinguishing two ways in which the term 'logic' is understood in Bolzano's text and we analyze his thesis that logic is synthetic (**2.2.2**). **Section 2.3** offers a global evaluation of Bolzano's principle of the syntheticity of logic. We highlight an apparent contradiction in Bolzano's system and we concentrate on the pragmatics of analyticity (**2.3.1**). We reason about the place of Bolzano's thesis in the history of analyticity and review the main interpretative trend on this topic (**2.3.2**).

Chapter 3 focuses on Frege and logical positivism. **Section 3.1** considers the principle of analyticity of logic. We introduce Frege's revolution in logic and his conception of the discipline within his logicist project (**3.1.1**). We examine Frege's analytic-synthetic distinction through an analysis of the §3 of the *Foundations of Arithmetic* and of its connection to Kant's conception (**3.1.2**). We focus on Frege's notion of analysis based on the function-argument distinction and we underline its differences with respect to the traditional theory of concepts (**3.1.3**). We investigate whether logical truths are, according to Frege, analytic and provide an overview of the different positions in the literature (**3.1.4**). We consider the principle of analyticity of logic as expressed by the manifesto of the Vienna Circle and make some observations on Quine and Carnap's approach to the issue (**3.1.5**). **Section 3.2** focuses on the related idea that logic is tautologous. We provide a historical overview on the paradox of analysis and we underline that both Wittgenstein and the Vienna Circle accept that logic is tautologous (**3.2.1**). We show Frege's position on the fruitfulness of analysis explicated in the *Grundlagen* (**3.2.2**) and its radical change after the introduction of the *Sinn-Bedeutung* distinction (**3.2.3**). We then consider the psychologistic solution to the paradox of analysis defended by Hahn, Hempel and Ayer (**3.2.4**) and Wittgenstein's employment of the myth of the perfect language to explain that logical deduction is uninformative (**3.2.5**).

Chapter 4 analyses Hintikka's work and it is divided into three main sections. **Section 4.1** focuses on Hintikka's peculiar interpretation of Kant's theory of the mathematical method. We first examine Hintikka's thesis that mathematical intuitions are, according to Kant, singular representations (**4.1.1**). Then, we deal with Hintikka's idea that the mathematical method, characterized by the use of geometrical constructions, plays a foundational role in the Kantian distinction between analytic and synthetic judgments (**4.1.2**). We investigate Hintikka's standpoint that Kant is an heir to the constructional sense of analysis, which is distinguished from the directional sense of analysis (**4.1.3**). **Section 4.2** is dedicated to the status of logical truths. We examine the conceptual kernel of Hintikka's analytic-synthetic distinction (**4.2.1**) and we investigate the actual influence of Kant's conception of syntheticity and Hintikka's interpretation of it over his theory (**4.2.2**). **Section 4.3** concerns Hintikka's arguments against logical positivism elaborated through formal instruments. We provide a simple reconstruction of Hintikka's theory of distributive normal forms for first-order logic and of the analytic-synthetic distinction defined in these terms (**4.3.1**). We present Hintikka's theory of probability and semantic information and we deal in particular with the notion of surface information (**4.3.2**). We conclude by offering an evaluation of Hintikka's overall work on the epistemological status of logic (**4.3.3**).

Chapter 5 presents Depth Bounded First-Order Logics and is organized into three main parts. **Section 5.1** is dedicated to the exposition of Depth Bounded Boolean Logics put forward by D'Agostino and Floridi. We analyze the consequences of the probable intractability of propositional logic (**5.1.1**). We introduce the informational semantics through the truth tables and the negative constraints on admissible partial evaluation (**5.1.2**). We discuss the notion of virtual information and the logics in which its bounded use is allowed (**5.1.3**). We exhibit the proof-theoretical characterization of these logics based on the so-called *intelim*-rules (**5.1.4**). We examine the relations between, on the one hand, Depth Bounded Boolean Logics and, on the other hand, Hintikka's work and Kant's conceptions (**5.1.5**). **Section 5.2** elaborates the idea of extending Hintikka's approach to the propositional case. We outline the project of formulating Depth Bounded First-Order Logics (**5.2.1**). We present two attempts made to obtain the desired result, namely, the idea of reducing the quantificational case to the propositional one (**5.2.2**) and of using Skolem functions to provide individuals with a structure (**5.2.3**). We propose our notion of quantificational depth and discuss four senses in which individuals might be said to be reciprocally related (**5.2.4**). **Section 5.3** introduces Depth Bounded First-Order Logics. We first define derivability relations in which no use of nested virtual information is allowed, but it is possible to employ the introduction of a bounded number of individuals (**5.3.1**). Then, we define derivability relations in which the bounded use of both virtual information and new individuals is allowed (**5.3.2**). We give a formal definition of quantificational depth of an inference (**5.3.3**). We conclude offering some examples of derivations (**5.3.4**).

Chapter 6 introduces Depth Bounded Epistemic Logics and is divided into two main parts. **Section 6.1** is devoted to the discussion of the problem of logical omniscience and of classical epistemic logics. We first provide the basic definitions of classical epistemic logics and we show the formal characterization of the principle of logical omniscience that they satisfy (**6.1.1**). We introduce, as a case study, the muddy children puzzle and its analysis through classical means (**6.1.2**). We reason about ideal and realistic agents, as well as on the possible way of representing their reasoning (**6.1.3**). We present the philosophical motivations that justify the notion of degrees of logical omniscience (**6.1.4**). **Section 6.2** is dedicated to the presentation of Depth Bounded Epistemic Logics. We first introduce the structure of this family of logics (**6.2.1**). Then, we enunciate the definitions of language and model, which are common to every logic of every hierarchy, paying specific attention to the comparison with the analogous definitions for classical epistemic logics and Depth Bounded Boolean Logics (**6.2.2**-**6.2.5**). We provide three notions of validity in a model, each of which corresponds to a specific hierarchy of logics (**6.2.7**) and we give the definitions of validity, each of which characterizes a specific logic (**6.2.8**). We discuss the relationships between the logics of this family (**6.2.9**). We give a formalization of the muddy children puzzle in Depth Bounded Epistemic Logics in order to clarify the definitions (**6.2.6**-**6.2.10**) and we conclude by pointing out the way in which the classical solution to the puzzle varies with our family of logics (**6.2.11**).

Part I

Historical and philosophical reconstruction

Chapter 1

Kant and the foundations of the analytic-synthetic distinction

1.1 The Kantian analytic-synthetic distinction

The analytic-synthetic distinction is fundamental for Kant's critical philosophy, whose main purpose is to show the possibility of synthetic *a priori* knowledge[1]. Although ancestors of the distinction can be found in several authors[2], the novelty

[1]The importance of the distinction is certainly not restricted within the limits of Kant's philosophy. As Hanna (2001) points out, the notion of analyticity, which finds in Kant its founder, is not only crucial for the development of the so-called analytic philosophy, but it also represents the *trait d'union* of that multifarious tradition. In Hanna (2001, p. 121)'s words: "The history of analytic philosophy from Frege to Quine is the history of the rise and fall of the concept of analyticity, whose origins and parameters both lie in Kant's first Critique".

[2]In a passage of the *Prolegomena* (Prol., p. 22), Kant recognizes that he has found a 'hint' of the analytic-synthetic distinction only in Locke's *An Essay Concerning Human Understanding.* In the paragraph referred to by Kant (Locke, 1975, Book IV, Chapter 3, parr. 9 and ff.), Locke distinguishes the connections of ideas via identity and via co-existence. However, as Anderson (2015, p. 35 and ff.) pinpoints, this 'hint' is rather opaque if compared to another passage of the same work (Locke, 1975, Book IV, Chapter 8, parr. 1 and ff.) that Kant seems to have neglected, in which Locke identifies a class of truths, called 'trifling propositions', which do not extend knowledge and are based on containment and contradiction.

Although his work is not mentioned by Kant in this respect, Leibniz admits a non-epistemic distinction between 'truths of reason', whose opposite is impossible, and 'truths of facts', whose opposite is possible. Next to these two kinds of truths, Leibniz recognizes also an intermediate sort truths, which are called 'mixed truths'. Moreover, he holds that any proposition is true if and only if the concept of the predicate is somehow contained in that of the subject: in other words, as we will see in Section 1.1.2, a proposition is true if and only if it is analytic *sensu* Kant. The distinction of kinds of truths and the definition of truth as containment lead Leibniz to face the problem of reconciling analyticity of truth on the one side and contingency on the other: the proposed solutions, which are not free of difficulties, consist in the analogy with the infinitesimal calculus and the distinction between absolute and hypothetical necessity.

of mature Kant's proposal is strongly claimed by the author[3] and can be mostly appreciated along two fronts. First, Kant employs the terms 'analytic' and 'synthetic' not only to qualify different methods of proof, but also, and crucially, to designate two different kinds of *judgments*. The latter use of these terms is only indirectly related to the former, which has an older tradition in mathematics and especially in geometry. Unless otherwise specified, with the expression 'analytic-synthetic distinction' we are going to refer to the latter and new employment of the terms. Second, the use Kant makes of this distinction far outstrips that of his predecessors. As Anderson (2015) claims, Kant's distinction is not merely verbal, but it rather hides a substantial thesis directed against the German rationalist metaphysics, especially against Leibniz and the Wolffian tradition: truth is not exhausted by the containment relation; on the contrary, all important cognition[4] cannot be explained in terms of analytic (in the Kantian sense) judgments and falls on the synthetic side. This thesis, which has probably sounded revolutionary to contemporary readers, together with the Kantian interest in synthetic *a priori* knowledge, gives pride of place in Kant's philosophy to the notion of syntheticity. This is one of the reasons for which we are going to conclude this part with a brief recapitulation of Kant's notion of synthetic knowledge[5] (Section 1.1.4), even if our main concern is Kant's theory of analyticity (Sections 1.1.1-1.1.3), the definition of which is essential to deal with the issue of this Chapter.

Last, Hume (1975, Section IV, Part I) in his *An Enquiry concerning Human Understanding* distinguishes between 'relations of ideas' and 'matters of facts'. While the former can be known through intuition or demonstration, the latter can be justified only through experience; while the negation of the former implies a contradiction and cannot be clearly conceived, the denial of the latter does not lead to a contradiction and can be easily conceived. This distinction, known as 'Hume's fork', receives no further development in that work, which is mainly focused on reasoning about matters of fact.

[3]For example, in the *Introduction* to the second edition of the first *Critique*, Kant says: "That metaphysics has until now remained in such a vacillating state of uncertainty and contradictions is to be ascribed solely to the cause that no one has previously thought of this problem [i.e. the possibility of synthetic *a priori* judgments] and perhaps even of the distinction between **analytic** and **synthetic** judgments" (CPR, B 19).

[4]This is clearly stated in the *Introduction* to the first *Critique*, where Kant says that: "**Judgments of experience, as such, are all synthetic**" (CPR A 7-8/B 11-12); "**Mathematical judgments are all synthetic**" (CPR, B 14); "**Natural science [. . .] contain within itself synthetic a priori judgments as principles**" (CPR, B 17-18) and "**In metaphysics [. . .] synthetic a priori cognitions are supposed to be contained**" (CPR, B 18).

[5]Hanna (2001, p. 181 and ff.) notices a common trend in the discussions of the analytic-synthetic distinction, which he calls the 'privileging of the analytic'. While the analytic tradition usually defines the synthetic as what is not analytic, this is not the case for Kant. We think that a reason for this peculiarity of Kant's thought, which goes beyond Kant's interest in the synthetic *a priori* and beyond his thesis against rationalist metaphysicians, is that his distinction is not exhaustive. This will be clarified in the following.

1.1.1 Criteria for analyticity

The distinction between analytic and synthetic judgments in the *Critique of Pure Reason* is not unambiguous: instead of providing a unique definition, Kant seems to propose no less than four criteria of analyticity. Two are the main *loci* of the first *Critique* in which Kant extensively deals with this issue. The former presents the analytic-synthetic distinction and can be found in the fourth paragraph of the *Introduction*; the latter, situated in the second Chapter of the *Analytic of Principles*, concerns the "supreme principle of all analytic judgments". It is worth quoting these passages at length:

> In all judgments in which the relation of a subject to the predicate is thought (if I consider only affirmative judgments, since the application to negative ones is easy) this relation is possible in two different ways. Either the predicate B belongs to the subject A as something that is (covertly) contained in this concept A; or B lies entirely outside the concept A, though to be sure it stands in connection with it. In the first case I call the judgment **analytic**, in the second **synthetic**. Analytic judgments (affirmative ones) are thus those in which the connection of the predicate is thought through identity, but those in which this connection is thought without identity are to be called synthetic judgments. One could also call the former **judgments of clarification**, and the latter **judgments of amplification**, since through the predicate the former do not add anything to the concept of the subject, but only break it up by means of analysis into its component concepts, which were already thought in it (though confusedly); while the latter, on the contrary, add to the concept of the subject a predicate that was not thought in it at all, and could not have been extracted from it through any analysis (CPR, A 6-7/B 10-11).
>
> [...] **if the judgment is analytic**, whether it be negative or affirmative, its truth must always be able to be cognized sufficiently in accordance with the principle of contradiction. [...] we must also allow the **principle of contradiction** to count as the universal and completely sufficient **principle of all analytic cognition** (CPR, A 151-152/B 190-191).

In the excerpt taken from the *Introduction*, we find three criteria of analyticity, which can be roughly summarized as follows.

1. In an analytic judgment the concept of the predicate is contained in the concept of the subject (containment criterion).

2. In an analytic judgment the connection between the concept of the predicate and that of the subject is thought through the principle of identity (identity criterion).

3. In an analytic judgment the concept of the predicate does not add anything to that of the subject: the already existing components of the subject concept are clarified through the processes of analysis (clarification criterion).

Each of these definitions is contrasted with a correspondingly standard of syntheticity, which is not our concern here.

The quotation from the *Analytic* seems to add another definition to our list: an analytic judgment is one that can be known through the only means of the principle of non contradiction, which is said to be the supreme principle of analytic judgments (contradiction criterion).

The many-sided presentation of the notion of analytic judgments raises serious problems about the relationship between the criteria put forward by Kant. One key observation is that these definitions are not all equivalent: although Kant probably took them to coincide in extension, they (or at least some of them) individuate different classes of judgments. To make an example that will be further examined, the judgment 'man is man' seems to be analytic according to the identity criterion: however, it turns out to be non-analytic following the clarification criterion because, although the concept of the predicate does not add anything to that of the subject, still the clarification of the components of the subject concept does not need to take place through analysis. Even the relationship between the identity and contradiction criteria is still debated[6]. This remark about the differences in extension of the four definitions, together with the doubts about the number of non-redundant criteria[7], leads to some related issues. Critics have discussed on whether the set of Kant's formulations is after all consistent, possibly because each

[6]Hanna (2001, Chapter 3) proposes an interpretation of the contradiction criterion that makes its extension wider than that of the identity formulation (and that of containment as well). This means that, according to Hanna, there is a class of propositions that are analytic according to the contradiction criterion and non-analytic following the identity one. This class of propositions includes, in Hanna's view, all the logical truths of monadic predicate logic. Against the idea of a difference between the two criteria just mentioned, Proops (2005) talks of the 'identity-and-contradiction criterion'. Discussing about the motivations of the pre-eminence of the contradiction criterion over the identity one in Kant's reasoning about the principle of analytic judgments, Proops recognizes the contradiction formulation has a wider applicability than the identity one. Less problematic seems to be Anderson's (2015, p. 13) identification of the two criteria.

[7]There is no unanimous consensus about which criteria can be taken to be, strictly speaking, *definitions* of analyticity and which formulations are after all redundant. For example, Hanna (2001) does not include the clarification criterion in his list; Proops (2005) holds that the contradiction formulation is not a characterization of analyticity at all and the same perspective is taken by de Jong (1995); Anderson (2015) identifies the contradiction and the identity criteria.

definition merely specifies different points of view on the same matter[8], or rather Kant's criteria cannot be reconciled or fit together and stand as evidence of the historical development of the notion of analyticity in Kant's thought. The latter position usually invites interpreters to identify, among the criteria, one definition as fundamental, either because it would represent Kant's mature position or because of its conceptual centrality. Thus, the remaining formulations are read as alternative specifications of the central one or as elements that endured despite an alleged development of Kant's position on that matter[9].

[8] According to Hanna's (2001, p. 124) interpretation, "while Kant does indeed employ several distinct formulations of his doctrine of analyticity, his use of these formulations is not after all incoherent, because each merely brings out a different aspect of a single, internally consistent, defensible Kantian theory". Roughly put, Hanna arranges Kant's formulations in this order: containment, identity and contradiction; and holds that each criterion both includes and extends the definition that precedes it in the list. In other words, each formulation solves some problems that the previous in that hierarchy could not solve. As a result, Hanna maintains that the most comprehensive definition of analyticity is that based on contradiction (or rather on a peculiar interpretation of it). Hanna (2001)'s interpretation is quite peculiar in the review of the secondary literature: most of the other readings we have considered tend to reconcile these different formulations of analyticity through an historical inquiry. Most notably, Anderson (2015) proposes a particularly accurate analysis of the emergence of Kant's analytic-synthetic distinction in the second part of his book.

[9] Anderson (2015, p. 16) states that "it is hard to avoid the conclusion that concept containment [...] served Kant himself as the fundamental idea behind analyticity". This point is crucial for Anderson's main thesis, that we have already mentioned, according to which Kant's analytic-synthetic distinction underwrites a strong criticism against the expressive limitation of the rationalist pre-Kantian metaphysics. The centrality of the containment criterion is also maintained by de Jong (1995) who, as we will see, defends the clarity of this characterization. In the following, we are going to opt for this position too.

After having shown that the contradiction criterion does not count, properly speaking, as a definition of analyticity, Proops (2005) argues instead that "the identity-and-contradiction characterization must be recognized as the most central and fundamental conception of analyticity in the first *Critique*". This thesis is based on the fact that identity can account for a class of judgments wider than those delimited by the other criteria. However, Proops recognizes that from the *Prolegomena* onward Kant tends to put more weight on the clarification criterion.

Allison (2004, p. 89 and ff.) sides instead with the clarification criterion: "Although it hardly resolves all of these difficulties, the second version [i.e. that based on the opposition between clarification and ampliativeness] is superior to the first [i.e. that founded on containment] because in it the notion of a synthetic judgment, the real focus of Kant's concern, wears the trousers". According to Allison, Kant accords his preference to the third criterion, especially after Eberhard's critics, because it underlines the relationship between synthetic judgments and real objects.

Last, the contradiction criterion has often been considered very attractive on the ground that it is the most inclusive and is not restricted to a certain subclass of judgments (such as the categorical ones). This is for example the Kneale and Kneale's (1962, p. 357) interpretation: "Such characterization of analytic judgments [i.e. that based on the principle of contradiction] is undoubtedly more suitable for Kant's purpose than that with which he began [i.e. that grounded on containment], since it is with this definition to divide all true judgments between the headings

In this Section, we analyse Kant's theory of analyticity examining the four criteria one by one. The main thesis that will be supported is that the containment criterion is the fundamental definition of analyticity in Kant's first *Critique*.

1.1.2 Containment

The containment criterion seems to enjoy a definitional priority over the other versions of the analytic-synthetic distinction (at least in Kant's first *Critique*). As Anderson (2015, p. 16) points out, in the passage quoted above (CPR, A 6-7/B 10-11), the containment criterion benefits of a privileged position: it is announced first and the other two formulations of the distinction seem to be inferred from it. In particular, Kant's wording seems to highlight through the particle 'thus' that the role assigned to the identity criterion is that of a consequence of the containment one: "Analytic judgments [...] are *thus* those in which the connection is thought through identity"; similarly, the clarification criterion is introduced as an alternative version that can be simply derived from the first one: "*One could also call* the former [i.e. analytic judgments defined through containment] **judgments of clarification**" (emphasis added). Last, it's important to notice that the contradiction criterion is presented only much later in Kant's *Critique*. This textual evidence testifies of a priority of the containment criterion as the exposition requires: this is but a hint of its *conceptual* centrality, which however can be argued only through an analysis of the criterion itself and of the other formulations.

As we have seen, the containment definition asserts that the relation of the subject A to the predicate B in an analytic judgment is such that "the predicate B belongs to the subject A as something that is (covertly) contained in this concept A". It's crucial to observe that this criterion cannot be adopted on any proposition whatsoever: Kant imposes several important restrictions on its applicability.

First, although Kant is not explicit on this point, the analytic-synthetic distinction in terms of the containment criterion applies only to *true* judgments. This is because a sufficient ground for the truth of a judgment is its being analytic according to the containment criterion. As Kant explains[10], a judgment like 'All bodies are extended' is analytic because an analysis of the concept corresponding to 'body' reveals that the predicate 'extended' is contained in it. But this condition also guarantees the truth of that judgment, for it is impossible that the concept 'extended' be contained in the concept 'body' and *at the same time* that the judgment 'All bodies are extended' be false[11]. The idea of analytic falsehood,

---

analytic and synthetic".

[10]In CPR A 7/B 11.

[11]Proops (2005) argues for this thesis dealing also with synthetic *a priori* judgments. Kant, saying that "the predicate B *belongs* to the subject A *as something that is (covertly) contained*

although hints of it can be found in some Kantian reflexions[12], is thus completely absent from Kant's *Critique*.

The second restriction imposed on the applicability of the distinction defined in terms of containment is instead more evident, for Kant explicitly states: "I consider only affirmative judgments, since the application to negative ones is easy". Proops (2005, p. 591) maintains that "the idea of containment is merely one application of a more general idea that is equally applicable to negative judgments" and individuates that idea in the "thought that analytic truth can be characterized in terms of relations of containment *and exclusion*". Kant's introduction of exclusion as a complementary element of the notion of containment is clearly justified by the following passage of the *Analytic of Principles*:

> In the analytic judgment I remain with the given concept in order to discern something about it. If it is an affirmative judgment, I only ascribe to this concept that which is already thought in it; if it is a negative judgment, I only exclude the opposite of this concept from it (CPR, A 154/B 193).

As Proops explains, the opposite of a concept is the negation of one of its constituent marks; so, a negative judgment such as 'Every body is not simple' is analytic because it excludes from the concept 'body' the opposite of what is thought in it, that is the predicate 'simple', which is the opposite of one of the constituent marks of body, such as 'extended'. Proops' convincing proposal underlines that the containment criterion *per se* cannot be applied to negative judgments: only an extension of its definition could avoid this second restriction.

The third restriction, which is more problematic in many respects, asserts that the analytic-synthetic distinction formulated via containment can only be applied to *categorical* judgments, i.e. judgments of the subject-predicate form. This is the meaning of the very beginning of the passage quoted above (CPR, A 6/B 10), where Kant states: "*In all judgments in which the relation of a subject to the predicate is*

in this concept A" (emphasis added), suggests that a concept B may belong to a concept A *without* being contained in it. The problem of the possibility of synthetic *a priori* judgments is accordingly that of explaining the possibility of the *truth* of a judgment in which the predicate concept is not contained in that of the subject. From this, Proops concludes that to say that one concept 'belongs' to another means that they are related in a *true* universal judgment and that "the idea behind the containment criterion must be that an affirmative analytic *truth* is a judgment whose truth is owed to the obtaining of a relation of containment between the subject and predicate concepts, while an affirmative synthetic truth is an affirmative judgment whose truth is not so explained".

[12] Proops (2005) argues that the idea of analytic falsehood is present in Kant's essay *Attempt to Introduce the Concept of Negative Magnitude into Philosophy* and indicates that in the 6327 Reflexion Kant describes the judgment "a resting body is moved" as "analytic and false".

thought [...] this relation is possible in two different ways [...]" (emphasis added). This limitation is one of the major reasons for which Kant's theory of analyticity has been criticized: its formulation soon appeared too narrow. For example Frege (FA, par. 88, pp. 99-100), in his *The Foundations of Arithmetic*, finds in this restriction one of the reasons for what he took to be Kant's misunderstanding of the status of arithmetical judgments and concludes that "Kant obviously - as a result, no doubt, of defining them too narrowly - underestimated the value of analytic judgments". Moreover, Frege adds to his interpretation of Kant's position that 'What he is thinking of is the universal affirmative judgment; there, we can speak of a subject concept and ask - as his definition requires - whether the predicate concept is contained in it or not" and points at some cases in which there can be no question of a subject concept in Kant's sense.

This kind of criticisms led some scholars to deny the very fact that Kant intended the analytic-synthetic distinction to be applied only to categorical judgments[13]. This denial is carried out according to two main strategies. On the one hand, it has been maintained that, although it is surely restricted to categorical judgments, the containment criterion is but a part of Kant's theory of analyticity, which is extended by more comprehensive criteria. On this account, the limitation would turn out to be an accidental feature of the containment formulation[14]. On the other hand, some critics simply balance the weight of the textual evidence given by the passage quoted above (CPR, A 6/B 10) with other Kantian *loci*, which are usually interpreted in a way apt to justify Kant's supposed intention of applying his distinction to any kind of judgments[15]. The strongest texts that are usually cited against the third restriction are the following:

> [...] judgments may have any origin whatsoever, or be constituted in whatever manner according to their logical form, and yet there is nonetheless a distinction between them according to their content, by dint of which they are either merely *explicative* and add nothing to the

[13]For example, Kneale and Kneale (1962, p. 357): "As it stands, Kant's explanation refers only to judgments of the subject-predicate form; but he can scarcely have intended his distinction to be limited to them in its application, for he goes on to talk as though he were dealing with the whole field of possible knowledge".

[14]Hanna (2001, p. 145) holds that Kant's theory of analyticity via the *identity* formulation avoids the limitation to categorical judgments. In his words: "despite misleading appearances, Kant's focus on categorical propositions in his theory of analyticity is only an expository convenience, but not a necessary or substantive feature of the theory". He then concludes that "Kant grants a certain primacy to the subject/predicate structure in his theory of judgement by treating it as generatively basic. But his theory of analyticity, construed in terms of his identity formulation, does not entail that every analytic truth be categorical in its gross logical or grammatical form".

[15]Anderson (2015, p. 20) follows this path.

> content of the cognition, or *ampliative* and augment the given cognition; the first may be called *analytic* judgments, the second *synthetic* (Prol., p. 16).
>
> [...] every existential proposition is synthetic (CPR, A 598/B 626).

The first excerpt, taken from the second Section of the preamble of the *Prolegomena*, is read by most scholars as saying that the analytic-synthetic distinction is not grounded on the logical form of judgments, but rather on their intension[16]. In other words, Kant seems here to be asserting that the distinction can be applied not only to categorical, but also to other kinds of judgments. The second statement, which can be found in the *Transcendental Dialectic*, seems to suggest that there are some judgments that are at the same time not categorical, in so far as they have an existential import, and synthetic. Proops (2005) proposes alternative and convincing readings of the two quotations above. The phrase 'logical form' in the text of the *Prolegomena* is employed, according to Proops, not in its contemporary sense, but as meaning 'degree of distinctness'. This use is well attested through Kant's works, including also the first *Critique*, and fits with the context in which it is inserted.

As far as the second quotation is concerned, Proops argues that Kant does not take existential judgments to lack subject-predicate form. An immediate objection against this interpretative thesis put forward by Proops might be the observation that Kant criticizes the ontological argument for the existence of God exactly on the basis of the rejection of the claim that existence is a property of an object. Nevertheless, Proops points out that what Kant actually says is that existence is not "a real predicate, i.e., a concept of something that could add to the concept of a thing"[17]; but he does *not* say that existence is not a *logical* predicate. On the contrary, continues Proops, Kant maintains that "anything one likes can serve as a logical predicate"[18]. Regarding the logical form of existential judgments, Kant maintains, in *The Only Possible Argument in Support of a Demonstration of the Existence of God*, that when in common speech we appear to be predicating existence of a thing we are really predicating it of the concept of that thing: following this principle, every proposition of the form 'Existence belongs to x' is more perspicuously expressed by 'The concept x is a concept that represents an existent thing' or 'The concept x is instantiated'. But the latter two expressions are of the subject-predicate form: thus, concludes Proops, existential sentences are not counterexamples to the third restriction.

[16]This is, for example, Allison's (1973, p. 56) reading.
[17]CPR, A 599/B 627.
[18]CPR, A 598/B 626.

As far as textual evidence is concerned, we have seen on the one hand that the strongest excerpts taken in support of the thesis that Kant intended to apply the analytic-synthetic distinction via containment to judgments other than categorical are not overwhelming; and on the other hand that Kant explicitly confines his discussion to subject-predicate judgments (CPR, A 6/B 10). But, we think, the decisive motivation for maintaining Kant's acceptance of the third restriction on the containment criterion is of a conceptual nature: the crucial motivation is that *it is not possible* to apply the containment criterion to judgments that are not of the subject-predicate form. As de Jong (1995, p. 617) observes, "only in the case of a categorical judgment is the relation of thought in judgment that of subject to predicate". In disjunctive and hypothetical judgments, which are examples of non-categorical judgments, the relation of thought is that of judgment to judgment(s), not that of two concepts, those of the subject and of the predicate[19].

Now, if Kant had intended to apply his distinction via containment to all kinds of judgments, he could have worked out a strategy to reduce a judgment whatsoever to a categorical one. This move would not have been absurd: Leibniz, for example in his *Elementa Calculi*, holds that 'A is B' is the canonical form of any judgment and thus tries to formulate a systematic algorithm to turn every proposition into its canonical form[20]. But this, we think, is not Kant's position: hypothetical and disjunctive propositions are both enumerated under the heading 'relation' in the table of judgments and Kant insists that all the twelve forms of judgments must be recognized as primitive. As a result, hypothetical and disjunctive judgments, being not reducible to categorical judgments, cannot be said to be analytic on the basis of the containment criterion.

To sum up, the analysis carried out on the three kinds of restrictions leads us to maintain that Kant intended to apply the analytic-synthetic distinction defined through the containment criterion only to *true affirmative categorical judgments*. By extending the criterion with the notion of exclusion, analyticity can be attributed to negative judgments, but this is the only extension allowed. In other words, Kant's distinction via containment is not exhaustive and, as a consequence, there are some judgments which are neither analytic nor synthetic[21]. Kant's lack

[19]See CPR, A 73-74/B 98-100.

[20]Leibniz shows, following the scholastics, that, first, any proposition can be turned into subject-copula-predicate form: for instance, 'Socrates runs' and 'It rains' become, respectively, 'Socrates is running' and 'Rain is falling'. And then he puts forward, for example in the essay *De Abstracto et Concreto*, his new proposal of translating any proposition of the form 'if A is B, then C is D' into one of the form 'the being B of A is the being D of C'. The two components of this sentence are the so-called 'logical abstracts', the former of which, i.e. 'the being B of A' (or 'the B-ness of A'), functions as a subject and the latter of which, i.e. 'the being D of C' (or 'the D-ness of C'), functions as a predicate.

[21]This conclusion, which is highly contrasted by the scholars, has been recently held by de Jong (1995) and Proops (2005). The latter underlines Kant's awareness of the consequences of

of interest in the exhaustiveness of the analytic-synthetic distinction can be justified starting from the employments he designed for it. First, as Proops (2005, p. 610) underlines, Kant's "chief concern is to argue for the *syntheticity* of certain judgments", such as the claims of mathematics, natural sciences and metaphysics, "that in his days would have been assumed to have subject-predicate form. The need for a classification that applies to all judgments is very much a late nineteenth and early twentieth-century concern". Second, if Anderson (2015)'s thesis that Kant's aim is to reject the metaphysics based on the Leibnizian predicate-in-subject theory is correct, and given the strict link between containment and subject-predicate judgments, then it is sufficient for Kant to focus on categorical proposition, to show that their truth cannot be justified on the basis of the containment criterion alone.

The charge of narrowness against Kant's theory of analyticity due to its confinement to categorical judgments is not the only criticism moved against it: the containment criterion has been accused of both psychologism and obscurity. An analysis of these two criticisms will lead us to gain important elements of the containment characterization.

According to the first charge, which has been advanced soon after the publication of the *Critique*, the analyticity or the syntheticity of a certain judgment depends on the subject that considers that judgment. In particular, it relies on the features that the individual involved associates both to the subject and to the predicate concepts of that judgment, on the basis of which it can be deduced whether the relation of containment holds or not. This charge can be easily dismissed by recalling, as Hanna (2001, p. 155 and ff.) does, the important Kantian distinction between subjective (or phenomenological) and objective (or semantic) elements of representations. Beside the subjective act of attaching to a given concept some characteristics, which of course varies from individual to individual, there is also the objective element of the representation of that concept, which instead determines the conceptual marks that are objectively contained in it: the analyticity or syntheticity of a judgment depends on this latter aspect.

The second charge has a long history: it has been put forward by Kant's contemporaries and frequently evoked even in recent times. For example, Bolzano (1973, par. 148, p. 196), in his *Wissenschaftslehre*, states that "the explications of this distinction one encounters, whether in Kant's own writings or those of others, still fall somewhat short of logical precision" and, talking about the author of the *Critique*, he adds that "these are in part merely figurative forms of expression that do not analyze the concept to be defined, in part expressions that admit of too wide an interpretation". The obscurity of Kant's containment criterion for

his definition: "the distinction is intended to have a narrow scope: it applies - and is intended to apply - only to judgments of subject-predicate form" (Proops, 2005, p. 589).

distinguishing analytic from synthetic judgments, which is said to be caused by a figurative or metaphorical language, has become a stereotype after Quine (1951, p. 21)'s attack in his influential paper *Two Dogmas of Empiricism*: "Kant conceived of an analytic statement as one that attributes to its subject no more than is already conceptually contained in the subject. This formulation [...] appeals to a notion of containment which is left at a metaphorical level".

Against the charge of obscurity some scholars have recently argued that the containment criterion, far from being a metaphorical formulation, is instead a precise notion. Anderson (2015, Part I), who, together with de Jong (1995), can be considered as the main representative of this interpretational trend, maintains that for Kant and his contemporaries the standard notion of containment is rigorously articulated through the appeal to the theory of logical division of concepts and to the Porphyrian concept hierarchies. This thesis deserves a closer examination. As Anderson explains, the standard notion of containment is twofold: on the one hand, each genus is said to be 'contained in' its species; on the other, each specie is said to be 'contained under' its genus. A concept's content is what is contained in it, i.e. more general concepts; a concept's extension is what is contained under it, i.e. more specific concepts[22]. For the traditional theory of concepts, then, concepts have the same extension if and only if they have the same content and containment relations establish a hierarchy. In Anderson's (2015, p. 55) words, conceptual content and logical extension are 'strongly reciprocal' and 'hierarchically ordered'. Higher and lower concepts are identified as genera and species:

> The higher concept, in respect to its lower one, is called genus, the lower concept in regard to its higher ones species (JL, par. 96, p. 594).

As a result, containment relations are ordered in a hierarchy of genera and species, where each genus is contained in its species and each species is contained under its genus. It is precisely due to this link between containment and the theory of genus and species that, according to Anderson, the rules of logical division can be applied to the standard notion of containment, with the positive consequence of regulating and specifying a quite technical notion of containment. The divisions, which are based on the Aristotelian definitions, are governed by the rules that the species exhaust the divided genus and that the species exclude one another: in other words, divisions are exclusive and exhaustive disjunctions. Therefore, the relation of two concepts whatever is either that of complete inclusion or that of total exclusion: partial overlaps are not admitted in these concepts hierarchies. Thus, as Anderson

[22] Anderson (2015, p. 50) underlines that Kant treats logical extension *intensionally*, in the sense that a concept's extension does not consist of individuals as in the modern interpretations, but rather of concepts. As a result, "the entire theory of conceptual contents and logical extensions is in fact based on ideas about concept containment".

underlines, the judgment that connects any two concepts will be either true, in the case of complete inclusion, or false, in the case of total exclusion: *tertium non datur*. The rules of divisions not only govern the relation between concepts, but also help to identify the content of a given concept: it is sufficient to add from its genus a differentia that marks its peculiarity. The process of division then turns out to be the inverse of that of logical abstraction via analysis. The picture conveyed by de Jong (1995) and Anderson (2015), although it is not free of difficulties[23], is, on balance, convincing: containment, for Kant and his contemporaries, is not a mere metaphor, but a quite technical notion.

1.1.3 Clarification, identity and contradiction

We have seen in the previous paragraph that the containment criterion applies only to true, affirmative, categorical judgments and is neither psychologistic nor obscure. We now turn to the remaining criteria of analyticity in the *Critique*, focussing in particular on their relation with the containment version. For ease of exposition, we will start from the clarification version of the distinction; we will move then to the identity criterion and conclude with the contradiction one.

As Proops (2005) emphasises, the characterization of analyticity in terms of clarification, as it is put forward by Kant in the excerpt of the *Introduction* to the *Critique* quoted above (CPR, A 7/B 11), combines both a negative and a positive requirement. The negative point is that analytic judgments "through the predicate [...] do not add anything to the concept of the subject"; the positive feature is that analytic judgments break the concept of the subject up "by means of analysis into its component concepts, which were already thought in it (though confusedly)"[24].

We hold that the clarification criterion can be reduced to the containment one, which constitutes its fundamental idea. The deep link between the two versions of analyticity can be mostly appreciated considering the positive feature of the definition above: the clarification of the concepts' intensions involved in a certain analytic judgment, which is obtained through conceptual analysis, consists of showing that the predicate concept is *contained* in that of the subject. In spite of the immediacy of this argument, some scholars have suggested that clarification

[23]Proops (2005, p. 599) argues that the notion of containment that emerges from this picture, far from being clear *tout court*, is "as clear as the general notion of the relation of a genus or differentia to a species", which, according to him, "is not well explained". Proops consequently concludes that "Quine's charge that Kant's explanation of analyticity rests on unclear metaphor is, on balance, justified".

[24]This duplicity of the clarification criterion is reaffirmed in the *Prolegomena* (Prol., p. 16), where Kant explains that "Analytic judgments say nothing in the predicate except what was actually thought already in the concept of the subject, though not so clearly nor with the same consciousness".

is not completely reducible to containment because the former would be characterized by an epistemic flavour that the latter would lack[25]. For example, Proops (2005, p. 602) maintains that clarification restricts the range of applicability of the containment criterion: while the latter applies to *true* judgments, the former would concern only judgments that are *known to be true*. He thus concludes that "the characterization in terms of the explicative-ampliative contrast is a classification of items of knowledge". The epistemic perspective of Kant's analyticity, which would be pointed out by the positive requirement of the clarification definition, is that analytic judgments are not cognitively empty, but are rather illuminating as they extend our knowledge.

While it is undeniable that analytic judgments according to Kant do have a cognitive content, we think that clarification merely emphasises the epistemic consequences of the analytic-synthetic distinction via containment that are *already* acting in the containment formulation. In other words, we concede the epistemic flavour of the clarification criterion: but this epistemic flavour is not given, in our opinion, by the epistemological force of the distinction itself, because the distinction via clarification (and amplification) is still a distinction between two kinds of propositional content, as it is for the containment criterion, and not of two kinds of cognitive procedures[26]. The point is that clarification is a characterization in epistemic terms of a logical distinction[27]. Moreover, the reason for which the pro-

[25]For example, Allison (2004, p. 90) on the one hand rejects the containment formulation precisely on the ground that this version of the analytic-synthetic distinction "suggests that the distinction is a logical one, concerning the relation between the subject and predicate concepts in a judgment"; on the other hand argues for the centrality of the clarification version of analyticity because "it indicates that the two species of judgments differ in their epistemic functions".

[26]This subtle distinction, which is due to Anderson (2015), has perhaps eluded also Kant's accusers of psychologism, for whom the exposition of the clarification criterion in CPR, A 7/B 11 has to be read as an evidence, even stronger than that of the containment criterion, that analyticity and syntheticity depend on the mental and subjective act of judgment. As for the case of containment, the charge of psychologism against clarification has to be rejected.

[27]The analytic-synthetic distinction is said to be 'logical' when it distinguishes between judgments or propositions on the basis of the logical relation among their constituents; as a result, analyticity and syntheticity turn out to be objective properties of the items considered, which remain fixed independently of the individual possibly involved in expressing a judgment or in thinking of a proposition. The logical conception of the distinction can be contrasted with both the methodological and the epistemological conceptions. This is what Anderson (2015, Chapter 1.3 and Part II) proposes, although he restricts himself to Kant's theory and developments of the distinction. The analytic-synthetic distinction of methodological nature is a distinction between two kinds of concept formations, analysis and synthesis, which are two traditional procedures or approaches for acquiring knowledge. This type of distinction can be extended only derivatively to judgments. The distinction is instead said to be 'epistemological' if it discerns between two different ways of knowing (and by extension, but again only derivatively, between two different types of judgments so known). This conception of the distinction concerns not only the generation of judgments, but also their epistemic justification. Nevertheless, both the methodological

cess of clarification has a cognitive content, that is, the analysis of the concepts involved in analytic judgments brings to light their conceptual marks that "were already thought in it (*though confusedly*)" (CPR, A 7/B 11), can be found ultimately and again in the containment criterion, in which the predicate-concept is explicitly said to be "*covertly*" (CPR, A 6/B 10) contained in the subject concept.

Now, if analytic judgments are illuminating and endowed with cognitive content, in so far as the process of analysis explicates the concepts involved by making distinct their conceptual marks, then analytic judgments are not trivial or tautologous, as they will be taken to be later on, especially in the Twenty-First century. As a result, this seems to require at least the predicate concept being different from the subject concept, for otherwise there is no room for any kind of clarification whatever. This is the main reason for which the identity criterion cannot be fully reduced to the containment one. For consider again, as an example, the judgment 'man is man': here "the connection of the predicate is thought through identity" (CPR, A 7/B 10) and thus the judgment is surely said to be analytic according to the identity criterion; nevertheless, the concept of the predicate is not 'covertly' contained in that of the subject and thus the judgment, strictly speaking, cannot be said analytic following the containment formulation. Both in the *Critique* (CPR, B 17) and in the *Prolegomena* (Prol., p. 19), Kant clearly asserts that the tautologous judgment '$a = a$' is analytic. The thesis of the analyticity of identical judgments, however, is explicitly rejected in other *loci* of Kant's work[28] and this has led scholars to quarrel on the interpretation of Kant's position[29]. Notice that

and the epistemological conceptions of the analytic-synthetic distinction, as Anderson repeatedly underlines, are founded on the way in which individuals relate to judgments and are thus subject-dependent. As a result, according to these two conceptions, analyticities can be turned to syntheticities and vice versa.

[28]For example, in a plan for a *Preisschrift über die Fortschritte der Metaphysik* (after 1791), Kant states that: "Analytic judgments are grounded on identity indeed and can be resolved in it, but they are not identical. Analytic judgments need an analysis and in this way they serve to explain concepts; while on the contrary identical judgments, *idem per idem*, do not explain anything" (PM, XX 322).

[29]Hanna (2001) includes in his list of the propositions said by Kant to be analytic also 'man is man', taken from the *Jäsche Logic*, thus assuming the analyticity of tautological propositions. De Jong (2010), reporting some doubts of the authenticity of that work and quoting as evidence the first text cited in the note above as well as others, goes against Hanna's conclusion. In particular, de Jong (1995, pp. 629-630) maintains that: "Tautological judgments are strictly speaking neither analytic nor synthetic [. . .] tautological judgments form according to Kant anything but ideal-typical examples of analytic judgments; to the extent that he regards such judgments as analytic, he sees them at most as dubious or degenerate cases of analyticity". Proops (2005) proposes a diachronical reading of Kant's position that accounts for his oscillations in his work: according to his interpretation, at the beginning Kant includes identical judgments among the analyticities because he understands his analytic-synthetic distinction as a classification of true judgments aimed against the Leibnizian; then, he comes to regard the distinction as focussed on knowledge-advancing judgments and thus he holds that identical judgments, being empty of

the issue at stake here is, once again, Kant's choice of one criterion as the fundamental basis for his theory of analyticity; for there is a class of judgments, that consisting of tautological judgments, whose analyticity or syntheticity depends on the selected criterion.

Although identity and containment differ for the treatment of tautologous judgments, they are nevertheless strictly connected. This is evident if we consider the fact that the identity criterion classifies as analytic not only tautologous judgments like '$a = a$', but also partial identities like 'all bodies are extended'. In a judgment of the latter kind the predicate concept is *partially* identical with the subject concept because the relation of *full* identity subsists only between the conceptual notes that constitute the predicate concept and *some but not all* the conceptual marks of the subject (in the example above, 'extension' is one of the conceptual notes of 'body'). But this is only another way of saying what is asserted by the containment criterion, because the predicate concept, being a part of the subject concept, is *contained* in the subject concept. Therefore, we can conclude that containment is the fundamental idea behind the identity criterion. The difference noted above between the two criteria is only a consequence of the fact that identity is a *purely* logical distinction in that it *exclusively* concerns the components of a judgment: as a result, it excludes any consideration of epistemic nature; but it is exactly on the basis of this kind of considerations that the containment criterion prevents tautologous judgments from being analytic.

As we will see, things are radically different for the contradiction criterion. The latter has been frequently indicated as the best among Kant's versions of analyticity and often recognized as the true Kantian account. The supposed reasons for its superiority are its inclusiveness, since, unlike the containment definition, it does not seem to be restricted to categorical propositions; and its affinity with the contemporary appeal to the class of logical truths in providing a definition of analyticity[30]. While both of these motivations are, as it will be clarified soon, unjustified, the latter is in addition anachronistic, in that it presumes to find in Kant's texts ideas conceived only later on. Kant introduces the relation between the principle of contradiction and the issue of analyticity in the second Chapter of the *Analytic of Principles*, in a paragraph entitled *On the Supreme Principle of*

cognitive content, cannot be said to be analytic.

[30]For example Kneale and Kneale (1962, pp. 357-358), talking about the contradiction criterion, maintain that "Such a characterization of analytic judgments is undoubtedly more suitable for Kant's purposes than that with which he began [i.e. containment], since it is possible with this definition to divide all true judgments between the headings analytic and synthetic". And conclude by saying: "we find it [i.e. the word 'analytic'] in many modern works with an explanation which makes it a synonym for 'true on logical grounds alone'. This usage can be defended by the argument that it maintains contact with tradition and renders Kant's intention better than he ever succeeded in doing for himself".

all Analytic Judgments, which is now time to quote in its context:

> Now the proposition that no predicate pertains to a thing that contradicts it is called the principle of contradiction, and is a general though merely negative criterion of all truth, but on that account it also belongs merely to logic, since it holds of cognitions merely as cognitions in general, without regard to their content, and says that contradiction entirely annihilates and cancels them.
> But one can also make a positive use of it, i.e., not merely to ban falsehood and error (insofar as it rests on contradiction), but also to cognize truth. For, **if the judgment is analytic**, whether it be negative or affirmative, its truth must always be able to be cognized sufficiently in accordance with the principle of contradiction. For the contrary of that which as a concept already lies and is thought in the cognition of the object is always correctly denied, while the concept itself must necessarily be affirmed of it, since its opposite would contradict the object.
> Hence we must also allow the **principle of contradiction** to count as the universal and completely sufficient **principle of all analytic cognition**; but its authority and usefulness does not extend beyond this, as a sufficient criterion of truth. For that no cognition can be opposed to it without annihilating itself certainly makes this principle into a *conditio sine qua non*, but not into a determining ground of the truth of our cognition (CPR, A 151-152/ B 190-191, underlining added).

In this excerpt, Kant is distinguishing between two functions of the contradiction criterion. On the one side, the principle is a "negative criterion of all truth": it is a negative criterion of truth because it serves to "ban falsehood and error" and, crucially, it applies to *all* kinds of truth, that is, to both analytic and synthetic judgments. In other words, the requirement of not violating the principle of non-contradiction is a necessary condition for truth in general, for opposing to it means "annihilating itself". However, Kant explicitly denies that that requirement serves as a sufficient reason for the truth of a judgment in general: that is, there exists a class of judgments that are not true even though they fulfill the principle. This possibility is explicitly acknowledged by Kant in the paragraph that precedes the quotation above, where he says that "even if there is no contradiction within our judgment, it can nevertheless combine concepts in a way not entailed by the object, or even without any ground being given to us either *a priori* or *a posteriori* that would justify such a judgment". On the other side, the principle functions as a "positive" criterion for the cognoscibility of analytic truth: it is positive because it serves to "cognize truth", and it does not apply to any kind of judgments, but only to analytic ones. Clearly this requirement is stronger and more restricted than the former. In particular, the sentence that "**if the judgment is analytic**, whether it be negative or affirmative, its truth must always be able to be cognized sufficiently in accordance with the principle of contradiction" can be paraphrased by saying that the sufficient cognoscibility in accordance with the principle of contradiction is a necessary condition for a judgment to be analytic. But this is only a more convoluted way of stating that the principle of contradiction is a necessary and sufficient condition for the cognoscibility of analytic judgments. The goodness of this reading is confirmed by the sentence at the beginning of the third paragraph: the principle has to count as the "universal", that is, necessary in this context, and "completely sufficient principle of all analytic cognition"[31].

In the textual analysis above, we have omitted to emphasize an important aspect of Kant's exposition: the principle of contradiction is invoked as an instrument for *knowing* the truth of analytic judgments and is appointed with a fundamental *epistemological function.* This aspect emerges clearly from the excerpt in the *Analytic*: the phrases that we have underlined show Kant's insistence on two main points. First, the principle is a criterion of truth: whether of truths in general or of analytic truths, this means that it is an instrument that establishes the truth of other propositions. Second, Kant is always careful in stating that the principle serves for the *cognoscibility* of something: in particular, he does never say that contradiction is the principle of analysis or of analyticity, but that it is the "principle of all analytic *cognition*" (emphasis added). From the epistemological function of contradiction some scholars have correctly inferred that, traditional interpretations notwithstanding, the principle is *not* a definitional criterion of analyticity. For example, de Jong (1995) distinguishes a "notion of analyticity" from an "epistemological criterion for analyticity" and Proops (2005, p. 603) explains that "Kant is thus making a point about the epistemology of analytic and synthetic judgments, and in doing so he is *presupposing* an understanding of the terms 'analytic' and 'synthetic'. He cannot therefore mean to be simultaneously *characterizing* analytic judgments as those that are cognizable in accordance with the principle of contradiction".

While the epistemological function of the principle is confirmed by the textual evidence shown above, the thesis that Kant would not intend to appeal to contradiction for *defining* analyticity is supported by a series of other elements. First,

[31] Proops (2005) seems to confine his interpretation to the hypothetical statement quoted above and concludes that "Kant does not say - or even imply - that analyticity consists in being knowable on the basis of the principle of contradiction. Instead, he states a necessary condition for analyticity: if something is an analytic judgment then it must be knowable (or "cognizable") in a certain way". We think that his deduction is wrong, because Kant explicitly holds that the truth of an analytic judgment also has to be *sufficiently* cognized according to the principle and he restates this point in closure of the passage we have quoted.

the claim that contradiction is "the principle of all analytic cognition" follows the assertion that contradiction is a necessary condition for truth in general; that is, that claim is inserted in a list of the uses that the principle can have, which conveys the idea of its instrumental nature. Second, Kant's explanation of the role of the principle of contradiction is not located in the *Introduction*, together with the other three definitions that we have examined; rather, it is isolated and arrives only later in the *Critique*[32]. Third, as Proops (2005) underlines, this definition would have been quite atypical if compared with the others, since it does not mention any restriction to categorical propositions: this does not imply, as the Kneales would have wanted, that Kant intended to extend his analytic-synthetic distinction to any kind of judgments; but rather this entails that the principle of contradiction is not appealed to for defining analyticity.

The exact way in which the principle of contradiction can be employed as an epistemic instrument for knowing the truth of analytic judgments is more perspicuous in the following excerpt taken from the *Prolegomena* than in that of the *Critique* quoted above:

> For since the predicate of an affirmative analytic judgment is already thought beforehand in the concept of the subject, it cannot be denied of that subject without contradiction; exactly so is its opposite necessarily denied of the subject in an analytic, but negative, judgment, and indeed also according to the principle of contradiction (Prol., p. 17).

This passage clearly excludes the widespread and anachronistic interpretation according to which that a judgment is known in accordance with the principle of contradiction means that it is possible to derive an explicit contradiction from the negation of the judgment involved. The idea is rather that in an affirmative analytic judgment the contradiction rests with the concept of the subject and the concept of the negation of the predicate; while, in the negative case, contradiction rests with the concepts of the subject and of the predicate. This is because the predicate is "already thought beforehand in the concept of the subject", for if the predicate were not thought in that of the subject, then the denial of the former would not contradict the latter. In other words, the ultimate reason for the epistemic function of the principle of contradiction in knowing the truth of analyticities is the relation of containment between the concepts involved in analytic judgments. Thus, although the principle of contradiction does not function,

[32]Actually, the principle of contradiction is mentioned in the fourth paragraph of the second edition of the *Introduction* to the *Critique*. But even this mention is separated from the definitions of the analytic-synthetic distinction and indicates its instrumental role, since it says that in analytical judgments the concept of the predicate is extracted from that of the subject via the principle of contradiction.

strictly speaking, as a definition of analyticities, the possibility of its employment as a tool of knowledge is granted by the fundamental idea of containment that grounds Kant's notion of analyticity.

1.1.4 Syntheticity and intuition

The importance of avoiding the 'privileging of analyticity' in a treatise about the Kantian analytic-synthetic distinction can be correctly justified by appealing to the need of a philological and conceptual adherence to the original texts. But the indisputable centrality of the synthetic *a priori* in Kant's system is not our concern here. Rather, we need to clarify the main defining features of syntheticity because we have seen that Kant's distinction is not exhaustive: as a result, in order to establish that a judgment is synthetic, it is no more sufficient to show that it is not analytic.

A natural point to start with are the counterparts of the four criteria of analyticity examined above. From the excerpt of the fourth paragraph of the *Introduction* to the *Critique* (CPR, A 6-7/B 10-11) we learn, first, that in synthetic judgments the concept of the predicate "lies entirely outside" the concept of the subject, "though to be sure it stands in connection with it"; second that this "connection is thought without identity" and third that synthetic judgments can also be called "judgments of amplification" because they "add to the concept of the subject a predicate that was not thought in it at all, and could not have been extracted from it through any analysis" (CPR, A 6-7/B 10-11). Moreover, Kant, in the paragraph entitled *On the Supreme Principle of all Synthetic Judgments* of the second Chapter of the *Analytic of Principles*, explains that any synthetic judgment can be denied without obtaining a contradiction and specifies that:

> In synthetic judgments, however, I am to go beyond the given concept in order to consider something entirely different from what is thought in it as in a relation to it, a relation which is therefore never one of either identity, or contradiction, and one where neither the truth nor the error of the judgment can be seen in the judgment itself (CPR, A 154-155/B 193-194).

The main idea behind these passages is that in synthetic judgments the concept of the predicate is not contained in the concept of the subject: rather, the former is 'outside' or 'beyond' the latter. Nevertheless, in order for grounding and justifying the truth of synthetic judgments, there must be some kind of connection between the two concepts involved. This connection cannot be thought through identity or contradiction and cannot be explicated through analysis exactly because it is not founded on containment. As a result, this connection cannot be but indirect in that it has to link the two concepts to one another by connecting them to a third and different element. In Kant's words[33]: "if it is thus conceded that one must go beyond a given concept in order to compare it synthetically with another, a third thing is necessary in which alone the synthesis of two concepts can originate" (CPR, A 155/B 194). But what does this third element consist of[34]? The third element that is always necessary for the truth of synthetic judgments is an object in which "the synthetic unity of their concepts could establish objective reality" (CPR, A 157/B 196). The relation of the concepts to the objects always has to be mediated by intuition[35], which for all human beings can only be sensible, in that it is necessarily related to the faculty of sensibility, and never intellectual[36]. For this reason, intuition dependence turns out to be the main feature of synthetic judgments as opposed to the analytic ones:

> If one is to judge synthetically about a concept, then one must go beyond this concept, and indeed go to the intuition in which it is given. For if one were to remain with that which is contained in the concept, then the judgment would be merely analytic, an explanation of what is actually contained in the thought (CPR, A 721/B 749).

Intuition, which plays a fundamental role in Kant's notion of syntheticity, is a kind of cognition[37], whose main defining features[38] are immediacy and singular-

[33]Kant affirms the need for a third element for connecting the two concepts involved in a synthetic judgment in several passages of the *Critique*. For example, in the *Introduction* he states that "in synthetic judgments I must have in addition to the concept of the subject something else (X) on which the understanding depends in cognizing a predicate that does not lie in that concept as nevertheless belonging to it" (CPR, A 8) and asks "What is the X here on which the understanding depends when it believes itself to discover beyond the concept of A a predicate that is foreign to it and that is yet connected with it?" (CPR, A 9/B 13).

[34]Or, in Kant's words: "Where is the third thing that is always requisite for a synthetic proposition in order to connect with each other concepts that have no logical (analytical) affinity?" (CPR, A 259/B 315).

[35]For example, at the beginning of the *Transcendental Aesthetic*, Kant maintains that "In whatever way and through whatever means a cognition may relate to objects, that through which it relates immediately to them, and at which all thought as a means is directed as an end, is intuition" (CPR, A 19/B 33).

[36]This is clearly stated in the following excerpt: "The capacity (receptivity) to acquire representations through the way in which we are affected by objects is called **sensibility**. Objects are therefore **given** to us by means of sensibility, and it alone affords us **intuitions**; but they are **thought** through the understanding, and from it arise **concepts**. But all thought, whether straightaway (*directe*) or through a detour (*indirecte*), must, ¡by means of certain marks,¿ ultimately be related to intuitions, thus, in our case, to sensibility, since there is no other way in which objects can be given to us" (CPR, A 19/B 33). On this point see also CPR, A 51/B 75.

[37]Kant explains that: "A **perception** that refers to the subject as a modification of its state is a **sensation** (*sensatio*); an objective perception is a **cognition** (*cognitio*). The latter is either an **intuition** or a **concept** (*intuitus vel conceptus*)" (CPR, A 320/B 377).

[38]Hanna (2001, p. 194 and ff.) identifies, through some textual evidence, five features that go

ity[39]. These characteristics can be better appreciated if compared to the opposite ones that go to identify concepts. First, while intuition refers to objects directly, the connection of the concept to the object is always "mediate, by means of a mark"; second, while intuition refers to one and only one object, a concept is a general representation, because the mark through which it refers to an object "can be common to several things" (CPR, A 320/B 377). It is possible to distinguish between two kinds of intuitions: empirical and pure. Judgments based on the former are synthetic a posteriori: this is the easiest case, which involves an appeal to experience. Judgments based on the latter are synthetic *a priori*: this is the most interesting case, because "synthetic *a priori* judgments are contained as principles in all theoretical sciences of reason" (CPR, A 10/B 14).

1.2 Kant's pure general logic

In the previous Sections, we have dealt with the Kantian analytic-synthetic distinction. We have argued that, despite the many-sidedness presentation of the distinction, the fundamental definition is based on containment (and possibly exclusion) and classifies as analytic all and only true, (affirmative), categorical judgments in which the concept of the predicate is (covertly) contained in the concept of the subject. The determination of the class of judgments that Kant regarded as analytic is not enough to establish the position Kant assigns to logic in connection to the analytic-synthetic distinction. It's clear that in order to define the status of logical truths, we first need to examine Kant's conception of logic. But which is the logic we are concerned with?

This question is not trivial, since in the introduction to the *Transcendental*

to identify Kantian intuitions. Beyond immediacy and singularity, we have that first, intuitions are related to sensibility because the reference to an object presupposes sensibility as the mode in which human beings are affected by objects; second, any intuition can be given prior to all thinking, since an object can be intuited without being conceptualized; third, intuitions depend upon the (current or former) presence of an object. From these characteristics, Hanna concludes that Kantian intuitions are essential indexical.

[39] Scholars have debated which of these two criteria works as fundamental. As we will examine later in details, Hintikka (1972) argues that Kantian intuitions are characterized in the first place by their singularity: upon this premise, he maintains the so-called 'logical interpretation' of the role of intuition in mathematics, according to which intuitions are essential for mathematical inferences. On the contrary, Parsons (1983) regards immediacy as the fundamental feature of intuition from which the other one simply follows; accordingly, he argues that intuitions, as far as they bring the spatio-temporal structure to the mind, provide the evidence and the certainty that are typical of mathematical reasoning. As Anderson (2015, p. 214 and ff.) accurately retraces, the logical and phenomenological interpretations of the role of intuition in mathematics have been later resumed and placed on a more sophisticated basis. Moreover, recent works have tried to reconcile different aspects of the two interpretations.

Logic of the *Critique*, Kant recognizes four different kinds of logic: pure general logic, applied general logic, special logic and transcendental logic. The first distinction introduced is that between general and special logics. While the former "contains the absolutely necessary rules of thinking, without which no use of the understanding takes place, [...] without regard to the difference of the objects to which it may be directed", the latter "contains the rules for correctly thinking about a certain kind of objects": as a result, we find that each special logic acts as an "organon of this or that science" (CPR, A 52/B 76) because "there is, for example, a use of the understanding in mathematics, in metaphysics, morals, etc." (JL, par. 12, p. 528). The second distinction proposed in the introduction to the *Transcendental Logic* is that between pure and applied logics. The former abstracts "from all empirical conditions under which our understanding is exercised, e.g., from the influence of the senses, from the play of imagination, the laws of memory, the power of habit, inclination, etc."; the latter undergoes "the subjective empirical conditions that psychology teaches us" (CPR, A 53/B 77). Applied logic is "a representation of the understanding and the rules of its necessary use *in concreto*, namely under the contingent conditions of the subject, which can hinder or promote this use, and which can all be given only empirically" (CPR, A 54/B 78-79): the term 'applied' seems to refer ultimately to empirical psychology. With the third distinction, which is the most fundamental one for the *Critique*, Kant introduces his radical innovation of a transcendental logic (as opposed to the general one). Transcendental logic investigates the origin and the objective validity of the cognition of pure understanding and pure reason, through which we think objects completely *a priori*. The central point here is that transcendental logic, unlike general logic, can be seen as a special science because it has content and it is related to some peculiar non-empirical objects, namely the pure concepts of the understanding[40].

Pure general logic[41] is the discipline that gets closer to our modern conception

[40]Tolley (2012, pp. 313-314) underlines the importance of transcendental logic next to traditional logic and holds that the introduction of the former testifies that Kant saw the necessity of providing a supplement of the latter. Tolley writes: "[...] having some sort of content ('relation to an object') forms part of the essence of thinking. Yet since it is the task of logic as such to provide the analysis of the essence of thinking 'in general', logic's task will remain deeply incomplete so long as it restricts itself to the approach delineated by the traditional logic, since it abstracted entirely from the content of thinking. The task of logic can only be completed, therefore, by the introduction of a science of the content of thought - that is, by Kant's new transcendental logic".

[41]Kant summarizes its main features as follows: "1) As general logic it abstracts from all contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing but the mere form of thinking. 2) As pure logic it has no empirical principles, thus it draws from psychology (as one has occasionally been persuaded), which therefore has no influence at all on the canon of the understanding. It is a proven doctrine, and everything in it

of logic and that is the focus of our discussion: from now on, unless specified otherwise, the term ‘logic’ will thus refer to pure general logic. Two are the issues that we are going to investigate in this Section. First, we provide an overview on both Kant’s logical notions and the topics that went in those times to constitute the discipline of logic. This will clarify which truths were considered as ‘logical’ truths in Kant’s view. Second, we are going to investigate Kant’s defining features and conception of this discipline. This will set the indispensable basis for our conclusion regarding the application and the applicability of the analytic-synthetic distinction to logic.

1.2.1 The topics of pure general logic

Logic is a constant presence in Kant’s intellectual life. During his academic education, Kant attended several logical courses and seminars; enhanced his logical notions while preparing the dissertation for obtaining the *venia legendi*; held numerous courses in logic in his forty-year teaching in Königsberg; and also wrote about this discipline[42]. The importance of logic in Kant’s cultural horizon seems also to emerge from his *Critique*: the division of the work into a *Doctrine of Elements* and a *Doctrine of Method* follows a logical partition; then, at the very beginning of the *Preface* to the second edition, logic is pointed at for having travelled the “secure course of a science” (CPR, B vii-viii); last, in the *Transcendental Analytic* Kant derives the table of categories from the classification of judgments

must be completely *a priori*” (CPR, A 54/B 78).

[42]In the *Collegium Federicianum*, during the years 1732-1740, Kant already received some kind of logical education: he was taught at least about the basic principle of the discipline. It was during his university period (1740-1746) in Knigsberg however that Kant could improve his skills thanks to a wide logical didactic offer (Wolffism enriched with several other trends). In particular, Martin Knutzen, *extraordinarius* professor in logic and metaphysics with a Wolffian preparation mitigated by a strongly critical spirit, taught Kant about more advanced instruments of logic, as it is proven by the *Elementa Philosophiae Rationalis seu Logicae*, which is the written outcome of his long teaching. After his degree and a brief work experience as preceptor, Kant obtains his *venia legendi* in 1755 with a dissertation, the *Nova Dilucidatio*, which mainly dealt with the logical principles of contradiction and sufficient reason. The readings of Kant’s aspiring teacher included the works of Christian August Crusius, a famous anti-Wolffian, and of Joachim Georg Darjes, a member of the logical school of Jena. Kant taught in Knigsberg from 1755 to 1796 and in 1770 obtained the chair in logic and metaphysics. Among the courses that he held in those forty years, those on logic were the most numerous: Emil Arnoldt’s reconstruction talks about no less than 56 biannual courses on logic, 32 of which have surely been held and 24 of which are certified only by their announcement. He adopted as handbook for his lessons the *Auszug aus der Vernunftlehre* by Georg Friedrich Meier. Kant’s logical courses produced almost 2000 *Reflexionen ber Logik*, the brief treatise *Die Falsche Spitzfindigkeit der vier syllogistischen Figuren erweisen* (1762), the *Nachricht* (1765), the numerous notes of his students and the handbook edited by Jsche. The reconstruction of the present footnote benefits from the work of Capozzi (2002, pp. 59-113).

according to their logical form.

These hints notwithstanding, scholars are unanimous in claiming that Kant's knowledge of logic, especially of its technical notions, is incomplete and elementary. For example, Kneale and Kneale (1962, p. 354) maintain that Kant's interest in traditional logic is "superficial"; Lapointe (2012, p. 11) agrees with the Kneales and adds that "his treatment of logical questions per se is often seen as relatively inconsequential"; then, Hazen (1999, p. 92) charges Kant of having "a terrifying narrow-minded, and mathematically trivial, conception of the province of logic". These accuses cannot be easily dismissed. The superficiality of Kant's interest in logic can be intuited through a closer examination of Kant's first *Critique*. First, the clues pointed at above are, on balance, quite restricted. Second, Kant dedicates only few pages of his main theoretical work to investigating pure general logic and his discussion aims primarily at introducing the innovation of a transcendental logic. Third, Kant's idea of the history of logic, which is sketched for example in the *Preface* of the second edition of the *Critique*, is totally static and completely unaware of the latest ideas and developments[43].

But the fundamental issue here is that Kant's knowledge of logic consists of (a restricted version of) the Aristotelian syllogistic with a simple theory of disjunctive and hypothetical propositions added on[44]. This latter point clearly arises from an

[43]In the *Preface* to the second edition of the *Critique*, Kant affirms that "What is further remarkable about logic is that until now it has also been unable to take a single step forward, and therefore seems to all appearance to be finished and complete" (CPR, B viii) and, in the *Jäsche Logic*, Kant arrives to dismiss any logical research project by holding that "In present times there has not been any famous logician, and we do not need any new inventions for logic, either, because it contains merely the form of thought" (JL, par. 21, p. 535). Kant's historical reconstruction from the antiquity to modern times follows a widespread stereotype. Aristotle is always recognized not only as the founder but also the finisher of logic and, at the same time, he is covertly accused of having introduced trivial subtleties in the discipline: "That from the earliest times **logic** has traveled this secure course can be seen from the fact that since the time of Aristotle it has not had to go a single step backwards, unless we count the abolition of a few dispensable subtleties or the more distinct determination of its presentation, which improvements belong more to the elegance than to the security of that science" (CPR, B viii). As Capozzi (2002, p. 271 and ff.) explains, the reason for this charge is the connection of the Aristotelian doctrines to the Aristotelianism of the scholastics, which was interpreted as the cultural mask of the Papist tyranny. The evaluation of the most recent logic is instead original and the judgments are quite variable in Kant's writings: this is the case, for example, of Locke's, Leibniz's and Wolff's estimate. The picture that emerges from Kant's historical reconstructions is at least reductive and for sure immune to the latest, new and promising ideas, such as that of a logical calculus.

[44]MacFarlane (2002, p. 26) underlines the expressive limitations of Kant's logic with respect to Frege's logic. He holds that "the most dramatic difference is that Frege's logic allows us to define concepts using nested quantifiers, while Kant's is limited to representing inclusion relations". The point is that Frege can express with a logical vocabulary what Kant thought could be represent only with the aid of intuition (*e.g.*, infinitude and natural numbers). As a result, Kant's logical

analysis of the so-called *Jäsche Logic*, a handbook that a student and friend of Kant, Gottlob Benjamin Jäsche, edited in 1800 on the basis of his teacher's notes and lessons[45].

The central part of the work, which is entitled *Universal Doctrine of Elements*, is divided into three Sections: *Of Concepts*, *Of Judgments*, *Of Inferences*. This was a common practice in Kant's times: as Anderson (2015, p. 51) shows, the same arrangement can be seen in Port Royal Logic, in the Wolffian *Deutsche Logik* and in Meier's *Auszug der Vernunftlehre*. The first Section is entirely devoted to the theory of concepts: among the topics treated in this context, we find all the tools that turned out to be necessary for the formulation of analyticity according to the containment criterion. In particular, we have the definitions of 'content' and 'extension' of concepts and of their relationship[46]. Moreover, we learn in an explicit way the connection between higher and lower concepts on the one hand and genus and species on the other. Last, we have that the processes of logical abstraction and logical determination are clearly compared and indicated as ways for obtaining in the former case higher concepts and in the latter lower concepts[47]. The theory of concepts enjoys an expositional priority because of its role of cornerstone for the topics dealt with in the remaining two Sections of the *Doctrine of Elements*.

The second part is dedicated to the theory of the logical form of judgments, which is traced back, as it is in the *Analytic of Concepts* in the *Critique*, to the four principal moments of quantity (universal, particular, singular), quality (affirmative, negative, infinite), relation (categorical, hypothetical and disjunctive) and modality (problematic, assertoric and apodeictic). Kant strongly emphasises the irreducibility of the hypothetical and disjunctive judgments to the categorical ones and justifies this assertion by appealing to the different logical functions of the understanding that each kind of judgment would request[48].

---

tool, which is the essentially monadic traditional logic, lacks the expressive power to represent arithmetic structure. And this will have important consequences, as we will see later on.

[45]The material that Jäsche used to compile the logical handbook is constituted of the notes that Kant wrote on his copy of Meier's *Auszug* during his forty-year teaching in logic.

[46]Kant's definition sounds as follows: "Every concept, *as partial concept*, is contained in the representation of things; as *ground of cognition*, i.e., as mark, these things are contained under it. In the former respect every concept has content, in the other an *extension*. The content and extension of a concept stand in inverse relation to one another. The more a concept contains *under* itself, namely, the less it contains *in* itself, and conversely" (JL, par. 95, p. 593).

[47]It is interesting to underline Kant's observation that "the greatest possible abstraction yields the highest or most abstract concept"; while the logical determination "can never be regarded as completed" because thoroughly determinate cognitions can only be given as intuitions and never as concepts (JL, par. 99, pp. 596-597).

[48]In an observation, Kant stresses: "Categorical judgments constitute the matter of the remaining judgments, to be sure, but one must not on this account believe, as several logicians do, that both hypothetical and disjunctive judgments are nothing more than various clothings of categoricals and hence may be wholly traced back to these latter. All three kinds of judg-

The third and last Section of the *Doctrine of Elements* concerns the theory of inferences. First, Kant discusses the inferences of the understanding, which are said to be characterized by their immediacy, since a judgment is here derived by another without the need of a third mediating judgment. The inferences discussed are those structured in the Aristotelian square of opposition, such as '*ab universali ad particulare valet consequentia*', or by conversion of the subject to the predicate. Second, Kant introduces the inferences of reason, which are "the cognition of the necessity of a proposition through the subsumption of its condition under a given universal rule" (JL, par. 120, p. 614). Here we find the Aristotelian syllogistic, which of course deals only with categorical judgments: after having introduced the appropriate terminology, Kant exposes the four figures and the reduction of the three latter figures to the first. Then, Kant suggests only *modus ponens* and *modus tollens* as far as hypothetical judgments are concerned and *modus ponendo tollens* and *modus tollendo ponens* as far as disjunctive judgments are concerned. The Section is closed by the third kind of inference, namely that of the power of judgment, which allows inferring from the particular to the universal and thus is not a function of the determinative power of judgment, but rather of the reflective one.

In the *Jäsche Logic*, the *Universal Doctrine of Elements* is followed by a shorter *Universal Doctrine of Method*, which deals with the form of a science in general, i.e. with the "ways of acting so as to connect the manifold of cognition in a science" (JL, par. 139, p. 630). Here we find, among several other topics, also some remarks on both notions of 'analytic and synthetic definitions' and of 'analytic and synthetic methods'[49]. The two are only indirectly related to the notion of analytic and synthetic judgments. It's interesting further to notice that in the *Doctrine of Method* we also find that the concept of logical division is defined as "the determination of a concept in regard to everything possible that is contained under it, insofar as things are opposed to one another, i.e., are distinct from one another" (JL, par. 146, p. 636). The general rules of logical division, which are put forward in the next paragraph, clearly prescribes exclusive and exhaustive disjunctions and dichotomy is said to be the only division based on *a priori* principles (while politomy is always empirical).

---

ments rest on essentially different logical functions of the understanding and must therefore be considered according to their specific difference" (JL, par. 105, p. 601).

[49] Analytic definitions are "of a concept that is *given*"; synthetic definitions are "of a concept that is *made*": in both cases however concepts can be given or made both *a priori* and *a posteriori* (see JL, par. 141, pp. 631-632). The two methods are instead contrasted on the basis that analytic method "begins with the conditioned and grounded and proceeds to principles", while synthetic method "goes from principles to consequences or from the simple to the composite" (see JL, par. 149, p. 639).

The two doctrines follow a long *Introduction*[50], in which Kant discusses several topics connected to the status of the discipline and gives the fundamental definitions. There we find a taxonomy of kinds of logic and a history of logic on which we have already paused; the position of logic in the horizon of philosophy and knowledge in general and the perfections of logic. But the paragraph that we need to investigate in depth is the first one, which is dedicated to the concept of pure general logic.

1.2.2 The conception of pure general logic

As we have seen in the introduction to the present Section, the two qualifying features of the kind of logic we are concerned with are its pureness and its generality. Kant argues with the former that logic is not concerned with the empirical conditions of the subject in his employment of the rules of the understanding and that it is distinguished from applied logic; with the latter that the discipline "contains the absolutely necessary rules of thinking" (CPR, A 52/B 76) and that it is different from special logics. The meaning of the term 'necessary' in describing the generality of logic is not univocal, as the following passages testify:

> All rules according to which the understanding operates are either *necessary* or *contingent*. The former are those without which no use of the understanding would be possible at all, the latter those without which a certain determinate use of the understanding would not occur. [...] The rules of this particular, determinate use of the understanding in the sciences mentioned are contingent, because it is contingent whether I think of this or that object, to which these particular rules relate.
> If now we put aside all cognition that we have to borrow from *objects* and merely reflect on the use just of the understanding, we discover those of its rules which are necessary without qualification, for every purpose and without regard to any particular objects of thought, because without them we would not think at all (JL, p. 528, par. 12).

> In logic, however, the question is not about *contingent* but about *necessary* rules; not how we do think, but how we ought to think. The rules

[50] The length of the *Introduction* (it is almost the half of the entire work) is mainly due to the fact that Jäsche, as he explains in the *Preface*, wanted in his edition to respect Kant's prescription that "nothing more may be taken up in the proper treatment of logic, and in particular in its *Doctrine of Elements*, than the theory of the three essential principal functions of thought: *concepts*, *judgments*, and *inferences*". Hence, Jäsche continues, "everything that deals with cognition in general and with its logical perfections, and which in Meier's textbook precedes the doctrine of concepts and takes in almost half of the whole, must accordingly be reckoned to the introduction" (JL, par. 4, p. 521).

> of logic must thus be derived not from the *contingent* but from the *necessary* use of the understanding, which one finds in oneself apart from all psychology. In logic we do not want to know how the understanding is and does think and how it has previously proceeded in thought, but rather how it ought to proceed in thought. Logic is to teach us the correct use of the understanding, i.e., that in which it agrees with itself (JL, p. 529, par. 14).

The rules of special logics are contingent not in the sense that they can be otherwise, but rather because they concern objects that can be thought or not and thus they are applied only in specific cases. We could infer by contraposition that the rules of general logic are necessary because they have to be applied no matter what are the objects we are thinking about. As Lapointe (2012, pp. 15-16) observes, Kant seems to give two interpretations to the meaning of the necessity of the logical rules just specified. On the one hand, as it is underlined in the first excerpt quoted above, the rules of logic are constitutive of the understanding, in the sense that we cannot think at all without them. On the other hand, as it is emphasised in the second passage above, the rules of logic are normative in the weaker sense that they prescribe how we have to think if we want to think correctly. While according to the second interpretation even violating the norms of logic we are still thinking, although committing mistakes, the first reading of the meaning of the term 'necessity' completely excludes the possibility of a thought elaborated without employing logical rules[51].

In a recent work, Tolley (2013) claims that according to Kant the generality of logic, understood in terms of its being constitutive for thought, is absolutely unrestricted. The starting point of this reading, which will be useful for our conclusions, is the observation that Kant is deeply committed to the essential unity of the theoretical and practical reason and that the objects of the theoretical understanding and of practical reason are both objects of the understanding. As a result, according to Tolley (2013), "Kant takes logic to be concerned with *anything* that

[51] Although scholars agree about the constitutive role of logic for thinking in Kant's view, some interpreters deny the weaker reading according to which Kant would claim the normativity of logic for thought. For example, Tolley (2006) maintains that there are good reasons for holding that normative interpretations of the generality of logic compel Kant to accept certain premises, which directly conflict with other Kantian beliefs. Tolley (2006, p. 375)'s conclusion is thus that "for Kant, normativity is at best an externally conferred, rather than essentially inherent, property of logical law". Mac Farlane (2002, p. 43 and ff.)'s interpretation of Kant's generality tends to line up with the constitutive reading. Although he talks about a *normative* generality, his explanation of the necessity of the rules of logic does not appeal to mistakes or correctness of thought, but rather to the possibility of thinking *tout court*: "The necessary rules are 'necessary', not in the sense that we cannot think contrary to them, but in the sense that they are *unconditionally binding* norms for thought - norms, that is, for thought *as such*".

can be *understood* [...] Kant clearly takes the scope of both understanding and also judgment to extend *well beyond* the theoretical sphere, to both practical acts of issuing and heeding commands and even to expressions of aesthetic satisfaction. Because of this, the sphere of logic itself - at least a truly 'universal or general [allgemeine]' logic - must also comprise within itself much more than the forms and laws of the theoretical use of understanding alone". Following this interpretation, Kant would have thus envisioned a peculiar notion of absolute generality for logic that his successors, despite a verbal agreement, would have lost: Frege for instance, subscribing a truth-theoretic understanding of the subject-matter of logic, proposes what Kant would have called a 'special logic'.

An aspect closely related to the generality of logic is its formality. According to Kant, logic is formal in the sense that it abstracts from the semantical content of thought. This is a feature that logic shares with grammar[52]. The latter is the science of the rules of a language and studies the form of a language; similarly, the former is the science of the rules of thinking and studies the form of thoughts and cognition:

> General logic abstracts [...] from all content of cognition, i.e. from any relation of it to the object, and considers only the logical form in the relation of cognitions to one another, i.e., the form of thinking in general (CPR, A 55/B 79).

An immediate consequence of the formality of logic, i.e. of the fact that logic completely overlooks the content of thoughts, is that logic alone cannot yield an extension of knowledge about reality or objects:

> But since the mere form of cognition, however well it may agree with logical laws, is far from sufficing to constitute the material (objective) truth of the cognition, nobody can dare to judge of objects and to assert anything about them merely with logic without having drawn on antecedently well-founded information about them from outside of logic [...] (CPR, A 60/B 85).

This point is further clarified by Kant's introduction of the division of general logic into analytic and dialectic[53]. The part of logic that concerns the form of thought and both the understanding and the reason in general is called 'analytic': it is said to be the negative condition of truth, because, as we have seen, it cannot deal with the content and the objects of knowledge. But when we forget that formal logic has no content, then we illegitimately turn it into an *organon* that produces

[52] JL, parr. 12-13, p. 528.

[53] CPR, A 57/B 82 - A 62/B 86.

completely arbitrary contents: this "logic of illusion" is called "dialectic". Kant is clear in stating that the presumption of employing general logic as a tool for extending our knowledge, i.e. its dialectical use, "comes down to nothing but idle chatter, asserting or impeaching whatever one wants with some plausibility" (CPR, A 61/B 86).

It has been recently argued that Kant was the first modern philosopher to claim that logic is formal[54]: MacFarlane (2002, pp. 44-46) shows that the tradition against which Kant was reacting accepted the generality, but not the formality of logic. In particular, on the neo-Leibnizian's perspective, although logic is said to be general in the sense that it is not restricted to a certain kind of objects, at the same time it is not completely formal, since, while it abstracts from particular contents (like those of 'red' or 'cat'), it does not abstract from the contents of highly general concepts (like those of 'being' or 'unity'). As the following passages suggest, the innovative thesis on the formality of logic turns out to be, according to Kant, only a consequence of the generality of this discipline:

> The former [i.e. the logic of the general use of the understanding] contains the absolutely necessary rules of thinking, without which no use of the understanding takes place, and it therefore concerns these rules without regard to the difference of the objects to which it may be directed (CPR, A 52/B 76).

> And from this [i.e. from the generality of logic] it follows at the same time that the universal and necessary rules of thought in general can concern merely *its form* and not in any way its matter. Accordingly, the science that contains these universal and necessary rules is merely a science of the form of our cognition through the understanding, or of thought (JL, p. 585, par. 12).

In the first excerpt the definition of the formality of logic is introduced through a 'therefore' after Kant's spelling of the feature of generality; in the second, formality is presented as a consequence of the necessity of the rules of the understanding. The logical dependence of the feature of formality on that of generality has led some scholars to argue that the two notions, given some Kantian premises, ultimately collapse[55].

[54] Lapointe (2012, p. 13) and especially MacFarlane (2002, pp. 44 - 46).

[55] For example, Lapointe (2012) argues that Kant does not provide independent criteria for generality and formality, because establishing the necessity of a rule amounts to establishing that that rule concerns the mere form of thought. As a result, according to Lapointe, the two notions collapse. MacFarlane (2002) claims that according to Kant the two features are ultimately different ways of expressing the same. He proposes a reconstruction of a Kantian

Another fundamental feature of pure general logic is that logic is, according to Kant, a canon for thinking. In the *Transcendental Doctrine of Method* of the *Critique*, Kant explains that the term 'canon' signifies in general a body of rules or *a priori* principles that govern the use of certain faculties and specifies that pure general logic in its analytical part can be said to be a canon, because it amounts to

> the sum total of the *a priori* principles of the correct use of [...] understanding and reason in general, but only as far as form is concerned (CPR, A 796/B 824).

The thesis that logic is a canon in the precise sense explicated above can be seen as a synthesis of Kant's conception of pure general logic, because its premises are, we think, exactly the three features of the discipline that we have just mentioned: its pureness, its generality and its formality[56]. In the first place, from the fact that logic is pure, i.e. that it is not concerned with the empirical conditions involved in applying the rules of the understanding and reason, it can be deduced that the rules of the correct use of those faculties are *a priori* and are not derivable from any kind of experience[57]. In the second place, the thesis that logic is general, i.e. that it contains the absolutely necessary rules of thinking, means, as we have seen, that it is both constitutive and normative for thinking. But this in turn indicates that logic is composed of the rules for the employment of the faculties involved. This point is made clear in Kant's Introduction to the *Jäsche Logic*:

> As a science of the necessary laws of thought, without which no use of the understanding or of reason takes place at all, laws which consequently are conditions under which the understanding can and ought to agree with itself alone - the necessary laws and conditions of its correct use - logic is, however, a *canon* (JL, par. 13, p. 529).

The normative side of the generality of this discipline clearly emerges from Kant's definition of logic as a canon: for not only logic is said to give the *correct* use of

---

argument from the generality to the formality of logic and concludes that "Formality is not, for Kant, an independent defining feature of logic, but rather a consequence of the Generality of logic, together with several auxiliary premises from Kant's critical philosophy" (MacFarlane, 2002, p. 32).

[56] Of course, for what has been said before, the premise that logic is formal is redundant in so far as it can be deduced from the fact that it is general.

[57] This is the meaning of the following excerpt: "A general but pure logic therefore has to do with strictly *a priori* principles, and is a **canon of the understanding** and reason, but only in regard to what is formal in their use, be the content what it may (empirical or transcendental)" (CPR, A 53/B 77).

the understanding and of reasoning, but also Kant used to call this discipline a "cathartic" and a "critic" before introducing the term 'canon'[58]. Last, from the fact that logic is formal, i.e. that it abstracts from the content, it follows that logic, in its analytical part, can only be a canon, not an *organon*[59]. As Kant explains in the *Jäsche Logic* (JL, par. 13, pp. 528-529), an *organon* is "a directive as to how certain cognition is to be brought about", which presupposes "exact acquaintance with the sciences, their objects and sources". But logic, being formal, cannot present any content and thus cannot anticipate the matter of any science. When used to produce effective knowledge, when used as an *organon*, pure general logic is no more analytic, but dialectic: the "logic of illusion".

The conception of logic as a canon and as a discipline characterized by the features of pureness, generality and formality has a fundamental role, as it will be soon clarified, in Kant's stance on the status of pure general logic in relation to the analytic-synthetic distinction.

1.3 The relationship between analyticity and logic

The issue about the relationship between analyticity and logic comprises, we think, two different matters, which must be kept distinct. On the one hand, we have to examine Kant's perspective on the role of *logic as an instrument* for both defining and applying the analytic-synthetic distinction (Section 1.3.1). On the other hand, we need to analyse the epistemological *status of logic* in order to provide an answer to the question of whether logical truths are, following Kant's definitions, analytic or synthetic *a priori* (or neither of them) (Section 1.3.2).

1.3.1 Logic as an instrument

Section 1.1.2 shows that recent scholars have brought to light the fact that Kant's notion of containment, far from being an obscure metaphor, is based on several concepts that were perfectly intelligible for Kant's contemporaries. The processes involved are, as we have seen, those of logical division of concepts and logical abstraction via analysis; notions of being 'contained in' and 'contained under' and

[58] Capozzi (2002, pp. 203-204) shows that the term "Kanon" appears quite late in the Kantian lexicon and that before its introduction (certified around the first half of the Seventies) Kant used expressions such as 'cathartic' or 'critic'. While a cathartic can only correct mistakes, a canon can also impose rules, which in turn do have a normative impact.

[59] This point is explicated in the Jäsche Logic: "Just because it does abstract wholly from all objects, however, it also cannot be an organon of the sciences" (JL, par. 13, p. 528).

the related ones of 'concepts containment' and 'concept extension'; last, the concepts of species and genera. Now, in Section 1.2.1, we have underlined, through our review of the *Jäsche Logic*, that all these elements, according to Kant, are part of pure general logic. Not only are these notions 'logical', but the theory of concepts, to which these notions belong, is essentially the theory of concept containment and, as Anderson (2015, pp. 51-54) explains, both the theories of judgments and of inferences are grounded on it. In other words, the fundamental notions of Kant's definition of analyticity via containment are at the core of (traditional and) Kantian logic. If we add to this result the facts that, as we have argued through all Section 1.1, the definition of analyticity via containment is the central one and that all the other criteria are based on the latter[60], we have all the elements to conclude that logic is the fundamental instrument that Kant employs for drawing his analytic-synthetic distinction, since it provides the basic notions for his definition[61].

Kant's pure general logic is an instrument not only for defining the analytic-synthetic distinction, but also for applying it. In Section 1.1.3, we have shown that the principle of non-contradiction, which is, needless to say, a fundamental principle of logic (even from Kant's point of view, as for example our review of the *Jäsche Logic* in Section 1.2.2 testifies), is an instrument for ascertaining the truth of analytic judgments and is appointed with an epistemological function. In other words, the principle of contradiction allows us to establish the truth of analytic judgments and, in so doing, to conclude that certain judgments are analytic: this is because, as we argued above, in an affirmative analytic judgment we find a contradiction between the concept of the subject and the concept of the negation of the predicate.

1.3.2 The status of logic

Once we have completed our investigation about logic as an instrument, we move to analyze the more complex issue about the status that Kant ascribes to pure general logic as far as the analytic-synthetic distinction is concerned. The majority of Kant's scholars maintains, and often takes for granted, that pure general logic is, according to Kant, analytic. Recent examples of this interpretational trend are those of Hanna (2001, p. 140), who claims that "Kant also holds that all the truths of logic - that is, all the truths of what he regarded as logic - are analytically true", and Anderson (2015, p. 103), who explicitly states, while examining the problems that emerge accounting for logical truth, that "Kant's formal general

[60]Either because they can be reduced to it (clarification); or because they are based on it (identity); or because they are not, strictly speaking, definitions (contradiction).

[61]This is the same conclusion attained by Anderson's (2015) and de Jong's (2010) works.

logic [...] is analytic". Moreover this reading of Kant's position, which soon became traditional, traces back to first eminent interpreters of the *Critique*, such as Bolzano and Frege. The former holds, in de Jong's (2010, p. 249) translation of an excerpt in Section 315 of his *Wissenschaftslehre*, that "as regards logic, Kant claims that it [that is, pure general logic] consists of nothing but analytic propositions and thus needs no intuitions for its cognition".

That logic is, according to Kant, analytic is a conclusion that is usually (and incorrectly) deduced either from the fact that logic is the fundamental instrument Kant uses to draw the analytic-synthetic distinction (see Section 1.3.1) or from the fact that the principle of contradiction, which is the grounding principle of logic, is said by Kant to be the "supreme principle of all analytic judgments" (see Section 1.1.3). But a closer inspection reveals that this kind of deductions is *non sequitur* and that neither of the two premises is sufficient to infer that logic is analytic.

Nor are the excerpts usually quoted in support of this view. Hanna (2001, p. 140) refers to the following passages of the *Critique*: CPR, A 59-60/B 83-84 and CPR, A 151-152/B 190-191. In the former Kant is dealing with the distinction between 'analytic' and 'dialectic' in pure general logic. After having said that logic is formal, Kant asserts that the part of logic that concerns the formal rules of the understanding and of reason "can therefore be called *an* analytic" (emphasis added). That part of logic is said to be an analytic in contraposition to the dialectic: here Kant follows the tradition, as he explicitly states (although the meaning of the term 'dialectic' that Kant attributes to the ancient philosophers is controversial). The second excerpt is that dedicated to the principle of contradiction as the supreme principle of analytic judgments, which we have already analyzed in Section 1.1.3.

Anderson (2015, p. 103) calls instead the following excerpts in support of the traditional thesis: CPR, A 65-66/B 90-91; CPR, A 76/B 102; and CPR, A 151-154/B 190-193. In the first passage, Kant is simply stating, we think, that logic concerns the decomposition via analysis of concepts: but, again, this is different from saying that logical judgments are analytic according to Kant. In the second passage, Kant claims that "general logic abstracts from all content of cognition, and expects that representations will be given to it elsewhere, wherever this may be, in order for it to transform them into concepts analytically". The point here is that logic is formal and that the process of forming concepts from representation does not need any appeal to intuitions. The third passage is again that about the principle of contradiction, the lines added concern the formulations of the principle and, again, do not affirm that logical judgments are analytic.

The reason why scholars cannot display excerpts in which Kant holds without a doubt that logic is analytic is, we argue, the fact that Kant does not apply the analytic-synthetic distinction to logic at all[62]. While he explicitly affirms that judgments of experience, mathematics, natural science and metaphysics are all synthetic (see note 4), Kant does not speak about the status of logic. We think, first, that this silence indicates that, according to Kant's view, the distinction *cannot* be applied to logic or that this application does not yield interesting results and, second, that the grounds for this reticence must be searched in Kant's peculiar conception of pure general logic as a canon (that is, as consequence of the features of generality, formality and pureness that we have described in Section 1.2.2). In particular, we propose two arguments: the former starts from the generality; the latter from the formality of logic.

First, recall Tolley (2013)'s interpretation of Kant's notion of the generality of logic (as it is explained in Section 1.2.2). If this reconstruction is right, then Kant does not share with Frege the idea that logic is concerned with thoughts understood as what is capable of being true and thus does not maintain that logic regards exclusively what can be true. On the contrary, Kant's logic contemplates for example also imperatives and aesthetic assessments. But we have seen in Section 1.1 that the analytic-synthetic distinction is meant to apply only to true judgments: as a result, one of the reasons for Kant's reticence in applying the distinction to logic is that true judgments are only a proper subset of logic. Second, as we have shown in Section 1.2.2, the formality of logic means that logic abstracts from the content of thinking and implies that logic cannot extend our knowledge of reality, of objects. Although it is a science, logic as propadeutic is not a kind of knowledge:

> [...] hence logic as a propadeutic constitutes only the outer courtyard, as it were, to the sciences; and when it comes to information, a logic may indeed be presupposed in judging about the latter, but its acquisition must be sought in the sciences properly and objectively so called (CPR, B ix).

Logic is a propadeutic because it precedes any kind of knowledge: its rules have to be learnt and respected as a *conditio sine qua non* of any cognitive enterprise. This could be another reason for Kant's silence on the status of logic, for in drawing his analytic-synthetic distinction, Kant's interests seem to stay with doctrines having a content of knowledge, such as mathematics or sciences[63].

Although Kant, as a matter of fact, does not apply his analytic-synthetic distinction to logic, we can still ask ourselves whether logical judgments are analytic

[62]This is the same conclusion put forward by de Jong (2010, p. 250), who states: "for Kant, in the end the distinction between analytic and synthetic judgments does not apply to logic".

[63]A similar perspective is put forward by de Jong (2010, p. 250).

or synthetic *a priori* following Kant's definitions of the terms and beyond his reasons for leaving the matter unsolved. In other words, we now move from the actual application to the possible applicability of the distinction to logical judgments: we are going to attempt an analysis that, as we have shown, Kant did not want to pursue. Two are the main theses that we support in this regard: following Kant's definitions, we have that, first, no logical judgment is synthetic; second, at least some logical truths are not analytic.

The first claim could seem trivial for what has been said until now, but it needs to be proved, for it has been strongly criticized. As we will see in due course, Hintikka (1973) holds that logical arguments deploying existential instantiation would count, according to Kant's perspective, as synthetic *a priori*: as a result, Kant would have classified as synthetic a well-defined subset of logical truths of what we now call 'polyadic quantification theory'. Analogous suggestions can be found in recent work put forward by D'Agostino (2013), who individuates a class of propositional truths that can be said to be synthetic and briefly hints that this conception could offer a "partial vindication" of Kant's view[64].

The thesis that according to Kant logical judgments are not synthetic is a conclusion that can be supported by the following two arguments[65]. The first piece of evidence results from checking whether Kant's definition of true synthetic *a priori* judgments applies to logical judgments. In Section 1.1.4, we have seen that in a synthetic judgment the relation of the subject to the predicate cannot be but indirect and it consists in linking the two concepts to one another by connecting them to a third different element. The third element, which is indispensable for the truth of any synthetic judgment, is for Kant an object. But the appeal to an object for a logical judgment is what is explicitly excluded by the feature of formality that characterizes logic: as we have seen in Section 1.2.2, that logic is formal according to Kant means that this discipline abstracts from any relation to objects.

The second argument is *ad absurdum* and shows that if logic were synthetic, then the analytic-synthetic distinction would collapse[66]. Assume that logic is synthetic: then it follows that the principle of non-contradiction is synthetic too. Now, we have seen in Section 1.1.3, that Kant regards the principle of non-contradiction

[64]The two works mentioned above will be amply discussed in this thesis. The point that we want to underline is that we do not hold, against Hintikka and D'Agostino, that logical truths are analytic. Rather, we maintain that that some logical truths are synthetic a priori is a Kantian position not in the strict sense that it is a thesis that Kant has or would have subscribed, but only in the loose sense that it employs notions and concepts that have been formulated in the *Critique*.

[65]In due course, we will also provide our reasons against Hintikka (1973)'s interpretation of Kant's position, which will strengthen our argument.

[66]A similar argument can be found in de Jong (2010, p. 249).

as the supreme principle of analytic judgments, since the judgments in that class can be proved on the basis of that principle. From the hypothesis ad absurdum together with this premises it follows that analytic judgments can be proved from a synthetic principle. Now we take, as a third assumption, something that Kant explicitly states in the *Introduction* to the first *Critique* (CPR, B 14), namely that what can be proved from a synthetic judgment is itself synthetic. But then we have as a conclusion an absurdity, namely that analytic judgments are synthetic.

The two arguments show that holding that logical judgments are synthetic explicitly contradicts some Kantian thesis: on the one hand, either the definition of synthetic judgments itself or the conception of logic as formal; on the other hand, either the idea that the principle of non-contradiction is the supreme principle of analytic judgments or the rule that what can be proved from a synthetic judgment is itself synthetic.

The second claim, that is, the thesis that according to Kant at least some logical truths are not analytic[67], has been variously acknowledged[68]. The crucial point is that Kant's definition of analytic judgments via containment is restricted, as we have argued in Section 1.1.2, only to *categorical* propositions, while there are logical truths that do not have[69] and cannot be reduced[70] to such a form. The first class of truths that are not analytic in so far as irreducible to the form 'S is P' includes all those validities turning essentially on relations. While this kind of truths is clearly part of modern logic, traditional logic was instead, as we have seen in Section 1.2.1, intrinsically monadic in character, since it was not equipped with dedicated instruments for handling relations. The obstacle to the development of a logic of relations in the proper sense must be searched in the ontology of relations. For, since the Middle Ages, it was generally refused that polyadic expressions of

[67]Notice that while it is possible to give a compact (and, as we have seen, negative) answer to the question of whether logical judgments are synthetic *a priori*, the answer to the issue of whether logical judgments are analytic requires a subdivision of logical judgments into classes. De Jong (2010, p. 249) proposes instead an argument for the whole class of logical judgment that, we maintain, is fallacious. De Jong holds that if logical principles were analytic, then there would be a violation of the rule that Kant puts forward in his *Introduction* to the *Critique* for which theoretical science *a priori* must be founded on synthetic principles a priori. However, we think that the assumption of the analyticity of logic does not lead to the conclusion that de Jong pinpoints, but rather to the conclusion, in accordance with the rule mentioned above, that logic is not a theoretical, but a practical science (since for sure it is *a priori*). But de Jong in the following page of his article uses the practical character of logic not to infer its analyticity, but rather to claim the inapplicability of the analytic-synthetic distinction to logic.

[68]See, for example, Anderson (2015, Section 4.2).

[69]The problem of accounting for logical truths is one of the main issues that led interpreters to reject the very restriction of Kant's definition using the strategies that we have indicated in Section 1.1.2. On this point see for example Hanna (2001, p. 145 and ff.).

[70]Kant's reasons for maintaining the irreducibility of non-categorical propositions to categorical ones has been explained in Section 1.1.2.

the language referred to some kind of polyadic property of the external world[71]. As a result, truths turning essentially on relations are not analytic, in so far as they cannot be reduced to categorical propositions; but it is still possible for them to be synthetic, in so far as they were excluded from the domain of the logical, which is, as we have argued above, the domain of the non-synthetic.

But non-analytic truths are not only truths that Kant would not have considered as logical: for there is a second class of truths that are not reducible to categorical propositions and at the same time do indeed belong to pure general logic even for Kant. This is the class that includes propositional truths such as *modus ponens*. Propositional truths cannot be turned into categorical judgments because the truth of the former relies on the relation between judgments independently of the concepts involved. Nevertheless, inferences as *modus ponens* or *modus tollens* and the like are for sure logical arguments: as our review of the *Jäsche Logic* in Section 1.2.2 has shown, Kant includes them into the theory of inferences that he presents in the *Universal Doctrine of Elements*. As a result, *there is at least one class of logical truths that are neither analytic nor synthetic* or, equivalently, to which the analytic-synthetic distinction does not apply[72].

Most scholars at this point plainly close affirming that all but non-categorical logical truths are analytic according to Kant[73]. However, we think that this conclusion is not completely justified. For example, we have seen in Section 1.1 that the fundamental criterion behind Kant's notion of analyticity is that of containment and we have underlined that judgments that are said to be analytic according to the containment criterion (as well as according to the clarification one) *must* have a cognitive content: they are not trivial or tautologous. As a consequence, not only non-categorical truths, but also categorical and identical truths would count, according to Kant, as neither analytic nor synthetic. And no argument so far has excluded the possibility that there are other classes of logical truths that are neither analytic nor synthetic.

To sum up, against the traditional view according to which Kant maintains that logic is analytic, we have argued that Kant does not apply his analytic-synthetic distinction to logic and that the grounds for this silence about the status of logic has to be searched in Kant's peculiar conception of the discipline. Even

[71]See Mugnai (2016).

[72]After having stated that logical truths based on relations and hypothetical truths as *modus ponens* are not analytic according to Kant's distinction founded on containment, Anderson (2015, p. 107) concludes that "many truths of our formal logic - especially, as we have seen, in polyadic quantification theory count as synthetic, given Kant's version of the distinction". We think that this conclusion is too quick (and, as we have claimed, wrong): in order to reach this result, Anderson should have proved that those kinds of inferences are synthetic and not simply non-analytic. The two are not equivalent exactly because the distinction is not exhaustive.

[73]This is for example Anderson (2015, p. 107)'s conclusion: "analyticities express only a fragment of general logic".

attempting an analysis that Kant did not think it was worth pursuing, we have maintained that no logical judgment is synthetic *a priori* and that some (if not all) logical judgments are not analytic. In other words, following Kant's definition of the analytic-synthetic distinction, we have the unexpected result that many logical judgments are neither analytic nor synthetic.

Chapter 2

Bolzano and the syntheticity of logic

2.1 Bolzano's analytic-synthetic distinction

2.1.1 Preliminary notions: the method of substitution, validity, derivability

One of the most innovative conceptions of Bolzano's work concerns the aim and the domain of logic. At his time, most philosophers did not recognize any clear boundary between logical investigations and psychological considerations on the way in which logical knowledge is to be attained. This was for example also Kant's case. Against this psychologistic trend in the study of logic, Bolzano is very careful in distinguishing the purely logical domain from the epistemological perspective. This attention is also reflected in the articulation of his main logical text, the *Wissenschaftslehre*, published in 1837, which can be seen as divided into two halves: the former, made up of the *Fundamentallehre* and the *Elementarlehre*, concerns logic as a science independent of human understanding; the latter, constituted by the theory of knowledge, the art of discovery and the theory of science in the strict sense, focuses on logic as an object of human knowledge. As a result of his anti-psychologistic assumptions, Bolzano postulates a logical realm of purely logical objects, which has to be distinguished from both the world of material objects and the world of mental entities. Among the inhabitants of this 'World 3', which exist despite their non-mental and non-linguistic nature, we find the protagonists of the *Theory of Science*: 'propositions in themselves' (*Sätze an sich*) and their parts, 'ideas in themselves' (*Vorstellungen an sich*). Subjective propositions, such as judgments, and subjective ideas are of course related to propositions and ideas in themselves respectively, for the latter are the matter of the former; similarly, sentences, that are sequences of signs of a certain language, are but expression of the meaning of propositions. Nevertheless, logical objects in themselves have a different ontological status.

Bolzano finds the most interesting properties of propositions in themselves through a new and fruitful procedure: the method of substitution. It consists in studying how the variation of one or other of the ideas occurring in a proposition affects its truth value: the result of this analysis is the determination of different types of semantic regularities. According to Bolzano, this idea is just a systematization of a common way of reasoning about context-sensitive sentences[1]:

> [E]ven if [...] we do suppose certain ideas in a given proposition to be variable, often without being clearly aware of it, and then consider the relation to the truth that follows for this proposition upon filling those variable places with whatever different ideas, it is always worth the trouble to do this consciously and with the definite intention of becoming the more precisely acquainted with the nature of the given propositions by observing this relation of theirs to the truth. Namely, if we consider in a given proposition not merely whether it is itself true or false, but also what relation to the truth follows for all the propositions that develop out of it when we assume certain of the ideas present in it to be variable and permit ourselves to exchange them for whatever other ideas, we shall be led to discover many extremely remarkable properties of propositions (TS, §147).

It is necessary to make some observations to specify Bolzano's method of substitution. First of all, the logical properties that are defined through the variation procedure must to be predicated of sets of propositions, not of propositions taken singularly: in other words, these regularities characterize propositional forms, that is to say, entire sets of propositions that share the property of being the result of varying the same idea in the same proposition[2]. Moreover, it is worth noticing that every proposition admits of different propositional forms, for in a proposition there is usually more than just one idea that can be varied. Second, there must obviously be some constraints on admissible substitutions, although Bolzano is not always explicit on this issue[3]. For example, the result of replacing an idea for

[1]On Bolzano's account, sentences whose truth-value depends on particular contexts are underdetermined because they do not reveal their *Sinn* of the proposition completely. In order to make them explicit, they must be paraphrased by sentences that express their propositional content explicitly. More on Bolzano's paraphrastic approach will follow.

[2]Once we have specified this point, we are going to use the term 'proposition' to refer not only to single propositions, but also to propositional form. This abuse of terminology, which is widespread in the literature, is justified by expositional reasons.

[3]These observations are due to Künne (2006).

something else must still be a proposition: ideas cannot be replaced by a string of symbols whatever. But neither the idea to be replaced can be chosen randomly: in particular, a notion that has proper parts that occur in the proposition is not a good candidate. Then, in the *Wissenschaftslehre* it is usually admitted simultaneous substitutions of more than one notion at a time. Third, Bolzano imposes two requirements on what counts as a variant of a proposition with respect to some of its ideas. The former is that the subject idea of the proposition that results from the substitution must be 'objectual' (*gegenständlich*) or non-empty: this means that it must have, represent or refer to one or more objects. The objectuality constraint is a consequence of Bolzano's theory of truth, according to which every proposition with objectless ideas is simply false. The latter restriction is that in order to be counted as a variant of a proposition, ideas cannot be replaced by notions that are equivalent to the original ones or to the ones characterizing other variants[4].

The objectuality and non-equivalence constraints play a fundamental role in the first logical property that Bolzano defines on the basis of the method of substitution. The 'degree of validity' of a proposition relative to a certain set of variable notions is given as the ratio of the number of true variants to the number of objectual and non-equivalent substitution instances. To put it another way, the degree of validity amounts to the probability of a certain proposition and it is a fraction between 0 and 1. For example, the proposition 'the ball numbered 8 will be among those drawn in the next lottery' has degree of validity $5/90 = 1/18$, if five balls are drawn, "for then among all of the 90 propositions that come into the picture in this case, there are only 5 which are true"[5].

A proposition is said to be 'universally valid' (*allgemeingültig*) if its degree of validity equals 1 and it is said to be 'universally invalid' (*allgemeinungültig*) if its degree of validity is 0. So, the proposition 'The man Caius is mortal' is universally valid with respect to 'Caius', because all the objectual substitutions of 'Caius', such as 'Sempronius' and 'Titus', make the proposition true; while the proposition 'The man Caius is omniscient' is universally invalid with respect to 'Caius', because no matter what notion is substituted for 'Caius' the resulting proposition is false. Notice that without the objectuality constraint it would be possible in virtually every case to generate false variants of a true proposition, with the result that there would not be any proposition universally valid with respect to its subject idea. Similarly, the non-equivalence constraint is fundamental in computing the effective number of all possible and objectual substitution instances of a proposition, that is to say, the denominator of the ratio that defines the degree of validity of that

[4]Another feature of the variation procedure that it is worth to mention is that it requires the exhaustiveness and uniformity of substitutions. Nevertheless, Rusnock (2013, p. 324) shows that Bolzano's example includes cases where only some occurrences of a given idea are varied.

[5]TS, §147.

proposition: without this requirement, the variants with the same meaning could be counted several times to the effect of decreasing the degree of validity of the proposition.

Bolzano uses the method of substitution not only for defining logical properties of propositions (or propositional forms), as in the case of validity, but also for specifying inferential notions that concern sets of propositions (or sets of propositional forms). The first step of this generalization from propositions to sets of propositions is the notion of compatibility. Propositions $A, B, C, D, \ldots$ are 'mutually compatible' (*verträglich*) with respect to the ideas $i, j, \ldots$ that occur conjointly in them if there is a sequence of ideas that could be set in the place of $i, j, \ldots$ that make these propositions all true at the same time. The propositions 'This flower has a red bloom', 'This flower smells good' and 'This flower belongs to the 12th class of Linnaeus' system' are compatible with respect to 'This flower', for all three propositions become true if 'This flower' is replaced by the idea of a rose[6]. The fundamental logical notion of derivability (*Ableitbarkeit*) is given as a special case of compatibility:

> I say that the propositions $M, N, O, \ldots$ are ableitbar from the propositions $A, B, C, D, \ldots$ with respect to the variable parts $i, j, \ldots$ if every set of ideas that can be put in the place of $i, j, \ldots$ and that makes all of $A, B, C, D, \ldots$ true, also make all of $M, N, O, \ldots$ true (TS, §155).

Many are the differences between Bolzano's notion of derivability and the modern one[7]. First, as a consequence of the dependence of the notion of derivability on that of compatibility, it turns out that it is impossible to take as premises contradictory (and thus objectless) propositions, although it is still possible to deduce from false but non-contradictory premises. Second, since it allows not to vary non-logical concepts, Bolzano's relation of *Ableitbarkeit* does not distinguish between arguments that preserve truth and argument that do so by virtue of their form[8]. Third, the relation defined in the *Wissenschaftslehre* is not monotonic and, while it does not validate the law of contraposition, it guarantees the validity of the principle of subalternation[9].

2.1.2 Bolzano's conception of analyticity

Bolzano's anti-psychologistic conception of propositions in themselves, the method of substitution, the notions of validity and derivability are indispensable elements

[6]TS, §154.

[7]The historical significance and the interpretative trends of Bolzano's work is largely discussed in Section 2.3.2.

[8]More on this issue will be said when discussing the notion of 'logical analyticity'.

[9]These considerations are analyzed in Lapointe (2011, pp. 77-80).

to understand the analytic-synthetic distinction as defined in §148 of the *Theory of Science*. This Section is structured into three paragraphs and four notes. We are going to consider one portion of the text at a time: for each of them, we first provide a summary and then a commentary articulated in several observations.

In the first paragraph, Bolzano gives the definition of analytic and synthetic propositions in the following terms:

> But suppose there is just a *single* idea in it [i.e., in a proposition] which can be arbitrarily varied without disturbing its truth or falsity, i.e. if all the propositions produced by substituting for this idea any other idea we pleased are either true altogether or false altogether, presupposing only that they have denotation. This property of the proposition is already sufficiently worthy of attention to differentiate it from all those propositions for which this is not the case. I permit myself, then, to call propositions of this kind, borrowing an expression from Kant, *analytic*. All the rest, however, i.e. in which there is not a single idea that can be arbitrarily varied without affecting their truth or falsity, I call *synthetic* propositions (TS, §148).

According to this definition, a propositional form P is analytic with respect to the ideas $i, j, \ldots$ if and only if either every objectual variant of P with respect to $i, j, \ldots$ is true or every objectual variant of P with respect to $i, j, \ldots$ is false; it is synthetic otherwise. As a result, a proposition P is analytically true with respect to the ideas $i, j, \ldots$ if and only if it is universally valid with respect to the same ideas; similarly, a proposition P is analytically false with respect to the ideas $i, j, \ldots$ if and only if it is universally invalid with respect to the same ideas. So, for example, the propositions 'A morally evil man deserves no respect' and 'A morally evil man nevertheless enjoys eternal happiness' are both analytic with respect to the idea 'man' (the former is analytically true, the latter is analytically false), while the propositions 'God is omniscient' and 'A triangle has two right angles' are both synthetic (the former is true, the latter is false, but for neither of them there exist some ideas, which could be arbitrarily varied without affecting their truth value).

We organize our comment on the *incipit* of §148 around three main issues: first, the relation between analyticity and truth values; second, the connection between analyticity and syntheticity; third, the link between analyticity and conceptuality. The last point is the crucial one and requires a detour explaining Bolzano's conception of the *a priori*-*a posteriori* distinction.

Analyticity and truth values. A characteristic feature of Bolzano's definition is that the analyticity and syntheticity of a given proposition P depends on the behavior of P with respect to its truth value (once certain variations in P occur). As Proust observes, "the relationship of the analytic to truth is exactly reversed"[10]. In the *Critique*, not only the analytic-synthetic distinction applies solely to *true* propositions, but also the very fact of being analytic implies for a proposition that it is true, for a proposition in which the predicate is contained in the subject cannot be false. Thus, for Kant, analytic propositions are a subset of true propositions. On the contrary, Bolzano admits false propositions among analyticities (and syntheticities) and explicitly recognizes that this is an innovation proper of his account[11]. False analyticities and syntheticities are possible in Bolzano's approach because his definition does not link this distinction to the notion truth, but rather to the concepts of validity and, crucially, invalidity given through the substitution method.

Analyticity and syntheticity. Another distinctive consequence of Bolzano's definition is that there is a close relationship between analytic and synthetic propositions: in particular, analytic propositions can be derived from synthetic ones and, vice versa, synthetic propositions can be derived from analytic ones. As an example of the former type of derivation, Bolzano shows that the synthetic truth 'In each triangle the sum of its angles equals two rights' entails 'In each equilateral triangle the sum of its angles equals two rights'[12]. As an example of the latter type of derivation, Künne shows that the synthetic proposition 'There was at least one Roman Catholic' is *ableitbar* from the analytic truth 'Professor Bolzano who took part in a secret meeting in Tiechobus in September 1838 was Roman Catholic' with respect to 'Roman Catholic'[13]. The reciprocal *Ableitbarkeit* of analytic and synthetic propositions is even more remarkable if we remind that Kant in the *Introduction* of the first *Critique* clearly states that synthetic propositions can be derived only from synthetic truths. We will see in Section 2.3.1 that this characteristic of Bolzano's account plays a decisive role in his philosophy of logic.

As far as the relations between analyticity and syntheticity are concerned, it is worth mentioning two more facts already stressed in the literature. First, two propositions may necessarily have the same truth value although one is synthetic and the other analytic. For example, while the proposition 'In each 3-angle the sum of its angles equals 2 rights' is synthetic, the proposition 'In each 3-angle the sum of its angles equals $(3-2) \times 2$' is analytic with respect to '3'[14]. Second, the

[10]Proust (1989, p. 62).

[11]"I thought it useful to interpret both concepts, of analytic as well as synthetic propositions, so broadly that not only true but false propositions could be included under them" (TS, §148).

[12]TS, §116.

[13]Künne (2006, p. 195).

[14]This example is due to Künne's (2006, pp. 194-195) interpretation of a passage in Hugo Bergmann's monograph on Bolzano dated 1909.

result of incorporating an analytic statement as part of a more complex statement is likewise analytic[15]. For example, if P is analytic with respect to the idea i and Q is synthetic, then $P \vee Q$ is analytic provided that i does not occur in Q. With this feature, Bolzano distances himself once more from the Kantian tradition.

Analyticity and conceptuality. According to Bolzano, ideas, which are parts of propositions that are not themselves propositions, can be distinguished with respect to their extension (the objects that fall under them) and to their content (the parts of which they are composed). Intuitions are defined as ideas that are singular with respect to extension, that is to say, they only have one single object as their extension, and simple with respect to content, that is to say, they do not have parts. There are three kinds of complex ideas: first, complex ideas made up solely of intuitions and called 'pure intuitions'; second, complex ideas in which no intuition occurs and that are called 'concepts'; third, 'mixed ideas', that are complex ideas in which both intuitions and concepts occur. Two kinds of propositions can be individuated on the basis of the type of ideas that occur in them. On the one hand, 'conceptual propositions' are defined as propositions that are composed only out of concepts and do not contain any intuition; on the other hand, 'empirical propositions' are propositions in which at least one constituent is an intuition and that contain demonstratives, indexicals, proper names of natural kind terms.

As Bolzano explicitly acknowledges, Kant's *a priori-a posteriori* distinction corresponds to his conceptual-empirical division. With an essential proviso however. The Kantian concepts belong to the *ordo cognoscendi* and are predicated primarily of knowledge: cognitions are said to be *a priori* if they can take place without experience and *a posteriori* otherwise. As Roski correctly underlines, "Kant thus clearly starts his analysis on the level of judgments [...] and uses a distinction drawn there to introduce a related one on the level of propositions"[16]. On the contrary, Bolzano, who is completely aware of this feature of Kant's distinction, believes that the attention paid to the epistemological side should not "suppress another one which does not depend on the mere relationship of propositions to our cognitive faculties, but on their intrinsic character"[17]. For this reason, he elaborates the conceptual-empirical distinction, which applies primarily to propositions and depends on the constituents of propositions. Only once this objective distinction has been drawn, it can be used to explain the epistemic properties of those judgments, whose matter are conceptual or empirical propositions. This also explains how the Kantian and the Bolzanian distinctions are related. In Bolzano's

[15]This observation is given in Hale and Wright (2015, pp. 339-340).
[16]Roski (2013, p. 105).
[17]TS, §133.

words:

> Now, it happens, however, that this division of our knowledge almost coincides with the division of propositions into conceptual and empirical propositions, since the truth of most conceptual propositions can be decided by mere thought without any experience, while propositions that include an intuition can in general be judged only on the basis of experience (TS, §133).

On the one hand, the truth value of empirical propositions can be discovered only *a posteriori*: experience is indispensable to grasp intuitions because the latter have empirical import and imply causal epistemic transaction with the world. On the other hand, Bolzano suggests that conceptual truths are to be known *a priori*, if they can be known at all[18].

This detour was essential to understand a fundamental feature of the conception of analyticity in the *Wissenschaftslehre*: the analytic-synthetic distinction and the conceptual-empirical distinction are conceived in such a way that they cut across one another. As a result, in Bolzano's system there is space not only for conceptual analyticities and empirical syntheticities, but also for two other controversial kinds of propositions: conceptual syntheticities and, crucially, empirical analyticities. Examples of the former kind are propositions such as 'Each proposition contains at least three parts' and 'Some notions are complex': each of these propositions lacks an idea, which can be arbitrarily varied without affecting its truth-value, and, at the same time, can be known *a priori* because no intuition occurs in it. Although he harshly criticizes the doctrine of pure intuition of the *Critique*, Bolzano retains Kant's category of the synthetic *a priori* and, as we will see in Section 2.2.1, its central role for deductive sciences.

But the most innovative concept in Bolzano's system is the analytic *a posteriori*. Examples of this kind of propositions are the following 'This, which is a drake, is male' and 'This triangle has the property that the sum of its angles equals two right angles'[19]: the idea expressed by 'this' can be varied *salva veritate* (and thus the propositions are analytic) and, at the same time, is an intuition (and thus the propositions are empirical). Analytic *a posteriori* propositions can be true (or false) by virtue of whatever is the case, contingent facts or states of affairs. For instance, in order to know that the proposition 'Albino Luciani, who was a Pope of the Twentieth century, had white skin' is analytic with respect to 'Albino Luciani',

[18]As Lapointe (2014a, p. 102) has emphasized, this conclusion is not uncontroversial, because in certain passages Bolzano suggests that some mathematical truths can be known only by induction, that is, by a kind of experience. But excerpts like these can be read as concessions to the limitations of human capacity.

[19]The first example is due to Roski (2010, p. 108) and the second to Proust (1981, p. 226).

it is not sufficient to know the meaning of the terms, but it is also necessary to have some historical cognitions.

The category of the 'analytic *a posteriori*' represents a radical break within the philosophical tradition: in so doing, Bolzano interrupts the privileged connection between, on the one hand, analyticity and, on the other hand, both *apriority* (as far as the epistemological side is concerned) and necessity (if seen from the metaphysical perspective). The same destiny is shared, after all, also by the notion of *Ableitbarkeit*, that may pick out inferences that are merely materially valid or have mere empirical generality. This feature of the *Wissenschaftslehre* has been frequently read as the major mistake of Bolzano's logical work. The critics' disappointment has been efficaciously summarized by Lapointe in the following terms:

> By admitting 'analytic' propositions that would be known *a posteriori*, Bolzano seems to have missed an important point about what we usually take to be the nature of analytic knowledge. The notion of analyticity should aim at providing an objective criterion on the basis of which one may account for cognitions whose justification is entirely independent of empirical data. But this is precisely the insight which Bolzano's broader notion of analyticity would seem unable to capture (Lapointe, 2011, p. 66).

The appraisal and the interpretation of this characteristic of Bolzano's notion of analyticity has been at the core of a lively debate that we are going to examine in Section 2.3.2.

2.1.3 The notion of logical analyticity

We now return to §148 of the *Theory of Science*. In the second paragraph, Bolzano provides several examples of analytic propositions, namely, 'A is A', 'A, which is B, is A', 'A, which is B, is B', 'Every object is either B or not B', and acknowledges that propositions of the form 'A is A' are usually called identical or tautological. In the third paragraph, Bolzano maintains that the examples of the second paragraph are different from those of the first one, inasmuch:

> Nothing is necessary for judging the analytic nature of the former besides logical knowledge, because the concepts that make up the invariant part of these propositions all belong to logic.

He then proposes to call propositions such as those of the second paragraph 'logically analytic' or 'analytic in the narrow sense' and propositions such as those of the first paragraph 'analytic in the broader sense', although he recognizes that the distinction proposed is ambiguous "because the domain of concepts belonging to logic is not so sharply demarcated that no dispute could ever arise over it".

In our comment, we distinguish two different questions. The former concerns the way in which Bolzano gives this definition and specifies some features of the concept of logical analyticity. The latter deals with the role of logic and logical truths with respect to the narrow version of Bolzano's analyticity.

Bolzano's definition of logical analyticity. Some commentators suggest that the third paragraph of §148 "has no definitional value" because the distinction between analyticities in the broader sense and analyticities in the narrow sense is "founded not 'in itself' but with respect to our mode of acquiring knowledge"[20]. Now, we think that the reason brought in support for this thesis is mistaken. As we have seen in Section 2.1.1, Bolzano usually pays great attention to the distinction between the *ordo essendi* and the *ordo cognoscendi* and the way in which 'logical analyticity' is defined makes no exception. He offers both an epistemological and a logical criterion to individuate logical analyticities: the former states that logical knowledge is sufficient to judge about these entities; the latter affirms that "the concepts that make up the invariant part of these propositions all belong to logic". Moreover, Bolzano underlines with the conjunction 'because' the fact that the way in which logical analyticities are known is a consequence of the way in which these entities are defined.

The reason why Bolzano's discourse on logical analyticity might lack a definitional value is, as the author makes explicit, the difficulty in establishing which concepts belong to logic and which do not. Notice that by saying that "the concepts that make up the invariant part of these propositions all belong to logic", Bolzano is claiming, contrary to what some scholars have suggested, that only *but not necessarily all* logical concepts remain invariant. For example[21], the proposition 'A man, who is rich and powerful, is rich and powerful' is logically analytic with respect to 'man' and 'rich and powerful' because only logical concepts remain invariant, but notice that not all logical concepts are invariant: the conjunction 'and' that links the two adjectives 'rich' and 'powerful' is a logical concept, but occurs in a varying part of the proposition.

A consequence of this feature that has been variously acknowledged in the literature[22] is that a proposition can be logically analytic only if it contains at least one non-logical notion. For example, the propositions '$\exists x \exists y \neg(x = y) \vee \neg\exists x \exists y \neg(x = y)$', 'Every object is an object' and 'Every proposition, which is true, is true' are not logically analytic because, according to Bolzano, there are no non-

[20]Proust (1989, p. 81).

[21]This example is due to Rusnock (2013, p. 326).

[22]See, for instance, Rusnock (2013) and Künne (2006).

logical concepts occurring in them that can be varied. This is the case even if the respective propositional forms 'Either B or not B', 'A is A' and 'A, which is B, is B' are explicitly said to be logically analytic in the third paragraph of §148. These examples points out in addition that Bolzano's understanding of what amounts to be a logical concept is so broad (although not sharply demarcated) that the ideas 'object', 'true' and 'proposition' are included in the category. In general, Bolzano classifies as 'logical' not only concepts expressed by logical constants, but also notions of formal ontology and metalogical concepts[23].

A traditional and widespread interpretation of Bolzano's text tends to exaggerate the role of 'logical analyticity' at the expense of the notion of 'analyticity in the broader sense'. The reasons behind this critical trend and the historical problems it poses will be amply examined in Section 2.3.2. By now it is sufficient to underline that this kind of readings is hindered by some evident textual elements. First, Bolzano's talking of logical analyticity may be regarded as an imperfect kind of definition because the logical criterion is founded on a notion, that of logical concept, which is not clearly demarcated. Second, a proposition, which is logically analytic, is *ipso facto* analytic in the broader sense and not vice versa. Third, Bolzano's discussion of logical analyticity appears only in the third and last paragraph of the Section dedicated to the analytic-synthetic distinction. For these motives, we do agree with Proust's words:

> Set in the general context of §148, the third subsection appears to have the function of an explanatory commentary on the examples of analytic propositions given. If we take Bolzano literally, the distinction he introduces here has only a clarifying value [...] Consequently, in the third paragraph there is no indication of anything else but a casual remark, the equivalent of which we find again concerning derivability (Proust, 1989, p. 81).

Logical analyticity and logical truths. If we examine whether logical truths are logically analytic or, viceversa, logical analyticities are logically true, we arrive at the following results[24]. On the one hand, some propositions are logically analytic but do not belong to logic. This is the case for propositions, such as the one already mentioned 'A man, who is rich and powerful, is rich and powerful', that do not involve solely purely logical concepts. On the other hand, some propositions belong to logic but are not logically analytic. The examples Bolzano brings for this second class are not only propositions made up of concepts that we, unlike Bolzano, would not count as belonging to logic, such as 'There is at least one notion' and

[23]See Künne (2006, p. 200).
[24]This point has been deeply investigated by Künne (2006, p. 201).

'Some notions are complex'. But there are also cases that invoke propositions that are logical par excellence, such as the syllogistic rule of Barbara that "out of two propositions of the form A is B, and B is C, follows a third of the form A is C"[25], which is said to be synthetic (and thus not only non-logically analytic, but also non-analytic *tout court*). These observations lead us to a turning point of the present Chapter, which will be further specified in the following Sections: according to Bolzano, *logic and (logical) analyticity do not coincide*. This tenet is a characterizing feature of Bolzano's position and it can be surprising only if read from the logical positivistic perspective that logic is analytic.

But this is not the only feature of Bolzano's theory that would disappoint a logical positivist. According to the author of the *Wissenschaftslehre*, logical analyticity does not coincide with triviality. For sure, some logically analytic propositions are also trivial: this is the case for instances of the propositional forms 'A, which is A, is B' and 'A is A'. But it is clear that, if we stick to the definition, many logical analyticities turn out to be instructive: this is the case for any propositional form that exceeds certain levels of complexity. Bolzano recognizes this point when he claims that "[n]ot every analytic truth goes without saying, so that trying to communicate it to anyone would be entirely superfluous"[26]. The lack of any reliable connection between logical analyticities and triviality can be explained by the fact that Bolzano's logical analyticity is not an epistemic notion: it is primarily defined in objective terms, whose reflection in the *ordo cognoscendi* does not invoke the trivial-instructive opposition, but rather the distinction among logical and non-logical concepts.

2.1.4 Language independence and synonymy

We now return to the Bolzanian text and start analyzing the notes of §148 of the *Wissenschaftslehre*. In the first note, Bolzano warns that "[M]aking a judgment as to whether a proposition as it is expressed in language is analytic or synthetic often demands more than a cursory glance at the words". On the one hand, he points out that a proposition can be analytic without any indication of it by its verbal expression: this is the case of the proposition 'Every effect has a cause'. On the other hand, he recognizes that some propositions are synthetic in meaning even if they sound analytic or tautological: the meaning of the tautological proposition 'Even a learned man is a man' is that even a learned man is fallible. The second and third notes are devoted to identical propositions. After having specified that identity is an intrinsic property of some propositions, while equivalence is a relation that holds between propositions, Bolzano claims that the definition of identical

[25]TS, §315.
[26]TS, §115.

propositions in terms of propositions in which subject and predicate are the same ideas must be rejected because “the proposition, A is A, does not have the idea A for its predicate idea, but rather the corresponding abstractum”.

We now deal with the commentary of these passages. Bolzano assumes that every proposition, no matter of its linguistic expression, is of the form ‘A has b’ and therefore has exactly three parts: a subject idea ‘A’, a predicate idea ‘b’ and the copula ‘has’ (or another form of the verb ‘to have’). Bolzano chooses this copula instead of the more traditional forms of ‘to be’ because the former represents better than the latter that every proposition is the attribution of a property or a relation to a subject and therefore every predicate idea ‘b’ is the idea of an attribute (either a property or a relation). This explains the reason why Bolzano rejects the common definition of identical proposition. However, although Bolzano defines analyticity for propositions, not for sentences, and propositions are all conceived in the ‘A has b’ form, the first note of §148 makes clear that the property of analyticity as given in the *Wissenschaftslehre* is independent of language and of the way in which propositions are expressed. The point is that Bolzano’s analyticity is not bounded to a particular logical form and the recognition that a proposition is analytic is not constrained by a determinate syntactic structure.

Moreover, the observations concerning hidden and apparent analyticities make clear that the resources Bolzano uses to define analyticity are not restricted to the notions of truth and uniform substitution. For, in order to recognize that the proposition in itself expressed by the sentence ‘Every effect has a cause’ is analytic, it is necessary to understand that the meaning of the term ‘effect’ is ‘what is effected by something else’. But this would require Bolzano to explain why the latter expression is the meaning of the former or, in other words, it would require an account of synonymy. As we will show later on, the recognition of this fact would have subjected Bolzano’s theory to Quine’s famous attack against the analytic-synthetic distinction put forward in the well-known article *Two Dogmas of Empiricism.*

2.1.5 Criticisms against Kant’s analysis and analytic-synthetic distinction

In the fourth and last note of §148, Bolzano compares his conception of analyticity and syntheticity with some definitions of his predecessors and makes some observations on his own distinction. After having mentioned Aristotle, Bolzano discusses Locke’s concept of ‘trifling propositions’ and criticizes his idea on several points. The author of the *Wissenschaftslehre* then moves to analyze Kant’s work, to whom he recognizes an eminent role in the history of the distinction. Bolzano accuses Kant’s explanation of “fall[ing] somewhat short of logical precision”: while the identity criterion "is applicable to identical propositions at most", the containment criterion makes use of "figurative forms of expression that do not analyze the concept to be defined". In particular, Bolzano argues that the Kantian definition is too broad, because it includes among the class of analyticities also propositions that are intuitively synthetic. For example, the proposition 'The father of Alexander, King of Macedon, is King of Macedon' turns out to be analytic according to Kant, because the predicate idea, simply repeating one of the components of the subject idea, is a part of, and thus contained in, the subject. At the same time, Kant's definition is, according to Bolzano, too narrow, because its applicability is restricted to propositions of the subject-predicate form. Bolzano adds that the importance of analytic judgments rests on the fact that their truth or falsity "does not depend on the particular ideas of which they are constituted, but remains the same no matter what changes are made in some of them". Moreover, contrary to the positions of many authors before him, he maintains that the analytic-synthetic distinction is not subjective and does not depend on the definitional choice of the individuals.

The examination of the last portion of §148 of the *Theory of Science* is structured into two parts. First, we are going to examine Bolzano's conception of analysis as opposed to the Kantian decompositional one. Second, we offer an interpretation of Bolzano's criticisms of Kant's analytic-synthetic distinction.

Bolzano's criticisms of Kant's conception of analysis. As Lapointe has repeatedly shown[27], Bolzano criticizes the Kantian analytic-synthetic distinction, because he rejects the decompositional conception of analysis on which Kant's theory of containment is based. We have seen in Chapter 1 that the traditional theory of concepts assumes that concepts are made up of constituents. The constituents of a concept must be individuated through a process of analysis that is understood in terms of decomposition or resolution and that starts from the initial concept and arrives to its simple elements. This process of decomposition is framed and regulated by the traditional theory of logical division of concepts that aims at producing Porphyrian hierarchies, in which concepts and their constituents are organized with respect to the notion of containment or inclusion: each genus is contained in its species and each species is contained under its genus. An essential principle of this theory is that conceptual content and logical extension are connected by a relation of inverse proportion.

Bolzano, demonstrating a significant and modern view, rejects the latter tenet pointing out that it is not the case that every idea A, which has a larger content than an idea B, has a smaller extension than B and vice versa. He proposes

[27]See, for example, Lapointe (2010, p. 265), Lapointe (2014, p. 220 and ff.), Lapointe (2011, Chapter 2).

the following counterexample. The proposition ‘A man who understands all *living* European languages’ has at the same time a greater content and a larger extension than the proposition ‘A man who understands all European languages’. But the main flaw of the decompositional conception of analysis is found by Bolzano in one of its assumptions: the correspondence between a concept and its object. The idea Bolzano attacks is an unsophisticated form of representationalism, which assumes that concepts are pictures of the objects they represent and, in particular, that properties of objects correspond to constituents of concepts. Bolzano sides decisively against this perspective:

> I maintain not only that there are various parts of an idea that do not express properties of the corresponding object at all, but that in every object there are also properties, which - even though they belong to it of necessity insofar as it is supposed to fall under a certain idea as object - are by no means conceived of as among the idea's components (TS, §64).

As far as the first claim is concerned, Bolzano argues that the idea of an object requires not only the ideas of its properties, but also some other ideas that serve to connect the latter. In other words, the theory of concepts that results from this assumption is doomed to characterize conceptual contents in terms of unstructured entities, that is, of non-ordered sums of constituents. Numerous are arguments proposed by Bolzano to support the second part of his thesis, but the most efficacious is the following example. The concept of an equilateral triangle does not include the concept of equiangularity, although it may be the case that the latter occurs to us spontaneously in connection with the former.

Bolzano does not restrict himself to the *pars destruens*, but he also proposes an account of analysis alternative to the decompositional conception. As Lapointe has highlighted[28], Bolzano's idea consists in assuming that every sentence utterance can be paraphrased into a proposition that expresses its complete meaning: the process of *Auslegung* or interpretation provides a complete analysis of the initial sentence. Many are the requirements on adequate paraphrases, but the most important of them are the following ones. First, only a proposition, in which no context-relative elements occur, can express the complete meaning of sentences; second, an adequate paraphrase must have the standard form ‘A has b’; third, a proposition is an adequate translation of a certain sentence only if it is not redundant.

Bolzano's criticisms of Kant's analytic-synthetic distinction. We now examine the criticism against Kant's notion of analyticity that Bolzano puts for-

[28]See, in particular, Lapointe (2007).

ward in §148 of the *Wissenschaftslehre* and that we have summarized above. The charge of "fall[ing] somewhat short of logical precision" and the accuse of being too broad are closely interconnected. We have seen that Bolzano, in order to support his latter claim, shows that the proposition 'The father of Alexander, King of Macedon, is King of Macedon' would be regarded as synthetic by everyone and analytic only following Kant's definition. On the one hand, everyone would agree that the phrase 'King of Macedon' that occurs in the subject idea 'The father of Alexander, King of Macedon' does not signify a property of Philip II, Alexander's father, but a property of Alexander. Moreover, the two properties of Philip II mentioned in the proposition under question, namely, being Alexander's father and being King of Macedon, do not seem to be interconnected from a conceptual point of view, but they can be predicated of the same man only as a matter of fact. For these reasons, the example shows a proposition that also Kant himself would have probably been ready to classify as synthetic. On the other hand, the idea 'King of Macedon' occurs in the proposition both as the predicate and as a part of the subject: as a result, the predicate is contained in the subject and the proposition is analytic in the Kantian sense of the term.

Bolzano's example aims to prove that Kant's definition of analyticity is underdetermined: it does not explain the manner in which the predicate must be contained in the subject and does not impose any restriction on the way in which a predicate can be a part of the subject in order for the resulting proposition to be analytic. In other words, according to Bolzano, Kant's notion of analyticity is too broad because the notion of containment on which it is based is left at a metaphorical level and is not completely explained. However, we think that, in elaborating this criticism, Bolzano probably misses the point. As we have seen in Chapter 1 and we have recapitulated above, Kant's notion of containment is not a metaphor, but a precise and technical notion founded on the traditional theory of concepts. In this theoretical frame, which probably prevents to call 'King of Macedon' a part of the concept 'The father of Alexander, King of Macedon', Bolzano's example would be mistaken.

Another charge that Bolzano moves against Kant's notion of analyticity is that it would be too narrow and could apply only to propositions of the form 'A, which is B, is B'. This criticism highlights a feature of Kant's definition that many scholars will regard as problematic: as we have already shown in Chapter 1, the analytic-synthetic distinction of the *Critique* applies only to propositions of the subject-predicate form and, as a consequence, many sentences turn out to be neither analytic nor synthetic. As we have shown in the previous Section, Bolzano does not accept this characteristic of Kant's theory: according to the author of the *Theory of Science*, analyticity and syntheticity must be independent of the logical form in which propositions are expressed and every proposition must be either analytic or synthetic.

Bolzano completes the review and criticisms of Kant's work drawing the following conclusion:

> In general it seems to me that all these definitions fail to place enough emphasis on what makes this sort of judgment really *important*. This, I believe, consists in the fact that their truth or falsity does not depend on the particular ideas of which they are constituted, but remains the same no matter what changes are made in some of them, presupposing only that the proposition's denotative character is not itself destroyed (TS, §148).

With this observation, Bolzano wants to underline the fundamental difference between Kant's and his own conception of analyticity. On the one hand, Kant assumes that there are two ways in which a subject and a predicate are connected in a sentence: in the analytic case, the predicate is contained in the subject concept; in the synthetic case, the two concepts are connected through an object given in intuition. It is the method of analysis, conceived in terms of the decomposition of concepts, that plays a central role and establishes that a sentence is analytic. In this approach, the content of the concepts occurring in a certain sentence is of course fundamental. On the other hand, Bolzano does not distinguish the ways in which concepts are connected in analytic and synthetic propositions. Analysis has no role in establishing which propositions are analytic and which are synthetic no matter if it is understood in the decompositional or the paraphrasitic terms. The real innovation of Bolzano's work on analyticity is that he defines this property of propositions through the substitutional method. A result of this procedure is that the analyticity of a proposition does not depend on the content of the ideas occurring in it, but, on the contrary, on the truth value of that proposition when some of its ideas are varied.

2.2 The science of logic

2.2.1 Grounding, deductive sciences and synthetic *a priori*

The notion of *Ableitbarkeit* that we have examined in Section 2.1.1 does not exhaust Bolzano's account of consequence. To play an equally important role in Bolzano's theory[29] is the concept of *Abfolge*, which is usually translated as 'grounding', 'ground and consequence relation' or 'entailment' and takes up much of the

[29] The relevance of this concept is immediately suggested by the way in which it is introduced: "Of all the relations that hold between truths, the one most worthy of attention in my opinion is that of ground and consequence, by virtue of which certain propositions are the *ground* of certain other propositions and the latter are *consequences*" (TS, §198).

third Chapter of the *Elementarlehre*. Despite its centrality in the *Theory of Science*, Bolzano was never satisfied with his own definition of grounding. He even arrived to suspect that this relation could not find a complete explication[30] and he settled for individuating its fundamental properties by comparison with other notions[31]. In this Section, we are interested in providing an overview of the main features of this notion and in showing the aim for which it has been devised, namely, that of ordering scientific truths into axiomatic systems.

Bolzano introduces the relation of grounding by way of the following example. Consider the propositions 'The thermometer is higher in summer than in winter' and 'It is warmer in summer than in winter'. Not only is the former *ableitbar* from the latter; but also vice versa, for no matter what ideas we substitute for them, only those which make the latter proposition true also make the former true. However, Bolzano argues:

> All the same, it could never occur to anyone to consider the latter of these two propositions as a consequence flowing from the former, and that as its ground, even if they are both true. No one will say that the true ground of its being warmer in the summer than in the winter is located in the fact that the thermometer mounts higher in the summer than in the winter. Instead, everyone regards the fact that the thermometer climbs as a consequence of the higher heat level, and not the other way around (TS, §162).

To sum up, the proposition 'It is warmer in summer than in winter' is the ground of the proposition 'The thermometer is higher in summer than in winter', which is its consequence, and not vice versa. Bolzano admitted that, in elaborating his notion of grounding, he was looking to Aristotle's distinction in the *Analytica Posteriora* between the demonstration *that* something is the case and the demonstration *why* something is the case or, to use the scholastics' terminology, between *demonstratio quia* and *demonstratio propter quid*. The example above, together

[30]For example, in §195, Bolzano admits that "Almost everything I advance in this part is tinged with uncertainty and, on many topics I have not reached any decision, and at best my inquires are only fragments and suggestions which will have attained their goal if they provide others with the stimulus to reflect further on these matters".

[31]In §203, the author of the *Wissenschaftslehre* argues the following: "If we are not in a position, whether because of ignorance or because it is impossible in itself, to analyze the concept of the relation of *ground and consequence* into other simpler concepts, it becomes all the more necessary for its correct interpretation for us to describe the distinctive features of this relation in a series of special theorems. And since it is the relation of derivability above all that could be confused with that of ground and consequence, because of the similarity between them, it will be useful to define the distinctive features of the latter concept by way of a comparison with those of the former".

with this historical antecedent, suggests that the main difference between *Abfolge* and *Ableitbarkeit* is thus that the former but not the latter has an explanatory pretension. But there are also other relevant differences.

First, while the relation of derivability can involve also false propositions, grounding holds solely between true propositions and, in particular, between a set of truths and their immediate consequences. Second, contrary to *Ableitbarkeit*, the grounding relation is unique. This is because each truth has only one ground, which is made up of all the truths of which the initial truth is a consequence. Each of these premises singularly considered is called a partial ground of the grounded truth. Moreover, although different grounds can have common partial consequences, they cannot have the same complete consequence. Third, *Abfolge* is anti-reflexive (for no truth can be the ground of itself) and anti-symmetrical (for no truth can be the ground of its own consequence), while derivability is reflexive and admits of symmetric instances. Fourth, simplicity and generality are the two features that often (but not always) characterize the ground with respect to its consequences and that do not pertain to the propositions connected by the relation of derivability. On the one hand, the ground of a proposition should not be more complex than its consequences, in the sense that it cannot have a higher number of simple concepts occurring in it. On the other hand, the ground should be more general than its consequences, in the sense that the extension of the subject and predicate ideas of the ground should be broader than that of its consequences. The requirement of simplicity is taken to be prior to that of generality. Last, as the example above has made clear, a certain proposition can be at the same time derivable from and grounded in the same premises: if this is the case, the grounding relation is said to be *formal*, otherwise it is said to be *material*.

If it is true that the notion of grounding must not be confused with the relation of derivability, it is also important to underline the differences between *Abfolge* and causality. According to Bolzano, the former concerns propositions, while the latter deals only with real things. As a result, the two notions can work in the same direction if propositions about empirical facts are taken into account. But of course the relation of causality plays no role at all when the focus is, for example, on mathematical truths standing in a grounding relation.

The grounding relation is the fundamental notion of the account of scientific explanation proposed in the *Wissenschaftslehre*, for it is at the core of both Bolzano's conception of deductive sciences and his definition of axiomatic structures[32]. The

[32]The tight connection between grounding and axiomatic systems is argued by Bolzano in the following explicit terms: "I occasionally doubt whether the concept of ground and consequence, which I have above claimed to be simple, is not complex after all; it may turn out to be none other than the concept of an ordering of truths which allows us to deduce from the smallest number of simple premises the largest possible number of the remaining truths as conclusions" (TS, §221).

relation of *Abfolge* (and, notice, not that of derivability) is what turns a body of truths into a theory or a science. The truths of deductive sciences are ordered according to the grounding relation, which assigns every truth to its proper place with respect to the remaining propositions of that science. In particular, each proposition is either a basic truth or is grounded on basic truths. Basic truths are conceived no more as evident propositions, but just as starting points of proofs in deductive theories. Bolzano was one of the first thinkers to introduce an axiomatic perspective in the conception of science and devises axiomatic systems as theories characterized by a grounding order, that is to say, as theories in which propositions relate as grounds to their consequences. The properties of *Abfolge* that we have mentioned above determine the axiomatic structure of sciences ordered by the grounding relation. For example, each proposition has a unique *Abfolge* hierarchy that leads to fundamental propositions. In this context, Bolzano proposes, probably for the first time in the history of logic, proof trees, that is to say, diagrams showing the dependence of propositions with respect to their grounds.

According to Bolzano, deductive sciences, understood as axiomatic systems ordered by the relation of grounding, are mainly synthetic *a priori* and, as it will be clarified in the next Section, analytic propositions may occur in them, but only in a subordinated role. This is for sure a Kantian stance on Bolzano's side, since also the author of the *Critique* famously argued that all important cognition cannot be explained in terms of analytic judgments and falls on the synthetic side. Nevertheless, we cannot forget that the two philosophers have two different answers to the questions of what the synthetic *a priori* is and how it can be justified. The possibility of this kind of judgments is the central issue of the first *Critique* and Kant's solution relied on the concept of pure intuition; while Bolzano firmly rejects the Kantian postulation of pure intuitions[33] and solves the problem raised by Kant's work in a rather hasty way:

> Especially here, where K. envisages a difficulty, there seems nothing incomprehensible to me. "What justifies the understanding to connect a subject A with a predicate B foreign to the concept of A?" Nothing else, I say, than that the understanding *has* and *knows* the concepts A and B. In my opinion, from the mere fact that we have certain concepts, we must also be in a position to judge about them. For to say that someone has certain concepts A, B, C,... is indeed to say that he knows and differentiates them. But to say that he knows and differentiates them is again only to say that he asserts something about the one that

[33]According to Bolzano, the reason why Kant falls for the seduction of the doctrine of pure intuition is that he misunderstood the notion of grounding and held that intuitions could serve as grounds for synthetic truths (TS §315).

> he does not want to assert about the other; it means therefore to say that he judges about them (TS, §380).

As Roski's reading makes clear[34], Bolzano in this passage is claiming that one can come to know a conceptual truth of the form 'A is B' if one knows the concepts A and B and that to know the concepts A and B means to form judgments of a certain kind about A and B. Bolzano's point here is, we think, that the truth of *a priori* propositions can be apprehended by virtue of the concepts occurring in them only if they are embedded in axiomatic structures (and, as a consequence of this fact, are synthetic). Since basic truths defines concepts implicitly, to know the concepts that occur in a true proposition amounts to know the grounding chain that leads from that proposition to some given basic truths. This simple observation is what should probably be taken as Bolzano's justification of the synthetic *a priori*.

2.2.2 Bolzano's thesis that logic is synthetic

Bolzano conceives logic as a theory of science that aims at formulating rules "by which we must proceed in the division of truth generally into particular sciences and their presentation in their own treatises, if we would proceed in a manner genuinely suited to the purpose"[35]. The structure of this discipline is reflected in the index of Bolzano's *Wissenschaftslehre*, where the *Theory of Science Proper* is but the summit of a stratified construction made up of a sequence of disciplines, each of which is founded on the preceding. First, the guidance of the theory of science proper for writing scientific treatises and classifying sciences "would be superfluous if we were not clever enough to acquire knowledge in the first instance of a significant quantity of truths that belong to this or that science"[36]. As a result, the *Theory of Science Proper* depends first of all on the *Art of Discovery* or *Heuristic*. Second, both of these disciplines presuppose the *Theory of (Human) Knowledge*, whose business is to give the laws that govern the recognition of truth and to explain the nature of human cognitive capacity. Third, the inquires of the *Wissenschaftslehre* examined up to this point crucially depend on the *Theory of Elements*, which collects the properties of propositions and truths in themselves. Bolzano claims that:

> Without having acquired knowledge of the various relations of derivability and consequentiality that hold between propositions generally; without having heard all about that particular manner of connection

[34]Roski (2010, p. 110).
[35]TS, §1.
[36]TS, §15.

> that obtains between truths when they are related to one another as premises and conclusions; without having any acquaintance with the different kinds of propositions and with the different kinds of ideas as those components into which propositions are directly analyzed; we are surely in no position to define the rules as to how new truths are to come to be known from truths that are given [...]; whether it belongs to this or that science; in what order and connection to other propositions it has to be adduced in a scholarly treatise [...] and so on (TS, §15).

Last, all of these investigations presuppose the *Theory of Fundamentals*, which constitutes the beginning of any course and proves that there are truths as such.

As this overview has clarified, Bolzano's theory of science is a discipline characterized by broad boarders. In particular, the second half of the book, that is made up of the *Theory of Knowledge*, the *Art of Discovery* and the *Theory of Science Proper*, includes in the realm of logic also methodological, pedagogical and epistemological considerations. The three disciplines just mentioned are however subordinated to the observations proposed in the first half of the work, which is free from any psychological insight and concerns propositions and ideas taken in themselves, that is, from an objective point of view. Now it is necessary to underline that, beyond the broad conception of logic understood as a theory of science that is typical of his times, Bolzano individuates also a narrow notion of logic, which is closer to the modern conception of formal logic. The latter coincides with the *Theory of Elements* and, as the quotation above specifies, treats ideas, propositions, truths and inferences.

Bolzano maintains that logic in the narrow sense is a deductive science. This assumption, together with the principle that axiomatic systems are constituted by synthetic *a priori* truths, naturally leads Bolzano to derive the inevitable (but, for some, surprising) conclusion that logic is synthetic:

> In my opinion not even one principle in logic, or in any other science, should be a merely analytic truth. For I look upon merely analytic propositions as much too unimportant to be laid down in any science as a proper theorem of it. Who would want to replenish geometry, for example, with propositions like: an equilateral triangle is a triangle, or is an equilateral figure, etc.? (TS, §12)

Here we see that Bolzano's thesis that logic and analyticity do not coincide is developed in an extreme way: in logic there seems to be no place for analytic propositions. Similarly, not only logical analyticity does not coincide with triviality; but, according to this quotation, trivial propositions seems to be completely excluded from this discipline. The theses that emerged in our investigation of Bolzano's conception of logical analyticity in Section 2.1.3 seem to experience a radicalization if we examine the same matter from the perspective of Bolzano's theory of deductive sciences. This twist raises some interpretative problems, that we are going to discuss in the next Section.

2.3 An evaluation

2.3.1 A contradiction in Bolzano's system? The pragmatics of analyticity

In the previous Sections, we have suggested that Bolzano's account of consequence is twofold. On the one hand, we have explained in Section 2.1.1 that the notion of *Ableitbarkeit* or derivability is, according to Bolzano, the "most important concept of logic"[37]. This relation is meant to provide an account of truth-preservation, be it by virtue of form or not, and is defined through the substitution procedure as a special case of compatibility. On the other hand, we have shown in Section 2.2.1 that the notion of *Abfolge* or grounding plays a fundamental role in the Bolzanian epistemology. It is aimed at defining axiomatic structures in which propositions relate as grounds to consequences and show their explanatory order. The grounding relation, exhibiting the objective connection among true propositions, is what turns a mere collection of truths into a theory or a deductive science.

Many are the differences that Bolzano individuates between the two conceptions of consequence, the most evident of which is that the relation of *Abfolge*, contrary to derivability, has an explanatory function. Nevertheless, the exact connections between the two are not as clearly defined and Bolzano himself recognizes the point[38]. For sure, grounding is a kind of derivability and not vice versa. But this observation does not seem to resolve the question and finds no demonstration. According to Lapointe, "Bolzano gives the impression that he is seeking a notion to bridge the gap between the two notions when he defines the notion of 'exact' *(genaue) Ableitbarkeit*", but one cannot avoid to admit that "[i]n the *Theory of Science*, Bolzano's definition of exact *Ableitbarkeit* remains incidental"[39]. As a result, the account of consequence in the *Wissenschaftslehre* seems to be irremediably split into two parts. This observation may drag an unwelcome doubt. Since *Ableitbarkeit* and *Abfolge* are the two most important notions of Bolzano's logic and epistemology respectively, isn't it the case that also these two areas of the theory of science are not properly connected?

[37]Bolzano (2004, p. 54).
[38]TS, §200.
[39]Lapointe (2011, pp. 88-89).

This insinuation may seem to find support in considering the status of logical propositions in relation to the analytic-synthetic distinction that emerges in Bolzano's system. As we have seen in Section 2.1, analyticity is a notion of invariance under some classes of transformations, that, like universal (in)validity and *Ableitbarkeit*, is defined on the basis of the substitutional procedure. According to this account of analyticity, there seems to be several propositions that at the same time are analytic (or logically analytic) and belong to the discipline of logic. This is the case, for example, of the following propositions, whose analyticity is explicitly mentioned in the *Theory of Science*[40]: 'A is A' is analytic with respect to 'A'; 'Every object is either B or non-B' with respect to 'B' and 'If all men are mortal and Caius is a man, then Caius is mortal' with respect to 'men', 'mortal' and 'Caius'. However, we have shown in Section 2.2 that if we turn to the epistemological observations relative to deductive sciences and based on the notion of grounding, we learn that "not even one principle in logic [...] should be a merely analytic truths"[41]. On this point Bolzano is crystal-clear and his words cannot be misunderstood.

Appearances notwithstanding, there is no contradiction in the *Wissenschaftslehre* as far as the issue mentioned above is concerned. But how can Bolzano keep the two perspectives together? The idea is simple. Bolzano distinguishes in logic between the *rule* and its *application*, or, in other words, between the general statements and their particular instantiations. Now, we have seen in Section 2.1.2 that analytic propositions can be derived from synthetic ones. As a special case of this property, consider the proposition $Ac \rightarrow Bc$, which is an instantiation of the general principle $\forall x(Ax \rightarrow Bx)$. The former is derivable (*ableitbar*) from the latter and is analytic, because c can be arbitrarily varied without affecting the truth value of that proposition. The latter is instead synthetic, for we may assume that there are substitutions on A and B that change the truth value of that proposition. In general, we have that any instance of a true general statement is at the same time a consequence of that general principle and an analytic proposition even if the premise is synthetic, for the singular term occurs inessentially in it. To clarify this point, Bolzano deals with the following example:

> Logic too contains a considerable number of synthetical propositions. [...] Even the well-known rules of syllogistics are wrongly conceived as analytical propositions. It is true that the proposition: if all men are mortal, and Caius is a man, then also C[aius] is mortal may be called analytic in the broad sense [...]; but the rule itself, that out of two propositions of the form A is B, and B is C, follows a third of the form A is C, is a synthetical truth (TS, §315).

[40]The former two examples are taken from §148 of the TS; the latter from §315.
[41]TS, §12.

Bolzano's idea is that, although both the general rules (e.g., the rules of the syllogistics) and their instantiations (e.g., the particular instances of the syllogisms) belong to the discipline of logic, analyticities and syntheticities play two definitely distinct roles. It is the pragmatics of analyticity and syntheticity, which will be discussed below, that explains why Bolzano does not contradict himself although he both offers some examples of analytic and logical propositions and maintains that logic is synthetic.

We have shown in Section 2.1.1 that one of the tools that Bolzano uses against the psychologistic trend in logic is the distinction between what it is and what it is known and we have highlighted in Section 2.2.2 that Bolzano's *Theory of Science* comprises, beyond the study of logic in the narrow sense, also an *Erkenntnislehre*. De Jong (2001) and Proust (1989), whose researches represent interesting viewpoints in the literature, stress that the metaphysical and epistemological levels must be distinguished also when discussing the roles of analytic and synthetic propositions in deductive sciences. From the perspective of the *ordo essendi*, Bolzano agrees with Kant in holding that scientific propositions are mainly synthetic and that analyticities play a secondary and modest role: to use Proust's words, "[t]he Bolzanian definition of analytic propositions presents synthetic truths as forming the final authority of the truth of a theory"[42]. Analytic propositions are usually grounded in synthetic ones[43] and only the latter manage to express scientific principles and theorems.

Analyticities acquire a more valuable position if we move to consider the level of the *ordo cognoscendi*: in the presentation and in the development of a deductive science, analytic propositions are needed, according to de Jong, for "the clarification of the objective structure of propositions and the finding of possible grounds from which a truth could be demonstrated"[44]. Proust adds the observation that "in the realm of knowledge, it is not possible to be satisfied with universal statements; a delicate balance has to be found between impenetrable generality and redundant detail [. . .], a balance in which analytic propositions constitute an essential element"[45]. He maintains that the freedom with which analytic propositions are used depends on three factors: first, the nature of discipline under study; second, the capacity of the subject; third, the objective of the authors of treatises.

To sum up, in the *Theory of Science* two different perspectives coexist: the logico-metaphysical and the epistemological. Sometimes the two levels don't seem

[42]Proust (1989, p. 106).

[43]De Jong (2001) suggests that Bolzano initially intended to link the analytic-synthetic distinction to the notion of grounding assuming that any analytic truth finds its (complete) ground in a synthetic proposition. The reasons why Bolzano did not complete his reasoning must be searched in his difficulties in defining the notion of grounding.

[44]De Jong (2001, p. 348).

[45]Proust (1989, p. 105).

to be properly connected: as we have seen, this is the case for example of the two notions of consequence, *Ableitbarkeit* and *Abfolge*. But what is important to underline is that, appearances notwithstanding, there are no reasons to maintain that the two perspectives go to contradict one another: they just complement each other when dealing with the same object. This is probably the case for the status of logical propositions in connection to the analytic-synthetic distinction. On the one hand, the examples of analyticities belonging to logic presented by Bolzano are only instances of more general and synthetic propositions or, in any case, propositions that play a role only in the presentation of logic; on the other hand, when Bolzano asserts that logic is synthetic, he is just assuming the metaphysical perspective, omitting the way in which the discipline of logic has to be presented and taught.

This conclusion raises two other relevant questions. First, we have seen that analyticity, being detached from apriority and necessity, cannot give an account of knowledge by virtue of meaning. This could lead to suppose that the notion of analyticity serves a rather trivial purpose in Bolzano's system. But this is not true. For we have already shown, following de Jong (2001) and Proust (1989), that "for Bolzano the analytic-synthetic distinction has to distinguish the proper propositions of a science from propositions that have some role in its presentation, especially in demonstration, but are not constitutive for the science concerned"[46]. For the ones who think that this is a still too insignificant objective, we could notice with Lapointe (2014b) that Bolzano's analyticity is the first systematic account of generality and quantification in general, because 'is analytic with respect to ...' is, like the universal quantifier of first-order logic, an operator that binds a variable to express generality. The pragmatics of analyticity is thus an essential aspect in Bolzano's theory.

Second, the objectives of analyticity in the *Theory of Science* may suggest how to answer a question that is ubiquitous in the literature: why did Bolzano introduce two notions of analyticity instead of one and did not restrict himself to logical analyticity? First, the two kinds of analyticity have the same role in the presentation of two different kinds of science: if a proposition is a logical analyticity, then its ground will be a synthetic proposition of logic; if a proposition is a non-logical analyticity, then its ground will be a synthetic proposition belonging to any science. In this sense, it is obvious that logical analyticity is only a special kind of analyticity in the broader sense. Second, Bolzano introduces his notion of analyticity in the broader sense not only because of his "habitual striving for maximal generality"[47], but also because, as de Jong (2010) suggests, "it is precisely the notion of analyticity 'in its broad sense' which would be important for

[46]De Jong (2010, p. 255).

[47]Rusnock (2013, p. 323).

the presentation of a science"[48]: the epistemological and pedagogical relevance rests with non-logically analytic truths, for their grounds in synthetic and general propositions are usually more hidden than what happens for logical analyticities[49].

2.3.2 A deceptive terminological choice? Bolzano's place in the history of analyticity

Bolzano's analytic-synthetic distinction enjoys an eminent role in the literature for two classes of reasons. First, the significancy of this definition consists in the novelty with respect to the modern tradition: as we have highlighted above, Bolzano's substitutional procedure represents for sure a great breakthrough and a herald of important anticipations. Second, much of the attention that Bolzano's definition received is due to its bizarre and apparently incomprehensible features, especially if these characteristics are compared with the perspectives put forward by other philosophers on the same matter.

The traditional interpretative trend cannot make sense of the fact that Bolzano's analyticity does not entertain any reliable connection with apriority and necessity and maintain that Bolzano's admission of empirical analyticities demonstrates a relevant deficiency in the understanding of the core of Kant's notion. It cannot be read but as a great mistake in the *Wissenschaftslehre*. The worry that troubles this interpretation is Bolzano's alleged misuse of the new and powerful logical tools he elaborated: his substitutional procedure and his insights in the theory of propositions could have served Bolzano better in his elaboration of central concepts such as that of analyticity. Moreover, this narrative classifies Bolzano's view on analyticity as an anomaly in a supposed linear and progressive history of this notion from Kant to the logical positivists. It assumes that every philosopher that is part of this story improves the understanding of the analytic-synthetic distinction and moves closer and closer to the acme of this development that is individuated in the positivists' conception of this notion. This picture, however, cannot assimilate Bolzano's theory, because the latter seems to point toward a different direction. To avoid a complete isolation of Bolzano's theory in this kind of historical reconstructions, most of these critics exaggerate the importance of logical analyticity in the *Theory of Science* at the expense of the broader notion of analyticity offered in Section 148: with this interpretative stretching, Bolzano's work is forcefully included into the tension towards formality that characterizes this supposed history of the notion of analyticity.

---

[48]De Jong (2010, p. 253).

[49]This perspective is put forward also by Proust (1981, p. 224): "Bolzano mentions the difference between the two kinds of analytic propositions because he wants to show the epistemological relevance of those analytic propositions whose analyticities may remain unnoticed. It is when the analyticity is hidden that it becomes truly fruitful to bring it to the fore".

Examples of this interpretational trend are Kneale and Kneale (1962) and Bar-Hillel (1970). The former dub as "curious" a number of features proper to Bolzano's notion of analyticity, such as for example the definition of false analyticities, and, as far as some employments of the related notion of derivability is concerned, the Kneales say that "it is interesting to ask why a philosopher who was so obviously gifted for logical studies should have failed to make progress in this part of his work"[50]. The latter holds the anachronistic thesis that "Bolzano's aim, in §148, was to define a concept which could serve as an adequate explication for what is now commonly termed logical truth"[51] and, as far as the notion of analyticity in the broader sense is concerned, comments as follows:

> The reader [...] must have become convinced of the almost ridiculous inadequacy of Bolzano's definition of 'analytic'. [...] We can only wonder about the lack of perspective which caused him to believe that his definition is only "somewhat broader", but even in this case, we must ask, what prevented him from looking for a more adequate definition? (Bar-Hillel, 1970, p. 10)

On the basis of these presuppositions, Bar-Hillel assembles a rather complicate narrative around the two notions of analyticity that appear in Bolzano's text. He conjectures that the author of the *Wissenschaftslehre* realized the inadequacy of the broader notion he offered and found only "at the very last moment, perhaps during the printing[52]" that the conception of logical analyticity represented the solution to all his problems. For this reason the interpreter thinks that "we are fully entitled" and "morally obliged"[53] to refer to Bolzano's logical analyticity with the term 'analyticity' *tout court*. According to Bar-Hillel, Bolzano thus managed to insert the third subsection of §148 dedicated to logical analyticity without having enough time to revise his work. As a result, not only "several parts, written at different times, probably years apart, were embodied side by side, without the necessary adjustments", but also the "far-reaching consequences" of Bolzano's notion of logical analyticity "for the whole section, even for the whole Bolzanian logic, were not worked out"[54].

This interpretational trend has been attacked by recent scholars on two main fronts: the anachronism of the historical presuppositions and the negligence in reading the Bolzanian texts. The latest critics agree in the reconstruction sketched above: while Proust (1981) calls Bar-Hillel's interpretation into question and highlights the contradiction of the reading proposed, Rusnock (2013) warns that the

[50] Kneale and Kneale (1962, p. 366 and p. 371).
[51] Bar-Hillel (1970, p. 3).
[52] Bar-Hillel (1970, p. 11).
[53] Bar-Hillel (1970, p. 12).
[54] Bar-Hillel (1970, p. 13).

problem of finding a role for Bolzano's work in the development of the notion of analyticity does not lie with the author of the *Wissenschaftslehre*, but rather with the historical frame presupposed above. In general, recent interpreters are careful about underlining the peculiarity and the specificity of Bolzano's proposal. Nevertheless, even this approach is not free of risks. In some but not all cases, this trend achieves the opposite result of depicting Bolzano's work on the analytic-synthetic distinction as something completely different from his predecessors and successors. In so doing, the notions exposed in the *Theory of Science* turn out to be isolated in the philosophical landscape. As a result, it is sometimes suggested that Bolzano's terminological choice is deceptive and tendentious[55]: he employs a Kantian term, but with a different meaning. This is the radical position held by an acute reader of Bolzano such as Künne:

> Bolzano's explanation of how 'analytic' in his mouth is to be understood and Kant's explanation(s) of how he wants this word to be understood are explanations of different concepts (with different extensions). Unlike 'true' and 'necessary', the word 'analytic' is a philosopher's term of art (Künne, 2006, p. 219).

The extreme consequences put forward by some recent scholars threaten the relevance of one of the most interesting aspects of Bolzano's theory, namely, his thesis that logical theorems are synthetic *a priori*. For if it is true that the meaning Bolzano attaches to the term 'synthetic' is so distant from our conception of this notion, then also the thesis of the syntheticity of logic maintained in the *Wissenschaftslehre* risks to have a different meaning than what it is usually assumed.

Against this conclusion, we hold the following theses that we support with several observations. First, taken for granted its peculiarities, Bolzano's notion of analyticity is less 'alien' than what it is supposed to be, but it rather shares decisive features with the definitions elaborated by other philosophers included in the 'semantic tradition'. Second, Bolzano's thesis that logic is synthetic is a substantial view, which is grounded in fundamental convictions on the nature and aims of the discipline of logic.

In order to support the former thesis, we now focus on the filiation, analogies and connections between Bolzano's and Kant's notions of analyticity. First of all, we must consider that, in early writings such as the *Beyträge zu einer begründeteren Darstellung der Mathematik* (1810) and *Etwas aus der Logik* (1812), Bolzano endorses with no objections Kant's analytic-synthetic distinction[56]. Then, if it is true, as we have shown in Section 2.1.5, that Bolzano criticizes many aspects of the distinction as it is put forward in the *Critique*, it is also fair to underline that

[55] See Rusnock (2013, p. 333).

[56] This aspect has been emphasized by Lapointe (2011, p. 59).

in several occasions the author of the *Theory of Science* praises and recognizes his debt with Kant's work[57]. For example, in Section 148, he holds that "[E]ven if it is true that this distinction was mentioned before at times, nevertheless it was never properly pinned down and fruitfully applied. The merit of having been the first to have done that indisputably belongs to Kant"[58]. Moreover, as it has been noted by several scholars[59], sometimes Bolzano prefers to make use of Kant's definition instead of his own: this happens even after Section 148 of the *Wissenschaftslehre*, that is, after Bolzano has already introduced and discussed his version of the same notion.

But beyond these considerations, which may be also interpreted as homages paid by Bolzano to his bright predecessor, the important point is that every analyticity *à la* Kant turns out to be analytic according to Bolzano's definition, but not vice versa: in other words, the effect of Bolzano's definition is simply to widen the extension of Kant's notion. For if in a Kantian analyticity the predicate is contained in the subject, then that predicate occurs inessentially in the proposition under examination, while, for example, every Bolzanian analyticity that is not of the subject-predicate form cannot be said to be analytic according to Kant. That's not all. If it is true that, generally speaking, analyticity and apriority are not correlated in any predictable way, things seem to be different if we focus on logical analyticities[60]. On the one hand, "nothing is necessary for judging the analytic nature" of them "besides logical knowledge"; on the other hand, logic is said to be a conceptual science. As a consequence, one may conclude that logical analyticities can only be known *a priori*, because the invariant parts of these propositional forms are made up by logical concepts (and not intuitions). This result, if correct[61], may be counted as another similarity between Kant's and Bolzano's analytic-synthetic distinction[62].

Not only is Bolzano's notion of analyticity strictly interwoven with Kant's

[57]On this point, see Siebel (2011, p. 100).

[58]TS, §148.

[59]Siebel (2011, p. 100), de Jong (2001, p. 334) and Künne (2006, p. 235).

[60]See Rusnock (2013, pp. 329-330).

[61]The only objection against this conclusion that we manage to think of is the following. In order to check that the objectuality constraint is satisfied by a certain variant, it may be the case that the recourse to experience turns out to be essential. The proposition 'A table, which is an object, is an object' is both logically analytic with respect to 'table' and conceptual, for it does not contain any intuition. However, in order to check the truth value of its variant 'This, which is an object, is an object' one has to resort to experience, because of the indexical 'this'. Nevertheless, this counter-argument goes against Bolzano's assertion that logical knowledge is sufficient to recognize that a certain proposition is logically analytic, and not against the idea that if a proposition is logically analytic, then it is conceptual.

[62]At the same time, it gives an additional reason why the traditional interpretational trend focuses on Bolzano's narrow notion of analyticity at the expense of the broader conception.

definition. The conception exposed in the *Wissenschaftslehre* anticipates in many respects fundamental insights of successive philosophers that are usually included in the history of the notion in question, although it is necessary to point out that the modest impact of Bolzano's theories on intellectual developments prevents to speak of a direct influence produced by his anticipatory insights and that most of the innovative tools and conceptions he elaborated have been re-discovered independently[63].

First, as we have underlined in Section 2.2.2, Bolzano's attention to the distinction between the *ordo essendi* and the *ordo cognoscendi* finds an important expression in his conception of logic. The discipline, understood in the narrow sense, has nothing to do with psychology and deals only with properties of propositions and ideas in themselves. In so doing, Bolzano anticipates the anti-psychologistic stances of Frege and Husserl. Moreover, it is this conception of logic that leads the author of the *Wissenschaftslehre* to draw an analytic-synthetic distinction that, as he himself is proud to underline at the end of §148, is completely independent from the individuals' definitional choices and is characterized by an objective nature.

Second, Bolzano's criticism of the decompositional conception of analysis at the basis of the *Critique* is an innovative viewpoint, which has been independently reformulated in the subsequent decades. Bolzano was probably the first post-Kantian author to feel uncomfortable with the decompositional conception of analysis, that had become dominant by the end of the early modern period and which ties Kant to the rationalist perspective he wanted to criticize and to the Leibnizian theories. As we have highlighted in Section 2.1.5, several are the anticipatory insights connected with Bolzano's rejection of that paradigm of analysis.

Not only did Bolzano attack the naïve representationalism at the basis of Kant's approach, but he also extended the analytic-synthetic distinction beyond the limits of categorical judgments, so as to release it from the boundaries of a particular syntactical form and language. As we have seen before, Bolzano complains that Kant's explication "fall[s] somewhat short of logical precision" and makes use of "figurative forms of expression that do not analyze the concept to be defined" and

[63]In the Introduction to her book, Lapointe (2011, p. 1 and ff.) lists a number of reasons that explain why Bolzano's work has not enjoyed a favorable critical fortune. First of all, the Austrian academic policy was characterized by obscurantism and absolutism: teachers had no intellectual freedom and were obliged to use certain textbooks. Moreover, Bolzano was accused of subversive activities: he was discharged from the University and was banned from public scientific and clerical activities. To this oppressive situation, one must also add a bad management of his unpublished works and Bolzano's own obsolete literary style. As a result, it is not a surprise that, despite the undeniable similarities, it is difficult to document any direct or indirect connection between on the one hand Bolzano's ideas and on the other Frege's work on meaning, Tarski's study of logical consequence and Quine's reflections on logical truths. However, as Morscher (2013) is careful to emphasize, it is fair to mention that Bolzano's philosophy had some indirect impact on the Polish school of logic and on Brentano's students.

of "expressions that admit of too wide an interpretation"[64]; similarly, as we have already shown in the previous Chapter, Quine's *Two Dogmas of Empiricism* famously maintain that the Kantian formulation "appeals to a notion of containment which is left at a metaphorical level"[65] and Frege[66] accuses Kant of having misunderstood the status of arithmetical judgments because of this restriction, that would have led the author of the *Critique* to underestimate the value of analytic judgments. Moreover, Bolzano's alternative to Kant's decompositional account of analysis, the so-called paraphrastic approach, is a revolutionary intuition in the history of analysis and anticipates in many respects the transformative or interpretative dimension of analysis that is commonly assumed to characterize analytic philosophy[67].

Third, Bolzano's analyticity is a formal property of propositions. To be a formal property, in the *Theory of Science*, simply means to be a property defined through the method of substitution[68]. The substitutional account of form establishes that formal properties convey certain features that are common to a whole kind of propositions. For this reason, to say that logic is formal amounts to maintain that that discipline studies certain semantic regularities that can be found in classes of propositions and not in individual propositions:

> Logic (at least in its doctrines - it can be otherwise in the examples) never considers a fully determinate proposition, i.e. one in which the subject, predicate, and copula are fully specified, but, rather, a whole

[64] TS, §148.

[65] Quine (1951, p. 21).

[66] Frege (1980, par. 88, pp. 99-100).

[67] See, for example, the classification put forward by Beaney (2018).

[68] Bolzano distinguishes several meanings of the word 'form' (TS §81) and accuses Kant and his followers of vagueness and evasiveness on this point. In the sense specified above, the term 'formal' can be predicated of properties, such as validity and analyticity, only if the latter are defined through the substitutional method. This meaning of the word 'formal' must be distinguished, we think, from another sense, which is nowadays more wide-spread than the Bolzanian one and says that a proposition (not a property) is formal only if it contains solely concepts that express logical constants.

While, according to Bolzano's meaning of the term, analyticity turns out to be formal, propositions that are analytic or even logically analytic, following Bolzano's definitions, may not be formal if we take the meaning of the latter term to be the one just sketched. For example, the proposition 'Every long proposition is a proposition' is logically analytic with respect to 'long', because every objectual variant of it is true and the only invariant concepts of it belong to Bolzanian logic. But at the same time, this proposition is not formal because of the occurrence of the term 'proposition', which is not a logical constant. In other words, Bolzano's notion of analyticity does not provide an account of what it means to be true or valid by virtue of form alone, because Bolzano has a wide understanding of the realm of logical concepts.

As a result, the efforts made by the traditional interpretational trend to introduce Bolzano's work into a progressive history that tends towards formality are doomed to encounter resistance.

> class of propositions, i.e. all propositions collectively, which, if some of their components are completely fixed, the remainder can be read in this or that way. [...] If one calls such classes of propositions general forms of propositions [...] then one can say that logic concerns only the forms of propositions, never individual propositions (TS, §12).

To consider analyticity as a formal property amounts to a great revolution. Kant held that a judgment was analytic because of its content, that is, because the predicate was contained into the subject; on the contrary, Bolzano maintains that analyticity is a matter of form, that is, what is important now is not the content of a proposition, but the invariance of truth-value under variation of content, namely, under variation of certain ideas occurring in the proposition[69].

After Bolzano, other philosophers have used the substitutional procedure to define formal notions in logic. One of them is Tarski[70], whose definition of logical consequence has been initially given in variational terms and seems to be a paraphrase of Bolzano's definition of derivability. To associate Bolzano with Tarski is also the common concern regarding a definitional detail of the substitution method, namely, the problem to individuate which concepts can be classified as 'logical' and can be held as fixed[71]. Another example of this procedure is Quine's definition of the notion of logical truth[72]. The analogy between Quine's notion and Bolzano's conception of logically analytic propositions can be spelled out in the following terms: "for Bolzano, a proposition is logically analytically true if and only if it is true and only logical ideas or concepts occur in it essentially; while for Quine, a statement is logically true iff it is true and contains only logical expressions essentially"[73]. This similarity is founded on the substitutional method on which the concepts of occurring and containing essentially are based. The recent scholars' observations about the differences between, on the one hand, Bolzano's derivability and Tarski's logical consequence and, on the other hand, Bolzano's logical analyticities and Quine's logical truths[74] do not undermine the point that most interests

[69] This observation has been noted by Rusnock (2011, p. 483).

[70] Tarski (1936).

[71] See Section 2.1 above.

[72] Quine (1960).

[73] Hale and Wright (2015, p. 329).

[74] The main differences can be summarized as follows. As far as Tarski's logical consequence is concerned, Šebestik (2017) observes that, first, while Tarski defined logical consequence for formalized languages, Bolzano's propositions and ideas that occur in the definition of *Ableitbarkeit* are expressed in natural language; second, Tarski, unlike Bolzano, rejected the condition of compatibility of the premises, that is, the requirement that they represent at least one object in common; third, Bolzano, unlike Tarski, does not generalize both over interpretations and domain. With regard to Quine, Künne (2006, p. 226) notes that first, while certain hidden analyticity are logically analytic according to Bolzano, they are not logical truths following Quine's definition; second, unlike Quine, Bolzano defines logical analyticity not for sentences but

us here, namely the fact that the substitutional account of formal properties is an instrument widely employed by the logicians of the Twentieth century.

To sum up, we have offered some observations on the similarities between Bolzano's notion of analyticity and both Kant's conception of the analytic-synthetic distinction and the perspective put forward by later analytic philosophers. These considerations are not meant to let one forget that Bolzano's analyticity (in the broader sense) is not directly linked to necessity and apriority and thus does not amount to the positivistic notion of truth by virtue of meaning. On the contrary, these are essential features that mark the specificity of the definition proposed in the *Wissenschaftslehre*. Nevertheless, the remarks above should have demonstrated that Bolzano belongs with full right to the same *coté* that gathers the authors we are examining in this thesis: for sure, at least, the way in which the word 'analytic' is used by Bolzano echoes in many respects the manners in which it is employed by the thinkers of this tradition. Then, to establish whether 'analytic' is a "philosopher's term of art" is, we hold, just a question of appraisal.

Once we have proved that the analytic-synthetic distinction in the *Theory of Science* is less *unheimlich* than what it is supposed to be, also the doubts sketched above on the significance of Bolzano's thesis that logic is synthetic turn out to be less pressing. However, we would like to add some more observations on this issue. In Section 2.2.2, we have shown that Bolzano agrees with Kant in holding that deductive sciences are mainly synthetic *a priori* in the sense that the analytic propositions that occur in them play at best a modest role. Bolzano not only fully accepts the controversial Kantian notion of synthetic *a priori* and the centrality of this concept in deductive sciences. He also pushes Kant's reasoning even beyond Kant's own conclusions and extends this insight to logic[75]. This is the sense in which Morscher maintains that "Thus - and this certainly comes as a surprise - concerning the fundamental problem of the synthetic a priori, Bolzano turns out to be ultimately not anti-Kantian at all, but - quite the contrary - even more Kantian than Kant himself, i.e. a Super-Kant, so to speak"[76].

The reason why Bolzano's insight that logic is synthetic *a priori* turns out to be a substantial conception and not a mere terminological trick is that it hides an important thesis on the nature of this discipline. By saying that logical theorems are not analytic, what Bolzano is actually holding is that logic is a body of truths like any other deductive science. This intuition finds the following textual support: every time Bolzano puts forward the thesis that logic is synthetic, he always makes

for propositions; third, Quine's logical particles are expressions (connectives, quantifiers and the identity predicate), whereas Bolzano's logical particles are notions.

[75]This issue has been already emphasized by de Jong (2010, p. 251): "Like Kant, Bolzano conceives of scientific propositions first and foremost as synthetic, and he too allocates a very restricted role to analytic propositions. But unlike Kant, he also applies this insight to logic".

[76]Morscher (2006, p. 261).

reference to other sciences. This is the case, for example, in this passage of the *New Anti-Kant*:

> [...] we have to agree with him [i.e., Kant] when he claims that "synthetic a priori judgements are contained as principles in all theoretical sciences of reason" (46/B14). However, judgements of this kind are not only to be found in mathematics, the pure natural sciences, and metaphysics, as Kant demonstrates uncontestably. They are also to be found in logic, and indeed not only among those doctrines which only belong to it according to a broader concept of the discipline, i.e. if one conceives of it, with Bolzano, as a theory of science, but even in that part of it which is called analytic and which has been worked on since Aristotle (Přihonský, 2014, pp. 52-53)[77].

Unlike Kant, who, as we have seen in Chapter 1, maintains that logic is a *canon*, a set of rules that govern our thinking, Bolzano's viewpoint is that logic is a science on its own. Moreover, Bolzanian logic cannot even be said an *organon* in Kant's sense, for it does not direct certain cognitions that deal with given objects, but it primarily concerns propositions in themselves and, only in its broader form, it considers how propositions relate to judgments and cognitions. In Chapter 1, we have suggested that one of the reasons that motivates Kant's lack of interest in applying his analytic-synthetic distinction to logic was that this discipline was a sort of propaedeutic that precedes any kind of knowledge, but it is not itself a kind of knowledge. What allows Bolzano to apply the analytic-synthetic distinction also to logic and its theorems and to maintain that logic is synthetic is precisely that the author of the *Theory of Science* has managed to release logic from the Kantian verdict on its alleged special status.

[77]See also the quotations from the *Wissenschaftslehre* in Section 2.2.2.

Chapter 3

Frege and logical positivism: the golden age of the principle of analyticity of logic

3.1 Logic is analytic

3.1.1 Logic in Frege's logicism

Frege motivates his work on the foundations of arithmetic with the belief that the most fundamental concepts of that discipline are not properly understood by the mathematicians and philosophers of his days. The paradigmatic case of this situation is that, according to the German philosopher, no one can give a coherent definition of the notion of natural number. The same happens also for the nature of calculation. But Frege sees it as a "scandal" that arithmetic cannot give a proper definition of its fundamental and simplest concepts and he feels obliged to perform the "imperative task" to investigate the matter more closely until these difficulties are overcome[1].

Famously this investigation leads Frege to devote his intellectual life to the realization of his logicist project of reducing arithmetic to logic. This program can be seen as made up of the following different theses[2]. First, every proposition of arithmetic can be shown to be derivable from logical axioms; second, the rules used by mathematicians to prove their theorems can be shown to be logical in character; and third, arithmetical concepts can be shown to be definable in terms of logical concepts. But Frege's logicist project has also an important philosophical counterpart. As we shall see in the next sections, the program of reducing

[1]FA, *Introduction*, p. ii.

[2]See de Jong (1996).

arithmetic to logic amounts to showing that the truths of arithmetic are analytic *a priori*. This is a radical thesis. On the one hand, in holding that arithmetic is analytic, Frege criticizes the well-known Kantian thesis that arithmetic is synthetic and founded on the pure intuition of time. On the other hand, in maintaining that arithmetic is *a priori*, Frege is rejecting the empiricist position of John Stuart Mill that mathematical truths are known on the basis of experience and are thus *a posteriori*.

It soon becomes clear to Frege that his project of reducing arithmetic to logic could not have the slightest chance of success unless logic itself were not properly developed. The dominant logic of the time was still Aristotle's syllogistic based on subject-predicate judgments and, beyond this tradition, Boole's algebra of logic. However, these logical tools proved to be completely inadequate for expressing certain kinds of mathematical reasoning. As a result, the first obstacle for the realization of his logicist program was to create a new and more powerful logical system.

This is what Frege manages to achieve in his *Begriffsschrift* (1879). In this book, he provides the basis of modern logic, proposing essentially what is known today as classical second-order logic with identity and embracing both the logic of propositions and the logic of quantification. In particular, Frege presents his calculus in an innovative symbolism, the concept script, for he believes that natural language is not precise enough for establishing the true nature of arithmetic. Moreover, he intends to devise a symbolic notation that can be useful in every field of knowledge in which logical proofs were required. However, the idiosyncrasy and bi-dimensionality of this notation is perhaps responsible for the scarce success of his work. One of Frege's fundamental ideas in his *Begriffsschrift* is to reject the traditional analysis of judgments into a subject and a predicate and to substitute these notions with the logical concepts of argument and function. Borrowing a mathematical usage, he analyzes propositions into a variable and a constant part and, in so doing, provides a more flexible method for bringing out logically relevant similarities between sentences beyond their linguistic presentation.

The introduction of the function-argument model allows Frege to formulate the first complete systematization of logical quantification and to overcome one of the principal difficulties of traditional syllogistic, viz. to deal with judgments with nested quantifiers. To this purpose, he presents a new notation to express generality, while he takes the existential quantifier as a derivative notion. Moreover, he defines also the negation, the conditional and the identity signs, as well as the notion of second-order function. Frege formulates his calculus in nine propositions that are declared to be axioms. The former three are devoted to the conditional; three other propositions are dedicated to the negation sign; two axioms govern the identity relation and the latter law concerns the universal quantifier. Axioms are justified, in Frege's eyes, by their expressing self-evident truths. All other logical propositions can be derived from Frege's axioms by virtue of two inference rules, namely *modus ponens* and the rule of generalization, that the author states explicitly, together with another rule, i.e. substitution, that Frege invokes implicitly.

Logic is for Frege the powerful system that he himself proposed in the *Begriffsschrift*. It is this notion of logic that lies, at the same time, both at the basis of his logicist project and, as we shall see in the next sections, at the core of his definition of analyticity. But what is Frege's conception of the discipline of logic[3]? According to Frege, logic has the task to elaborate and prove certain true statements, the logical laws. A law can be normative, if it prescribes what one ought to do and it represents a standard that has to be achieved; or it can be descriptive, if it describes the way in which things are. For Frege, logical laws are descriptive in their content, but imply norms for thinking[4]. On the one hand, logical laws are descriptive in the same way physical laws are: they do not prescribe how things ought to be, but they rather illustrate how they are. Any logical law is a claim about concepts and objects and their relations: it does not affirm that such and such ought to be the case. On the other hand, logical laws imply prescriptions about thinking, judging and asserting: they represent an evaluative standard of rationality to which individuals ought to relate.

The characterizing feature of logical laws is, according to Frege, their generality. How should the term 'generality' be understood in connection with Frege's logic? MacFarlane[5] rejects two tempting but anachronistic explanations. First, it is not the case that Frege, by characterizing logic as the maximally general science, intends that its truths are not about anything in particular. On the contrary, logic is concerned with particular concepts and relations, namely identity, negation, conditional. The point is that although these notions are used in every discipline, they are investigated only in the discipline logic. Second, it is not the case that with generality Frege means that logical laws are insensitive to the differences between particular objects. This standpoint would be incompatible with his logicism: since each number has its own peculiarities, logical notions cannot be distinguished by their permutation invariance.

But what, then, does Frege mean by saying that logic is general? The generality of logic for Frege consists in the unlimited range of applicability of its norms. Logical laws are general because they pertain to the universal domain and apply to the set of all thinkable objects. This feature distinguishes logic from all the special sciences, which deal only with restricted domains. A consequence of this conception of generality is that, in Frege's eyes, the traditional distinction between

[3]The answer to this question that we propose below greatly benefited from two masterful articles: Goldfarb (2010) and MacFarlane (2002).

[4]This observation is due to MacFarlane (2002, p. 36).

[5]MacFarlane (2002, pp. 33-34).

domain-specific and non-specific principles amounts to the distinction between non-logical and logical principles[6]. Examples of special sciences are, according to Frege, geometry, that deals with the laws of space, and physics, that concerns the physical world. On the contrary, Frege's thesis that arithmetic can be reduced to logic is a substantial thesis on the nature of arithmetic[7], namely, that this discipline is marked by generality and can be applied to all thinkable objects:

> The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? (FA, §14, p. 21)

As we have seen above, Frege's thesis that arithmetic is analytic contradicts Kant's conception of that discipline. This contradiction is substantial and it is not a merely verbal change in subject because, as MacFarlane (2002) has shown in his persuasive text, the two philosophers essentially share the same conception of logic. The basis of this agreement can be found in the generality of logic. For also Kant believed that logic is general. Generality for the author of the *Critique* means, as we have shown in Chapter 1 of this thesis, that logic "contains the absolutely necessary rules of thinking"[8] or, in other words, that it contains the rules that have to be applied no matter of what are the objects we are thinking about.

However, while Kant from the generality of logic, together with other principles, infers that logic is formal, Frege is clear in denying the formality of the discipline[9]. The author of the *Critique* maintains the innovative claim that logic is formal in the sense that it abstracts from the semantical content of thought and from all content of cognition. A consequence of this perspective is that pure general logic for Kant cannot yield an extension of knowledge about reality or objects. Contrary to Kant, Frege cannot hold that logic is formal without contradiction. For, as we will clarify below, he believes that logic can supply with substantive knowledge about objects such as numbers and that logical deduction can extend our knowledge.

[6]See de Jong (1996, pp. 314-317).

[7]This is the reason why Dummett (1991, p. 43) holds that "*Grundlagen* in fact advances two distinguishable theses about arithmetical truths: that they are analytic, and that they are expressible in purely logical terms. On his own principles, neither implies the other".

[8]CPR, A 52/B 76.

[9]MacFarlane (2002) shows that although Kant holds that logic is formal and Frege denies this, they share the same conception of logic based on generality. The reason for this is that it is just in the context of Kant's other philosophical commitments that generality implies formality. But Frege can reject Kant's commitments and hold that logic is general in the same sense as Kant's, but, at the same time, that it is not formal.

According to Frege, logic cannot abstract from all semantic content, but it must attend at least to the semantic content of the logical expressions. This is explicitly mentioned in an article on geometry written against the formalist position put forward by David Hilbert:

> Logic is not unrestrictedly formal [...] If it were, then it would be without content. Just as the concept *point* belongs to geometry, so logic, too, has its own concepts and relations; and it is only in virtue of this that it can have a content. Toward what is thus proper to it, its relation is not at all formal. No science is completely formal; but even gravitational mechanics is formal to a certain degree [...] To logic, for example, there belong the following: negation, identity, subsumption, subordination of concepts (FG, p. 338).

As this quotation makes clear, the feature of formality in a science contradicts the requirement of having its own objects and contents. While Kant, accepting the formality of logic, considers that discipline to be a canon, a body of rules, Frege rejects formality in favor of content. Logic, in his conception, is a body of substantive truths and not of empty schemata. In one word, logic is a proper science: its distinctive mark is unrestricted generality.

To sum up, we have seen that logic has a central role in Frege's logicist project. In order to realize the latter, Frege has to create a new and powerful system in the *Begriffsschrift*, characterized by the function-argument analysis of judgments and by a complete account of quantification. Generality (and not formality) is the defining characteristic of Frege's logic, which is conceived as a body of substantial truths. In the following, we are going to show that logic is fundamental also for Frege's conception of analyticity.

3.1.2 Frege's analytic-synthetic distinction

The words 'analytic' and 'synthetic' appear very rarely in Frege's published writings. Except for two more occurrences in the remaining works[10], Frege employs these terms only in the *Grundlagen der Arithmetik* (1884). However, in the third paragraph of this book, the analytic-synthetic distinction is not only mentioned, but it is also given the following definition:

> The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in

[10]De Jong (1996, pp. 291-292) examines the two passages (the former taken from the *Begriffschrift* and the latter from Frege's article *Über Sinn und Bedeutung*) and notes that both the excerpts regard the proper interpretation of the identity sign and the problem of the cognitive content of this kind of expressions.

> carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all the propositions upon which the admissibility of any of the definitions depends. If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one (FA, §3, p. 4).

The common reading of this passage suggests that, according to Frege, a proposition is analytic if it can be proved with help of general logical laws from definitions only[11]; otherwise, that is, if the laws of logic are not sufficient to go back to the definitions and the proof requires extra-logical truths, the proposition is synthetic. This interpretation finds further support in a letter that Frege wrote in 1882 probably to his colleague Anton Marty, where, talking about the principles of computation, he says that they "can be proved from definitions by means of logical laws alone" and so they can be regarded as analytic judgments[12]. However, in reading Frege's reflection on the analytic-synthetic distinction, several considerations must be taken in account.

First of all, the context in which this definition is stated. The passage quoted above is part of a discussion on the notion of proof in mathematics and the propositions of which a proof has to be found, that are mentioned there, belong to the sphere of mathematics. This is certainly no coincidence. Frege introduces his analytic-synthetic distinction as the philosophical counterpart of his logicist project[13]: demonstrating that arithmetic can be reduced to logic amounts to showing that it is analytic. As a result and despite appearances to the contrary, analyticity plays a decisive role in Frege's thought, because, as Proust rightly observes, "the significance of analyticity is now attached to a concrete project: it must be the mortar for building up the system of arithmetic"[14]. But this observation may be pushed one step further. The way in which Frege distinguishes between analytic and synthetic propositions might be seen as strongly influenced by his logicist program: analytic propositions are defined as those that can be proved through logical means alone, *because* Frege's aim was to show that arithmetic could be proved from logic alone. A consequence of this hypothesis, that will be examined below, is that Frege's use of the words 'analytic' and 'synthetic' might be more faithful to his intellectual project, than to the traditional meaning of the terms.

A second important issue is whether the third paragraph of the *Foundations of*

[11]See, for example, Mayer (2003, p. 67).

[12]LTM, p. 79.

[13]This point has been variously acknowledged. See for example de Jong (1996, p. 321) and Mayer (2003, p. 67).

[14]Proust (1989, p. 111).

Arithmetic contains a proper definition of 'analiticity' and 'syntheticity'. Doubts might arise not only considering, as we have just seen, that Frege is talking here about a specific kind of propositions, namely the mathematical ones. A more serious problem derives from observing that Frege uses a conditional, or, in other words, that he seems to give a sufficient but not a necessary condition for a proposition to be analytic or synthetic. Burge[15] proposes several reasons for thinking contrary to the literal reading of the passage in question. For example, he points out that the author of the *Grundlagen* elsewhere uses 'if' where 'if and only if' is meant and he holds that Frege's conception is close to the view on analyticity put forward by Leibniz, whose characterization was necessary and sufficient. However, we think that these motivations are not convincing and do not overweigh the textual evidence. As a consequence, we agree with Bar-Elli in holding that it is left open "the possibility of there being an analytic truth that is not proved, and may even be unprovable"[16].

Third, in order to understand Frege's conception of analyticity, it is necessary to make sense of the specification that "we must take account also of all the propositions upon which the admissibility of any of the definitions depends"[17]. What are admissible definitions? And what are the propositions upon which their admissibility depend? These questions are not easy to answer for at least two related reasons. First, the author's silence on that matter: as Dummett[18] puts it, "in *Grundlagen*, Frege simply takes it for granted that we know a correct definition when we see one". Second, as we will see in detail below, Frege's conception of what counts as an admissible definition significantly varies with the development of his thought, so that we are not allowed to use another text to explain what he really meant in this passage of the *Foundations*. Although it is impossible to rely on Frege's explicit statements on this point, we can nevertheless examine his practice throughout his book. This leads us to acknowledge that most of Frege's efforts are directed towards a definition of arithmetical concepts in terms of logical concepts. This is the case, for example, of the well-known notion of natural number. This observation suggests, as de Jong[19] has rightly recognized, that a definition is admissible only if it is expressed in logical terms. As a result, Frege's logicist program requires to show not only that propositions of arithmetic can be proved from logical truths and through logical methods, but also, and crucially, that the fundamental concepts of arithmetic can be defined in terms of logical concepts[20].

[15]Burge (2005, pp. 359-360).
[16]Bar-Elli (2010, p. 167).
[17]FA, §3, p. 4.
[18]Dummett (1991, p. 31).
[19]See de Jong (1996, pp. 309-310 and pp. 313-314), de Jong (2010, p. 257).
[20]De Jong (1996, p. 314) observes that Frege explicitly recognizes this point for the first

More on Frege's analytic-synthetic distinction can be understood examining the remarks expressed by the author of the *Grundlagen* on the relations of his conception with the one put forward by Kant. The first clue in this sense can be found again in the third paragraph of the *Foundations of Arithmetic*, short before the reflections on the analytic-synthetic distinction:

> Now these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it*, not the content of the judgment but the justification for making the judgment. Where there is no such justification, the possibility of drawing the distinctions vanishes. [...] When a proposition is called a posteriori or analytic in my sense, [...] it is a judgment about the ultimate ground upon which rests the justification for holding it to be true.
>
> * By these I do not, of course, mean to assign a new sense to these terms, but only to state accurately what earlier writers, Kant in particular, have meant by them (FA, §3, p. 3).

The difference between Frege and Kant on this point is not without importance. While the two philosophers agree in believing that the *a priori-a posteriori* distinction concerns the way in which propositions are justified, they disagree as far as the analytic-synthetic distinction is concerned. As we have seen in Chapter 1, Kant maintains that the latter distinction concerns the relationship between the subject and the predicate of a statement, or, in other words, the content of a judgment. In particular, in an analytic judgment, the predicate is (covertly) contained in the subject concept; in a synthetic judgment, the predicate lies entirely outside the subject concept. Here Frege is instead claiming that the analytic-synthetic distinction concerns the justification of judgments, namely, not the way in which people do in fact know the proposition to be true, but the justification that could be given for that proposition.

Why, then, does Frege feel the need to add a footnote claiming that his conception is simply an accurate reformulation of the traditional definition of these terms? The first thing that we would like to underline is that Frege's footnote is restricted to his insight that the analytic-synthetic distinction concerns the justification of judgments and does not apply to the analytic-synthetic distinction as a whole. This observation, which is missed rather frequently by interpreters of Frege, is essential to understand the relationship between Frege and Kant's conceptions.

---

time one year after the publication of the *Grundlagen* in his *On Formal Theories of Arithmetic* (1885), where he says: "if arithmetic is to be independent of all particular properties of things, this must also hold true of its building blocks: they must be of a purely logical nature. From this there follows the requirement that everything arithmetical be reducible to logic by means of definitions" (FTA, p. 114).

Now, one persuasive answer to question regarding Frege's footnote has been suggested by de Jong, who maintains that the author of the *Grundlagen* "is more likely to have Kant's epistemological criterion for analyticity in mind than his original characterization of the distinction"[21]. We have seen in Chapter 1 that Kant's contradiction criterion is not, strictly speaking, a definition, but it is rather endowed with an epistemological function: it is the necessary and sufficient condition for the cognoscibility of analytic judgments. If Frege took the contradiction criterion to be equivalent to Kant's definition of analyticity based on the containment criterion, then his conception based on justification would have been not so distant from the traditional one. Moreover, he could have underestimated the difference between his proposal and the definition of the *Critique* on the basis of the strict relationship between the general laws of logic that contribute to Frege's definition of analyticity and the principle of contradiction that occur as the the supreme principle of analytic judgments in Kant.

If it is now clear that the analytic-synthetic distinction is a matter of justification, it is time to ask what counts as a justification for the author of the *Foundations*. Although it is difficult to find open remarks on this issue, the interpreters unanimously agree on this point[22], except for the interesting theory of Bar-Elli (2010) that we shall examine below: the only kind of justification is, for Frege, proofs that are deductive in character. If this were the case, only propositions that admit of a deductive proof might be classified as analytic, for Frege is explicit that his analytic-synthetic distinction applies only to propositions that can find a justification: "Where there is no such justification, the possibility of drawing the distinctions vanishes". This feature of Frege's definition shall bear with it an apparent difficulty in understanding his texts.

Although it is based on the notion of justification, Frege's analytic-synthetic distinction does not coincide with the other epistemological contraposition *par excellence*, namely that between *a priori* and *a posteriori*. The latter is defined in the *Foundations of Arithmetic* using the following terms:

> For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general, since they contain assertions about particular objects. But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori (FA, §3, p. 3).

Frege's conception is very close to Kant's theory, for in the *Critique*, *a priori* knowledge is said to be necessary and independent of experience, while *a pos-*

[21] De Jong (1996, p. 296).

[22] See, for example, Dummett (1991, p. 23).

teriori knowledge is instead cognized empirically[23]. The two philosophers agree that analytic truths cannot be but *a priori*: according to Frege, since 'general logical laws' are a kind of 'general laws', it turns out that if a proposition can be derived from definitions with only the help of general logical laws, then its proof can be obtained exclusively from general laws. Similarly, the author of the *Grundlagen* agrees with Kant that syntheticity and *a posteriori* knowledge do not coincide. For, although both of them concern, according to Frege, the justification of propositions, their difference consists in the level of generality. For example, Frege accepts with Kant that geometrical propositions are synthetic *a priori*[24]: they are said to be *a priori*, because they can be proved exclusively from general laws; but, at the same time, they cannot be classified as analytic, because these laws are not the ones of general logic, but rather belong to the sphere of a special science, geometry.

3.1.3 Frege's notion of analysis

The affinity with Kant's thought that Frege expresses in the footnote of the third paragraph of his *Grundlagen* is balanced by some critical remarks put forward in the conclusion of the same book. Frege's critical attitude towards Kant's work takes the steps from a widespread charge, that we have already examined in Chapter 1:

> Kant obviously - as a result, no doubt, of defining them too narrowly - underestimated the value of analytic judgments [...] On the basis of his definition, the division of judgments into analytic and synthetic is not exhaustive. What he is thinking of is the universal affermative judgment; there, we can speak of a subject concept and ask - as his definition requires - whether the predicate concept is contained in it or not. But how can we do this, if the subject is an individual object? Or if the judgment is an existential one? In these cases there can simply be no question of a subject concept in Kant's sense (FA, §88, pp. 99-100).

Like Bolzano before him[25], Frege complains that Kant's definition in terms of containment is too narrow because it is restricted to judgments of the subject-predicate form and can be applied only to categorical judgments: but what about

[23] "Such universal cognitions, which at the same time have the character of inner necessity, must be clear and certain for themselves, independently of experience; hence one calls them *a priori* cognitions: whereas that which is merely borrowed from experience is, as it is put, cognized only *a posteriori* or empirically" (CPR, A2/B2).

[24] "In calling the truths of geometry synthetic and a priori, he [Kant] revealed their true nature. And this is still worth repeating, since even to-day it is often not recognized" (FA, §3, pp. 101-102).

[25] See Chapter 2.

other kinds of judgments? This criticism is, of course, related to the enormous advancements in logic presented in the *Begriffsschrift*. Kant's definition had been elaborated in the logical epoch of the traditional syllogistic, where judgments were all of the subject-predicate form. Now that Frege has developed his logical tools, he feels that he cannot profitably employ Kant's distinction for his logicist program. But inventing a new logic was not enough for Frege. He needed to invent a new kind of analysis. And this is the deepest reason for criticizing Kant's approach.

In Chapter 1 we have seen that Kant's conception of analysis is founded on the traditional theory of logical division of concepts. According to this perspective, each concept is assumed to be made up by constituents, each of which finds its place in a hierarchy organized with respect to the notions of containment and inclusion: each genus is contained in its species and each species is contained under its genus. Analysis is thus understood in terms of a decompositional or resolutive process that, starting from the initial concepts, aims at arriving at its simple elements. This kind of analysis is based on the Aristotelian definitions and divisions are taken to be exclusive and exhaustive disjunctions. A characteristic feature of this theory of analysis, that we have not underlined before, is that the division is a univocal process: each propositional content admits of a unique ultimate analysis into simple constituents.

Frege's attack against Kant's conception of analysis is different from Bolzano's criticisms, according to which, as we have seen in Chapter 2, the traditional theory of concepts was founded on an unsophisticated form of representationalism that had to be rejected. Two are the starting points of Frege's reflections on Kant's notion of analysis. First, the traditional theory of conceptual analysis takes the steps from concepts and, only at a later stage, considers the judgments in which they occur. In other words, this theoretical framework prescribes first of all the decomposition of the subject and predicate concepts: only once this passage has been completed, it is possible to determine the relation between the two[26]. This observation is a further element that confirms the strong dependence of the traditional conception of analysis on the Aristotelian syllogistic: the judgments contemplated are only categorical propositions of the subject-predicate form. Thus, Frege's critical consideration can be traced back to the general logic used by Kant.

Second, Frege notices that Kant "seems to think of concepts as defined by giving a simple list of characteristics in no special order": but, according to the author of the *Grundlagen*, "of all way of forming concepts, this is one of the least fruitful"[27]. Later shall we examine in detail the fundamental issue regarding the fruitfulness of concept formation. By now, it is sufficient to underline that, in

[26]See, for instance, the unpublished text entitled *Boole's logical Calculus and the Concept-script* that Frege wrote in 1880-1881 (BLC).

[27]FA, §88, p. 100.

Frege's eyes, the constituents that form a given concept according to this framework are not linked by particular connections and are given a linear but random disposition. This criticism, we think, does not justice to the sophisticated and technical principles that founded the traditional conception of analysis, namely the so-called Porphyrian concept hierarchies that we have discussed in Chapter 1. Nevertheless, Frege restates this point on several occasions and in different shapes. For example, in the unpublished text entitled *Boole's logical Calculus and the Concept-script* written in 1880-1881, Frege discusses Boole's logical notation and deductive machinery. But his treatment of Boole's concept formation anticipates the spirit and the main kernel of the criticism against Kant's conception of analysis put forward in the *Grundlagen* in 1884. In fact, Frege notices that Boole's language allows to form concepts only by taking the logical sums, products and complements of already existing concepts.

The conception of analysis that Frege sets against the traditional and Kantian framework is probably better introduced by seeing it at work in an example taken from the *Begriffsschrift*. The proposition 'Cato killed Cato' can be analyzed into a constant and a variable part, a function and an argument respectively. But this analysis can be carried out in different ways:

> If we here think of 'Cato' as replaceable at its first occurrence, 'to kill Cato' is the function; if we think of 'Cato' as replaceable at its second occurrence, 'to be killed by Cato' is the function; if, finally, we think of 'Cato' as replaceable at both occurrences 'to kill oneself' is the function (BS, §9, p. 22).

Frege replaces the traditional analysis of propositions into subject and predicate concepts with a function-argument analysis that is possible thanks to the logical advancements that he made in the *Begriffsschrift*. Changing the underlying logic implies changing the conception of analysis. But that's not all. As it is made clear by the example, Frege's analysis starts out from judgements and their contents and not from concepts. This might be seen as a natural consequence of the context principle, one of the three fundamental tenets that the author of the *Grundlagen* enunciates in the introductory section of his book, viz. "never to ask for the meaning of a word in isolation, but only in the context of a proposition"[28]. Another essential feature of Frege's proposal is that, unlike the traditional and Kantian conception, the analysis of a proposition is not univocal: as in the case of 'Cato killed Cato', every propositional content admits of distinct decompositions, none of which has a privileged role over the other. We shall come back to this characteristic of Frege's analysis, for it shall be essential in understanding the notion of fruitfulness of mathematical definitions. Last, it is easy to check that the result of

[28]FA, *Introduction*, p. x.

Frege's analysis is different from the Kantian one in that it is structured and not merely a juxtaposition of constituents.

It is worth mentioning at this point the well-known interpretation put forward by Michael Dummett on Frege's conception of analysis[29]. Dummett's starting point is the recognition that Frege endorses the idea that the thought expressed by a sentence is a whole, whose parts are sense expressed by the words in that sentence. This perspective, according to the English philosopher, forces Frege to commit to the principle that the only way for grasping the thought expressed by a sentence is by grasping the senses expressed by the words in that sentence. But, according to Dummett and other interpreters, this part-whole model is not compatible with the function-argument model[30]. This leads Dummett to distinguish in Frege's thought between two sorts of analysis of propositional content: the former, like the Kantian one, would yield unique results and accomodate the part-whole model; the latter, as the one that we have attributed to Frege above, would produce different results and accomodate the function-argument model[31]. Although recent scholars have rejected Frege's commitment to both conceptions of analysis, and in particular to the one connected to the part-whole model[32], Dummett has the merit of having shown the peculiarity of the analysis based on the function-argument pattern.

We have seen that Frege formulates his analytic-synthetic distinction in a way that is adjusted to his logicist program, in which the concept of analyticity plays a significant philosophical role. His intellectual project conditioned Frege's reformulation of the distinction so as to differentiate it in many respects from the Kantian one. To sum up, we have shown that the first important difference consists in Frege's concern with the justification of a proposition as opposed to the Kantian interest in the content of a judgment: in other words, Frege could have taken Kant's epistemological criterion based on the principle of contradiction as a definition of the terms 'analytic' and 'synthetic'. Second, Frege criticizes Kant's distinction for being too narrow: its applicability only to categorical judgments is seen as an overwhelming flaw that must be solved. Third, Frege does not accept Kant's conception of analysis, because it is strictly interwoven with the traditional subject-predicate logic. His *pars construens* is a notion of analysis that is founded

[29]See, for example, Dummett (1981, pp. 62-66). In what follows, I rely on Levine (2002)'s reconstruction of Dummett's argument.

[30]To see this point is sufficient to notice that the parts of a given propositional content cannot be the functions and arguments into which that content may be decomposed unless we are ready to accept the idea that analysis is not univocal.

[31]Dummett calls the first method simply 'analysis' and the second 'decomposition'. To be more precise, he interprets Frege as accepting the part-whole model and rejecting the function-argument pattern. However, he holds more broadly that once analysis and decomposition are distinguished, there is no conflict in Frege's using both the part-whole and the function-argument model: the problem is in using the latter pattern for talking about analysis.

[32]See, for example, Levine (2002).

on the function-argument distinction, gives priority to judgments over concepts and yields different results, each of which is on the same level with the others.

But did Frege conceive his definition as a clarification of Kant's conception? We have seen that in a footnote to the third paragraph of the *Grundlagen*, Frege minimizes the distance between Kant's and his own conceptions saying that the analytic-synthetic distinction concerns the justification, rather than the content, of a judgment. A similar behaviour can be highlighted as far as a more radical distance between the two philosophers is concerned:

> I have no wish to incur the reproach of picking petty quarrels with a genius to whom we must all look up with grateful awe; I feel bound, therefore, to call attention also to the extent of my agreement with him, which far exceeds any disagreement. To touch only upon what is immediately relevant, I consider Kant did great service in drawing the distinction between synthetic and analytic judgments. In calling the truths of geometry synthetic and a priori, he revealed their true nature. And this is still worth repeating, since even to-day it is often not recognized. If Kant was wrong about arithmetic, that does not seriously detract, in my opinion, from the value of his work. His point was, that there are such things as synthetic judgments a priori; whether they are to be found in geometry only, or in arithmetic as well, is of less importance (FA, §89, pp. 101-102).

Frege's thesis that arithmetic is reducible to logic contradicts Kant's idea that arithmetic is synthetic *a priori*. Both these principles are at the core of the philosophers' thoughts and, as we have seen in Section 3.1.1, there is no change in subject. Nevertheless, in this passage Frege acknowledges the value of Kant's work and underplays the difference. Why? An interesting explanation comes again from de Jong[33]. He suggests that Frege was "less than fully aware" of the implications of his reformulation of the Kantian analytic-synthetic distinction and realizes the deep dissonance between the two conceptions of analyticity only later on. This would clarify the fact that after the *Grundlagen* Frege never returned to his analytic-synthetic distinction and preferred to describe the goal of his logicist program as that of reducing arithmetic to logic, instead of showing that arithmetic is analytic.

The most evident element of continuity between the two conceptions of analyticity is that both of them find in the process of analysis the method for discovering analytic propositions and their justifications. As we will see below, Frege treats definitions as a kind of concept formation and deduction as the process that renders explicit what is contained in a concept[34]. In other words, as we will discuss

[33]De Jong (1996, p. 322).

[34]This observation is due to Horty (1992, p. 236).

in the next sections, Frege identifies Kant's analysis of the subject concept with the analysis needed in the definition of this concept. However, despite this important point of similarity, it cannot be forgotten that Frege's underlying notion of analysis is different from Kant's traditional conception of analysis. This, we think, together with the requirement that admissible definitions must be expressed in logical terms, explains why the most part of Kant's examples of analyticities turns out to be synthetic according to Frege.

3.1.4 Frege on the analyticity of general logical laws

As it is widely recognized in the literature[35], Frege's definition of analyticity of the third paragraph of the *Grundlagen* does not seem to specify whether general logical laws, through which analytic propositions are brought back to definitions, are themselves analytic or synthetic. Nor does Frege return explicitly to the issue elsewhere. Two are the main readings of this fact proposed by the scholars.

The first interpretative trend holds that, although he is not explicit on this point, Frege maintains that logical truths are analytic. This is, for example, Dummett's perspective. The English philosopher believes that "With uncharacteristic carelessness, Frege has framed his definition so as not to cover the initial premises themselves", but adds that "an obvious extension of his definition would rate [...] the general logical laws as analytic"[36]. Burge recognizes that Frege "neglects to formulate his notions of analyticity and apriority so as to either include or rule out the foundations of logic", but he agrees with Dummett that Frege's behaviour can be interpreted as a "harmless oversight"[37]. A slightly different explanation of Frege's silence on the analyticity of logical laws is provided by Proust, who holds that the "analyticity of *logical* propositions is not itself in question, but is rather presupposed by the problem Frege has to solve: he is not concerned with knowing what an analytic truth is [...] he is concerned whether the truths of a science can be clearly identified as analytic or synthetic"[38]. The idea is that Frege's logical laws are analytic, but he takes this point for granted, because his interest rests with the status of truths of other sciences.

The second reading of Frege's treatment of logic in connection to the issue of analyticity is that, according to the author of the *Grundlagen*, general logical laws are neither analytic nor synthetic. This is the position put forward by Burge, who, in his *Postscript to 'Frege on Apriority'*[39], withdraws his previous thesis

[35]See, for example, Dummett (1991, p. 24), Burge (2005, p. 322, pp. 388-389), Shieh (2008, p. 1010), Bar-Elli (2010), Proust (1989, p. 112).

[36]Dummett (1991, p. 24).

[37]Burge (2005, p. 322, p. 388).

[38]Proust (1991, p. 112).

[39]Burge (2005, pp. 388-389).

that Frege's silence was an oversight of no great significance. He now observes that, as we have seen in Chapter 1, "omission of basic logical axioms from the category *analyticity* goes back to Kant's own formulations" and maintains that Frege intentionally follows Kant in saying that the analytic-synthetic distinction does not apply to logic. The reason for excluding the class of logical laws from analyticities is, according to Burge, that "such laws are not subject of analysis", while "both Kant's definition and Frege's, however, take analyticity to consist in being subject to analysis".

We think that this second reading must be rejected for several reasons. First of all, we have seen in Chapter 1 that the main reason why Kant is not interested in applying the analytic-synthetic distinction to logic must be found in his conception that logic is a general body of rules that govern the use of certain faculties. In being a canon, logic is, according to Kant, a propedeutic and not a kind of knowledge. But we have shown in Section 3.1.1 that Frege does not follow Kant on this point. For the author of the *Grundlagen*, as for Bolzano[40], logic is a body of truths and a science in the strict sense of the term. Why, then, should Frege have avoided to apply the analytic-synthetic distinction to logic, given his concern with *a priori* sciences? The second motivation[41] against Burge's proposed interpretation is that Frege was aware of the possibility of equivalent but alternative axiom systems[42]. But that, which is a logical axiom or a general logical law in a certain system, may become a provable theorem in another axiomatic system to the effect that the same proposition turns out to be neither analytic nor synthetic in the first system, while analytic in the second one. However, it is highly implausible that Frege's analyticity (as well as of apriority) is a notion relative to a system. The third reason Burge puts forward in order to support his thesis is that logical laws are not subject to analysis: but, as we will see below, according to Frege the process of analysis amounts to the process of concept formation and it is not clear why logical laws should be extraneous to analysis understood in this sense.

Once we have rejected the second interpretation of Frege's reticency on the analyticity of logic, we now turn to the first reading. However, before embracing the thesis that, according to Frege, logical laws are analytic, we must get rid of the following problem that arises in its connection. On the one hand, as we have highlighted above, the author of the *Grundlagen* affirms that the possibility of drawing the analytic-synthetic distinction, such as the *a priori*-*a posteriori* distinction, vanishes when there is no justification for making the judgment. In

[40]See Chapter 2.

[41]This reflection is quite common in the literature. See, for example, Bar-Elli (2010, p. 168).

[42]In the *Begriffsschrift*, §13, p. 29, talking about his logical system, Frege says: "Now it must be admitted, certainly, that the way followed here is not the only one in which reduction can be done [...] There is perhaps another set of judgments from which, when those contained in the rules are added, all laws of thought could likewise be deduced".

other words, the notions of analyticity and apriority can be applied only to those propositions that can be justified. On the other hand, in the very same passage in which the analytic-synthetic distinction is introduced, Frege states that general laws "neither need nor admit of proof"[43]. This means that logical laws, that are a special case of general laws, do not admit of proof and that logical laws cannot be seen as special cases of one-line proof[44]. If we put together the two observations, it seems that we obtain the conclusion that general laws are neither analytic, nor synthetic, neither *a priori*, nor *a posteriori*. In particular, it seems that we reach the unacceptable conclusion that logical laws are neither analytic nor *a priori* and that geometrical axioms are neither synthetic nor *a priori*.

The unwelcome conclusion is unavoidable unless we are able to show that justification is not a synonym of proof, or, in other words, that proof is not taken to be the only pertinent way of justification. For suppose that logical laws are justified by some kind of justification j that is not a proof. In this case, Frege can affirm at the same time that 1) the analytic-synthetic and the *a priori-a posteriori* distinctions apply only to justifiable judgments; 2) general laws do not admit of proof and 3) logical laws, being justifiable by j, are analytic and *a priori*. But what does j stand for? The possibility of the existence of such a kind of justification j that is different from deductive proof is left open by Frege's choice to use, as we have seen, a conditional instead of a biconditional in giving the definition of the analytic-synthetic distinction suggesting that there may exist analytic truths that cannot be proved.

For sure, deductive proof is the kind of mathematical and logical justification *par excellence*. However, in his compelling article, Bar-Elli (2010) suggests that justification is not, for Frege, only deductive or inferential, but that there is also an epistemic kind of justification, in which statements are justified "in expressing aspects or features of the ways 'their objects' are given to us, or, in other words, by the modes of presentation or senses (*Sinne*) of the things their are about"[45]. The idea is that the way in which fundamental objects are given is central to the justification of the axioms of a system. This is what Frege suggests in the geometrical case in a posthumous essay: "So long as I understand the words 'straight line', 'parallel' and 'intersection' as I do, I cannot but accept the parallel axiom [...] Their sense is indissolubly bound up with the axiom of parallels"[46]. Bar-Elli standpoint is the more convincing the more we read it as a clarification of Frege's Euclideanism about axioms[47]: to put it roughly, the justification of general

[43]FA, §3, p. 4.

[44]This is observed by Dummett (1991, p. 24).

[45]Bar-Elli (2010, p. 173).

[46]LM, p. 247.

[47]Burge (2005)'s tenth chapter entitled *Frege on Knowing the Foundations* underlines, beyond the familiar picture of Frege's indebtedness to the Euclidean tradition, also the elements in Frege's

laws (and, in particular, of logical laws) is their self-evidence or, to avoid seemly psychological allusions, their independence of other truths.

To sum up, we believe that, in Frege's theory, logical laws are analytic, despite the absence of an explicit statement of the author. We have shown that, first, what discriminates Frege's position from the Kantian one and leads us to conclude that logic is analytic only for the former and not for the latter, although both of them are not explicit on this point, is the difference in the underlying conception of logic. Second, the laws of logic that are chosen as axioms of the system are analytic because of their self-evidence; logical theorems are instead analytic in that they can be proved through logical laws only. Third, the analyticity of logic is taken for granted by the author of the *Grundlagen*. Frege has solved in the initial paragraphs of his book the problem of reformulating Kant's analytic-synthetic distinction in a way that was suitable to his logicist program, but his real concern, viz. the thesis that he believes in need of proof, is the status of arithmetical propositions, not that of basic logical laws[48].

3.1.5 Analyticity of Logic in Logical Empiricism

The fundamental role of the principle of analyticity of logic in the theories of the logical empiricist movement is clearly expressed in the manifesto of the Vienna Circle written in 1929 by Rudolf Carnap, Hans Hahn and Otto Neurath. The *Wissenschaftliche Weltauffassung* is characterized by two features: it holds that there is knowledge only from experience and it finds in logical analysis the method of clarification of philosophical problems. Metaphysics is rejected because it relies on the ambiguity of natural language and it claims that it can produce knowledge on its own sources without using any empirical material. Logical analysis shall overcome not only traditional forms of metaphysics, but also "the hidden metaphysics of Kantian and modern *apriorism*"[49]. Kant's synthetic *a priori*, which had already been impoverished by Frege's thesis that arithmetic is analytic, is here rejected *in toto*:

> The scientific world-conception knows no unconditionally valid knowledge derived from pure reason, no 'synthetic judgments a priori' of the kind that lies at the basis of Kantian epistemology [...] It is precisely in the rejection of the possibility of synthetic knowledge a priori that the basic thesis of modern empiricism lies. The scientific world-conception knows only empirical statements about things of all kinds, and analytical statements of logic and mathematics (VC, p. 308).

---

thought that apparently differ from this framework.

[48] As we have mentioned above, this observation is due to Proust (1989).

[49] VC, p. 308.

Following the neo-positivistic perspective, two aspects that from Kant on were kept separated do now coincide: the analytic-synthetic distinction on the one hand, the *a priori-a posteriori* distinction on the other. This is because synthetic statements are always grounded in facts and analytic statements are known *a priori*. What is crucial for our reconstruction is that logical laws, together with mathematical statements, are, according to this movement, the paradigmatic examples of analytic judgments and this is true despite the definition of analyticity is not exactly the same for all the authors that belong to this cultural *milieu*[50]. The principle of analyticity of logic, which was implicitly accepted by Frege, finds in this philosophical movement the most fertile ground.

Willard Van Orman Quine's well-known paper *Two Dogmas of Empiricism*[51] represents the strongest attack against the logical positivists' epistemology and, in particular, against the analytic-synthetic distinction and the theory of reductionism. As far as the first dogma is concerned, the American philosopher gives the following characterization of analytic statements:

> Statements which are analytic by general philosophical acclaim [...] fall into two classes. Those of the first class, which may be called logically true, are typified by: (1) No unmarried man is married. [...] But there is also a second class of analytic statements, typified by: (2) No bachelor is married. The characteristic of such a statement is that it can be turned into a logical truth by putting synonyms for synonyms; thus (2) can be turned into (1) by putting 'unmarried man' for its synonym 'bachelor'. We still lack a proper characterization of this second class of analytic statements, and therewith of analyticity generally, inasmuch as we have had in the above description to lean on a notion of synonymy which is no less in need of clarification than analyticity itself (Quine, 1951, p. 23).

The definition that a statement is analytic if it is either a logical truth or can be turned into a logical truth by putting synonyms for synonyms can be taken as a good representative of the modern and empiricist account of the matter, which is different not only to Kant's analytic-synthetic distinction, but also to Frege's approach. The problem rests with the so-called material analyticities[52], of which the judgment 'All bachelors are unmarried man' is the standard example. For,

---

[50]As anticipated in the introduction to this thesis, we are not interested here in the reconstruction of the different conceptions of analyticity held by the philosophers that declare themselves to be logical empiricists, for our main focus is on those accounts that contrast this traditional paradigm and argue that logic is not analytic.

[51]Quine (1951).

[52]On this point see Mayer (2003, p. 68) and de Jong (1996, pp. 317-318 and 322-323).

as we have seen in Section 3.1.2, Frege's analyticities are propositions that can be derived from definitions with the help of logical laws only; but the definitions themselves must be expressed through logical means. However, concepts of natural language are usually ambiguous: therefore, their definition cannot be taken as admissible according to Frege's requirements. This difference can be explained as a result of the context and the aims that pushed Frege to formulate his analytic-synthetic distinction, namely, his logicist project to reduce arithmetic to logic. This is the reason why he, unlike logical positivists, restricts this notion to formal proofs.

As the text above makes clear, Quine's criticism against the notion of analyticity in general is motivated by the impossibility of giving a non-circular characterization of the second class of analytical statements, the material ones: for this kind of sentences, the concept of analyticity as truth in virtue of meaning presupposes the idea of synonymy, but the latter in turn cannot be defined without the former. In other words, Quine's epochal criticism spares completely the first class of analyticities made up of logical truths. In holding that logical truths are non-questionable cases of analyticities, the author of the *Two Dogmas* agrees with his main critical target, Rudolf Carnap.

Although his account of analyticity underwent several significant modifications, Carnap's formulations are characterized by a common feature, namely, the idea that the notion of analytic truth is relative to a certain language or, to be more precise, that analytic truths define what makes something into a language. The principle of tolerance[53], which is one of the most radical changes of *The Logical Syntax of Language*, implies the thesis that analyticity is a relative notion and that analytic truths are the rules and the consequences of a given particular language. In Carnap's conception, 'logical' is simply treated as a synonym with 'analytic': "By means of the concept 'analytic', an exact understanding of what is usually designated as 'logically valid' or 'true on logical grounds' is achieved [...] In material interpretation, an analytic sentence is absolutely true whatever the empirical facts may be. Hence, it does not state anything about facts"[54]. This attitude towards the relationship between logic and analyticity does not change during the development of Carnap's philosophy: his move to the semantic setting is accompanied by the abandonment of his *Syntax* method for defining 'analytic', but not of the belief that logical truths belong to the pre-philosophical *explicandum* of analyticity. According to his final account of analyticity, which can be found in his paper *Meaning Postulates* (1951), a sentence of a specified formal language is analytic in

[53] "*In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments" (Carnap, 1934, p. 52).

[54] Carnap (1934, p. 41).

that language just in case it is logically implied by the meaning postulates of the language, which are part of the specification of the linguistic framework.

3.2 Logic is tautologous

3.2.1 The paradox of analysis

The term 'paradox of analysis' was first used by Cooper H. Langford in his 1942 article entitled *The Notion of Analysis in Moore's Philosophy.* After having noticed that analysis plays a decisive role in determining the character of Moore's philosophy, Langford supposes that the significance of analysis may be denied on the ground of the paradox of analysis, which may be formulated as follows:

> Let us call what is to be analyzed as the analysandum, and let us call that which does the analyzing the analysans. The analysis then states an appropriate relation of equivalence between the analysandum and the analysans. And the paradox of analysis is to the effect that, if the verbal expression representing the analysandum has the same meaning as the verbal expression representing the analysans, the analysis states a bare identity and is trivial; but if the two verbal expressions do not have the same meaning, the analysis is incorrect (Langford, 1942, p. 323)[55].

The paradox of analysis is here described as the incompatibility of two desirable properties, correctness and informativity, and its scope is the widest possible: it affects any kind of analysis and the whole philosophy, if this discipline is interpreted in terms of analysis. The paradox notwithstanding, many would be tempted to say that we have numerous examples of successful analysis, which are both intuitively correct and significantly informative: this puzzle must find a solution. The natural way out of the paradox seems to be the distinction between two levels of discourse, the former aimed at justifying the correctness of analysis and the latter its informativeness, and this strategy founds many attempts made to solve the problem.

Being so pervasive, the paradox of analysis is as old as Western philosophy and Langford's formulation has many historical precedents. We now mention four of

[55] In his reply to Langford, Moore confesses he is unable to solve the puzzle and gives the following suggestion: "I think that, in order to explain the fact that, even if 'To be a brother is the same thing as to be a male sibling' is true, yet nevertheless this statement is *not* the same as the statement 'To be a brother is to be a brother', one *must* suppose that both statements are in *some* sense about the expressions used as well as about the concept of being a brother. But in *what* sense they are about the expressions used I cannot see clearly; and therefore I cannot give any clear solution to the puzzle" (Moore, 1968, p. 666).

them. In one of the most well-known passages of Plato's *Meno*, Socrates proposes to Meno to inquire into what they do not know and his interlocutor's answer is to pose an epistemological challenge to the very possibility of inquiry: "And how will you inquire into a thing when you are wholly ignorant of what it is? Even if you happen to bump right into it, how will you know it is the thing you didn't know?"[56]. Socrates rephrases the question and says: "[A] man cannot search either for what he knows or for what he does not know [...] He cannot search for what he knows - since he knows it, there is no need to search - nor for what he does not know, for he does not know what to look for"[57]. Meno's challenge is but an epistemological version of the paradox of analysis and it prompts Socrates to introduce the theory of recollection: in learning something, we do not come to know something we did not already know; what we do is recollect and this is proven by Socrates through the examination of the slave boy.

Another formulation of the paradox of analysis can be read in the work *Quod nihil scitur*, written by the doctor and philosopher Francisco Sanchez in 1581. Founding his skepticism on a rejection of Aristotelianism and an epistemological analysis of knowledge, he holds that all definitions are merely nominal and are not related with the object considered. A man is a single thing and yet it is described by several names, such as being, substance, body, animal. But, he continues, "if they refer to the same thing, then they are too many of them; but if they mean different things, then a man is not a single thing possessing identity"[58].

The third statement of the paradox of analysis that we would like to mention is put forward by Frege in his 1894 review of Husserl's *Philosophie der Arithmetik*. Frege first concentrates on ideas and then on definitions:

> If words and combinations of words mean ideas, then for any two of them there are only two possibilities: either they designate the same idea or they designate different ideas. In the former case it is pointless to equate them by means of a definition: this is 'an obvious circle'; in the latter case it is wrong [...] A definition is also incapable of analysing the sense, for the analysed sense just is not the original one. In using the word to be explained, I either think clearly everything I think when I use the defining expression: we then have the 'obvious circle'; or the defining expression has a more richly articulated sense, in which case I do not think the same thing in using it as I do in using the word to be explained: the definition is then wrong (RH, p. 199).

Both Sanchez and Frege's formulations focus on definitions. But definitions do not exhaust the paradox of analysis, for, as we have seen, the problem affects

[56]Meno, 80D.
[57]Meno, 80E.
[58]Sanchez, p. 175.

everything that is essentially based on analysis. This might be the case also for logical inferences, if we read the conclusion of an inference as resulting from the analysis of its premises. In 1934, Cohen and Nagel express the paradox of inference in the following terms:

> If in an inference the conclusion is not contained in the premises, it cannot be valid; and if the conclusion is not different from the premises, it is useless; but the conclusion cannot be contained in the premises and also possess novelty; hence inferences cannot be both valid and useful (Cohen and Nagel, 1934, p. 173).

It is clear that this paradox is nothing more than a consequence of the paradox of analysis: an inference cannot be both valid and useful at the same time, because analysis cannot be both correct and informative at the same time.

The connection between the paradox of analysis and our research on the epistemological status of logic is immediate. In Section 3.1 we have seen the development of the principle of analyticity of logic from Frege to the Vienna Circle. Holding that logic is analytic amounts to say that logic is the result of some kind of analysis: for example, the conclusion of a logical inference is the result of the analysis of its premises. But here the paradox does show all of its force: if logic is analytic, then (or so it seems) it must be trivial. In other words, the principle of analyticity of logic, together with the paradox of analysis, implies the principle of tautologicity of logic: since it is correct, logic cannot yield new information.

The logical empiricist movement accepts the paradox and its seemly inescapable consequence that logic and mathematics are tautologous. In the *Wissenschaftliche Weltauffassung*, Carnap, Hahn and Neurath write:

> Logical investigation [...] leads to the result that all thought and inference consists of nothing but a transition from statements to other statements that contain nothing that was already in the former (tautological transformation) [...] The conception of mathematics as tautological in character, which is based on the investigation of Russell and Wittgenstein, is also held by the Vienna Circle (VC, pp. 308, 311).

This thesis held by the Vienna Circle is influenced by the *Tractatus*' analysis of logical truth. While Frege treats logical truths as universal laws applying to any statement, Wittgenstein believes that the laws of logic are tautologies, which, in themselves, do not say anything. According to the Austrian philosopher, tautologies, which cannot be false, do not tell us how the world in fact is: "tautologies and contradictions show that they say nothing. A tautology has no truth-conditions, since it is unconditionally true [...] (For example, I know nothing about the weather when I know that it is either raining or not raining.)"[59]. As a result, tautologies, as well as contradictions, lack sense.

The thesis that logic is tautologous raises some problems. The conclusion obtained through a long deductive chain might appear as an actual novelty with respect to its premises and the recognition that a particularly complex sentence is a tautology might appear as a true discovery. Similarly, it is difficult to regard the results of mathematics as sterile and to dismiss the fact that it is impossible to know all the logical consequences of what we know. Thus, a legitimate question is then how do philosophers, who hold that logic is analytic and trivial, deal with the intuitive idea that logical deduction is fruitful? Or how can the paradox of analysis be overcome?

In Section 3.2.2, we point out that in the *Grundlagen*, Frege implicitly denies the paradox of analysis: he maintains that logic is analytic, correct and informative at the same time. Roughly put, the idea is that 'A is B' is correct in so far as A and B have the same content and fruitful to the extent that the content of A and B is split up differently. In Section 3.2.3, we show that Frege's position on the fruitfulness of definitions undergoes a radical change after the introduction of the *Sinn-Bedeutung* distinction. Initially, he tries to use this distinction to solve the paradox of analysis: in brief, 'A is B' is correct because A and B share the same reference and it is informative because A and B have different senses. Then, he seems to recognize the inadequacy of this answer and tries to avoid definitions based on analysis. In Section 3.2.4, we examine a solution common in the logical positivistic *milieu*. Both the principle of analyticity of logic and the paradox of analysis are accepted and, as a consequence, logic is said to be tautologous. 'A is B' is correct and trivial, because A and B have the same meaning: the only kind of novelty contained in an analytical statement is psychological in character. In Section 3.2.5, we consider Wittgenstein's solution: the author of the *Tractatus* accepts that logic is both analytic and tautologous, but he finds in the language an objective reason why logical inferences seem to be informative despite their triviality.

3.2.2 Frege against the "legend of the sterility of pure logic"

Frege offers an original solution to the problem posed by the paradox of analysis. In the *Grundlagen der Arithmetik*, he maintains that "propositions which extend our knowledge can have analytic judgments for their content"[60] and, as a special case of this fact, he holds the thesis that logic is, at the same time, analytic and

[59] Wittgenstein (1921, 4.461).
[60] FA, §91, p. 104.

informative.

His starting point seems to be, once more, his logicist position that arithmetic can be reduced to logic. According to the author of the *Grundlagen*, if his logicist bet proved to be winning, "the truths of arithmetic would then be related to those of logic in much the same way as the theorems of geometry to the axioms"[61]. In other words, the relationship between logic and arithmetic would be so close that the former would contain, albeit in a concentrated format, all the theorems of the latter discipline. At the same time, Frege assumes as a matter of fact that arithmetic cannot be charged of sterility: the results of this discipline are so evident and exceptional that nobody could seriously maintain that arithmetical propositions are uninformative or tautologous. The natural result of combining these two premises is that the fruitfulness of arithmetic expands to cover logic, so that also the latter discipline must be equally seen as capable of extending our knowledge:

> Can the great tree of the science of number as we know it, towering, spreading, and still continually growing, have its roots in bare identities? And how do the empty forms of logic come to disgorge so rich a content? (FA, §16, p. 22)
>
> [...] the prodigious development of arithmetical studies, with their multitudinous applications, will suffice to put an end to the widespread contempt for analytic judgments and to the legend of the sterility of pure logic (FA, §17, p. 24).

However, the connection between logic and arithmetic cannot be considered as an explanation in itself and a way out of the paradox of analysis. How does Frege manage to keep analyticity and informativeness together? The answer must be searched in the notion of definition proposed at the beginning of the Eighties. In this period, Frege characterizes definitions as a kind of concept-formation through the process of analysis. We have seen above that Frege's conception of analysis can be read as opposed to the Kantian paradigm: while the latter is characterized by the subject-predicate pattern, Frege's notion of analysis is based on the function-argument model, takes the steps from judgments and not from concepts and it is not univocal. It is now time to see these two conceptions of analysis at work in constructing definitions, viz. in forming concepts.

In the already mentioned unpublished text *Boole's Logical Calculus and the Concept-Script*, Frege criticizes Boole's concept formation with the following words:

> In this sort of concept formation, one must, then, assume as given a system of concepts, or speaking metaphorically, a network of lines.

[61] FA, §17, p. 24.

> These really already contain the new concepts: all one has to do is to use the lines already there to demarcate complete surface areas in a new way. It is the fact that attention is principally given to this sort of formation of new concepts from old ones, while other more fruitful ones are neglected which surely is responsible for the impression one easily gets in logic that for all our to-ing and fro-ing we never leave the same spot (BLC, p. 34).

Frege explains that, according to Boole's perspective, the definition of *homo* in terms of *animal rationale* corresponds to logical multiplication and illustrates that the extension of *homo* is the intersection of two circles, the former representing the extension of the concept *animal* and the latter of *rationale*. Moreover, he notices that concepts can be formed in Boolean logic not only through multiplication, but also through addition. This is the case, for example, of the the definition of 'capital offence' as 'murder or the attempted murder of the Kaiser or of the rules of one's own *Land* or of a German prince in his own *Land*'. The extension of the concept 'capital offence' is given as the union of two circles, the former representing the extension of 'murder' and the latter of the second disjunct of the definition above. Addition and multiplication are the familiar ways of forming concepts but, as the quotation makes clear, they are characterized by the same method of using old lines to demarcate new surfaces. This kind of concept-formation is, in Frege's eyes, responsible for the idea that logic is sterile.

We have underlined in Section 3.1.3 that one of the most original feature of Frege's notion of analysis is that it can yield more than one result. This is a consequence of the possibility of choosing in different ways which parts of a judgment to consider as variable and which to consider as constant. Now, the function-argument analysis not only yields a plurality of results. It also provides a fruitful method of concept-formation. One of Frege's examples[62] considers the equation $2^4 = 16$. This judgment can be decomposed in several ways. First, if we consider number 2 to be variable, which may be indicated as $x^4 = 16$, we obtain the concept '4th root of 16'. Second, if we consider number 4 to be variable, so that we may write $2^x = 16$, we obtain the concept 'logarithm of 16 to the base 2'. Third, if we consider both 2 and 16 to be replaceable and we indicate this by the expression $x^4 = y$, we obtain the relation of a number to its 4th power. The point is that the concepts that have been formed in this way are actually new and fruitful:

> There's no question here of using the boundary lines of concepts we already have to form the boundaries of new ones. Rather, totally new boundary lines are drawn by such definitions - and these are the scientifically fruitful ones. Here too, we use old concepts to construct new

[62]BLC, pp. 16-17.

> ones, but in so doing we combine the old ones together in a variety of ways by means of the sign for generality, negation and the conditional (BLC, p. 34).

In modern terms, we could say that the fruitfulness of the process described by Frege consists in the individuation of a quantificational structure in logically unstructured judgments or, in other words, in the recognition of certain patterns, such as predicates and relations, in given propositions[63]. The problem is to establish which is the pattern, among the many options available in every proposition, that better suits the demonstration. Moreover, it is the work of extracting these structures from judgments that yields an extension of knowledge.

In the *Grundlagen*, Frege's criticism of Kant's notion of analysis echoes the words he spent for Boole in the text written in 1880-1881[64]. He states that Kant's definitions are mere lists of unordered characteristics composing concepts and complains that this concept formation is not fruitful. On the contrary, every element of the definitions presented in the *Foundations* are, according to Frege, "intimately" and "almost organically" connected one to the other[65]. In this work, Frege makes one step further, for he says that it is precisely due to this new concept-formation that produces fruitful definitions that analytic judgments can extend our knowledge:

> What we shall be able to infer from it [i.e., from the more fruitful type of definition], cannot be expected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they

[63]Dummett explains this point with the following example: "If we define 'x is intermediate in size between y and z' to mean 'Either y is larger than x and x is larger than z, or z is larger than x and x is larger than y', we need, if we are to draw the conclusion 'There is a body intermediate in size between Jupiter and Mars', to be able to recognize the complex three-place predicate as extractable from the proposition 'Either Jupiter is larger than Neptune and Neptune is larger than Mars, or Mars is larger than Neptune and Neptune is larger than Jupiter': we have to discern that pattern in it" (Dummett, 1991, p. 42).

[64]In particular, Frege uses again the geometrical metaphor: "If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this, therefore, what we do - in terms of our illustration - is to use the lines already given in a new way for the purpose of demarcating an area. Nothing essentially new, however emerges in the process. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all" (FA, §88, p. 100).

[65]FA, §88, p. 100.

> are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house (FA, §88, pp. 100-101).

Here, Frege is explaining two main points: first, the passage from the fruitfulness of definitions to the fertility of deductions; second, the link between analyticity and informativeness. As far as the first issue is concerned, Frege is saying that conclusions that can be drawn from fruitful definitions extend our knowledge exactly because of the fertility of those definitions and despite the fact that conclusions can be proved by purely logical means alone. In other words, in this excerpt Frege maintains that the fruitfulness of definitions makes deductions that take the steps from them informative and this happens even in the case in which deductions employ only general logical laws and, thus, produce analytic conclusions.

The second matter receives another, albeit metaphorical, explanation. On the one hand, deductions are knowledge-extending processes because conclusions are contained in the premises, i.e. the initial definitions, only *in posse*, but not *in esse*. It is in the transition from potentiality to actuality, viz. from seeds to plants, that the informativeness of deductive reasoning finds its place. This is a resource-consuming process: as Dummett puts it, this is not a mechanical procedure, but it has a rather creative component due to the requirement of pattern recognition[66]. On the other hand, however, the conclusion of a deductive process is analytic: there is no need of any other tool than the logical ones to prove it, since the conclusion is, although only potentially, already contained in the given definitions. Informativeness and analyticity are thus two unavoidable consequences of the way Frege characterizes the notion of deduction and its dependence on fruitful definitions. This is well explained by Dummett in the following terms:

> Since it has this creative component, a knowledge of the premisses of an inferential step does not entail a knowledge of the conclusion, even when we attend to them simultaneously; and so deduction can yield new knowledge. Since the relevant patterns need to be discerned, such reasoning is fruitful; but, since they are there to be discerned, its validity is not called in question (Dummett, 1991, p. 42).

The solution to the paradox of analysis that Frege offers at the beginning of the Eighties is fascinating. It has the merit of breaking the connection between tautologicity and analyticity. The basic idea is that the recognition of a certain pattern and the extraction of a quantificational structure from a given judgment is, by itself, a creative process. Logical deduction is seen as a knowledge-extending procedure because theorems are concentrated into basic definitions and a resource-consuming procedure of extraction is needed.

[66]Dummett (1991, p. 42).

3.2.3 Frege after the *Sinn-Bedeutung* distinction

Between the *Grundlagen der Arithmetik* (1884) and the first volume of the *Grundgesetze der Arithmetik* (1893), Frege introduces his well-known distinction between *Sinn* (sense) and *Bedeutung* (reference, meaning, significance). First presented in the 1891 article *Function and Concept*, the distinction was developed in another paper entitled *Sense and Reference* (1892). Roughly put, the reference of a proper name is the objects it indicates, while its sense is said to be the expression of the name. Thus, the sense of different names is different even if they have the same referent, as in the case of 'Hesperus' and 'Phosphorus'. Similarly, the reference of a sentence is its truth value, the True or the False, while its sense is the thought it expresses. For instance, the two sentences 'The morning star is a body illuminated by the Sun' and 'The evening star is a body illuminated by the Sun' express different thoughts, but have the same reference, because the morning star is the evening star.

The introduction of the distinction between sense and reference has important consequences on the *Grundlagen*'s conception of definitions. In particular, the fruitfulness of good definitions devised in the *Foundations* is substituted by a view that confines their usefulness to an abbreviatory and simplificatory function. In other words, after 1884, definitions are, from a logical point of view, "wholly inessential and dispensable"[67] and are devoid of any creative power: "Just as a geographer does not create a sea when he draws boundary lines and say: the part of the ocean's surface bounded by these lines I am going to call the Yellow Sea, so too the mathematician cannot really create anything by his defining"[68]. But this radical change in view as far as definitions are concerned brings along an equally drastic change in Frege's response to the paradox of analysis. Two are the main sources in which Frege explicitly addresses the problem: the Review of Husserl's *Philosophie der Arithmetik I* published in 1894 and the posthumous text entitled *Logic in Mathematic* dated 1914.

In his work, Husserl criticizes the cases of equivalence of the *Grundlagen* on the basis that they would not embody identity of *content*, but only equivalences in *extension*. In Frege's definitions of the *Grundlagen*, the *definiendum* and the *definiens* are only logically equivalent, but are articulated through different concepts and are thus neither intensionally nor epistemically equivalent. To use Frege's terminology, Husserl's point seems to be that a definition is correct just in case the sense of the *definiens* is identical to that of the *definiendum*. After having clearly stated the paradox of analysis with the words quoted above, Frege replies to Husserl's attack by distinguishing in the treatment of definitions the attitudes of psychological logicians and of mathematicians. The former, among which Husserl

[67]LM, p. 208
[68]GGA I, p. xiii.

seems to be included, are interested in the sense of the words, which they fail to discern from the ideas. On the contrary, mathematicians' main concern is with the thing itself, viz. the meaning of the words:

> The reproach that what is defined is not the concept but its extension actually affects all mathematical definitions. For the mathematician, it is no more right and no more wrong to define a conic as the line of intersection of a plane with the surface of a circular cone than to define it as a plane curve with an equation of the second degree in parallel coordinates. His choice of one or the other of these expressions have neither the same sense nor evoke the same ideas. I do not mean by this that a concept and its extension are one and the same, but that coincidence in extension is a necessary and sufficient criterion for the occurrence between concepts of the relation that corresponds to the identity between objects (RH, p. 200).

According to Frege, mathematical definitions are correct if the *definiens* and the *definiendum* share the same reference, but it is not necessary that they share the same sense. Frege's distinction between sense and reference, which disambiguates the notion of meaning, is here exploited in the obvious way to solve the paradox of analysis: definitions are correct because the two terms share the same reference, but at the same time they are informative because they express different senses. This suggests that the fruitfulness of mathematical definitions understood in this way consists in the discovery that two different concepts expressed through two different senses have the same referent. This can be regarded as a genuine extension of knowledge.

However, this attempt to solve the paradox of analysis through the sense-reference distinction proves to be inadequate. As several scholars underline[69], the problem is that analyses and definitions seem to capture more than just sameness of reference: the sameness of truth-values between the sentence A and B cannot be sufficient to conclude that the latter is an analysis or a definition of the former. Consider the key example Frege proposes in the *Grundlagen*[70] in order to introduce his definition of number, namely the sentences 'Line a is parallel to line b' and 'The direction of line a is identical with the direction of line b'. When the former is true, then the latter is true and viceversa: the two statements share the same *Bedeutung*. But this cannot be all of the story: they seem to share the same sense, on some conception of sense. The same happens, according to Dummett[71], for Frege's definition of equinumerosity: here again, the sameness of *Bedeutung*

[69] See, for example, Beaney (2005, p. 295) and Shieh (2008, p. 1001).
[70] FA, §64, p. 74.
[71] Dummett (1991, pp. 148-154).

is not sufficient. Frege acknowledges this conclusion in several passages of his production and an explicit endorsement of this thesis is offered in the first volume of the *Grundgesetze*: “By means of a *definition* we introduce a new name by stipulating that it is to have the same sense and the same *Bedeutung* as a name composed of already known signs”[72].

In his posthumous writing *Logic in Mathematics*, Frege distinguishes between two kinds of definitions, which he calls ‘constructive’ (*aufbauende*) and ‘analytic’ (*zerlegende*) respectively. In the first kind of definitions, “we construct a sense out of its constituents and introduce an entirely new sign to express this sense”[73]. Constructive definitions are in fact stipulations for shortenings of complicated expressions and are thus clearly uninformative: they are at the same time “wholly inessential and dispensable”[74] and the only proper kind of definitions, as it is suggested by Frege’s proposal to call them “definitions *tout court*”[75].

Analytic definitions are definitions of terms “with a long established use”[76], whose senses, albeit partially known, need to be clarified. This kind of definitions are the results of logical analysis, not of mere stipulations, because the *definiendum* already has a sense. Here, Frege supposes the identity not only of reference, but also of sense. But two cases must be distinguished in this respect. First, the sense of the *definiens* and of the *definiendum* is identical and this fact can be recognized by an “immediate insight”[77]. As a result, “it is better to eschew the word ‘definition’ altogether in this case, because what we should here like to call a definition is really to be regarded as an axiom”[78]. Thus, the first kind of analytic definitions are, properly speaking, axioms.

The second kind of analytic definitions is characterized by the fact that the two terms A and C that are equated in the definition ‘A is C’ do not obviously have the same sense. In this case, “we are not certain whether the analysis is successful”[79] and “we do not have a clear grasp”[80] of A, although it is a term that has been in use for a long time. What we have to do here, according to Frege, is to choose a fresh sign B and to stipulate its sense through the constructive definition that ‘B is C’. Then, in order to bypass the question of whether A and B have the same sense, we have to construct a new system from the bottom up by replacing all the occurrence of A in the old system with the new term B.

[72]GG, §27.
[73]LM, p. 210.
[74]LM, p. 208.
[75]LM, p. 210.
[76]LM, p. 210.
[77]LM, p. 210.
[78]LM, p. 210.
[79]LM, p. 210.
[80]LM, p. 221.

What does Frege aim to achieve with this procedure? First of all, he rejects definitions based on logical analysis. For we have seen that constructive definitions are a kind of stipulation; the first kind of analytic definitions are not, properly speaking, definitions, but axioms; and the second kind of analytic definitions is again turned into a kind of constructive definitions. By rejecting definitions based on logical analysis, Frege hopes to give a response to the paradox of analysis. However, what he actually does is to avoid the paradox. For, although analysis is now precedent to systematic construction, the question of the relationship between the old term in use, A, and the fresh stipulated one, B, is still legitimate and has not been given a proper answer. As Beaney puts it, "The building stones may have to be prepared carefully, by conferring clear sense on the simple terms, but do these senses not have to bear some recognized relation to the senses (however vague or inadequately grasped they may be) of the ordinary terms that the new terms replace?"[81] The inadequacy of the response given by Frege through the notion of analytic definitions is even more evident if we consider his logicist project. Assume that Frege has replaced existing arithmetic, the fundamental notions of which were unclear to mathematicians of his days, with a new science that assigns new senses to the key arithmetical terms. Then, what is the epistemological significance of showing that this entirely new logic is analytic and can be reduced to logic?

Frege's notion of analytic definitions does not solve the paradox of analysis. But what about constructive definitions? At first sight, this kind of definitions is, as we have seen, entirely uninformative, for they result from a mere stipulation. Nonetheless, in the same manuscript *Logic in Mathematics*, Frege seems to suggest a way in which stipulative definitions may be useful:

> Definition is, after all, quite inessential. In fact considered from a logical point of view it stands out as something wholly inessential and dispensable. [...] I want to stress the following point. To be without logical significance is still by no means to be without psychological significance. [...] So if from a logical point of view definitions are at bottom quite inessential, they are nevertheless of great importance for thinking as this actually takes place in human beings (LM, pp. 208-209).

In this passage, the distinction between two levels, the logical and the psychological, allows Frege to propose another solution to the paradox of analysis. Definitions are correct, because the *definiens* and the *definiendum* are taken to be identical through a stipulative act; at the same time, they find their usefulness at the psychological level. Frege's starting point of his surprising suggestion is the (reasonable) assumption that when a word is present to our consciousness, we

[81] Beaney (2005, p. 303).

are not able "to call to mind everything appertaining to the sense of this word", because "our minds are simply not comprehensive enough"[82]. Definitions prove useful exactly in solving this problem. Signs are interpreted as a "receptacle for the sense", which may very well be complex. The intermediary between signs and consciousness are definitions, which are needed "so that we can cram this sense into the receptacle and also take it out again"[83].

As Horty has underlined[84], this psychologistic solution to the paradox of analysis is a consequence of the representationalist view of thinking that emerges in Frege's later writings, albeit never with a systematic treatment. According to this perspective, our grasping of thoughts is not directed, but mediated through the linguistic items with which they are associated. This standpoint explains the sense in which our minds are not comprehensive enough to accomodate complex senses: the size of their linguistic expression is simply too large to fit our mental capacities. Definitions are thus useful because they manage to compact and modify the structure in which we represent thoughts.

3.2.4 Logical empiricism and the psychologistic solution

The psychologistic solution to the paradox of analysis that Frege seems to suggest in *Logic in Mathematics* is extensively developed and explicitly defended in the logical positivistic *milieu*. In this Section, we provide an analysis of the contributions to the discussion offered by Hans Hahn, Carl Gustav Hempel and Alfred Jules Ayer.

In his pamphlet *Logic, Mathematics and Knowledge of Nature* written in 1933, Hans Hahn recognizes that there are two sources of knowledge, experience and thinking, and reconstructs the controversy in the history of philosophy about the relationship between the two. While rationalism failed because its fruits lacked nourishing value, the earlier empiricists committed the error of interpreting the propositions of logic and mathematics as mere facts of experience. Thus, according to the neo-empiricist mathematician, a different conception of logic and mathematics is needed. Hahn argues that "logic does not by any means treat of the totality of things, it does not treat of objects at all but *only of our way of speaking about objects*"[85]: in other words, logic is generated by language. Moreover, influenced by Wittgenstein, he holds that the statements of logic, that express the way in which the rules that govern the application of words to facts depend upon each other, are tautologies: "they say nothing about objects and are for this very

[82]LM, p. 209.
[83]LM, p. 209.
[84]Horty (1992, p. 243 and ff.).
[85]Hahn (1959, p. 152).

reason certain, universally valid, irrefutable by observation"[86]. In conveying this interpretation of logic, Hahn is thus choosing one horn of the paradox of analysis, namely, that logic is correct and uninformative. But, at the same time, he offers an answer to the question as to what purpose logic serves:

> Thus logical propositions, though being purely tautologous, and logical deductions, though being nothing but tautological transformations, have significance for us because we are not omniscient. Our language is so constituted that in asserting such and such propositions we implicitly assert such and such other propositions – but we do not see immediately all that we have implicitly asserted in this manner. It is only logical deduction that makes us conscious of it. [...] The propositions of mathematics are of exactly the same kind as the propositions of logic (Hahn, 1959, pp. 157-158).

According to Hahn, logic and mathematics are not objectively new or informative. They are instruments that simply compensate for our limitations and our inability to see immediately the consequences of what we know: "an omniscient being has no need for logic and mathematics"[87]. Using the words that Hintikka chooses to criticize this position[88], we could say that "all that is involved is merely psychological conditioning, some sort of intellectual psychoanalysis, calculated to bring us to see better and without inhibitions what objectively speaking is already before our eyes"[89]. Logic and mathematics make us conscious of the consequences of our premises that we are not intelligent enough to recognize by mere inspection.

A similar position is put forward by the mathematician and philosopher Carl Gustav Hempel. In his article *On the Nature of Mathematical Truth* first published in 1945, he holds that the validity of mathematics depends neither on it alleged self-evidential character, nor on any empirical basis, but derives by virtue of definitions and stipulations which determine the meaning of the terms. Mathematical and logical statements are called analytic and their truth is independent of any experiential evidence. However, the price paid for the theoretical certainty of these disciplines is very high: they convey no factual information. In another text of the same year, *Geometry and Empirical Science*, Hempel suggests that logic might be psychologically useful along the same lines indicated by Hahn:

> Logical deduction – which is the one and only method of mathematical proof – is a technique of conceptual analysis: it discloses what assertions are concealed in a given set of premises, and it makes us realize

[86] Hahn (1959, p. 153).
[87] Hahn (1959, p. 159).
[88] See Section 4.3.2.
[89] Hintikka, 1973, X, p. 223.

> to what we committed ourselves in accepting those premises; but none of the results obtained by this technique ever goes by one iota beyond the information already contained in the initial assumptions [...] a mathematical theorem, such as the Pythagorean theorem in geometry, asserts nothing that is *objectively* or *theoretically new* as compared with the postulates from which it is derived, although its content may well be *psychologically new* in the sense that we were not aware of its being implicitly contained in the postulates (Hempel, 2001, pp. 20-21).

A slightly different perspective on the same issue is given by Alfred Jules Ayer, who dedicates the fourth chapter of his book entitled *Language, Truth and Logic* written in 1936 to answer to the traditional objection that it is impossible on empiricist principles to account for our knowledge of necessary truths. After having recognized the problem posed by the paradox of analysis[90], he argues that, on the one hand, analytic propositions do not give us new knowledge, for they are devoid of factual content and consequently they say nothing. Nevertheless, he maintains, on the other hand, that there is a sense in which analytic propositions might add something to our knowledge: "although they give us no information about any empirical situation, they do enlighten us by illustrating the way in which we use certain symbols"[91]. Ayer's idea is that logical deduction calls attention to the implications of a certain linguistic usage, such as the convention which governs our employment of the connectives, of which we might otherwise not be conscious. Hempel and Hahn found that the power of logic to surprise us depended on the recognition of the consequences that could not be immediately grasped because of human limitations; Ayer is more specific and maintains that logical deduction sheds new light in particular on the functioning of certain linguistic items and, ultimately, of our logical systems. This difference notwithstanding, Ayer solution to the paradox of analysis is as psychologistic in character as the previous ones, because also the British philosopher holds that the novelty of the result of a logical deduction depends on the limitation of our reason:

> A being whose intellect was infinitely powerful would take no interest in logic and mathematics. For he would be able to see at a glance everything that his definitions implied, and, accordingly, could never learn anything from logical inference which he was not fully conscious

[90] "The empiricist must deal with the truths of logic and mathematics in one of the two following ways: he must say either that they are not necessary truths, in which case he must account for the universal conviction that they are; or he must say that they have no factual content, and then he must explain how a proposition which is empty of all factual content can be true and useful and surprising. If neither of these courses proves satisfactory, we shall be obliged to give way to rationalism" (Ayer, 1958, p. 73).

[91] Ayer (1958, p. 80).

> of already. But our intellects are not of this order. It is only a minute proportion of the consequences of our definitions that we are able to detect at a glance (Ayer, 1958, pp. 85-86).

We now propose some observations. First, Ayer's thesis that logical deduction provides us with new information concerning our logical system does not, *by itself*, imply that linguistic and conceptual information cannot be perfectly objective and non-psychological. The viability of this theoretical option is shown by the work of Hintikka, which will be discussed in the next chapter. Second, the idea that logic is useful in so far as it makes us conscious of consequences that we are not able to grasp immediately due to our limitations might be read as a reformulation of Wittgenstein's theory that philosophy is the activity of clarification of thought[92]. This conception of philosophy, which is attested also in the second Wittgenstein[93] and is unconditionally accepted by the Vienna Circle[94], will be distinctive of the analytic tradition. Third, the psychologistic flavor of Hahn, Hempel and Ayer's solution is not completely satisfying. The obstacle that prevents us to derive all the consequences of what we know cannot be a subjective incapability of the individuals, but it rather seems to be an objective barrier. The insight at the basis of this criticism, which will be fully developed in the next Chapters of this thesis, is connected to certain results coming from the theories of computation and computational complexity.

3.2.5 Wittgenstein and the myth of the perfect language

Frege's reflection on the sterility of analysis put forward in *Logic in Mathematics* might be read as an anticipation not only of the psychologistic solution strenuously

[92]For example, proposition number 4.112 of the *Tractatus* states that: "Philosophy aims at the logical clarification of thoughts. Philosophy is not a body of doctrine but an activity. A philosophical work consists essentially of elucidations. Philosophy does not result in 'philosophical propositions', but rather in the clarification of propositions. Without philosophy thoughts are, as it were, cloudy and indistinct: its task is to make them clear and to give them sharp boundaries".

[93]As Kuusela (2011) observes, Wittgenstein writes in the *Big Typescript* in the early 1930s: "As I practice philosophy, its entire task consists in expressing myself in such a way that certain disquietudes [...] disappear", "If I am correct, philosophical problems must be completely solvable" and "The problems are solved in the actual sense of the word, like a lump of sugar in the water".

[94]In the manifesto of the movement written in 1929, Carnap, Hahn and Neurath argue that "Clarification of the traditional philosophical problems leads us partly to unmask them as pseudo-problems, and partly to transform them into empirical problems and thereby subject them to the judgment of experimental science. The task of philosophical work lies in the clarification of problems and assertions, not in the propounding of special philosophical pronouncements. The method of this clarification is that of *logical analysis*" (VC, 1973, p. 306).

defended by the logical positivists, but also of another way out of the paradox of analysis that finds an eminent supporter in Ludwig Wittgenstein. As before, we are not suggesting a direct influence of Frege's passage on Wittgenstein's perspective on the paradox of analysis, but only, so to speak, a 'family resemblance' between the two points of view[95].

We have seen that in the unpublished manuscript dated 1914 Frege's starting point is a representationalist view of thinking, according to which linguistic items are essential mediators of our grasping thoughts and human beings do think in language. Moreover, he holds that in signs "we conceal a very complex sense as in a receptacle"[96] and that signs as such play a fundamental role in "calling to mind everything appertaining to a word"[97]. A radicalization of the idea that signs are essential for our thinking might be seen in the insight that a suitable logical or mathematical notation makes our thinking itself superfluous. This observation has been already proposed in the *Introduction* to the *Grundlagen*:

> It is possible, of course, to operate with figures mechanically, just as it is possible to speak like a parrot: but that hardly deserves the name of thought. It only becomes possible at all after the mathematical notation has, as a result of genuine thought, been so developed that it does the thinking for us, so to speak (FA, p. iv).

As Carapezza and D'Agostino (2010) have underlined, Frege's words express the ideal of the logically perfect language and its relation to the myth of instant rationality. In a suitable notation, all logical relations become visible and thinking turns out to be superfluous. Frege's standpoint is clearly influenced by the Leibnizian tradition, to whom he explicitly refers. In the preface of the *Begriffsschrift*, he points out that Leibniz recognized the advantages of an adequate system of notation and holds that his own ideography is a characterization of Leibniz's "universal characteristic" and of "a *calulus philosophicus* or *ratiocinator*"[98]. Moreover, after being accused by Ernst Schröder of creating not a universal language, but a *calculus ratiocinator*, he restates that his own *Begriffsschrift* is a *lingua characteristica* in Leibniz's sense[99].

[95]For the Fregean influences on Wittgenstein's thought see Kienzler (2001).

[96]LM, p. 209.

[97]LM, p. 209.

[98]BF, p. 6.

[99]The dispute between Frege and Schröder has been interpreted in different ways. According to Van Heijenoort (1967), Frege holds that his *Begriffsschrift* is a *lingua characteristica* and that Boole's calculus is a *calculus ratiocinator*, because, while in the latter the propositions remain unanalyzed, the former, with help of predicate letters, variables and quantifiers, manages to articulate propositions so as to let them express a meaning. Sluga (1987) challenges Van Heijenoort's interpretation and argues that the distinction between the two approaches is that in

There are manifold Leibnizian echoes also in Wittgenstein's thought. In particular, the author of the *Tractatus Logico-Philosophicus* deals with the problem of an adequate logical notation that he characterizes in terms of the perfect coincidence of the grammatical and the logical structures of a sentence. Each sentence expressed through an adequate notation shall immediately show its sense, which is given by the conditions in which it is true or false. As a result of the use of a perfect language, every tautology shall be recognized immediately as a sentence that is true no matter the interpretation, and the validity of every inference shall be clear at first sight. With an adequate notation, logical deduction shall be completely superfluous, for a mere inspection of even complex sentences shall be sufficient to grasp their sense. As Wittgenstein (1921) puts it:

> 5.13 When the truth of one proposition follows from the truth of others, we can see this from the structure of the propositions.
>
> 6.122 [...] in a suitable notation we can in fact recognize the formal properties of propositions by mere inspection of the propositions themselves.
>
> 6.127 [...] Every tautology itself shows that it is a tautology.

But does anything like an adequate logical notation exist according to the author of the *Tractatus*? As Carapezza and D'Agostino (2012) show, Frege and Russell's proof system is criticized by Wittgenstein because the criterion according to which some tautologies are chosen to play the role of axioms, namely, self-evidence, is shared by every single tautology and is thus useless. Moreover, not only the same proposition, that for the Austrian philosopher is identifiable with the possibility of its being true or false, can be expressed through different signs, due to the interdefinabilty of logical connectives; but also logical derivability itself might turn into an extremely complex task. According to Wittgenstein, these features of Frege and Russell's proof system are harmful to the immediate visibility of propositions and this means that they didn't manage to find an adequate logical notation.

But Wittgenstein's reflection on the adequate notation is also endowed with a *pars construens*. The so-called 'truth tables', introduced in the *Tractatus* for the first time[100], are, according to their inventor, an adequate way to express propositions. The *Tractatus* distinguishes between elementary and complex propositions,

the *Begriffsschrift* concepts results from analyzing judgments and not viceversa as in the case of Boole's logic. Peckhaus (2004) maintains that the distinction is based on the superior expressive power of Frege's concept-script, while Korte (2010) argues that Frege's reason for regarding his work as a *lingua characteristica* must be searched in his logicist program and in his thesis that judgments of arithmetic are analytic.

[100] See Wittgenstein 1921, 4.31 and 4.442.

the latter being made up by the former: while the truth of an elementary proposition depends on the existence of certain facts of the world, the truth value of a complex proposition depends on the elementary constituents of which it is constructed. The truth table of a complex proposition not only shows in an explicit way the truth conditions of that proposition, but it is in itself a propositional sign, viz. it serves as a proposition.

In which way does the myth of the perfect language offer a solution to the paradox of analysis? As we have seen in Section 3.2.1, Wittgenstein, as well as logical positivists, holds that logic is both analytic and tautological. However, while the psychologistic perspective *à la* Hempel, although denying an objective usefulness of the deductive reasoning, accepts that the conclusion of a valid inference might be psychologically new with respect to its premises, Wittgenstein rejects the usefulness of logical deduction *tout court*. According to the insights conveyed in the *Tractatus*, once propositions are expressed through an adequate notation, i.e., in terms of their truth tables, logical deduction shall be replaced by the mere inspection of the propositions. Wittgenstein has thus chosen one horn of the dilemma, namely, analysis is not informative. At the same time, he offers an explanation of the apparent novelty of the conclusion of an inference with respect to its premises in terms of the inadequacy of our logical language.

However, Wittgenstein position might be attacked on two fronts. On the one side, the solution proposed in the *Tractatus* is restricted to propositional logic and, crucially, Church-Turing's undecidability theorem (1936) excludes the possibility of finding a similar perfect notation for first-order logic[101]. On the other side, there is probably no feasible translation from the ordinary language to the adequate notation, for its existence would also imply the existence of an efficient deterministic algorithm to solve the tautology problem. But unfortunately, the currently accepted conjecture in theoretical computer science that $\mathsf{P} \neq \mathsf{NP}$, which shall be examined in Section 6.1.1, excludes the tractability of the tautology problem.

[101]Carapezza and D'Agostino (2010) underline that this negative result does not exclude the possibility of an 'almost-perfect' language working reasonably well in all practical contexts.

Chapter 4

Hintikka's modern reconstruction of Kant's theory: against the traditional paradigm

According to Creath[1], "W. V. O. Quine's *Two Dogmas of Empiricism* is perhaps the most famous paper in the twentieth-century philosophy". Be that as it may, Quine's paper has raised a huge interest in the analytic-synthetic distinction, which nevertheless mainly focused on material analyticities, rather than on the status of logical truths. Beyond Quine's followers, who rejected the notion of analyticity as a whole, and Quine's critics, such as Grice and Strawson (1956)'s, among the major trends in this discussion we find the upsurge in the scientific study of natural languages that led to defend a version of analyticity[2]; the externalist theories of meaning that concentrate on studying the relations between words and phenomena[3]; the rehabilitation of metaphysics together with the insistence upon an inner faculty of intuition[4].

However, our interest concerns mainly the first class of supposed analyticities, which are called by Quine 'logically true'. During the Seventies, Quine[5] himself seems to be open to change his mind on this point. The persistent disagreement on some logical laws (such as the excluded middle) leads the American philosopher to doubt that logical validities can be learnt simply by the learning of words and to suggest that logical laws "should perhaps be seen as synthetic"[6]. However, far from developing this idea further, in some retrospective positions Quine (1991)

[1]Creath (2004, p. 47).

[2]Chomsky (1968) and, later, Katz (1992) could be read along these lines.

[3]Putnam (1975) and Kripke (1972) are two of the most relevant contributions to this trend.

[4]On this point, see, for example, Lewis (1972).

[5]Quine (1974).

[6]Quine (1974, p. 80).

restates his initial commitment to the analyticity of logical laws.

The most thorough post-Quinean investigation on the epistemological status of logic has been carried out by Hintikka, although his work is usually excluded by the most common reconstructions of the history of the notion of analyticity[7]. The Finnish philosopher attacks the principle of analyticity of logic and proposes a modern reconstruction of Kant's analytic-synthetic distinction according to which there exists a class of quantified and polyadic logical truths, which express a kind of mathematical reasoning, that are synthetic and not tautological. Hintikka's philosophical premises consists of a peculiar interpretation of Kant's conception of analyticity and syntheticity, but his work not only gets back the Kantian heritage; it also adds to the debate some modern means and direct them against the logical empiricists. This Chapter is entirely devoted to analyze the different components of Hintikka's work and the way in which they interact.

4.1 Kantian premises of Hintikka's work

The philosophical foundation of the thesis that a definite class of first-order logical truths are synthetic must be searched in the peculiar interpretation that Hintikka offers of Kant's theory of the mathematical method. This sophisticated critical work has been presented by Hintikka in a series of papers and books mainly written during the Sixties and the first part of the Seventies, as well as in some later responses and retrospective considerations in the Eighties[8]. The kernel of this interpretational theory has been summarized in the usually clear style by Hintikka himself as follows:

> By intuition (*Anschauung*), Kant meant a representative (*Vorstellung*) of a particular entity in the human mind. By *construction*, Kant meant the introduction of such a particular to instantiate a general concept. The gist of the mathematical method *apud* Kant was the use of such

[7]See, for example, Rey (2018), Ebbs (Forthcoming), Juhl and Loomis (2010) and the historical background provided in the *Introduction* in Russell (2008). Reasons for this reticency may be the strong use of formal tools and, above all, the choice to defend a minor position.

[8]Hintikka's first study on Kant's conception of the mathematical method has been published in 1959 ("Kantin oppi matematiikasta: tutkimuksia sen perusksitteist, rakenteesta ja esikuvista," Ajatus 22: 5-85. "Kant's Theory of Mathematics: Studies in its Basic Concepts, Structure, and Precedents"), while the last systematic work on this interpretational theory appears in 1984 ("Kant's Transcendental Method and His Theory of Mathematics," Topoi 3: 99-108). In the discussion of the present Section, we will consider with particular attention, beyond extemporaneous references to other texts, the following contributions: Hintikka (1966), Hintikka (1967), Hintikka (1969), Hintikka (1972), Hintikka (1973, IX), Hintikka (1974, VI), Hintikka (1982) and Hintikka (1984).

> constructions (a modern logician would say 'the use of rules of instantiation'). A mathematical argument is synthetic if it involves the use of 'auxiliary constructions,' i.e., the introduction of new particulars over and above those given in the conditions of the argument (sometimes given in the premises and sometimes given in the premises *or* in the purported conclusion). A mathematical truth is synthetic if it can be established only by such synthetic arguments (Hintikka, 1982, p. 201).

The Hintikkian reading of Kant's work is based, we think, on two main elements. As it will be soon pointed out, both of the two premises have proved to be controversial and have been widely criticized by several scholars for different reasons. The first one concerns the nature of Kantian intuitions. Hintikka holds that Kant defines intuitions as singular representations and that, in so doing, he maintains that intuitions always represent (or even are) individual objects (Section 4.1.1). The second point concerns Kant's conception of syntheticity and the tradition which his thought would refer to. Hintikka holds that Kant is "an heir to the constructional sense of analysis"[9] and that the notion of syntheticity is founded on the necessary employment of constructions (Section 4.1.3). These two theses are not only the basis of Hintikka's interpretation of the Kantian material, but they are also indispensable as philosophical premises for Hintikka's modern theory regarding the syntheticity of logic. The way in which these two readings are employed in the modern framework and the relation between the interpretative and the logical works presented by Hintikka will be clarified only in the next Section (Section 4.2).

4.1.1 Mathematical intuitions as singular representations

Hintikka's well-known interpretation of Kant's notion of mathematical intuitions can be reconstructed, we think, as consisting in the following moments:

1. There are two stages in the development of Kant's conception of mathematical intuition. The former is reflected in the *Prize Essay*, in the first paragraphs of his *Lectures of Logic* and, above all, in the *Critique Introduction* and its *Doctrine of Method*; the latter is represented by the *Critique Transcendental Aesthetic*. The former stage is logically prior to the latter.

2. In the former stage, mathematical intuitions are simply defined as representations of individuals.

3. In the latter stage, mathematical intuitions are connected to sensibility.

[9] Hintikka (1973, IX, p. 205).

4. The former stage can be detached from the latter, which represents a mistake on Kant's side and must be criticized.

We now analyze the four steps of Hintikka's reading one by one. The thesis expressed by the first point is what Hintikka regards as his "main suggestion" toward an interpretation of the Kantian materials: the idea is that Kant's theory of the mathematical method described at the end of the first *Critique*, together with the conception of intuition on which it is based, is "not posterior but rather systematically prior to the *Transcendental Aesthetic*"[10]. The reasons Hintikka invokes in support of the "primacy thesis"[11] are the following[12]. First, the dependence of the latter step on the former is made explicit in the *Prolegomena*, that is the text in which Kant wanted to clarify the structure of his theoretical philosophy: in this work, the argument that corresponds to the *Transcendental Aesthetic* is articulated through the explicit reference to the theory of the mathematical method as exposed at the end of the *Critique*. Second, in the *Aesthetic* the definition of intuition as individuality is taken for granted and used as a premise by Kant. Third, the author of the *Critique*'s starting point is his precritical position expressed in the 1764 *Prize Essay* that the mathematical method is based on the use of general concept in the form of individual instances: therefore, Hintikka argues, the introduction of intuition for explaining the mathematical method, which is proper to the critical framework, could not be but related to the notion of individuality. Beyond the question regarding the persuasiveness of the textual evidence presented, the primacy thesis is taken by Hintikka as the fundamental premise of his reading of Kant's theory: "Whether or not intuitions there means something more than a particular idea, in any case this reading is the one which we have to start from in trying to understand Kant's view on mathematics"[13].

The content of the Kantian definition of intuition that characterizes the first and prior stage depicted above is expressed by the second point of our reconstruction. In Hintikka's words: "For Kant, an intuition is simply anything which represents or stands for an individual object as distinguished from general concept"[14] and "Everything [...] which in the human mind represents an individual is an intuition"[15]. Hintikka acknowledges that the notion of intuition as individuality gives rise to the paradoxical situation that "the 'intuitions' Kant contemplated were not necessarily very intuitive"[16], in so far as they do not display any connection to imagination or to direct perceptual evidence. But after all these features,

[10]Hintikka (1967, p. 355).
[11]This name is due to Russell (1990).
[12]These reasons are put forward in Hintikka (1967, pp. 356 and ff.).
[13]Hintikka (1967, p. 357).
[14]Hintikka (1974, VI, p. 130).
[15]Hintikka (1967, pp. 354-355).
[16]Hintikka (1974, VI, p. 130).

according to Hintikka's reading, are not required by Kant's explanation of the mathematical method nor of its syntheticity: on the contrary, the assumption that intuition simply means individuality would facilitate the clarification of some obscure passages of Kant's philosophy of mathematics, such as the idea that algebra itself is based on intuitions. As Hintikka makes clear, "because the gist of the mathematical method according to Kant is to deal with particular objects, the relevant objects of mathematical knowledge are for him particulars"[17].

Nevertheless, Hintikka registers that later in the system Kant makes intuitions intuitive again, by holding that all human intuitions are bound up with sensible perception: this is precisely the second stage of the Hintikkian reading of Kant's development of intuitions as described by point three of our reconstruction. How does Hintikka explain Kant's need to create a connection between sensibility and mathematical intuition? He argues that Kant needed to justify the reason why mathematical results obtained through the employment of intuitions could be applied to all experience. But why sensibility? To understand this point, Hintikka recalls Kant's Copernican revolution and its transcendental assumption that reason "must adopt as its guide [...] that which has itself put into nature"[18]. From this premise, it follows, according to Hintikka, that the "only satisfactory explanation of the applicability of our mathematical arguments to all experience is to assume that we have ourselves put into objects the properties and relations with which these arguments deal"[19]. On the one hand, as we have seen, the relevant objects of mathematics are particulars; on the other, a long tradition wants that sense perception is the only way in which we can come to know individuals. The outcome of this proof is Kant's famous conclusion that the knowledge obtained through mathematical means applies to objects only in so far as they are objects of sensation.

We now turn to Hintikka's evaluation of the two stages of Kant's development of intuition, which is the fourth and last point of our reconstruction. The Finnish interpreter regards the latter stage as a mistake: Kant's assumption that sense-perception is the process by means of which we become aware of the existence of individuals is considered by Hintikka as "deeply wrong" and pointed at as "Kant's basic fallacy in the first *Critique*"[20]. In assuming the connection between sensibility and individuality, Kant would have surrendered himself to the classical tradition, in particular to the Aristotelian conception put forward in the *Analytica Posteriora*, at the expense of his own principles: the fallacy he has succumbed to is profoundly un-transcendental, for it assumes that knowledge is a passive affair and that we can "in general sit back and wait until particular objects show up

[17]Hintikka (1987, p. 29).
[18]CPR, B xiv.
[19]Hintikka (1974, VI, p. 131).
[20]Hintikka (1984, p. 102).

in our passive sense-perception"[21]. To repair to Kant's un-Kantian mistake, Hintikka proposes the activity of seeking and finding instead of sense-perception for the role of reaching information about individuals and develops a corresponding game-theoretical approach. Hintikka is careful in showing that this difficulty does not threaten the whole construction, because the conception of intuition as individuality (first stage), which represents the core of Kant's theory of the mathematical method, can be detached from its later relation to sensibility (second stage). In particular, Hintikka underlines that the "connection between sensibility and intuition was for Kant something to be proved, not something to be assumed"[22] and concludes that this link can only be assumed in those parts of Kant's system that are logically posterior to the *Transcendental Aesthetic*, where this proof is exhibited.

Hintikka's interpretation of Kant's mathematical intuitions has been at the core of an enduring debate, a famous chapter of which is represented by Charles Parsons' criticism and the Finnish philosopher's responses. Against Hintikka's thesis that the essential feature of intuition must be searched in its individuality, Parsons maintains that intuitions are defined on the basis of the immediacy criterion. Both of the two features invoked find textual confirmation in the *Critique*, where Kant, defining intuitions in contrast to concepts, affirms "The former [i.e., intuition] is immediately related to the object and is singular; the latter [i.e., concept] is mediate, by means of a mark, which can be common to several things"[23]. On the one side, Hintikka's position about the relationship between the two criteria remains essentially anchored to the notion of individuality, although it's possible to underline a kind of development motivated by the received criticisms. Initially, Hintikka does not distinguish the two features and holds that "another way of saying that *Anschauungen* have an immediate relation to their objects is to say that they are particular ideas"[24]; then, he maintains that immediacy is just a 'corollary'[25] of the individuality criterion, because immediacy is the proper mode of reference of singular objects. On the other side, Parsons argues that the reason for which Kant added the feature of immediacy next to that of individuality in his definition of intuition is that the former is significantly different from the latter. The scholar goes on to describe immediacy in perceptual terms, as a phenomenological presence to the mind, and accuses Hintikka's reading and its reduction of intuitions to particulars of being responsible for not doing justice to the spatio-temporal content of intuition and to the distinctive role it plays in mathematics.

Hintikka and Parsons' positions have come to be known respectively as the

[21] Hintikka (1987, p. 29).
[22] Hintikka (1967, p. 355).
[23] CPR, A 320/B 377. See Section 1.1.4 of this work for the nature of intuitions *apud* Kant.
[24] Hintikka (1969, p. 42).
[25] Hintikka (1982, p. 202).

'logical' and 'phenomenological' interpretations of Kant's intuitions[26]. Different scholars have sided with one or the other reading, presenting acute arguments to strengthen the respective position: to mention the most important of them, Michael Friedman has developed further the logical tradition and answered to Parsons' criticisms, while Emily Carson has sided with the phenomenological side. However, we are not interested in the details of this famous debate, but rather in one of the outcomes it offered, namely that the position originally proposed by Hintikka turned out to be too radical and in need of assimilating some insights of the competing phenomenological view. Friedman's most recent works express the belief that "this dichotomy is artificial" and that "a truly adequate interpretation of Kant's philosophy of mathematics [...] must make room for elements from both the 'logical' and 'phenomenological' approaches"[27] and propose an attempt in the direction of healing the two perspectives. His reconciliation is based on the idea that geometrical intuition is fundamentally kinematical and, in so doing, connects the geometrical space to the perspectival space: the former can be acquainted through geometrical constructions, the latter is the form of sensibility.

Beyond the content of Friedman's interpretation, the strongest reason for charging Hintikka of radicalism is, we argue, the way in which he treats the Kantian text. Parsons is right, we think, in noting that "Many of the passages Hintikka cites also mention the immediacy criterion, and it is not clear why Hintikka thinks it nonessential"[28]. We consider a single, but telling example. As we have seen above, in supporting the primacy thesis (first point of our reconstruction), Hintikka maintains that (second textual evidence) in the *Critique Transcendental Aesthetic* Kant's reasoning proceeds from the assumption that intuitions are individual representations. Nevertheless, Kant opens that Chapter of his work by a clear reference to the immediacy criterion, which has been completely overlooked by Hintikka: "In whatever way and through whatever means a cognition may relate to objects, that through which it relates immediately to them, and that which all thought as a means is directed as an end, is intuition"[29]. Although in a late article Hintikka complains that the criticisms his interpretation received payed no attention to the doctrinal context, but rather focused on his reading of particular passages[30], we hold that these textual criticisms cannot be dismissed as a kind of pedantry because of the role that these passages play in a proper understanding of the Kantian materials. We now move to analyze the second element of Hintikka's reading of Kant's theory of the mathematical method on which the Finnish philosopher founds his modern proposal about the syntheticity of logic.

[26]The debate has been organized along these lines by Friedman (2000).
[27]Friedman (2010, p. 586).
[28]Parsons (1969, p. 570).
[29]CPR, A 19/B 33.
[30]Hintikka (1982, pp. 202-203).

4.1.2 From the mathematical method to the analytic-synthetic distinction

The premise at the basis of Hintikka's interpretation is that the model Kant employs in developing his theory of mathematics is provided by Euclid's system of elementary geometry and its eighteenth-century variants. This assumption is justified by the allusion to the readings and interests of the young Kant and to the historical background characterized by the centrality of that paradigm[31].

According to Proclus, the solution to a problem or a proof of a theorem in Euclid's *Elements* consisted of six main parts (although not all of them always appeared). First, we find the enunciation or *protasis* of the general proposition, which is followed by the application of its content to a particular figure that is usually drawn. The fundamental proviso in the choice of the figure prescribed by this step, which is called setting-out or *ecthesis*, is that its particular determinations must be completely indifferent to the proof of the proposition or, in other terms, we cannot assume anything about the particular figure introduced except what is already given in the enunciation. The third passage is the definition or specification (*diorismos*), in which the figure set out in the *ecthesis* is determined more precisely. The fourth passage, which is closely related to the previous ones, is the so-called preparation or machinery. It consists in completing the figure drawn in the setting-out with certain additional points, lines and circles: these supplements are going to be essential for the proof proper or *apodeixis*, in which the construction of any other element is completely forbidden. In the last passage the conclusion about the particular figure is extended to the general case.

Not only does Hintikka recognize that most of these steps turn up also in Kant's conception of the geometrical procedures. He also maintains that Kant's overall picture of the mathematical method is influenced by two specific moments of the six parts in which an Euclidean proof was articulated: the *ecthesis* and the machinery. Hintikka gleans textual evidence in support of his point from some passages of the *Doctrine of Method*, where Kant draws a comparison between philosophical and mathematical methods. The centrality of the setting-out is demonstrated by Kant's idea that mathematics cannot deal with general concepts, but needs to reason *in concreto* and nonetheless reaches results that hold in general[32]. The relevance of the machinery is shown by Kant's claim that the mathematical method proves to be better than philosophical procedures in certain tasks (such as deter-

[31] Hintikka (1967, pp. 360-361) and Hintikka (1982, pp. 203-204).

[32] Kant claims that mathematics "cannot do anything with the mere concepts but hurries immediately to intuition, in which it considers the concept *in concreto*, although not empirically, but rather solely as one which it has exhibited a priori, i.e., constructed, and in which that which follows from the general conditions of the construction must also hold generally of the object of the constructed concept" (CPR, A 715/B743 - A 716/B744).

mining the relation between the sum of the angles of a triangle and right angle) thanks to the fact that geometers, unlike philosophers, can draw actual figures and carry out proofs in terms of such figures[33].

What is exhibited by the two moments of setting-out and machinery that occur in an Euclidean proof are constructions in the usual sense of the term: the significance these two steps assume in Kant's perspective on geometrical procedures determines, according to Hintikka, that the essence of the geometrical method in Kant's theory is exactly the use of constructions. But that's not all: constructions are the characteristic feature not only of geometry, but also of mathematics as a whole. Using words from the *Prolegomena*: "The essential feature of pure mathematical cognition, differentiating it from all other a priori cognition, is that it must throughout proceed not from concepts, but always and only through the construction of concepts"[34]. On which basis does Kant apply the Euclidean paradigm of geometrical constructions also to arithmetic and algebra? The missing ingredient is, according to Hintikka, Descartes' new geometry: "Descartes sees the essence of his mathematical method in a systematic and comprehensive *analogy between geometrical constructions and algebraic operations*. It was precisely by means of this analogy that Kant was able to think of the concepts he had first formulated by reference to geometrical constructions as being applicable to all mathematics"[35]. In support of this interpretation, Hintikka shows that Kant, in his analysis of simple arithmetical equations such as 7+5=12, employs the Euclidean model based on constructions.

Hintikka's interpretation of Kant's theory of the mathematical method does not restrict itself to the thesis that the essential feature of Kant's mathematical method is the use of geometrical constructions. It goes on to claim that the theory described in these terms plays a foundational role in the Kantian distinction between analytic and synthetic judgments. We think that it is precisely in this second step, that is, in this leap from the mathematical method to the analytic-synthetic distinction, that Hintikka's interpretation of the Kantian framework shows hesitations and weaknesses[36]. In order to reach his conclusion on the connection between

[33]Kant's description goes as follows: "he [i.e., the geometer] begins at once to construct a triangle [. . .] he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. in such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question" (CPR, A 716/B 745 - A 717/B 745).

[34]*Prol.*, p. 20.

[35]Hintikka (1982, p. 206).

[36]This point has been recognized and deeply analyzed by de Jong (1997), who reaches this conclusion following a different line of reasoning: "The point of my argument concerns the way in which Hintikka self-evidently connects these theses regarding, in particular, the method of

the method of proof of mathematics and the Kantian sense of the terms ‘analytic’ and ‘synthetic’, Hintikka’s reasoning needs two interpretational premises:

> P1. The use of constructions makes a method synthetic.
> P2. The use of synthetic (analytic) methods in their proof makes judgments synthetic (resp. analytic).

The need of these two assumptions is evident. The former guarantees that, according to Kant, there is a linkage between the methods that use constructions, such as the mathematical one, and the notions of analyticity and syntheticity. The latter provides a solid bridge that connects (the analyticity and syntheticity of) methods to (the analyticity and syntheticity of) judgments. But are these two interpretational premises faithful readings of Kant’s texts or necessary consequences of Kant’s theory?

We first consider premise P2. Hintikka’s assumption of this premise, which has already been emphasized by some scholars[37], amounts to suppose that the Kantian distinction between analytic and synthetic judgments is founded on the distinction between analytic and synthetic methods. In other words, the idea is that according to Kant the difference between the two kinds of judgment must be explained in terms of the methods according to which these two sorts of propositions would be proved: synthetic (analytic) judgments are those which can be shown to be true by synthetic (resp., analytic) methods. Hintikka is neither explicit in assuming P2, nor careful in justifying his ascription of it to Kant: nonetheless, P2 is a widespread and essential element in Hintikka’s interpretation of the Kantian analytic-synthetic distinction. In several context, Hintikka does not even appeal directly to the notion of method or proof, but rather speaks of analytic and synthetic ‘argument steps’[38].

This assumption highlights, we think, an important difficulty: Hintikka’s work is in the first place an interpretation of Kant’s theory of mathematics, but, at the same time, it presumes to explain the analytic-synthetic distinction not only for mathematical judgments. Now, P2 is, as de Jong rightly emphasizes, “fairly plausible” in the domain of mathematics; however, “in the light of Kant’s entire philosophical edifice a serious difficulty immediately presents itself: namely, mutatis mutandis a number of things seem to fit in much less well for metaphysics or philosophy”[39]. Moreover, although it seems to be compatible especially with some Kantian passages about mathematical judgments[40], no overwhelming textual evi-

---

proof of mathematics with Kant’s use of the paired terms ‘analytic’ - ‘synthetic’ and ‘analysis’ - synthesis’, or, as the case may be, interprets this method of proof with the help of these terms”.

[37]See, for example, de Jong (1997, p. 145) and Brittan (2015, p. 55).

[38]See, for example, the different senses of analyticity that are listed in Hintikka (1973, VI, p. 148).

[39]De Jong (1997, p. 146).

[40]See, for example, CPR, B 15-16.

dence seems to confirm the ascription of P2 to the author of the *Critique*[41]. This means that the second premise of Hintikka's argument is neither completely justified by Kant's assertions, nor free from difficulties. We now turn to examine the most complex question regarding premise P1.

4.1.3 Kant as "an heir to the constructional sense of analysis"

P1 is the result of a generalization. Hintikka assumes that the use of geometrical constructions makes the mathematical method synthetic and extends this insight by holding that the use of constructions makes a method synthetic. As we have seen above, he needs this extension because he wants his reconstruction of the analytic-synthetic distinction to apply not only to mathematical judgments, but also to other kinds of judgments. This generalization is based on the insight that geometrical constructions are analogous to another wider and abstract notion of construction that Kant employs throughout his work[42]. This second 'abstract' concept of construction is based on the notion of intuition and is defined as follows: "to construct a concept means to exhibit a priori the intuition corresponding to it"[43]. Through constructions, it is possible to move from a general concept to a

[41]See, for example, the pertinent passages in the *Transcendental Doctrine of Method* of the *Critique* (CPR, A 734-735/ B 762-763) and de Jong's considerations on this point (de Jong, 1997, pp. 161-162).

[42]Although the analogy between abstract and geometrical constructions seems to be rather clear, a legitimate question is, we maintain, the following: which of the two notions of construction has a conceptual or logical priority over the other one? On the one hand, Hintikka suggests that Kant, in formulating his general and abstract notion of construction, has primarily in mind the constructions of geometers (Hintikka, 1967, p. 353) and claims that "Kant's wider notion of a construction is thus nothing but a generalization from the constructions which make geometrical arguments synthetic" (Hintikka, 1973, IX, p. 207). On the other hand, Hintikka argues that it is the reference to intuition occurring in the abstract notion of construction that allows Kant to justify the use and the possibility itself of geometrical constructions in the *Transcendental Aesthetic* or, in Hintikka's (1967, p. 353) words, "Kant's appeal to intuition is designed to furnish a better foundation to the geometrical constructions". At a certain point, Hintikka (1967, p. 362) seems even to hint that the two notions are independent or, at least, not connected by a logical link of priority, when he says that "setting-out and preparation were the two parts of a Euclidean proposition where constructions in the usual sense of the word were made; and [...] these two parts were also the ones in which constructions in Kant's abstract sense of the word were needed". So, Hintikka's explanations of the conceptual priority regulating the two notions of construction is not univocal: the abstract conception is, at the same time, a generalization and a foundation of geometrical constructions, as well as an unrelated idea. Nevertheless, we think that the only possible relation between the two notions is that geometrical constructions are a kind of abstract constructions: this is not only a consequence of the definitions of the two concepts, but also a clarification needed for the sake of Hintikka's argument itself.

[43]CPR, A713/B 741.

non-empirical intuition that represents that concept. The analogy between the two notions of construction is quite evident: geometrical constructions introduce in the argument geometrical figures, which are singular and individual objects, in order to represent the general concept of a certain kind of figure; similarly, constructions in general introduce in the argument intuitions, which, according to the Hintikkian reading of Kant that we have emphasized in Section 4.1.1, are individuals, in order to represent general concepts. The analogy notwithstanding, the generalization from the mathematical domain to an unrestricted field seems to be in need of further justifications, which are nevertheless not provided by Hintikka. In what follows we are not going to distinguish between P1 and its restriction anymore.

The analogy between the two kinds of construction, together with Kant's claim that intuition dependence is the main feature of synthetic judgments[44], could lead to think that P1 is not a further premise of Hintikka's reconstruction, but rather a consequence of his reading of Kant's mathematical method. To be more specific, from the premises that abstract constructions, in so far as they are analogous to the geometrical ones, are at the basis of the mathematical method and are at the same time what makes a judgment synthetic, one could infer that the mathematical method itself is synthetic. But this deduction is not correct, because, again, nothing justifies the leap between methods and judgments: we know that constructions make judgments, not methods, synthetic. One could reply that the needed bridge is provided by P2. But this is not the case. P2 not only establishes a connection between methods and judgments, but it also suggests the direction of this linkage: from methods to judgments. Here, nevertheless, the notion of syntheticity should be transferred from judgments to methods and thus the direction sought would be the opposite one. However, that P1 is not a deductive consequence of Kant's theory of the mathematical method is not a sufficient reason to reject it. Hintikka could assume P1 in his interpretation on the basis of strong textual evidence found in the Kantian materials. This is exactly what Hintikka does, this time explicitly, in at least two occasions. But both of them deserve a close examination, because the Kantian texts he appeals to do not admit a univocal interpretation.

The first occasion is given by his article entitled *Kant and the Tradition of Analysis*, in which Hintikka maintains that Kant is "an heir to the constructional

[44]See the analysis drawn in Chapter 1, especially as far as the link between syntheticity and intuitions is concerned (Section 1.1.4), as well as Kant's explanation of this point: "If one is to judge synthetically about a concept, then one must go beyond this concept, and indeed go to the intuition in which it is given. For if one were to remain with that which is contained in the concept, then the judgment would be merely analytic, an explanation of what is actually contained in the thought" (CPR, A 721/B 749).

sense of analysis"[45] and opposes the constructional sense of analysis to the directional one. Both conceptions originate from the same text, Pappus' *Mathematical Collection*[46], which, although composed around 300 A.D., is the most complete and reliable exposition of the meaning assigned to these words by the ancient Greeks. One of the main interpretative difficulties that this text poses regards the direction of analysis as compared with the direction of relations of logical consequence: but in his work with Unto Remes, Hintikka claims that, appearances notwithstanding, Pappus' reconstruction is after all consistent. Analysis is understood in the regressive sense as the route in the direction of the principles or axioms: it assumes the desired end ("what is sought") and argues backwards, following the opposite direction drawn by relations of logical or causal consequence. Synthesis follows a progressive path along the same steps as analysis but in the other way around, that is, from principles towards what is grounded on them: synthesis follows logical or causal relation in the usual order.

However, Hintikka regards the directional sense of analysis and synthesis as "a pale reflection of the richness of the ideas involved in the original Greek concepts" and claims that it "has nothing to do with the heuristic method of analysis". According to Hintikka, the preoccupation with the direction of analysis has caused the serious side effect that subtler ingredients of the Greek method of analysis and synthesis have been overlooked for centuries, at least until the end of the Middle Ages[47]. The most prominent element among those that have been obscured by directional problems is, of course, the role of constructions and the importance of the 'machinery' needed to carry out the demonstration. Following the constructional tradition then, "a method or a procedure is analytic if in it we do not introduce any new geometrical entities, in brief, if we do not carry out any constructions. A procedure is synthetic if such constructions are made use of, i.e., if new geometrical entities are introduced into the argument"[48]. The constructional interpretation of the terms 'analysis' and 'synthesis' is suggested, according to Hintikka, not only

---

[45] Hintikka (1973, IX, p. 205).

[46] The excerpt considered is the following: "Now analysis is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as been a solution backwards.
But in synthesis, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis" (Hintikka, 1973, IX, pp. 199-200).

[47] Hintikka and Remes (1974, Chapter II).

[48] Hintikka (1973, IX, p. 203).

by the general definition of analysis provided by Pappus, in which it is explicitly stated that 'what is sought' must be assumed as admitted, but also by the description he gives of the so-called 'theoretical analysis' (i.e., the analysis applied to theorems)[49], where it is restated that the desired conclusion must be assumed as true and existent. Hintikka explains that in these formulations the assumption of the sought end could amount to the admission that the sufficient auxiliary constructions have already been made.

Three are the Kantian *loci* that Hintikka invokes in support of his thesis that Kant is "an heir to the constructional sense of analysis", that is to say, in support of the ascription of P1 to Kant. In the former two, Hintikka actually shows that the directional sense is rejected by Kant; in the latter, he positively proves that Kant is inspired by the constructional tradition. The first text is taken from a footnote of the *Prolegomena*[50]. Here Kant is pointing out at the risks that arise when traditional terms are employed for describing new concepts: he distinguishes between the meanings that the words 'analytic' and 'synthetic' assume if applied to judgments rather than methods and proposes, in order to avoid confusions, to refer to the latter with the adjectives 'regressive' and 'progressive'. From the fact that the two methods are described through directional terms, Hintikka concludes that, in applying the analytic-synthetic distinction to judgments, Kant is referring to the constructional tradition of the terms. Now, Hintikka's conclusion seems to be liable, we think, of two charges that aim to opposite directions. First, it illegitimately assumes, beyond P2, the premise that there are only two traditions of analyticity to which Kant could appeal. Second, as de Jong[51] underlines, it does not take Kant at his words when he says that his distinction between analytic and synthetic judgments founds a new meaning of the terms. No matter which way is chosen, Hintikka's reading of this excerpt is far from being persuasive.

The second passage is taken from a footnote at the beginning of the *Inaugural Dissertation*[52], where Kant is dealing with a double meaning of the words 'anal-

[49] Pappus' description goes as follows: "In the theoretical kind, we suppose what is sought to exist and to be true, and then we pass through its consequences in order, as though they were also true and established by our hypothesis, to something which is admitted" (Hintikka, 1973, IX, p. 200). Theoretical analysis is contrasted with problematical analysis, i.e. analysis as applied to problems. The difference between the two is not relevant in this context, for Hintikka maintains that the constructional sense applies to theoretical as well as problematical analysis and the distinction between the two thus vanishes.

[50] "The analytic method, insofar as it is opposed to the synthetic, is something completely different from a collection of analytic propositions; it signifies only that one proceeds from that which is sought as if it were given, and ascends to the conditions under which alone it is possible. In this method one often uses nothing but synthetic propositions, as mathematical analysis exemplifies, and it might better be called the regressive method to distinguish it from the synthetic or progressive method" (*Prol.*, p. 28).

[51] De Jong (1997, p. 159).

[52] "A double meaning is commonly assigned to the words 'analysis' and 'synthesis'. Thus

ysis' and 'synthesis': the qualitative and quantitative senses. The structure of Hintikka's interpretation is similar to the reading of the previous passage. Since qualitative synthesis and analysis are described along the lines of the directional sense and their use is plainly rejected by Kant, Hintikka thinks it to be 'obvious' that the meaning chosen by the author of the *Dissertation* for the terms 'analysis' and 'synthesis' cannot be but the constructional one. Again, de Jong is right in emphasizing that not only the qualitative, but also the quantitative sense of analysis and synthesis mirrors the directional meaning of the terms. This conclusion is suggested by the use of the notions of 'progression' and 'regression' in talking of the qualitative acceptation and by the context in which no proof or method is mentioned. For the second time, we must admit that Hintikka's interpretation is not satisfying.

Last, the only positive evidence in support of the ascription of the constructional sense of analysis and synthesis to Kant comes from the interpretation of a particularly debated passage of the *Critique*[53] (CPR, B14) where he maintains that "the inferences of the mathematicians all proceed in accordance with the principle of contradiction". But this claim raises some problems, since the principle of contradiction is, as we have seen in Chapter 1 of this thesis, the supreme principle of analytical judgments. Hintikka holds that the affirmation in question can be clarified along the constructional interpretation, if we understand it as referring only to a single part in the proof of a proposition, namely, the *apodeixis*: the proof proper is in fact analytic, because the synthetical element of a geometrical proposition rests only with the auxiliary constructions.

Hintikka's reading seems to be consistent, but nevertheless it is not overwhelming: several are the alternative, competing and persuasive readings proposed for this sentence. For example, according to the 'evidentialist' interpretation of this passage[54], Kant is appealing here to the syntheticity of the axioms of mathematics

synthesis is either qualitative, in which case it is a progression through a series of things which are subordinate to each other, the progression advancing from the ground to that which is grounded, or the synthesis is quantitative, in which case it is a progression within a series of things which are co-ordinate with each other, the progression advancing from a given part, through parts complementary to it, to the whole [...] Here we use both 'synthesis' and 'analysis' only in their second sense" (*Diss.*).

[53] "**Mathematical judgments** are all synthetic. This proposition seems to have escaped the notice of the analysts of human reason until now, indeed to be diametrically opposed to all of conjectures, although it is incontrovertibly certain and is very important in the sequel. For since one found that the inferences of the mathematicians all proceed in accordance with the principle of contradiction (which is required by the nature of any apodictic certainty), one was persuaded that the principles could also be cognized from the principle of contradiction, in which, however, they erred; for a synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself" (CPR, B14).

[54] With the adjective 'evidentialist', Brittan (2006) refers to those classical and widespread

as a clarification for the syntheticity of all mathematical theorems: this would explain Kant's claim that "a synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself". Another interesting perspective is that in this passage Kant is underlining, against the Leibnizians, that the principle of non-contradiction can be used only as a negative criterion of truth. A virtue of this position is that B14 is read in the same interpretative framework that is provided by Kant's discussion of the principle of non-contradiction in the paragraph of the *Critique* entitled *The System of the Principle of Pure Understanding*[55].

The second occasion in which Hintikka appeals to textual evidence in support of P1 can be found in his article *An Analysis of Analyticity*. Here Hintikka maintains that the following definition "approximates rather closely Kant's notion of analyticity": "An argument step is analytic if and only if it does not introduce any new individuals into the discussion"[56]. He then contrasts this definition with other conceptions of analyticity and syntheticity. The most widespread meaning of the terms is the notion of analyticity as conceptual truth: although it has been specified in numerous and different directions, Hintikka holds that its basic idea is that "a sentence is analytically true if and only if its truth can be established by the sole means of conceptual analysis, without recourse to experience"[57]. The Finnish philosopher argues that, although there is a certain similarity between Kant's notion of analyticity and analyticity as conceptual truth, "this similarity is largely an illusion"[58]. He supports his thesis by invoking another debated passage of the *Critique*[59] (CPR, B17), where Kant seems to claim that the truth of mathematical judgments depends on a relationship between concepts ("necessity already

readings of Kant's theory of the mathematical method according to which "intuitions provide indispensable evidence for the truth of mathematics". These positions are contrasted by the 'objectivist' interpretations, according to which intuitions provide semantic vehicles of singular reference and objective reality.

[55]CPR, A 148/B 187 and ff. Recall also the interpretation of these passages given in Section 1.1.1. of this thesis. See on this point also de Jong (1997, pp. 164-165).

[56]Hintikka (1973, VI, pp. 136-137). The claim that Kant's notion of analyticity is the one expressed above clearly amounts to assume again premise P1.

[57]Hintikka (1973, VI, p. 126).

[58]Hintikka (1973, VI, p. 129).

[59]"What usually makes us believe here that the predicate of such apodictic judgments already lies in our concept, and that the judgment is therefore analytic, is merely the ambiguity of the expression. We **should**, namely, add a certain predicate to a given concept in thought, and this necessity already attaches to the concepts. But the question is not what we **should think** in addition to the given concept, but what we **actually think** in it, though only obscurely, and there it is manifest that the predicate certainly adheres to those concepts necessarily, though not as thought in the concept itself, but by means of an intuition that must be added to the concept" (CPR, B 17).

attaches to the concepts") and that nonetheless they are said to be synthetic. In other words, Hintikka thinks, there are conceptual truths that are not analytic and this is a sufficient reason for Hintikka to conclude that Kant's notion of analyticity cannot be understood in terms of conceptual truth.

Now, Hintikka's conclusion is neither justified nor convincing. First of all, the obscurity of B17 does not make completely clear that the synthetic truths Kant is speaking of are conceptual truths, because, at the end of the same paragraph, the author of the *Critique* specifies that "it is manifest that the predicate certainly adheres to those concepts necessarily, though not as thought in the concepts itself, but by means of an intuition that must be added to the concept". This specification added to the description of conceptual truths raise some doubts about the nature of the result. Nevertheless, even if we concede that in this excerpt he is really claiming that at least some conceptual truths are synthetic truths, we must emphasize that Kant is by no means holding that there are analytic propositions that are not conceptual truths. This observation is crucial: B17 does not support Hintikka's thesis. What may perhaps go against the interpretation of Kant's analyticity in terms of conceptual truth can only be premise P2, that is to say, Hintikka's interpretation is internally coherent. Nonetheless, we have shown in Chapter 1 of this thesis that Kant's theory of analytic and synthetic judgments receives its justification exactly from its appeal to the theory of logical division of concepts: the containment criterion, which is the essential notion that founds Kant's analyticity, loses its apparent obscurity only in this framework, in which analytic truths are defined in terms of conceptual truths.

To sum up, we have proved that all the Kantian passages that Hintikka invokes to support premise P1, that is, his thesis that Kant's conception of analyticity is inscribed in the constructional tradition, are not persuasive. Neither the directional sense of analysis nor the notion of analyticity as conceptual truth can be said to have been ruled out by Hintikka's evidence. On the other hand, premise P2, which is strictly interwoven with the former assumption, cannot be said to have been vindicated by textual evidence or arguments. We have to conclude that the connection Hintikka wanted to establish between Kant's theory of the mathematical procedures and his distinction between analytic and synthetic judgments is not free of difficulties and to underline that also the second premise of Hintikka's reading of Kant, as well as the first one on the nature of intuitions, is not grounded on solid bases. Nevertheless, Hintikka's reading of Kant's theory of the mathematical method remains convincing and his attempt to inscribe Kant in the constructional tradition of analysis and synthesis, though not persuasive, impressive.

4.2 Hintikka's main thesis: synthetical logical truths

We now focus on Hintikka's main thesis that there exists a class of quantified and polyadic logical truths, which are synthetic and express a kind of mathematical reasoning. This thesis can be considered from two different perspectives. First, it can be regarded as a modern theory that Hintikka defends on independent logical grounds. Second, following the repeated indications of the Finnish philosopher, it can be considered as a modern reconstruction of Kant's philosophy of mathematics. In the present Section, we take into account both of these points of view: first, we are going to analyze the conceptual kernel of Hintikka's main thesis, which is simply the result of a peculiar definition of the analytic-synthetic distinction; then, we are going to show the actual influence of Kant's conception of syntheticity and Hintikka's interpretation of it over this theory. We conclude by pointing out at some difficulties.

4.2.1 Hintikka's analytic-synthetic distinction

Hintikka's thesis that some logical truths are synthetic is but a consequence of the way in which the analytic-synthetic distinction has been defined. For this reason we are going to concentrate on the different ways in which Hintikka gives this definition and in which this notion can be explained and understood. The first formulation of Hintikka's analytic-synthetic distinction that we are going to examine is the following, which we are going to call D_1:

> Synthetic steps are those in which new individuals are introduced into the argument; analytic ones are those in which we merely discuss the individuals which we have already introduced (Hintikka, 1973, IX, p. 210)[60].

In choosing this sense of the terms, Hintikka wants to avoid the difficulties that would arise when concepts are involved. Instead of talking about analysis of concepts, this proposal deals with the analysis of individuals and "the new sense can be understood better than the old one to the extent the notion of an individual object can be understood better than that of a general concept"[61].

At several points of his work[62], Hintikka adds to this characterization a further

[60] This is the third sense of analyticity discussed in Hintikka (1973, VI, p. 136): "An argument step is analytic if and only if it does not introduce any new individuals into the discussion".

[61] Hintikka (1973, VI, pp. 136-137).

[62] For example, Hintikka (1965a, p. 199): "Natural deduction methods are interesting from our point of view because the synthetic element in them may be reduced to a single rule". See also Hintikka (1973, IX, p. 210).

explanation by suggesting that the paradigmatic synthetic method in first-order logic is the natural deduction rule of existential instantiation[63]. By this rule, a free individual symbol is introduced to replace the occurrences of a certain bound variable: this rule allows to infer from an existentially quantified sentence $(\exists x)p$ a sentence instantiating it, *e.g.* $p(a/x)$, where a is a free individual symbol and $p(a/x)$ the result of replacing x by a in p. The fact that this rule introduces new representatives of individuals in the proof is due to the requirement that the instantiating symbol a must be different from all the free individual symbols occurring earlier in the proof[64]. Hintikka remarks that "the problem about the use of such instantiation methods is that in them we introduce a representative of a particular entity *a priori*, without there being any such entity present or otherwise given to us"[65].

The appeal to the natural deduction rule of existential instantiation does not exhaust Hintikka's investigation of the way in which individuals are introduced into logical arguments. In his article *An Analysis of Analyticity*, he maintains that "if individuals are introduced into one's logical arguments by free singular terms, they are likewise introduced by quantifiers, too"[66]. Hintikka explains his claim by suggesting that "the existential quantifier $(\exists x)$ should be read somewhat as follows: 'there is at least one individual (call it x) such that' and the universal quantifier $(\forall x)$ should be read: 'each individual (call it x) is such that'. This observation should support the conclusion that "although the bound variable 'x' does not stand for any particular individual, each quantifier invites us to consider one individual in addition to the other ones which may have been introduced earlier" however indefinite this individual may be. Two are the consequences that can be drawn from the unspecified character of the individuals to which quantifiers refer. First, a reasonable question about a certain argument step that involves quantifiers is not if it introduces new individuals into the argument, but rather if it increases the number of individuals we have considered so far in their relations to each other. Second, quantifiers, whose scopes do not overlap, do not add to the number of individuals that have to be considered in their relation to one another.

The recognition that the quantifiers refer to the individuals, together with the two remarks above, leads Hintikka to define the maximal number of individuals that are considered together in a quantificational sentence F, which is called the degree of F, as the sum of two numbers: 1. the number of the free singular terms of F; 2. the depth of F, that is the maximum of the lengths of nested sequences of quantifiers in F. The notion of depth of a formula is fundamental for the next Chapters of the present work and we will discuss it further. By now, it is sufficient

[63]Beth talked about the rule of universal generalization, but the two are of course closely allied.
[64]Later on we consider further specifications on this requirement.
[65]Hintikka (1984, p. 101).
[66]Hintikka (1973, VI, p. 140).

to notice that the depth of a sentence is just the number of bound individual symbols that are needed to understand that sentence, provided that quantifiers with overlapping scopes always have different variables bound to them. Another explanation of the notion of depth comes from underlining its difference with the number of individuals the existence of which is asserted in a sentence. As Hintikka shows[67], in the sentence 'all men admire Buddha', $\forall x(Mx \rightarrow Axb)$, the number of the individuals considered in their relation to each other is two (Buddha, b, and each man at a time, just take an arbitrary individual, say a, to instantiate the variable x), while reference is somehow made to all individuals of the domain ($\forall x$).

The notions of depth and degree of a sentence, which have been formulated to clarify the issue of the introduction of individuals into arguments, allow a further specification of D_1, that is, the sense of the analytic-synthetic distinction that Hintikka has chosen as a basis for his thesis. The definition that results is the following one, which we are going to call D_2:

> A proof of F_2 from F_1 is analytic if and only if the degree of each of the intermediate stages is smaller than, or equal to, the degree of either F_1 or F_2 (Hintikka, VI, 1973, p. 144).

This is just a reformulation of the definition D_1, because in asking that the degree of each intermediate step is not bigger than those of the premises[68], it requires not to add any new individual into the argument. Nevertheless, it both highlights some problems of the explanation based on the rule of existential instantiation and makes another important aspect of Hintikka's notion of analyticity come to light: the idea of analysis as 'analysis of configurations'. Let's examine the two consequences.

If it is true that quantifiers invite us to consider exactly one individual, no matter how indefinite it may be, then, from this perspective, it is not always the case that an application of the rule of existential instantiation, that Hintikka indicates as the paradigm of the synthetic method, introduces new individuals into our reasoning. For in the sentence $(\exists x)Px$, the quantifier does somehow already invite you to consider an object that has the property P and all the rule does is but to give it a name, usually expressed by a parameter, and to allow to reason in 'propositional terms', once the quantifier has been removed. But the

[67] Hintikka (1965a, p. 187).

[68] The reason why it is required that the degree of the intermediate stages of a proof is not bigger also than the degree of the conclusion and so the reason why the direction of the proof is not taken into account is a simple and technical matter. The point is that by contraposing all the steps of a proof of F_2 from F_1 we obtain a proof of $\neg F_1$ from $\neg F_2$ and, without the condition involving the conclusion, we could reach the undesirable situation in which one of the two proofs is analytic and the other synthetic.

information that a certain individual has the property P is, in a certain sense, already contained in the premise to which the rule applies and, as a consequence, it is not necessary for the rule to increase the number of individuals considered. In other words, the definition D_2 reveals that the explanation based on the rule of existential instantiation is more complex than what it seemed and in need of further specification[69].

Nonetheless, D_1 indicates in the idea of configurations a replacing intuitive guide in understanding Hintikka's use of the analytic-synthetic distinction and its key notion of introduction of individuals. This concept is presented by Hintikka in his article with Unto Remes *Ancient Geometrical Analysis and Modern Logic*[70], where, talking about the ancient method of analysis as described by Pappus, it is said that the proofs obtained by the analytical method "can be thought of as dealing with one specific kind of constellation of individuals (member of our universe of discourse) [...] [and] conceived of as a study of the interdependencies within this configuration of individuals" and that "this thesis may be dubbed analysis as analysis of configurations"[71]. Here analysis is seen as the study of the configurations of individuals given by the free singular terms and the quantifiers occurring in the premises and the conclusion of a certain argument. However, it could happen that these configurations do not suffice, in the sense that during the deduction it is necessary to resort to argument steps, in which the configurations of individuals are more complex than those that describe the problem. These argument steps makes the proof synthetic, for, following the definition D_2, their degree is bigger than the degrees of the premises and the conclusion of the deduction.

An example[72] will clarify the role of configurations in understanding the difference between analytic and synthetic methods. Consider the argument from the premises P_1, P_2 and P_3 to the conclusion C:

$$P_1 : \forall x \forall y (Rxy \to \exists z (Gxz \land Gzy))$$
$$P_2 : \forall x \forall y (Gxy \to \exists z (Bxz \land Bzy))$$
$$P_3 : \forall x \forall y ((Bxy \land Cx) \to Cy)$$
$$C : \forall x \forall y ((Rxy \land Cx) \to Cy).$$

The first premise says that every time two points are connected by a red arrow (the relation R) from the left to the right, then we have to interpolate another

[69]We are going to specify the cases in which the rule of existential instantiation adds a genuinely new individual through formal tools in the next Chapter of this work.

[70]Hintikka and Remes (1976). In this interesting article, the authors put forward the thesis that the old method of analysis is almost a special case of the modern techniques in symbolic logic called natural deduction methods.

[71]Hintikka and Remes (1976, p. 266). The authors holds that it is the configurative nature of analytical arguments rather than their direction that is heuristically essential.

[72]This example is a simplification of the case presented in Hazen (1999, p. 86 and ff.), which in turn is a miniaturized version of the example from Boolos (1984).

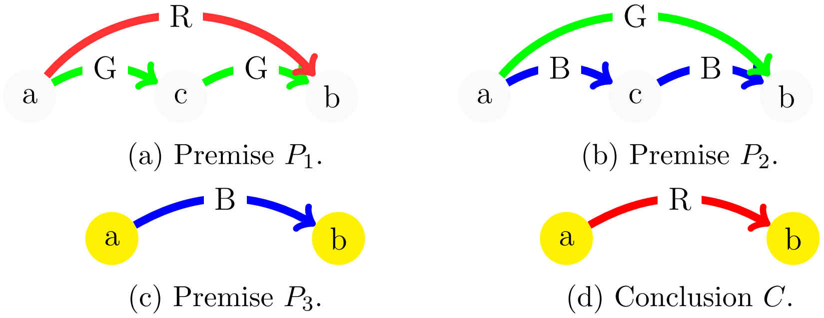
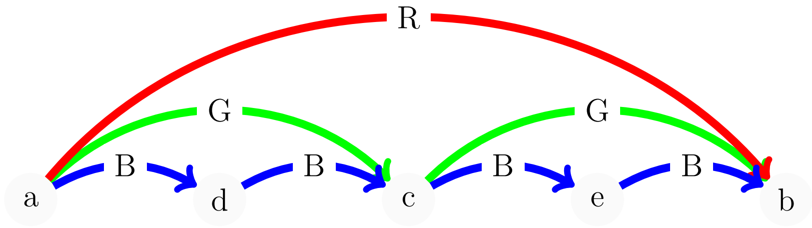


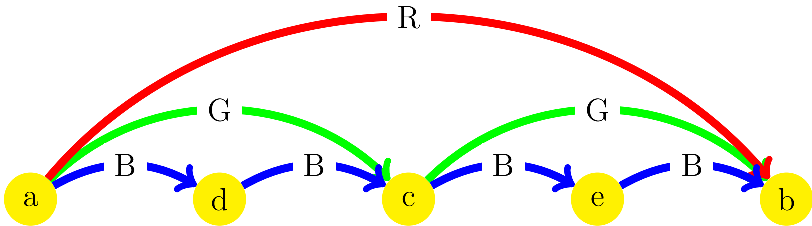
Figure 4.1: Configurations of the premises and conclusion of the example.

point, with green arrows (the relation G) from the first original point to the new one and from the new one to second original point. In order to understand P_1, it is convenient to draw a configuration of three individuals that are in relation to each other, where, for example, a is an arbitrary individual instantiating the variable x, b instantiates y and c instantiates z, while the red arrow represents the relation R and the green arrows the relations G. The result may be thus the one in Figure 1(a). The second premise is similar to the first one, but now the relations, and so the colors of the arrows, are different: P_2 can be understood thanks to the diagram in Figure 1(b), that represents a configuration of three individuals. The third premise says that the colored marker ink spreads along blue arrows, that is to say, whenever two points are connected by a blue arrow from the left to the right and the leftward point is colored (or has the property C), then also the rightward point is colored. This premise can be depicted by a configuration of three individuals as in Figure 1(c). From the premises P_1, P_2 and P_3 it follows the conclusion C, which says that the colored marker ink spreads along red arrows too, that is to say, whenever two points are linked by a red arrow from the left to the right and the leftward point is colored, then also the rightward point is colored. The conclusion can be represented by the configuration of two individuals in Figure 1(d), that is similar to that of P_3 except for the color of the arrow.

Which is the reasoning that leads us from the premises to the conclusion? How can we represent it? We could start from the configuration described by the first premise and drawn in Figure 1(a). Then, we could use premise P_2 and reason as follows. Since a and c are connected by a green arrow, then there is another individual d, which is linked to a and c by blue arrows; similarly, since also c and b are connected by a green arrow, then there is a fifth point e, which is linked to c and b by blue arrows. The visual counterpart of this reasoning is given in Figure 2(a). Then, given the premise P_3, if a is colored, then also d is colored (because they are connected by a blue arrow); for the same reason and given that d is colored, then also c is colored; again, since c is colored then also e is colored;



(a) Intermediate step 1.



(b) Intermediate step 2.

Figure 4.2: Configurations of the intermediate steps of the example.

last, we get that b is also colored. In this way and since we didn't assume anything about the instantiating individuals, we reach the general conclusion that that the colored marker ink spreads along red arrows too. This last step can be depicted as in Figure 2(b). This argument is synthetic, because the complexity of at least one of the intermediate steps (in this case of both of them) exceeds that of the configurations that depict the premises and the conclusion. The degree of the two intermediate steps is five, because five is the number of individuals in relation to each other that are considered, while the degree of P_1 and P_2 is three and the degree of P_3 and of C is two.

This example completes our exposition of the conceptual kernel of the analytic-synthetic distinction that grounds Hintikka's thesis that some logical validities are synthetic. This proposal, as we explain it in the next Section, will be extended by a theory of information and a logical counterpart. But before moving to these topics, we need to investigate the link between Hintikka's main thesis and his historical and philosophical work on Kant. In which sense is Hintikka's theory a modern reconstruction and a vindication of Kant's conception?

4.2.2 A vindication of Kant's conception?

In Section 4.1, we have argued that Hintikka's interpretation of Kant's philosophy of mathematics is based on the two theses that Kant conceived the analytic-synthetic distinction in constructional terms and that intuitions are first and foremost defined in the *Critique* as singular individuals. We now show the way in which these two interpretational theses are employed and translated in Hintikka's modern theory about logic.

First, the theory that it is the use of constructions that makes a mathematical argument step synthetic, which is premise P1 of Hintikka's reasoning as reconstructed in Section 4.1, is translated in modern terms by the theory that it is the use of the rule of existential instantiation of modern first-order logic that makes a logical argument step synthetic. To focus on the other side of the analytic-synthetic distinction, we could say, using Hintikka's formulation, that "a geometrical argument in the course of which no new geometrical entities are 'constructed' - that is, introduced into the discussion - will normally be converted into a quantificational argument in the course of which no new free individual symbols are introduced"[73]. Hintikka's interpretation of Kant's analytic-synthetic distinction is translated into modern logic thanks to the parallelism between, on the one hand, the use of construction, interpreted as a tool for shifting from a general concept to an intuition which represents the concept, and, on the other hand, the use of the rule of existential instantiation, understood as a tool for introducing new individuals. What is essential for Hintikka's parallelism (and, *a fortiori*, for his employment of Kant's conception of the analytic-synthetic distinction in logic) is that the notion of construction is not considered in its representative character, but rather in its power to introduce new individuals. But the step of a solution to a problem or a proof of a theorem in Euclid's *Elements* in which new individuals are introduced is, as we have seen in Section 4.1, the setting-out or *ecthesis*, which requires that nothing can be assumed about the particular figure introduced except for what is already contained in the enunciation. As a result, the parallelism between constructions and applications of the rule of existential instantiation that grounds Hintikka's loan of Kant's notion of syntheticity in the context of modern logic can be made even more precise by saying, as Hintikka repeatedly does, that *ecthesis* becomes "identical"[74] with the rule of existential instantiation.

Second, the theory that what is exhibited in the mathematical constructions, that is, intuitions, are simply individuals is translated in modern terms by the fact that the things introduced through the rule of existential instantiation are simply individuals. It is due to Hintikka's choice for the singularity criterion of intuition that synthetical arguments in first-order logic are identified as arguments involving singular terms. The identification between, on the one hand, Kant's intuitions and, on the other hand, singular terms of first-order logic can be expressed by saying that quantificational reasoning is, according to Hintikka, an intuitive kind

[73]Hintikka (1973, VII).

[74]Hintikka (1967, pp. 368-369).

of reasoning.

These two steps allow Hintikka to conclude that both Kant's constructional sense of the analytic-synthetic distinction and his own formulation D_2 that grounds his logical theory are but different versions of one and the same idea D_1 that an argument is synthetic when new individuals are introduced into it[75]. This means that Hintikka's distinction between analytic and synthetic proofs in first-order logic wants to be a modern employment of Kant's original distinction. But this is not all. The employment of the Kantian heritage in the formulation of the analytic-synthetic distinction leads Hintikka to defend one of the most characteristic and controversial principles of Kant's philosophy of mathematics, that is to say, its syntheticity: "We can now vindicate Kant. What he meant when he held that mathematical arguments are normally synthetic was quite right".

Why should Hintikka's talking of logic be a vindication of Kant's talking of mathematics? Simply because "by mathematical arguments he [i.e., Kant] meant modes of reasoning which are now treated in quantification theory"[76]. In other words, the point is that the contemporary boundaries between mathematics and logic are not the Kantian ones. Modern first-order logic includes modes of reasoning that Kant wouldn't have called logical, but mathematical, and it is precisely this kind of derivations that Hintikka considers to be synthetic. As we have seen in Chapter 1 of this thesis, Kant distinguished between pure general logic, that consisted in the Aristotelian syllogistic and some propositional pattern of reasoning, and mathematics, the inferences of which are paradigmatic examples of the synthetic reasoning. Similarly, according to Hintikka, the inferences of modern first-order logic can be divided in two classes: the former is composed by the analytic inferences of monadic logic; the latter by the synthetic inferences that translated a typical mathematical way of reasoning. To sum up, Kant's influence on Hintikka's work regarding modern logic does not only involve the definition of analyticity and syntheticity, but it also concerns the conclusion regarding the status of mathematics.

There are no doubts that Hintikka's work is Kantian in spirit and probably captures Kant's fundamental intentions. Nevertheless, we would like to underline some distances and important gaps between the original framework and the proposed modern reconstruction. First, some inferences that are synthetic according to Kant are analytic following Hintikka's definition. This is also the case of the well-known Kantian example '7+5=12', that can be turned into a quantificational derivation in which the degree of the conclusion is higher than the degree of any intermediate stage. However, if one notices that this difference can be attributed

[75]This is one of the theses Hintikka (1973, VI) puts forward in his article *An Analysis of Analyticity.* Here (p. 137), he maintains that "Sense III [i.e., our D_1] approximates rather closely Kant's notion of analyticity".

[76]Hintikka (1973, VIII, p. 182).

to technicalities that do not involve the sameness of inspiration of the two distinctions[77] and accepts, as Hazen does, the fact that "logical theory has advanced too much since Kant's day for his views to have any precise application to it"[78], this observation turns out to be less alarming than what it seemed. Second, in Chapter 1 of this thesis, we have shown that Kant does not apply his analytic-synthetic distinction to logic and, attempting the analysis that Kant did not think it was worth pursuing, the result is that many logical judgments are neither analytic nor synthetic. This suggests that Hintikka's vindication of Kant is only partial and confined to the status of mathematics: as far as logic is concerned, the Finnish philosopher ascribes to the author of the *Critique* perspectives that he never put forward, that is, that monadic first-order logic (or, in Kantian terms, pure general logic) is analytic. The third difficulty we would like to highlight is the most serious one and regards the interpretational basis of Hintikka's work. In Section 4.1, we have pointed at the hesitations and weaknesses of Hintikka's reading of the Kantian material: in particular, the characterization of Kantian intuitions in terms of singular representations was too radical; the leap from the mathematical method to the analytic-synthetic distinction found no convincing basis on the Kantian texts; and Hintikka's thesis that Kant's conception of analyticity was inscribed in the constructional tradition was unjustified. In this Section, we have shown that the same interpretational passages are at the core of Hintikka's modern translation of Kant. As a result, these considerations cannot be but an important point against the view that Hintikka's work is a faithful reconstruction of the Kantian positions.

4.3 Against logical positivism through modern means

We have seen in Section 3.2.1 that the authors of the *Wissenschaftliche Weltauffassung* reject the existence of synthetic *a priori* judgments and hold that logic and mathematics are both analytic and tautological. Hintikka's thesis that there exists a class of *synthetic* quantified logical truths, or, in other words, that the 'synthetic *a priori*' is a non-empty category, is thus a vindication through modern means of the main principles of Kant's philosophy against the criticisms moved by the logical positivists: it is at the same time a (supposed) reconstruction of Kant's theory and an open attack against the perspectives held by the Vienna Circle.

[77]The technicalities to which we are referring are the ones exposed in note 69. Hintikka (1973, VIII, pp. 195-196) explains this gap in terms of the difference between the following two questions: 1. In thinking of the number 12 distinctly are we already thinking of the numbers 5 and 7?; 2. Can we from the concept of 7 and 5 analytically arrive at the concept of 12? Only the latter question can be attributed to Kant according to Hintikka's reading.

[78]Hazen (1999, p. 97).

It is this second aim of Hintikka's theory that explains, we think, the two most serious weaknesses of his interpretation of the *First Critique* that we have individuated above. On the one hand, Hintikka's ascription to Kant of the idea that pure general logic (or, in modern terms, monadic first-order logic) is analytic is nothing but a piece of the positivistic heritage spared by the Finnish philosopher's critical fury. On the other hand, the interpretative 'stretching' that we have highlighted on several points of Hintikka's reconstruction of the Kantian material can be explained as an anachronistic way to let Kant speak the same modern language of the logical positivists: in this way, the modern tools used by Hintikka to answer to the criticisms of the Vienna Circle can be passed off as Kantian means and Hintikka's work can be seen as Kant's defense of himself through his own means.

However, the synthetic and *a priori* character of a class of first-order logical inferences is vigorously argued by Hintikka on strong grounds that could seem to be independent from Kant's premises or, at least, distant developments of seeds proposed in the *Critique*. This Section wants to provide a complete overview of these arguments. Section 4.3.1 provides a simple reconstruction of Hintikka's theory of distributive normal forms for first-order logic and of the analytic-synthetic distinction defined in these terms. Section 4.3.2 expounds Hintikka's theory of probability and semantic information and deals in particular with the notion of surface information. Section 4.3.3 draws conclusions and offers an evaluation of Hintikka's work on the epistemological status of logic.

4.3.1 The theory of distributive normal forms

The first issue that we have to consider concerns the problem of finding a plausible linguistic counterpart to a certain possible world. This question is only apparently far from our starting point, namely the epistemological status of logic, because, together with the criticism Hintikka puts forward against Carnap's proposal, it leads the Finnish philosopher to define the theory of distributive normal forms for first-order logic that gives a procedure to discern which logical inferences are analytic and which are synthetic according to the sense of the terms exposed by definition D_2.

At the very beginning of his book entitled *Meaning and Necessity*[79], Carnap, in order to introduce the key concept of 'L-truth'[80], defines the notion of 'state description' as follows. First, a set of singular terms and predicative symbols is assumed. Second, the relevant set of atomic sentences is given in the obvious way

[79]Carnap (1947, p. 9). In this work, Rudolf Carnap lays the foundations for the semantic of modal logics and proposes a new method for the semantical analysis of the meaning of linguistic expressions, according to which the latter designate their intentions and extensions.

[80]According to Carnap, this notion is meant as an *explicatum* for what Leibniz called necessary truth and Kant analytic truth.

as the set of all the atomic sentences which can be formed from these singular terms and predicate symbols. Then, a state description is said to be a class of sentences, which contains for every atomic sentence defined above either this sentence or its negation, but not both, and no other sentence. Carnap's state descriptions are the most complete descriptions that can be given of a possible state of the universe of individuals and extend to first-order logic the same approach used in propositional logic.

In several articles[81], Hintikka criticizes Carnap's notion of 'state description' on the basis that, in order to be able to specify a state description, we should be able first of all to name each individual in our universe[82]. But this requirement proves to be a great disadvantage from many perspectives. First of all, it shows that Carnap's proposal has no concrete applications. Moreover, "as long as one sticks to such full descriptions, one gains no major advantage by considering descriptions of worlds instead of worlds themselves"[83]. Last, Carnap's conception goes against one of the most important principles of Hintikka's philosophy, namely that part of the scientific enterprise "is to come to know the universe in which we live, and a part of this task is to find out what individuals there are in the world and how many they are"[84].

Hintikka individuates two paths to overcome the drawbacks of Carnap's notion[85]. The former proposal abandons the scope of giving an exhaustive description of the possible worlds in favor of a partial one, which anyway suffices to show that the state represented is not contradictory. This is the idea that grounds the model set technique. The latter alternative is to provide a description of the states of affairs that is as complete as possible given a suitable restriction on the expressive means. Hintikka's acute choice of the restrictions to impose on the means of expression is what provides the premise for linking his theory of distributive normal forms to his reasoning on the analytic-synthetic distinction. The idea is to restrict the complexity of the configurations of individuals that can be considered: as we have seen above, this amounts to limit the length of sequence of quantifiers of a given sentence or, in Hintikka's term, its depth. This is the principle that

[81] In particular, in Hintikka (1973, I); Hintikka (1973, VII) and Hintikka (1987).

[82] Hintikka (1973, VII, p. 158) specifies this point as follows: "formally speaking, the whole approach, including the notion of a state-description, is relative to a given set of free singular terms. (It is also relative to a set of predicates, but that fact is of lesser importance.) Informally speaking, the possible worlds or possible kinds of worlds which Carnap contemplates are specified by specifying the individuals there are in the world in question. In order to describe such a world, we must know all its individuals; moreover, we must know that they are all the individuals there are in the world in question. In order to say what such a description looks like, we have to know how many individuals there are, i.e., what the size of our universe of discourse is".

[83] Hintikka (1987, p. 13).

[84] Hintikka (1973, VII, p. 158).

[85] See, for example, Hintikka (1973, I).

guides the theory of distributive normal forms for first-order logic, which is our concern here[86].

This theory is an extension of the theory of distributive normal forms for propositional logic and monadic first-order logic. Let F be a first-order formula characterized by the following parameters:

P1) the set of all the predicates occuring in it;
P2) $\{a_1, \ldots, a_k\}$, which is the set of all the free individual symbols occurring in it;
P3) d, which is the maximal length of sequences of nested quantifiers occurring in it (its depth).

The distributive normal form of F with the same parameters is a disjunction of conjunctions called constituents, C_i^d:

$$F^d = DNF^d(F) = \bigvee_{i=1}^{j} C_i^d(a_1, \ldots, a_k). \tag{4.1}$$

Each constituent $C_i^d(a_1, \ldots, a_k)$ characterized by certain parameters P1), P2) $= \{a_1, \ldots, a_k\}$ and P3) $= d$ represents a possible world that can be described by the sole means of these parameters. The disjuncts that occur in the distributive normal form of the formula F represent all and only the descriptions of the possible worlds that are not excluded by F. Before giving a definition of the constituents, we first need to explicate some notations and to define the crucial notion of attributive constituent. Given a set of predicates P1), let $A_i(a_1, \ldots, a_k)$ be all the atomic formulae that can be formed by the members of P1) and from the individual symbols $\{a_1, \ldots, a_k\}$ and $B_i(a_1, \ldots, a_k)$ be all the atomic formulae so defined that contain at least one occurrence of a_k. It follows[87] that:

$$\bigwedge_{i=1}^{r} A_i(a_1, \ldots, a_k) = \bigwedge_{i=1}^{s} A_i(a_1, \ldots, a_{k-1}) \wedge \bigwedge_{i=1}^{t} B_i(a_1, \ldots, a_k). \tag{4.2}$$

An attributive constituent, written $Ct^d(a_1, \ldots, a_k)$, with parameters P1), P2) $= \{a_1, \ldots, a_k\}$ and P3) $= d$ can be recursively defined in terms of attributive constituents of depth $d-1$ as follows[88]:

[86] The works in which the author presents and discusses the theory of distributive normal forms for first-order logic are the following ones: Hintikka (1953), Hintikka (1965b), Hintikka (1970a), Hintikka (1970b), Hintikka (1973 X), Hintikka (1973 XI) and Hintikka (1973).

[87] For each given r and for suitably chosen (with respect to r) s and t.

[88] Here it is assumed that indices are used to distinguish different attributive constituents with the same parameters from one another. The index r is a function of s and t. This dependence will be assumed also in the expressions that follow.

$$Ct_r^d(a_1, \dots, a_k) = \bigwedge_{i=1}^{s} B_i(a_1, \dots, a_k) \wedge \bigwedge_{i=1}^{t} \exists x Ct_i^{d-1}(a_1, \dots, a_k, x). \tag{4.3}$$

Intuitively speaking, if constituents represent possible worlds, attributive constituents describe possible kinds of individuals. The attributive constituent shown in the expression 4.3 represents the complex attribute of the individual referred to by a_k and describes this individual by the sole means of P1), the 'reference-point' individuals $a_1, \dots, a_k$ and at most d layer of quantifiers. We now examine the components of the equation 4.3 one by one. The first conjunct $\bigwedge_{i=1}^{s} B_i(a_1, \dots, a_k)$ specifies the way in which a_k is in relation with the individuals $a_1, \dots, a_{k-1}$: it is a conjunction of all the atomic formulae that can be formed from P1) and P2) that contain a_k. Then, for every i, $Ct_i^{d-1}(a_1, \dots, a_k, x)$ provides a list of all the kinds of individuals that can be specified through the parameters P1), the individuals $a_1, \dots, a_k$ and $d-1$ layers of quantifiers. For each such kind of individual, the expression $\bigwedge_{i=1}^{t} \exists x$ specifies whether individuals of that particular kind exist or not.

The notion of a constituent with parameters P1), P2) $= \{a_1, \dots, a_k\}$ and P3) $= d$ is simply defined on the basis of attributive constituents with the same parameters as follows[89]:

$$C^d(a_1, \dots, a_k) = \bigwedge_{i=1} A_i(a_1, \dots, a_{k-1}) \wedge Ct^d(a_1, \dots, a_k). \tag{4.4}$$

Hintikka proves that every first-order formula F with certain parameters can be converted into its distributive normal form with the same parameters or with certain fixed larger ones. As a special case, every constituent with depth d and some parameters P1) and P2) can be converted into a disjunction of constituents, called 'subordinate', with the same parameters P1) and P2) but greater depth $d+e$ for some natural number e.

The theory of distributive normal forms, the elementary notions of which we have just sketched above[90], is very articulated and has several employments. We

[89]Here the indexes of C^d, $\bigwedge$ and Ct^d have not been specified, but of course the first depends on the other two.

[90]Hintikka provides a second and equivalent formulation of the key notions of constituents and attributive constituents. Following this alternative, an attributive constituent with the parameters P1), P2) $= \{a_1, \dots, a_k\}$ and P3) $= d$ is defined as:

$$\begin{aligned} Ct_r^d(a_1, \dots, a_k) = \bigwedge_{i=1}^{s} B_i(a_1, \dots, a_k) \wedge \prod_{i=1}^{t} \exists x Ct_i^{d-1}(a_1, \dots, a_k, x) \wedge \\ \wedge \forall x \sum_{i=1}^{t} Ct_i^{d-1}(a_1, \dots, a_k, x) \end{aligned} \tag{4.5}$$

have introduced it as an instrument that Hintikka elaborates in conflict with Carnap's proposal and we now want to show the way in which Hintikka uses it to prove, against the positivists, that some inferences of first-order logic are synthetic. The Finnish philosopher's starting point is a result of the computability theory, namely, Church's theorem that establishes the undecidability of first-order logic:

> In propositional logic and in monadic first-order logic distributive normal forms yield a decision method: if a formula has a non-empty normal form, it is satisfiable, and vice versa; it is logically true if and only if its normal form contains all the constituents with the same parameters as it. In view of Church's undecidability result they cannot do this in the full first-order logic (with or without identity). It is easily seen that this failure is possible only if some of our constituents are in this case inconsistent. In fact, the decision problem of first-order logic is seen to be equivalent to the problem of deciding which constituents are inconsistent. More explicitly, the decision problem for formulae with certain fixed parameters is equivalent to the problem of deciding which constituents with these parameters are inconsistent (Hintikka, 1973, XI, p. 255).

The bridge built from the decidability problem to the problem of finding which constituents are inconsistent is fundamental. Hintikka thus distinguishes between inconsistent constituents that are trivially inconsistent and inconsistent constituents that are not trivially inconsistent. While the former are blatantly self-contradictory and satisfy some of the inconsistency conditions put forward by Hintikka, the inconsistency of the latter can be detected only by increasing their depth. In other words, it is shown that for every inconsistent constituent C^d there is some natural number e such that all the subordinate constituents of depth $d + e$ are trivially inconsistent. The point is that we do not know which depth we should reach in order to acknowledge that a certain constituent is inconsistent because first-order logic is undecidable.

---

where the Π-operator and the Σ-operator indicate the conjunction and the disjunction of the non-negated members of $\bigwedge_{i=1}{}^{t}\ \exists x Ct_i^{d-1}(a_1, \ldots, a_k, x)$ respectively. Instead of listing all the different kinds of individuals that exists and also all the different kinds of individuals that do not exist as in the first formulation 4.3, 4.5 lists all the existing ones and then adds that they are all the existing ones. On the basis of this second formulation attributive constituent, an alternative definition of the notion of constituent can be given in the obvious way:

$$\begin{aligned} C^d(a_1, \ldots, a_k) = \bigwedge_{i=1} A_i(a_1, \ldots, a_k) \wedge \prod_{i=1}{}^{t}\ \exists x Ct_i^{d-1}(a_1, \ldots, a_k, x) \wedge \\ \wedge \forall x \sum_{i=1}{}^{t}\ Ct_i^{d-1}(a_1, \ldots, a_k, x). \end{aligned} \tag{4.6}$$

Once the issue of consistency has been clarified, a disproof procedure and a method of proof from assumptions can be easily defined. First, the disproof procedure for inconsistent constituent can be described in the following terms. Suppose that we want to disproof a certain constituent C^d. If it is not trivially inconsistent, we have to expand it into a disjunction of a number of subordinate constituents of depth $d + 1$. If some of them are not trivially inconsistent, we have to keep on expanding C^d into a disjunction of subordinate constituents of greater depth $d+e$. Then, for a certain natural number e, if C^d is inconsistent, then all its subordinate constituents of depth $d + e$ will turn out to be trivially inconsistent.

Second, in order to prove G from F, we combine the parameters P1) and P2) of F and G, take the maximum (say d) of their depths, and convert F and G into their distributive normal forms F^d and G^d with the parameters just obtained. Then we expand F^d and G^d by splitting their constituents into disjunctions of deeper and deeper constituents and at each step we omit all the trivially inconsistent constituents. Then, if G follows from F, there will be an e such that all the non-trivially inconsistent members of F^{d+e} obtained by these procedure are among the non-trivially inconsistent members of G^{d+e} obtained through the same procedure. A proof of G from F will thus follow the steps described below:

$$F \leftrightarrow F^d \leftrightarrow F^{d+1} \leftrightarrow \cdots \leftrightarrow F^{d+e} \rightarrow G^{d+e} \leftrightarrow G^{d+e-1} \leftrightarrow \cdots \leftrightarrow G^d. \tag{4.7}$$

What interests us most here is that this method of proof from assumptions gives us a way to discern which inferences are analytic and which are synthetic according to the sense D_2 put forward by Hintikka. In the former, no increase in depth is needed: the elimination of trivially inconsistent constituents of depth d is sufficient to show that all the constituents of F^d are among those of G^d. In the latter, in order to bring out the desired relationship between F and G, an increase of depth is required and we need to consider configurations of individuals of greater complexity than those that represent the premises and the conclusion of the argument. This classification of logical inferences allows Hintikka to give a characterization of syntheticity as a matter of degree: an inference from G from F is synthetic of degree e if and only if we need to expand the normal forms of F and G by splitting their constituents into disjunctions of depth $d + e$. In other words, the degree of syntheticity of an inference counts the number of individuals that we need to include in the initial configurations in order to be able to derive the conclusion from the premises.

4.3.2 Probability and semantic information

The theory of distributive normal forms can be used as a basis to define a theory of probability and a theory of semantic information, which provide an answer to the question of how much information is conveyed by a certain first-order sentence. The conceptual steps to obtain the desired result with those means can be roughly summarized as follows, where parameters P1 and P2 are assumed to be fixed:

1. For every depth d, let X^d be the set of constituents in the polyadic predicate calculus that can be described at depth d: it is clear that the probability of the disjunction of the members of X^d is one, $p(\vee X^d) = 1$, because the disjunction of the members of X^d includes all the possible alternatives that can be specified with that expressive restriction[91].

2. The probability of sentence F^d, written $p(F^d)$, is thus the sum of the probabilities assigned to the constituents in X^d that are included in its normal form $DNF^d(F^d)$, for the latter specifies what the basic alternatives are by listing all the possible worlds that are admitted by F and can be specified with no more than d individuals mutually related.

3. Once the probability of every sentence F has been defined, the content measure, $cont(F)$, and the information measure, $inf(F)$, of F can be easily defined using the following equations, which are very common in the theories of information and embodies the principle that information equals elimination of uncertainty[92]:
$$inf(F) = -\ log\ p(F) \tag{4.8}$$
$$cont(F) = 1 - p(F). \tag{4.9}$$

The procedure that we have outlined above lacks an important specification. While it is obvious that trivially inconsistent constituents must be excluded from X^d (or, equivalently, must be assigned a zero weight), because they do not represent viable alternatives, it is not clear whether non-zero weights must be assigned only to consistent constituents or also to constituents that are not trivially inconsistent at a certain depth (but can possibly reveal their inconsistency at a higher depth). The observation, which is made possible by the theory of distributive normal forms, that there are two possibilities concerning the probability assignment to first-order constituents leads Hintikka to define two notions of information conveyed by a certain polyadic sentence.

---

[91]The way in which the probability to each constituent is assigned does not matter here. We could assume a 'purely logical' probability, that is, we could simply let all the constituents have an equal weight.

[92]The idea is that the more alternatives a sentence admits of, the more probable it is and the less information it gives us. For example, the sentence 'either it rains or it does not rain' admits of all the alternatives, it has probability one and it gives us no information (as Wittgenstein puts it, 'I know nothing about the weather when I know that it is either raining or not raining').

'Depth information' is the measure obtained by assigning a positive probability weight only to the consistent constituents of the polyadic calculus. It is a reconstruction of the notion of semantic information chosen by Bar-Hillel and Carnap[93] for their theory, although the latter is based on state descriptions and thus suffers from the problems connected with this notion[94]. An essential feature of depth information is its non-recursive character. This amounts to say that depth information is not calculable in practice and that in general there is no decision procedure through which the initial distribution of probability can be assigned. The reason for the non-recursive character of depth information is that we cannot effectively isolate the inconsistent constituents because first-order logic is undecidable[95]. Hintikka considers this feature a major disadvantage of depth information:

> But measures of information which are not effectively calculable are well-nigh absurd. What realistic use can there be for measure of information which are such that we in principle cannot always know (and cannot have a method of finding out) how much information we possess? One of the purposes the concept of information is calculated to serve is surely to enable us to review what we know (we have information about) and what we do not know. Such a review is in principle impossible, however, if our measure of information are non-recursive (Hintikka, 1973, X, p. 228).

The non-recursive character of depth information is seen by Hintikka as a good reason to react against the conception of information elaborated by Bar-Hillel and Carnap that become the traditional option and to formulate an alternative measure of information that he calls 'surface information'. The latter is obtained by assigning non-zero weights to all the constituents that are not trivially inconsistent at a certain depth[96]. The result of this assignment is a measure of information that

[93] Bar-Hillel and Carnap (1953).

[94] Carnap (1962, p. 294 and ff..) assigns an equal probability weight to every 'structure-descriptions', that is to say, to every disjunction of all the state-descriptions which can be transformed into each other by permuting free singular terms; this probability weight is then divided evenly among the members of the structure-description in question. The semantic information conveyed by a certain sentence can then be defined as the sum of the weights of all the state-descriptions excluded by that sentence.

[95] Another name Hintikka uses to refer to depth information is 'post-logical' information, because, in order to assign this measure, we need to use the tools of logic to discover the constituents whose consistency was only apparent. We think however that the adjective coined by Hintikka is quite misleading, for it does not take into account the practical limits of the logical tools we are considering.

[96] It is possible for all the constituents of depth $d+1$, that are subordinate to the non-trivially inconsistent constituent C^d, to be trivially inconsistent. In this case, Hintikka (1973, X, p. 229) prescribes to redistribute probability weights in the following way. We have to find the

is calculable in practice and that is more realistic than the depth one, because it takes into serious account the fact that it is practically impossible to individuate among the non-trivially inconsistent constituents those that are inconsistent *tout court* and accepts this restriction[97].

An important feature of surface information, which is not shared by depth information, is that it can be increased by logical deduction: this tool enables us to find that certain alternatives about the world were only apparently viable or, to put it more formally, it enables us to find that certain non-trivially inconsistent constituents were nevertheless inconsistent at a greater depth. But what is surface information information about? According to the Finnish philosopher[98], surface information has a double nature: on the one hand, it is information about reality, because it allows to exclude the existence of (mutually related) individuals; on the other hand, it is conceptual information, because the existence of non-trivially inconsistent constituents is a feature proper to the conceptual system, that is to say, to the relation between the first-order sentences and the reality they speak of. This observation leads Hintikka to maintain that in first-order logic, due to its undecidability, conceptual information is inseparable from factual information, for the elimination of only apparently consistent constituents, which cannot be effectively isolated, improves our knowledge both of the reality and of our conceptual system.

In Chapter 4, we have seen that the logical positivists' favorite solution to the paradox of analysis was psychologistic in nature and affirmed that the conclusion of a valid inference is not objectively new or informative with respect to its premises: the incompatibility between validity and informativity highlighted by the paradox was solved in favor of the former. One of the consequences of Hintikka's main thesis that (part of) logic is synthetic *a priori* is of course the rejection of the positivists' answer to the paradox of analysis, that the Finnish philosopher openly attacks dubbing it "the scandal of deduction", and the claim that logic is informative:

> If no objective, non-psychological increase of information takes place in deduction, all that is involved is merely psychological conditioning, some sort of intellectual psychoanalysis, calculated to bring us

constituent, call it C^e, to which C^d is subordinate, that has the greater possible depth $e < d$ and that has some non-trivially inconsistent constituent of depth $d + 1$. Once we have isolated C^e, we have to redistribute the weight of C^d between all the subordinate constituents of C^e of depth $d + 1$.

[97]Sometimes Hintikka uses a different name for surface information: 'pre-logical information'. The reason for this is that "the depth information of a sentence is its surface information after we have subjected it to the whole treatment logic puts at our disposal" (Hintikka, 1973, X, p. 230). Again, the couple 'post-logical' and 'pre-logical' is, we think, not appropriate in a context in which logical tools play a decisive role.

[98]See, for example, Hintikka (1973, X, p. 230 and ff.).

> to see better and without inhibitions what objectively speaking is already before our eyes. Now most philosophers have not taken to the idea that philosophical activity is a species of brainwashing. They are scarcely any more favourably disposed towards the much far-fetched idea that all the multifarious activities of a contemporary logician or mathematician that hinge on deductive inference are as many therapeutic exercises calculated to ease the psychological blocks and mental cramps that initially prevented us from being, in the words of one of these candid positivists, 'aware of all that we implicitly asserted' already in the premises of the deductive inference in question (Hintikka, 1973, X, p. 223).

Hintikka's solution to the paradox of analysis makes use of the two notions of information that he had elaborated on the basis of the theory of distributive normal forms. While the logical positivists' perspective is vindicated by the recognition that depth information is not increased by logical deduction, the idea that logic is not merely an 'intellectual psychoanalysis' is justified by the fact that surface information can be increased during a deduction. The latter kind of information provides an objective and non-psychological sense in which logic is informative. As it can be proved if the corresponding notions are suitably defined, the depth information of a sentence is the limit to which its surface information converges when it is expanded into deeper normal forms or, equivalently, when all the possible logical work has been carried out.

4.3.3 An evaluation

In this Chapter, we have deeply analyzed Hintikka's attack against the logical positivists' tenet that logic is analytic. We have seen that the Finnish philosopher's main thesis that there exists a class of polyadic logical inferences that are synthetic is defended through a modern reconstruction of Kant's positions and the elaboration of a theory of semantic information based on the theory of distributive normal forms. In Section 4.1, we have highlighted what we regarded as the main difficulties of Hintikka's peculiar reading of the Kantian material: a too radical interpretation of the nature of Kantian intuitions; an unjustified leap from the *Critique* account of the mathematical method to the Kantian analytic-synthetic distinction; and an unpersuasive positioning of Kant's work into the constructional tradition. In Section 4.2, we have argued that there are important gaps and distances between the original Kantian framework and Hintikka's supposed modern reconstruction of it: first, Hintikka's vindication of Kant's theory is confined to the status of mathematics and, as far as logic is concerned, misses completely the point; second, the interpretative difficulties shown above are at the core of Hintikka's translation of Kant.

It's now time to evaluate the most modern part of Hintikka's proposal, which has been presented in this Section. To sum up, we have shown that, first, a criticism against the notion of state-description that Carnap has offered to provide a linguistic counterpart of possible worlds leads Hintikka to formulate the theory of distributive normal forms for first-order logic. This theory allows Hintikka to give a formal definition of the analytic-synthetic distinction based on the principle exhibited in D_2 and to show, against one of the dogmas of the Vienna Circle, that some polyadic inferences are synthetic *a priori*. Second, against Bar-Hillel and Carnap's theory of semantic information and on the basis of his theory of distributive normal forms, Hintikka elaborates the notion of surface information, which gives a non-psychologistic solution to the paradox of analysis and a tool to avoid the positivists' scandal of deduction.

We believe that two are the main limits of Hintikka's formal equipment. First of all, the disproof method via distributive normal forms is rather complex or, to use Rantala and Tselishchev words[99], it is "not very practical" in practice. Now, this feature calls Hintikka's success in having explicated the amount of semantic information generated by deductive inferences into question. Even if we agreed with Sequoiah-Grayson's *caveat* that "Hintikka understands his disproof method as an *auxiliary process* that allows us to identify and measure the informativeness of deductive inferences and logical truths *irrespectively of the actual proof procedure used* in their derivation"[100], still Hintikka's proposal would not be free of difficulties. The calculation of the surface information of an inference depends on the form of the particular formula involved: as a result, we could make one inference more informative than another by simply adding in irrelevant steps. Moreover, surface probability is only assigned to closed constituents: this means that Hintikka does not define the notion of surface information for formulae containing free variables[101].

The second worry about Hintikka's theory is perhaps more serious than the former one. The set of formulae that turn out to be analytic following Hintikka's definitions is much less restricted than what it might first appear[102]. It includes,

[99]Rantala and Tselishchev (1987, p. 89).

[100]Sequoiah-Grayson (2008, p. 87, emphasis added). This observation corrects Rantala and Tselishchev (1987, p. 87)'s claims that the connection between surface information and the method of disproof must be understood as being very strict: "So it seems that what is given to us by Hintikka by means of surface information is a way to measure [...] *only* the kind of information which is attainable by means of this specific method of proof".

[101]The two technical points mentioned above have been noticed by Rantala and Tselishchev (1987, pp. 87-88).

[102]This observation has been mentioned by both Sequoiah-Grayson (2008, p. 88 and ff.) and D'Agostino and Floridi (2009, p. 278).

beyond many polyadic deductions, also the entire set not only of propositional logic, but also of monadic logic, because both the propositional and the monadic calculus contain only consistent constituents. The inferences included in this set fail to increase surface information and are said by Hintikka to be analytic. This conclusion seems however too strict and liable to be attacked by the criticisms that Hintikka himself directed against the logical positivists. Are these kind of inferences really uninformative? Isn't this conclusion still a (perhaps more restricted) "scandal of deduction"? Is philosophical activity in this context really a "species of brainwashing"?

The two main limits of Hintikka's formal proposal, together with the questions it leaves open, will be addressed in the next Chapter. Now, we would like to underline an important aspect of Hintikka's work. The kernel of the formal theory comes from the field of computability: the undecidability of first-order logic is a fundamental observation for the development of both the theory of distributive normal forms and the theory of semantic information. The fact that we have to expand a given constituent at a certain depth in order to acknowledge its inconsistency grounds Hintikka's notion of degree of syntheticity and the fact that we do not know which depth the expansion has to reach in order to achieve the desired result represents the main motivation towards Hintikka's formulation of the notion of surface information. The brilliant idea behind these constructions is that the definition of the analytic-synthetic distinction must take into account the result that some inferences are 'more difficult' than others and require a greater computational effort and that useful measures of information must be realistic and must envision that in general there is no decision procedure for determining which constituents are inconsistent.

Needless to say, the undecidability result of first-order logic was completely unknown to Kant. What is more interesting to notice perhaps is that even the intuitive idea of the 'difficulty of a reasoning pattern' seems to be at the periphery of the analytic-synthetic distinction presented in the *Critique*[103]. This observation confirms our suggestion that a certain component of Hintikka's theory, which is more important than what the Finnish philosopher would probably be ready to admit, is not Kantian and has been elaborated as a direct response to some principles put forward by the positivists of the Vienna Circle.

[103]On this point, see Chapter 1.

Part II

Formal proposals

Chapter 5

Depth Bounded First-Order Logics

5.1 Depth Bounded Boolean Logics

5.1.1 Is propositional logic really uninformative?

In Section 4.3.3, we have seen that Hintikka's work classifies as analytic a wide class of logical inferences that includes many polyadic deductions, as well as the entire sets of propositional and monadic inferences. On that occasion, we have raised doubts on the completeness of this result: on the one hand, it seemed to be only a *partial* vindication of the intuitive idea that logical deduction can increase our knowledge; on the other hand, it could be charged of the same criticisms that Hintikka himself formulated against the logical positivists' dogma of the analyticity of logic.

D'Agostino and Floridi (2009) have recently argued that these doubts concerning the analyticity of *propositional* logic find a confirmation in the theory of computational complexity, a branch of the theory of computation in theoretical computer science that at the time of Hintikka's proposal was at the beginning of its flourishing[1]. In this context, decision problems can be classified according to their resource-based complexity. The class P includes all the decision problems that can be solved in polynomial time by a deterministic Turing machine and that are said to be tractable or solvable in practice, while the class NP is made up by decision problems that can be solved in polynomial time by a non-deterministic Turing machine. The most important unsolved problem in theoretical computer science concerns the relationship between these two classes and asks whether P is identical with NP or not. The most part of researchers assume that the two classes

[1]See Garey and Johnson (1979).

are not identical, that is to say, that $\mathsf{P} \neq \mathsf{NP}$ and that no deterministic Turing machine can be found to solve problems in the class NP.

As far as Boolean logic is concerned, it is possible to identify three decision problems that are strictly connected. First, the Boolean satisfiability problem (SAT), which is the problem of determining whether there exists an interpretation that satisfies a given propositional formula, was proven[2] to be NP-complete, that is to say, one of the most difficult problems in NP. Second, the problem of determining whether a given Boolean formula is a tautology (TAUT) is NP-hard, that is to say, it is not known whether it belongs to NP and every problem in NP can be reduced to it in polynomial time. Third, the problem of determining whether a Boolean inference is correct or not can be reduced to TAUT from both a deterministic and a non-deterministic point of view. This means that, if the widely accepted conjecture $\mathsf{P} \neq \mathsf{NP}$ turns out to be true, then SAT, TAUT and the inference problem are intractable, viz. not decidable in practice. As D'Agostino (2010) underlines, this amounts to say that any real agent, even if equipped with an up-to-date computer running a decision procedure for Boolean logic, may never be able to feasibly recognize that certain Boolean sentences logically follow from sentences that she regards as true. Hintikka considered the undecidability of first-order logic as a strong reason to hold that polyadic logical truths are not analytic. Similarly, this conjecture of the computational complexity is a reasonable justification to reject the logical positivists' and Hintikka's thesis on propositional logic: if the decision problem for Boolean logic is (most probably) intractable, how is it possible to maintain that it is uninformative and analytic?

The probable intractability of propositional logic leads D'Agostino and Floridi (2009) to formulate an innovative non-classical semantic, called 'informational semantic', according to which the class of synthetic propositional inferences is not empty. Following this account, the conclusion of an analytic inference depends solely on the *informational* meaning of the logical operators occurring in its premises and conclusion, while synthetic inferences are characterized by the use of some intuitions, called 'virtual information', which represent the kind of temporary assumptions needed in every reasoning by cases and in the well-known natural deduction rule for the introduction of the conditional. The notion of synthetic inference is moreover given a gradual characterization. The *degree of syntheticity of an inference* or the *depth of an inference* is said to be k if and only if k is the lowest number of nested pieces of virtual information needed to obtain the conclusion from the premises. The depth of an inference is thus a formal translation of the intuitive idea of the degree of difficulty of an inference and depends on the cognitive effort and on the computational resources needed to recognize the validity of that inference. The fundamental idea of the authors is that the intractability of

[2]Cook (1971).

classical logic depends exactly on the unbounded use of virtual information.

From a formal point of view, D'Agostino and Floridi (2009)'s approach consists of the formulation of a hierarchy of new logical systems, which are called 'Depth Bounded Boolean Logics'. These logics, which are based on the informational semantics, provide an incremental characterization of classical propositional logic that results as the limit of this infinite sequence of weaker and tractable logics. This hierarchy of logics represents increasing levels of depth or informativeness of classical reasoning, where the increase of the degree of computational complexity is associated with the depth at which the use of virtual information is allowed.

We shall now discuss the basic notions of the semantics (Section 5.1.2 and Section 5.1.3), as well as the a particularly elegant proof-theoretic presentation of these logics (Section 5.1.4). We shall rely on the expositions carried out in the following articles: D'Agostino and Floridi (2009), D'Agostino (2010), D'Agostino (2013a), D'Agostino (2013b), D'Agostino (2014a), D'Agostino (2014b) and D'Agostino (2015).

5.1.2 Informational semantics

The informational semantics is based on two primitive notions: 'agent a actually possesses the information that B is true' and 'agent a actually possesses the information that B is false'. These two notions replace the classical and alethic ones: 'B is true' and 'B is false'. According to its primitive notions, the informational semantics is grounded on the following principle:

> **(IS)** The informational meaning of an n-ary logical operator $\star$ is determined by specifying the necessary and sufficient conditions for an agent a to actually hold the information that a sentence of the form $\star(\varphi_1, \ldots, \varphi_n)$ is true, respectively false, in terms of the information that a actually holds about the truth or falsity of $\varphi_1, \ldots, \varphi_n$ (D'Agostino, 2013a, p. 50).

The informational meaning of the logical operators is given by the informational 3-valued matrices for the Boolean operators represented in Figure 5.1. These matrices have been anticipated by Quine (1974) in his dispositional theory of the meaning of the logical operators. As the matrices make clear, the informational semantics is three-valued, not truth-functional and weaker than the classical one. These three features are justified by the following reasons.

First, the classical principle of bivalence, interpreted in informational terms, says that for any sentence B, every agent possesses either the information that B is true or the information that B is false: indeed, it is too strong and thus it is rejected. This is conveyed by the use of a third truth value, $*$, which indicates that agent a neither possesses the information that B is true, nor that B is false.

$\neg$	
1	0
0	1
$*$	$*$

$\wedge$	1	0	$*$
1	1	0	$*$
0	0	0	0
$*$	$*$	0	$*$,0

$\vee$	1	0	$*$
1	1	1	1
0	1	0	$*$
$*$	1	$*$	$*$,1

$\rightarrow$	1	0	$*$
1	1	0	$*$
0	1	1	1
$*$	1	$*$	$*$,1

Figure 5.1: Informational three-valued matrices for the Boolean operators (D'Agostino, 2013a, p. 39).

Second, according to the authors, truth-functionality is an unwelcome property for an informational semantics. Consider the necessary and sufficient classical conditions, interpreted in informational terms, for the truth of a disjunction: 1) if agent a actually possesses the information that $B \vee C$ is true, then a actually possesses either the information that B is true or the information that C is true; 2) if agent a actually possesses the information that B is true or that C is true, then a actually possesses the information that $B \vee C$. While the second clause respects **(IS)**, the first requirement is too strong: as the authors suggest, it is possible for a to actually have the information that the sentence 'either the roulette ball will fall into a red pocket or it will fall into a black pocket' is true, even when a does not hold any information about the truth or falsity of its immediate components. Similar arguments can be given both for the falsity of a conjunction and for the truth of an implication. The informational matrices of Figure 5.1, rejecting the principle of truth-functionality, represent an example of a non-deterministic semantics, whose general theory has been extensively studied by Avron and Zamansky (2001). This feature can be easily seen by detecting that in some cells of the matrices there are two admitted values.

Third, the informational meaning of the logical operators is weaker than the classical one because the informational matrices are an extension of the classical truth tables.

Let $\mathcal{LP}$ be the set of Boolean formulae. D'Agostino[3] defines a 3ND-valuation as a function $v : \mathcal{L} \longrightarrow \{1, 0, *\}$, which satisfies the following conditions for every $B, C \in \mathcal{LP}$:

i. $v(\neg B) = \hat{f}_\neg(v(B))$

ii. $v(B \circ C) \in \hat{f}_\circ(v(B), v(C))$

where $\circ$ stands for $\wedge, \vee$ or $\rightarrow$; $\hat{f}_\neg$ is the deterministic function defined by the informational matrix for $\neg$ and $\hat{f}_\circ$ is the non-deterministic function[4] defined by

[3] D'Agostino (2014a, p. 39).

[4] By saying that $\hat{f}_\circ$ is a non-deterministic function, we mean that $\hat{f}_\circ : V \times V \longrightarrow 2^V \setminus \emptyset$.

$\neg A$	A
1	1
0	0

$A \vee B$	A	B
1	0	0
0	1	1
0	1	0
0	1	$*$
0	0	1
0	$*$	1

$A \wedge B$	A	B
1	0	1
1	0	0
1	0	$*$
1	1	0
1	$*$	0
0	1	1

$A \rightarrow B$	A	B
1	1	0
0	1	1
0	0	1
0	$*$	1
0	0	0
0	0	$*$

Figure 5.2: Negative constraints on admissible partial evaluations. Each line represents a forbidden configuration of values. D'Agostino (2010, p. 257).

the informational matrix for $\circ$ in Figure 5.1. A 3ND-valuation is called a 0-depth informational state or, equivalently, shallow informational state. A sentence C is a 0-depth consequence of a set of formulae Γ, written $\Gamma \vDash_0 C$, if and only if $v(C) = 1$ for every shallow state v such that $v(B) = 1$ for every $B \in \Gamma$. The 0-depth consequence relation is Tarskian. The logic $\vDash_0$ is the basic element of the hierarchy of Depth Bounded Boolean Logics. $\vDash_0$ does not allow to use virtual information and thus includes all and only those classical inferences that can be derived solely in virtue of the informational meaning of the logical operators. In other words, the inferences valid in $\vDash_0$ are *analytic* according to the informational meaning of the operators. Crucially, D'Agostino e Floridi (2009) prove that this logic is tractable, that is, decidable in practice: this means that there exists a feasible procedure to establish whether C is an analytical consequence or, equivalently, a 0-depth consequence of a set of sentences Γ.

The informational semantics, which we have presented in this Section through the informational three-valued matrices proposed in D'Agostino (2013a), D'Agostino (2014b) and D'Agostino (2015), is expressed in a different manner in D'Agostino and Floridi (2009) and D'Agostino (2010). We briefly point out the main features of this alternative formulation, which is essentially based on the so-called single candidate principle.

A partial evaluation $v : \mathcal{LP} \longrightarrow \{0, 1\}$ is said to be admissible if and only if it does not violate the negative constraints shown in Figure 5.2, each line of which represents a forbidden configuration of values in agreement with the positive constraints of the informational three-valued matrices of Figure 5.1. Let $\mathcal{A}$ be the set of all the admissible partial evaluation of $\mathcal{LP}$ and let $\mathcal{LP}^*$ be the evaluated language based on $\mathcal{LP}$, i.e. the set of all ordered pairs $\langle B, i\rangle$ such that $B \in \mathcal{LP}$ and $i \in \{0, 1\}$. Then, $\Vdash_0$ is defined as a relation on $\mathcal{A} \times \mathcal{LP}^*$ that satisfies the following condition, that is called the single candidate principle: $v \Vdash_0 \langle B, i\rangle$ if and only if $v \cup \{\langle B, |i - 1|\rangle\} \notin \mathcal{A}$. A 0-depth informational state is defined as an admissible partial evaluation that is closed under $\Vdash_0$ and the 0-depth consequence relation is defined as truth preserving over 0-depth informational states. As D'Agostino (2010) points out, the single candidate is a structural principle that prescribes to infer that B is true (false) if the other option is immediately ruled out by some of the accepted constraints that define the meaning of the logical operators.

The semantics resulting from these definitions is essentially the same as the informational semantics based on the informational three-valued matrices. The only difference concerns the idempotent laws: B turns out to be a 0-depth consequence of $B \wedge B$ ($B \vee B$) according to the semantics based on the single candidate principle, but not according to the formulation through the matrices. This gap between the two formulations is slight, due to the dubious semantic sense made by formulae like $B \wedge B$ and $B \vee B$, and can be easily closed following two strategies. First, one might add a side condition to the formulation through the matrices, which says that if A and B are indeterminate, then $A \wedge B$ ($A \vee B$) is false (resp. true) if and only if $A \neq B$. Second, one might define logical language so as to exclude formulae such as $B \wedge B$ and $B \vee B$ for any formula B. This can be done by saying that the well-formed formulae of the language comprise all, and only, the strings of symbols that can be generated recursively from the propositional parameters by the following rule:

$$\text{If } A \text{ and } B \text{ are formulae, then} \begin{cases} \neg A \text{ is a formula} \\ A \wedge B \text{ is a formula provided that } A \neq B \\ A \vee B \text{ is a formula provided that } A \neq B \\ A \to B \text{ is a formula.} \end{cases}$$

5.1.3 Virtual information

Since $\vDash_0$ is weaker than Boolean logic, there are some classically valid propositional inferences that are not valid in $\vDash_0$. For instance, while $p \vee q, \neg p \vee q \vDash_C q$, the inference from $p \vee q, \neg p \vee q$ to q is not valid in $\vDash_0$: a counterexample can be given by any 3ND-valuation such that $v(p \vee q) = v(\neg p \vee q) = 1$ and $v(p) = v(q) = *$. Although it does not follow immediately from the informational meaning of the logical operators that occur in the premises, the truth of q seems to be implicitly contained in the truth of $p \vee q$ and $\neg p \vee q$. To obtain the information that q is true, an agent is compelled to go temporarily beyond the information that she has about the truth values of the premises and to reason by cases in the following way:

i. Proposition p is either objectively true or objectively false, although this piece of information is not available.

ii. Assume that p is true. Then, $\neg p$ is false. Since $\neg p \vee q$ is true and $\neg p$ is false, q is true.

iii. Assume that p is false. Since $p \vee q$ is true and p is false, q is true.

iv. In both cases q is true, independently of the objective truth value of p: the information that q is true is implicitly contained in the premises.

The sense in which the truth of q is 'implicitly contained' in the truth of $p \vee q$ and $\neg p \vee q$ is different from the sense in which, for instance, the truth of s is 'implicitly contained' in the truth of $r \vee s$ and $\neg r$. In the latter case, s is an analytical consequence of the premises, since it is derivable only in virtue of the informational meaning of the operators $\neg$ and $\vee$ that occur in the premises. On the contrary, in the former case, the truth of q is obtained introducing virtual information (in steps ii. and iii.), viz. through the temporal assumption of the truth of p and the falsity of p. As D'Agostino (2010) says:

> These steps [ii. - iii.] cannot be internally justified on the basis of the agent's actual information state, but involve simulating the possession of definite information about the objective truth-value of p, by enumerating the two possible outcomes of the process of acquiring such information, *neither of which* is deterministically dictated by v. The inference displays, intuitively, a deeper reasoning process than the one displayed by disjunctive syllogism, and we relate this depth to the necessity of manipulating virtual information concerning p (D'Agostino, 2010, p. 259).

Inferences that employ pieces of virtual information are said to be synthetic because they increase the initial information requiring the use of some intuitions, pieces of virtual information, which go beyond the informational meaning of the logical operators and which are essential to obtain the conclusion.

The notion of synthetic inference admits a gradual characterization, which formally translates the intuitive idea of the degree of logical difficulty of an inference or depth of deductive process. In order to deduce the truth of q from the truth of $p \vee q$ and $\neg p \vee q$, one piece of virtual information, which concerns the objective value of p, is needed. This inference turns out to be valid because $p, p \vee q, \neg p \vee q \vDash_0 q$ and $\neg p, p \vee q, \neg p \vee q \vDash_0 q$: the depth of this inference, that is its degree of syntheticity, is one because one is the number of pieces of virtual information necessary to obtain the conclusion from its premises. If both steps ii. and iii. employed in turn one piece of virtual information to obtain a shared conclusion, then that process of inference would have depth two. Iterating this reasoning, the authors obtain a classification of classical inferences according to their logical depth or their degree of syntheticity.

From a formal point of view, D'Agostino and Floridi (2009) recursively introduce the notion of k-depth consequence relation, written $\vDash_{\mathsf{k}}$, for every $\mathsf{k} \in \mathbb{N}$. In the logic $\vDash_{\mathsf{k}}$, all the analytic inferences and the inferences of depth at most k are valid. For every $\mathsf{k} \in \mathbb{N}$, a sentence C is k-depth consequence of a set of sentences Γ, written $\Gamma \vDash_{\mathsf{k}} C$, if and only if there exists a propositional letter p such that $\Gamma \cup \{p\} \vDash_{\mathsf{k}-1} C$ and $\Gamma \cup \{\neg p\} \vDash_{\mathsf{k}-1} C$. Again, each k-depth consequence relation is Tarskian.

Since $\vDash_0$ is monotonic, $\vDash_{\mathsf{j}} \subseteq \vDash_{\mathsf{k}}$ for every $\mathsf{j} \leq \mathsf{k}$. The transition from $\vDash_{\mathsf{k}}$ to $\vDash_{\mathsf{k}+1}$ corresponds to the increasing of the depth at which the employment of virtual information is allowed. The authors prove the fundamental result that the deducibility of depth k, for every fixed k, is a tractable problem, that is to say, a problem that is decidable in practice, although its complexity grows with the increasing of k.

Last, classical propositional logic is defined as the limit of the infinite sequence of logics of depth k, each of which is tractable and weaker than the following one in the hierarchy:

$$\vDash_C = \bigcup_{\mathsf{k} \in \mathbb{N}} \vDash_{\mathsf{k}}.$$

Therefore, classical propositional logic allows an unbounded use of virtual information.

As D'Agostino (2015) underlines, the notion of k-depth consequence depends not only on the depth at which the use of virtual information is allowed, but also on the definition of the virtual space, that is the subset of formulae on which the introduction of virtual information is allowed. It is essential that the dimension of the virtual space should be given some constraints: otherwise, every classical validity turns out to be derivable at depth one. The definition above, which requires p to be a propositional letter, is not the only possibility: for example, in D'Agostino (2014a), we find the virtual space of an inference defined as the set of subformulae of the premises and the conclusion. The former proposal is more restrictive than the latter and turns out not to be structural: while it validates $\vDash_1 \; p \vee \neg p$, the minimum depth at which $\vDash_1 \; \sigma(p \vee \neg p)$ holds depends on the substitution σ. The result presented in D'Agostino, Finger and Gabbay (2013) and in D'Agostino (2015) generalizes the one mentioned above, which is taken from D'Agostino and Floridi (2009), establishing that every k-depth consequence is tractable provided that the size of the virtual space, defined as a function of the set consisting of the premises and of the conclusion of a given inference, is polynomially bounded.

5.1.4 Tableaux rules for Depth Bounded Boolean Logics

We now move from the discussion of the semantics to the presentation of the main features of the proof-theoretical characterization for Depth Bounded Boolean Logics.

First of all, we have to pinpoint that the main reason why proof systems for propositional classical logic cannot be adequate for Depth Bounded Boolean Logics is that they allow an unbounded use of virtual information, through the so-called 'discharge rules'. The latter admit temporary assumptions of some propositions that are discharged before the derivation has come to an end. Examples of such rules in natural deduction system are the elimination rule for the disjunction $[\vee E]$ and the introduction rule for the conditional $[\rightarrow I]$:

$$\dfrac{\begin{matrix}\Gamma\\ \Pi_0\\ A\vee B\end{matrix}\qquad \begin{matrix}\Delta,[A]^x\\ \Pi_1\\ C\end{matrix}\qquad \begin{matrix}\Lambda,[B]^y\\ \Pi_2\\ C\end{matrix}}{C}\,{\scriptstyle \vee E(x,y)} \qquad\qquad \dfrac{\begin{matrix}\Gamma,[A]^w\\ \Pi_0\\ B\end{matrix}}{A\rightarrow B}\,{\scriptstyle \rightarrow I(w)}$$

where the sentences in square brackets ($[A]$ and $[B]$) are information that are not contained in the premises of the derivations (X, Δ e Λ).

As a result of this observation, no discharge rule is admitted in the proof-theoretical characterization of the 0-depth consequence relation $\vDash_0$ given by D'Agostino and Floridi (2009) by means of a set of introduction and elimination rules for the connectives. These rules are called *intelim* rules and are shown in Figures 3 and 4. They are formulated in terms of signed formulae, that are expressions of the kind TA and FA and that are intuitively interpreted as meaning that formula A is, respectively, true and false[5]. Let φ^u be the unsigned translation of the signed formula φ, namely, A if and only if $\varphi = TA$; $\neg A$ if and only if $\varphi = FA$. Therefore, the translation of the *intelim* rule in terms of unsigned formulae is immediate. D'Agostino (2014a) defines the notion of 0-depth derivability, written $\vdash_0$, in the following way:

- An *intelim* sequence for a set of signed formulae X is a sequence of signed formulae $\psi_1, \ldots, \psi_n$ such that, for every $i = 0, \ldots, n$, either $\psi_i \in X$ or is the conclusion of the application of an *intelim* rule to preceding formulae.
- An *intelim* proof of φ from X is an *intelim* sequence for X such that φ is the last formula in the sequence.
- A signed formula φ is *intelim* derivable from X if and only if there is an *intelim* proof of φ from X.
- φ is 0-depth derivable from X, written $X \vdash_0 \varphi$, if and only if φ is *intelim* deducible from X.

[5]Hereinafter, following the convention adopted by D'Agostino (2010), we shall use capital Greek letters, $\Gamma, \Delta, \Lambda, \ldots$, as variables for sets of unsigned formulae; first capital letters of the latin alphabet, $A, B, C, \ldots$, for arbitrary unsigned formulae; lower case letters of the latin alphabet, $p, q, r, \ldots$, for atomic propositions; last capital letters of the latin alphabet, $W, X, Y, \ldots$, for sets of signed formulae and lower case Greek letters, $\varphi, \psi, \chi, \ldots$, for arbitrary signed formulae.

$$\frac{FA}{TA \to B}[T \to I_1] \qquad \frac{TB}{TA \to B}[T \to I_2] \qquad \frac{\begin{array}{c}TA\\FB\end{array}}{FA \to B}[F \to I]$$

$$\frac{TA}{TA \vee B}[T \vee I_1] \qquad \frac{TB}{TA \vee B}[T \vee I_2] \qquad \frac{\begin{array}{c}FA\\FB\end{array}}{FA \vee B}[F \vee I]$$

$$\frac{\begin{array}{c}TA\\TB\end{array}}{TA \wedge B}[T \wedge I] \qquad \frac{FA}{FA \wedge B}[F \wedge I_1] \qquad \frac{FB}{FA \wedge B}[F \wedge I_2]$$

$$\frac{TA}{F\neg A}[F\neg I] \qquad \frac{FA}{T\neg A}[T\neg I]$$

Figure 5.3: Introduction rules for the connectives (D'Agostino, 2010, p. 264).

Consider, by way of example, the 0-depth derivation of the *ex falso quodlibet* principle ($Tp \wedge \neg p \vdash_0 Tq$):

1.	$Tp \wedge \neg p$	[Ass.]
2.	Tp	$[T \wedge E\ 1]$
3.	$T\neg p$	$[T \wedge E\ 1]$
4.	Fp	$[T\neg E\ 3]$
5.	$Tp \vee q$	$[T \vee I\ 2]$
6.	Tq	$[T \vee E\ 4{,}5]$

Three are the fundamental features of the derivability relation $\vdash_0$ demonstrated by the authors. First, the relations $\vDash_0$ and $\vdash_0$ are coextensional, that is to say, for every set of signed formulae X and every signed formula φ, $X^u \vDash_0 \varphi^u$ if and only if $X \vdash_0 \varphi$. This amounts to say that the *intelim* rules reflect the informational meaning of the logical operators, in an analogous way in which, for example, Gentzen's NK system reflect the classical meaning of the connectives. Second, the notion of 0-depth derivability allows for a particularly strong normalization procedure: every 0-depth derivation that do not use explicitly contradictory premises do satisfy the subformula property. Third, the 0-depth derivability problem is tractable. Telling whether a formula φ is derivable or not at depth 0 from a set of premises X is a problem that can be decided in time $\mathcal{O}(n^2)$.

$$\frac{\begin{array}{c} TA \to B \\ TA \end{array}}{TB}[T \to E_1] \qquad \frac{\begin{array}{c} TA \to B \\ FB \end{array}}{FA}[T \to E_2]$$

$$\frac{FA \to B}{TA}[F \to E_1] \qquad \frac{FA \to B}{FB}[F \to E_2]$$

$$\frac{\begin{array}{c} TA \vee B \\ FA \end{array}}{TB}[T \vee E_1] \qquad \frac{\begin{array}{c} TA \vee B \\ FB \end{array}}{TA}[T \vee E_2] \qquad \frac{TA \vee A}{TA}[T \vee E_3]$$

$$\frac{FA \vee B}{FA}[F \vee E_1] \qquad \frac{FA \vee B}{FB}[F \vee E_2]$$

$$\frac{TA \wedge B}{TA}[T \wedge E_1] \qquad \frac{TA \wedge B}{TB}[T \wedge E_2]$$

$$\frac{\begin{array}{c} FA \wedge B \\ TA \end{array}}{FB}[F \wedge E_1] \qquad \frac{\begin{array}{c} FA \wedge B \\ TB \end{array}}{FA}[F \wedge E_2] \qquad \frac{FA \wedge A}{FA}[F \wedge E_3]$$

$$\frac{F\neg A}{TA}[F\neg E] \qquad \frac{T\neg A}{FA}[T\neg E]$$

Figure 5.4: Elimination rules for the connectives (D'Agostino, 2010, p. 264).

In the preceding Sections, we have seen that the notion of k-depth consequence relation is introduced in a recursive way. Something similar happens also for the general proof-theoretic presentation of the logics $\vdash_{\mathsf{k}}$ with $\mathsf{k} > 0$, whose definition is given by D'Agostino (2014) in the following terms:

> φ is k-depth derivable from X, written $X \vdash_{\mathsf{k}} \varphi$, if and only if $X \cup \{TA\} \vdash_{\mathsf{k}-1} \varphi$ and $X \cup \{FA\} \vdash_{\mathsf{k}-1} \varphi$ for some formula A that is a subformula of the unsigned parts of the signed formulae in $X \cup \{\varphi\}$[6].

Derivations of depth $\mathsf{k} > 0$ admit of the use of k nested applications of the following rule, called PB after the principle of bivalence:

$$
\begin{array}{c}
\vdots \\
\begin{array}{|ll|ll|}
\hline
TA & [\text{PB}] & FA & [\text{PB}] \\
\vdots & & \vdots & \\
\psi & & \psi & \\
\hline
\end{array} \\
\psi \\
\vdots
\end{array}
$$

The general structure of the derivation of φ from X at depth $k \geq 0$ can be conveniently represented as illustrated in Figure 5.5; while an example of a derivation of depth 1 is shown below, where q is derived from $\neg p \vee q$ and $p \vee q$:

$$
\begin{array}{|l|}
\hline
1.\ T\neg p \vee q \quad [\text{Ass.}] \\
2.\ Tp \vee q \quad [\text{Ass.}] \\
\begin{array}{|ll|ll|}
\hline
3.\ Tp & [\text{PB}] & Fp & [\text{PB}] \\
4.\ F\neg p & [T\neg I\ 3] & Tq & [T\vee E\ 2{,}3] \\
5.\ Tq & [T\vee E\ 2{,}4] & & \\
\hline
\end{array} \\
6.\ Tq \\
\hline
\end{array}
$$

As for the derivability relation $\vdash_0$, D'Agostino proves, for every $\mathsf{k} \in \mathbb{N}$, that, first, the relations $\vDash_{\mathsf{k}}$ and $\vdash_{\mathsf{k}}$ are coextensive, viz. for every X and every φ, $X^u \vDash_{\mathsf{k}} \varphi^u$ if and only if $X \vdash_{\mathsf{k}} \varphi$, and, second, the k-depth derivability problem is tractable, for telling whether a formula φ is derivable or not at depth k from a set of premises X is a problem that can be decided in time $\mathcal{O}(n^{2\mathsf{k}+2})$.

[6]The definition of the formula A depends on the determination of the virtual space, viz. the subset of formulae on which the introduction of virtual information is allowed. A definition alternative to the one proposed above and put forward in D'Agostino and Floridi (2009, p. 381) is to require A to be a propositional formula. On this point see the conclusive remarks of Section 5.1.3.

$$X$$

$$\vdots$$

$$\boxed{\begin{array}{ll} TA_1 & \quad [\mathrm{PB}] \\ \vdots & \\ \psi_{m+1} & \end{array}} \boxed{\begin{array}{ll} FA_1 & \quad [\mathrm{PB}] \\ \vdots & \\ \psi_{m+1} & \end{array}}$$

$$\psi_{m+1}$$

$$\vdots$$

$$\psi_{m+i}$$

$$\boxed{\begin{array}{ll} TA_{i+1} & \quad [\mathrm{PB}] \\ \vdots & \\ \psi_{m+i+1} & \end{array}} \boxed{\begin{array}{ll} FA_{i+1} & \quad [\mathrm{PB}] \\ \vdots & \\ \psi_{m+i+1} & \end{array}}$$

$$\psi_{m+i+1}$$

$$\vdots$$

$$\varphi$$

Figure 5.5: General structure of the derivation of φ from X at depth $k \geq 0$. Each box leading to φ_{m+j} (with $j = 1, \ldots, n$) contains a derivation of depth $k - 1$.

5.1.5 Observations on the relation with Hintikka's work and Kant's conceptions

At several points of his work, D'Agostino suggests that his proposal is, on the one hand, "orthogonal" to Hintikka's view on the informativity of quantification logic[7] and, on the other hand, "a partial vindication of the Kantian notion of 'synthetic a priori' even in the allegedly trivial domain of propositional logic"[8]. We now examine these two statements in order to place D'Agostino's approach inside the historical and philosophical reconstruction of the principle of analyticity of logic that we have offered in the previous part of this thesis.

As far as the first claim is concerned, several observations confirm the idea that D'Agostino's approach is a continuation in the field of propositional logic of the work Hintikka carried out in the context of the polyadic calculus. First, D'Agostino draws a *methodological* distinction between analytic and synthetic inferences and, in so doing, accepts the hypothesis P2 that Hintikka formulated in reading the Kantian materials, according to which synthetic judgments (or inferences) are those that can be proved by synthetic methods[9]. As a result, both Hintikka and D'Agostino speak in the first place of analytic and synthetic inferences (or inference steps) and then, thanks to the deduction theorem, of analytic and synthetic validities. Second, D'Agostino follows the Finnish philosopher in employing results belonging to the computability theory to attack the neo-positivistic dogma of the analyticity of logic and in linking the notions of syntheticity and informativity to the cognitive and computational resources that are needed in carrying out logical inferences.

Third, both the approaches impose definite limits on the complexity of the resources that are available: on the one side, Hintikka restricts the complexity of the configurations of individuals or, equivalently, the number of the individuals mutually related, that can be employed in giving the linguistic counterpart of possible worlds; on the other side, D'Agostino limits the complexity of the nested pattern of virtual information that can be used in obtaining a conclusion from a given set of premises. The term 'depth', that in Hintikka's work indicated the maximum of the lengths of nested sequences of quantifiers of a certain sentence[10], is used in D'Agostino's account to refer to the lowest number of nested pieces of virtual information needed to obtain the conclusion from the premises of a certain inference.

Fourth, the two notions of 'synthetic argument steps' have a common structure: they stand for a preparatory phase that is followed by an analytical proof. In the

[7] D'Agostino (2016).
[8] D'Agostino (2013b).
[9] On this point see Section 4.1.2.
[10] See Section 4.2.1.

Finnish philosopher's definition, it is first necessary to introduce an appropriate new individual into the reasoning and then it is possible to expand the constituents to a higher depth and to check the relations between the expanded constituents of the premises and of the conclusion. In D'Agostino's construction, it is first necessary to introduce an appropriate piece of virtual information and then it is possible to employ it in the derivation. Fifth, D'Agostino follows Hintikka's insight that syntheticity and informativity are matters of degree linked to the amount of computational and cognitive effort.

The relations that D'Agostino's approach maintains with Kant's conceptions are more complex than those it has with Hintikka's work and there are several elements that need to be discussed in this regard. As far as the notion of syntheticity and the role of virtual information are concerned, the author makes the following suggestions:

> This use of virtual information, which is not contained in the data and so may not be actually held by any agent who holds the information carried by the data, appears to qualify this kind of argument as 'synthetic' in a sense close to Kant's sense, in that it forces the agent to consider potential information that is not included in the information 'given' to him. [...] synthetic ones [i.e., synthetic inferences] are those that are 'augmentative', involving some intuition that goes beyond this meaning, i.e., involving the consideration of virtual information (D'Agostino, 2013a, pp. 55-56).

Virtual information in D'Agostino's theory plays the role that intuitions have in Kant's conception: both of them go beyond the concept of the subject (or beyond what is contained in the premises), cannot be found through analysis and are essential elements to prove the truth of a synthetic judgment (or the validity of a synthetic inference step). While the *role* of virtual information is surely Kantian, their *nature* cannot be compared to the intuitions presented in the *Critique*. Pieces of virtual information are simply propositional formulae and this cannot be reconciled neither with Parsons' phenomenological interpretation, nor with Hintikka's logical reading[11] of Kantian intuitions. Second, unlike Hintikka, Depth Bounded Boolean Logics do not defend Kant's debated principle that mathematics is synthetic *a priori*. Third, although D'Agostino contributes in the struggle against the logical positivists and in favor of the synthetic *a priori*, we must remember that Kant does not apply the distinction to classical propositional logic at all, because Kant's notion of analyticity as conceptual truth does not apply to sentential logic (where the relation studied is not between concepts, but between judgments)[12].

[11]See Section 4.1.1.
[12]See Chapter 1.

To sum up, while it is clear that D'Agostino's approach is the development of Hintikka's results in the field of propositional logic, its relations with the Kantian materials is more troubled. As we have shown, Depth Bounded Boolean Logics are based on a fruitful and constant dialogue with Kant's principles and motivations and, nevertheless, some of their details turn out to be profoundly un-Kantian: they can be regarded as a strong vindication of Kant's synthetic *a priori* even beyond and sometimes against Kant.

5.2 The idea of extending Hintikka's approach

5.2.1 The project of Depth Bounded First-Order Logics

On the one hand, we have seen[13] that one of the main drawbacks of Hintikka's work is that it seemed to be only a partial vindication of the idea that logical deduction is informative, for it classified as analytic the entire set of propositional validities. On the other hand, we have analyzed[14] the numerous reasons why D'Agostino's approach concerning propositional logic can be said to be orthogonal to Hintikka's standpoint on quantificational logic. At this point, our task is clear enough. In order to provide logical systems that represent a complete vindication of the thesis that logic is synthetic, we need to unify Hintikka and D'Agostino's approach, viz. to extend Hintikka's view on quantificational logic to the propositional case or to extend D'Agostino's perspective on propositional logic to the first-order case. The logics we want to provide shall obviously be called 'Depth Bounded First-Order Logics'. But how should these logics look like?

Our idea is to construct Depth Bounded First-Order Logics as an infinite hierarchy of logics. In this thesis, we are going to identify each logic with its corresponding derivability relation: we are going to propose a proof-theoretical account of this family of logical systems leaving aside issues connected with the semantics. Each derivability relation (viz. each logic) is identified by two parameters, k and q, that measure two different kinds of syntheticity. As a result, each derivability relation will be indicated in the following way: $\vdash_{\mathsf{k},\mathsf{q}}$, for some $\mathsf{k}, \mathsf{q} \in \mathbb{N}$. What do these parameters stand for?

- Parameter k measures the propositional depth of a derivation, namely, the number of nested pieces of virtual information that can be introduced in a derivation. It represents the maximum propositional depth of the derivations that are valid in logics $\vdash_{\mathsf{k},\mathsf{q}}$ for any q. As it is clear, this parameter is inherited from Depth Bounded Boolean Logics. As such, it will be defined

[13]See Section 4.3.3.
[14]See Section 5.1.5.

$\dots$	$\vdash_{0,\dots}$	$\vdash_{1,\dots}$	$\vdash_{2,\dots}$	$\vdash_{3,\dots}$	$\vdash_{\dots,\dots}$
$\mathsf{q} = 3$	$\vdash_{0,3}$	$\vdash_{1,3}$	$\vdash_{2,3}$	$\vdash_{3,3}$	$\vdash_{\dots,3}$
$\mathsf{q} = 2$	$\vdash_{0,2}$	$\vdash_{1,2}$	$\vdash_{2,2}$	$\vdash_{3,2}$	$\vdash_{\dots,2}$
$\mathsf{q} = 1$	$\vdash_{0,1}$	$\vdash_{1,1}$	$\vdash_{2,1}$	$\vdash_{3,1}$	$\vdash_{\dots,1}$
$\mathsf{q} = 0$	$\vdash_{0,0}$	$\vdash_{1,0}$	$\vdash_{2,0}$	$\vdash_{3,0}$	$\vdash_{\dots,0}$
	$\mathsf{k} = 0$	$\mathsf{k} = 1$	$\mathsf{k} = 2$	$\mathsf{k} = 3$	$\dots$

Figure 5.6: The hierarchy of Depth Bounded First-Order Logics.

in a recursive way and will correspond to the maximum number of nested applications of the bivalence rule PB (see Section 5.3.2).

- Parameter q measures the quantificational depth of a derivation. The definition of the latter notion will be the result (Section 5.3.3) of several attempts and discussions (Sections 5.2.2, 5.2.3 and 5.2.4). By now, it is sufficient to say that this notion aims to capture Hintikka's idea of the number of new and related individuals that must be introduced in a derivation to obtain the conclusion from the premises.

Although each logic will present the two parameters at the same time, we have chosen to keep these two measures distinct. The reason why we have presented them in the same context is they share the same structure and describe a synthetic pattern of reasoning; the motivation why we have kept them distinct is that we think that these two parameters refer to two different kinds of computational and cognitive efforts with different contents. This should be clear from our analysis of the relations between Hintikka and D'Agostino's approaches in the previous Section.

However, to sum up, the idea is to construct definitions of derivability relations along the following lines. The formula φ can be derived from the set of formulae

X in logic $\vdash_{\mathsf{k},\mathsf{q}}$, written $X \vdash_{\mathsf{k},\mathsf{q}} \varphi$, if and only if we need to use at most k pieces of virtual information and at most q new individuals, that is, individuals that are not necessary to represent the premises and the conclusion, in order to obtain the conclusion from the premises. To put it in another way, we will say that $X \vdash_{\mathsf{k},\mathsf{q}} \varphi$ if and only if the derivation of φ from X has propositional depth at most k and quantificational depth at most q. The problem now is the formal translation of this project and the greatest difficulty concerns the notion of quantificational depth.

The most straightforward way to construct Depth Bounded First-Order Logics may seem, at first glance, to resort to Hintikka's theory of distributive normal forms and, in particular, the refutation procedure put forward in *Logic, Language Games and Information*. An idea in this direction is to say that $\Gamma \vdash_{\mathsf{k},\mathsf{q}} A$ if and only if:

- The distributive normal forms at depth d of the elements in Γ and A, $DNF^d(\Gamma)$[15] and $DNF^d(A)$, must be expanded to reach depth $d + \mathsf{q}$ in order to show that all the members of $DNF^{d+\mathsf{q}}(\Gamma)$ are included in those of $DNF^{d+\mathsf{q}}(A)$, where d is the maximum depth of the formulae in $\Gamma \cup \{A\}$.
- In order to convert Γ and A into their distributive normal forms at depth $d + \mathsf{q}$, it necessary to use at most k nested pieces of virtual information.

This method would probably be the most faithful to Hintikka's work. Nevertheless, as we have underlined in Section 4.3.3, this disproof method is too complicated to be used in practice and we believe that this is a sufficient reason to search for an alternative way.

A natural option is to look at the proof-theoretical approach proposed by D'Agostino and to implement the set of *intelim* rules with those for the quantifiers. For what has been said in Chapter 4, the definition of the elimination rule of the existential quantifier is the most delicate one: it is (primarily but not exclusively) through this rule that new individuals can be introduced during derivations. Two are the major difficulties in giving this definition. First, finding the right 'dosage', namely, allowing the introduction of actually necessary individuals and avoiding that of superfluous ones; second, finding a mechanism such that new individuals introduced in this manner can be 'counted'. In the following, we propose two methods to solve this problem (Sections 5.2.2 and 5.2.3). Both of them satisfy the desiderata just mentioned, but cannot be accepted for other reasons. However, the analysis of these proposals will draw us near the final solution and comprehension of quantificational depth (Section 5.2.4).

[15]Let this notation be a shorthand for $DNF^d(B_1), DNF^d(B_2), \ldots, DNF^d(B_n)$, where $B_1, B_2, \ldots, B_n \in \Gamma$.

5.2.2 First attempt: Propositionalization

The first method that could be employed to define Depth Bounded First-Order Logics is to reduce the quantificational case to the propositional one. Consider Depth Bounded First-Order Logics with quantificational depth $\mathsf{q} = 0$, that is to say, logics in which no new individuals can be employed. The feature that characterizes these logical systems is that the domain of a derivation is bounded and contains all and only the names of the terms that occur in the premises and the conclusion of that derivation. In these contexts, the universal quantifier can be defined in terms of a conjunction and the existential quantifier can be defined in terms of a disjunction as follows:

$$T\forall x A(x) =_{\text{def}} TA(c_1) \wedge A(c_2) \wedge \cdots \wedge A(c_n)$$
$$F\forall x A(x) =_{\text{def}} FA(c_1) \vee A(c_2) \vee \cdots \vee A(c_n)$$

$$T\exists x A(x) =_{\text{def}} TA(c_1) \vee A(c_2) \vee \cdots \vee A(c_n)$$
$$F\exists x A(x) =_{\text{def}} FA(c_1) \wedge A(c_2) \wedge \cdots \wedge A(c_n)$$

where $\{c_1, \ldots, c_n\}$ is the set of all and only the terms that denote the individuals in the finite domain. As a result of the abbreviations above, the rules for the introduction and elimination of the quantifiers are nothing but the rules for the introduction and elimination of the conjunction and the disjunction.

However, three are the main drawbacks of the propositionalization approach. First, the method proposed, being nothing more than a translation from the quantificational to the propositional case, is not particularly interesting. The role that the quantifiers play in this context turns out to be limited: they are necessary only if the domain of discourse is unknown before going through the derivation. Second, following this approach, we are compelled to take into account in our reasoning all the individuals of the domain even in situations in which it is not necessary. For example, the derivation in the propositionalization approach of the formula $T\forall x Ax$ from the premise $T\forall x(Ax \wedge Bx)$ would look like as follows:

$$\begin{gathered}
T\forall x(Ax \wedge Bx)\\
T(Ac_1 \wedge Bc_1) \wedge (Ac_2 \wedge Bc_2) \wedge \cdots \wedge (Ac_n \wedge Bc_n)\\
T(Ac_1 \wedge Bc_1)\\
TAc_1\\
T(Ac_2 \wedge Bc_2)\\
TAc_2\\
\vdots\\
T(Ac_n \wedge Bc_n)\\
TAc_n\\
TAc_1 \wedge Ac_2 \wedge \cdots \wedge Ac_n\\
T\forall x Ax
\end{gathered}$$

Not only does this derivation increase with the increasing of n. But it is also counterintuitive, for the natural kind of reasoning in this case is to take into account a generic individual, to show that if it has both the properties A and B, then it has also the property A and to deduce that the conclusion is valid for any individual.

Third, in the propositionalization approach, we have defined the existential quantifier in terms of a disjunction; as a result, the elimination of the existential quantifier $\exists x Ax$ amounts to simplify the expression $TA(c_1) \vee A(c_2) \vee \cdots \vee A(c_n)$. Now, it might be the case that the formula $TA(c_1)$ does not occur above in the derivation. In this situation, we are compelled to use an instance of the PB rule and to go through the following reasoning by cases: either A is satisfied by c_1 ($TA(c_1)$) or it is not ($FA(c_1)$). The same obviously might happen also for c_2, c_3 and so on. This observation suggests that the use of the propositionalization method might increase the propositional depth of an inference even in cases in which this is not necessary, for the natural kind of reasoning for eliminating the existential quantifier, especially when the domain is unknown, does not involve the propositional depth of an inference, but amounts to consider just one and fresh individual and not every term one by one. These motivations are strong enough to reject the propositionalization method and to look for something else.

5.2.3 Second attempt: Skolem functions

A second promising approach consists in providing the individuals with a structure capable of showing their genealogy through the elimination rule for the existential quantifier. This plan could be efficiently fulfilled using Skolem functions[16] in the following manner:

$$\frac{F\forall x A(x)}{FA(f(t_1,\dots,t_n))}[F\forall E] \qquad \frac{T\exists x A(x)}{TA(f(t_1,\dots,t_n))}[T\exists E]$$

where $\{t_1,\dots,t_n\}$ is the set of all open terms (variables) and closed terms (constants and Skolem terms) that occur in the formula A and f is a fresh function, viz. a function not occurring on the branch. The fundamental idea of these rules is quite simple. An unknown individual that has the property A is denoted by a Skolem term that is a function of the terms occurring in A: $f(t_1,\dots,t_n)$.

[16]If $A(x_1,\dots,x_n,y)$ is a predicate formula with individual variables $x_1,\dots,x_n,y$ whose domains are sets $X_1,\dots,X_n,Y$, respectively, then a function $f : X_1 \times \cdots \times X_n \longrightarrow Y$ is called a Skolem function of the formula $\exists y(A(x_1,\dots,x_n,y))$ if and only if, for all $x_1 \in X_1, \dots, x_n \in X_n$, $\exists y(A(x_1,\dots,x_n,y)) \longrightarrow A(x_1,\dots,x_n,f(x_1,\dots,x_n))$. f is called a Skolem function and $A(x_1,\dots,x_n,f(x_1,\dots,x_n))$ is called a Skolem term.

Thanks to the structure of the individuals introduced through the elimination rules expressed by Skolem functions, it is possible to formulate the new notions of 'depth of an individual' and 'weighted degree of a formula' (points 3 and 4), that we enunciate after having recalled Hintikka's definitions of 'depth of a formula' and 'degree of a formula' (points 1 and 2):

1. The notion of 'depth of a formula A', $d(A)$, is recursively defined[17] as follows:
   - $d(A) = 0$ whenever A is atomic;
   - $d(\neg A) = d(A)$;
   - $d(A_1 \wedge A_2) = d(A_1 \vee A_2) = d(A_1 \rightarrow A_2) =$ the greater of the numbers $d(A_1)$ and $d(A_2)$;
   - $d(\forall x A(x)) = d(\exists x A(x)) = d(A) + 1$.
2. The notion of 'degree of a formula A' is given by Hintikka[18] as the sum of the depth of A and the number of terms that occur in A.
3. The notion of 'depth of an individual a', $\delta(a)$, could be recursively defined as follows:
   - $\delta(a) = 0$ if and only if a is a constant or a variable;
   - $\delta(a) = n + 1$ if and only if, for some function f, $a = f(t_1, \dots, t_k)$ and n is the maximum depth of the terms $t_1, \dots, t_k$.
4. The notion of 'weighted depth of a formula A' could be given as the sum of the depth of A and of the depths of all the individuals occurring in A.

The idea of using Skolem functions to structure individuals gives the possibility to contemplate several and interesting ways to define the quantificational depth of a derivation, that is, to count the individuals that are effectively introduced in that derivation. However, upon closer inspection, we encounter some problems that lead to the conclusion that all of these proposals are not completely satisfying.

The most immediate way to define the notion of quantificational depth within the approach based on Skolem functions is focused on the structure of the individuals introduced through the elimination rule for the existential quantifier. This idea can be expressed in several forms:

1. First, we could define the quantificational depth of an inference as the *maximum depth of new individuals* that have been introduced in the derivation. Despite appearances, this solution does not work. The problem is that the

[17] Cf. Hintikka (1973, VI, pp. 141-142).

[18] Hintikka (1973, VI, p. 141).

depth of a certain individual measures the number of applications of the rules $T\exists E$ and $F\forall E$ that are necessary to define that individual. But not every application of the rules $T\exists E$ and $F\forall E$ introduces in the derivation effectively new individuals or, equivalently, individuals that are not already thought of by thinking of the premises.

2. Second, we could define the quantificational depth of an inference as the *maximum of the arities of the functions* that have been introduced in the derivation. However, also this proposal must be rejected. The arity of a Skolem function f is the number of the individuals on which f depends. But this number is nothing else than the number (minus one) of the nested quantifiers (or the depth) of the premise, which originates f through the elimination of the quantifiers, plus (possibly) the number of individual terms occurring in that premise.

3. Third, we could define the quantificational depth of an inference as the *number of individuals with the maximum depth among the individuals that have been introduced* or as the *number of individuals that are represented by Skolem function with the maximum arity among the functions that have been introduced.* For the same reasons expressed above, also these measures are not satisfying.

These proposals share the same fundamental difficulty: the structure of the individuals involved is not sufficient *by itself* to reveal the quantificational depth of a derivation. The number of new individuals that are necessary for the derivation seems to be rather the result of a comparison between, on the one hand, the number of the individuals that are already thought of in the premises and conclusion and, on the other hand, the number of individuals introduced during the derivation. This observation motivates the following set of attempts, which takes advantage of the peculiarities of the approach based on Skolem functions and, at the same time, takes into account those individuals that have already been thought in the premises and conclusion of a certain derivation.

1. First, we could define the quantificational depth of an inference as the *difference between the maximum degree of the intermediate steps and the maximum degree of premises and conclusion.* This measure does not work because, even if a certain derivation is synthetic and requires the construction of configurations of individuals that are more complex than the initial ones, this does not imply that there exists a step in the derivation in which all the new individuals occur at the same time.

2. Second, we could define the quantificational depth of an inference as the *difference between the maximum weighted degree of the intermediate steps*

and the maximum weighted degree of premises and conclusion. But this proposal cannot be accepted for the same reasons as above.

3. Third, we could define the quantificational depth of an inference as the *number of steps with (weighted) degree greater than the (weighted) maximum degree of premises and conclusion.* This latter attempt must be rejected, because there usually are several steps in a derivation that are, at the same time, not particularly significative and marked by a high (weighted) degree.

This second set of proposals is marred by the following problem: it is difficult to find among the steps of a derivation a single step in which all the new individuals occur at the same time. On the contrary, a derivation usually presents steps in which a proper subset of the terms occurs or different steps in which the same terms occur.

5.2.4 Final attempt: What is quantificational depth?

We have seen the way in which two promising approaches did not succeed in giving a definition of quantificational depth. How, then, is it possible to count actually new individuals introduced in a derivation?

Intuitively, we could say that an inference is characterized by quantificational depth $\mathsf{q} > 0$ if, during the derivation, the configuration of individuals representing one or more premises is reiterated so as to produce a new configuration that is more complex than the initial one. Recall the explanation of the example given in Section 4.2.1. Following this suggestion, we could say that the derivation from premises:

$$P_1 : \forall x \forall y(Rxy \rightarrow \exists z(Gxz \wedge Gzy))$$
$$P_2 : \forall x \forall y(Gxy \rightarrow \exists z(Bxz \wedge Bzy))$$
$$P_3 : \forall x \forall y((Bxy \wedge Cx) \rightarrow Cy)$$

to the conclusion:

$$C : \forall x \forall y((Rxy \wedge Cx) \rightarrow Cy)$$

has quantificational depth equal to two, because the configuration of premise P_2 has been reiterated twice and with a common individual to the effect that a new and more complex configuration has been introduced during the derivation (see Figure 1 and Figure 2 of Section 4.2.1).

Nevertheless, this intuition can hardly be translated into a general definition. One attempt in this sense could be to define the quantificational depth of an inference as the number of times in which the premise with the maximum degree has been used. But this proposal does not work, because it assumes two unjustified hypotheses: first, that the premise that has to be reiterated is the one characterized by the maximum degree; second, that the reiterated configurations have individuals in common. Similarly, for the other attempts. To conclude, this intuition cannot be generalized, because the reiterated premises and the way in which they have been reiterated must be checked case by case.

Once also this hypothesis has been discussed and rejected, we enunciate at last the definition of quantificational depth that we think must be preferred over the other proposals by virtue of its faithfulness to Hintikka's reasoning (see also Definition 18, Section 5.3.3):

> The quantificational depth of a derivation is the difference between the number of related individuals used in that derivation and the maximum degree of premises and conclusion. If this number is negative, then the quantificational depth of the corresponding derivation is zero.

So, for example, the quantificational depth of the inference examined in Section 4.2.1 is two, because its derivation needs to consider five related individuals, while the maximum degree of premises and conclusion is three.

Of course, this definition is incomplete until we do not fix the conditions in which individuals can be said to be related. This issue is not trivial. To simplify the matter, we start by discussing when *two* individuals are related. Moreover, we restrict to a logical language with no free variables and no individual constants and, as a result of this assumption, we take individuals to be represented only by individual variables. We hold that there are at least four senses in which two individual variables might be said to be mutually related:

> **Sense 1**: Two individual variables are related if and only if they occur in the same sentence.
>
> **Sense 2**: Two individual variables are related if and only if the quantifiers bounding them are nested.
>
> **Sense 3**: Two individual variables are related if and only if the quantifiers bounding them are nested and connected.
>
> **Sense 4**: Two individual variables are related if and only if they occur as arguments of the same relation.

The four senses are described from the weakest to the strongest. As we show below, this means that if two individual variables are related according to sense n, for $n = 2, 3, 4$, then they are also related according to sense $n - 1$. Moreover, which of these four senses has to occur in our definition of quantificational depth is purely a matter of choice. Each of them is legitimate. For this reason, we are going to choose the criterion that we believe to be the most faithful to Hintikka's original idea.

Sense 1 is the most trivial one. Unlike the other senses specified above, according to the first meaning, individual variables x and y are related even in a formula like $F_1 = \exists x \exists z Pxz \wedge \exists y Qy$. But, we think, sense 1 is too broad, because it seems to relate individuals merely on the 'subjective' ground of being thought or mentioned in the same sentence and not on the basis of the 'objective' relations among them. For example, in the sentence 'there are two individuals that play tennis together and there exists an individual that loves painting', the only relation among the couple of individuals and the third agent is that they have been mentioned in the same sentence.

Sense 2 is close to what Hintikka has proposed by defining the depth of a formula as the the maximum of the lengths of nested sequences of quantifiers (see Section 4.2.1). Two quantifiers are nested if and only if the scope of one of them is included in the scope of the other. On the one hand, according to sense 2, variables x and y are not related in formula F_1, because the existential quantifier bounding y is not included in the scope of the quantifier bounding x. On the other hand, variables x and y are related in the formula $F_2 = \exists x \exists z \exists y(Pxz \wedge Qy)$, because the third quantifier bounds y and is within the scope of the first quantifier bounding x. However, the latter example poses some problems.

Although in formula F_2 the three existential quantifiers are nested, the way in which y is related to x seems to be somehow artificial. In particular, it seems to be too close to the way in which x and y were said to be related according to sense 1 in formula F_1, of which F_2 is a consequence. In other words, the way in which one individual of the couple is related to the third agent in the sentence 'there are three individuals, two of them play tennis together and the third loves painting' is not different, from an intuitive point of view, to the way in which they are related in the sentence 'there are two individuals that play tennis together and there exists an individual that loves painting'. Hintikka himself recognized that sense 2 is too broad and this led him to formulate sense 3, which is based on the notion of connectedness.

Sense 3 is explicitly introduced by Hintikka in the footnote number 33 on page 142 of his *Logic, Language-Games and Information*:

> It is obvious, however, that the individuals that nested quantifiers introduce into our considerations may not be related to each other in any direct or indirect way in the sentence F in question. Hence a sharper definition may be obtained by considering only such bound variables x_1, x_k as invite us to consider individuals that are related to each other in the sentence in the sense that there is a sequence of

> bound variables $x_1, x_2, \ldots, x_{k-1}, x_k$ with the following properties: for each $i = 1, \ldots, k-1$, x_i and x_{i+1} occur in the same atomic subsentence or identity of F; each variable x_i is bound to a quantifier occurring within the scope of the wider of the two quantifiers to which x_1 and x_k are bound. Let us call such variables and the quantifiers to which they are bound *connected.* (It may be assumed for simplicity that different quantifiers have typographically different variables to bound them in F) (Hintikka, 1973, VI, p. 142).

In this passage, Hintikka observes that sense 2 contemplates individuals that are not 'properly' connected and proposes what we have called sense 3 of the notion of 'being related to'. According to the latter, in order for x and y to be related, it is not sufficient that the quantifiers that bound x and y are nested, they must also be connected. The notion of connectedness can be clarified constructing a graph for each formula, in which vertices are individuals and edges are logical relations between individuals. Two individuals are said to be connected in a certain formula if and only if there exists a path made up of edges that connects the vertices that represent those individuals. A similar definition obtains for connectedness in a derivation instead of a formula. As a result, individuals x and y are connected according to sense 3 in formula $F_3 = \exists x \exists z \exists y (Pxz \wedge Qzy)$, but are not connected in formula F_2. For, consider the graphs of the two formulas represented in Figure 5.7. In the graph for F_2, no path can be traced from x to y: individual y is somehow isolated. In the graph for F_3 instead, there is a path from x to y, which is made up by the edges P from x to z and Q from z to y. We choose sense 3 of the expression 'being related to' for our definition of quantificational depth, because we believe that it embodies Hintikka's final point of view on the matter. In Section 5.3.3, we are going to drop the limitations we have imposed above, namely, that the relation is restricted to two individuals and that individuals are represented only by variables, and thus we generalize the definition of 'being related to' (see Definition 13).

Sense 4 is indeed too strict to be chosen. It says that two individual variables are related if and only if they occur as arguments of the same relation. If we use the graphs we have explicated above, this amounts to asking that the vertices are related if and only if there exists an edge linking them. It turns out that, according to sense 4, x and y are related in formula $F_4 = \exists x \exists y \exists z (Pxz \wedge Qxy)$, due to the relation Q in which both x and y occur, but the two individual variables are not related in formula F_3, for no edge directly connects x and y.

To conclude, notice that the rejection of sense 2 as a basis for our definition of quantificational depth amounts to reject once more and in a final way the usefulness of Skolem functions for the true existential quantifier. This is because the structure of a Skolem function indicates precisely which are the individuals that are related

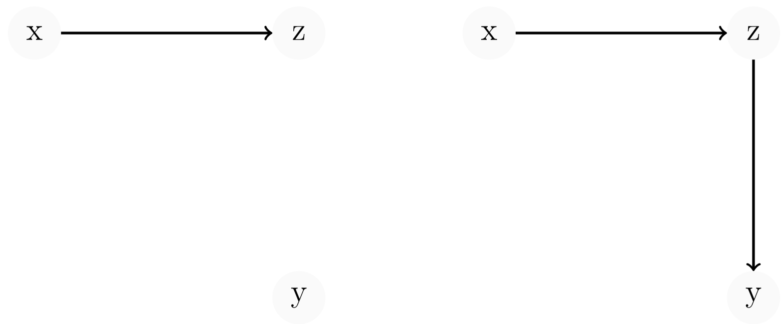


Figure 5.7: Graphs for $F_2 = \exists x \exists z \exists y(Pxz \wedge Qy)$ and $F_3 = \exists x \exists z \exists y(Pxz \wedge Qzy)$ respectively.

according to sense 2 with that function. For example, the function $f(a, b)$ says that the individual f is related according to sense 2 with the individuals a and b. But why is this relation, in general, defined according to sense 2? Because the function $f(a, b)$, that we assume occurs in the formula A, has been introduced by the elimination rule for the existential quantifier from the formula $\exists xA$, in which not only x, but also the individuals a and b occur:

$$\frac{T\exists xA[x, a, b]}{TA[f(a, b), a, b]}[T\exists E]$$

but it is clear that in the formula A the individuals $f(a, b)$, a and b could occur in whatever way, even in an unconnectedness way, because it is necessary neither that $f(a, b)$, a and b occur as arguments of the same logical relation nor that there exists a path of edges connecting them. Consider, as an example, the following application of the elimination rule for the existential quantifier:

$$\frac{T\exists x(Pxa \wedge Qb)}{TP(f(a, b), a) \wedge Qb}[T\exists E]$$

The function $f(a, b)$ suggests that the individual f is related with the individual b, but this is the case only if the notion of relation is understood following sense 2 and not sense 3, because, if we examine the conclusion of this rule, we find that $f(a, b)$ and b never occur as arguments of the same logical relation nor as vertices of a path.

At this point, once we have individuated the best way to measure the quantificational depth of a derivation, we are ready to define at last Depth Bounded First-Order Logics.

Sense of ‘being related to’	Are x and y related in $\exists x \exists z Pxz \wedge \exists y Qy$?	$\exists x \exists z \exists y (Pxz \wedge Qy)$?	$\exists x \exists z \exists y (Pxz \wedge Qzy)$?	$\exists x \exists z \exists y (Pxz \wedge Qxy)$?
Sense 1: Two individual variables are related if and only if they occur in the same sentence	Yes	Yes	Yes	Yes
Sense 2: Two individual variables are related if and only if the quantifiers bounding them are nested	No	Yes	Yes	Yes
Sense 3: Two individual variables are related if and only if the quantifiers bounding them are nested and connected	No	No	Yes	Yes
Sense 4: Two individual variables are related if and only if they occur as arguments of the same relation	No	No	No	Yes

Figure 5.8: Four senses of ‘being related to’.

5.3 Depth Bounded First-Order Logics

As we have anticipated in Section 5.2.1, Depth Bounded First-Order Logics are a hierarchy of logical systems, each of which is characterized by the two parameters k and q, measuring the propositional and the quantificational depth respectively. In this thesis, we are going to characterize this family of logics solely through the proof-theoretical account based on the tableaux rules. The exposition of Depth Bounded First-Order Logics is structured as follows:

1. In Section 5.3.1 we define *derivability relations* $\vdash_{0,\mathsf{q}}$ *with* $\mathsf{q} \geq 0$, that is to say, derivability relations in which no use of nested virtual information is admitted, but it is possible to employ the introduction of q new individuals, for a fixed q greater than or equal to zero. In particular, we will show that a derivation with propositional depth $\mathsf{k} = 0$ and quantificational depth $\mathsf{q} \geq 0$ is a sequence of signed formulae, each of which is either a premise, the conclusion or the result of applying one of the *intelim* rules. These rules are classified as introduction and elimination rules for connectives and quantifiers.

2. In Section 5.3.2 we define *derivability relations* $\vdash_{\mathsf{k},\mathsf{q}}$ *with* $\mathsf{k} > 0$ *and* $\mathsf{q} \geq 0$, that is to say, derivability relations in which it is possible to use both $\mathsf{k} > 0$ nested pieces of virtual information and $\mathsf{q} \geq 0$ new individuals for some k and q fixed. The definition proposed has a recursive character and specifies that, given a fixed quantificational depth q, a derivation that has propositional depth $\mathsf{k}+1$ can be obtained by a derivation of propositional depth k with an application of the bivalence rule PB, that introduces in the derivation one piece of virtual information.

3. In Section 5.3.3 the exposition of Depth Bounded First-Order Logics is completed by giving a definition of the *parameter* q, that is to say, of quantificational depth. Unlike parameter k, q is not defined in a recursive way. As we have established above, q measures the difference between the number of related individuals used in a derivation and the maximum degree of premises and conclusion and indicates the number of actually new individuals that must be introduced in a derivation in order to get the conclusion from the premises.

4. In Section 5.3.4 we propose some examples and derivations in order to clarify the definitions expressed above.

5.3.1 Derivability relations $\vdash_{0,\mathsf{q}}$ with $\mathsf{q} \geq 0$ and *intelim* rules

In this Section, we provide the basic definitions of the derivability relations with propositional depth equal to zero. As a matter of fact, many of the following definitions are taken from D'Agostino's exposition of Depth Bounded Boolean Logics: the choice of using signed formulae, the definition of derivation and the *intelim* rules for the connectives. What is peculiar to the first order family of logics is, of course, the introduction and the elimination rules for the quantifiers and the related notions of general and critical terms. As we have anticipated above, the most delicate definition is that of the elimination rule for the true existential quantifier.

Definition 1 (Signed formulae). Let $\mathcal{L}_s$ be the signed language based on $\mathcal{L}$, that is to say, the set of all the expressions of the kind TA and FA with $A \in \mathcal{L}$. The elements of $\mathcal{L}_s$ are called 'signed formulae': intuitively, TA means that A is true and FA that A is false. Given a signed formula SA, with S equal to T or F, let $\overline{S}A$ be its conjugate, that is, FA if $S = T$ and TA if $S = F$. Following the convention adopted by D'Agostino (2010), we shall use capital Greek letters, $\Gamma, \Delta, \Lambda, \ldots$, as variables for sets of unsigned formulae; first capital letters of the latin alphabet, $A, B, C, \ldots$, for arbitrary unsigned formulae; last capital letters of the latin alphabet, $W, X, Y, \ldots$, for sets of signed formulae and lower case Greek letters, $\varphi, \psi, \chi, \ldots$, for arbitrary signed formulae.

Definition 2 (Derivations $\vdash_{0,\mathsf{q}}$ with $\mathsf{q} \geq 0$). For every $\mathsf{q} \geq 0$, a derivation in the logic $\vdash_{0,\mathsf{q}}$ of the signed formula φ from the set of signed formulae X is a sequence of signed formulae $\psi_1, \ldots, \psi_n$ such that:

- $\psi_n = \varphi$;

and for every intermediate element of the sequence ψ_i such that $1 \leq i < n$, one of the following conditions is satisfied:

- $\psi_i \in X$;
- ψ_i is the conclusion of an application of an *intelim* rule, whose premises precede ψ_i in the sequence.

Definition 3 (*Intelim* rules). *Intelim* rules are the set of introduction and elimination rules for connectives and quantifiers specified in the following paragraphs (Definitions 4-7).

Definition 4 (Introduction rules for the connectives).

$$\frac{FA}{TA \to B}[T \to I_1] \qquad \frac{TB}{TA \to B}[T \to I_2] \qquad \frac{\begin{array}{c}TA\\FB\end{array}}{FA \to B}[F \to I]$$

$$\frac{TA}{TA \lor B}[T \lor I_1] \qquad \frac{TB}{TA \lor B}[T \lor I_2] \qquad \frac{\begin{array}{c}FA\\FB\end{array}}{FA \lor B}[F \lor I]$$

$$\frac{\begin{array}{c}TA\\TB\end{array}}{TA \land B}[T \land I] \qquad \frac{FA}{FA \land B}[F \land I_1] \qquad \frac{FB}{FA \land B}[F \land I_2]$$

$$\frac{TA}{F\neg A}[F\neg I] \qquad \frac{FA}{T\neg A}[T\neg I]$$

At this point, one could think that we have to introduce some restrictions to avoid that these rules are used to introduce in the reasoning new individuals. We could require, for the rules $T \to I_2$, $T \lor I_2$ and $F \land I_2$, that all the individuals occurring in A have previously occurred on the branch; and, for the rules $T \to I_1$, $T \lor I_1$ and $F \land I_1$, that all the individuals occurring in B have previously occurred on the branch. At a closer look, however, these requirements turn out to be too restrictive.

Consider the following example, although its details could be completely understood only after Definition 12. The derivation of the formula $\exists y \exists x(A(x) \to B(y))$ from premise $\exists x B(x)$, with no restriction imposed on the introduction rules for the connectives, would have propositional depth $\mathsf{k} = 0$:

1. $T\exists x B(x)$	[Ass.]
2. $TB(c)$	[$T\exists E$ 1]
3. $TA(d) \to B(c)$	[$T \to_2 I$ 2]
4. $T\exists x(A(x) \to B(c))$	[$T\exists I$ 3]
5. $T\exists y \exists x(A(x) \to B(y))$	[$T\exists I$ 4]

On the contrary, if we choose to impose the restrictions discussed above, the same derivation would have propositional depth $\mathsf{k} = 1$. The third step would be blocked by those restrictions. As a result, from the fact that $B(c)$ is true, we could not derive that $B(c)$ is true even in the case in which $A(d)$ is true. Therefore, in order to obtain even in these conditions the truth of $A(d) \to B(c)$, we should resort, in quite a counter-intuitive way, to the reasoning by cases and to the PB rule. In other words, instead of deriving the truth of the implication from the truth of the consequent, with these restrictions it would be necessary to assume, first, the truth and, then, the falsity of the antecedent, in order to conclude that in both of these cases the implication is true. However, the latter way seems to be a trick, rather than the reproduction of the kind of reasoning at the base of this derivation.

Definition 5 (Elimination rules for the connectives).

$$\frac{\begin{array}{c}TA \to B\\ TA\end{array}}{TB}[T\to E_1] \qquad \frac{\begin{array}{c}TA \to B\\ FB\end{array}}{FA}[T\to E_2]$$

$$\frac{FA \to B}{TA}[F\to E_1] \qquad \frac{FA \to B}{FB}[F\to E_2]$$

$$\frac{\begin{array}{c}TA \vee B\\ FA\end{array}}{TB}[T\vee E_1] \qquad \frac{\begin{array}{c}TA \vee B\\ FB\end{array}}{TA}[T\vee E_2] \qquad \frac{TA \vee A}{TA}[T\vee E_3]$$

$$\frac{FA \vee B}{FA}[F\vee E_1] \qquad \frac{FA \vee B}{FB}[F\vee E_2]$$

$$\frac{TA \wedge B}{TA}[T\wedge E_1] \qquad \frac{TA \wedge B}{TB}[T\wedge E_2]$$

$$\frac{\begin{array}{c}FA \wedge B\\ TA\end{array}}{FB}[F\wedge E_1] \qquad \frac{\begin{array}{c}FA \wedge B\\ TB\end{array}}{FA}[F\wedge E_2] \qquad \frac{FA \wedge A}{FA}[F\wedge E_3]$$

$$\frac{F\neg A}{TA}[F\neg E] \qquad \frac{T\neg A}{FA}[T\neg E]$$

Definition 6 (Introduction rules for the quantifiers).

- c is a *general term* (for a clarification of this notion see Definition 8):

$$\frac{TA(c)}{T\forall x A(x)}[T\forall I] \qquad \frac{FA(c)}{F\exists x A(x)}[F\exists I]$$

- c is whatever term:

$$\frac{FA(c)}{F\forall x A(x)}[F\forall I] \qquad \frac{TA(c)}{T\exists x A(x)}[T\exists I]$$

Definition 7 (Elimination rules for the quantifiers).

- c is whatever term:

$$\frac{T\forall x A(x)}{TA(c)}[T\forall E] \qquad \frac{F\exists x A(x)}{FA(c)}[F\exists E]$$

- c is a term occurring on the branch satisfying simultaneously the following requirements:

  1. c has not been previously introduced through $F\forall E$ or $T\exists E$;
  2. c does not occur in A;
  3. no term occurring in A has been previously introduced through $F\forall A$ or $T\exists E$;

  if such a term does not exists, then c is a fresh term, viz. a term not occurring previously on the branch, called *critical term* (for a clarification of this notion see Definition 8):

$$\frac{F\forall x A(x)}{FA(c)}[F\forall E] \qquad \frac{T\exists x A(x)}{TA(c)}[T\exists E]$$

Discussion on the rules F∀E and T∃E. The formulation of the rules $F\forall E$ and $T\exists E$ just given makes sure that new individuals are introduced in a derivation only if they are really necessary. As Smullyan (1995, p. 55) explains, the idea is as follows. Suppose in the course of an argument we prove the sentence $T\forall xP(x)$ and then we conclude $TP(c)$ through an application of the rule $T\forall E$. In this case, the term c is not the name of a particular individual because $P(c)$ holds for every value of c. So if we subsequently prove that $T\exists xQ(x)$, we are in the position to apply the rule $T\exists E$, to choose again the same term c and to conclude that $TQ(c)$:

$$\boxed{\begin{array}{l} \vdots \\ T\forall xP(x) \\ TP(c) \\ \dots \\ T\exists xQ(x) \\ TQ(c) \\ \vdots \end{array}}$$

The requirements that the term c has to satisfy in order to be a term that, at the same time, occurs on the branch and is introduced by the elimination rules above prevent the following patterns of wrong reasoning:

1. The first requirement prevents to obtain the conclusion that 'someone invented penicillin and landed on the moon' from the premise that 'someone invented penicillin and someone landed on the moon'. Formally, the last step of this derivation must be rejected because the term c has been previously introduced through an application of the rule $T\exists E$ (step 4):

1. $T\exists xP(x) \wedge \exists xQ(x)$	[Ass.]
2. $T\exists xP(x)$	$[T \wedge E\ 1]$
3. $T\exists xQ(x)$	$[T \wedge E\ 1]$
4. $TP(c)$	$[T\exists E\ 2]$
~~5. $TQ(c)$~~	~~$[T\exists E\ 3]$~~

2. The second requirement prevents to obtain the conclusion that 'someone loves himself' from the premise that 'everybody loves somebody'. Formally, the last step of this derivation cannot be accepted because the term c occurs in $R(c, x)$:

1. $T\forall x\exists yR(x, y)$	[Ass.]
2. $T\exists yR(c, y)$	$[T\forall E\ 1]$
~~3. $TR(c, c)$~~	~~$[T\exists E\ 2]$~~

3. The third requirement prevents to obtain the conclusion that 'x has been robbed by c' from the premise that 'c is a thief and x has been robbed by y'. Formally, the last step of this derivation does not work because the term d occurs in $R(d, y)$ and has been previously introduced by an application of the rule $T\exists E$:

1. $TQ(c) \wedge \exists x\exists yR(x, y)$	[Ass.]
2. $TQ(c)$	$[T \wedge E\ 1]$
3. $T\exists x\exists yR(x, y)$	$[T \wedge E\ 1]$
4. $T\exists yR(d, y)$	$[T\exists E\ 3]$
~~5. $TR(d, c)$~~	~~$[T\exists E\ 4]$~~

Definition 8 (General and critical terms).

- A term is called *critical* if and only if it is fresh (that is to say, it does not previously occur on the branch) and has been introduced by the rules $F\forall E$ and $T\exists E$.
- The elimination rules $F\forall E$ and $T\exists E$ are thus called *critical rules*.

- A term is *general* if and only if it is not critical and does not occur in the premises of the derivation. Asking that a general term be not critical prevents the step from the elimination of the existential quantifier to the introduction of the universal one, that is to say, this requirement blocks the last step of the following wrong derivation:

1. $T\exists xA(x)$	[Ass.]
2. $TA(c)$	[$T\forall E$ 1]
~~3. $T\forall xA(x)$~~	~~[$T\exists I$ 2]~~

- We say that *term c is general for A* when we apply rule $T\forall I$ on $TA(c)$ to obtain $T\forall xA(x)$. Similarly, *term c is general for $\neg A$* when we apply rule $F\exists I$ on $FA(c)$ to obtain $F\exists xA(x)$. Intuitively, saying that a certain term c is general for A means that all the individuals in the domain satisfy the property A.

- The introduction rules $T\forall I$ and $F\exists I$ are called *general rules.*

5.3.2 Derivability relations $\vdash_{\mathsf{k,q}}$ with $\mathsf{k} > 0$ and $\mathsf{q} \geq 0$ and the PB rule

In this Section, we provide the definitions for the remaining derivability relations, namely, $\vdash_{\mathsf{k,q}}$ with $\mathsf{k} > 0$ and $\mathsf{q} \geq 0$. As above, the definition of derivation, the introduction of the PB rule and the notion of propositional depth are due to D'Agostino's work. In the first-order, however, an important restriction on the introduction rule for the true universal quantifier must be introduced. This restriction permits to avoid fallacies in Depth Bounded First-Order Logics with propositional depth greater than zero.

Definition 9 (Derivations $\vdash_{\mathsf{k,q}}$ with $\mathsf{k} > 0$ and $\mathsf{q} \geq 0$). For every $\mathsf{k} > 0$ and $\mathsf{q} \geq 0$, a derivation in the logic $\vdash_{\mathsf{k,q}}$ of the signed formula φ from the set of signed formulae X is a sequence of signed formulae $\psi_1, \ldots, \psi_n$ such that:

- $\psi_n = \varphi$;

and for every element of the sequence ψ_i such that $1 \leq i < n$, one of the following conditions is satisfied:

- $\psi_i \in X$,

- ψ_i is the conclusion of an application of an *intelim* rule, whose premises precede ψ_i in the sequence;

- ψ_i is derivable in the logic $\vdash_{\mathsf{k}-1,\mathsf{q}}$ from $\psi_1, \ldots, \psi_{i-1}, TA$ and from $\psi_1, \ldots, \psi_{i-1}$, FA for some formula A that is a subformula of the unsigned parts of the signed formulae in $X \cup \{\varphi\}$;
- ψ_i is derivable in the logic $\vdash_{\mathsf{k}-1,\mathsf{q}}$ from $\psi_1, \ldots, \psi_{i-1}, SA$ for some formula A that is a subformula of the unsigned parts of the signed formulae in $X \cup \{\varphi\}$ such that $\psi_1, \ldots, \psi_{i-1}, \overline{S}A$ is inconsistent.

Definition 10 (PB rule). A derivation with propositional depth $\mathsf{k} \geq 0$ can be defined in a non-recursive but equivalent way as a derivation in which k nested applications of the following rule, called PB rule, are needed:

$$\begin{array}{c} \vdots \\ \boxed{\begin{array}{ll} TA & [\mathrm{PB}] \\ \vdots & \\ \psi & \end{array}} \boxed{\begin{array}{ll} FA & [\mathrm{PB}] \\ \vdots & \\ \psi & \end{array}} \\ \psi \\ \vdots \end{array}$$

Similarly to what has been said about the introduction rules of the connectives (see Definition 4), one could think that we should avoid that the PB rule introduces fresh terms that have not occurred above. But also in this case, the restriction turns out to be counter-intuitive, because reasoning by cases works independent of the nature of the chosen formula and of the terms occurring in it.

Definition 11 (Propositional depth: parameter k). Parameter k measures the propositional depth of a derivation of a formula φ from a set of formulae X, that is, the number of nested application of the principle of bivalence (PB rule) or, equivalently, the number of nested pieces of virtual information needed to derive the conclusion from the premises.

Definition 12 (Restriction on T∀I and F∃I for the derivations $\vdash_{\mathsf{k},\mathsf{q}}$ with $\mathsf{k} \neq 0$). In Depth Bounded First-Order Logics with propositional depth $\mathsf{k} \neq 0$, namely, when the PB rule is admitted, it is necessary to impose the following restriction on general rules, that is, on $T\forall I$ and $F\exists I$. For every formula A and every term c:

$$\begin{array}{c} T\forall I \text{ is applied on } TA(c) \ \underline{\text{or}}\ F\exists I \text{ is applied on } FA(c) \\ \underline{\text{if and only if}} \\ F\exists I \text{ is } \underline{\text{not}} \text{ applied on } FA(c) \ \underline{\text{and}}\ T\forall I \text{ is } \underline{\text{not}} \text{ applied on } T\neg A(c) \end{array}$$

The formulation of this restriction seems to be complicated only because we have chosen to use signed formulae. On the contrary, if we had employed unsigned formulae, the condition would have sounded as follows. For every formula A and every term c:

$$\forall I \text{ is applied on } A(c) \text{ } \underline{\text{if and only if}} \text{ } \forall I \text{ is } \underline{\text{not}} \text{ applied on } \neg A(c)$$

This remark should shed light on the meaning of this restriction, that is, the requirement that terms cannot be used as general for a formula and for its negation also on different branches of the same derivation.

Discussion on the restriction on T∀I and F∃I for the derivations $\vdash_{\mathsf{k,q}}$ with $\mathsf{k} \neq 0$. The restriction on $T\forall I$ and $F\exists I$ for the derivations with $\mathsf{k} \neq 0$ is necessary to avoid wrong patterns of reasoning like the following one, where c is a fresh term:

$$\begin{array}{c}
\vdots \\
\begin{array}{|ll|ll|}
\hline
TA(c) & [\mathrm{PB}] & FA(c) & [\mathrm{PB}] \\
T\forall x A(x) & [T\forall I] & T\neg A(c) & [T\neg I] \\
\vdots & & T\forall x \neg A(x) & [T\forall I] \\
 & & \vdots & \\
\hline
\end{array} \\
\vdots
\end{array}$$

In this case, the PB rule introduces in the derivation the fresh term c and the general rule $T\forall I$ is thus applied on both the formula $TA(c)$ that occurs on the left branch and on its negation $T\neg A(c)$ that occurs on the right branch. Why is this pattern of reasoning mistaken? The rule of bivalence allows, by its nature, to consider two complementary cases, such that the latter assumption is the negation of the former. In the example that we are examining, the application in sequence of the PB and $T\forall I$ rules on the two branches violates this fundamental feature of the rule of bivalence, because, on the first branch, we assume that c is general for A, while, on the second branch, we assume that c is general for $\neg A$: but the two assumptions are not contrary to each other. Indeed, denying that c is general for A, that is, denying that all the individuals have the property A, means affirming that there exists an individual that does not satisfy the property A. But this is different from assuming that c is general for $\neg A$, viz. from assuming that all the individuals do not satisfy A. In other words, the correct derivation would sound as follows:

$$
\begin{array}{c}
\vdots \\
\begin{array}{|ll|ll|}
\hline
TA(c) & [\mathrm{PB}] & FA(c) & [\mathrm{PB}] \\
T\forall x A(x) & [T\forall I] & T\neg A(c) & [T\neg I] \\
\vdots & & T\exists x \neg A(x) & [T\exists I] \\
 & & \vdots & \\
\hline
\end{array} \\
\vdots
\end{array}
$$

Notice that the restriction just discussed does not block the following derivations, since the $T\forall I$ rule is applied on the same formulae on both branches:

$$
\begin{array}{c}
\vdots \\
\begin{array}{|ll|ll|}
\hline
TA(c) & [\mathrm{PB}] & FA(c) & [\mathrm{PB}] \\
\vdots & & \vdots & \\
TB(c) & [\ldots] & TB(c) & [\ldots] \\
T\forall x B(x) & [T\forall I] & T\forall x B(x) & [T\forall I] \\
\vdots & & \vdots & \\
\hline
\end{array} \\
\vdots
\end{array}
$$

$$
\begin{array}{c}
\vdots \\
\begin{array}{|ll|ll|}
\hline
TA(c) & [\mathrm{PB}] & FA(c) & [\mathrm{PB}] \\
TA(c) \vee \neg A(c) & [T \vee I] & T\neg A(c) & [T\neg I] \\
T\forall x (A(x) \vee \neg A(x)) & [T\forall I] & TA(c) \vee \neg A(c) & [T \vee I] \\
\vdots & & T\forall x (A(x) \vee \neg A(x)) & [T\forall I] \\
 & & \vdots & \\
\hline
\end{array} \\
\vdots
\end{array}
$$

Moreover, the proposed restriction cannot be substituted by requiring that each term be general for only one formula also on different branches. This formulation seems to be too general and to exclude correct derivations like this one:

1. $T\forall x(A(x) \wedge B(x))$	[Ass.]
2. $TA(c) \wedge B(c)$	[$T\forall E$ 1]
3. $TA(c)$	[$T \wedge E$ 2]
4. $TB(c)$	[$T \wedge E$ 2]
5. $T\forall xA(x)$	[$T\forall I$ 3]
6. $T\forall xB(x)$	[$T\forall I$ 4]
7. $T\forall xA(x) \wedge \forall xB(x)$	[$T \wedge I$ 5,6]

5.3.3 Quantificational depth and parameter q

In this Section, we complete the definition of Depth Bounded First-Order Logics. In particular, we focus on the notion of quantificational depth: up to this point, we have always taken parameter q to be fixed without specifying its reference. The definition of q requires some preliminary notions. On the one hand, we determine when a group of individuals can be said to be reciprocally related. This definition is the result of generalizing sense 3 of the notion of 'being related to' that we have discussed in Section 5.2.4: we extend it from two to n individuals and take constants into account. On the other hand, we work on Hintikka's notion of degree of a formula. In particular, we include in his definition of depth of a formula the requirement of connectedness, that Hintikka himself has suggested and that we have clarified in Section 5.2.4. As a result, the notions of depth of a formula and degree of a formula that we use for our definition of parameter q is slightly different from those used in *Logic, Language Games and Information.*

Definition 13 (Related individuals). Individuals $a_1, \ldots, a_m$ are related in the derivation (of any depth) of φ from X if and only if, for any pair of individuals $a_i, a_j \in \{a_1, \ldots, a_m\}$, there exists a sequence of logical relations of the following kind:

$$R_1(c_{\langle 1,1\rangle}, c_{\langle 2,1\rangle}, \ldots, c_{\langle p,1\rangle})$$
$$R_2(c_{\langle 1,2\rangle}, c_{\langle 2,2\rangle}, \ldots, c_{\langle q,2\rangle})$$
$$\ldots$$
$$R_n(c_{\langle 1,n\rangle}, c_{\langle 2,n\rangle}, \ldots, c_{\langle r,n\rangle})$$

such that:

- the sequence of logical relations has one or more elements ($n \geq 1$);
- every relation occurs in the derivation of φ from X and is preceded by the prefix T;
- $a_i = c_{\langle k,1\rangle}$ for some k such that $1 \leq k \leq p$;

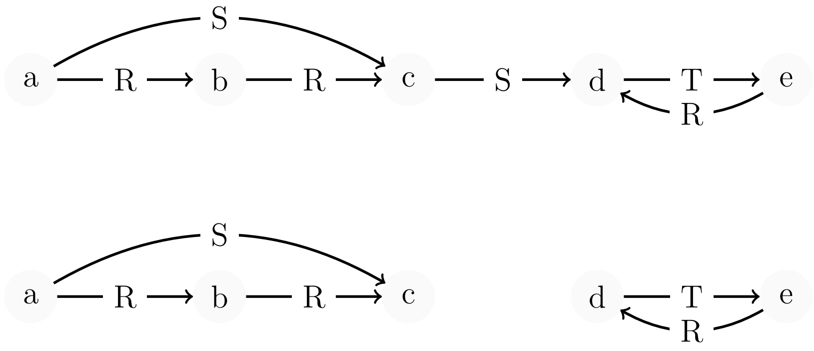


Figure 5.9: The individuals a, b, c, d, e are connected in the former, but not in the latter configuration.

- $a_j = c_{\langle k,n \rangle}$ for some k such that $1 \leq k \leq r$;
- for every relation of the sequence R_i, such that $1 \leq i < n$, there exists an individual c, such that c occurs as an argument in both R_i and R_{i+1}.

Discussion on the definition of related individuals. Intuitively, Definition 13 holds that the individuals of a group are related if there exists a path from the former to the latter individual of any pair of individuals arbitrarily chosen from that group. The elements of the path are relations occurring in the derivation. The former individual of the pair must occur in the first element of the path; the latter individual must occur in the latter element of the path. Moreover, two elements of the path are in succession if and only if there exists an individual that occurs in both of the relations. In order to be related, it is not sufficient for the elements of a group to be such that each of them is related with some other individuals. For example, the individuals a, b, c, d, e are related only in the former of the configurations depicted in Figure 5.9, but not in the latter, where it is impossible to move from d to c, although each individual is related to someone else.

Definition 14 (Connectedness). The variables $x_1, \ldots, x_k$ and the quantifiers bounding these variables are connected in the formula A if and only if these requirements are simultaneously satisfied:

- for each $i = 1, \ldots, k-1$, x_i and x_{i+1} occur in the same atomic subsentence of A;
- each variable x_i is bound to a quantifier occurring within the scope of the wider of the two quantifiers to which x_1 and x_k are bound.

Definition 15 (Depth of a formula). Absorbing the connectedness requirement in Hintikka's definition of depth (see Section 4.2.1 and Section 5.2.3), we obtain the following alternative definition of depth of a formula A:

- $d(A) = 0$ whenever A is atomic;
- $d(\neg A) = d(A)$;
- $d(A_1 \wedge A_2) = d(A_1 \vee A_2) = d(A_1 \rightarrow A_2) =$ the greater of the numbers $d(A_1)$ and $d(A_2)$;
- $d(Q_1 x_1 Q_2 x_2 \ldots Q_n x_n B) = d(B) + n$, if there exists a number n of variables $x_i \in \{x_1, \ldots, x_k\}$ that are connected to the quantifiers that bound them and B is any formula that does not start with a quantifier Q.

Definition 16 (Degree of a formula). The degree of a formula A is the sum of the depth of A together with the number of constants and free variables occurring in A.

Definition 17 (Quantificational depth: parameter q). Parameter q measures the quantificational depth of a derivation of a formula φ from a set of formulae X, that is, the difference between the number of related individuals used in that derivation and the maximum degree of premises and conclusion. If this difference is negative, then the quantificational depth of the corresponding derivation is assumed to be zero.

5.3.4 Examples

In this Section, we are going to present and discuss some derivations in Depth Bounded First-Order Logics. The examples that we have chosen are summarized in Table 5.1.

The first set of derivations (Examples 1-3) are all carried out in the logic $\vdash_{0,0}$, viz. the basic element of Depth Bounded First-Order Logics. In this logical system, no virtual information and no new individual is allowed: only analytical inferences are valid. In the Example 1, individual a has been introduced at step 2 through an application of the elimination rule for the existential quantifier. But this individual does not count as an actually new individual, because it is somehow already thought in the premise affirming that *there is* some individual that has both the property A and B. This case confirms the importance of our observation made in Section 5.2.3 that counting the individuals occurring in a derivation is not *by itself* sufficient to determine the quantificational depth of an inference: we rather need to compare that result with the information contained in the premises

Example	Premises	Conclusion	Logic
Example 1	$T\exists x(Ax \wedge Bx)$	$T\exists xAx \wedge \exists xBx$	$\vdash_{0,0}$
Example 2	$T\exists x\exists yPxy$	$T\exists y\exists xPxy$	$\vdash_{0,0}$
Example 3	$T\exists y\forall xPxy$	$T\forall x\exists yPxy$	$\vdash_{0,0}$
Example 4	-	$T\exists y(\exists xPx \rightarrow Py)$	$\vdash_{1,0}$
Example 5	$T\forall x\forall y(Rxy \rightarrow Ryx)$ $T\forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz)$	$T\forall x\forall y(Rxy \rightarrow Rxx)$	$\vdash_{1,0}$
Example 6	$T\forall x\forall y(Pxy \rightarrow Pyx)$ $T\forall x(Rx \leftrightarrow \exists yPxy)$ $T\forall x(Bx \leftrightarrow \exists yPyx)$	$F\exists x(Rx \wedge \neg Bx)$	$\vdash_{1,0}$
Example 7	$T\exists x(\exists yAy \rightarrow Bx)$	$T\exists yAy \rightarrow \exists xBx$	$\vdash_{1,0}$
Example 8	$T\forall x\forall y(Rxy \rightarrow \exists z(Gxz \wedge Gzy))$ $T\forall x\forall y(Gxy \rightarrow \exists z(Bxz \wedge Bzy))$ $T\forall x\forall y((Bxy \wedge Cx) \rightarrow Cy)$	$T\forall x\forall y((Rxy \wedge Cx) \rightarrow Cy)$	$\vdash_{2,2}$
Example 9	$\forall x\forall y\forall z((Bxy \wedge Byz) \rightarrow Bxz)$ $\forall x\forall y(Cxy \leftrightarrow \exists w\exists z(Fwx \wedge Fzy \wedge Bwz))$ $\forall x\forall y\forall z((Fxy \wedge Fzy) \rightarrow Bxz)$	$\forall x\forall y\forall z((Cxy \wedge Cyz) \rightarrow Cxz)$	$\vdash_{5,3}$

Table 5.1: Examples of derivations in Depth Bounded First-Order Logics.

and conclusion. After all, the quantificational analyticity of this derivation should come as no surprise, for the monadic part of first order logic cannot be synthetic.

Therefore, we now move to examine some cases involving polyadic predicates. Consider Example 2 and Example 3, which are the only valid quantifier shifts involving the existential quantifier. Again, in both of them, we find that the individuals occurring in the derivations do not exceed the number of those already thought in the premises. To put it in another way, the number of individuals that is necessary to understand the premise 'there exists somebody that loves someone else' is also sufficient to derive from this sentence the conclusion that 'there exists somebody that is loved by someone else'. However, it is fair to mention that this result, viz. that the quantifier shifts are analytic inferences from not only the propositional but also the quantificational side, contradicts one passage in Hintikka (1973, VIII, 8, p. 193). Here, the Finnish philosopher suggests that the only 'easy' inferences of first-order logic that are also synthetic are exactly the quantifiers shifts, but he does not explain in details this affirmation that, we think, seems to be against his point of view on this matter that prevails in his corpus.

Example 3 gives us the opportunity to underline another important issue. Individual b is introduced at step 3 through an application of the $T\forall E$ rule, which, recall, can be used with whatever term. If we had introduced a instead of b, we would have obtained the formula $TPaa$: but then, at step 4, we would have obtained the formula $T\exists yPyy$ because of the uniformity requirement on substitutions. And if that wasn't enough, we could not have applied rule $T\forall I$ at step 5, because the individual a was critic and not generic.

The second set of derivations (Examples 4-7) are all carried out in the logic $\vdash_{1,0}$. Since there are no premises, the derivation shown in the Example 4 cannot start but with an application of the PB rule introducing a new individual, that we have named a. This derivation shows the importance of not having given in to the temptation of imposing restrictions on the terms occurring in the formula introduced by PB that we have discussed in Section 5.3.2. For here the introduction of a through PB is the only reasonable way to start this deduction. Moreover, this example underlines that in the definition of the notion of quantificational depth it is essential to consider the maximum degree not only of the premises, but also of the conclusion of the derivation. Examples 5, 6 and 7 propose other cases in which the employment of one piece of virtual information cannot be avoided, but, at the same time, the quantificational depth is still equal to zero.

The third set of derivations (Examples 8-9) is the most interesting one, because not only propositional depth, but also quantificational depth is greater than zero. The first of these derivations is the formal translation of the kind of reasoning that solved the example that we have discussed in Section 4.2.1:

$P_1 : \forall x \forall y(Rxy \to \exists z(Gxz \wedge Gzy))$
$P_2 : \forall x \forall y(Gxy \to \exists z(Bxz \wedge Bzy))$
$P_3 : \forall x \forall y((Bxy \wedge Cx) \to Cy)$
$C : \forall x \forall y((Rxy \wedge Cx) \to Cy)$.

This confirms the heuristic value of the reasoning pattern through configurations of individuals that has been suggested by Hintikka. Example 9[19] proposes a slightly more complicated situation, for the quantificational depth is five instead of three:

P1: $\forall x \forall y \forall z((Bxy \wedge Byz) \to Bxz)$
P2: $\forall x \forall y(Cxy \leftrightarrow \exists w \exists z(Fwx \wedge Fzy \wedge Bwz))$
P3: $\forall x \forall y \forall z((Fxy \wedge Fzy) \to Bxz)$
C: $\forall x \forall y \forall z((Cxy \wedge Cyz) \to Cxz)$.

But the representation of the configurations of the individuals involved in the premises, the conclusion and the intermediate step could be used as a guide through the derivation (see below).

[19] The justification for each step is not reported due to space constraints.

1.	$T\exists x(Ax \wedge Bx)$	[Ass.]
2.	$TAa \wedge Ba$	$[T\exists E\ 1]$
3.	TAa	$[T \wedge E\ 2]$
4.	TBa	$[T \wedge E\ 2]$
5.	$T\exists xAx$	$[T\exists I\ 3]$
6.	$T\exists xBx$	$[T\exists I\ 4]$
7.	$T\exists xAx \wedge \exists xBx$	$[T \wedge I\ 5{,}6]$

Example 1

1.	$T\exists x\exists yPxy$	[Ass.]
2.	$T\exists yPay$	$[T\exists E\ 1]$
3.	$TPab$	$[T\exists E\ 2]$
4.	$T\exists xPxb$	$[T\exists I\ 3]$
5.	$T\exists y\exists xPxy$	$[T\exists I\ 4]$

Example 2

1.	$T\exists y\forall xPxy$	[Ass.]
2.	$T\forall xPxa$	$[T\exists E\ 1]$
3.	$TPba$	$[T\forall E\ 2]$
4.	$T\exists yPby$	$[T\exists I\ 3]$
5.	$T\forall x\exists yPxy$	$[T\forall I\ 4]$

Example 3

1.	TPa	[PB]	FPa	[PB]
2.	$T\exists xPx \to Pa$	$[T \to I\ 1]$	$F\exists xPx$	$[F\exists I\ 1]$
3.	$T\exists y(\exists xPx \to Py)$	$[T\exists I\ 2]$	$T\exists xPx \to Pa$	$[T \to I\ 2]$
4.			$T\exists y(\exists xPx \to Py)$	$[T\exists I\ 3]$
5.	$T\exists y(\exists xPx \to Py)$			

Example 4

1.	$T\forall x\forall y(Rxy \to Ryx)$	[Ass.]		
2.	$T\forall x\forall y\forall z((Rxy \land Ryz) \to Rxz)$	[Ass.]		
3.	$T\forall y(Ray \to Rya)$	$[T\forall E\ 1]$		
4.	$TRab \to Rba$	$[T\forall E\ 3]$		
5.	$T\forall y\forall z((Ray \land Ryz) \to Raz)$	$[T\forall E\ 2]$		
6.	$T\forall z((Rab \land Rbz) \to Raz)$	$[T\forall E\ 5]$		
7.	$T(Rab \land Rba) \to Raa$	$[T\forall E\ 6]$		
8.	$TRab$	[PB]	$FRab$	[PB]
9.	$TRba$	$[T \to E\ 4{,}8]$	$TRab \to Raa$	$[T \to I\ 8]$
10.	$TRab \land Rba$	$[T \land I\ 8{,}9]$		
11.	$TRaa$	$[T \to E\ 7{,}10]$		
12.	$TRab \to Raa$	$[T \to I\ 11]$		
13.	$TRab \to Raa$			
14.	$T\forall y(Ray \to Raa)$	$[T\forall E\ 1]$		
15.	$T\forall x\forall y(Rxy \to Rxx)$	$[T\forall I\ 14]$		

Example 5

1.	$T\forall x\forall y(Pxy \to Pyx)$	[Ass.]		
2.	$T\forall x(Rx \leftrightarrow \exists yPxy)$	[Ass.]		
3.	$T\forall x(Bx \leftrightarrow \exists yPyx)$	[Ass.]		
4.	$T\forall y(Pay \to Pya)$	[$T\forall E$ 1]		
5.	$TPab \to Pba$	[$T\forall E$ 4]		
6.	$TRa \leftrightarrow \exists yPay$	[$T\forall E$ 2]		
7.	$TBa \leftrightarrow \exists yPya$	[$T\forall E$ 3]		
8.	TRa	[PB]	FRa	[PB]
9.	$T\exists yPay$	[$T \leftrightarrow E$ 6,8]	$FRa \wedge \neg Ba$	[$T \wedge I$ 8]
10.	$TPab$	[$T\exists E$ 9]		
11.	$TPba$	[$T \to E$ 5,10]		
12.	$T\exists yPya$	[$T\exists I$11]		
13.	TBa	[$T \leftrightarrow E$ 7,12]		
14.	$F\neg Ba$	[$F\neg I$ 13]		
15.	$FRa \wedge \neg Ba$	[$F \wedge I$ 14]		
16.	$FRa \wedge \neg Ba$			
17.	$F\exists x(Rx \wedge \neg Bx)$	[$T\exists I$ 16]		

Example 6

1.	$T\exists x(\exists yAy \to Bx)$	[Ass.]		
2.	$T\exists yAy \to Ba$	[$T\exists E$ 1]		
3.	$T\exists yAy$	[PB]	$F\exists yAy$	[PB]
4.	TBa	[$T \to E$ 2,3]	$T\exists yAy \to \exists xBx$	[$T \to I$ 3]
5.	$T\exists xBx$	[$T\exists I$ 4]		
6.	$T\exists yAy \to \exists xBx$	[$T \to I$ 5]		
7.	$T\exists yAy \to \exists xBx$			

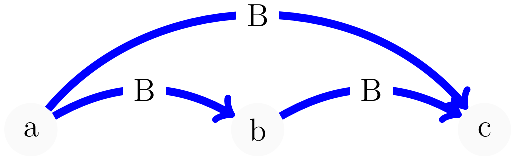
Example 7

1. $T\forall x\forall y(Rxy \to \exists z(Gxz \wedge Gzy))$ [Ass.]
2. $T\forall x\forall y(Gxy \to \exists z(Bxz \wedge Bzy))$ [Ass.]
3. $T\forall x\forall y((Bxy \wedge Cx) \to Cy)$ [Ass.]
4. $T\forall y(Ray \to \exists z(Gaz \wedge Gzy))$ [$T\forall E$ 1]
5. $TRab \to \exists z(Gaz \wedge Gzb)$ [$T\forall E$ 4]

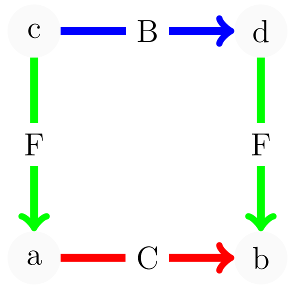
<table>
<tr>
<td>
6. $TRab$ [PB]
7. $T\exists z(Gaz \wedge Gzb)$ [$T \to E$ 5,6]
8. $TGac \wedge Gcb$ [$T\exists E$ 7]
9. $TGac$ [$T \wedge E$ 8]
10. $TGcb$ [$T \wedge E$ 8]
11. $T\forall y(Gay \to \exists z(Baz \wedge Bzy))$ [$T\forall E$ 2]
12. $TGac \to \exists z(Baz \wedge Bzc)$ [$T\forall E$ 11]
13. $T\exists z(Baz \wedge Bzc)$ [$T \to E$ 9,12]
14. $TBad \wedge Bdc$ [$T\exists E$ 13]
15. $TBad$ [$T \wedge E$ 14]
16. $TBdc$ [$T \wedge E$ 14]
17. $T\forall y(Gcy \to \exists z(Bcz \wedge Bzy))$ [$T\forall E$ 2]
18. $TGcb \to \exists z(Bcz \wedge Bzb)$ [$T\forall E$ 17]
19. $T\exists z(Bcz \wedge Bzb)$ [$T \to E$ 10,18]
20. $TBce \wedge Beb$ [$T\exists E$ 19]
21. $TBce$ [$T \wedge E$ 20]
22. $TBeb$ [$T \wedge E$ 20]
23. $T\forall y((Bay \wedge Ca) \to Cy)$ [$T\forall E$ 3]
24. $T(Bad \wedge Ca) \to Cd$ [$T\forall E$ 23]
<table>
<tr>
<td>
25. TCa [PB]
26. $TBad \wedge Ca$ [$T \wedge I$ 15,25]
27. TCd [$T \to E$ 24,26]
28. $T\forall y((Bdy \wedge Cd) \to Cy)$ [$T\forall E$ 3]
29. $T(Bdc \wedge Cd) \to Cc$ [$T\forall E$ 28]
30. $TBdc \wedge Cd$ [$T \wedge I$ 16,27]
31. TCc [$T \to E$] 29,30
32. $T\forall y((Bcy \wedge Cc) \to Cy)$ [$T\forall E$ 3]
33. $T(Bce \wedge Cc) \to Ce$ [$T\forall E$ 32]
34. $TBce \wedge Cc$ [$T \wedge I$ 21,31]
35. TCe [$T \to E$] 33,34
36. $T\forall y((Bey \wedge Ce) \to Cy)$ [$T\forall E$ 3]
37. $T(Beb \wedge Ce) \to Cb$ [$T\forall E$ 36]
38. $TBeb \wedge Ce$ [$T \wedge I$ 22,35]
39. TCb [$T \to E$ 37,38]
40. $T(Rab \wedge Ca) \to Cb$ [$T \to I$ 39]
</td>
<td>
FCa [PB]
$FRab \wedge Ca$ [$F \wedge I$ 25]
$T(Rab \wedge Ca) \to Cb$ [$T \to I$ 26]
</td>
</tr>
</table>
41. $T(Rab \wedge Ca) \to Cb$
</td>
<td>
$FRab$ [PB]
$FRab \wedge Ca$ [$F \wedge I$ 6]
$T(Rab \wedge Ca) \to Cb$ [$T \to I$ 7]
</td>
</tr>
</table>

42. $T(Rab \wedge Ca) \to Cb$
43. $T\forall y(Ray \wedge Ca) \to Cy$ [$T\forall I$ 42]
44. $T\forall x\forall y(Rxy \wedge Cx) \to Cy$ [$T\forall I$ 43]

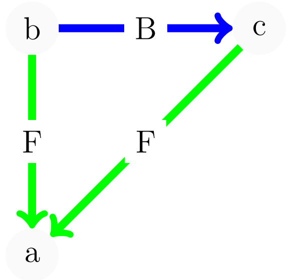
Example 8



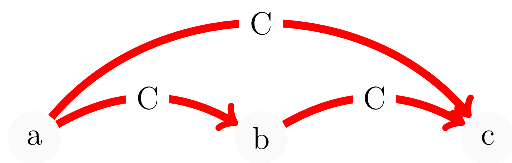
(a) Premise P_1.



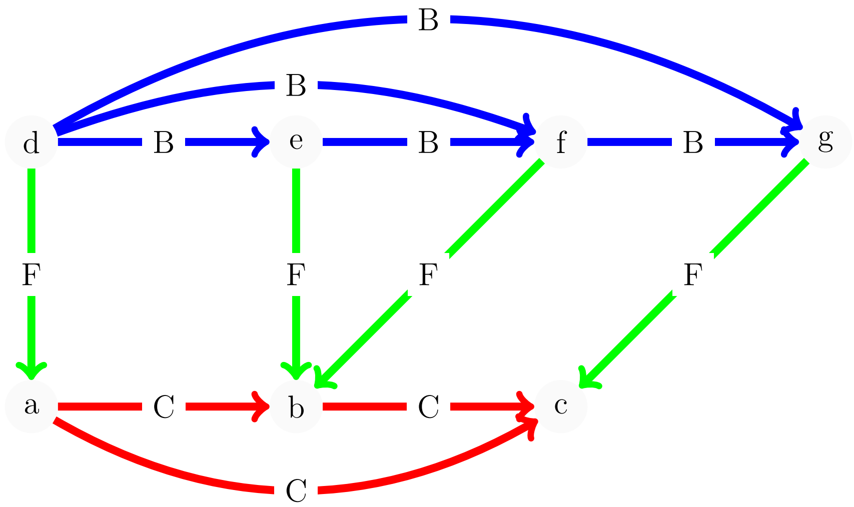
(b) Premise P_2.



(c) Premise P_3.



(d) Conclusion C.



(e) Intermediate step

Configuration of individuals of the Example 9.

1. $T\forall x\forall y\forall z(Bxy \land Byz \to Bxz)$ [Ass.]
2. $T\forall x\forall y(Cxy \leftrightarrow \exists w\exists z(Fwx \land Fzy \land Bwz))$ [Ass.]
3. $T\forall x\forall y\forall z((Fxy \land Fzy) \to Bxz)$ [Ass.]
4. $T\forall y(Cay \leftrightarrow \exists w\exists z(Fwa \land Fzy \land Bwz))$
5. $TCab \leftrightarrow \exists w\exists z(Fwa \land Fzb \land Bwz)$

6. $TCab$
7. $\exists w\exists z(Fwa \land Fzb \land Bwz)$
8. $\exists z(Fda \land Fzb \land Bdz)$
9. $Fda \land Feb \land Bde$
10. $TFda$
11. $TFeb$
12. $TBde$
13. $T\forall y(Cby \leftrightarrow \exists w\exists z(Fwb \land Fzy \land Bwz))$
14. $TCbc \leftrightarrow \exists w\exists z(Fwb \land Fzc \land Bwz)$

15. $TCbc$
16. $T\exists w\exists z(Fwb \land Fzc \land Bwz)$
17. $T\exists z(Ffb \land Fzc \land Bfz)$
18. $TFfb \land Fgc \land Bfg$
19. $TFfb$
20. $TFgc$
21. $TBfg$
22. $T\forall y\forall z((Fey \land Fzy) \to Bez)$
23. $T\forall z((Feb \land Fzb) \to Bez)$
24. $T(Feb \land Ffb) \to Bef)$
25. $TFeb \land Ffb$
26. $TBef$
27. $T\forall y\forall z((Bdy \land Byz) \to Bdz)$
28. $T\forall z((Bde \land Bez) \to Bdz)$
29. $T(Bde \land Bef) \to Bdf)$
30. $TBde \land Bef$
31. $TBdf$
32. $T\forall z((Bdf \land Bfz) \to Bdz)$
33. $T(Bdf \land Bfg) \to Bdg)$
34. $TBdf \land Bfg$
35. $TBdg$
36. $TCac \leftrightarrow \exists w\exists z(Fwa \land Fzc \land Bwz)$

37. $TCac$
38. $T(Cab \land Cbc) \to Cac$
39.
40.
41.
42.
43.
44.
45.
46.
47.

$FCac$
$F\exists w\exists z(Fwa \land Fzc \land Bwz)$
$F\exists z(Fda \land Fzc \land Bdz)$
$F(Fda \land Fgc \land Bdg)$

$TFda$
$F(Fgc \land Bdg)$

$TFgc$
$FBdg$
x

$FFgc$
x

x

$FFda$
x

x

48. $T(Cab \land Cbc) \to Cac$

$FCbc$
$FCab \land Cbc$
$T(Cab \land Cbc) \to Cac$

49. $T(Cab \land Cbc) \to Cac$

$FCab$
$FCab \land Cbc$
$T(Cab \land Cbc) \to Cac$

50. $T(Cab \land Cbc) \to Cac$
51. $T\forall z((Cab \land Cbz) \to Caz)$
52. $T\forall y\forall z((Cay \land Cyz) \to Caz)$
53. $T\forall x\forall y\forall z((Cxy \land Cyz) \to Cxz)$

Example 9

Chapter 6

Depth Bounded Epistemic Logics

6.1 Analyticity of logic and logical omniscience

The assumption of the traditional tenet that logic is analytic and tautological amounts, in the epistemic context, to the assumption of the principle of logical omniscience, which states that:

> **(LO)** Individuals know all the logical consequences of what they know.

If we admit that logical inferences are true in virtue of the meaning of the logical operators and do not have informational content because the information conveyed by their conclusion is contained in the premises, then we shall assume that agents are able to derive immediately all the consequences of the information they possess and, thus, to know them. In other words, we are obliged to accept that individuals are logical omniscient if we maintain that carrying out logical inferences does not require any kind of cognitive effort. If the conclusion of an inference is contained in the premises, why should an agent ignore the conclusion once she is aware of the premises?

In the previous Chapter we have proposed a family of logical systems that vindicated the idea that logic is synthetic. We now aim at realizing the same project in the epistemic field, that is, we shall criticize the principle of logical omniscience and shall provide a characterization of knowledge free of that assumption.

6.1.1 Classical epistemic logics and logical omniscience

In this Section, we provide the basic definitions of classical epistemic logics and we show the formal characterization of the principle of logical omniscience that they satisfy.

The idea behind classical epistemic logics is that of representing the ignorance of an agent in terms of the situations that she considers possible, which are called the epistemic alternatives of the agent. So, for instance, the fact that individual i doesn't know whether she has turned off the gas or not is described by saying that i considers exactly two situations as possible, the former in which the gas is on, the latter in which it is off. This intuition is formally conveyed using Kripke's semantics: thus classical epistemic logics are extensions of the normal modal system K, which interprets modal operators in epistemic terms.

The alphabet A of classical epistemic logics consists of: a finite set of agents $\mathcal{A} = \{1, 2, 3, \ldots, n\}$; an enumerable set of propositional parameters $\mathcal{P} = \{p_1, p_2, \ldots\}$; a set of connectives $\mathcal{C} = \{\neg, \wedge, \vee, \rightarrow\}$; a set of epistemic operators $\mathcal{O} = \{\mathcal{K}_i \mid i \in \mathcal{A}\}$ and a set of auxiliary symbols $\mathcal{S} = \{(\, , \,)\}$. The set $\mathcal{L}$ of well-formed formulae is recursively defined as follows:

- $\mathcal{L}^0 = \mathcal{P}$
- $\mathcal{L}^{n+1} = \{\neg B, \mathcal{K}_i B, B \vee C, B \rightarrow C, B \wedge C | B, C \in \mathcal{L}^n \text{ and } i \in \mathcal{A}\}$
- $\mathcal{L} = \bigcup_{n \in \mathbb{N}} \mathcal{L}^n$

A Kripke model for n agents over $\mathcal{P}$ is defined as a tuple $\mathsf{M} = (S, v, R_1, \ldots, R_n)$, where S is a set of states of affairs or possible worlds; v is function relative to a state that assigns to each propositional parameter a truth value (true or false); for every agent i, R_i is i's accessibility relation defined as a binary relation on S. Given a certain situation $s \in S$, the epistemic alternatives of an agent i at u are represented by all those states u that are accessible for i from s through the relation R_i.

The systems we are considering are distinguished by the requirements imposed on the accessibility relations R_i:

- If we require that the relations R_i are reflexive ($\forall i \in \mathcal{A}, \forall s \in S$, $(s, s) \in R_i$), we obtain the logical system T. This constraint amounts to the introduction of the T-axiom: $\vdash_T \mathcal{K}_i B \rightarrow B$. This axiom requires the truth of what is known.

- If we require that the relations R_i are not only reflexive, but also transitive ($\forall i \in \mathcal{A}, \forall s, t, u \in S$, if $(s, t) \in R_i$ and $(t, u) \in R_i$, then $(s, u) \in R_i$), we obtain the logical system $S4$. This constraint amounts to the introduction, beyond the T-axiom, also of the $S4$-axiom: $\vdash_{S4} \mathcal{K}_i B \rightarrow \mathcal{K}_i \mathcal{K}_i B$. This is called 'positive introspection axiom' and requires agents to know what they know.

- If we require that the relations R_i are not only reflexive and transitive, but also symmetric ($\forall i \in \mathcal{A}, \forall s, t \in \mathcal{S}$, if $(s, t) \in R_i$, then $(t, s) \in R_i$), we obtain the logical system $S5$. This constraint amounts to the introduction, beyond the T-axiom and the $S4$-axiom, also of the $S5$-axiom: $\vdash_{S5} \neg\mathcal{K}_i B \to \mathcal{K}_i \neg\mathcal{K}_i B$. This is called 'negative introspection axiom' and requires agents to know the information they ignore[1].

In what follows, we will employ the system $S5$. This logic is particularly simple and intuitive: accessibility relations turns out to be equivalence relations that split the set of states of affairs into classes of equivalence, whose members are reciprocally accessible. Each class of equivalence represents a set of worlds that are epistemic alternatives one of the other.

The recursive definition of what it means for a formula A to be satisfied (or to hold) at a given state s in a model M (written $(\mathsf{M}, s) \vDash A$) is as follows:

- $(\mathsf{M}, s) \vDash p$ (for $p \in \mathcal{P}$) if and only if $v(s)(p) =$ true
- $(\mathsf{M}, s) \vDash \neg A$ if and only if $(\mathsf{M}, s) \nvDash A$
- $(\mathsf{M}, s) \vDash A \vee B$ if and only if $(\mathsf{M}, s) \vDash A$ or $(\mathsf{M}, s) \vDash B$
- $(\mathsf{M}, s) \vDash A \to B$ if and only if $(\mathsf{M}, s) \nvDash A$ or $(\mathsf{M}, s) \vDash B$
- $(\mathsf{M}, s) \vDash A \wedge B$ if and only if $(\mathsf{M}, s) \vDash A$ and $(\mathsf{M}, s) \vDash B$
- $(\mathsf{M}, s) \vDash \mathcal{K}_i A$ if and only $(\mathsf{M}, u) \vDash A$ for all $u \in S$ such that $(s, u) \in R_i$.

Although she could be uncertain about the nature of the actual world, agent i has no doubt about the truth value of formula A at state s because A is true in all her epistemic alternatives. In this sense, we could really say that i knows that A is true.

Then, a formula A is said to be satisfiable in a model M if and only if it holds at some state in M and it is said to be valid in a model M, written $\mathsf{M} \vDash A$, if and only if it holds at all states in M. For $\Gamma \subseteq \mathcal{L}$ and $A \in \mathcal{L}$, the notions of logical consequence and logical validity are defined as follows:

- A is a logical consequence of Γ, written $\Gamma \vDash A$, if and only if $\mathsf{M} \vDash A$ for all models M such that $\mathsf{M} \vDash B$ for every $B \in \Gamma$;
- A is a logical validity, written $\varnothing \vDash A$, if and only if $\mathsf{M} \vDash A$ for all models M.

Every classical epistemic logic that we have examined so far satisfies the principle of omniscience, which has been formally defined by van Ditmarsch, van der Hoek and Kooi (2007) as the set of the following propositions:

(LO1) $\vdash (\mathcal{K}_i B \wedge \mathcal{K}_i (B \to C)) \to \mathcal{K}_i C$

[1]However, $S4$-axiom is redundant and can be derived from the other ones.

(LO2) If $\vdash B$, then $\vdash \mathcal{K}_i B$

(LO3) If $\vdash B \to C$, then $\vdash \mathcal{K}_i B \to \mathcal{K}_i C$

(LO4) If $\vdash B \leftrightarrow C$, then $\vdash \mathcal{K}_i B \leftrightarrow \mathcal{K}_i C$

(LO5) $\vdash (\mathcal{K}_i B \wedge \mathcal{K}_i C) \to \mathcal{K}_i(B \wedge C)$

(LO6) $\vdash \mathcal{K}_i B \to \mathcal{K}_i(B \vee C)$

These assumptions are inevitable consequences of the use of normal modal logics to characterize the notion of knowledge. For LO1 is equivalent to the distribution axiom ($\vdash \mathcal{K}_i(B \to C) \to (\mathcal{K}_i B \to \mathcal{K}_i C)$) that, together with the necessitation rule LO2, is at the basis of all the systems we have mentioned so far and LO3-LO6 follow from the combination of LO1 and LO2.

6.1.2 The muddy children puzzle in *S5*

In this Section, we introduce, as a case study, the muddy children puzzle, which is the subject of a considerable amount of logical and philosophical literature and a good example of the subtleties that can arise when considering knowledge in groups of individuals. This puzzle, which is a variant of the well known wise men and the cheating wives puzzles[2], goes as follows:

> **The muddy children puzzle.** Imagine n children playing together. The mother of these children has told them that if they get dirty there will be severe consequences. So, of course, each child wants to keep clean, but each would love to see the others get dirty. Now it happens during their play that some of the children, say k of them, get mud on their foreheads. So, of course, no one says a thing. Along comes the father, who says, 'At least one of you has mud on your forehead', thus expressing a fact known to each of them before he spoke (if $k > 1$). The father then asks the following question, over and over: 'Does any of you know whether you have mud on your forehead?' Assuming that all the children are perceptive, intelligent, truthful, and that they answer simultaneously, what will happen?
>
> **Classical solution.** The first $k-1$ times the father asks the question, all the children will say 'No', but then the k^{th} time the children with muddy foreheads will answer 'Yes' (Fagin, Halpern, Moses and Vardi, 1995, p. 4).

[2] See Dolev, Halpern and Moses (1986).

We now explain the classical solution to the puzzle through the means of the logical system $S5$. This will give us the opportunity to clarify the definitions given in Section 6.1.1 and, above all, to show the assumption of logical omniscience at work. This analysis is taken from Fagin, Halpern, Moses and Vardi (1995) with minor modifications. We are going to consider the case in which there are four children, so $\mathcal{A} = \{1, 2, 3, 4\}$.

Since we are interested in whether or not a given child's forehead is muddy, we take $\mathcal{P} = \{p_i \mid i \in \mathcal{A}\}$, where each p_i stands for 'child i has a muddy forehead', and we use P as a shorthand notation for $(((p_1 \vee p_2) \vee p_3) \vee p_4) \in \mathcal{L}$, which says 'at least one child has a muddy forehead'.

Then, we have to distinguish the following moments of the story:

t_0 is before the father speaks
t_1 is after the father has said that P
t_2 is after all the children have answered 'No' to q_1
t_3 is after all the children have answered 'No' to q_2
t_4 is after all the children have answered 'No' to q_3

where q_m is the father m^{th} question 'Does any of you know whether you have mud on your forehead?'.

First, the situation before the father speaks (time t_0) can be characterized by a Kripke model $\mathsf{M}_{t_0} = (S, v, R_1, R_2, R_3, R_4)$ defined as follows:

- A possible state $s \in S$ can be described as a tuple of 0's and 1's of the form $(x_1x_2x_3x_4)$, where $x_i = 1$ if child i is muddy and $x_i = 0$ otherwise. Thus, for example, (0101) says that precisely child 2 and child 4 are muddy. The set S consists of 2^4 states, one for each of the possible tuples $(x_1x_2x_3x_4)$.

- The evaluation v is defined so that $(\mathsf{M}, (x_1x_2x_3x_4)) \vDash p_i$ if and only if $x_i = 1$; it follows that $(\mathsf{M}, (x_1x_2x_3x_4)) \vDash P$ if and only if $x_i = 1$ for some x_i.

- Last, we have to define the R_i relations. Suppose that the actual situation is described by the tuple (0101). Since child 1 can see the foreheads of all the children except herself, her only doubt is about whether she has mud on her own forehead: thus, child 1 considers two situations as possible, (0101) and (0100). In general, at a certain state s, child i has two epistemic alternatives, which agree in all components except the i^{th} component. As a result, we take $(s, u) \in R_i$ if and only if s and u agree in all components except the i^{th} component. This definition makes every R_i an equivalence relation.

M_{t_0} has the graphical representation given in Figure 6.1. Each node represents a state $(x_1x_2x_3x_4)$; a green edge linking the states s and u indicates that both (s, u) and (u, s) are included in R_1; an orange edge indicates that both (s, u) and (u, s)

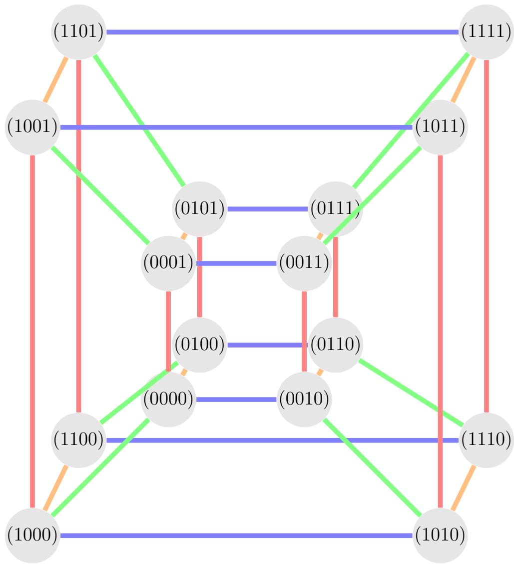


Figure 6.1: Kripke model M_{t_0} for the muddy children puzzle with $n = 4$ at time t_0.

are included in R_2; a blue edge indicates that both (s, u) and (u, s) are included in R_3 and a red edge indicates that both (s, u) and (u, s) are included in R_4. We have omitted self loops: nonetheless, each couple (s, s) is included in each R_i.

It turns out that in model M_{t_0}, according to the intuitions, each child knows which of the other children have muddy foreheads, but none of the children knows whether her own forehead is muddy or not, since at any state every child considers the other alternative possible. Thus, if the father does not announce that P, the muddy children will never be able to conclude that their foreheads are muddy. For instance, we have that $(\mathsf{M}_{t_0}, (1010)) \vDash \mathcal{K}_1 p_3$, since when the actual situation is (1010), child 3 is muddy in both worlds that child 1 considers possible. Nonetheless, we have that $(\mathsf{M}_{t_0}, (1010)) \vDash \neg\mathcal{K}_1 p_1$: child 1 does not know that she is muddy because at the other world that she considers possible, (0010), her forehead is not muddy.

Consider then what happens at time t_1, that is, after the father speaks. Assume that at the actual state there is only one child with a muddy forehead: that child sees that no one else is muddy and, since she knows that there is at least one muddy child, she concludes that she must be the one. The situation at time t_1 can be represented by model M_{t_1}, which is exactly the same as M_{t_0} except for the accessibility relations, which get 'truncated'. All the edges between the node (0000) and the nodes with exactly one 1 disappear, since after the father speaks every child will not consider it possible that no one has a muddy forehead. M_{t_1} is represented in the first graph of Figure 6.2. As a result, we have that $(\mathsf{M}_{t_1}, (1000)) \vDash \mathcal{K}_1 p_1$,

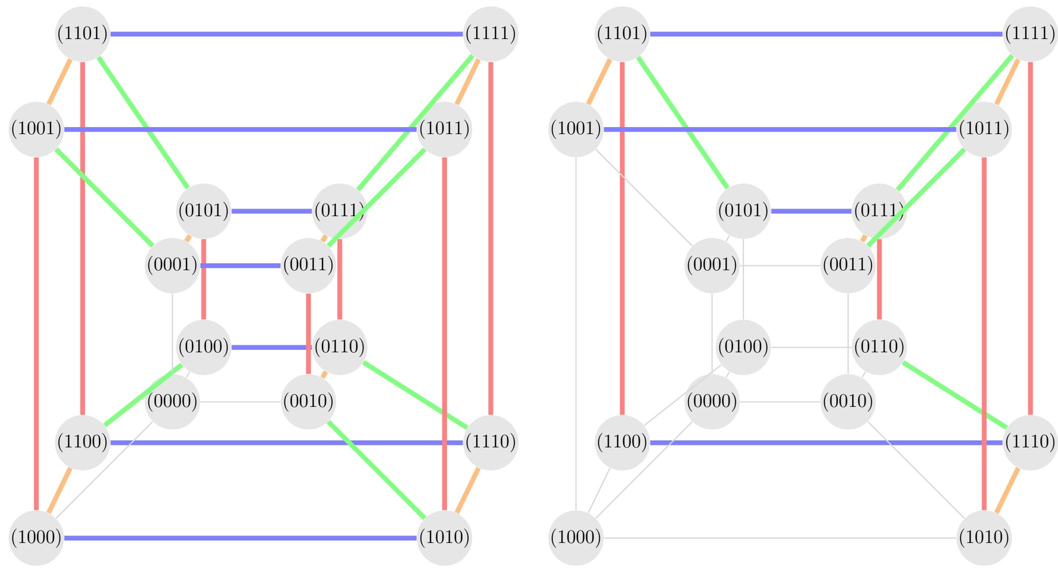


Figure 6.2: Kripke models M_{t_1} and M_{t_2} for the muddy children puzzle with $n = 4$ at time t_1 and t_2 respectively.

$(\mathsf{M}_{t_1}, (0100)) \vDash \mathcal{K}_2 p_2$, $(\mathsf{M}_{t_1}, (0010)) \vDash \mathcal{K}_3 p_3$ and $(\mathsf{M}_{t_1}, (0001)) \vDash \mathcal{K}_4 p_4$.

Notice that the public nature of the father's announcement not only let every child know that P, but it also let every child know that every child knows that P, but it also let every child know that every child knows that every child knows that P and so on. In one word, after the father speaks it becomes *common knowledge* that at least one child has a muddy forehead. To appreciate the difference between simple knowledge and common knowledge notice that in the states in which there are at least two muddy children, every child knows that at least one child has a muddy forehead even before the father speaks: however, at time t_1, the information that P becomes common knowledge among the children. This point is crucial for the next moments of the story.

Consider what happens at time t_2, that is, after all the children have answered 'No' to the father's first question. This information changes the state of knowledge. The result of this change can be represented through another model, M_{t_2}, which is exactly the same as M_{t_1}, except for the accessibility relations, which get 'truncated'. All the edges between all the nodes with exactly one 1 and all the nodes with exactly two 1 all disappear. M_{t_2} is represented in the second graph of Figure 6.2.

The justification goes as follows. If the actual situation were described by the tuple (1000), then before the father speaks child 1 would consider both (1000) and (0000) possible. Once the father speaks it is common knowledge that (0000) is not possible, so every child knows that child 1 would then know that the situation is described by (1000), that is to say, that her forehead is muddy. Once everyone

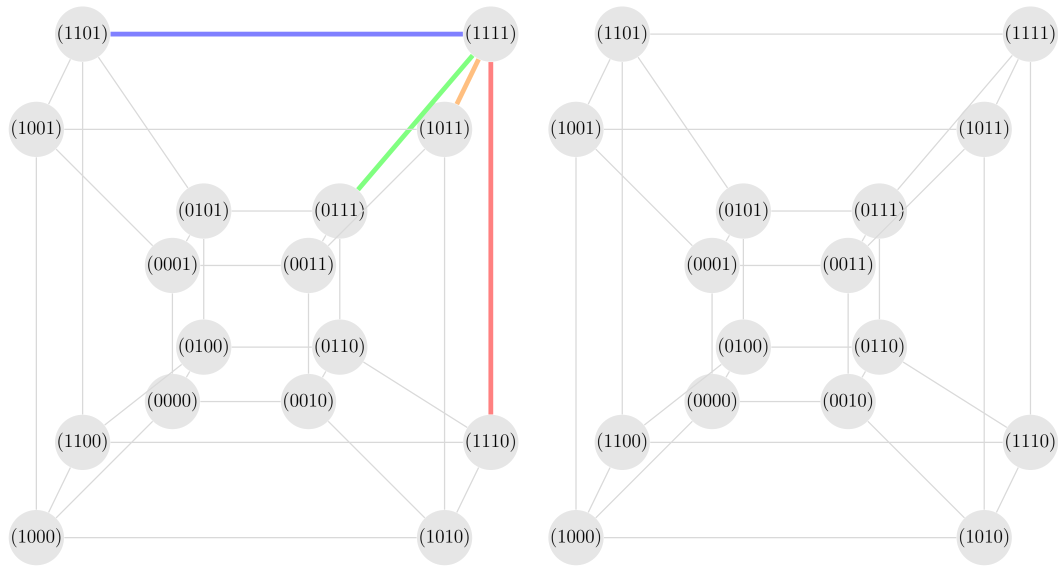


Figure 6.3: Kripke models M_{t_3} and M_{t_4} for the muddy children puzzle with $n = 4$ at time t_3 and t_4 respectively.

answers 'No' to the father's first question, it is common knowledge that the situation cannot be (1000), for otherwise child 1 would have spoken. The same goes for the other nodes with exactly one 1.

So, for example, $(\mathsf{M}_{t_2}, (1100)) \vDash \mathcal{K}_1 p_1$ and $(\mathsf{M}_{t_2}, (1100)) \vDash \mathcal{K}_2 p_2$: in state (1100), child 1 and child 2 recognize the fact that they are muddy. Both 1 and 2 answer 'No' to q_1, because of the mud on the other. But when 2 says 'No', 1 realizes that she must be muddy, for otherwise 2 would have known that the mud was on her forehead and would have answered 'Yes' the first time. 2 goes through the same reasoning. The same happens for all the other states in which there are exactly two muddy children.

As a result, it turns out that after all the children have answered 'No' to the father's first question, it is common knowledge that at least two children have muddy foreheads.

As Fagin, Halpern, Moses and Vardi (1995) point out, similar arguments can be used to show that the model M_{t_k} that characterizes the situation at time t_k (which is after the children have answered 'No' to the father's $k-1^{th}$ question) can be obtained from the model $\mathsf{M}_{t_{k-1}}$ that characterizes the situation at time t_{k-1} (which is after the children have answered 'No' to the father's $k-2^{th}$ question) simply disconnecting the nodes with at most $k-1$ ones from the rest of the graph. M_{t_3} and M_{t_4} are depicted in Figure 6.3. If, in some node s, it becomes common knowledge that a node t is impossible, then for every node u reachable from s, the edge from u to t is eliminated. So, at time t_k, that is, after $k-1$ rounds of questioning, it is common knowledge that at least k children have mud on their foreheads and all of them will recognize that they are dirty and will speak. This result holds for an arbitrary n.

The analysis of the muddy children puzzle through the means of classical epistemic logic S5 is made possible by a number of assumptions over and above the hypothesis explicitly stated in the story that 'the children are perceptive, intelligent and truthful'. Indeed, the analysis just concluded makes also use of the following assumptions:

1) It's common knowledge among the children that the father is truthful;

2) It's common knowledge among the children that every child is truthful;

3) It's common knowledge among the children that every child can and does see the others;

4) It's common knowledge among the children that every child can and does hear the father and the others;

5) It's common knowledge among the children that none of the children can see her forehead;

6) It's common knowledge among the children that none of the children tells the others whether they are muddy or not.

For suppose that 1) does not hold because the father is not truthful: then, at time t_1 the children cannot use the information that P. If 1) does not hold because the information that the father is truthful is not common knowledge, then at time t_2 and at state (1100), child 1 cannot realize that she is muddy: she cannot exclude that 2 has answered 'No' to q_1 because she lacks the information that P. Then, suppose that it's not common knowledge that every child can and does see the others, against 5). Again, at time t_2 and at state (1100), child 1 cannot realize that she is muddy, that is, she cannot exclude her epistemic alternative (0100): indeed, 2 could have answered 'No' to q_1 because 2 could have not seen that all the other children were clean. Similar arguments can be used to show that the other features are indeed assumed.

But, of course, the assumption that interests most us here is the following:

> **($\mathbf{LO_c}$)**: It's common knowledge that every individual is logically omniscient.

Roughly put, it is clear that if the children were not omniscient, they could not conclude that their forehead is muddy; at the same time, if their logical omniscience were not common knowledge, they could not employ the negative answer that the others give to the father as the sign that it is their forehead to be muddy. The precise result of weakening the (LO_c) assumption will be amply discussed in Section 6.2 (see especially Section 6.2.11).

6.1.3 Ideal and realistic agents

It is clear that classical epistemic logics, in so far as they validate the principle of logical omniscience, characterize a notion of knowledge which is possible only for ideal agents, viz. imaginary entities with unbounded resources, time, memory and computational capacities. It is a matter of fact that no concrete individual can know all the logical consequences of what she knows. For instance, an individual could know the rules of chess without knowing whether the White has a winning strategy or could know all the axioms of Peano arithmetic without knowing all of its theorems. The muddy children puzzle itself is a clear example of the unrealistic feature of this assumption: it is sufficient to take an arbitrary large number of muddy children k to imagine that concrete individuals will not be able to figure out that they 'knew' that their foreheads were muddy, even though in principle they have enough information to do so.

However, it is sometimes suggested[3] that the study of logically omniscient agents could be indirectly useful to understand the behavior of realistic individuals. First, any idealization could be interpreted as the equilibrium state that a certain system could attain without the pressure of external forces: this is the case, for example, of the study of frictionless planes in physics. Seen in this perspective, concrete individuals might fail to derive the consequences of what they know because of external obstacles, such as the arrival of new information, but, nevertheless, they tend toward the equilibrium at which they satisfy conditions of perfect rationality. Second, the behavior of ideal agents might be taken as the normative ideal that concrete agents should approximate: the divergence between ideal and real is interpreted as a deficiency that concrete individuals should minimize.

These motivations are not particularly convincing. The limits on computational resources cannot be simply considered as external forces: realistic agents not only are not omniscient, but also *cannot* be omniscient. In other words, the standard represented by ideal agents is a normative ideal that cannot be attained in practice. Some scholars go one step further and maintain that logical omniscience cannot even play the role of a regulative ideal, because it is not a desirable property:

> Even if a genie could grant us the capacity for arithmetical omniscience, it's not clear we'd have reason to accept it. Only a small number of the theorems are likely to be of any practical or theoretical use to us; why

[3]See Stalnaker (1991).

> must we clutter up our minds with all the rest? (MacFarlane, 2004, p. 11)

The so-called 'clutter avoidance principle', which has been put forward by Harman (1986) to support his rejection of the normative character of logic for human reasoning, holds that an agent should not clutter his mind with trivial consequences derived from the information she possesses. A similar insistence on the practical consequences of logical omniscience can be found in Gabbay and Woods (2003), who maintain that the logical validity of an inference is not a sufficient reason for its normativity:

> Consider the case in which John comes home and sees smoke pouring from an open door (S, for short). John then reasons as follow: 'Since S, then S or 2+2=4; since S or 2+2=4, then S or Copenhagen is the capital of Denmark'. Meanwhile John's house burns to the ground (Gabbay and Woods, 2003, p. 608).

Although the latter criticisms seem to be too extreme to be accepted[4] (for who would really deny the opportunity to be logically omniscient?), it should be clear at this point that classical epistemic logics can be profitably used to study neither directly nor indirectly realistic agents. This is what led numerous scholars to formulate nonstandard logics that aim at characterizing the kind of knowledge held by bounded individuals. We now consider three of them by way of example.

First, in order to avoid the assumptions of logical omniscience, Rantala (1982) modifies the notion of possible worlds and introduces, among the states, a set of worlds that are called 'impossible' or 'non-normal'[5], in which classical logical rules do not hold and everything can happen. For instance, it could be the case that in an impossible world two sentences are true, but their conjunction is not. The idea is that these worlds, although logically impossible, could be regarded as possible. Impossible worlds are interpreted as fictions created by the agents and, as such, are used only as epistemic alternatives, while the notions of logical consequence and validity are defined only on possible worlds. Here, the assumption of logical omniscience does not work. For suppose that C is a logical consequence of B and that an individual knows that B: in this case, B is true in every epistemic alternative of the individual, but in an impossible world C could be false, notwithstanding the truth of B.

Second, Fagin, Halpern and Vardi (1995) propose the 'Nonstandard Propositional System' (NPL) as an attempt to weaken the 'logical' aspects of logical omniscience. NPL modifies the classical notion of truth through a nonstandard

[4]It is not clear to us why a logically omniscient agent should have storage capacity problems and difficulties in recalling the information she needs.

[5]This notion is due to Hintikka (1975).

propositional logic that is strictly interwoven with relevant and four-values logics; at the same time, it preserves the traditional definition of knowledge as truth in all possible worlds. The result is a logical system that assumes the principles of logical omniscience, but these principles are now relative to a nonstandard logic.

Third, the 'systems of awareness' presented by Fagin and Halpern (1987) introduce a new modal operator for every agent of the system, where $C_i A$ means that 'agent i is aware that A'. The idea is that truth in every possible world is a necessary, but not a sufficient condition for knowledge: in order to know that certain sentence is true, agents must be aware of that sentence. As a result, two are the notions of knowledge for every agent proposed by these systems: on the one hand, 'implicit knowledge', which corresponds to the definition given by classical epistemic logics; on the other hand, 'explicit knowledge', which is defined through the notion of awareness. Depending on the restrictions imposed on the operator C, it is clearly possible to avoid the principles of logical omniscience.

In what follows, we are going to propose a nonstandard system to avoid the classical assumptions of logical omniscience. In particular, we shall follow Fagin, Halpern and Vardi (1995) in weakening the principles of logical omniscience by relying on a nonstandard propositional logic; at the same time, the systems we present can be interpreted as the result of imposing certain requirements on the notion of awareness introduced by Fagin and Halpern (1987).

6.1.4 Degrees of logical omniscience

At the beginning of this Chapter, we have seen that the principles of logical omniscience are epistemic consequences of the traditional tenet that logic is analytic and tautological. In agreement with the thesis of this work that logic is not analytic, we are now going to present a new family of logical systems, that we call 'Depth Bounded Epistemic Logics', where the classical assumptions of logical omniscience are rejected. How could this project be carried out? The answer to this question can be found in the basic idea of depth bounded logics.

On the one hand, we have seen in the previous Chapter that not every logical inference can be said to be analytic, because most of them require a remarkable computational effort; for this reason, we have assumed that syntheticity is a matter of degree and that every logical inference is characterized by a certain degree of syntheticity. On the other hand, if we move to the epistemic context, we could say that not every individual can be said to be logically omniscient, because most of them do not have the necessary resources to derive all the logical consequences of what they know; for this reason, we could assume that logical omniscience itself is a matter of degree and that every agent is characterized by a certain degree of logical omniscience. Intuitively, the idea is that an agent's ability of computing the information that she actually possesses can be given a gradual characterization:

the greater an agent's ability, the more similar to an ideal agent. In other words, the assumption of logical omniscience holding in classical epistemic logics:

> **(LO)** Individuals know all the logical consequences of what they know.

shall thus be replaced in Depth Bounded Epistemic Logics by this assumption:

> **($\mathbf{LO}_{\mathsf{DBELs}}$)** Every individual's degree of logical omniscience is at least k, for some fixed $k \in \mathbb{N}$.

How then could the 'degree of omniscience of an individual' be defined? Again, we turn to the notion of 'depth of an inference' put forward by D'Agostino and Floridi (2009) to obtain the following *equivalent* definitions:

> **(DO)** Agent i has a degree of logical omniscience equal to k or, equivalently, agent i's depth is k if and only if:
>
> - i can carry out all the propositional synthetic inferences of degree k;
> - i can carry out all the propositional inferences of depth k;
> - i can manage in her reasoning at most k nested pieces of virtual information;
> - i's propositional reasoning follows the consequence relation of depth k.

The relationship between the notions of 'depth of an inference' and of 'agent's depth' can be established as follows. While the depth of an inference is a measure of the *objective* difficulty of an inference, the agent's depth is a measure of the *subjective* capability of an individual. Notice that, although the two primitive notions of the informational semantics do have an epistemic flavour (see Section 5.1.2), Depth Bounded Boolean Logics are propositional logics: they lack both a definition of knowledge and a definition of the interaction between agents' knowledge. Nonetheless, these two features are necessary to analyze situations, such as that of the muddy children puzzle, focused on the notion of knowledge in group of agents.

As we have seen in Section 6.1.2, classical epistemic logics assume not only **(LO)**, but also:

> **($\mathbf{LO}_{\mathsf{c}}$)**: It's common knowledge that every individual is logically omniscient.

But also this hypothesis is an idealization. Therefore, in line with our handling of assumption **(LO)**, we are going drop assumption **($\mathbf{LO}_{\mathsf{c}}$)** and we are going to study each of the following cases:

(u) None of the individuals knows that every individual's depth is at least k, for some fixed $k \in \mathbb{N}$;
(e) Every individual knows that every individual's depth is at least k, for some fixed $k \in \mathbb{N}$;
(c) It's common knowledge that every individual's depth is at least k, for some fixed $k \in \mathbb{N}$.

According to (u), each agent ignores that the other agents' depth is at least k and assumes that it is zero. (e) assumes that every agent knows the other agents' depth, but does not know whether this is common knowledge or not: that is to say, the agents' depth has been privately announced. Therefore every agent, while knowing that the other agents' depth is k, assumes that the other agents do not have this information and that they assume that the other agents' depth is zero. According to (c), the agents' depth is common knowledge, that is to say, it has been publicly announced. Therefore every agent knows that the other agents' depth is at least k and knows that all the other agents have this piece of information.

6.2 Depth Bounded Epistemic Logics

6.2.1 The structure

Depth Bounded Epistemic Logics are both modal epistemic logics, because they characterize the notion of knowledge employing a semantics with accessibility relations and states, and depth bounded logics, because they characterize the notion of knowledge in a gradual way on the basis of the computational resources of the individuals and thus avoid the problem of logical omniscience. The structure of this family can be described as follows.

We propose three infinite hierarchies of logics: DBELu, DBELe and DBELc. Every hierarchy DBELx (where x is one of u, e and c) consists of an infinite chain of logics, which starts with a basic element, called DBELx_0, and continues with DBELx_1, DBELx_2, ..., DBELx_k, DBELx_{k+1}, ...

Every logic shares *with all the other logics in each hierarchy* the same definitions of language and model. In particular, it is useful to say in advance some features of these models. First, every model characterizes each agent with two parameters: her set of initial information and her depth. Second, the class of all the models is called $\mathcal{M}_0$. Third, for every $k \in \mathbb{N}$, $\mathcal{M}_k$ is the class of models in which each agent has at least depth k. Thus it follows that for every $k \in \mathbb{N}$, $\mathcal{M}_{k+1} \subseteq \mathcal{M}_k$.

We then define three notions of validity in a model: DBELu-validity in M, DBELe-validity in M and DBELc-validity in M. Every logic shares *with all the other logics in the same hierarchy* the same notion of validity in a model. The first notion of validity rests on the assumption (u); the second on (e); the third on (c).

Logics			Common notions		
DBEL	DBELu	DBELu$_0$	Language Model	DBELu validity in M	DBELu$_0$ validity
		DBELu$_1$			DBELu$_1$ validity
		DBELu$_{\dots}$			DBELu$_{\dots}$ validity
		DBELu$_n$			DBELu$_n$ validity
	DBELe	DBELe$_0$		DBELe validity in M	DBELe$_0$ validity
		DBELe$_1$			DBELe$_1$ validity
		DBELe$_{\dots}$			DBELe$_{\dots}$ validity
		DBELe$_n$			DBELe$_n$ validity
	DBELc	DBELc$_0$		DBELc validity in M	DBELc$_0$ validity
		DBELc$_1$			DBELc$_1$ validity
		DBELc$_{\dots}$			DBELc$_{\dots}$ validity
		DBELc$_n$			DBELc$_n$ validity

Figure 6.4: Notions shared by Depth Bounded Epistemic Logics.

The first hierarchy, DBELu, whose logics share the same notion of validity in a model, DBELu-validity in M, is construed as follows. Each logic in the hierarchy is characterized by a different notion of validity. In the basic logic of DBELu, called DBELu$_0$, the notion of validity, called DBELu$_0$-validity, is defined as DBELu-validity in all the models $\mathsf{M} \in \mathcal{M}_0$. Then, in the first logic of DBELu, called DBELu$_1$, the notion of validity, called DBELu$_1$-validity, is defined as DBELu-validity in all the models $\mathsf{M} \in \mathcal{M}_1$. In general, in the k^{th} logic of DBELu, called DBELu$_k$, the notion of validity, called DBELu$_k$-validity, is defined as DBELu-validity in all the models $\mathsf{M} \in \mathcal{M}_k$. It follows that every logic DBELu$_{k+1}$ is an extension of previous logic DBELu$_k$ in the hierarchy: the set of models for DBELu$_{k+1}$, $\mathcal{M}_{k+1}$, is a subset of the set of models for DBELu$_k$, $\mathcal{M}_k$ and the set of all the DBELu$_k$-valid formulae is a subset of the set of all the DBELu$_{k+1}$-valid formulae.

The other two hierarchies of logics are construed in the same way. Each logic in the hierarchy DBELe, while sharing the same notion of validity in a model with the other logics in the hierarchy, DBELe-validity in M, is characterized by a specific notion of validity, DBELe$_k$-validity. Every logic DBELe$_{k+1}$ is an extension of previous logic DBELe$_k$ in the hierarchy DBELe. And each logic in the hierarchy DBELc, while sharing the same notion of validity in a model with the other logics in the hierarchy, DBELc-validity in M, is characterized by a specific notion of validity, DBELc$_k$-validity. Every logic DBELc$_{k+1}$ is an extension of previous logic DBELc$_k$ in the hierarchy DBELc.

Each logic DBELx$_k$ is thus characterized by two assumptions. The first assumption is one of (u), (e) and (c) and indicates to which hierarchy DBELx$_k$ belongs. The second assumption, which determines at which level of the hierarchy DBELx$_k$

stands, regards the lowest depth admitted. To sum up:

> DBELx_k is the Depth Bounded Epistemic Logic, whose assumption regarding the knowledge of the agents' depth is (x), for (x) one of (u), (e) and (c) and whose assumption regarding the agents' depth is that every child's depth is at least k.

Last, Depth Bounded Epistemic Logics assume a prescriptive point of view, that is to say, they characterize the kind of knowledge that a rational individual could and ought to have given some initial information. This characterization accounts for the computational and cognitive limits of the agents involved and thus, unlike classical epistemic logics, it is not a normative ideal unattainable for realist agents. In particular, the assumptions of logical omniscience are unproblematic: they ask an individual to know the logical consequences that she is able to derive from what she knows in virtue of her computational resources. This characterization is neither descriptive: the focus is not on what realistic agents do actually know, but rather on what they could and thus ought to know.

6.2.2 Language and grammar

In this and in the following Sections (6.2.2, 6.2.3, 6.2.4 and 6.2.5), we introduce the notions of language and model for Depth Bounded Epistemic Logics paying specific attention to the comparison with the analogous definitions for classical epistemic logics on the one hand and Depth Bounded Boolean Logics on the other. All the definitions and the propositions of the Sections just mentioned are common to *every logic of each of the three hierarchies.*

The alphabet A and the language $\mathcal{L}$ for Depth Bounded Epistemic Logics are exactly the same as those for classical epistemic logic (see Section 6.1.1), except that the alphabet is augmented with a set of prefixes $\mathcal{T} = \{t, f\}$. These two notions will be referred to as **Definition 1** and **Definition 2** respectively. Moreover, we will use the following abbreviations:

1. $\neg^n$ is a shorthand notation for $\overbrace{\neg \ldots \neg}^{n \text{ times}}$ for any $n \in \mathbb{N}$.

2. LA is a shorthand notation for $\bigcup\limits_{m=1}^{n} \mathcal{A}^m = \mathcal{A} \times \mathcal{A}^2 \times \cdots \times \mathcal{A}^n = \mathcal{A} \cup \{\mathcal{A} \times \mathcal{A}\}$ $\cup \cdots \cup \{\mathcal{A} \times \cdots \times \mathcal{A}\}$. So for instance $(3) \in \mathsf{LA}$ and $(7, 4) \in \mathsf{LA}$. Intuitively, LA is the set containing all the singletons, couples, triples, ..., n-tuples of agents.

3. For every $i \in \mathcal{A}$, CAP_i is a shorthand notation for $(\wp(\mathcal{P}) \times \{i\}) \times (\wp(\mathcal{P}) \times \mathcal{A})$. So for instance $((\varnothing, 1), (\{p, q\}, 3)) \in \mathsf{CAP}_1$ and $((\{p, q\}, 3), (\varnothing, 1)) \in \mathsf{CAP}_3$.

Intuitively, for every agent i, CAP_i is a set of couples, whose elements are themselves couples. The first couple is composed by a subset of propositional parameters and agent i; the second couple is composed by a subset of propositional parameters and an agent in $\mathcal{A}$.

4. For every $i \in \mathcal{A}$, LAP_i is a shorthand notation for $\bigcup_{m=1}^{n} (\wp(\mathcal{P}) \times \{i\}) \times (\wp(\mathcal{P}) \times \mathcal{A})^m$. So for instance $((\{p, q\}, 3), (\varnothing, 1)) \in \mathsf{LAP}_3$ and $((\varnothing, 8), (\{q, r, s\}, 7), (\{r\}, 4)) \in \mathsf{LAP}_8$. Intuitively, for ever agent i, LAP_i is a set of n-tuples, whose first element is a couple composed by a subset of propositional parameters and agent i and the other elements are couples composed by a subset of propositional parameters and an agent in $\mathcal{A}$. As a result, for each $i \in \mathcal{A}$, $\mathsf{CAP}_i \subseteq \mathsf{LAP}_i$.

5. For any list of agents $(j_1, j_2, \ldots, j_m) \in \mathsf{LA}$, $\mathsf{K}_{(j_1,j_2,\ldots,j_m)}$ is a shorthand notation for $\mathcal{K}_{j_1}\mathcal{K}_{j_2}\ldots\mathcal{K}_{j_m}$. So for instance $\mathsf{K}_{(3)}$ abbreviates $\mathcal{K}_3$ and $\mathsf{K}_{(7,4)}$ abbreviates $\mathcal{K}_7\mathcal{K}_4$.

6. K is a shorthand notation for any (possibly empty) string $\mathcal{K}_{j_1}\mathcal{K}_{j_2}\ldots\mathcal{K}_{j_m}$ where $j_1, j_2, \ldots, j_m \in \mathcal{A}$.

7. L is a shorthand notation for any (possibly empty) string $\mathsf{L}_{j_1}\mathsf{L}_{j_2}\ldots\mathsf{L}_{j_m}$ where $j_1, j_2, \ldots, j_m \in \mathcal{A}$ and each L_{j_i} is one of $\mathcal{K}_{j_i}$ and $\neg\mathcal{K}_{j_i}$.

We prefer to shift from formulae to signed formulae, that is to say, expressions of the kind tB and fB where B is any sentence in $\mathcal{L}$. The symbols in $\mathcal{T}$ are not new logical operators, since they cannot be used inside a sentence, nor can they be iterated. They can only be used to prefix a sentence: they stand in front of a sentence and their scope is the entire sentence that follows. Their intuitive meaning is that the prefixed sentence is evaluated as true and false respectively. So for instance tB is read it's true that B and $f\mathcal{K}_iB$ is read it's false that i knows that B. Formally:

Definition 3 (Set of signed well-formed formulae). The set $\mathcal{L}^s$ of signed well-formed formulae based on $\mathcal{L}$ consists of all and only the expressions of the form sB such that $B \in \mathcal{L}$ and $s \in \mathcal{T}$. $\bar{s}B$ is the conjugate of sB, that is, fB if $s = t$ and tB if $s = f$.

6.2.3 Intermezzo I

Before introducing the notion of a model, we need some preliminary definitions.

Definition 4 (Coherent set). A coherent set Γ is a set of signed well-formed formulae ($\Gamma \subseteq \mathcal{L}^s$), such that for no $B \in \mathcal{L}$, sB and $\bar{s}B$ are both in Γ. $\mathbb{G}$ is the set of the coherent sets Γ.

Definition 5 (Admissible set). For any $B, C \in \mathcal{L}$ and for any $j \in \mathcal{A}$, a coherent set Γ is an admissible set if and only if Γ violates all of the following conditions:

1. $t\neg B \in \Gamma$ and $tB \in \Gamma$
2. $f\neg B \in \Gamma$ and $fB \in \Gamma$

3. $tB \vee C \in \Gamma$, $fB \in \Gamma$ and $fC \in \Gamma$
4. $fB \vee C \in \Gamma$ and $tB \in \Gamma$
5. $fB \vee C \in \Gamma$ and $tC \in \Gamma$

6. $tB \rightarrow C \in \Gamma$, $tB \in \Gamma$ and $fC \in \Gamma$
7. $fB \rightarrow C \in \Gamma$ and $fB \in \Gamma$
8. $fB \rightarrow C \in \Gamma$ and $tC \in \Gamma$

9. $fB \wedge C \in \Gamma$, $tB \in \Gamma$ and $tC \in \Gamma$
10. $tB \wedge C \in \Gamma$ and $fB \in \Gamma$
11. $tB \wedge C \in \Gamma$ and $fC \in \Gamma$

12. $t\mathsf{K}\mathcal{K}_j\mathsf{L}B \in \Gamma$ and $f\mathsf{KL}B \in \Gamma$
13. $t\mathcal{K}_jB \in \Gamma$ and $f\mathcal{K}_j\mathcal{K}_jB \in \Gamma$

$\mathbb{A} \subseteq \mathbb{G}$ is the set of the admissible sets Γ.

A coherent set $\Gamma \in \mathbb{G}$ is a set of sentences that is not immediately contradictory and an admissible set $\Gamma \in \mathbb{A}$ is a set of sentences which is not immediately inconsistent. The conditions listed above define in a negative way the informational meaning of the connectives and the operators following the main idea that grounds the second formulation of the informational semantics for Depth Bounded Boolean Logics presented in Section 5.1.2. In order to count as admissible, a set cannot include any instantiation of these conditions, because each of these instantiations is a case of explicit or analytic inconsistency, that is to say, an inconsistency that anyone, who understands the informational meaning of the connectives and of the operators, can and has to recognize as such. In other words, each condition represents an inconsistency that can be detected without drawing upon any reasoning based on virtual information. In particular, conditions 1.-11. determine the informational meaning of the connectives rephrasing the definition offered by D'Agostino and

Floridi (2009) for signed sentences and we argue that conditions 12. and 13. are natural completions of the previous ones in a modal epistemic context, as far as they define the informational meaning of the epistemic operators.

Let's start with condition 12. First, suppose that both K and L are empty. In this case, the requirement, saying that it is inadmissible a situation in which it's true that some agent j knows that B ($t\mathcal{K}_jB$) and it's false that B (fB), rules out as inadmissible the falsity of one's and others' knowledge. In this specific case, the analogy with the T axiom of classical epistemic logic is made evident. Axiom T, also known as the knowledge or truth axiom, characterizes the weakest system of epistemic interest: it demands the truth of what is known ($\mathcal{K}_iB \to B$) and corresponds to the reflexivity of the accessibility relation ($\forall s \in S, (s, s) \in R_i$). Now, requirement 12. is more general: it also classifies as inadmissible cases such as the one in which $t\mathcal{K}_1\mathcal{K}_2\neg\mathcal{K}_3B$ and $f\mathcal{K}_1\neg\mathcal{K}_3B$. If it's true that 1 knows that 2 knows that 3 does not know that B, then it's inadmissible that it's false that 1 knows that 3 does not know that B, and vice versa.

We turn now to condition 13. This requirement rules out as inadmissible situations in which it's true that some agent j knows that B ($t\mathcal{K}_jB$) and it's false that j knows that she knows that B ($f\mathcal{K}_j\mathcal{K}_jB$). This condition is analogous to the $S4$ axiom of classical epistemic logic. Axiom $S4$, also known as the positive introspection axiom, captures another desirable property of knowledge: it demands knowledge of what is known ($\mathcal{K}_iB \to \mathcal{K}_i\mathcal{K}_i$ B) and corresponds to the transitivity of the accessibility relation ($\forall s, t, u \in S$, if $(s, t) \in R_i$ and $(t, u) \in R_i$, then $(s, u) \in R_i$).

Notice that the property of negative introspection, for which an agent has to know that she doesn't know that B and which is captured by axiom $S5$ of classical epistemic logic ($\neg\mathcal{K}_iB \to \mathcal{K}_i\neg\mathcal{K}_iB$), is not incorporated into the inadmissibility requirement. Negative introspection does not contribute to define the informational meaning of the epistemic operator because it's a too strong requirement: it is indeed plausible that agent i has both the piece of information that it's false that j knows that B and that it's false that agent j knows that she doesn't know that B. However we will see that once agent i can derive from her set of information that $f\mathcal{K}_iB$, that is to say, once that she is aware of her ignorance regarding the truth value of B, i has to conclude that $t\mathcal{K}_i\neg\mathcal{K}_iB$.

Notice that a set Γ can be admissible without being classically consistent. For instance $\Lambda_1 = \{tp \wedge q, fp \vee q\}$ and $\Lambda_2 = \{tp \wedge q, t\neg p\}$ are admissible, because they do not satisfy any of the conditions above, although they are both classically inconsistent.

Proposition 1 establishes the properties of reflexivity and monotonicity for admissible sets.

Proposition 1. For any $\Gamma, \Delta \subseteq \mathcal{L}^s$ and $B \in \mathcal{L}$:

1. if $sB \in \Gamma$, then $\Gamma \cup \{\bar{s}B\} \notin \mathbb{A}$;
2. if $\Gamma \cup \{sB\} \notin \mathbb{A}$, then $\Gamma \cup \Delta \cup \{sB\} \notin \mathbb{A}$.

Definition 6 is essential to introduce the notion of an accessibility relation.

Definition 6 (Refinement of an admissible set).

1. For any $\Gamma \in \mathbb{A}$, the *set of the refinements of* Γ, R_Γ, is defined as follows:
$$\mathsf{R}_\Gamma = \{\Delta \in \mathbb{A} \mid \Gamma \subseteq \Delta\}$$
2. For any $\Gamma \in \mathbb{A}$ and for any (possibly empty) set of propositional parameters $J = \{q_1, \ldots, q_j\} \in \wp(\mathcal{P})$, the *set of the refinements of* Γ *on* J, R_Γ^J, is defined as follows:
$$\mathsf{R}_\Gamma^J = \{\Delta \in \mathsf{R}_\Gamma \mid s_1 q_1, \ldots, s_j q_j \in \Delta \text{ for } s_i \in \mathcal{T} \text{ and } q_i \in J\}$$
3. For any $\Gamma \in \mathbb{A}$ and for any (possibly empty) set of propositional parameters $J = \{q_1, \ldots, q_j\} \in \wp(\mathcal{P})$, Δ is a *minimal refinement of* Γ *on* J if and only if $\Delta \in \mathsf{R}_\Gamma^J$ and $|\Delta| \leq |\Lambda|$ for every $\Lambda \in \mathsf{R}_\Gamma^J$.
4. For any $k \in \mathbb{N}$, the *set of sets of refinements of admissible sets on at most* k *propositional parameters*, $\mathbb{R}^k$, is defined as follows:
$$\mathbb{R}^k = \{\mathsf{R}_\Delta^J \mid J \in \wp(\mathcal{P}), 0 \leq |J| \leq k \text{ and } \Delta \in \mathbb{A}\}$$

Definition 6.1 says that a refinement of an admissible set Γ is a superset of Γ which is itself admissible. So for instance, let $\Gamma = \{tp \wedge q\}$. Then, $\Delta_1 = \Gamma$, $\Delta_2 = \{tp \wedge q, fr\}$, $\Delta_3 = \{tp \wedge q, fp \vee q\}$ are refinements of Γ, while $\Delta_4 = \{tp \wedge q, fp\}$ and $\Delta_5 = \varnothing$ are not. Definition 6.2 says that a refinement of Γ on a set of propositional parameters J is an admissible superset of Γ which includes an evaluation of J. So for instance, for Γ as above and $J = \{r\}$, $\Delta_2 = \{tp \wedge q, fr\}$, $\Delta_6 = \{tp \wedge q, tr\}$ and $\Delta_7 = \{tp \wedge q, fr, tr \vee s\}$ are refinements of Γ on J, while $\Delta_8 = \{tp \wedge q, tr \vee r\}$ is not. Definition 6.3 says that the minimal refinements of Γ on a set of propositional parameters J are smallest refinements of Γ on a set of propositional parameters J. So for instance, for Γ and J as above, Δ_2 and Δ_6 are minimal, while Δ_7 is not. Definition 6.4 says that $\mathbb{R}^k$ is a set that consists of all and only the sets of refinements of admissible sets on at most k propositional parameters. So for instance, for Γ as before and $k = 1$, $\mathsf{R}_\Gamma^{\{r\}}$, $\mathsf{R}_\Gamma^{\{p\}}$ and $\mathsf{R}_{\Gamma\cup\{tp\}}^{\{q\}}$ are all included in $\mathbb{R}^1$, while $\mathsf{R}_\Gamma^{\{p,q\}}$ is not.

Proposition 2 establishes some properties regarding the refinements that are going to be used later on. In particular, it states the properties of reflexivity (1.), monotonicity (2. and 3.) and transitivity (4.).

Proposition 2. Let $J = \{q_1, \ldots, q_j\} \in \wp(\mathcal{P})$, $L = \{r_1, \ldots, r_l\} \in \wp(\mathcal{P})$ and $N = \{r_1, \ldots, r_l, r_{l+1}, \ldots, r_n\} \in \wp(\mathcal{P})$, where $L \subseteq N$. For any $\Gamma, \Gamma \cup \{sA\}, \Delta, \Lambda \in \mathbb{A}$ and for any $A \in \mathcal{L}$:

1. $\Gamma \in \mathsf{R}_\Gamma^\varnothing$.
2. If $\Delta \in \mathsf{R}_{\Gamma\cup\{sA\}}^J$, then $\Delta \in \mathsf{R}_\Gamma^J$.
3. If $\Delta \in \mathsf{R}_\Gamma^N$, then $\Delta \in \mathsf{R}_\Gamma^L$.
4. If $\Delta \in \mathsf{R}_\Gamma^J$ and $\Lambda \in \mathsf{R}_\Delta^L$, then $\Lambda \in \mathsf{R}_\Gamma^J$, $\Lambda \in \mathsf{R}_\Gamma^L$ and $\Lambda \in \mathsf{R}_\Gamma^{J\cup L}$.

6.2.4 Models

Given the preliminary definitions of the first intermezzo, we are now ready to introduce the notion of a model.

Definition 7 (Model, M). A model for n agents over $\mathcal{P}$ is a tuple $\mathsf{M} = (\mathbb{A}, \varphi^{\mathsf{M}}, \delta^{\mathsf{M}}, \mathsf{I}_1, \mathsf{I}_2, \ldots, \mathsf{I}_n)$, where:

1. $\mathbb{A}$ is the set of admissible sets;
2. $\varphi^{\mathsf{M}} : \mathcal{A} \longrightarrow \mathbb{A}$.
   φ^{M} is a function that assigns to each agent i an admissible set Γ. We will write φ_i^{M} as a shorthand for $\varphi^{\mathsf{M}}(i)$.
3. $\delta^{\mathsf{M}} : \mathcal{A} \longrightarrow \mathbb{N}$.
   δ^{M} is a function that assigns to each agent i a natural number k. We will write δ_i^{M} as a shorthand for $\delta^{\mathsf{M}}(i)$.
4. For all $i \in \mathcal{A}$, $\mathsf{I}_i = (\varphi_i^{\mathsf{M}}, \mathbb{R}^{\delta_i^{\mathsf{M}}})$.

We now explain the elements that contribute to the definition of a model for Depth Bounded Epistemic Logics.

For every $i \in \mathcal{A}$, φ_i^{M} is interpreted as the initial set of information available to i. The codomain of φ^{M} is $\mathbb{A}$, that is to say, the set of initial information that an agent is given has to be admissible. For suppose that an agent is given from some external source an inadmissible set of information: since it is assumed her understanding of the informational meaning of the connectives and of the operators, she is going to recognize the inadmissible set as such and, to quote Dummett (1991, p. 209), "once the contradiction has been discovered, no one is going to go *through* it: to exploit it to show that the train leaves at 11:52 or that the next Pope will be a woman".

We now start a comparison between epistemic classical logics and the proposed approach focusing first of all on the incompleteness of the information possessed by an agent. Epistemic classical logics give a characterization of knowledge of ideal agents who have incomplete information and unbounded computational power. The incompleteness of the information possessed by an agent is represented in classical epistemic logics by the individual's uncertainty in recognizing the actual world (see Section 6.1.1): possible worlds depict epistemic alternatives and the accessibility relation of an agent indicates which are the epistemic alternatives of that individual at a certain state. In Depth Bounded Epistemic Logics instead, the possible incompleteness of an agent's information is represented by the fact that her initial set of information is by no means bounded to include for each sentence $C \in \mathcal{L}$ either tC or fC. In other words, any admissible set can be represented by an evaluation that satisfies certain constraints: the incompleteness of an agent's information is given by the fact that such an evaluation is partial, that is to say, it allows epistemic gaps. These gaps are exactly the pieces of information that the agent does not actually possess.

The second point of the comparison between the classical approach and the present account concerns the evaluations. Each model for classical epistemic logics includes an evaluation v (see Section 6.1.1), which is relative to a certain world and which assigns to every propositional parameter a truth value. In our context however the focus is not on the alethic property of a certain proposition, but rather on its informational ones, that is to say, on its being or not really possessed by certain agents. This is the reason for which the classical evaluation v, which is relative to a certain model, is substituted in our account by a plenty of evaluations, each of which is not only relative to a certain model, but also to a specific agent. These evaluations are expressed as admissible sets determined by the function φ.

Then, for every $i \in \mathcal{A}$, δ_i^{M} represents agent's i depth according to M as defined and discussed in Section 5.1.2. As a result, for every model M, the functions φ^{M} and δ^{M} together give a complete characterization of the agents in $\mathcal{A}$. The idea that this construction tries to convey is that an agent's knowledge depends not only on the information that she has, as in classical epistemic logics, but also on her capacity to compute the initial information available to her. This crucial point is reflected by the definition of an agent's interpretation, which is completely determined by the two functions discussed above. For all $i \in \mathcal{A}$, the interpretation of agent i is given as a couple: its former element is the initial set of information available to i, φ_i^{M}; its latter element is the set of the accessibility relations of agent i, which consists of the set of sets of refinements on at most δ_i^{M} propositional parameters. In other words, an agent, whose depth is k, is and has to be able to access to all the refinements on at most k propositional parameters.

The last point of our comparison between epistemic classical logics and the present approach regards the accessibility relations. In epistemic classical logics, possible worlds and accessibility relations are used to describe an agent's ignorance; as we have seen, in Depth Bounded Epistemic Logics, an agent's epistemic gaps are conveyed by the possible incompleteness of the initial set of information. Thus, in the latter logics, the accessibility relations are employed to reach a different aim, that of characterizing the computational resources available to an individual.

Definition 8 introduces different classes of models and it is essential to present the extensions of the basis of each of the three hierarchies.

Definition 8 (Class of models, M_k). For every $k \in \mathbb{N}$, $\mathcal{M}_k$ is the class of models M satisfying the following condition:

$$\text{for every } i \in \mathcal{A},\ \delta_i^{\mathsf{M}} \geq k.$$

$\mathcal{M}_k$ is the class of models in which each agent's depth is at least k and it follows by definition that for every $k \in \mathbb{N}$, $\mathcal{M}_{k+1} \subseteq \mathcal{M}_k$ and $\mathcal{M}_k \subseteq \mathcal{M}_0$.

6.2.5 Intermezzo II

In this second intermezzo, we introduce two concepts that are needed to define the three notions of validity in a model on which the three hierarchies are grounded respectively.

Definition 9 specifies which are the information in a set Γ that are available to a list of agents g.

Definition 9 (Set of information in Γ available to g). For any $\Gamma \subseteq \mathcal{L}^s$, $B \in \mathcal{L}$, $g \in \mathsf{LA}$ and $n, m \in \mathbb{N}$:

$$\Sigma_g(\Gamma) = \{t\neg^{2m}B \mid t\mathsf{K}_g\neg^{2n}B \in \Gamma\} \cup \{f\neg^{2m}B \mid t\mathsf{K}_g\neg^{2n+1}B \in \Gamma\}$$

Consider the case in which Γ is the initial set of information of some agent i of some model M ($\Gamma = \varphi_i^{\mathsf{M}}$). $\Sigma_j(\varphi_i^{\mathsf{M}})$ is the set of information that agent i can legitimately assume that j possesses: in other words, $\Sigma_j(\varphi_i^{\mathsf{M}})$ represents i's set of information concerning j's information or the information that i knows that j

possesses. So, in general, $\Sigma_g(\Gamma)$ is the set of information that Γ allows to suppose that the list of agents g has. For instance, let $\Gamma = \{t\mathcal{K}_3\mathcal{K}_2 B, t\mathcal{K}_3\neg C\}$. Then, $t\mathcal{K}_2 B, t\neg C, fC, f\neg\neg C \in \Sigma_{(3)}(\Gamma)$ and $tB \in \Sigma_{(3,2)}(\Gamma)$.

Definition 10 introduces in a recursive way the notion of the analytic consequences of an admissible set.

Definition 10 (Set of analytic consequences of Γ). For any $\Gamma \subseteq \mathcal{L}^s$, the set of analytic consequences of Γ, $\mathsf{W}(\Gamma)$, is the subset of $\mathcal{L}^s$ recursively defined as follows:

- $\mathsf{W}_0(\Gamma) \quad = \Gamma$
- $$\begin{aligned}\mathsf{W}_{n+1}(\Gamma) = {} & \{sB \mid \mathsf{W}_n(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}\} \cup \\ & \cup \bigcup_{g \in \mathsf{LA}} \{t\mathsf{K}_g \neg^{2n} B \mid \Sigma_g(\mathsf{W}_n(\Gamma)) \cup \{f\neg^{2m}B\} \notin \mathbb{A}\} \cup \\ & \cup \bigcup_{g \in \mathsf{LA}} \{t\mathsf{K}_g \neg^{2n+1} B \mid \Sigma_g(\mathsf{W}_n(\Gamma)) \cup \{t\neg^{2m}B\} \notin \mathbb{A}\}\end{aligned}$$
- $\mathsf{W}(\Gamma) \quad = \bigcup_{n \in \mathbb{N}} \mathsf{W}_n(\Gamma)$

$\mathsf{W}(\Gamma)$ includes only sentences that follow from Γ in virtue of the informational meaning of the connectives and the operators under the assumption that the other agents are aware of this meaning too. Consider the recursive step of this definition ($\mathsf{W}_{n+1}(\Gamma)$). The first set of the union is construed on the basis of the single candidate principle put forward by D'Agostino (2010) and presented in Section 5.1.2: it includes those sentences whose conjugate added to $\mathsf{W}_n(\Gamma)$ makes the resulting set inadmissible. The second and the third sets of the union that define $\mathsf{W}_{n+1}(\Gamma)$ rest on the assumption that the other agents are aware of the informational meaning of the connectives and of the operators and that they use the single candidate principle too. One has to conclude that it is true that j knows that A ($\neg A$) if the falsity (resp. the truth) of A is immediately ruled out by the information that j is known to possess together with some of the accepted constraints that define the meaning of the logical operators.

Consider the set $\Lambda = \{tp \wedge q, t\neg p\}$. $\Lambda \in \mathbb{A}$ because Λ does not satisfy any of the inadmissibility conditions of Definition 5. However, $\mathsf{W}(\Lambda) \notin \mathbb{A}$. Indeed, $fp \in \mathsf{W}_1(\Lambda)$ because $\mathsf{W}_0(\Lambda) \cup \{tp\} \notin \mathbb{A}$ since $tp \in \mathsf{W}_0(\Lambda) \cup \{tp\}$ and $t\neg p \in \mathsf{W}_0(\Lambda) \cup \{tp\}$ satisfy the inadmissibility condition 1. Nonetheless, $\mathsf{W}_1(\Lambda) \notin \mathbb{A}$, because $fp \in \mathsf{W}_1(\Lambda)$ and $tp \wedge q \in \mathsf{W}_0(\Lambda) \subseteq \mathsf{W}_1(\Lambda)$ satisfy the inadmissibility condition 10. By the property of monotonicity of admissible set (Proposition 1.2) it follows that $\mathsf{W}(\Lambda) \notin \mathbb{A}$. This is an example of an admissible set whose set of analytic consequence is inadmissible and we say that sets of this kind are analytically inconsistent.

Proposition 3 establishes the property of reflexivity (1.), monotonicity (2.) and transitivity (3.) of W, while Proposition 4 states several facts, which are going to be useful later on, regarding some interactions between the structures we have seen so far.

Proposition 3. For any $\Gamma, \Delta \subseteq \mathcal{L}^s$ and $B, C \in \mathcal{L}$:

1. If $sB \in \Gamma$, then $sB \in \mathsf{W}(\Gamma)$
2. If $sB \in \mathsf{W}(\Gamma)$, then $sB \in \mathsf{W}(\Gamma \cup \Delta)$
3. If $sB \in \mathsf{W}(\Gamma)$ and $sC \in \mathsf{W}(\Gamma \cup \{sB\})$, then $sC \in \mathsf{W}(\Gamma)$

Proposition 4. For any $\Gamma \subseteq \mathcal{L}^s$, $B \in \mathcal{L}$, $n \in \mathbb{N}$ and $g \in \mathsf{LA}$:

1. If $\Gamma \notin \mathbb{A}$, then $\mathsf{W}(\Gamma) \notin \mathbb{A}$
2. If $\mathsf{W}(\Gamma) \notin \mathbb{A}$, then $sB \in \mathsf{W}(\Gamma)$ for any $B \in \mathcal{L}$
3. If $sB \in \mathsf{W}_n(\Gamma)$, then $sB \in \mathsf{W}_{n+1}(\Gamma)$
4. $\mathsf{W}(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ if and only if $sB \in \mathsf{W}(\Gamma)$.
5. $\mathsf{W}(\Gamma) = \mathsf{W}(\mathsf{W}(\Gamma))$
6. $\Sigma_g(\Gamma) \subseteq \mathsf{W}(\Gamma)$
7. If $sB \in \mathsf{W}(\Sigma_g(\Gamma))$, then, for $s = t$, $t\mathsf{K}_g B \in \mathsf{W}(\Gamma)$ and, for $s = f$, $t\mathsf{K}_g \neg B \in \mathsf{W}(\Gamma)$
8. If $sB \in \mathsf{W}(\Delta)$ for any minimal refinement Δ of Γ on J, then $sB \in \mathsf{W}(\Lambda)$ for every $\Lambda \in \mathsf{R}^J_\Gamma$.

Proposition 4.1 says that an inadmissible set is analytically inconsistent. Proposition 4.2 states that anything follows from an analytically inconsistent set. Propositions 4.3, 4.4 and 4.5 clarify relevant features of the structure of W. Propositions 4.6 and 4.7 deal with interaction between $\Sigma_g(\Gamma)$ and $\mathsf{W}(\Gamma)$: the former states that the set of information that Γ allows to suppose that the list of agents g has is a subset of the analytic consequences of Γ itself; the latter says that if a sentence belongs to the analytic consequences of the set of information that Γ allows to suppose that the list of agents g has, then the fact that g knows that sentence is an analytic consequence of Γ. Proposition 4.8, which is essential for the three notions of satisfiability that follow, establishes that a sentence, which is an analytic consequence of every minimal refinement of Γ on J, is an analytic consequence of any refinement of Γ on J.

6.2.6 The muddy children puzzle in Depth Bounded Epistemic Logics I

In this Section, we are going to see the definitions above at work in the formalization of the muddy children puzzle in the case in which there are four agents. The assumptions regarding $\mathcal{A}$ and $\mathcal{P}$ are the same as those made for the system $\mathsf{S5}$ (see Section 6.1.1). Now, consider the following sets of signed sentences:

$$\begin{aligned}
U = {} & \{tP,\ \ fp_2,\ \ fp_3,\ \ fp_4\} \\
X = {} & \{t\mathcal{K}_1 P,\ \ t\neg p_2 \to \mathcal{K}_1\neg p_2,\ \ t\mathcal{K}_1\neg p_3,\ \ t\mathcal{K}_1\neg p_4,\ \ f\mathcal{K}_1 p_1\} \\
Y = {} & \{t\mathcal{K}_2\mathcal{K}_1 P,\ \ t\mathcal{K}_2(\neg p_2 \to \mathcal{K}_1\neg p_2),\ \ t\neg p_3 \to \mathcal{K}_2\mathcal{K}_1\neg p_3,\ \ t\mathcal{K}_2\mathcal{K}_1\neg p_4, \\
& \ t\mathcal{K}_2\neg\mathcal{K}_1 p_1,\ \ f\mathcal{K}_2 p_2\} \\
Z = {} & \{t\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3 P,\ \ t\mathcal{K}_3\mathcal{K}_2(\neg p_2 \to \mathcal{K}_1\neg p_2),\ \ t\mathcal{K}_3(\neg p_3 \to \mathcal{K}_2\mathcal{K}_1\neg p_3), \\
& \ t\neg p_4 \to \mathcal{K}_3\mathcal{K}_2\mathcal{K}_1\neg p_4,\ \ t\mathcal{K}_3\mathcal{K}_2\neg\mathcal{K}_1 p_1,\ \ t\mathcal{K}_3\neg\mathcal{K}_2 p_2,\ \ f\mathcal{K}_3 p_3\}
\end{aligned} \tag{6.1}$$

Each set in (6.1) is both coherent and admissible, since none of the inadmissibility conditions is satisfied. Moreover, we have that:

$$tp_1 \in \mathsf{W}(U) \tag{6.2}$$

This result comes from the following steps:

- $t(p_1 \vee p_2) \vee p_3 \in \mathsf{W}_1(U)$ because $\Lambda_1 = \mathsf{W}_0(U) \cup \{f(p_1 \vee p_2) \vee p_3\} \notin \mathbb{A}$, since $tP, fp_4, f(p_1 \vee p_2) \vee p_3 \in \Lambda_1$ satisfying the inadmissibility condition 3;
- $tp_1 \vee p_2 \in \mathsf{W}_1(U)$ because $\Lambda_2 = \mathsf{W}_1(U) \cup \{fp_1 \vee p_2\} \notin \mathbb{A}$, since $t(p_1 \vee p_2) \vee p_3, fp_3, fp_1 \vee p_2 \in \Lambda_2$ satisfying the inadmissibility condition 3;
- $tp_1 \in \mathsf{W}_2(U)$ because $\Lambda_3 = \mathsf{W}_2(U) \cup \{fp_1\} \notin \mathbb{A}$, since $tp_1 \vee p_2, fp_2, fp_1 \in \Lambda_3$ satisfying the inadmissibility condition 3;
- $tp_1 \in \mathsf{W}(U)$

The following example is slightly more complex:

$$tp_2 \in \mathsf{W}(\Delta) \text{ for } \Delta = X \cup \{fp_2\} \tag{6.3}$$

- $t\neg p_2 \in \mathsf{W}_1(\Delta)$ because $\Lambda_1 = \mathsf{W}_0(\Delta) \cup \{f\neg p_2\} \notin \mathbb{A}$, since $fp_2, f\neg p_2 \in \Lambda_1$ satisfying the inadmissibility condition 2;
- $t\mathcal{K}_1((p_1 \vee p_2) \vee p_3) \in \mathsf{W}_1(\Delta)$ because $tP, fp_4 \in \Sigma_1(\mathsf{W}_0(\Delta))$ and $\Lambda_2 = \Sigma_1(\mathsf{W}_0(\Delta)) \cup \{f(p_1 \vee p_2) \vee p_3\} \notin \mathbb{A}$, since $tP, fp_4, f(p_1 \vee p_2) \vee p_3 \in \Lambda_2$ satisfying the inadmissibility condition 3;

- $t\mathcal{K}_1\neg p_2 \in \mathsf{W}_2(\Delta)$ because $\Lambda_3 = \mathsf{W}_0(\Delta) \cup \{f\mathcal{K}_1\neg p_2\} \notin \mathbb{A}$, since $t\neg p_2, t\neg p_2 \to \mathcal{K}_1\neg p_2, f\mathcal{K}_1\neg p_2 \in \Lambda_3$ satisfying the inadmissibility condition 6;

- $t\mathcal{K}_1(p_1 \vee p_2) \in \mathsf{W}_2(\Delta)$ because $t(p_1 \vee p_2) \vee p_3, fp_3 \in \Sigma_1(\mathsf{W}_1(\Delta))$ and $\Lambda_4 = \Sigma_1(\mathsf{W}_1(\Delta)) \cup \{fp_1 \vee p_2\} \notin \mathbb{A}$, since $t(p_1 \vee p_2) \vee p_3, fp_3, fp_1 \vee p_2 \in \Lambda_4$ satisfying the inadmissibility condition 3;

- $t\mathcal{K}_1 p_1 \in \mathsf{W}_3(\Delta)$ because $tp_1 \vee p_2, fp_2 \in \Sigma_1(\mathsf{W}_2(\Delta))$ and $\Lambda_5 = \Sigma_1(\mathsf{W}_2(\Delta)) \cup \{fp_1\} \notin \mathbb{A}$, since $tp_1 \vee p_2, fp_2, fp_1 \in \Lambda_5$ satisfying the inadmissibility condition 3;

- $t\mathcal{K}_1 p_1, f\mathcal{K}_1 p_1 \in \mathsf{W}(\Delta)$: this means that, although $\Delta \in \mathbb{A}$, $\mathsf{W}(\Delta) \notin \mathbb{A}$. By Proposition 4.2, it follows that $tp_2 \in \mathsf{W}(\Delta)$.

A model for the muddy children puzzle with four agents is a structure of the kind $\mathsf{M} = (\mathbb{A}, \varphi^{\mathsf{M}}, \delta^{\mathsf{M}}, \mathsf{I}_1, \mathsf{I}_2, \mathsf{I}_3, \mathsf{I}_4)$. Every model has to characterize the information and the computational resources of each child at a specific time t (see Section 6.1.1). Notice that although each agent can individually fail to recognize that a given situation is the actual one, the group of children has distributed knowledge of it: if they had been allowed to talk to each other pooling their initial set of information together, the children would have recognized the actual situation. As a result, every model for the muddy children puzzle describes a specific situation: however, this feature is not valid in general. We are going to consider the following situations, which do not exhaust all the possibilities: $s_A = (1000)$, $s_B = (1100)$, $s_C = (1110)$ and $s_D = (1111)$.

In Section 6.2.10, which is again devoted to the muddy children puzzle, we are going to work with the models presented below, which again do not exhaust all the possibilities.

$$
\begin{array}{lllll}
\mathsf{M}^1 & s_A & t_1 & \varphi_1^{\mathsf{M}^1} = U,\, tp_1 \in \varphi_z^{\mathsf{M}^1}\ \forall z \in \mathcal{A} - \{1\} & \delta_1^{\mathsf{M}^1} = 0 \\
\mathsf{M}^2 & s_B & t_2 & \varphi_2^{\mathsf{M}^2} = X,\, tp_2 \in \varphi_z^{\mathsf{M}^2}\ \forall z \in \mathcal{A} - \{2\} & \delta_1^{\mathsf{M}^2} = \delta_2^{\mathsf{M}^2} = \delta_3^{\mathsf{M}^2} = \delta_4^{\mathsf{M}^2} = 0 \\
\mathsf{M}^3 & s_B & t_2 & \varphi_2^{\mathsf{M}^3} = X,\, tp_2 \in \varphi_z^{\mathsf{M}^3}\ \forall z \in \mathcal{A} - \{2\} & \delta_2^{\mathsf{M}^3} = 1 \\
\mathsf{M}^4 & s_C & t_3 & \varphi_3^{\mathsf{M}^4} = Y,\, tp_3 \in \varphi_z^{\mathsf{M}^4}\ \forall z \in \mathcal{A} - \{3\} & \delta_1^{\mathsf{M}^4} = \delta_2^{\mathsf{M}^4} = \delta_3^{\mathsf{M}^4} = \delta_4^{\mathsf{M}^4} = 1 \\
\mathsf{M}^5 & s_C & t_3 & \varphi_3^{\mathsf{M}^5} = Y,\, tp_3 \in \varphi_z^{\mathsf{M}^5}\ \forall z \in \mathcal{A} - \{3\} & \delta_3^{\mathsf{M}^5} = 2,\, \delta_1^{\mathsf{M}^5} = \delta_2^{\mathsf{M}^5} = 0 \\
\mathsf{M}^6 & s_C & t_3 & \varphi_3^{\mathsf{M}^6} = Y,\, tp_3 \in \varphi_z^{\mathsf{M}^6}\ \forall z \in \mathcal{A} - \{3\} & \delta_3^{\mathsf{M}^6} = 2,\, \delta_2^{\mathsf{M}^6} = 1 \\
\mathsf{M}^7 & s_D & t_4 & \varphi_4^{\mathsf{M}^7} = Z,\, tp_4 \in \varphi_z^{\mathsf{M}^7}\ \forall z \in \mathcal{A} - \{4\} & \delta_1^{\mathsf{M}^7} = \delta_2^{\mathsf{M}^7} = \delta_3^{\mathsf{M}^7} = \delta_4^{\mathsf{M}^7} = 2 \\
\mathsf{M}^8 & s_D & t_4 & \varphi_4^{\mathsf{M}^8} = Z,\, tp_4 \in \varphi_z^{\mathsf{M}^8}\ \forall z \in \mathcal{A} - \{4\} & \delta_1^{\mathsf{M}^8} = \delta_2^{\mathsf{M}^8} = \delta_3^{\mathsf{M}^8} = 1,\, \delta_4^{\mathsf{M}^8} = 3 \\
\mathsf{M}^9 & s_D & t_4 & \varphi_4^{\mathsf{M}^9} = Z,\, tp_4 \in \varphi_z^{\mathsf{M}^9}\ \forall z \in \mathcal{A} - \{4\} & \delta_2^{\mathsf{M}^9} = 1,\, \delta_3^{\mathsf{M}^9} = 2,\, \delta_4^{\mathsf{M}^9} = 3
\end{array}
\tag{6.4}
$$

Consider the first line of (6.4), which describes *only the relevant* features of M^1. M^1 is a model that characterizes situation s_A at time t_1, because child 1's initial set of information is U (see (6.1)) and the other children have the information that

tp_1. Since at time t_1 the information that at least one of the children is muddy has been publicly announced, we have that $tP \in \varphi_1^{\mathsf{M}^1}$. $fp_2, fp_3, fp_4 \in \varphi_1^{\mathsf{M}^1}$ because it is assumed that each child can see the others. Moreover, in M^1, child 1's depth is zero. The same goes for the other models.

So, for instance, M^6 is a model that characterizes situation s_C at time t_3, because child 3's initial set of information is Y (see (6.1)) and the other children have the information that tp_3. $t\mathcal{K}_2\mathcal{K}_1 P \in \varphi_3^{\mathsf{M}^6}$, because at time t_1 the information that at least one of the children is muddy has been publicly announced; $t\mathcal{K}_2(\neg p_2 \to \mathcal{K}_1\neg p_2), t\neg p_3 \to \mathcal{K}_1\mathcal{K}_2\neg p_3, t\mathcal{K}_2\mathcal{K}_1\neg p_4 \in \varphi_3^{\mathsf{M}^6}$ because of the assumption that it is common knowledge that each child can see the others; $t\mathcal{K}_2\neg\mathcal{K}_1 p_1 \in \varphi_3^{\mathsf{M}^6}$ because at time t_2 child 1 has publicly announced her ignorance about p_1 and $f\mathcal{K}_2 p_2 \in \varphi_3^{\mathsf{M}^6}$ because at time t_3 child 2 has publicly announced her ignorance about p_2. Notice that $\varphi_3^{\mathsf{M}^6}$ includes only those formulae, which are *relevant* for child 3's reasoning. Moreover, in M^6, child 3's depth is two and child 2's depth is one.

6.2.7 Three notions of validity in a model

In classical epistemic logics (see Section 6.1.1), once the notion of model $\mathsf{M} = (S, v, R_1, \ldots, R_n)$ has been given, it is defined what it means for a formula to hold at some state s in a model M. A formula is then said to be satisfiable in M if it holds at some state s of M and it is said to be valid in M if it holds at all state s of M. The definitions below for Depth Bounded Epistemic Logics follow the same path, except for two fundamental elements.

First, for what has been said in Section 6.2.4 concerning the comparison between models for classical epistemic logics and models for Depth Bounded Epistemic Logics, instead of the definition of holding at some state in a model, we need to determine the notion of holding in some interpretation in a model. The shift from states in a model to interpretations in a model is a coherent consequence of the shift of the focus from the alethic property of sentences to the informational ones. We are no more interested of what it means for a formula to hold at a certain state: our aim here is to define what it means for a formula to hold in the interpretation of an agent.

Second, instead of *one* notion of validity in a model, we define simultaneously *three* notions of validity in a model (Definition 15): DBELu-validity in a model, DBELe-validity in a model and DBELc-validity in a model.

The first notion is shared by all and only the logics of the first hierarchy DBELu and rests on the definition of DBELu-holding at some interpretation I_i (Definition 11): DBELu-validity in a model characterizes those situations in which every agent ignores the other agents' depth according to assumption (u) as clarified in Section 6.2.1. As a result, the only kinds of reasoning that an agent i can and ought to carry out are the following ones:

> **(W)** Analytical reasoning;
> **(V)** Synthetic reasoning according to i's depth.

The second notion of validity in a model, namely DBELe-validity in a model, is shared by all and only the logics of the second hierarchy DBELe and rests on the definition of DBELe-holding at some interpretation I_i (Definition 12): DBELe-validity in a model characterizes those situations in which every agent knows the other agents' depth according to assumption (e) as clarified in Section 6.2.1. As a result, the only kinds of reasoning that an agent i can and ought to carry out are **(W)**, **(V)** and

> **(S)** Synthetic reasoning that simulates the synthetic reasoning held by any agent j according to j and i's depth.

The third notion of validity in a model, namely DBELc-validity in a model, is shared by all and only the logics of the third hierarchy DBELc and rests on the definition of DBELc-holding at some interpretation I_i (Definition 13): DBELc-validity in a model characterizes those situations in which the agents' depth is common knowledge according to assumption (c) as clarified in Section 6.2.1. As a result, the only kinds of reasoning that an agent i can and ought to carry out are **(W)**, **(V)**, **(S)** and

> **(C)** Synthetic reasoning that simulates the synthetic reasoning held by any agent j that simulates the synthetic reasoning held by any agent k and so on, according to the agents involved and i's depth.

The formal definitions go as follows:

Definition 11 (Set of formulae DBELu-satisfied by the interpretation I_i, $Cnu(\mathsf{I}_i)$). For any model $\mathsf{M} \in \mathcal{M}_0$ and interpretation $\mathsf{I}_i \in \mathsf{M}$, the *set of formulae* DBELu-*satisfied by* I_i, $Cnu(\mathsf{I}_i)$, is recursively defined as follows:

- $Cnu_0(\mathsf{I}_i) \quad = \varphi_i^{\mathsf{M}}$
- $Cnu_{n+1}(\mathsf{I}_i) = \mathsf{W}(Cnu_n(\mathsf{I}_i)) \cup \mathsf{V}(Cnu_n(\mathsf{I}_i))$
- $Cnu(\mathsf{I}_i) \quad = \bigcup_{n \in \mathbb{N}} Cnu_n(\mathsf{I}_i)$

where, for every $n \in \mathbb{N}$, $\mathsf{V}(Cnu_n(\mathsf{I}_i))$ consists of all and only formulae of the kind $t\mathcal{K}_i B$ such that there exists some $J \in \wp(\mathcal{P})$ for which:

1. $|J| \leq \delta_i^{\mathsf{M}}$ and
2. for any $\Delta \in \mathsf{R}^J_{Cnu_n(\mathsf{I}_i)}$, $tB \in \mathsf{W}(\Delta)$.

For any model $\mathsf{M} \in \mathcal{M}_0$, interpretation $\mathsf{I}_i \in \mathsf{M}$ and formula $B \in \mathcal{L}$, we say that I_i DBELu-*satisfies* sB or equivalently that sB DBELu-*holds at* I_i if and only if $sB \in Cnu(\mathsf{I}_i)$.

The basic step, $Cnu_0(\mathsf{I}_i)$, is the initial set of information given to i in M. Consider the recursive step, $Cnu_{n+1}(\mathsf{I}_i)$. The former element of the union, $\mathsf{W}(Cnu_n(\mathsf{I}_i))$, is the set of the analytic consequences of the previous step which formally translates the kind of reasoning described in **(W)**.

The latter element of the union, $\mathsf{V}(Cnu_n(\mathsf{I}_i))$, is the set of the synthetic consequences of the previous step that i can and has to derive in virtue of her depth, that is, the kind of reasoning indicated by **(V)**. A formula $t\mathcal{K}_iB$ belongs to $\mathsf{V}(Cnu_n(\mathsf{I}_i))$ if and only if tB is an analytic consequence of all the refinements of $Cnu_n(\mathsf{I}_i)$ on a certain set of propositional parameters J, whose cardinality cannot be greater than i's depth. The idea behind this definition is that in order for B to be known by agent i there should exists a set J which, no matter how its formulae are evaluated, allows i to derive by analytical means that B is true from $Cnu_n(\mathsf{I}_i)$. Intuitively, J represents the set of virtual information needed to derive B from $Cnu_n(\mathsf{I}_i)$. Notice that $|J|$ cannot be greater than δ_i^{M}: this clause makes sure that i's set of accessibility relations $\mathbb{R}^{\delta_i^{\mathsf{M}}}$ includes $\mathsf{R}^J_{\varphi_i^{\mathsf{M}}}$ and $\mathsf{R}^J_{Cnu_k(\mathsf{I}_i)}$ for any $k \in \mathbb{N}$. Moreover, according to Proposition 4.8, in order to check whether a formula tB is an analytical consequence of all the refinements of a certain $Cnu_k(\mathsf{I}_i)$ on a given J, it is sufficient to check whether tB is an analytical consequence of all the *minimal* refinements of Γ on J. The last step of the definition assures that $Cnu(\mathsf{I}_i)$ is closed under the operations described by the recursive step.

Definition 12 (Set of formulae DBELe-satisfied by the interpretation I_i, $Cne(\mathsf{I}_i)$). For any model $\mathsf{M} \in \mathcal{M}_0$ and interpretation $\mathsf{I}_i \in \mathsf{M}$, the *set of formulae* DBELe-*satisfied by* I_i, $Cne(\mathsf{I}_i)$, is recursively defined as follows:

- $Cne_0(\mathsf{I}_i) \quad = \varphi_i^{\mathsf{M}}$
- $Cne_{n+1}(\mathsf{I}_i) = \mathsf{W}(Cne_n(\mathsf{I}_i)) \cup \mathsf{SV}(Cne_n(\mathsf{I}_i))$
- $Cne(\mathsf{I}_i) \quad = \bigcup_{n \in \mathbb{N}} Cne_n(\mathsf{I}_i)$

where, for every $n \in \mathbb{N}$, $\mathsf{SV}(Cne_n(\mathsf{I}_i))$ consists of all and only formulae of the kind $t\mathcal{K}_iB$ such that there exists some $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$ for which:

1. $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$
   $\delta_i^{\mathsf{M}} \geq |L| + |J|$
   $\delta_j^{\mathsf{M}} \geq |J|$

2. $\exists A_2 \in \mathcal{L} \Big(tB \in \mathsf{W}(Cne_n(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\}) \wedge \forall \Lambda \in \mathsf{R}^L_{Cne_n(\mathsf{I}_i)} \Big(tA_2 \in \mathsf{W}(\Lambda) \vee \exists A_1 \in \mathcal{L} \Big(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_j A_1\}) \wedge \forall \Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))} \Big(tA_1 \in \mathsf{W}(\Delta)\Big)\Big)\Big)\Big)$.

For any model $\mathsf{M} \in \mathcal{M}_0$, interpretation $\mathsf{I}_i \in \mathsf{M}$ and formula $B \in \mathcal{L}$, we say that I_i DBELe-*satisfies* sB or equivalently that sB DBELe-*holds at* I_i if and only if $sB \in Cne(\mathsf{I}_i)$.

The latter element of the union of the recursive step, $\mathsf{SV}(Cne_n(\mathsf{I}_i))$, contains both the synthetic consequences that i can and has to derive in virtue of her depth (the kind of reasoning described by **(V)**) and the synthetic consequences that i can and has to derive simulating the synthetical reasoning that any agent is granted to perform in virtue of her depth, which is known by i (the kind of reasoning described by **(S)**).

In order for $t\mathcal{K}_i B$ to be included in $\mathsf{SV}(Cne_n(\mathsf{I}_i))$ the following conditions have to hold. The second requirement says that, for some A_2 (which can well be B itself), tB follows analytically from $Cne_n(\mathsf{I}_i)$ together with the fact that i knows that A_2 and for every refinement Λ of $Cne_n(\mathsf{I}_i)$ on L, one of the following two holds. Either tA_2 is an analytical consequence of Λ (in which case tB is a synthetic consequence of the information i has); or, (and in this case i simulates j's synthetic reasoning) for some A_1 (which again can well be B itself), tA_2 analytically follows from Λ together with the fact that j knows that A_1 and for all the refinements Δ of the set of information that Λ allows to suppose that j has, $\Sigma_j(\mathsf{W}(\Lambda))$, tA_1 is an analytical consequence of Δ. The first condition establishes that: first, i's depth has to be greater than j's depth, that is the depth of the agent whose synthetic reasoning is simulated by i; second, i's depth has to be greater than the sum of the cardinalities of J and L, which represent the sets of virtual information on which i refines and j refines respectively; third, j's depth has to be greater than the cardinality of J, on which j refines. This clause makes sure that i's set of accessibility relations $\mathbb{R}^{\delta_i^{\mathsf{M}}}$ includes $\mathsf{R}^L_{Cne_k(\mathsf{I}_i)}$ for any $k \in \mathbb{N}$ and $\mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}$ for any Λ in any $\mathsf{R}^L_{Cne_k(\mathsf{I}_i)}$. Moreover, according to Proposition 4.8, in order to check whether a formula tB is an analytical consequence of all the refinements of a certain set on a given J, it is sufficient to check whether tB is an analytical consequence of all the *minimal* refinements of that set on J.

Definition 13 (Set of formulae DBELc-satisfied by the interpretation I_i, $Cnc(\mathsf{I}_i)$). For any model $\mathsf{M} \in \mathcal{M}_0$ and interpretation $\mathsf{I}_i \in \mathsf{M}$, the *set of formulae* DBELc-*satisfied by* I_i, $Cnc(\mathsf{I}_i)$, is recursively defined as follows:

- $Cnc_0(\mathsf{I}_i) \quad = \varphi_i^{\mathsf{M}}$
- $Cnc_{n+1}(\mathsf{I}_i) = \mathsf{W}(Cnc_n(\mathsf{I}_i)) \cup \mathsf{CSV}(Cnc_n(\mathsf{I}_i))$

- $Cnc(\mathsf{I}_i) \quad = \bigcup_{n\in\mathbb{N}} Cnc_n(\mathsf{I}_i)$

where, for every $n \in \mathbb{N}$, $\mathsf{CSV}(Cnc_n(\mathsf{I}_i))$ consists of all and only formulae of the kind $t\mathcal{K}_iB$ such that there exists some $\mathbf{J} = ((L,i),(J_1,j_1),\ldots,(J_n,j_n)) \in \mathsf{LAP}_i$ for which:

1. $\delta_i^{\mathsf{M}} \geq \delta_{j_1}^{\mathsf{M}}$
   $\delta_i^{\mathsf{M}} \geq |L|$
   $\forall m = 1,\ldots,n,\ \delta_m^{\mathsf{M}} \geq \sum_{k=m}^{n} |J_k|$;
2. and for $A_n \in \mathcal{L}$:
   $\exists A_{n+1}\Big(tB \in \mathsf{W}(Cnc_n(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge \forall\Delta_0 \in \mathsf{R}^{L}_{Cnc_n(\mathsf{I}_i)}\Big(tA_{n+1} \in \mathsf{W}(\Delta_0)$
   $\vee$
   $\exists A_n\Big(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1}A_n\}) \wedge \forall\Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}\Big(tA_n \in \mathsf{W}(\Delta_1) \vee$
   $\vdots$
   $\exists A_3\Big(tA_4 \in \mathsf{W}(\Delta_{n-3} \cup \{t\mathcal{K}_{j_{n-2}}A_3\}) \wedge \forall\Delta_{n-2} \in \mathsf{R}^{J_{n-2}}_{\Sigma_{j_{n-2}}(\mathsf{W}(\Delta_{n-3}))}\Big(tA_3 \in \mathsf{W}(\Delta_{n-2}) \vee$
   $\exists A_2\Big(tA_3 \in \mathsf{W}(\Delta_{n-2} \cup \{t\mathcal{K}_{j_{n-1}}A_2\}) \wedge \forall\Delta_{n-1} \in \mathsf{R}^{J_{n-1}}_{\Sigma_{j_{n-1}}(\mathsf{W}(\Delta_{n-2}))}\Big(tA_2 \in \mathsf{W}(\Delta_{n-1}) \vee$
   $\exists A_1\Big(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n}A_1\}) \wedge \forall\Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}\Big(tA_1 \in \mathsf{W}(\Delta_n)\Big)\Big)$
   $\ldots\Big)$.

For any model $\mathsf{M} \in \mathcal{M}_0$, interpretation $\mathsf{I}_i \in \mathsf{M}$ and formula $B \in \mathcal{L}$, we say that I_i DBELc-*satisfies* sB or equivalently that sB DBELc-*holds at* I_i if and only if $sB \in Cnc(\mathsf{I}_i)$.

The latter element of the union of the recursive step, $\mathsf{CSV}(Cnc_n(\mathsf{I}_i))$, contains the synthetic consequences that i can and has to derive in virtue of her depth (the kind of reasoning described by **(V)**), the synthetic consequences that i can and has to derive simulating the synthetical reasoning that any other agent is granted to perform in virtue of her depth, which is known by i (see **(S)**), and the synthetic consequences that i can and has to derive simulating the synthetical reasoning of any other agent j_1 who is simulating the synthetic reasoning of any other agent j_2 and so on (see **(C)**). The dimension of this chain of simulations depends in the first place on i's depth and also on the other agents' depth, which is common knowledge among the individuals. This definition is just an extension of the previous one for $\mathsf{SV}(Cne_n(\mathsf{I}_i))$.

Proposition 5. For any $\mathsf{I}_i = (\varphi_i^{\mathsf{M}}, \mathbb{R}^{\delta_i^{\mathsf{M}}})$ in any model $\mathsf{M} \in \mathcal{M}_0$, let:

- $\mathsf{M}^1 \in \mathcal{M}_0$ be exactly the same as M except for the fact that $\mathsf{I}_i^1 \in \mathsf{M}^1 \neq \mathsf{I}_i \in \mathsf{M}$ since $\varphi_i^{\mathsf{M}^1} = \varphi_i^{\mathsf{M}} \cup \{sA\}$ for some $A \in \mathcal{L}$, although $\mathbb{R}^{\delta_i^{\mathsf{M}^1}} = \mathbb{R}^{\delta_i^{\mathsf{M}}}$.
- $\mathsf{M}^2 \in \mathcal{M}_0$ be exactly the same as M except for the fact that $\mathsf{I}_i^2 \in \mathsf{M}^2 \neq \mathsf{I}_i \in \mathsf{M}$ since $\mathbb{R}^{\delta_i^{\mathsf{M}^2}} = \mathbb{R}^{\delta_i^{\mathsf{M}}+1}$, although $\varphi_i^{\mathsf{M}^2} = \varphi_i^{\mathsf{M}}$.

Fix x as one of u, e, c. Then, for any $B \in \mathcal{L}$:

1. If $sB \in \varphi_i^{\mathsf{M}}$, then $sB \in Cnx(\mathsf{I}_i)$
2. If $sB \in Cnx(\mathsf{I}_i)$, then $sB \in Cnx(\mathsf{I}_i^1)$
3. If $sB \in Cnx(\mathsf{I}_i)$, then $sB \in Cnx(\mathsf{I}_i^2)$

This proposition is valid for each of the three definitions of DBELx-holding at some interpretation. Proposition 5.1 states that the initial information that i has in M is included in set of formulae DBELx-satisfied by her interpretation. Proposition 5.2 shows that every sentence DBELx-satisfied by i's interpretation in M is also DBELx-satisfied by i's interpretation in M^1, where i is given a richer initial set of information than before. Proposition 5.3 shows that every sentence DBELx-satisfied by i's interpretation in M is also DBELx-satisfied by i's interpretation in M^2, where i is assigned a greater depth than before.

We conclude this Section with the simultaneous formal definition of the three notions of DBELx-satisfiability in a model and DBELx-validity in a model.

Definition 14 (sB is DBELx-satisfiable in M). Fix x as one of u, e, c. For any model $\mathsf{M} \in \mathcal{M}_0$ and formula $sB \in \mathcal{L}^s$, we say that sB *is* DBELx-*satisfiable in* M if and only if, for some $\mathsf{I}_i \in \mathsf{M}$, $sB \in Cnx(\mathsf{I}_i)$.

Definition 15 (sB is DBELx-valid in M, $sB \in Cnx(\mathsf{M})$). Fix x as one of u, e, c. For any model $\mathsf{M} \in \mathcal{M}_0$ and formula $sB \in \mathcal{L}^s$, we say that sB *is* DBELx *valid in* M, written $sB \in Cnx(\mathsf{M})$, if and only if, for every $\mathsf{I}_i \in \mathsf{M}$, $sB \in Cnx(\mathsf{I}_i)$.

6.2.8 Infinite notions of validity and the logics DBELx$_k$

Each logic DBELx$_k$ shares *with all of the other Depth Bounded Epistemic Logics* the same notions of language and models and *with all of the other logics of the same hierarchy* the same notion of validity in a model. What distinguishes DBELx$_k$ from the other logics of the same hierarchy is, of course, a notion of logical consequence and a derived concept of validity. We simultaneously define all of these notions as follows:

Definition 16 (sB is a DBELx_k-consequence of Γ, $sB \in Cnx^k(\Gamma)$). Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For any $sB \in \mathcal{L}^s$ and $\Gamma \in \mathbb{A}$, we say that sB *is a* DBELx_k*-consequence of* Γ, written $sB \in Cnx^k(\Gamma)$, if and only if, for all $\mathsf{M} \in \mathcal{M}_k$, if $\Gamma \subseteq Cnx(\mathsf{M})$, then $sB \in Cnx(\mathsf{M})$.

Definition 17 (sB is DBELx_k-valid, $sB \in Cnx^k(\varnothing)$). Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For any $sB \in \mathcal{L}^s$, we say that sB *is* DBELx_k*-valid*, written $sB \in Cnx^k(\varnothing)$, if and only if, for all $\mathsf{M} \in \mathcal{M}_k$, $sB \in Cnx(\mathsf{M})$.

We now establish some properties of the relations defined above:

Proposition 6. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For any $sB \in \mathcal{L}^s$ and $\Gamma \in \mathbb{A}$, sB is a DBELx_k-consequence of Γ if and only if for all $\mathsf{M} \in \mathcal{M}_k$ and for all $\mathsf{I}_i \in \mathsf{M}$, if $\Gamma \subseteq Cnx(\mathsf{I}_i)$, then $sB \in Cnx(\mathsf{I}_i)$.

Proposition 6 intuitively means that sB is a DBELx_k-consequence of Γ if and only if any agent, whose interpretation is included in any model of the class $\mathcal{M}_k$, is able to derive sB from Γ under the assumption (x).

The following Proposition 7 shows that each notion of DBELx_k-consequence, Cnx^k, is Tarskian: it is reflexive, monotonic and transitive.

Proposition 7. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For all $\Gamma, \Gamma \cup \{sA\} \in \mathbb{A}$ and $sA, sB \in \mathcal{L}^s$:

1. If $sB \in \Gamma$, then $sB \in Cnx^k(\Gamma)$
2. If $sB \in Cnx^k(\Gamma)$, then $sB \in Cnx^k(\Gamma \cup \{sA\})$
3. If $sA \in Cnx^k(\Gamma)$ and $sB \in Cnx^k(\Gamma \cup \{sA\})$, then $sB \in Cnx^k(\Gamma)$

Proposition 8. Fix x as one of u, e, c. There are no DBELx_0-valid formulae.

Proposition 8 states that, for x as any of u, e, c, DBELx_0 has not valid formulae (tautologies). This is a feature that the notions of DBELx_0 share with the basic logic of the hierarchy of Depth Bounded Boolean Logics, $\vDash_0$. D'Agostino (2010)'s explanation for the lack of tautologies in his system can be rephrased to justify the inexistence of DBELx_0-valid formulae:

> $\vDash_0$ [read: DBELx_0], like Belnap's four-valued logic and the NPL system of Fagin, Halpern and Vardi (1995), has no tautologies. This is not surprising, however, since a tautology is a sentence that is a 'logical

> consequence of the empty set of assumptions' and so, in order to establish its truth [...], we must make essential use of virtual information, to the effect that the information state itself cannot be of depth 0 [read: an agent whose depth is zero is not able to derive it].

Proposition 9 formally expresses the principle of logical omniscience, for which if tB is a DBELx_k-consequence of Γ and i, whose depth is at least k, knows the sentences in Γ, then i knows that B is true under the assumption (x).

Proposition 9. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For any $B \in \mathcal{L}$, $\Gamma \in \mathbb{A}$, $\mathsf{M} \in \mathcal{M}_k$ and $\mathsf{I}_i \in \mathsf{M}$, let $\mathcal{K}_i\Gamma = \{t\mathcal{K}_iC \mid tC \in \Gamma\} \cup \{t\mathcal{K}_i\neg C \mid fC \in \Gamma\}$. Then: if $tB \in Cnx^k(\Gamma)$ and $\mathcal{K}_i\Gamma \subseteq Cnx(\mathsf{I}_i)$, then $t\mathcal{K}_iB \in Cnx(\mathsf{I}_i)$.

D'Agostino and Floridi (2009) have shown that the property of logical omniscience of classical logic turns out to be problematic precisely because the consequence relation of classical logic is intractable. And they have also shown that the intractability of classical propositional logic is determined by an unbounded use of virtual information. Now, since it makes use of *bounded* virtual information, each DBELx_k is tractable: this points out that the property of logical omniscience in DBELx_k is not problematic, in that it simply asks agents to know all the consequences of what they know, which they are able to derive in virtue of their computational resources.

6.2.9 Relationships between the logics DBELx_k

The relationships between Depth Bounded Epistemic Logics are summarized by the following propositions:

Proposition 10. Fix x as one of u, e, c. For all $k \in \mathbb{N}$, $\Gamma \in \mathbb{A}$ and $B \in \mathcal{L}$: if $sB \in Cnx^k(\Gamma)$, then $sB \in Cnx^{k+1}(\Gamma)$.

Proposition 11. For all $\Gamma \in \mathbb{A}$ and for all $k \in \mathbb{N}$:

1. $Cnu^k(\Gamma) \subseteq Cne^k(\Gamma) \subseteq Cnc^k(\Gamma)$
2. $Cnu^0(\Gamma) = Cne^0(\Gamma) = Cnc^0(\Gamma)$

Proposition 10 concerns the relationship between logics of the *same* hierarchy. It states that every logic DBELx_{k+1} is an extension of previous logic DBELx_k in the hierarchy: the set of models for DBELx_{k+1}, $\mathcal{M}_{k+1}$, is a subset of the set of models for DBELx_k, $\mathcal{M}_k$ and, for every $\Gamma \in \mathbb{A}$, the set of all the DBELx_k-consequences of Γ is a subset of the set of all the DBELx_{k+1}-consequences of Γ. Intuitively, this means that a formula true for all the agents with depth at least k is true for all the agents with depth at least $k+1$, under the same assumption (x).

Proposition 11 deals with the relationship between logics of *different* hierarchies. Proposition 11.1 says that formula DBELu_k-valid is also DBELe_k-valid and that a formula DBELe_k-valid is also DBELc_k-valid. The former part intuitively means that a formula true for all agents that do not know the other agents' depth is also true for all agents that know the other agents' depth, under the same assumption that the agents' depth is at least k. The latter part intuitively means that a formula true for all agents that know the other agents' depth is also true for all agents among which the other agents' depth is common knowledge, under the same assumption that the agents' depth is at least k.

Proposition 11.2 says that the basic logics of the three hierarchies are the same logic, which is defined in three different ways. This result corresponds to the intuitions. The former part, which says that $Cnu^0(\Gamma) = Cne^0(\Gamma)$, means that that a formula true for all agents, whose depth is zero, that do not know the other agents' depth is also true for all agents, whose depth is zero, that know the other agents' depth. This is because an agent, who can reason only analytically, cannot simulate any synthetic reasoning of any other individual: thus, the information that she has concerning the other agents depth is useless. The latter part, which says that $Cne^0(\Gamma) = Cnc^0(\Gamma)$, means that that a formula true for all agents, whose depth is zero, that know the other agents' depth is also true for all agents, whose depth is zero, under the assumption that the agents' depth is common knowledge. This is because an agent, who can reason only analytically, cannot simulate any synthetic reasoning of any other agent who is simulating any other synthetic reasoning: thus, the information that the agents' depth is common knowledge is useless.

6.2.10 The muddy children puzzle in Depth Bounded Epistemic Logics II

In order to clarify the definitions of the previous Sections, we consider again the muddy children puzzle with four agents. As far as the set of formulae DBELx-satisfied by an interpretation is concerned (Def. 11, Def. 12 and Def. 13), we have that:

$$tp_1 \in Cnu(\mathsf{I}_1^1) \text{ for } \mathsf{I}_1^1 \in \mathsf{M}^1 \tag{6.5}$$

$$tp_2 \notin Cnu(\mathsf{I}_2^2) \text{ for } \mathsf{I}_2^2 \in \mathsf{M}^2 \tag{6.6}$$

$$tp_2 \in Cnu(\mathsf{I}_2^3) \text{ for } \mathsf{I}_2^3 \in \mathsf{M}^3 \tag{6.7}$$

$$tp_3 \notin Cne(\mathsf{I}_3^4) \text{ for } \mathsf{I}_3^4 \in \mathsf{M}^4 \tag{6.8}$$

$$tp_3 \notin Cne(\mathsf{I}_3^5) \text{ for } \mathsf{I}_3^5 \in \mathsf{M}^5 \tag{6.9}$$

Depth Bounded Epistemic Logics				
Hierarchy DBELu		Hierarchy DBELe		Hierarchy DBELc
$\mathsf{DBELu_0}$	$=$	$\mathsf{DBELe_0}$	$=$	$\mathsf{DBELc_0}$
$\cap$		$\cap$		$\cap$
$\mathsf{DBELu_1}$	$\subset$	$\mathsf{DBELe_1}$	$\subset$	$\mathsf{DBELc_1}$
$\cap$		$\cap$		$\cap$
$\mathsf{DBELu_2}$	$\subset$	$\mathsf{DBELe_2}$	$\subset$	$\mathsf{DBELc_2}$
$\cap$		$\cap$		$\cap$
$\vdots$		$\vdots$		$\vdots$
$\cap$		$\cap$		$\cap$
$\mathsf{DBELu_n}$	$\subset$	$\mathsf{DBELe_n}$	$\subset$	$\mathsf{DBELc_n}$
$\cap$		$\cap$		$\cap$
$\vdots$		$\vdots$		$\vdots$

Figure 6.5: The relationships between Depth Bounded Epistemic Logics.

$$tp_3 \in Cne(\mathsf{I}_3^6) \text{ for } \mathsf{I}_3^6 \in \mathsf{M}^6 \tag{6.10}$$
$$tp_4 \notin Cnc(\mathsf{I}_4^7) \text{ for } \mathsf{I}_4^7 \in \mathsf{M}^7 \tag{6.11}$$
$$tp_4 \notin Cnc(\mathsf{I}_4^8) \text{ for } \mathsf{I}_4^8 \in \mathsf{M}^8 \tag{6.12}$$
$$tp_4 \in Cnc(\mathsf{I}_4^9) \text{ for } \mathsf{I}_4^9 \in \mathsf{M}^9 \tag{6.13}$$

(6.5) is justified as follows. We have seen that $\varphi_1^{\mathsf{M}^1} = U$ (6.4) and that $tp_1 \in \mathsf{W}(U)$ (6.2). By Def. 11, we have that $\varphi_1^{\mathsf{M}^1} = U = Cnu_0(\mathsf{I}_1^1)$ and that $tp_1 \in \mathsf{W}(U) \subseteq Cnu_1(\mathsf{I}_1^1) \subseteq Cnu(\mathsf{I}_1^1)$. (6.5) shows that the interpretation of child 1 in model M^1 DBELu-satisfies tp_1, because tp_1 analytically follows from 1's initial set of information. (6.7) is a case of synthetic reasoning, which results from the following steps.

- $X = \varphi_2^{\mathsf{M}^3} = Cnu_0(\mathsf{I}_2^3)$ because of (6.4) and Def. 11
- $t\mathcal{K}_2p_2 \in \mathsf{V}(Cnu_0(\mathsf{I}_2^3)) \subseteq Cnu_1(\mathsf{I}_2^3)$ because for $J = \{p_2\}$, we have that:
  1) $|J| = 1$ is equal to $\delta_2^{\mathsf{M}^3} = 1$ (see (6.4))
  2) $tp_2 \in \mathsf{W}(\Lambda)$ for $\Lambda = X \cup \{tp_2\}$ by Prop. 3.1
  3) $tp_2 \in \mathsf{W}(\Delta)$ for $\Delta = X \cup \{fp_2\}$ because of (6.3)
  4) Λ and Δ are the only minimal refinements of $Cnu_0(\mathsf{I}_2^3)$ on J
  5) for all $\Phi \in \mathsf{R}^J_{Cnu_0(\mathsf{I}_2^3)}$, $tp_2 \in \mathsf{W}(\Phi)$ because of 2), 3), 4) and Prop. 4.8.
- $tp_2 \in \mathsf{W}(Cnu_1(\mathsf{I}_2^3)) \subseteq Cnu_2(\mathsf{I}_2^3)$ because $\Sigma = Cnu_1(\mathsf{I}_2^3) \cup \{fp_2\} \notin \mathbb{A}$, since $t\mathcal{K}_2p_2, fp_2 \in \Sigma$ satisfying the inadmissibility condition 12.

- $tp_2 \in Cnu(\mathsf{I}_2^3)$

(6.6) is a case of a failure of a synthetic reasoning due to the fact that $\delta_2^{\mathsf{M}^2} = 0$. In other words, the interpretation of child 2 defined in model M^2 cannot DBELu-satisfy tp_2, because 2 can reason only analytically, while in order to conclude that tp_2 it is necessary to use a virtual information, as (6.7) shows.

(6.10) can be obtained noticing that first, $Cne_0(\mathsf{I}_3^6) = \varphi_3^{\mathsf{M}^6} = Y$; second, $t\mathcal{K}_3p_3 \in \mathsf{SV}(Y)$ and thus $tp_3 \in Cne(\mathsf{I}_3^6)$. The second step is justified as follows. For $\mathbf{J} = ((\{p_3\}, 3), (\{p_2\}, 2)) \in \mathsf{CAP}_3$ we have that:

1. $\delta_3^{\mathsf{M}^6} = 2$ is greater than $\delta_2^{\mathsf{M}^6} = 1$; $\delta_3^{\mathsf{M}^6} = 2$ is equal to $|\{p_3\}| + |\{p_2\}| = 2$; $\delta_2^{\mathsf{M}^6} = 1$ is equal to $|\{p_2\}| = 1$

2. and for $p_i \in \mathcal{L}$:
   for $p_3(tp_3 \in \mathsf{W}(Y \cup \{t\mathcal{K}_3p_3\}) \wedge \forall\Delta \in \mathsf{R}_Y^{\{p_3\}}(tp_3 \in \mathsf{W}(\Delta)\vee$
   for $p_2(tp_3 \in \mathsf{W}(\Delta \cup \{t\mathcal{K}_2p_2\}) \wedge \forall\Lambda \in \mathsf{R}_{\Sigma_2(\mathsf{W}(\Delta))}^{\{p_2\}}(tp_2 \in \mathsf{W}(\Lambda)))))$

The last fact can be shown through the following steps:

1. $\Delta_1 = Y \cup \{tp_3\}$ and $\Delta_2 = Y \cup \{fp_3\}$ are the only minimal refinements in $\mathsf{R}_Y^{\{p_3\}}$

2. $\Lambda_1 = \Sigma_2(\mathsf{W}(\Delta_2)) \cup \{tp_2\}$ and $\Lambda_2 = \Sigma_2(\mathsf{W}(\Delta_2)) \cup \{fp_2\}$ are the only minimal refinements in $\mathsf{R}_{\Sigma_2(\mathsf{W}(\Delta_2))}^{\{p_2\}}$

3. $tp_3 \in \mathsf{W}(Y \cup \{t\mathcal{K}_3p_3\})$ because $\Psi_1 = Y \cup \{t\mathcal{K}_3p_3\} \cup \{fp_3\} \notin \mathbb{A}$ since $t\mathcal{K}_3p_3, fp_3 \in \Psi_1$ satisfying condition 12

4. $tp_3 \in \mathsf{W}(\Delta_1)$

5. $tp_3 \in \mathsf{W}(\Delta_2 \cup \{t\mathcal{K}_2p_2\})$ because $t\mathcal{K}_2p_2, f\mathcal{K}_2p_2 \in \Delta_2 \cup \{t\mathcal{K}_2p_2\}$ and Prop. 4.2

6. $tp_2 \in \mathsf{W}(\Lambda_1)$

7. $tp_2 \in \mathsf{W}(\Lambda_2)$

(6.10) shows an example of an agent who reasons synthetically simulating another agent who is reasoning synthetically. (6.8) is a case of failure of a simulation of a synthetic reasoning due to the fact that $\delta_3^{\mathsf{M}^4} = 1$. In other words, the interpretation of child 3 defined in model M^4 cannot DBELe-satisfy tp_3 because child 3 can use at most one piece of nested virtual information, while in order to conclude that tp_3 it is necessary to employ two pieces of nested virtual information, the second of which is used to simulate child 2's synthetic reasoning. (6.9) is another case of a failure, which is now determined by the fact that $\delta_2^{\mathsf{M}^5} = 0$. In other words, the interpretation of child 3 cannot DBELe-satisfy tp_3 because in model M^4 child 2 can only reason analytically: as a result, child 3, who is informed of 2's depth, cannot use the public announcement of child 2 that 2 doesn't know whether she is muddy or not to rule out the possibility that 3 is not muddy. For, if child 3 had been not muddy, child 2, because of her depth, wouldn't have known that 2 was muddy.

(6.13) can be obtained noticing that first, $Cnc_0(\mathsf{I}_4^9) = \varphi_4^{\mathsf{M}^9} = Z$; second, $t\mathcal{K}_4 p_4 \in \mathsf{CSV}(Z)$ and thus $tp_4 \in Cnc(\mathsf{I}_4^9)$. The second step is justified as follows. For $\mathbf{J} = ((\{p_4\}, 4), (\{p_3\}, 3), (\{p_2\}, 2)) \in \mathsf{LAP}_4$ we have that:

1. $\delta_4^{\mathsf{M}^9} = 3$ is greater than $\delta_3^{\mathsf{M}^9} = 2$; $\delta_4^{\mathsf{M}^9} = 3$ is greater than $|\{p_4\}| = 1$; $\delta_3^{\mathsf{M}^9} = 2$ is equal to $|\{p_3\}| + |\{p_2\}| = 2$; $\delta_2^{\mathsf{M}^9} = 1$ is equal to $|\{p_2\}| = 1$
2. and for $p_i \in \mathcal{L}$:
   for $p_4(tp_4 \in \mathsf{W}(Z \cup \{t\mathcal{K}_4 p_4\}) \wedge \forall\Phi \in \mathsf{R}_Z^{\{p_4\}}(tp_4 \in \mathsf{W}(\Phi) \vee$
   for $p_3(tp_4 \in \mathsf{W}(\Phi \cup \{t\mathcal{K}_3 p_3\}) \wedge \forall\Delta \in \mathsf{R}_{\Sigma_3(\mathsf{W}(\Phi))}^{\{p_3\}}(tp_3 \in \mathsf{W}(\Delta) \vee$
   for $p_2(tp_3 \in \mathsf{W}(\Delta \cup \{t\mathcal{K}_2 p_2\}) \wedge \forall\Lambda \in \mathsf{R}_{\Sigma_2(\mathsf{W}(\Delta))}^{\{p_2\}}(tp_2 \in \mathsf{W}(\Lambda))))))$

The last fact can be shown through the following steps:

1. $\Phi_1 = Z \cup \{tp_4\}$ and $\Phi_2 = Z \cup \{fp_4\}$ are the only minimal refinements in $\mathsf{R}_Z^{\{p_4\}}$
2. $\Delta_1 = \Sigma_3(\mathsf{W}(\Phi_2)) \cup \{tp_3\}$ and $\Delta_2 = \Sigma_3(\mathsf{W}(\Phi_2)) \cup \{fp_3\}$ are the only minimal refinements in $\mathsf{R}_{\Sigma_3(\mathsf{W}(\Phi_2))}^{\{p_3\}}$
3. $\Lambda_1 = \Sigma_2(\mathsf{W}(\Delta_2)) \cup \{tp_2\}$ and $\Lambda_2 = \Sigma_2(\mathsf{W}(\Delta_2)) \cup \{fp_2\}$ are the only minimal refinements in $\mathsf{R}_{\Sigma_2(\mathsf{W}(\Delta_2))}^{\{p_2\}}$
4. $tp_4 \in \mathsf{W}(Z \cup \{t\mathcal{K}_4 p_4\})$ because $\Psi_1 = Z \cup \{t\mathcal{K}_4 p_4\} \cup \{fp_4\} \notin \mathbb{A}$, since $t\mathcal{K}_4 p_4, fp_4 \in \Psi_1$ satisfying condition 12
5. $tp_4 \in \mathsf{W}(\Phi_1)$
6. $tp_4 \in \mathsf{W}(\Phi_2 \cup \{t\mathcal{K}_3 p_3\})$ because $t\mathcal{K}_3 p_3, f\mathcal{K}_3 p_3 \in \Phi_2 \cup \{t\mathcal{K}_3 p_3\}$ and Prop. 4.2
7. $tp_3 \in \mathsf{W}(\Delta_1)$
8. $tp_3 \in \mathsf{W}(\Delta_2 \cup \{t\mathcal{K}_2 p_2\})$ because $f\mathcal{K}_2 p_2 \in \Sigma_3(\mathsf{W}(\Phi_2))$, $t\mathcal{K}_2 p_2, f\mathcal{K}_2 p_2 \in \Delta_2 \cup \{t\mathcal{K}_2 p_2\}$ and Prop. 4.2
9. $tp_2 \in \mathsf{W}(\Lambda_1)$
10. $tp_2 \in \mathsf{W}(\Lambda_2)$

(6.13) shows an example of an agent who reasons synthetically simulating another first agent who is reasoning synthetically simulating another second agent who is reasoning synthetically. (6.11) is a case of failure of a simulation of a synthetic reasoning due to the fact that $\delta_4^{\mathsf{M}^7} = 2$. In other words, the interpretation of child 4 defined in model M^7 cannot DBELc-satisfy tp_4 because child 4 can use at most two piece of nested virtual information, while in order to conclude that tp_3 it is necessary to employ three pieces of nested virtual information, the second of which is used to simulate child 3's synthetic reasoning and the third of which is used to simulate child 3's reasoning that simulates child 2's reasoning. (6.12) is another case of a failure, which is now determined by the fact that $\delta_2^{\mathsf{M}^8} = \delta_3^{\mathsf{M}^8} = 1$. In other words, the interpretation of child 4 cannot DBELc-satisfy tp_4 because in model M^8 child 3 can only use one piece of virtual information and thus cannot simulate 2's synthetic reasoning: as a result, child 4, who is informed of 3's depth, cannot use the public announcement of child 3 that 3 doesn't know whether she is muddy or not to rule out the possibility that 4 is not muddy. For, if child 4 had been not muddy, child 3, because of her depth, wouldn't have known that 3 was muddy.

As far as DBELx-satisfiability and DBELx-validity in a model are concerned (Def. 14 and Def. 15), we have that:

$$tp_1 \text{ is DBELu-satisfied by } \mathsf{M}^1 \text{ and } tp_1 \in Cnu(\mathsf{M}^1) \tag{6.14}$$

$$tp_2 \text{ is DBELu-satisfied by } \mathsf{M}^2 \text{ and } tp_2 \notin Cnu(\mathsf{M}^2) \tag{6.15}$$

$$tp_3 \text{ is DBELe-satisfied by } \mathsf{M}^6 \text{ and } tp_3 \in Cne(\mathsf{M}^6) \tag{6.16}$$

$$tp_3 \text{ is DBELe-satisfied by } \mathsf{M}^5 \text{ and } tp_3 \notin Cne(\mathsf{M}^5) \tag{6.17}$$

$$tp_4 \text{ is DBELc-satisfied by } \mathsf{M}^9 \text{ and } tp_4 \in Cnc(\mathsf{M}^9) \tag{6.18}$$

$$tp_4 \text{ is DBELc-satisfied by } \mathsf{M}^8 \text{ and } tp_4 \notin Cnc(\mathsf{M}^8) \tag{6.19}$$

(6.14) shows an example of a formula that is both DBELu-satisfied in a model (see (6.5)) and DBELu-valid in a model (see (6.5) and (6.4)). (6.15) shows a case in which a formula is DBELu-satisfied in a model, since for instance $tp_2 \in Cnu(\mathsf{I}_1^2)$ (see (6.4)), but it is not DBELu-valid in a model (see (6.6)). Similarly for the other couples of cases proposed.

Then, regarding the notion of DBELx$_k$-consequence relation (Def. 16), we can state several facts:

$$tp_1 \in Cnu^0(U) \tag{6.20}$$

$$tp_2 \notin Cnu^0(X) \tag{6.21}$$

$$tp_2 \in Cnu^1(X) \tag{6.22}$$

$$tp_3 \notin Cnu^k(Y) \text{ for any } k \in \mathbb{N} \tag{6.23}$$

$$tp_3 \in Cne^2(Y) \tag{6.24}$$

$$tp_3 \notin Cne^1(Y) \tag{6.25}$$

$$tp_4 \notin Cne^k(Z) \text{ for any } k \in \mathbb{N} \tag{6.26}$$

$$tp_4 \in Cnc^3(Z) \tag{6.27}$$

$$tp_4 \notin Cnc^2(Z) \tag{6.28}$$

(6.20) is an example of a DBELu_0-consequence: $\forall \mathsf{M} \in \mathcal{M}_0$ and $\forall \mathsf{I}_i \in \mathsf{M}$, if $U \subseteq Cnu(\mathsf{I}_i)$, then for some $n \in \mathbb{N}$, $U \subseteq Cnu_n(\mathsf{I}_i)$. And, since $tp_1 \in \mathsf{W}(U)$ (see (6.2)), $tp_1 \in \mathsf{W}(Cnu_n(\mathsf{I}_i)) \subseteq Cnu_{n+1}(\mathsf{I}_i) \subseteq Cnu(\mathsf{I}_i)$. So $tp_1 \in Cnu^0(U)$. (6.21) is an example of a non DBELu_0-consequence, which is justified by (6.4), for which we have that $\mathsf{M}^2 \in \mathcal{M}_0$, and by the fact that $X \subseteq Cnu(\mathsf{I}^2_2)$ and $tp_2 \notin Cnu(\mathsf{I}^2_2)$ (see (6.6)). (6.22) says that in the situation in which there are two muddy children and at the time after all the individuals have answered 'No' to the father's first question, the second child recognizes that she is muddy if and only if her depth is equal or greater than 1. It can be proved using Prop. 5.3 and (6.6). (6.23) states that in the situation in which there are three muddy children and at the time after all the individuals have answered 'No' to the father's second question, the third muddy child cannot recognize that she is muddy *if she doesn't know the other agents depth.*

(6.24) says that in the situation in which there are three muddy children and at the time after all the individuals have answered 'No' to the father's second question, the third child recognizes that she is muddy *if* her depth is equal or greater than 2 and *if* she knows the other agents' depth. (6.24) can by proved using Prop. 5.3 and (6.10). (6.25) says that the former conditions are also necessary and it is justified by (6.4), for which we have that $\mathsf{M}^4 \in \mathcal{M}_1$, and by the fact that $Y \subseteq Cnu(\mathsf{I}^4_3)$ and $tp_3 \notin Cnu(\mathsf{I}^4_3)$ (see (6.8)). (6.26) states that in the situation in which there are four muddy children and at the time after all the individuals have answered 'No' to the father's third question, the fourth muddy child cannot recognize that she is muddy *if the agents' depth is not common knowledge.*

(6.27) says that in the situation in which there are four muddy children and at the time after all the individuals have answered 'No' to the father's third question, the fourth child recognizes that she is muddy *if* her depth is equal or greater than 3 and *if* she the agents' depth is common knowledge. (6.27) can by proved using Prop. 5.3 and (6.13). (6.28) says that the former conditions are also necessary and is justified by (6.4), for which we have that $\mathsf{M}^7 \in \mathcal{M}_2$, and by the fact that $Z \subseteq Cnu(\mathsf{I}^7_4)$ and $tp_4 \notin Cnu(\mathsf{I}^7_4)$ (see (6.11)).

Agents' depth	**Logic**	**Muddy children**	**Solution**
0	DBELu_0	1	The only muddy child answers 'Yes' to q_1
		$m > 1$	The m muddy children do not recognize that they are muddy
$k \geq 1$	DBELu_k	$m \leq 2$	The m muddy children answer 'Yes' to q_m
		$m > 2$	The m muddy children do not recognize that they are muddy

Table 6.1: Solution to the muddy children puzzle under the assumption that every child ignores the other children's depth.

6.2.11 Different conclusions

In this Chapter, we have moved to the epistemic context and we have considered the notion of knowledge. We have criticized the classical assumptions of logical omniscience with the same reasons that led us in the present work to reject the traditional tenet that logic is analytic, namely, the cognitive and computational effort required by propositional inferences. In so doing, we have focused on the behavior of realistic individuals, rather than ideal agents, and we hope to have suggested that our study on the analyticity of logic provides fruitful applications beyond the strictly philosophical field.

We have chosen as a case study the muddy children puzzle because it presents a number subtleties regarding reasoning about knowledge in groups of agents. We have seen that the classical solution of the puzzle, which says that the first $k-1$ times the father asks whether anyone is aware of being muddy, all the children will say 'No', but then the k^{th} time the children with muddy foreheads will answer 'Yes', is obtained through the means of classical epistemic logics precisely because these systems assume that it's common knowledge among the agents involved that every individual is logically omniscient. Now, what happens in the story if the children are taken to be realistic, rather than idealized agents? After the analysis with four children brought through the means of Depth Bounded Epistemic Logics (see Sections 6.2.6 and 6.2.10), we are now ready to give an answer for the general case.

First, under the assumption that every child ignores the other children's depth, we have that if there is only one muddy child, she will recognize that she is muddy and will answer 'Yes' to the father's first question. This is because, as we have seen in Section 6.2.10, $tp_1 \in Cnu^0(U)$ and this result holds in DBELu_0, *mutatis mutandis*, for every child involved. Notice that, since DBELu_0 is the basic element of the first hierarchy, this result holds also for agents with higher depth and, since DBELu_0 is equivalent to DBELe_0 and DBELc_0, this result holds also if we strengthen the assumption regarding the state of knowledge of the children about the others' depth. The kind of reasoning involved in the one muddy child case is analytic: any muddy individual, who knows the informational meaning of the logical operators, can and has to derive that she is muddy from the fact that she sees that all the other children are clean.

Then, again under the assumption that every child ignores the other children's depth, we have that if every child's depth is one and if there are at most two muddy children, the muddy children will recognize that they are muddy and will answer 'Yes' to the father's first question or to the father's second question, if there is only one muddy child or two muddy children respectively. This is because, as we have seen in Section 6.2.10, $tp_1 \in Cnu^1(U)$ and $tp_2 \in Cnu^1(X)$, and these results hold in DBELu_1, *mutatis mutandis*, for every child involved. Notice that, since DBELu_k for any $k > 1$ extends DBELu_1, these results hold also for agents with higher depth and, since DBELe_1 and DBELc_1 extend DBELu_1, these results hold also if we strengthen the assumption regarding the state of knowledge of the children about the others' depth. The kind of reasoning involved in the two muddy children case is synthetic of degree one and does not require any reasoning about the other children's knowledge. For this reason, if a child can reason only analytically, then she will not recognize that she is muddy in the two muddy children case.

Last, it turns out that if the muddy children are more than two, they will never recognize that they are muddy if every child ignores the other children's depth. This is because, as it is made clear in the second hierarchy of logics, the kind of reasoning involved in the case in which there are more than two muddy children requires reasoning about the other children's knowledge, which depends on their depth.

The situation under the assumption that every child ignores the other children's depth is represented in Table 6.1. We now move to examine the solutions to the muddy children puzzle under the assumption that every child knows the other children's depth, which are depicted in Table 6.2. The cases in which the muddy children are at most two, which are represented in the first four rows of the table, are justified by the discussion above. Under the assumption that every child knows the other children's depth, if every child's depth is at least two and there are three muddy children, then the muddy children will recognize that they are muddy and will answer 'Yes' to the father's third question. This is because, as we have seen in Section 6.2.10, $tp_3 \in Cne^2(Y)$ and this result holds in DBELe_2, *mutatis mutandis*, for every child involved. The kind of reasoning involved in the three muddy children case is synthetic of degree two and requires simulating another child's synthetic reasoning: thus, the assumption that every child knows the other

Agents' depth	**Logic**	**Muddy children**	**Solution**
0	DBELe_0	1	The only muddy child answers 'Yes' to q_1
		$m > 1$	The m muddy children do not recognize that they are muddy
1	DBELe_1	$m \leq 2$	The m muddy children answer 'Yes' to q_m
		$m > 2$	The m muddy children do not recognize that they are muddy
$k \geq 2$	DBELe_k	$m \leq 3$	The m muddy children answer 'Yes' to q_m
		$m > 3$	The m muddy children do not recognize that they are muddy

Table 6.2: Solution to the muddy children puzzle under the assumption that every child knows the other children's depth.

children's depth is essential. For this reason, if a child's depth is lower than two or if she is not informed about the other children's depth, then she will not recognize that she is muddy in the three muddy children case.

Moreover, if the muddy children are more than three, they will never recognize that they are muddy if the other children's depth is not common knowledge. This is because, as it is made clear in the third hierarchy of logics, the kind of reasoning involved in the case in which there are more than three muddy children requires simulating another child's reasoning, which is simulating in turn another child's reasoning.

The situation under the assumption that the other children's depth is common knowledge is represented in Table 6.3. The cases in which the muddy children are at most three, which are represented in the first six rows of the table, are justified by the discussions above. Under the assumption that the other children's depth is common knowledge, if every child's depth is three and there are four muddy children, then the muddy children will recognize that they are muddy and will answer 'Yes' to the father's fourth question. This is because, as we have seen in Section 6.2.10, $tp_4 \in Cnc^3(Z)$ and this result holds in DBELc_3, *mutatis mutandis*, for every child involved. The kind of reasoning involved in the four muddy children case is synthetic of degree three and requires simulating another child's reasoning, which is simulating in turn another child's reasoning: thus, the assumption that the other children's depth is common knowledge is essential. For this reason, if a child's depth is lower than three or if she is not informed that all the other children

Agents' depth	**Logic**	**Muddy children**	**Solution**
0	DBELc_0	1	The only muddy child answers 'Yes' to q_1
		$m > 1$	The m muddy children do not recognize that they are muddy
1	DBELc_1	$m \leq 2$	The m muddy children answer 'Yes' to q_m
		$m > 2$	The m muddy children do not recognize that they are muddy
2	DBELc_2	$m \leq 3$	The m muddy children answer 'Yes' to q_m
		$m > 3$	The m muddy children do not recognize that they are muddy
3	DBELc_3	$m \leq 4$	The m muddy children answer 'Yes' to q_m
		$m > 4$	The m muddy children do not recognize that they are muddy
$k > 3$	DBELc_k	$m \leq k+1$	The m muddy children answer 'Yes' to q_m
		$m > k+1$	The m muddy children do not recognize that they are muddy

Table 6.3: Solution to the muddy children puzzle under the assumption that the other children's depth is common knowledge.

Muddy children	**Weakest logic to obtain CS**	**Weakest assumptions to obtain CS**
1	DBELu_0	Every child's depth is zero and this is unknown to the others.
2	DBELu_1	Every child's depth is one and this is unknown to the others.
3	DBELe_2	Every child's depth is two and this is known to the others.
4	DBELc_3	Every child's depth is three and this is common knowledge among the others.
$k \geq 4$	DBELc_{k-1}	Every child's depth is $k-1$ and this is common knowledge among the others.

Table 6.4: Assumptions needed to obtain the classical solution CS.

know the others' depth, then she will not recognize that she is muddy in the four muddy children case.

This result can be easily generalized: under the assumption that the other children's depth is common knowledge, if every child's depth is k and there are $k+1$ muddy children, then the muddy children will recognize that they are muddy and will answer 'Yes' to the father's $k+1$ question. Otherwise, this result does not hold.

We can compare the solution of the muddy children puzzle obtained through the means of Depth Bounded Epistemic Logics to that proposed through classical epistemic logics. The classical solution can be obtained, in the one muddy child case, only if every child's depth is at least zero and even if the other children's depth is unknown to the children. It can be obtained, in the two muddy children case, only if every child's depth is at least one and even if the other children's depth is unknown to the children. It can be obtained, in the three muddy children case, only if every child's depth is at least two and only if every child knows the others' depth. It can be obtained, in the four muddy children case, only if every child's depth is at least three and only if the others' depth is common knowledge among the children. And, in the general case, the classical solution holds in the k muddy children case for $k \geq 4$, only if the children's depth is at least $k-1$ and only if the others' depth is common knowledge among the children. This comparison is summarized in Table 6.4.

Conclusions

Prelude

In the *Introduction* to this work, we have argued that the clarification of the notions of analyticity and of logic provides in itself an answer to the question of whether logic is analytic. This is why we now reason on the conclusions we have reached as far as some micro-stories are concerned, which are unavoidable premises to decide on the principle of analyticity of logic. In other words, the history of the analytic-synthetic distinction, of the notion of analysis, of the conceptions of logic and of the principle of tautologicity of logic stand in as the prelude to the discussion about the principle of analyticity of logic, which will be the focus of the conclusion in the strict sense of the term.

Analytic-synthetic distinction. Our starting point has been Kant's analytic-synthetic distinction. Although ancestors of this distinction can be found in several authors, the major novelty of Kant's approach must be searched in the leading role it plays in his philosophical system. Kant provides four criteria of analyticity: containment, clarification, identity and contradiction. We have argued that the former criterion, which states that in analytic judgments the predicate is (covertly) contained in the subject, applies only to judgments that are true, affirmative and, crucially, categorical: this means that Kant's distinction via containment is not exhaustive and, as a consequence, there are some judgments which are neither analytic nor synthetic. Against the criticisms that have been moved against it, we have held that the containment criterion is neither psychological nor obscure: on the contrary, it is a technical notion founded on the theory of concepts.

Containment is the fundamental criterion for Kant's conception of analyticity, for we have shown that the other ones might be generally reduced to it. First, clarification of the concepts' intensions involved in a certain analytic judgment, which is obtained through conceptual analysis, consists in showing that the predicate concept is contained in that of the subject. Moreover, clarification is a characterization in epistemic terms of a logical distinction. Second, although the identity criterion cannot be fully reduced to the containment one due to the class of tautologous judgments, the latter amounts to the former as far as partial identities are concerned. Third, in an affirmative analytic judgment the contradiction rests with the concept of the subject and the concept of the negation of the predicate because the predicate is already thought beforehand in the concept of the subject. The principle of contradiction is a necessary and sufficient condition for the cognoscibility of analytic judgments and we have pointed out that it is invoked as an epistemological instrument for knowing the truth of analytic judgments. Traditional interpretations notwithstanding, the principle is not a definitional criterion of analyticity.

The true innovation of Bolzano's work on analyticity is, we think, that he defines this property of propositions through the substitutional method, which is a fruitful instrument that will be used by other philosophers, such as Tarski and Quine, to define other formal notions in logic. According to Bolzano, a proposition is analytically true with respect to some ideas if and only if every objectual variant of that proposition with respect to those ideas is true. While in Kant's approach the content of the concepts occurring in a certain sentence is fundamental for establishing its status, which depends on the kind of connection between its subject and its predicate, we have shown that the analytical character of a proposition, for Bolzano, does not involve the content of the ideas occurring in it, but, on the contrary, only the truth value of that proposition when some of its ideas are varied. As a result, Bolzano's analyticity is not bound to a particular logical form and the recognition that a proposition is analytic is not constrained by a determinate syntactic structure.

The analytic-synthetic distinction that emerges in the *Wissenschaftslehre* has some peculiarities, of which traditional interpretations often could not make sense. First, contrary to Kant, Bolzano admits false propositions among analyticities (and syntheticities) because his definition does not link this distinction to the notion of truth, but rather to the concepts of validity and invalidity given through the substitution method. Second, in Bolzano's system there is room not only for conceptual syntheticities, but also for empirical analyticities: analyticity does not entertain any reliable connection with apriority and necessity. Along with false and empirical analyticities, we have argued that Bolzano's reflections produced also some ideas that will play an eminent role in the subsequent literature. First of all, the Bohemian philosopher complained about the obscurity of Kant's containment criterion: this criticism, which is strictly interwoven with the attack against the traditional conception of analysis, will be restated by both Frege and Quine. Second, Bolzano, like Frege and many others after him, underlines the narrowness of Kant's analytic-synthetic distinction. Not only does Bolzano attack the naïve representationalism at the basis of Kant's approach, but he also extends the analytic-synthetic distinction beyond the limits of categorical judgments, so as to release it from the boundaries of a particular syntactical form and language. Last, Bolzano's notion of 'logical analyticity', whose position in his thought has been erroneously overstated, anticipates in many respect Frege's definition, although, following the account of the *Wissenschaftslehre*, some propositions belong to logic but are not logically analytic and some logical analyticities do not belong to logic.

We have shown that the way in which Frege distinguishes between analytic and synthetic propositions is strongly influenced by his logicist program: analytic propositions are defined as those that can be proved from definitions with help of logical laws, *because* Frege's aim was to show that arithmetic could be proved from logic alone. This explains, we think, the differences between Kant and Frege's analytic-synthetic distinction. The most important of them is the disagreement of the underlying notion of analysis. Another essential difference is that Frege is concerned with the justification of a proposition as opposed to the Kantian interest in the content of a judgment. We have noticed that, although it is based on the notion of justification, Frege's analytic-synthetic distinction does not coincide with the other epistemological contraposition *par excellence*, namely that between *a priori* and *a posteriori* that, this time, is defined in Kantian terms, for the two differ in the level of generality.

As Bolzano before him, Frege complains about the narrowness of Kant's distinction and this criticism is, of course, related to the enormous advancements in logic presented in the *Begriffsschrift*. Frege seems to believe that his position is an accurate restatement of Kant's distinction, although it is not. This is probably due to the fact that he took the contradiction criterion, with its epistemological flavor, to be at the core of Kant's characterization. Nevertheless, the most evident element of continuity between the two conceptions of analyticity is that both of them find in the process of analysis the method for discovering analytic propositions and their justifications: Frege identifies Kant's analysis of the subject concept with the analysis needed in the definition of this concept. A position that is different not only to Kant and Bolzano's approaches, but also to Frege's standpoint is the one held by the logical empiricists' movement, that can be represented by the idea that a statement is analytic if it is either a logical truth or can be turned into a logical truth by putting synonyms for synonyms. In this way, the new linkage between analyticity and meaning introduces in the picture also the so-called material analyticities.

Hintikka's analytic-synthetic distinction applies primarily to the steps of a proof: synthetic steps are those in which new individuals are introduced into the argument; analytic ones are those in which we merely discuss the individuals which we have already introduced. The philosophical premise of Hintikka's distinction is an interpretation of Kant's mathematical method, according to which intuitions are defined as singular representations and constructions are necessarily used in synthetic arguments. We have argued that the first thesis is too radical and that the leap from the mathematical method to the analytic-synthetic distinction finds no convincing basis on the Kantian texts. Nevertheless, both of these readings are building blocks of Hintikka's conception. In particular, the theory that it is the use of constructions that makes a mathematical argument step synthetic is translated in modern terms by the theory that it is the use of the rule of existential instantiation of modern first-order logic that makes a logical argument step synthetic and the theory that what is exhibited in the mathematical constructions, viz. intuitions, are simply individuals is translated in modern terms by the fact that the things introduced through the rule of existential instantiation are simply individuals.

In this way, Hintikka manages to propose also a formal characterization of the analytic-synthetic distinction, according to which a proof is analytic if and only if the degree of each intermediate stage is smaller than, or equal to, the degree of either the premises or the conclusion, where the degree of a formula measures the maximal number of individuals that are considered together in it. A proof method elaborated through the theory of distributive normal forms allows to discern which inferences are analytic and which are synthetic in this sense and to give a gradual characterization of the notion of syntheticity: the more new individuals are needed to prove the conclusion from the premises, the higher the degree of syntheticity of the proof.

Analysis

Kant's conception of analysis is founded on the traditional theory of logical division of concepts. According to this perspective, each concept is assumed to be made up by constituents, each of which finds its place in a hierarchy organized with respect to the notions of containment and inclusion: each genus is contained in its species and each species is contained under its genus. Analysis is thus understood in terms of a decompositional or resolutive process that, starting from the initial concepts, aims at arriving at its simple elements. This kind of analysis is based on the Aristotelian definitions and divisions are taken to be exclusive and exhaustive disjunctions. The decompositional conception of analysis, that had become dominant by the end of the early modern period and which ties Kant to the rationalist perspective, has been attacked by both Bolzano and Frege and has been wrongly interpreted by Hintikka.

Bolzano criticizes the unsophisticated form of representationalism at the basis of Kant's conception, which assumes that concepts are pictures of the objects they represent and, in particular, that properties of objects correspond to constituents of concepts. He formulates an alternative kind of analysis that has been called 'paraphrastic analysis', according to which every sentence utterance can be paraphrased into a proposition that expresses its complete meaning: the process of *Auslegung* or interpretation provides a complete analysis of the initial sentence. Bolzano's approach is a revolutionary intuition in the history of analysis and anticipates in many respects the transformative or interpretative dimension of analysis that is commonly assumed to characterize analytic philosophy. The reason why Frege does not accept Kant's conception of analysis is that it is strictly interwoven with the traditional subject-predicate logic. He proposes a notion of analysis that is founded on the function-argument distinction and that gives priority to judgments over concepts and yields different results, each of which is on the same level with the others. Hintikka ascribes to Kant and, at the same time, uses in his own work, the so-called 'constructional conception of analysis', according to which analysis does not introduce any individual entity, while synthetic procedures are marked by the use of constructions, which allow to move from a general concept to a non-empirical intuition that represents that concept.

Tautologicity of logic

We have shown that Kant's position on tautologous judgments is not univocal. In particular, it depends on which criterion of analyticity is considered as fundamental. While according to the identical criterion tautologous statements are of course analytic, the clarification criterion, as well as the containment one, requires at least the predicate concept being different from the subject concept, otherwise there is no room for any kind of clarification whatever. Kant seems to gradually abandon the idea that identical judgments are analytic as he comes to regard the analytic-synthetic distinction as focused on knowledge-advancing judgments. On the contrary, Bolzano is clear in affirming that analyticity does not coincide with triviality and that many analyticities turn out to be instructive. This is because, as we have underlined, Bolzano's analyticity is not an epistemological notion and does not invoke the trivial-instructive opposition.

We have pointed out that, in the *Grundlagen*, Frege holds that logic is, at the same time, analytic and informative. Assuming, on the one hand, that logic contains, albeit in a concentrated format, all the theorems of arithmetic and, on the other hand, that arithmetic cannot be charged of sterility, Frege concludes that logic is not tautologous. The key point of his explanation is his conception of definition in terms of concept-formation through the process of analysis and the idea that the recognition of a certain pattern and the extraction of a quantificational structure from a given judgment is, by itself, a creative process. Logical deduction is thus a knowledge-extending procedure because theorems are concentrated into basic definitions and a resource-consuming procedure of extraction is needed. We have argued that the introduction of the *Sinn-Bedeutung* distinction produces a radical change in view on the usefulness of definitions and, as a result, on the informativity of logical deduction. In the review to Husserl, Frege maintains that mathematical definitions are correct if the *definiens* and the *definiendum* share the same reference, but it is not necessary that they share the same sense. After having realized the inadequacy of this formulation, in the unpublished text *Logic in Mathematics* he tries to reject definitions based on logical analysis and proposes to distinguish between the logical and psychological level in talking of the fruitfulness of definitions.

Logical empiricism borrows from Frege the principle of analyticity of logic, but, unlike the author of the *Grundlagen*, argues that logic is tautologous. The idea that logical deduction is uninformative is a consequence of the paradox of analysis together with the thesis that logic is analytic. The conception that depicts logic and mathematics as sterile seems to clash with our common intuition that the result of a complex or long inference does indeed add something to our knowledge. The latter fact is explained in the Vienna Circle with a strongly psychologistic stance. Hahn holds that logical propositions, albeit sterile, have significance for us because we are not omniscient and Hempel argues that, although they are not objectively informative, logic and mathematics help us in disclosing what is already contained in the premises of an inference. Similiarly, Ayer's idea is that logical deduction calls attention to the implications of a certain linguistic usage, such as the convention which governs our employment of the connectives, of which we might otherwise not be conscious. A different solution is proposed by Wittgenstein, who denies any utility to the process of logical deduction. According to the insight conveyed in the *Tractatus*, once propositions are expressed through an adequate notation, such as the one offered by truth tables, logical deduction shall be replaced by the mere inspection of the propositions.

Hintikka offers a radically new solution to the paradox of analysis, which is an open attack against the logical empiricists' conception. He uses the theory of distributive normal forms as a basis to define the theory of probability and a theory of semantic information, which provides an answer to the question of how much information is conveyed by a certain first-order sentence. He thus distinguishes between two kinds of information. On the one hand, depth information is the measure obtained by assigning a positive probability weight only to the consistent constituents of the polyadic calculus and is not effectively calculable. On the other hand, surface information is obtained by assigning non-zero weights to all the constituents that are not trivially inconsistent at a certain depth. According to Hintikka, logical deduction is not sterile because it can increase surface information and it enables us to find that certain non-trivially inconsistent constituents were nevertheless inconsistent at a greater depth.

Conceptions of logic

In the introduction to the *Transcendental Logic* of the *Critique*, Kant recognizes four different kinds of logic: pure general logic, applied general logic, special logic and transcendental logic. The former, on which we have focused our attention, consists of a restricted version of the Aristotelian syllogistic with a simple theory of disjunctive and hypothetical judgments added on. We have pointed out that four are the fundamental features of this discipline. First, it is pure, namely, it is not concerned with the empirical conditions of the subject in her employment of the rules of the understanding. Second, it is general, namely, it contains the absolutely necessary rules of thinking, where these rules are said to be necessary in the sense that they have to be applied no matter what are the objects we are thinking about. Third, it is formal, namely, it abstracts from the semantical content of thought and, as a result, it cannot yield an extension of knowledge about realty. Formality turns out to be a consequence of the generality of logic. Fourth, it is a canon for thinking, namely, a body of rules or *a priori* principles. The thesis that logic is a canon for thinking can be derived from the features of pureness, generality and formality of the discipline.

Bolzano's distinction between two notions of logic is a consequence of the attention he paid to the differences between the *ordo essendi* and the *ordo cognoscendi*. On the one hand, the theory of science is characterized by broad borders and includes also methodological, pedagogical and epistemological considerations. On the other hand, logic in the narrow sense has nothing to do with psychology and is a deductive science. For the author of the *Wissenschaftslehre*, deductive sciences are ordered according to the grounding relation, which assigns every truth to its proper place with respect to the remaining propositions of that science. This means that, unlike Kant's pure general logic, this discipline is, for Bolzano, a body of truths and not a body of rule. We have underlined that this shift, which amounts to release logic from an alleged special status, is crucial and that a similar position will be held by Frege.

We have seen that Frege radically innovates Aristotelian logic: through a peculiar symbolism, in his *Begriffsschrift* he proposes a classical second-order logic with identity expressing both the logic of propositions and the logic of quantification. A characterizing feature of his work is the introduction of the function-argument model, which allows Frege to overcome the difficulties of traditional syllogistic. The formulation of a new logic is the first requirement of Frege's logicist project, which aims at reducing arithmetic to logic or, equivalently, at showing that the truths of arithmetic are analytic. His project obviously criticizes Kant's thesis that mathematics is synthetic *a priori*. Frege's logicist program requires to show not only that propositions of arithmetic can be proved from logical truths and through logical methods, but also, and crucially, that the fundamental concepts of arithmetic can be defined in terms of logical concepts. Frege conceives of logic as a body of laws, being descriptive in their content and implying norms for thought. We have shown that Frege agrees with Kant in holding that logic is general, in that it allows an unrestricted applicability of its norms, but contradicts the author of the *Critique* as far as formality is concerned. Logic for Frege cannot abstract from all semantic content and must attend at least to the semantic content of the logical expression. Like Bolzano, the author of the *Begriffsschrift* rejects Kant's thesis that logic is simply a *canon* for thought.

We have assumed that, with Quine, conceptions of logic are all inscribed in the same framework, one in which the subject matter of logic is taken to be logical properties of sentences and logical relations among sentences defined by logical forms understood as simply schemata.

Against the principle of analyticity of logic

We have started our dissertation with the question of whether logic is analytic. The thesis we have argued for in this work is that logic is neither analytic nor tautologous. In giving this answer, we have taken logic to be classical first-order logic and we have assumed that the analytic-synthetic distinction, which applies primarily to inferences, concerns a distinction between degrees of computational complexity, where analyticity and tractability are reconciled. The idea that most of the logical inferences are synthetic has been a result of a philosophical reconstruction, which has shown in the first place that the history of the principle of analyticity of logic is not a linear and progressive narrative from Kant to the logical empiricists' perspective. On the contrary, we have argued that the standpoint of the Vienna Circle, although soon became traditional, is an exception, rather than the rule, and does not represent a climax of a supposed development or improvement. We have seen that both Bolzano and Hintikka clearly break this positivistic paradigma: the former by detaching analyticity from necessity, the latter by resorting to Kant's conceptual apparatus against the thesis of the Vienna Circle. The stages individuated in our work and the main conclusions we have reached might be summarized as follows.

First, we have argued that Kant does not apply his analytic-synthetic distinction to logic, because logic is seen as a *canon* and not as a body of truths. This is the case although the traditional view depicts Kant as maintaining that logic is analytic, probably relying on the fact that logic is indeed the fundamental instrument that Kant employs for both drawing and applying his analytic-synthetic distinction, since it provides the basic notions for his definitions and the central tool for the determination of the truth of analytic judgments. But the issue of the role of logic as an instrument for defining and applying the analytic-synthetic distinction must be kept separate from the question of the epistemological status of logic. Moreover, even attempting an analysis that Kant did not think it was worth pursuing, we have shown that no logical judgment is synthetic *a priori* and that some (if not all) logical judgments are not analytic. This is the case not only for validities turning essentially on relations, but also for propositional truths such as *modus ponens* and identical truths. In other words, following Kant's definition of the analytic-synthetic distinction, we have the unexpected result that many logical judgments are neither analytic nor synthetic.

Second, we have shown that Bolzano applies his analytic-synthetic distinction to logic and obtains the result that logic is synthetic. The conclusion that logic is synthetic is derived by Bolzano from the assumptions that logic is a deductive science and that deductive sciences are mainly synthetic *a priori*. We think that the reason why Bolzano's insight of the syntheticity of logic turns out to be a substantial conception and not a mere terminological trick is that it hides an important thesis on the nature of this discipline, namely, that logic is a body of truths like any other deductive science. The thesis that logic is synthetic assumes the *ordo essendi* perspective. Moving to the epistemic side, we find instead that some logical propositions are analytic. The latter play a crucial role in the presentation and in the development of logic and are characterized by their being instantiations of general rules, that, according to Bolzano, are always synthetic.

Third, we have pointed out that Frege maintains that logical laws are analytic, but not tautologous, and that this conclusion depends on his logicist program. In particular, the laws of logic that are chosen as axioms of the system are analytic because of their self-evidence, while logical theorems are instead analytic in that they can be proved through logical laws only. We have shown that what discriminates Frege's position from the Kantian one and leads us to conclude that logic is analytic only for the former and not for the latter, although both of them are not explicit on this point, is the difference in the underlying conception of logic: for the author of the *Grundlagen*, as for Bolzano, logic is a body of truths and a science in the strict sense of the term. Why, then, should Frege have avoided to apply the analytic-synthetic distinction to logic, given his concern with *a priori* sciences?

Fourth, we have underlined that the logical empiricist movement follows Frege in holding the principle of analyticity of logic, but goes beyond the author of the *Grundlagen* in saying that this discipline is tautologous. Kant's synthetic *a priori*, which had already been impoverished by Frege's thesis that arithmetic is analytic, is rejected *in toto* by the Vienna Circle. As a result, the analytic-synthetic distinction and the *a priori*-*a posteriori* distinction do not cut across one another: synthetic statements are always grounded on facts and analytic statements are known *a priori*. Quine's *Two Dogmas*, which represents the strongest attack against the logical empiricists' epistemology, does not criticize the idea that logic is analytic, but only the lack of a proper characterization of statements that are materially analytic. On this issue, Quine agrees with Carnap, who, despite the significant modifications of his account of analyticity, mainly treats 'logical' simply as a synonym for 'analytic'.

Fifth, we have shown that, restoring to Kant's conception of the mathematical method, Hintikka holds that there exists a class of quantified logical truths that are synthetic *a priori*. In holding that the synthetic *a priori* is a non-empty category, he vindicates through modern means the main principles of Kant's philosophy against the criticisms moved by the logical positivists. Hintikka's work is at the same time a (supposed) reconstruction of Kant's theory and an open attack against the perspectives held by the Vienna Circle. It is this second aim that explains, we think, the most serious weaknesses of his reading of the *Critique*, namely the interpretative stretching of his reconstruction and his ascription to Kant of the idea that pure general logic is, contrary to what we have argued, analytic. The reason why Hintikka's talk of logic is a vindication of Kant's talk of mathematics is that contemporary boundaries between mathematics and logic are not the Kantian ones. Modern first-order logic includes modes of reasoning that Kant wouldn't have called logical, but mathematical, and it is precisely this kind of derivations that Hintikka considers to be synthetic.

The conceptual kernel of Hintikka's formal theory comes from the field of computability: the undecidability of first-order logic is a fundamental observation for the development of both the theory of distributive normal forms and the theory of semantic information. The fact that we have to expand a given constituent at a certain depth in order to acknowledge its inconsistency grounds Hintikka's notion of degree of syntheticity and the fact that we do not know which depth the expansion has to reach in order to achieve the desired result represents the main motivation towards Hintikka's formulation of the notion of surface information. The brilliant idea behind these constructions is that the definition of the analytic-synthetic distinction must take into account the result that some inferences are 'more difficult' than others and require a greater computational effort and that useful measures of information must be realistic and must envision that in general there is no decision procedure for determining which constituents are inconsistent. The intuitive idea of the 'difficulty of reasoning pattern' seems to be at the periphery of both Kant and Bolzano's conceptions of the analytic-synthetic distinction. Nevertheless, Frege's reasoning about the fruitfulness of an inference seems very close to this insight: in the *Grundlagen*, he seems to recognize that the extraction of a quantificational structure from a given judgment is, by itself, a creative and resource-consuming procedure.

However, Hintikka's work classifies as analytic a wide class of logical inferences that includes many polyadic deductions as well as the entire set of propositional and monadic inferences. As a consequence, Hintikka's approach is only a partial vindication of the intuitive idea that logical deduction can increase our knowledge. D'Agostino and Floridi have recently argued that these doubts concerning the analyticity of propositional logic find a confirmation in the theory of computational complexity: if the decision problem for Boolean logic is (most probably) intractable, how is it possible to maintain that it is uninformative and analytic? D'Agostino and Floridi formulate an innovative non-classical semantics according to which the class of synthetic propositional inferences is not empty. Following this account, the conclusion of an analytic inference depends solely on the informational meaning of the logical operators occurring in its premises and conclusion, while synthetic inferences are characterized by the use of virtual information.

With Depth Bounded First-Order Logics, we have unified Hintikka's reasoning on first-order logic with D'Agostino and Floridi's account of propositional logic in order to provide a complete vindication of the thesis that the most part of logical inferences are synthetic. Our approach primarily applies the analytic-synthetic distinction to inferences and, crucially, provides a classification of classical first-order inferences according to two notions of syntheticity. The two criteria of syntheticity refer to two different kinds of computational and cognitive efforts with different contents, namely, the idea that an inference is synthetic if, in obtaining the conclusion from the premises, it is necessary to employ pieces of virtual information, and the idea that an inference is synthetic if it is necessary to use in the derivation individuals that were not involved in the configurations of the premises. At the basis of our hierarchy of logics, there is the class of first-order logical inferences that are analytic from both the propositional and the quantificational case. Synthetic inferences are classified according to two parameters: propositional depth and quantificational depth. The more the number of nested pieces of virtual information are needed to obtain the conclusion from the premises, the higher the level in the hierarchy as far as the propositional depth is concerned. Similarly, the more the number of individuals that are not contained in the premises are needed to obtain the conclusion, the higher the level in the hierarchy as far as the quantificational depth is concerned.

The assumption of logical omniscience is an epistemic translation of the principle that logic is analytic and tautologous. For this reason, after having vindicated the idea that most of classical first-order inferences are synthetic, we have proposed a taxonomy of classical epistemic logic according to the degree of omniscience of the agents involved and of the status of knowledge of the other's knowledge. The former is defined as the depth of the synthetic inferences that the agent is able to carry out; the latter might be that the individuals' degree of omniscience is unknown, known or commonly known. The result of this taxonomy is given by Depth

Bounded Epistemic Logics. The shift to the epistemic context and our discussion on the muddy children puzzle should have made clear that the characterization of the analytic-synthetic distinction in terms of computational complexity might have important applications. In particular, we have shown that the solution to the puzzle through our family of logic is much more realistic than that obtained through classical epistemic logic.

Appendix

This Appendix consists of the proofs of the propositions enunciated in Chapter 6.

Proposition 1.1 For any $\Gamma \subseteq \mathcal{L}^s$ and $B \in \mathcal{L}$:

1) $sB \in \Gamma$ [Hypothesis]

2) $sB \in \Gamma \cup \{\bar{s}B\}$ e $\bar{s}B \in \Gamma \cup \{\bar{s}B\}$ [1) and construction]

3) $\Gamma \cup \{\bar{s}B\} \notin \mathbb{G}$ [2) and Def. 4]

4) $\mathbb{A} \subseteq \mathbb{G}$ [Def. 5]

5) $\Gamma \cup \{\bar{s}B\} \notin \mathbb{A}$ [3) and 4)]

Proposition 1.2 For any $\Gamma, \Delta \subseteq \mathcal{L}^s$ and $B \in \mathcal{L}$:

1) $\Gamma \cup \{sB\} \notin \mathbb{A}$ [Hypothesis]

2) $\Gamma \cup \{sB\}$ satisfies at least one of the conditions, call it $\mathcal{C}$, for some $F, G \in \mathcal{L}$ [Def. 5 e 1)]

3) $\Gamma \cup \Delta \cup \{sB\}$ satisfies condition $\mathcal{C}$ for $F, G \in \mathcal{L}$ [2) and construction]

4) $\Gamma \cup \Delta \cup \{sB\} \notin \mathbb{A}$ [Def. 5 and 3)]

Proposition 2.1 For any $\Gamma \in \mathbb{A}$, $\Gamma \subseteq \Gamma$ and, by Def. 6, Γ is a refinement of Γ on $\varnothing$, that is to say, $\Gamma \in \mathsf{R}_{\Gamma}^{\varnothing}$.

Proposition 2.2 For any $\Gamma, \Gamma \cup \{sA\}, \Delta \in \mathbb{A}$ and $J = \{q_1, \dots, q_j\} \in \wp(\mathcal{P})$:

1) $\Delta \in R_{\Gamma \cup \{sA\}}^{J}$ [Hypothesis]

2) Δ is a refinement of $\Gamma \cup \{sA\}$ on J [1) and Def. 6]

3) $\Gamma \cup \{sA\} \subseteq \Delta$ [2) and Def. 6]

4) $s_1q_1, \ldots, s_jq_j \in \Delta$ for $s_i \in \{t, f\}$ [2) and Def. 6]

5) $\Gamma \subseteq \Gamma \cup \{sA\}$ [Construction]

6) $\Gamma \subseteq \Delta$ [3) and 5)]

7) Δ is a refinement of Γ on J [4), 6) and Def. 6]

8) $\Delta \in R^J_\Gamma$ [7) and Def. 6]

Proposition 2.3 Let $L = \{r_1, \ldots, r_l\} \in \wp(\mathcal{P})$ and $N = \{r_1, \ldots, r_l, r_{l+1}, \ldots, r_n\} \in \wp(\mathcal{P})$, where $L \subseteq N$. For any $\Gamma, \Delta \in \mathbb{A}$:

1) $\Delta \in \mathsf{R}^N_\Gamma$ [Hypothesis]

2) Δ is a refinement of Γ on N [1) and Def. 6]

3) $\Gamma \subseteq \Delta$ [2) and Def. 6]

4) $s_1r_1, \ldots, s_lr_l, s_{l+1}r_{l+1}, \ldots, s_nr_n \in \Delta$ per $s_i \in \{t, f\}$ [2) and Def. 6]

5) $L \subseteq N$ [Hypothesis]

6) $s_1r_1, \ldots, s_lr_l \in \Delta$ for $s_i \in \{t, f\}$ [4) and 5)]

7) Δ is a refinement of Γ on L [3), 6) and Def. 6]

8) $\Delta \in \mathsf{R}^L_\Gamma$ [7) and Def. 6]

Proposition 2.4 For any $J = \{q_1, \ldots, q_j\} \in \wp(\mathcal{P})$, $L = \{r_1, \ldots, r_l\} \in \wp(\mathcal{P})$, $\Gamma, \Delta, \Lambda \in \mathbb{A}$:

1) $\Delta \in \mathsf{R}^J_\Gamma$ [Hypothesis]

2) Δ is a refinement of Γ on J [1) and Def. 6]

3) $\Gamma \subseteq \Delta$ [2) and Def. 6]

4) $s_1q_1, \ldots, s_jq_j \in \Delta$ for $s_i \in \{t, f\}$ [2) and Def. 6]

5) $\Lambda \in \mathsf{R}^L_\Delta$ [Hypothesis]

6) Λ is a refinement of Δ on L [5) and Def. 6]

7) $\Delta \subseteq \Lambda$ [6) and Def. 6]

8) $s_1q_r, \ldots, s_lr_l \in \Lambda$ for $s_i \in \{t, f\}$ [6) and Def. 6]

9) $\Gamma \subseteq \Lambda$ [3) and 7)]

10) $s_1q_1, \ldots, s_jq_j \in \Lambda$ for $s_i \in \{t, f\}$ [4) and 7)]

11) $s_1q_1, \ldots, s_jq_j, s_{j+1}r_1, \ldots, s_mr_l \in \Lambda$ for $s_i \in \{t, f\}$ [8) and 10)]

12) Λ is a refinement of Γ on J [9), 10) and Def. 6]

13) $\Lambda \in \mathsf{R}_\Gamma^J$ [12) and Def. 6]

14) Λ is a refinement of Γ on L [8), 9) and Def. 6]

15) $\Lambda \in \mathsf{R}_\Gamma^L$ [14) and Def. 6]

16) Λ is a refinement of Γ of $J \cup L$ [9), 11) and Def. 6]

17) $\Lambda \in \mathsf{R}_\Gamma^{J \cup L}$ [16) and Def. 6]

Proposition 3.1 For any $i \in \mathcal{A}$, $\Gamma \subseteq \mathcal{L}^s$ and $B \in \mathcal{L}$:

1) $sB \in \Gamma$ [Hypothesis]

2) $sB \in \mathsf{W}_0(\Gamma)$ [2) and Def. 10]

3) $sB \in \mathsf{W}(\Gamma)$ [3) and Def. 10]

Proposition 3.2 For any $\Gamma, \Delta \subseteq \mathcal{L}^s$ and $B \in \mathcal{L}$:

1) $sB \in \mathsf{W}(\Gamma)$ [Hypothesis]

2) $\exists j \geq 0 \mid sB \in \mathsf{W}_j(\Gamma)$ [1) and Def. 10]

3) $sB \in \mathsf{W}_j(\Gamma \cup \Delta)$ [2) and Lemma 1 (following)]

4) $sB \in \mathsf{W}(\Gamma \cup \Delta)$ [3) and Def. 10]

Lemma 1. For any $\Gamma, \Delta \subseteq \mathcal{L}^s$, $B \in \mathcal{L}$ and $j \geq 0$, if $sB \in \mathsf{W}_j(\Gamma)$, then $sB \in \mathsf{W}_j(\Gamma \cup \Delta)$.
Proof by induction on $j \geq 0$:

- Basic case: $j = 0$. We prove that if $sB \in \mathsf{W}_0(\Gamma)$, then $sB \in \mathsf{W}_0(\Gamma \cup \Delta)$.

    1) $sB \in \mathsf{W}_0(\Gamma)$ [Hypothesis]
    2) $sB \in \Gamma$ [1) and Def. 10]
    3) $sB \in \Gamma \cup \Delta$ [2) and construction]

4) $sB \in \mathsf{W}_0(\Gamma \cup \Delta)$ [3) and Def. 10]

- Inductive hypothesis: $j = n$. We assume that if $sB \in \mathsf{W}_n(\Gamma)$, then $sB \in \mathsf{W}_n(\Gamma \cup \Delta)$.

- Inductive step: $j = n + 1$. We prove that if $sB \in \mathsf{W}_{n+1}(\Gamma)$, then $sB \in \mathsf{W}_{n+1}(\Gamma \cup \Delta)$.

1) $sB \in \mathsf{W}_{n+1}(\Gamma)$ [Hypothesis]

2) There are three cases to be considered: i. $\mathsf{W}_n(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$, ii. $sB = t\mathsf{K}_g\neg^{2n}C$ and $\Sigma_g(\mathsf{W}_n(\Gamma)) \cup \{f\neg^{2m}C\} \notin \mathbb{A}$ and iii. $sB = t\mathsf{K}_g\neg^{2n+1}C$ and $\Sigma_g(\mathsf{W}_n(\Gamma)) \cup \{t\neg^{2m}C\} \notin \mathbb{A}$ [1) and Def. 10]

3) Case i. $\mathsf{W}_n(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$

3.1) $\mathsf{W}_n(\Gamma) \subseteq \mathsf{W}_n(\Gamma \cup \Delta)$ [Inductive hypothesis]

3.2) $\mathsf{W}_n(\Gamma \cup \Delta) \cup \{\bar{s}B\} \notin \mathbb{A}$ [3), 3.1) and Prop. 1.2]

3.3) $sB \in \mathsf{W}_{n+1}(\Gamma \cup \Delta)$ [3.2) and Def. 10]

4) Case ii. $sB = t\mathsf{K}_g\neg^{2n}C$ and $\Sigma_g(\mathsf{W}_n(\Gamma)) \cup \{f\neg^{2m}C\} \notin \mathbb{A}$

4.1) $\mathsf{W}_n(\Gamma) \subseteq \mathsf{W}_n(\Gamma \cup \Delta)$ [Inductive hypothesis]

4.2) $\Sigma_g(\mathsf{W}_n(\Gamma)) \subseteq \Sigma_g(\mathsf{W}_n(\Gamma \cup \Delta))$ [4.1) and Def. 9]

4.3) $\Sigma_g(\mathsf{W}_n(\Gamma \cup \Delta)) \cup \{f\neg^{2m}C\} \notin \mathbb{A}$ [4), 4.2) and Prop. 1.2]

4.4) $sB = t\mathsf{K}_g\neg^{2n}C \in \mathsf{W}_{n+1}(\Gamma \cup \Delta)$ [4.3) and Def. 10]

5) Case iii. $sB = t\mathsf{K}_g\neg^{2n+1}C$ and $\Sigma_g(\mathsf{W}_n(\Gamma)) \cup \{t\neg^{2m}C\} \notin \mathbb{A}$

5.1) $\mathsf{W}_n(\Gamma) \subseteq \mathsf{W}_n(\Gamma \cup \Delta)$ [Inductive hypothesis]

5.2) $\Sigma_g(\mathsf{W}_n(\Gamma)) \subseteq \Sigma_g(\mathsf{W}_n(\Gamma \cup \Delta))$ [5.1) and Def. 9]

5.3) $\Sigma_g(\mathsf{W}_n(\Gamma \cup \Delta)) \cup \{t\neg^{2m}C\} \notin \mathbb{A}$ [5), 5.2) and Prop. 1.2]

5.4) $sB = t\mathsf{K}_g\neg^{2n+1}C \in \mathsf{W}_{n+1}(\Gamma \cup \Delta)$ [5.3) and Def. 10]

6) $sB \in \mathsf{W}_{n+1}(\Gamma \cup \Delta)$ [3.3), 4.4) and 5.4)]

Proposition 3.3 For any $\Gamma \subseteq \mathcal{L}^s$ and $B, C \in \mathcal{L}$:

1) $sB \in \mathsf{W}(\Gamma)$ [Hypothesis]

2) $\exists n \geq 0 \mid sB \in \mathsf{W}_n(\Gamma)$ [1) and Def. 10]

3) $sC \in \mathsf{W}(\Gamma \cup \{sB\})$ [Hypothesis]

4) $\exists k \geq 0 \mid sC \in \mathsf{W}_k(\Gamma \cup \{sB\})$ [3) and Def. 10]

5) $sC \in \mathsf{W}_{n+k}(\Gamma)$ [2), 4) and Lemma 2 (following)]

6) $sC \in \mathsf{W}(\Gamma)$ [5) and Def. 10]

Lemma 2. For any $\Gamma \subseteq \mathcal{L}^s$, $B, C \in \mathcal{L}$ and $n, k \geq 0$, if $sB \in \mathsf{W}_n(\Gamma)$ and $sC \in \mathsf{W}_k(\Gamma \cup \{sB\})$, then $sC \in \mathsf{W}_{n+k}(\Gamma)$.
Proof by induction on $k \geq 0$:

- Base case: $k = 0$. We prove that if $sB \in \mathsf{W}_n(\Gamma)$ and $sC \in \mathsf{W}_0(\Gamma \cup \{sB\})$, then $sC \in \mathsf{W}_n(\Gamma)$.

  1) $sC \in \mathsf{W}_0(\Gamma \cup \{sB\})$ [Hypothesis]
  2) $sC \in \Gamma \cup \{sB\}$ [1) and Def. 10]
  3) $sC \in \Gamma$ (Obviously, if $sC = sB$, the desired result simply is the first hypothesis) [2) and construction]
  4) $sC \in \mathsf{W}_0(\Gamma)$ [3) and Def. 10]
  5) $sC \in \mathsf{W}_n(\Gamma)$ [4) and Prop. 4.3]

- Inductive hypothesis: $k = m - 1$. We assume that if $sB \in \mathsf{W}_n(\Gamma)$ and $sC \in \mathsf{W}_{m-1}(\Gamma \cup \{sB\})$, then $sC \in \mathsf{W}_{n+m-1}(\Gamma)$.

- Inductive step: $k = m$. We prove that if $sB \in \mathsf{W}_n(\Gamma)$ and $sC \in \mathsf{W}_m(\Gamma \cup \{sB\})$, then $sC \in \mathsf{W}_{n+m}(\Gamma)$.

  1) $sB \in \mathsf{W}_n(\Gamma)$ [Hypothesis]
  2) $sC \in \mathsf{W}_m(\Gamma \cup \{sB\})$ [Hypothesis]
  3) There are three cases to be considered: i. $\mathsf{W}_{m-1}(\Gamma \cup \{sB\}) \cup \{\bar{s}C\} \notin \mathbb{A}$, ii. $sC = t\mathsf{K}_g\neg^{2n}D$ and $\Sigma_g(\mathsf{W}_{m-1}(\Gamma \cup \{sB\})) \cup \{f\neg^{2m}D\} \notin \mathbb{A}$ and iii. $sC = t\mathsf{K}_g\neg^{2n+1}D$ and $\Sigma_g(\mathsf{W}_{m-1}(\Gamma \cup \{sB\})) \cup \{t\neg^{2m}D\} \notin \mathbb{A}$ [2) and Def. 10]
  4) Case i. $\mathsf{W}_{m-1}(\Gamma \cup \{sB\}) \cup \{\bar{s}C\} \notin \mathbb{A}$
     4.1) $\mathsf{W}_{m-1}(\Gamma \cup \{sB\}) \subseteq \mathsf{W}_{n+m-1}(\Gamma)$ [1) and Inductive hypothesis]
     4.2) $\mathsf{W}_{n+m-1}(\Gamma) \cup \{\bar{s}C\} \notin \mathbb{A}$ [4), 4.1) and Prop. 1.2]
     4.3) $sC \in \mathsf{W}_{n+m}(\Gamma)$ [4.2) and Def. 10]
  5) Case ii. $sC = t\mathsf{K}_g\neg^{2n}D$ and $\Sigma_g(\mathsf{W}_{m-1}(\Gamma \cup \{sB\})) \cup \{f\neg^{2m}D\} \notin \mathbb{A}$
     5.1) $\mathsf{W}_{m-1}(\Gamma \cup \{sB\}) \subseteq \mathsf{W}_{n+m-1}(\Gamma)$ [1) and Inductive hypothesis]
     5.2) $\Sigma_g(\mathsf{W}_{m-1}(\Gamma \cup \{sB\})) \subseteq \Sigma_g(\mathsf{W}_{n+m-1}(\Gamma))$ [5.1) and Def. 9]
     5.3) $\Sigma_g(\mathsf{W}_{n+m-1}(\Gamma)) \cup \{f\neg^{2m}D\} \notin \mathbb{A}$ [5), 5.2) and Prop. 1.2]
     5.4) $sC = t\mathsf{K}_g\neg^{2n}D \in \mathsf{W}_{n+m}(\Gamma)$ [5.3) and Def. 10]
  6) Case iii. $sC = t\mathsf{K}_g\neg^{2n+1}D$ and $\Sigma_g(\mathsf{W}_{m-1}(\Gamma \cup \{sB\})) \cup \{t\neg^{2m}D\} \notin \mathbb{A}$

6.1) $\mathsf{W}_{m-1}(\Gamma \cup \{sB\}) \subseteq \mathsf{W}_{n+m-1}(\Gamma)$ [1) and Inductive hypothesis]
6.2) $\Sigma_g(\mathsf{W}_{m-1}(\Gamma \cup \{sB\})) \subseteq \Sigma_g(\mathsf{W}_{n+m-1}(\Gamma))$ [6.1) and Def. 9]
6.3) $\Sigma_g(\mathsf{W}_{n+m-1}(\Gamma)) \cup \{t\neg^{2m}D\} \notin \mathbb{A}$ [6), 6.2) and Prop. 1.2]
6.4) $sC = t\mathsf{K}_g\neg^{2n+1}D \in \mathsf{W}_{n+m}(\Gamma)$ [6.3) and Def. 10]

7) $sC \in \mathsf{W}_{n+m}(\Gamma)$ [4.3), 5.4) and 6.4)]

Proposition 4.1 For any $\Gamma \subseteq \mathcal{L}^s$:

1) $\Gamma \notin \mathbb{A}$ [Hypothesis]

2) $\Gamma \subseteq \mathsf{W}(\Gamma)$ [Def. 10]

3) $\mathsf{W}(\Gamma) \notin \mathbb{A}$ [1) and Prop. 1.2]

Proposition 4.2 For any $\Gamma \subseteq \mathcal{L}^s$:

1) $\mathsf{W}(\Gamma) \notin \mathbb{A}$ [Hypothesis]

2) $\exists j \geq 0 \mid \mathsf{W}_j(\Gamma) \notin \mathbb{A}$ [1) and Def. 10]

3) $\mathsf{W}_j(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ for any $B \in \mathcal{L}$ [2) and Prop. 1.2]

4) $sB \in \mathsf{W}_{j+1}(\Gamma)$ for any $B \in \mathcal{L}$ [3) and Def. 10]

5) $sB \in \mathsf{W}(\Gamma)$ for any $B \in \mathcal{L}$ [4) and Def. 10]

Proposition 4.3 For any $\Gamma \subseteq \mathcal{L}^s$, $B \in \mathcal{L}$ and $n \in \mathbb{N}$:

1) $sB \in \mathsf{W}_n(\Gamma)$ [Hypothesis]

2) $\mathsf{W}_n(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ [1) and Prop. 1.1]

3) $sB \in \mathsf{W}_{n+1}(\Gamma)$ [2) and Def. 10]

Proposition 4.4 We split the proof into two parts.

1. For any $\Gamma \subseteq \mathcal{L}^s$ and $B \in \mathcal{L}$, if $\mathsf{W}(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$, then $sB \in \mathsf{W}(\Gamma)$.

1) $\mathsf{W}(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ [Hypothesis]

2) there are at least two formulae, $s\alpha$ and $s\beta$, that satisfy one of the inadmissibility conditions and belong to $\mathsf{W}(\Gamma) \cup \{\bar{s}B\}$ [1) and Def. 5)]

3) there are two cases to be considered: i. one of the two is $\bar{s}B$ or ii. neither of the two is $\bar{s}B$ [2)]

4) Case i. $s\alpha = \bar{s}B$ and $s\beta \in \mathsf{W}(\Gamma)$.

    4.1) $s\alpha = \bar{s}B$ [Assumption]

    4.2) $s\beta \in \mathsf{W}(\Gamma)$ [Assumption]

    4.3) $s\beta \in \mathsf{W}_n(\Gamma)$ for some $n \in \mathbb{N}$ [4.2) and Def. 10]

    4.4) $s\alpha, s\beta \in \mathsf{W}_n(\Gamma) \cup \{\bar{s}B\}$ [4.1) and 4.3)]

    4.5) $\mathsf{W}_n(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ [2), 4.4) and Def. 5]

5) Case ii. $s\alpha \neq \bar{s}B$, $s\beta \neq \bar{s}B$ and $s\alpha, s\beta \in \mathsf{W}(\Gamma)$.

    5.1) $s\alpha, s\beta \in \mathsf{W}(\Gamma)$ [Assumption]

    5.2) $s\alpha \in \mathsf{W}_j(\Gamma)$ for some $j \in \mathbb{N}$ [5.1) and Def. 10]

    5.3) $s\beta \in \mathsf{W}_m(\Gamma)$ for some $m \in \mathbb{N}$ [5.1) and Def. 10]

    5.4) let n =max$\{j, m\}$ [construction]

    5.5) $\mathsf{W}_j(\Gamma) \subseteq \mathsf{W}_n(\Gamma)$ and $\mathsf{W}_m(\Gamma) \subseteq \mathsf{W}_n(\Gamma)$ [5.4) and Prop. 4.3]

    5.6) $s\alpha, s\beta \in \mathsf{W}_n(\Gamma) \cup \{\bar{s}B\}$ [5.2) and 5.3)]

    5.7) $\mathsf{W}_n(\Gamma) \notin \mathbb{A}$ [2), 5.6) and Def. 5]

    5.8) $\mathsf{W}_n(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ [5.7) and Prop. 1.2]

6) $\mathsf{W}_n(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ for some $n \in \mathbb{N}$ [4) and 5)]

7) $sB \in \mathsf{W}_{n+1}(\Gamma)$ for some $n \in \mathbb{N}$ [6) and Def. 10]

8) $sB \in \mathsf{W}(\Gamma)$ [7) and Def. 10]

2. For any $\Gamma \subseteq \mathcal{L}^s$ and $B \in \mathcal{L}$, if $sB \in \mathsf{W}(\Gamma)$, then $\mathsf{W}(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$.
It follows by Prop. 1.1.

Proposition 4.5 We split the proof in two parts:

1. For any $sB \in \mathcal{L}^s$, if $sB \in \mathsf{W}(\Gamma)$, then $sB \in \mathsf{W}(\mathsf{W}(\Gamma))$.
It follows by Prop. 3.1.

2. For any $sB \in \mathcal{L}^s$, if $sB \in \mathsf{W}(\mathsf{W}(\Gamma))$, then $sB \in \mathsf{W}(\Gamma)$.
If $sB \in \mathsf{W}(\mathsf{W}(\Gamma))$, then there exists some $n \in \mathbb{N}$ such that $sB \in \mathsf{W}_n(\mathsf{W}(\Gamma))$. We prove by induction on n that if $sB \in \mathsf{W}_n(\mathsf{W}(\Gamma))$, then there exists some $l_n \in \mathbb{N}$ such that $sB \in \mathsf{W}_{l_n}(\Gamma)$.

- Base case: $n = 0$. By Def. 10, $\mathsf{W}_0(\mathsf{W}(\Gamma)) = \mathsf{W}(\Gamma)$.
- Inductive hypothesis: $n = m$. We assume that $\mathsf{W}_m(\mathsf{W}(\Gamma)) \subseteq \mathsf{W}_{l_m}(\Gamma)$.
- Inductive step: $n = m+1$. We assume that $sB \in \mathsf{W}_{m+1}(\mathsf{W}(\Gamma))$. Then there are three cases to be considered:
  - $\mathsf{W}_m(\mathsf{W}(\Gamma)) \cup \{\bar{s}B\} \notin \mathbb{A}$
    1) $\mathsf{W}_{l_m}(\Gamma) \cup \{\bar{s}B\} \notin \mathbb{A}$ [Prop. 1.2 and Inductive hypothesis]
    2) $sB \in \mathsf{W}_{l_m+1}(\Gamma)$ [1) and Def. 10]
  - $sB = t\mathsf{K}_g C$ and $\Sigma_g(\mathsf{W}_m(\mathsf{W}(\Gamma))) \cup \{fC\} \notin \mathbb{A}$
    1) $\Sigma_g(\mathsf{W}_m(\mathsf{W}(\Gamma))) \subseteq \Sigma_g(\mathsf{W}_{l_m}(\Gamma))$ [Def. 9 and Inductive hypothesis]
    2) $\Sigma_g(\mathsf{W}_{l_m}(\Gamma)) \cup \{fC\} \notin \mathbb{A}$ [1) and Prop. 1.2]
    3) $t\mathsf{K}_g C \in \mathsf{W}_{l_m+1}(\Gamma)$ [2) and Def. 10]
  - $sB = t\mathsf{K}_g \neg C$ and $\Sigma_g(\mathsf{W}_m(\mathsf{W}(\Gamma))) \cup \{tC\} \notin \mathbb{A}$
    1) $\Sigma_g(\mathsf{W}_m(\mathsf{W}(\Gamma))) \subseteq \Sigma_g(\mathsf{W}_{l_m}(\Gamma))$ [Def. 9 and Inductive hypothesis]
    2) $\Sigma_g(\mathsf{W}_{l_m}(\Gamma)) \cup \{tC\} \notin \mathbb{A}$ [1) and Prop. 1.2]
    3) $t\mathsf{K}_g \neg C \in \mathsf{W}_{l_m+1}(\Gamma)$ [2) and Def. 10]

Proposition 4.6 We prove that for any $sB \in \mathcal{L}^s$, if $sB \in \Sigma_g(\Gamma)$, then $sB \in \mathsf{W}(\Gamma)$. We consider the following two cases:

Case i. $sB = t\neg^{2n} C \in \Sigma_g(\Gamma)$.

1) $t\mathsf{K}_g \neg^{2n} C \in \Gamma = \mathsf{W}_0(\Gamma)$ [Def. 9 and Def. 10]

2) $\mathsf{W}_0(\Gamma) \cup \{f\neg^{2n} C\} \notin \mathbb{A}$ because satisfies inadmissibility condition 12 [1) and Def. 5]

3) $t\neg^{2n} C \in \mathsf{W}_1(\Gamma)$ [2) and Def. 10]

4) $t\neg^{2n} C \in \mathsf{W}(\Gamma)$ [3) and Def. 10]

Case ii. $sB = f\neg^{2n} C \in \Sigma_g(\Gamma)$.

1) $t\mathsf{K}_g \neg^{2n+1} C \in \Gamma = \mathsf{W}_0(\Gamma)$ [Def. 9 and Def. 10]

2) $\mathsf{W}_0(\Gamma) \cup \{f\neg^{2n+1} C\} \notin \mathbb{A}$ because satisfies inadmissibility condition 12 [1) and Def. 5]

3) $t\neg^{2n+1} C \in \mathsf{W}_1(\Gamma)$ [2) and Def. 10]

4) $\mathsf{W}_1(\Gamma) \cup \{t\neg^{2n}C\} \notin \mathbb{A}$ because satisfies inadmissibility condition 1 [3) and Def. 5]

5) $f\neg^{2n}C \in \mathsf{W}_2(\Gamma)$ [4) and Def. 10]

6) $f\neg^{2n}C \in \mathsf{W}(\Gamma)$ [5) and Def. 10]

Proposition 4.7 If $sB \in \mathsf{W}(\Sigma_g(\Gamma))$, then, for some $n \in \mathbb{N}$, $sB \in \mathsf{W}_n(\Sigma_g(\Gamma))$. We prove by induction on n that if $sB \in \mathsf{W}_n(\Sigma_g(\Gamma))$, then there exists some $l_n \in \mathbb{N}$ such that, for $s = t$, $t\mathsf{K}_g B \in \mathsf{W}_{l_n}(\Gamma)$ and, for $s = f$, $t\mathsf{K}_g\neg B \in \mathsf{W}_{l_n}(\Gamma)$.

- Base case: $n = 0$.
  1) $sB \in \mathsf{W}_0(\Sigma_g(\Gamma))$ [Hypothesis]
  2) $sB \in \Sigma_g(\Gamma)$ [1) and Def. 10]
  3) for $s = t$, $t\mathsf{K}_g B \in \Gamma$ and, for $s = f$, $t\mathsf{K}_g\neg B \in \Gamma$ [2) and Def. 9]
  4) $t\mathsf{K}_g B \in \mathsf{W}_0(\Gamma)$ [3) and Def. 10]
- Inductive hypothesis: we assume the conclusion for $n = m$
- Inductive step: $n = m + 1$.
  If $sB \in \mathsf{W}_{m+1}(\Sigma_g(\Gamma))$, then there cases to be considered:
  1) Case i: $\Lambda = \mathsf{W}_m(\Sigma_g(\Gamma)) \cup \{\bar{s}B\} \notin \mathbb{A}$
     It follows that for some $\Delta = \{sG, sF, (sH)\} \subseteq \Lambda$, the formulae in Δ satisfy the inadmissibility condition C. We have to consider two subcases:
     - $\bar{s}B \notin \Delta$ and $\Delta \subseteq \mathsf{W}_m(\Sigma_g(\Gamma))$
       1) for each $sC \in \Delta$, $t\mathsf{K}_g C \in \mathsf{W}_{l_m}(\Gamma)$ for $s = t$ and $t\mathsf{K}_g\neg C \in \mathsf{W}_{l_m}(\Gamma)$ for $s = f$ by the Inductive hypothesis
       2) $\Delta \subseteq \Sigma_g(\mathsf{W}_{l_m}(\Gamma))$ by Def. 9
       3) $\Sigma_g(\mathsf{W}_{l_m}(\Gamma)) \notin \mathbb{A}$
       4) $\Sigma_g(\mathsf{W}_{l_m}(\Gamma)) \cup \{\bar{s}B\} \notin \mathbb{A}$ by 3) and Prop. 1.2
       5) $t\mathsf{K}_g B \in \mathsf{W}_{l_m+1}(\Gamma)$ for $s = t$ and $t\mathsf{K}_g\neg B \in \mathsf{W}_{l_m+1}(\Gamma)$ for $s = f$
     - $\bar{s}B \in \Delta$
       1) for each $sC \in \Delta - \{\bar{s}B\}$, $t\mathsf{K}_g C \in \mathsf{W}_{l_m}(\Gamma)$ for $s = t$ and $t\mathsf{K}_g\neg C \in \mathsf{W}_{l_m}(\Gamma)$ for $s = f$ by the Inductive hypothesis
       2) $\Delta \subseteq \Sigma_g(\mathsf{W}_{l_m}(\Gamma)) \cup \{\bar{s}B\}$ by Def. 9
       3) $\Sigma_g(\mathsf{W}_{l_m}(\Gamma)) \cup \{\bar{s}B\} \notin \mathbb{A}$ by 2)
       4) $t\mathsf{K}_g B \in \mathsf{W}_{l_m+1}(\Gamma)$ for $s = t$ and $t\mathsf{K}_g\neg B \in \mathsf{W}_{l_m+1}(\Gamma)$ for $s = f$

2) Case ii. $\Lambda = \Sigma_f(\mathsf{W}_m(\Sigma_g(\Gamma))) \cup \{fD\} \notin \mathbb{A}$ and $sB = t\mathsf{K}_f D$
   It follows that for some $\Delta = \{sG, sF, (sH)\} \subseteq \Lambda$, the formulae in Δ satisfy some inadmissibility condition. Notice that:
   $(*)$ if $sD \in \Sigma_f(\mathsf{W}_m(\Sigma_g(\Gamma)))$, then $sD \in \Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma))$, because:
   - for $s = t$:
     $tD \in \Sigma_f(\mathsf{W}_m(\Sigma_g(\Gamma)))$
     $t\mathsf{K}_f D \in \mathsf{W}_m(\Sigma_g(\Gamma))$ by Def. 9
     $t\mathsf{K}_g\mathsf{K}_f D \in \mathsf{W}_{l_m}(\Gamma)$ by Inductive hypothesis
     $tD \in \Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma))$ by Def. 9
   - *Mutatis mutandis* for $s = f$

   We have to consider two sub-cases:
   - $fD \notin \Delta$ and $\Delta \subseteq \Sigma_f(\mathsf{W}_m(\Sigma_g(\Gamma)))$
     1) $\Delta \subseteq \Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma))$ by $(*)$
     2) $\Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma)) \notin \mathbb{A}$
     3) $\Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma)) \cup \{fD\} \notin \mathbb{A}$ by 2) and Prop. 1.2
     4) $t\mathsf{K}_g\mathsf{K}_f D \in \mathsf{W}_{l_m+1}(\Gamma)$ by Def. 9
   - $fD \in \Delta$
     1) $\Delta - \{fD\} \subseteq \Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma))$ by $(*)$
     2) $\Delta \subseteq \Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma)) \cup \{fD\}$ by 2)
     3) $\Sigma_{(g,f)}(\mathsf{W}_{l_m}(\Gamma)) \cup \{fD\} \notin \mathbb{A}$
     4) $t\mathsf{K}_g\mathsf{K}_f D \in \mathsf{W}_{l_m+1}(\Gamma)$ by Def. 9

3) Case iii. $\Lambda = \Sigma_f(\mathsf{W}_m(\Sigma_g(\Gamma))) \cup \{tD\} \notin \mathbb{A}$ and $sB = t\mathsf{K}_f\neg D$. *Mutatis mutandis* as in 2.

4) $t\mathsf{K}_g B \in \mathsf{W}(\Gamma)$ for $s = t$ and $t\mathsf{K}_g\neg B \in \mathsf{W}(\Gamma)$ for $s = f$

Proposition 4.8 For any $J = \{q_1, \ldots, q_j\}$, $\Gamma \in \mathbb{A}$ and $A \in \mathcal{L}$, assume that $sA \in \mathsf{W}(\Delta)$ for any minimal refinement of Γ on J. By Def. 6, every $\Lambda \in \mathsf{R}^J_\Gamma$ is such that $\Gamma \in \Lambda$ and $J_\Lambda = \{s_1q_1, \ldots, s_jq_j\} \in \Lambda$ for certain $s_i \in \mathcal{T}$. Now, let $\Delta_\Lambda = \Gamma \cup J_\Lambda$. By construction, $\Delta_\Lambda \subseteq \Lambda$. Moreover, Δ_Λ is a minimal refinement of Γ on J and, by hypothesis, $sA \in \mathsf{W}(\Delta_\Lambda)$. By Prop. 3.1, we can conclude that $sA \in \mathsf{W}(\Lambda)$ and, since we haven't made any specific assumption on Λ, the result holds for every $\Lambda \in \mathsf{R}^J_\Gamma$.

Proposition 5.1 Fix x as one of e, u, c. For any $\mathsf{I}_i = (\varphi^{\mathsf{M}}_i, \mathbb{R}^{\delta^{\mathsf{M}}_i})$ in any model $\mathsf{M} \in \mathcal{M}_0$ and for any $B \in \mathcal{L}$:

1) $sB \in \varphi^{\mathsf{M}}_i$ [Hypothesis]

2) $sB \in Cnx_0(\mathsf{I}_i)$ [1) and Def. 11, Def. 12, Def. 13]

3) $sB \in Cnx(\mathsf{I}_i)$ [2) and Def. 11, Def. 12, Def. 13]

Proposition 5.2 Fix x as one of e, u, c. If $sB \in Cnx(\mathsf{I}_i)$, then, by Def. 11, Def. 12 and Def. 13, there exists some $m \in \mathbb{N}$ such that $sB \in Cnx_m(\mathsf{I}_i)$. We prove by induction on m that if $sB \in Cnx_m(\mathsf{I}_i)$, then $sB \in Cnx_m(\mathsf{I}_i^1)$.

- Base case: $m = 0$. We prove that if $sB \in Cnx_0(\mathsf{I}_i)$, then $sB \in Cnx_0(\mathsf{I}_i^1)$.
  1) $sB \in Cnx_0(\mathsf{I}_i)$ [Hypothesis]
  2) $sB \in \varphi_i^{\mathsf{M}}$ [1) and Def. 11, Def. 12, Def. 13]
  3) $sB \in \varphi_i^{\mathsf{M}} \cup \{sA\}$ [2) and Construction]
  4) $sB \in \varphi_i^{\mathsf{M}^1}$ [3) and Def. of $\varphi_i^{\mathsf{M}^1}$]
  5) $sB \in Cnx_0(\mathsf{I}_i^1)$ [4) and Def. 11, Def. 12, Def. 13]
- Inductive hypothesis: $m = n-1$. We assume that $Cnx_{n-1}(\mathsf{I}_i) \subseteq Cnx_{n-1}(\mathsf{I}_i^1)$.
- Inductive step: $m = n$. We prove that if $sB \in Cnx_n(\mathsf{I}_i)$, then $sB \in Cnx_n(\mathsf{I}_i^1)$.
  1) $sB \in Cnx_n(\mathsf{I}_i)$ [Hypothesis]
  2) There are four cases to be considered: i. $sB \in \mathsf{W}(Cnx_{n-1}(\mathsf{I}_i))$, ii. if $x = u$, $sB = t\mathcal{K}_iC \in \mathsf{V}(Cnu_{n-1}(\mathsf{I}_i))$, iii. if $x = e$, $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_{n-1}(\mathsf{I}_i))$ and iv. if $x = c$, $sB = t\mathcal{K}_iC \in \mathsf{CSV}(Cnc_{n-1}(\mathsf{I}_i))$.
  3) Case i. $sB \in \mathsf{W}(Cnx_{n-1}(\mathsf{I}_i))$.
     3.1) $Cnx_{n-1}(\mathsf{I}_i) \subseteq Cnx_{n-1}(\mathsf{I}_i^1)$ [Inductive hypothesis]
     3.2) $sB \in \mathsf{W}(Cnx_{n-1}(\mathsf{I}_i^1))$ [3), 3.1) and Prop. 3.2]
     3.3) $sB \in Cnx_n(\mathsf{I}_i^1)$ [3.2) and Def. 11, Def. 12, Def. 13]
  4) Case ii. If $x = u$, $sB = t\mathcal{K}_iC \in \mathsf{V}(Cnu_{n-1}(\mathsf{I}_i))$.
     4.1) there exists some $J \in \wp(\mathcal{P})$ for which:
        i. $|J| \leq \delta_i^{\mathsf{M}}$ and
        ii. for any $\Delta \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i)}$, $tC \in \mathsf{W}(\Delta)$.
     4.2) for $J \in \wp(\mathcal{P})$, $|J| \leq \delta_i^{\mathsf{M}^1}$ [4.1.i) and Def. of $\delta_i^{\mathsf{M}^1}$]
     4.3) $Cnu_{n-1}(\mathsf{I}_i) \subseteq Cnu_{n-1}(\mathsf{I}_i^1)$ [Inductive hypothesis]
     4.4) $\mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i^1)} \subseteq \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i)}$ [4.3) and Prop. 2.2]
     4.5) for all $\Lambda \in \mathbb{A}$, if $\Lambda \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i^1)}$, then $\Lambda \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i)}$ [4.4)]
     4.6) for all $\Lambda \in \mathbb{A}$, if $\Lambda \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i^1)}$, then $tC \in \mathsf{W}(\Lambda)$ [4.1.ii) and 4.5)]

4.7) $sB = t\mathcal{K}_iC \in \mathsf{V}(Cnu_{n-1}(\mathsf{I}_i^1))$ [4.2) and 4.6)]

4.8) $sB = t\mathcal{K}_iC \in Cnu_n(\mathsf{I}_i^1)$ [4.7)]

5) Case iii. If $x = e$, $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_{n-1}(\mathsf{I}_i))$

5.1) There exists some $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$ for which:

i. $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$
$\delta_i^{\mathsf{M}} \geq |L| + |J|$
$\delta_j^{\mathsf{M}} \geq |J|$

ii. $\exists A_2(tC \in \mathsf{W}(Cne_{n-1}(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_2\}) \wedge \forall\Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}_i)}\ (tA_2 \in \mathsf{W}(\Lambda)\vee$
$\exists A_1(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_jA_1\}) \wedge \forall\Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta))))).$

5.2) Since by definition of M^1 for all $j \in \mathcal{A}$ $\delta_j^{\mathsf{M}^1} = \delta_j^{\mathsf{M}}$, we have that for that $\mathbf{J}$:
$\delta_i^{\mathsf{M}^1} \geq \delta_j^{\mathsf{M}^1}$
$\delta_i^{\mathsf{M}^1} \geq |L| + |J|$
$\delta_j^{\mathsf{M}^1} \geq |J|$ [5.1.i)]

5.3) $Cne_{n-1}(\mathsf{I}_i) \subseteq Cne_{n-1}(\mathsf{I}_i^1)$ [Inductive hypothesis]

5.4) for A_2 $(tC \in \mathsf{W}(Cne_{n-1}(\mathsf{I}_i^1) \cup \{t\mathcal{K}_iA_2\})$ [5.1.ii), 5.3) and Prop. 3.2]

5.5) for all $\Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}_i^1)}$, $\Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}_i)}$ [5.3) and Prop. 2.2)]

5.6) for $A_2(tC \in \mathsf{W}(Cne_{n-1}(\mathsf{I}_i^1) \cup \{t\mathcal{K}_iA_2\}) \wedge \forall\Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}_i^1)}(tA_2 \in \mathsf{W}(\Lambda)\vee$
for $A_1(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_jA_1\}) \wedge \forall\Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta))))).$
[5.1.ii), 5.4) and 5.5)]

5.7) $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_{n-1}(\mathsf{I}_i^1))$ [5.2), 5.6) and Def. 12]

5.8) $sB = t\mathcal{K}_iC \in Cne_n(\mathsf{I}_i^1)$ [5.7) and Def. 12]

6) Case iv. If $x = c$, $sB = t\mathcal{K}_iC \in \mathsf{CSV}(Cnc_{n-1}(\mathsf{I}_i))$
There exists some $\mathbf{J} = ((L, i), (J_1, j_1), \ldots, (J_m, j_m)) \in \mathsf{LAP}_i$ for which:

i. $\delta_i^{\mathsf{M}} \geq \delta_{j_1}^{\mathsf{M}}$
$\delta_i^{\mathsf{M}} \geq |L|$
$\forall m = 1, \ldots, n,\ \delta_m^{\mathsf{M}} \geq \sum\limits_{k=m}^{n} |J_k|;$

ii. and for $A_n \in \mathcal{L}$:
$\exists A_{n+1}(tC \in \mathsf{W}(Cnc_{n-1}(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge \forall\Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}_i)}(tA_{n+1} \in \mathsf{W}(\Delta_0)\vee$
$\exists A_n(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1}A_n\}) \wedge \forall\Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}(tA_n \in \mathsf{W}(\Delta_1)\vee$
$\vdots$

$\exists A_1(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n}A_1\}) \wedge \forall \Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}(tA_1 \in \mathsf{W}(\Delta_n)))\ldots)$.

6.1) Since by definition of M^1 for all $j \in \mathcal{A}$ $\delta_j^{\mathsf{M}^1} = \delta_j^{\mathsf{M}}$, we have that for that $\mathbf{J}$:
$\delta_i^{\mathsf{M}^1} \geq \delta_{j_1}^{\mathsf{M}^1}$
$\delta_i^{\mathsf{M}^1} \geq |L|$
$\forall m = 1, \ldots, n,\ \delta_m^{\mathsf{M}^1} \geq \sum\limits_{k=m}^{n} |J_k|$

6.2) $Cnc_{n-1}(\mathsf{I}_i) \subseteq Cnc_{n-1}(\mathsf{I}_i^1)$ [Inductive hypothesis]

6.3) for A_{n+1}, $tC \in \mathsf{W}(Cnc_{n-1}(\mathsf{I}_i^1) \cup \{t\mathcal{K}_i A_{n+1}\})$ [6), 6.2) and Prop. 3.2]

6.4) for all $\Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}_i^1)}$, $\Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}_i)}$ [6.2) and Prop. 2.2]

6.5) for A_{n+1}, $(tC \in \mathsf{W}(Cnc_{n-1}(\mathsf{I}_i^1) \cup \{t\mathcal{K}_i A_{n+1}\}) \wedge \forall \Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}_i^1)}(tA_{n+1} \in \mathsf{W}(\Delta_0) \vee$
for A_n $(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1}A_n\}) \wedge \forall \Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}$ $(tA_n \in \mathsf{W}(\Delta_1) \vee$
$\vdots$
for A_1 $(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n}A_1\}) \wedge \forall \Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}$ $(tA_1 \in \mathsf{W}(\Delta_n)\ )\ )\ \ldots)$.
[6), 6.3) and 6.4)]

6.6) $sB = t\mathcal{K}_i C \in \mathsf{CSV}(Cnc_{n-1}(\mathsf{I}_i^1))$ [6.1), 6.5) and Def. 13]

6.7) $sB = t\mathcal{K}_i C \in Cnc_n(\mathsf{I}_i^1)$ [6.6) and Def. 13]

7) $sB \in Cnx_n(\mathsf{I}_i^1)$ [3.3), 4.8), 5.8) and 6.7)]

Proposition 5.3 Fix x as one of u, e, c. If $sB \in Cnx(\mathsf{I}_i)$, then, by Def. 11, Def. 12 and Def. 13, there exists some $m \in \mathbb{N}$ such that $sB \in Cnx_m(\mathsf{I}_i)$. We prove by induction on m that if $sB \in Cnx_m(\mathsf{I}_i)$, then $sB \in Cnx_m(\mathsf{I}_i^2)$.

- Base case: $m = 0$. We prove that if $sB \in Cnx_0(\mathsf{I}_i)$, then $sB \in Cnx_0(\mathsf{I}_i^2)$.

  1) $sB \in Cnx_0(\mathsf{I}_i)$ [Hypothesis]
  2) $sB \in \varphi_i^{\mathsf{M}}$ [1) and Def. 11, Def. 12, Def. 13]
  3) $sB \in Cnx_0(\mathsf{I}_i^2)$ [2) and Def. of $\varphi^{\mathsf{M}_i^2}$]

- Inductive hypothesis: $m = n - 1$. We assume that $Cnx_{n-1}(\mathsf{I}_i) \subseteq Cnx_{n-1}(\mathsf{I}_i^2)$.

- Inductive step: $m = n$. We prove that if $sB \in Cnx_n(\mathsf{I}_i)$, then $sB \in Cnx_n(\mathsf{I}_i^2)$.

  1) $sB \in Cnx_n(\mathsf{I}_i)$ [Hypothesis]

2) There are four cases to be considered: i. $sB \in \mathsf{W}(Cnx_{n-1}(\mathsf{I}_i))$, ii. if $x = u$, $sB = t\mathcal{K}_iC \in \mathsf{V}(Cnu_{n-1}(\mathsf{I}_i))$, iii. if $x = e$, $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_{n-1}(\mathsf{I}_i))$ and iv. if $x = c$, $sB = t\mathcal{K}_iC \in \mathsf{CSV}(Cnc_{n-1}(\mathsf{I}_i))$.

3) Case i. $sB \in \mathsf{W}(Cnx_{n-1}(\mathsf{I}_i))$.

 3.1) $Cnx_{n-1}(\mathsf{I}_i) \subseteq Cnx_{n-1}(\mathsf{I}_i^2)$ [Inductive hypothesis]

 3.2) $sB \in \mathsf{W}(Cnx_{n-1}(\mathsf{I}_i^2))$ [3), 3.1) and Prop. 3.2]

 3.3) $sB \in Cnx_n(\mathsf{I}_i^2)$ [3.2) and Def. 11, Def. 12, Def. 13]

4) Case ii. If $x = u$, $sB = t\mathcal{K}_iC \in \mathsf{V}(Cnu_{n-1}(\mathsf{I}_i))$.

 4.1) there exists some $J \in \wp(\mathcal{P})$ for which:

  i. $|J| \leq \delta_i^{\mathsf{M}}$ and

  ii. for any $\Delta \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i)}$, $tC \in \mathsf{W}(\Delta)$.

 4.2) for $J \in \wp(\mathcal{P})$, $|J| \leq \delta_i^{\mathsf{M}^2}$ [4.1.i) and Def. of $\delta_i^{\mathsf{M}^2}$]

 4.3) $Cnu_{n-1}(\mathsf{I}_i) \subseteq Cnu_{n-1}(\mathsf{I}_i^2)$ [Inductive hypothesis]

 4.4) $\mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i^2)} \subseteq \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i)}$ [4.3) and Prop. 2.2]

 4.5) for all $\Lambda \in \mathbb{A}$, if $\Lambda \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i^2)}$, then $\Lambda \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i)}$ [4.4)]

 4.6) for all $\Lambda \in \mathbb{A}$, if $\Lambda \in \mathsf{R}^J_{Cnu_{n-1}(\mathsf{I}_i^2)}$, then $tC \in \mathsf{W}(\Lambda)$ [4.1.ii) and 4.5)]

 4.7) $sB = t\mathcal{K}_iC \in \mathsf{V}(Cnu_{n-1}(\mathsf{I}_i^2))$ [4.2) and 4.6)]

 4.8) $sB = t\mathcal{K}_iC \in Cnu_n(\mathsf{I}_i^2)$ [4.7)]

5) Case iii. If $x = e$, $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_{n-1}(\mathsf{I}_i))$

 5.1) There exists some $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$ for which:

  i. $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$
  $\delta_i^{\mathsf{M}} \geq |L| + |J|$
  $\delta_j^{\mathsf{M}} \geq |J|$

  ii. $\exists A_2(tC \in \mathsf{W}(Cne_{n-1}(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_2\}) \wedge \forall \Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}_i)}\ (tA_2 \in \mathsf{W}(\Lambda) \vee$
  $\exists A_1(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_jA_1\}) \wedge \forall \Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta)))))$.

 5.2) Since, by definition of M^2, $\delta_i^{\mathsf{M}^2} = \delta_i^{\mathsf{M}} + 1$ and for all $j \in \mathcal{A} - i$ $\delta_j^{\mathsf{M}^2} = \delta_j^{\mathsf{M}}$, we have that for that $\mathbf{J}$:
 $\delta_i^{\mathsf{M}^2} \geq \delta_j^{\mathsf{M}^2}$
 $\delta_i^{\mathsf{M}^2} \geq |L| + |J|$
 $\delta_j^{\mathsf{M}^2} \geq |J|$ [5.1.i)]

 5.3) $Cne_{n-1}(\mathsf{I}_i) \subseteq Cne_{n-1}(\mathsf{I}_i^2)$ [Inductive hypothesis]

 5.4) for A_2 $(tC \in \mathsf{W}(Cne_{n-1}(\mathsf{I}_i^2) \cup \{t\mathcal{K}_iA_2\})$ [5.1.ii), 5.3) and Prop. 3.2]

5.5) for all $\Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}^2_i)}$, $\Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}_i)}$ [5.3) and Prop. 2.2)]

5.6) for $A_2(tC \in \mathsf{W}(Cne_{n-1}(\mathsf{I}^2_i) \cup \{t\mathcal{K}_i A_2\}) \wedge \forall \Lambda \in \mathsf{R}^L_{Cne_{n-1}(\mathsf{I}^2_i)}(tA_2 \in \mathsf{W}(\Lambda) \vee$
for $A_1(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_j A_1\}) \wedge \forall \Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta))))))$.
[5.1.ii), 5.4) and 5.5)]

5.7) $sB = t\mathcal{K}_i C \in \mathsf{SV}(Cne_{n-1}(\mathsf{I}^2_i))$ [5.2), 5.6) and Def. 12]

5.8) $sB = t\mathcal{K}_i C \in Cne_n(\mathsf{I}^2_i)$ [5.7) and Def. 12]

6) Case iv. If $x = c$, $sB = t\mathcal{K}_i C \in \mathsf{CSV}(Cnc_{n-1}(\mathsf{I}_i))$
There exists some $\mathbf{J} = ((L, i), (J_1, j_1), \ldots, (J_m, j_m)) \in \mathsf{LAP}_i$ for which:

i. $\delta^{\mathsf{M}}_i \geq \delta^{\mathsf{M}}_{j_1}$
$\delta^{\mathsf{M}}_i \geq |L|$
$\forall m = 1, \ldots, n$, $\delta^{\mathsf{M}}_m \geq \sum\limits_{k=m}^{n} |J_k|$;

ii. and for $A_n \in \mathcal{L}$:
$\exists A_{n+1}(tC \in \mathsf{W}(Cnc_{n-1}(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_{n+1}\}) \wedge \forall \Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}_i)}(tA_{n+1} \in \mathsf{W}(\Delta_0) \vee$
$\exists A_n(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1} A_n\}) \wedge \forall \Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}(tA_n \in \mathsf{W}(\Delta_1) \vee$
$\vdots$
$\exists A_1(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n} A_1\}) \wedge \forall \Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}(tA_1 \in \mathsf{W}(\Delta_n))) \ldots)$.

6.1) Since, by definition of M^2, $\delta^{\mathsf{M}^2}_i = \delta^{\mathsf{M}}_i + 1$ and for all $j \in \mathcal{A} - i$ $\delta^{\mathsf{M}^2}_j = \delta^{\mathsf{M}}_j$, we have that for that $\mathbf{J}$:
$\delta^{\mathsf{M}^2}_i \geq \delta^{\mathsf{M}^2}_{j_1}$
$\delta^{\mathsf{M}^2}_i \geq |L|$
$\forall m = 1, \ldots, n$, $\delta^{\mathsf{M}^2}_m \geq \sum\limits_{k=m}^{n} |J_k|$

6.2) $Cnc_{n-1}(\mathsf{I}_i) \subseteq Cnc_{n-1}(\mathsf{I}^2_i)$ [Inductive hypothesis]

6.3) for A_{n+1}, $tC \in \mathsf{W}(Cnc_{n-1}(\mathsf{I}^2_i) \cup \{t\mathcal{K}_i A_{n+1}\})$ [6), 6.2) and Prop. 3.2]

6.4) for all $\Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}^2_i)}$, $\Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}_i)}$ [6.2) and Prop. 2.2]

6.5) for A_{n+1}, $(tC \in \mathsf{W}(Cnc_{n-1}(\mathsf{I}^2_i) \cup \{t\mathcal{K}_i A_{n+1}\}) \wedge \forall \Delta_0 \in \mathsf{R}^L_{Cnc_{n-1}(\mathsf{I}^2_i)}(tA_{n+1} \in \mathsf{W}(\Delta_0) \vee$
for A_n $(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1} A_n\}) \wedge \forall \Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}$ $(tA_n \in \mathsf{W}(\Delta_1) \vee$
$\vdots$
for A_1 $(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n} A_1\}) \wedge \forall \Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}$ $(tA_1 \in \mathsf{W}(\Delta_n)\ )\ )\ \ldots)$.

[6), 6.3) and 6.4)]

6.6) $sB = t\mathcal{K}_iC \in \mathsf{CSV}(Cnc_{n-1}(\mathsf{I}_i^2))$ [6.1), 6.5) and Def. 13]

6.7) $sB = t\mathcal{K}_iC \in Cnc_n(\mathsf{I}_i^2)$ [6.6) and Def. 13]

7) $sB \in Cnx_n(\mathsf{I}_i^2)$ [3.3), 4.8), 5.8) and 6.7)]

Proposition 6. We split the proof into two parts.

1. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For any $sB \in \mathcal{L}^s$ and $\Gamma \in \mathbb{A}$, if sB is a DBELx_k-consequence of Γ, then for all $\mathsf{M} \in \mathcal{M}_k$ and for all $\mathsf{I}_i \in \mathsf{M}$, if $\Gamma \subseteq Cnx(\mathsf{I}_i)$, then $sB \in Cnx(\mathsf{I}_i)$.

We prove the contrapositive.

1) For $\mathsf{M}^1 \in \mathcal{M}_k$ and for $\mathsf{I}_i^1 \in \mathsf{M}^1$, $\Gamma \subseteq Cnx(\mathsf{I}_i^1)$ and $sB \notin Cnx(\mathsf{I}_i^1)$ [Hypothesis]

2) Let $\mathsf{M}^2 = (\mathbb{A}, \varphi^{\mathsf{M}^2}, \delta^{\mathsf{M}^2}, \mathsf{I}_1^2, \ldots, \mathsf{I}_n^2) \in \mathcal{M}_k$ such that:

2.1) $\varphi_j^{\mathsf{M}^2} = \Gamma$ for all $j \in \mathcal{A} - \{i\}$

2.2) $\varphi_i^{\mathsf{M}^2} = \varphi_i^{\mathsf{M}^1}$

2.3) $\delta_i^{\mathsf{M}^2} = \delta_i^{\mathsf{M}^1}$

3) For $\mathsf{M}^2 \in \mathcal{M}_k$ and for all $\mathsf{I}_j^2 \in \mathsf{M}^2$, $\Gamma \subseteq Cnx(\mathsf{I}_j^2)$ [1), 2) and Prop. 5.1]

4) For $\mathsf{M}^2 \in \mathcal{M}_k$, $\Gamma \subseteq Cnx(\mathsf{M}^2)$ [3) and Def. 15]

5) For $\mathsf{M}^2 \in \mathcal{M}_k$ and $\mathsf{I}_i^2 \in \mathsf{M}^2$, $sB \notin Cnx(\mathsf{I}_i^2)$ [1) and 2)]

6) For $\mathsf{M}^2 \in \mathcal{M}_k$, $sB \notin Cnx(\mathsf{M}^2)$ [5) and Def. 15]

7) For $\mathsf{M}^2 \in \mathcal{M}_k$, $\Gamma \subseteq Cnx(\mathsf{M}^2)$ and $sB \notin Cnx(\mathsf{M}^2)$ [4) and 6)]

8) $sB \notin Cnx^k(\Gamma)$ [7) and Def. 16]

2. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For any $sB \in \mathcal{L}^s$ and $\Gamma \in \mathbb{A}$, if for all $\mathsf{M} \in \mathcal{M}_k$ and for all $\mathsf{I}_i \in \mathsf{M}$, if $\Gamma \subseteq Cnx(\mathsf{I}_i)$, then $sB \in Cnx(\mathsf{I}_i)$, then sB is a DBELx_k-consequence of Γ.

We prove the contrapositive.

1) $sB \notin Cnx^k(\Gamma)$ [Hypothesis]

2) For some $\mathsf{M} \in \mathcal{M}_k$, $\Gamma \subseteq Cnx(\mathsf{M})$ and $sB \notin Cnx(\mathsf{M})$ [1) and Def. 16]

3) For some $\mathsf{M} \in \mathcal{M}_k$, $\Gamma \subseteq Cnx(\mathsf{I}_j)$ for all $\mathsf{I}_j \in \mathsf{M}$ and $sB \notin Cnx(\mathsf{I}_i)$ for some $\mathsf{I}_i \in \mathsf{M}$ [2) and Def. 15]

4) For some $\mathsf{M} \in \mathcal{M}_k$ and $\mathsf{I}_i \in \mathsf{M}$, $\Gamma \subseteq Cnx(\mathsf{I}_i)$ and $sB \notin Cnx(\mathsf{I}_i)$ [3)]

Proposition 7.1. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For all $\Gamma \in \mathbb{A}$ and $sB \in \mathcal{L}^s$:

1) $sB \in \Gamma$ [Hypothesis]

2) For any $\mathsf{M} \in \mathcal{M}_k$ such that $\Gamma \subseteq Cnx(\mathsf{M})$, $sB \in Cnx(\mathsf{M})$ [1)]

3) $sB \in Cnx^k(\Gamma)$ [2) and Def. 16]

Proposition 7.2. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For all $\Gamma, \Gamma \cup \{sA\} \in \mathbb{A}$ and $sA, sB \in \mathcal{L}^s$:

1) $sB \in Cnx^k(\Gamma)$ [Hypothesis]

2) For all $\mathsf{M} \in \mathcal{M}_k$, if $\Gamma \subseteq Cnx(\mathsf{M})$, then $sB \in Cnx(\mathsf{M})$ [1) and Def. 16]

3) For all $\mathsf{M}' \in \mathcal{M}_k$, if $\Gamma \cup \{sA\} \subseteq Cnx(\mathsf{M}')$, then $\Gamma \subseteq Cnx(\mathsf{M}')$ [Construction]

4) For all $\mathsf{M}' \in \mathcal{M}_k$, if $\Gamma \cup \{sA\} \subseteq Cnx(\mathsf{M}')$, then $sB \in Cnx(\mathsf{M}')$ [2) and 3)]

5) $sB \in Cnx^k(\Gamma \cup \{sA\})$ [4) and Def. 16]

Proposition 7.3. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For all $\Gamma, \Gamma \cup \{sA\} \in \mathbb{A}$ and $sA, sB \in \mathcal{L}^s$:

1) $sA \in Cnx^k(\Gamma)$ [Hypothesis]

2) For all $\mathsf{M} \in \mathcal{M}_k$, if $\Gamma \subseteq Cnx(\mathsf{M})$, then $sA \in Cnx(\mathsf{M})$ [1) and Def. 16]

3) For all $\mathsf{M} \in \mathcal{M}_k$, if $\Gamma \subseteq Cnx(\mathsf{M})$, then $\Gamma \cup \{sA\} \subseteq Cnx(\mathsf{M})$ [2)]

4) $sB \in Cnx^k(\Gamma \cup \{sA\})$ [Hypothesis]

5) For all $\mathsf{M} \in \mathcal{M}_k$, if $\Gamma \cup \{sA\} \subseteq Cnx(\mathsf{M})$, then $sB \in Cnx(\mathsf{M})$ [4) and Def. 16]

6) For all $\mathsf{M} \in \mathcal{M}_k$, if $\Gamma \subseteq Cnx(\mathsf{M})$, then $sB \in Cnx(\mathsf{M})$ [3) and 5)]

7) $sB \in Cnx^k(\Gamma)$ [6) and Def. 16]

Proposition 8. Fix x as one of u, e, c. Prop. 8 is equivalent to say that $Cnx^0(\varnothing) = \varnothing$. Suppose ad absurdum that $sB \in Cnx^0(\varnothing)$. By Def. 16 and Def. 15, the hypothesis ad absurdum is equivalent to assume that for all $\mathsf{M} \in \mathcal{M}_0$ and for all $\mathsf{I}_i \in \mathsf{M}$, $sB \in Cnx(\mathsf{I}_i)$. Let $\mathsf{M}' \in \mathcal{M}_0$ such that for all $i \in \mathcal{A}$, $\mathsf{I}'_i = (\varnothing, \mathbb{R}^0)$. We now show that $Cnx(\mathsf{I}'_i) = \varnothing$, contradicting the hypothesis ad absurdum and thus proving the thesis.

We prove by induction on n that for all $n \in \mathbb{N}$, $Cnx_n(\mathsf{I}'_i) = \varnothing$:

- Base case: $n = 0$. $Cnx_0(\mathsf{I}'_i) = \varnothing$ since $\varphi_i^{\mathsf{M}'} = \varnothing$ by construction, by Def. 11, by Def. 12 and by Def. 13 $Cnx_0(\mathsf{I}'_i) = \varphi_i^{\mathsf{M}'}$.
- Inductive hypothesis: $n = m$. We assume that $Cnx_m(\mathsf{I}'_i) = \varnothing$.
- Inductive step: $n = m + 1$. We prove that $Cnx_{m+1}(\mathsf{I}'_i) = \varnothing$.
  Due to Def. 11, Def. 12 and Def. 13, we have to prove four facts: i. $\mathsf{W}(Cnx_m(\mathsf{I}'_i)) = \varnothing$, ii. if $x = u$, $\mathsf{V}(Cnu_m(\mathsf{I}'_i)) = \varnothing$, iii. if $x = e$, $\mathsf{SV}(Cne_m(\mathsf{I}'_1)) = \varnothing$ and iv. if $x = c$, $\mathsf{CSV}(Cnc_m(\mathsf{I}'_1)) = \varnothing$.

  i. $\mathsf{W}(Cnx_m(\mathsf{I}'_i)) = \varnothing$. By the Inductive hypothesis, this amounts to show that $\mathsf{W}(\varnothing) = \varnothing$. We prove by induction that for all $n \in \mathbb{N}$, $\mathsf{W}_n(\varnothing) = \varnothing$.

  - Base case: n= 0. $\mathsf{W}_0(\varnothing) = \varnothing$ by Def. 10.
  - Inductive hypothesis: n= m-1. We assume that $\mathsf{W}_{m-1}(\varnothing) = \varnothing$.
  - Inductive step: n=m.
    For any $B \in \mathcal{L}$, $\mathsf{W}_i^{m-1}(\varnothing) \cup \{\bar{s}B\} \in \mathbb{A}$, because each admissibility condition involves at least two formulae, while $\mathsf{W}_i^{m-1}(\varnothing) \cup \{\bar{s}B\}$ is a singleton due to the Inductive hypothesis.
    For all $g \in \mathsf{LA}$, $\Sigma_g(\mathsf{W}_{m-1}(\varnothing)) = \varnothing$. And again for any $s\neg^{2m}B \in \mathcal{L}$, $\Sigma_g(\mathsf{W}_{m-1}(\varnothing)) \cup \{s\neg^{2m}B\} \in \mathbb{A}$, because each admissibility condition involves at least two formulae, while $\Sigma_g(\mathsf{W}_{m-1}(\varnothing)) \cup \{\bar{s}B\}$ is a singleton.
    Thus, we can conclude that $\mathsf{W}_m(\varnothing) = \varnothing$.

  ii. $\mathsf{V}(Cnu_m(\mathsf{I}'_i)) = \varnothing$.
  By Def. 16, $\mathsf{V}(Cnu_m(\mathsf{I}'_i))$ consists of all and only formulae of the kind $t\mathcal{K}_iB$ such that there exists some $J \in \wp(\mathcal{P})$ for which:

  1. $|J| \leq \delta_i^{\mathsf{M}'}$ and
  2. for any $\Delta \in \mathsf{R}^J_{Cnu_m(\mathsf{I}'_i)}$, $tB \in \mathsf{W}(\Delta)$.

The only possible J here is $\varnothing$, since $\delta_i^{\mathsf{M}'} = 0$; by the Inductive hypothesis we know that $Cnu_m(\mathsf{I}'_i) = \varnothing$ and the minimal refinement of $\varnothing$ on $\varnothing$ is $\varnothing$ itself. This amounts to say that $\mathsf{V}(Cnu_m(\mathsf{I}'_i)) = \{t\mathcal{K}_i B$ such that tB $\in \mathsf{W}(\varnothing)\}$. We have just proved that $\mathsf{W}(\varnothing) = \varnothing$. Thus, we can conclude that $\mathsf{V}(Cnu_m(\mathsf{I}'_i)) = \varnothing$.

iii. $\mathsf{SV}(Cne_m(\mathsf{I}'_1)) = \varnothing$.
By Def. 12, $\mathsf{SV}(Cne_m(\mathsf{I}'_1))$ consists of all and only formulae of the kind $t\mathcal{K}_i B$ such that there exists some $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$ for which:

1. $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$
   $\delta_i^{\mathsf{M}} \geq |L| + |J|$
   $\delta_j^{\mathsf{M}} \geq |J|$
2. $\exists A_2 \in \mathcal{L}(tB \in \mathsf{W}(Cne_m(\mathsf{I}'_1) \cup \{t\mathcal{K}_i A_2\}) \wedge \forall \Lambda \in \mathsf{R}^L_{Cne_m(\mathsf{I}'_1)}(tA_2 \in \mathsf{W}(\Lambda) \vee$
   $\exists A_1 \in \mathcal{L}(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_j A_1\}) \wedge \forall \Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta))))))$.

Due to the conditions 1 and due to the construction of M', the only possible $\mathbf{J}$s here are such that $L = J = \varnothing$. The only minimal refinement Λ of $Cne_m(\mathsf{I}'_1) = \varnothing$ on $L = \varnothing$ is $\varnothing$ and $tA_2 \notin \mathsf{W}(\varnothing)$. The only minimal refinement Δ of $\Sigma_j(\mathsf{W}(\Lambda)) = \varnothing$ on $J = \varnothing$ is $\varnothing$ and $tA_1 \notin \mathsf{W}(\varnothing)$. So we can conclude that $\mathsf{SV}(Cne_m(\mathsf{I}'_1)) = \varnothing$.

iv. $\mathsf{CSV}(Cnc_m(\mathsf{I}'_1)) = \varnothing$.
By Def. 13, $\mathsf{CSV}(Cnc_m(\mathsf{I}_i))$ consists of all and only formulae of the kind $t\mathcal{K}_i B$ such that there exists some $\mathbf{J} = ((L, 1), (J_1, j_1), \ldots, (J_m, j_m)) \in \mathsf{LAP}_i$ for which:

1. $\delta_1^{\mathsf{M}} \geq \delta_{j_1}^{\mathsf{M}}$
   $\delta_1^{\mathsf{M}} \geq |L|$
   $\forall m = 1, \ldots, n,\ \delta_m^{\mathsf{M}} \geq \sum\limits_{k=m}^{n} |J_k|$;
2. and for $A_n \in \mathcal{L}$:
   $\exists A_{n+1}(tB \in \mathsf{W}(Cnc_m(\mathsf{I}'_1) \cup \{t\mathcal{K}_i A_{n+1}\}) \wedge \forall \Delta_0 \in \mathsf{R}^L_{Cnc_m(\mathsf{I}'_1)}(tA_{n+1} \in \mathsf{W}(\Delta_0) \vee$
   $\exists A_n(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1} A_n\}) \wedge \forall \Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}(tA_n \in \mathsf{W}(\Delta_1) \vee$
   $\vdots$
   $\exists A_1(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n} A_1\}) \wedge \forall \Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}(tA_1 \in \mathsf{W}(\Delta_n))) \ldots)$.

Due to the conditions 1 and due to the construction of M', the only possible $\mathbf{J}$s here are such that $L = \varnothing$ and $\forall m = 1, \ldots, n,\ J_m = \varnothing$. The only minimal refinement Δ_0 of $Cnc_m(\mathsf{I}'_1) = \varnothing$ on $L = \varnothing$ is $\varnothing$ and $tA_{n+1} \notin \mathsf{W}(\varnothing)$. The only minimal refinement Δ_1 of $\Sigma_{j_1}(\mathsf{W}(Cnc_m(\mathsf{I}'_1))) = \varnothing$ on $J_1 = \varnothing$ is $\varnothing$ and again $tA_n \notin \mathsf{W}(\varnothing)$. This reasoning can be repeated until we get that $tA_1 \notin \mathsf{W}(\varnothing)$. So we can conclude that $\mathsf{CSV}(Cnc_m(\mathsf{I}'_1)) = \varnothing$.

Proposition 9. Fix x as one of u, e, c and fix $k \in \mathbb{N}$. For any $B \in \mathcal{L}$, $\Gamma \in \mathbb{A}$, $\mathsf{M} \in \mathcal{M}_k$ and $\mathsf{I}_i \in \mathsf{M}$:

1) $tB \in Cnx^k(\Gamma)$ [Hypothesis]

2) For all $\mathsf{M}' \in \mathcal{M}_k$ and for all $\mathsf{I}'_i \in \mathsf{M}'$, if $\Gamma \subseteq Cnx(\mathsf{I}'_i)$, then $tB \in Cnx(\mathsf{I}'_i)$ [1) and Prop. 6]

3) $\mathcal{K}_i\Gamma \subseteq Cnx(\mathsf{I}_i)$ [Hypothesis]

4) There exists some $m \in \mathbb{N}$ such that $\mathcal{K}_i\Gamma \subseteq Cnx_m(\mathsf{I}_i)$ [3) and Def. 11]

5) $\Gamma \subseteq \mathsf{W}(Cnx_m(\mathsf{I}_i))$ because [5.1) and 5.2)]

   5.1) For any $tC_j \in \Gamma$:
   $\Delta = Cnx_m(\mathsf{I}_i) \cup \{fC_j\} \notin \mathbb{A}$ since $fC_j, t\mathcal{K}_iC_j \in \Delta$ satisfy the inadmissibility condition 12. Thus, by Def. 10, $tC_j \in \mathsf{W}_1(Cnx_m(\mathsf{I}_i)) \subseteq \mathsf{W}(Cnx_m(\mathsf{I}_i))$.

   5.2) For any $fC_j \in \Gamma$:
   $\Delta = Cnx_m(\mathsf{I}_i) \cup \{f\neg C_j\} \notin \mathbb{A}$ since $f\neg C_j, t\mathcal{K}_i\neg C_j \in \Delta$ satisfy the inadmissibility condition 12. Thus, by Def. 10, $t\neg C_j \in \mathsf{W}_1(Cnx_m(\mathsf{I}_i))$.
   $\Delta_1 = \mathsf{W}_1(Cnx_m(\mathsf{I}_i)) \cup \{tC_j\} \notin \mathbb{A}$ since $t\neg C_j, tC_j \in \Delta_1$ satisfy the inadmissibility condition 1. Thus, by Def. 10, $fC_j \in \mathsf{W}_2(Cnx_m(\mathsf{I}_i)) \subseteq \mathsf{W}(Cnx_m(\mathsf{I}_i))$.

6) $\Gamma \subseteq Cnx_{m+1}(\mathsf{I}_i)$ [5) and Def. 11]

7) $\Gamma \subseteq Cnx(\mathsf{I}_i)$ [6) and Def. 11]

8) $tB \in Cnx(\mathsf{I}_i)$ [2) and 7)]

9) There exists some $l \in \mathbb{N}$ such that $tB \in Cnx_l(\mathsf{I}_i)$ [8) and Def. 11]

10) We have to consider three sub-cases:

   i) If $x = u$, then for $\varnothing \in \wp(\mathcal{P})$:
   - $0 = |\varnothing| \leq \delta_i^{\mathsf{M}}$
   - $Cnu_l(\mathsf{I}_i)$ is the only minimal refinement of $Cnu_l(\mathsf{I}_i)$ on $\varnothing$
   - $\forall\Delta \in \mathsf{R}^{\varnothing}_{Cnu_l(\mathsf{I}_i)}, tB \in \mathsf{W}(\Delta)$ [9), 10) and Prop. 4.8]

   $t\mathcal{K}_iB \in \mathsf{V}(Cnu_l(\mathsf{I}_i))$ [10.i) and Def. 11]

ii) If $x = e$, then for $\mathbf{J} = ((\varnothing, i), (\varnothing, j)) \in \mathsf{CAP}_i$:

- $\delta_i^{\mathsf{M}} \geq \delta_i^{\mathsf{M}}$
  $\delta_i^{\mathsf{M}} \geq |\varnothing| + |\varnothing|$
  $\delta_i^{\mathsf{M}} \geq |\varnothing|$
- for $B \in \mathcal{L}$, $tB \in \mathsf{W}(Cne_l(\mathsf{I}_i) \cup \{t\mathcal{K}_i B\}) \wedge \forall \Lambda \in \mathsf{R}^{\varnothing}_{Cne_l(\mathsf{I}_i)}$, $tB \in \mathsf{W}(\Lambda)$

$t\mathcal{K}_i B \in \mathsf{SV}(Cne_l(\mathsf{I}_i))$ [10.ii) and Def. 12]

iii) If $x = c$, then for $\mathbf{J} = ((\varnothing, i), (J, j)) \in \mathsf{LAP}_i$:

- $\delta_i^{\mathsf{M}} \geq |\varnothing| = 0$
- $tB \in \mathsf{W}(Cnc_l(\mathsf{I}_i) \cup \{t\mathcal{K}_i B\}) \wedge tB \in \mathsf{W}(Cnc_l(\mathsf{I}_i))$

$t\mathcal{K}_i B \in \mathsf{CSV}(Cnc_l(\mathsf{I}_i))$ [10.iii) and Def. 13]

11) $t\mathcal{K}_i B \in Cnx_{l+1}(\mathsf{I}_i)$ [10) and Def. 11, Def. 12, Def. 13]

12) $t\mathcal{K}_i B \in Cnx(\mathsf{I}_i)$ [11) and Def. 11, Def. 12, Def. 13]

Proposition 10. Fix x as one of u, e, c. For all $k \in \mathbb{N}$, $\Gamma \in \mathbb{A}$ and $B \in \mathcal{L}$, let $\alpha = \{\mathsf{M} \in \mathcal{M}_k \mid \Gamma \subseteq Cnx(\mathsf{M})\}$ and $\beta = \{\mathsf{M}' \in \mathcal{M}_{k+1} \mid \Gamma \subseteq Cnx(\mathsf{M}')\}$.

1) $sB \in Cnx^k(\Gamma)$ [Hypothesis]

2) $\forall \mathsf{M} \in \mathcal{M}_k$, if $\Gamma \subseteq Cnx(\mathsf{M})$, then $sB \in Cnx(\mathsf{M})$ [1) and Def. 16]

3) If $\mathsf{M} \in \alpha$, then $sB \in Cnx(\mathsf{M})$ [2)]

4) $\mathcal{M}_{k+1} \subseteq \mathcal{M}_k$ [Def. 8]

5) If $\mathsf{M}' \in \beta$, then $\mathsf{M}' \in \alpha$ [4)]

6) If $\mathsf{M}' \in \beta$, then $sB \in Cnx(\mathsf{M}')$ [3) and 5)]

7) $\forall \mathsf{M}' \in \mathcal{M}_{k+1}$, if $\Gamma \subseteq Cnx(\mathsf{M}')$, then $sB \in Cnx(\mathsf{M}')$ [6)]

8) $sB \in Cnx^{k+1}(\Gamma)$ [7) and Def. 16]

Proposition 11.1 We split the proof into two parts.

1. For all $\Gamma \in \mathbb{A}$, for all $k \in \mathbb{N}$ and for any $sB \in \mathcal{L}$, if $sB \in Cnu^k(\Gamma)$, then $sB \in Cne^k(\Gamma)$

1) $sB \in Cnu^k(\Gamma)$ [Hypothesis]

2) For all $\mathsf{M} \in \mathcal{M}_k$ and for all $\mathsf{I}_i \in \mathsf{M}$, if $\Gamma \subseteq Cnu(\mathsf{I}_i)$, then $sB \in Cnu(\mathsf{I}_i)$ [1) and Prop. 6]

3) For every interpretation I_i in any model $\mathsf{M} \in \mathcal{M}_k$ such that $\Gamma \subseteq Cne(\mathsf{I}_i)$, let I'_i in model $\mathsf{M}' \in \mathcal{M}_k$ be an interpretation, which is exactly the same as I_i except that $\varphi_i^{\mathsf{M}'} = \varphi_i^{\mathsf{M}} \cup \Gamma$ [Construction]

4) $\Gamma \subseteq Cnu(\mathsf{I}'_i)$ [3) and Prop. 5.1]

5) $sB \in Cnu(\mathsf{I}'_i)$ [2) and 4)]

6) $sB \in Cne(\mathsf{I}'_i)$ [5) and Lemma 3 (following)]

7) $sB \in Cne(\mathsf{I}_i)$ [3), 6) and Lemma 4 (following)]

8) $sB \in Cne^k(\Gamma)$ [3), 7) and Prop. 6]

Lemma 3. Let I_i be any interpretation in any model $\mathsf{M} \in \mathcal{M}_0$. Then, $Cnu(\mathsf{M}) \subseteq Cne(\mathsf{M})$.

If $sA \in Cnu(\mathsf{I}_i)$, then, by Def. 11, there exists some $m \in \mathbb{N}$ such that $sA \in Cnu_m(\mathsf{I}_i)$. We prove by induction on m that if $sA \in Cnu_m(\mathsf{I}_i)$, then there exists some $l_m \in \mathbb{N}$ such that $sA \in Cne_{l_m}(\mathsf{I}_i)$.

- Base case: $m = 0$. We prove that if $sA \in Cnu_0(\mathsf{I}_i)$, then there exists some $l_0 \in \mathbb{N}$ such that $sA \in Cne_{l_0}(\mathsf{I}_i)$.

  1) $sA \in Cnu_0(\mathsf{I}_i)$ [Hypothesis]
  2) $sA \in \varphi_i^{\mathsf{M}}$ [1) and Def. 11]
  3) $sA \in Cne_0(\mathsf{I}_i)$ [2) and Def. 12]

- Inductive hypothesis: $m = n$. We assume that if $sA \in Cnu_n(\mathsf{I}_i)$, then there exists some $l_n \in \mathbb{N}$ such that $sA \in Cne_{l_n}(\mathsf{I}_i)$, that is to say, $Cnu_n(\mathsf{I}_i) \subseteq Cne_{l_n}(\mathsf{I}_i)$.

- Inductive step: $m = n + 1$. We prove that if $sA \in Cnu_{n+1}(\mathsf{I}_i)$, then there exists some $l_{n+1} \in \mathbb{N}$ such that $sA \in Cne_{l_{n+1}}(\mathsf{I}_i)$.

  1) $sA \in Cnu_{n+1}(\mathsf{I}_i)$ [Hypothesis]
  2) There are two cases to be considered: i. $sA \in \mathsf{W}(Cnu_n(\mathsf{I}_i))$ and ii. $sA = t\mathcal{K}_iC \in \mathsf{V}(Cnu_n(\mathsf{I}_i))$
  3) Case i. $sA \in \mathsf{W}(Cnu_n(\mathsf{I}_i))$.
     3.1) $Cnu_n(\mathsf{I}_i) \subseteq Cne_{l_n}(\mathsf{I}_i)$ [Inductive hypothesis]
     3.2) $sA \in \mathsf{W}(Cne_{l_n}(\mathsf{I}_i))$ [3), 3.1) and Prop. 3.2]

3.3) $sA \in Cne_{l_n+1}(\mathsf{I}_i)$ [3.2) and Def. 12]

4) Case ii. $sA = t\mathcal{K}_iC \in \mathsf{V}(Cnu_n(\mathsf{I}_i))$.

4.1) there exists some $J \in \wp(\mathcal{P})$ such that:

i. $|J| \leq \delta_i^{\mathsf{M}}$ and

ii. for every $\Delta \in \mathsf{R}^J_{Cnu_n(\mathsf{I}_i)}$, $tC \in \mathsf{W}(\Delta)$

4.2) for $\mathbf{J} = ((J, i), (\varnothing, i))$: $\delta_i^{\mathsf{M}} \geq \delta_i^{\mathsf{M}}$; $\delta_i^{\mathsf{M}} \geq |J| + |\varnothing|$; $\delta_i^{\mathsf{M}} \geq |\varnothing|$ [4.1.i)]

4.3) $Cnu_n(\mathsf{I}_i) \subseteq Cne_{l_n}(\mathsf{I}_i)$ [Inductive hypothesis]

4.4) for all $\Lambda \in \mathsf{R}^J_{Cne_{l_n}(\mathsf{I}_i)}$, $\Lambda \in \mathsf{R}^J_{Cnu_n(\mathsf{I}_i)}$ [4.3) and Prop. 2.2]

4.5) for all $\Lambda \in \mathsf{R}^J_{Cne_{l_n}(\mathsf{I}_i)}$, $tC \in \mathsf{W}(\Lambda)$ [4.1.ii) and 4.4)]

4.6) $tC \in \mathsf{W}(Cne_{l_n}(\mathsf{I}_i) \cup \{t\mathcal{K}_iC\})$ [Construction and Def. 10]

4.7) $tC \in \mathsf{W}(Cne_{l_n}(\mathsf{I}_i) \cup \{t\mathcal{K}_iC\}) \wedge \forall\Lambda \in \mathsf{R}^J_{Cne_{l_n}(\mathsf{I}_i)}$, $tC \in \mathsf{W}(\Lambda)$ [4.5) and 4.6)]

4.8) $sA = t\mathcal{K}_iC \in \mathsf{SV}(Cne_{l_n}(\mathsf{I}_i))$ [4.2), 4.7) and Def. 12]

4.9) $sA = t\mathcal{K}_iC \in Cne_{l_n+1}(\mathsf{I}_i))$ [4.7) and Def. 12]

5) $sA \in Cne_{l_n+1}(\mathsf{I}_i)$ [3.3) and 4.9)]

Lemma 4. For every interpretation I_i in any model $\mathsf{M} \in \mathcal{M}_0$, let I'_i in model $\mathsf{M}' \in \mathcal{M}_0$ be the interpretation, which is exactly the same as I_i except that $\varphi_i^{\mathsf{M}'} = \varphi_i^{\mathsf{M}} \cup \Gamma$ for some $\Gamma \subseteq \mathcal{L}^s$. If $sB \in Cne(\mathsf{I}'_i)$ and $\Gamma \subseteq Cne(\mathsf{I}_i)$, then $sB \in Cne(\mathsf{I}_i)$.

By Def. 12, there exist some $k, m \in \mathbb{N}$ such that $sB \in Cne_k(\mathsf{I}'_i)$ and $\Gamma \subseteq Cne_m(\mathsf{I}_i)$. We prove by induction on k that if $sB \in Cne_k(\mathsf{I}'_i)$ and $\Gamma \subseteq Cne_m(\mathsf{I}_i)$, then $sB \in Cne_{k+m}(\mathsf{I}_i)$.

- Base case: $k = 0$. We have to prove that if $sB \in Cne_0(\mathsf{I}'_i)$ and $\Gamma \subseteq Cne_m(\mathsf{I}_i)$, then $sB \in Cne_m(\mathsf{I}_i)$.

1) $sB \in Cne_0(\mathsf{I}'_i)$ [Hypothesis]

2) $\Gamma \subseteq Cne_m(\mathsf{I}_i)$ [Hypothesis]

3) $sB \in \varphi_i^{\mathsf{M}} \cup \Gamma$ [1) and Def. 12]

4) There are two cases to be considered: i. $sB \in \varphi_i^{\mathsf{M}}$ and ii. $sB \in \Gamma$

5) Case i. $sB \in \varphi_i^{\mathsf{M}}$:

5.1) $sB \in Cne_0(\mathsf{I}_i)$ [5) and Def. 12]

5.2) $sB \in Cne_m(\mathsf{I}_i)$ [5.1) and Prop. 4.3]

6) Case ii. $sB \in \Gamma$:

   6.1) $sB \in Cne_m(\mathsf{I}_i)$ [2) and 6)]

7) $sB \in Cne_m(\mathsf{I}_i)$ [5.2) and 6.1)]

- Inductive hypothesis: $k = n$. We assume that if $sB \in Cne_n(\mathsf{I}'_i)$ and $\Gamma \subseteq Cne_m(\mathsf{I}_i)$, then $sB \in Cne_{n+m}(\mathsf{I}_i)$. This is logically equivalent to the following: if $\Gamma \subseteq Cne_m(\mathsf{I}_i)$, then $Cne_n(\mathsf{I}'_i) \subseteq Cne_{n+m}(\mathsf{I}_i)$.

- Inductive step: $k = n + 1$. We have to prove that if $sB \in Cne_{n+1}(\mathsf{I}'_i)$ and $\Gamma \subseteq Cne_m(\mathsf{I}_i)$, then $sB \in Cne_{n+m+1}(\mathsf{I}_i)$.

1) $sB \in Cne_{n+1}(\mathsf{I}'_i)$ [Hypothesis]

2) $\Gamma \subseteq Cne_m(\mathsf{I}_i)$ [Hypothesis]

3) There are two cases to be considered: i. $sB \in \mathsf{W}(Cne_n(\mathsf{I}'_i))$ and ii. $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_n(\mathsf{I}'_i))$

4) Case i. $sB \in \mathsf{W}(Cne_n(\mathsf{I}'_i))$

   4.1) $sB \in \mathsf{W}(Cne_{n+m}(\mathsf{I}'_i))$ [2), 4), Inductive hypothesis, Prop. 3.2]

   4.2) $sB \in Cne_{n+m+1}(\mathsf{I}_i)$ [4.1) and Def. 12]

5) Case ii. $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_n(\mathsf{I}'_i))$.

   5.1) There exists some $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$ for which:

      i. $\delta_i^{\mathsf{M}'} \geq \delta_j^{\mathsf{M}'}$; $\delta_i^{\mathsf{M}'} \geq |L| + |J|$ and $\delta_j^{\mathsf{M}'} \geq |J|$

      ii. $\exists A_2(tC \in \mathsf{W}(Cne_n(\mathsf{I}'_i)) \cup \{t\mathcal{K}_iA_2\}) \wedge \forall\Lambda \in \mathsf{R}^L_{Cne_n(\mathsf{I}'_i)}(tA_2 \in \mathsf{W}(\Lambda)\vee$
      $\exists A_1(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_jA_1\}) \wedge \forall\Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta)))))$.
      [5) and Def. 12]

   5.2) For $\mathbf{J}$ as in 5.1): $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$; $\delta_i^{\mathsf{M}} \geq |L| + |J|$ and $\delta_j^{\mathsf{M}} \geq |J|$ [5.1.i) and Def. of M]

   5.3) $tC \in \mathsf{W}(Cne_{n+m}(\mathsf{I}_i)) \cup \{t\mathcal{K}_iA_2\})$ [2), 5.1.ii), Inductive hypothesis and Prop. 3.2]

   5.4) If $\Lambda \in \mathsf{R}^L_{Cne_{n+m}(\mathsf{I}_i)}$, then $\Lambda \in \mathsf{R}^L_{Cne_n(\mathsf{I}'_i)}$ [2), Inductive hypothesis and Prop. 3.2]

   5.5) $\exists A_2(tC \in \mathsf{W}(Cne_{n+m}(\mathsf{I}_i)) \cup \{t\mathcal{K}_iA_2\}) \wedge \forall\Lambda \in \mathsf{R}^L_{Cne_{n+m}(\mathsf{I}_i)}(tA_2 \in \mathsf{W}(\Lambda)\vee$
   $\exists A_1(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_jA_1\}) \wedge \forall\Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta)))))$.
   [5.1.i), 5.3) and 5.4)]

   5.6) $sB = t\mathcal{K}_iC \in \mathsf{SV}(Cne_{m+n}(\mathsf{I}_i))$ [5.2), 5.5) and Def. 12]

   5.7) $sB \in Cne_{n+m+1}(\mathsf{I}_i)$ [5.6) and Def. 12]

6) $sB \in Cne_{n+m+1}(\mathsf{I}_i)$ [4.2) and 5.7)]

2. For all $\Gamma \in \mathbb{A}$, for all $k \in \mathbb{N}$ and for any $sB \in \mathcal{L}$, if $sB \in Cne^k(\Gamma)$, then $sB \in Cnc^k(\Gamma)$.

1) $sB \in Cne^k(\Gamma)$ [Hypothesis]

2) For all $\mathsf{M} \in \mathcal{M}_k$ and for all $\mathsf{I}_i \in \mathsf{M}$, if $\Gamma \subseteq Cne(\mathsf{I}_i)$, then $sB \in Cne(\mathsf{I}_i)$ [1) and Prop. 6]

3) For every interpretation I_i in any model $\mathsf{M} \in \mathcal{M}_k$ such that $\Gamma \subseteq Cnc(\mathsf{I}_i)$, let I'_i in model $\mathsf{M}' \in \mathcal{M}_k$ be an interpretation, which is exactly the same as I_i except that $\varphi_i^{\mathsf{M}'} = \varphi_i^{\mathsf{M}} \cup \Gamma$ [Construction]

4) $\Gamma \subseteq Cne(\mathsf{I}'_i)$ [3) and Prop. 5.1]

5) $sB \in Cne(\mathsf{I}'_i)$ [2) and 4)]

6) $sB \in Cnc(\mathsf{I}'_i)$ [5) and Lemma 5 (following)]

7) $sB \in Cnc(\mathsf{I}_i)$ [3), 6) and Lemma 6 (following)]

8) $sB \in Cnc^k(\Gamma)$ [3), 7) and Prop. 6]

Lemma 5. Let I_i be any interpretation in any model $\mathsf{M} \in \mathcal{M}_0$, $Cne(\mathsf{M}) \subseteq Cnc(\mathsf{M})$.

If $sA \in Cne(\mathsf{I}_i)$, then, by Def. 12, there exists some $m \in \mathbb{N}$ such that $sA \in Cne_m(\mathsf{I}_i)$. We prove by induction on m that if $sA \in Cne_m(\mathsf{I}_i)$, then there exists some $l_m \in \mathbb{N}$ such that $sA \in Cnc_{l_m}(\mathsf{I}_i)$.

- Base case: $m = 0$. We prove that if $sA \in Cne_0(\mathsf{I}_i)$, then there exists some $l_0 \in \mathbb{N}$ such that $sA \in Cnc_{l_0}(\mathsf{I}_i)$.

  1) $sA \in Cne_0(\mathsf{I}_i)$ [Hypothesis]
  2) $sA \in \varphi_i^{\mathsf{M}}$ [1) and Def. 12]
  3) $sA \in Cnc_0(\mathsf{I}_i)$ [2) and Def. 13]

- Inductive hypothesis: $m = n$. We assume that if $sA \in Cne_n(\mathsf{I}_i)$, then there exists some $l_n \in \mathbb{N}$ such that $sA \in Cnc_{l_n}(\mathsf{I}_i)$, that is to say, $Cne_n(\mathsf{I}_i) \subseteq Cnc_{l_n}(\mathsf{I}_i)$.

- Inductive step: $m = n + 1$. We prove that if $sA \in Cne_{n+1}(\mathsf{I}_i)$, then there exists some $l_{n+1} \in \mathbb{N}$ such that $sA \in Cnc_{l_{n+1}}(\mathsf{I}_i)$.

1) $sA \in Cne_{n+1}(\mathsf{I}_i)$ [Hypothesis]
2) There are two cases to be considered: i. $sA \in \mathsf{W}(Cne_n(\mathsf{I}_i))$ and ii. $sA = t\mathcal{K}_i C \in \mathsf{SV}(Cne_n(\mathsf{I}_i))$
3) Case i. $sA \in \mathsf{W}(Cne_n(\mathsf{I}_i))$.
   3.1) $Cne_n(\mathsf{I}_i) \subseteq Cnc_{l_n}(\mathsf{I}_i)$ [Inductive hypothesis]
   3.2) $sA \in \mathsf{W}(Cnc_{l_n}(\mathsf{I}_i))$ [3), 3.1) and Prop. 3.2]
   3.3) $sA \in Cnc_{l_n+1}(\mathsf{I}_i)$ [3.2) and Def. 13]
4) Case ii. $sA = t\mathcal{K}_i C \in \mathsf{SV}(Cne_n(\mathsf{I}_i))$.
   4.1) there exists some $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$ for which:
      i. $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$
      $\delta_i^{\mathsf{M}} \geq |L| + |J|$
      $\delta_j^{\mathsf{M}} \geq |J|$
      ii. $\exists A_2 (tC \in \mathsf{W}(Cne_n(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\}) \wedge \forall \Lambda \in \mathsf{R}^L_{Cne_n(\mathsf{I}_i)} (tA_2 \in \mathsf{W}(\Lambda) \vee$
      $\exists A_1\ (tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_j A_1\}) \wedge \forall \Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}\ (tA_1 \in \mathsf{W}(\Delta))))))$.
   4.2) $Cne_n(\mathsf{I}_i) \subseteq Cnc_{l_n}(\mathsf{I}_i)$ [Inductive hypothesis]
   4.3) for all $\Lambda \in \mathsf{R}^L_{Cnc_{l_n}(\mathsf{I}_i)}$, $\Lambda \in \mathsf{R}^L_{Cne_n(\mathsf{I}_i)}$ [4.2) and Prop. 2.2]
   4.4) $tC \in \mathsf{W}(Cnc_{l_n}(\mathsf{I}_i) \cup \{t\mathcal{K}_i C\})$ [Construction and Def. 10]
   4.5) For $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$:
      i. $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$
      $\delta_i^{\mathsf{M}} \geq |L|$
      $\delta_j^{\mathsf{M}} \geq |J|$
      ii. $\exists A_2 (tC \in \mathsf{W}(Cnc_{l_n}(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\}) \wedge \forall \Lambda \in \mathsf{R}^L_{Cnc_{l_n}(\mathsf{I}_i)} (tA_2 \in$
      $\mathsf{W}(\Lambda) \vee$
      $\exists A_1 \in \mathcal{L}\ (tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_j A_1\}) \wedge \forall \Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}\ (tA_1 \in$
      $\mathsf{W}(\Delta))))))$.
      [4.1), 4.3) and 4.4)]
   4.6) $sA = t\mathcal{K}_i C \in \mathsf{CSV}(Cnc_{l_n}(\mathsf{I}_i))$ [4.5) and Def. 13]
   4.7) $sA = t\mathcal{K}_i C \in Cnc_{l_n+1}(\mathsf{I}_i)$ [4.6) and Def. 13]
5) $sA \in Cnc_{l_n+1}(\mathsf{I}_i)$ [3.3) and 4.7)]

Lemma 6. For every interpretation I_i in any model $\mathsf{M} \in \mathcal{M}_0$, let I'_i in model $\mathsf{M}' \in \mathcal{M}_0$ be the interpretation, which is exactly the same as I_i except that $\varphi_i^{\mathsf{M}'} = \varphi_i^{\mathsf{M}} \cup \Gamma$ for some $\Gamma \subseteq \mathcal{L}^s$. If $sB \in Cnc(\mathsf{I}'_i)$ and $\Gamma \subseteq Cnc(\mathsf{I}_i)$, then $sB \in Cnc(\mathsf{I}_i)$.

By Def. 13, there exist some $k, m \in \mathbb{N}$ such that $sB \in Cnc_k(\mathsf{I}'_i)$ and $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$. We prove by induction on k that if $sB \in Cnc_k(\mathsf{I}'_i)$ and $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$, then $sB \in Cnc_{k+m}(\mathsf{I}_i)$.

- Base case: $k = 0$. We have to prove that if $sB \in Cnc_0(\mathsf{I}'_i)$ and $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$, then $sB \in Cnc_m(\mathsf{I}_i)$.

  1) $sB \in Cnc_0(\mathsf{I}'_i)$ [Hypothesis]
  2) $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$ [Hypothesis]
  3) $sB \in \varphi_i^{\mathsf{M}} \cup \Gamma$ [1) and Def. 13]
  4) There are two cases to be considered: i. $sB \in \varphi_i^{\mathsf{M}}$ and ii. $sB \in \Gamma$
  5) Case i. $sB \in \varphi_i^{\mathsf{M}}$:
     5.1) $sB \in Cnc_0(\mathsf{I}_i)$ [5) and Def. 13]
     5.2) $sB \in Cnc_m(\mathsf{I}_i)$ [5.1) and Prop. 4.3]
  6) Case ii. $sB \in \Gamma$:
     6.1) $sB \in Cnc_m(\mathsf{I}_i)$ [2) and 6)]
  7) $sB \in Cnc_m(\mathsf{I}_i)$ [5.2) and 6.1)]

- Inductive hypothesis: $k = n$. We assume that if $sB \in Cnc_n(\mathsf{I}'_i)$ and $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$, then $sB \in Cnc_{n+m}(\mathsf{I}_i)$. This is logically equivalent to the following: if $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$, then $Cnc_n(\mathsf{I}'_i) \subseteq Cnc_{n+m}(\mathsf{I}_i)$.

- Inductive step: $k = n + 1$. We have to prove that if $sB \in Cnc_{n+1}(\mathsf{I}'_i)$ and $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$, then $sB \in Cnc_{n+m+1}(\mathsf{I}_i)$.

  1) $sB \in Cnc_{n+1}(\mathsf{I}'_i)$ [Hypothesis]
  2) $\Gamma \subseteq Cnc_m(\mathsf{I}_i)$ [Hypothesis]
  3) There are two cases to be considered: i. $sB \in \mathsf{W}(Cnc_n(\mathsf{I}'_i))$ and ii. $sB = t\mathcal{K}_i C \in \mathsf{CSV}(Cnc_n(\mathsf{I}'_i))$
  4) Case i. $sB \in \mathsf{W}(Cnc_n(\mathsf{I}'_i))$
     4.1) $sB \in \mathsf{W}(Cnc_{n+m}(\mathsf{I}'_i))$ [2), 4), Inductive hypothesis, Prop. 3.2]
     4.2) $sB \in Cnc_{n+m+1}(\mathsf{I}_i)$ [4.1) and Def. 13]
  5) Case ii. $sB = t\mathcal{K}_i C \in \mathsf{CSV}(Cnc_n(\mathsf{I}'_i))$.
     5.1) There exists some $\mathbf{J} = ((L, i), (J_1, j_1), \ldots, (J_n, j_n)) \in \mathsf{LAP}_i$ for which:
        i. $\delta_i^{\mathsf{M}'} \geq \delta_{j_1}^{\mathsf{M}'}$; $\delta_i^{\mathsf{M}'} \geq |L|$ and $\forall m = 1, \ldots, n$, $\delta_m^{\mathsf{M}'} \geq \sum_{k=m}^{n} |J_k|$;

ii. and for $A_n \in \mathcal{L}$:
$\exists A_{n+1}(tC \in \mathsf{W}(Cnc_n(\mathsf{I}'_i)\cup\{t\mathcal{K}_i A_{n+1}\})\wedge\forall\Delta_0 \in \mathsf{R}^L_{Cnc_n(\mathsf{I}'_i)}(tA_{n+1} \in \mathsf{W}(\Delta_0)\vee$
$\exists A_n(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1} A_n\}) \wedge \forall\Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}(tA_n \in \mathsf{W}(\Delta_1)\vee$
$\vdots$
$\exists A_1(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n} A_1\}) \wedge \forall\Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}(tA_1 \in \mathsf{W}(\Delta_n)))\dots)$.
[5) and Def. 13]

5.2) For $\mathbf{J}$ as in 5.1): $\delta^{\mathsf{M}}_i \geq \delta^{\mathsf{M}}_{j_1}$; $\delta^{\mathsf{M}}_i \geq |L|$ and $\forall m = 1,\dots,n$, $\delta^{\mathsf{M}}_m \geq \sum\limits_{k=m}^{n} |J_k|$ [5.1.i) and Def. of M]

5.3) $tC \in \mathsf{W}(Cnc_{n+m}(\mathsf{I}_i))\cup\{t\mathcal{K}_i A_2\})$ [2), 5.1.ii), Inductive hypothesis and Prop. 3.2]

5.4) If $\Delta_0 \in \mathsf{R}^L_{Cnc_{n+m}(\mathsf{I}_i)}$, then $\Delta_0 \in \mathsf{R}^L_{Cnc_n(\mathsf{I}'_i)}$ [2), Inductive hypothesis and Prop. 3.2]

5.5) $\exists A_{n+1}(tC \in \mathsf{W}(Cnc_{n+m}(\mathsf{I}_i)\cup\{t\mathcal{K}_i A_{n+1}\})\wedge\forall\Delta_0 \in \mathsf{R}^L_{Cnc_{n+m}(\mathsf{I}_i)}(tA_{n+1} \in \mathsf{W}(\Delta_0)\vee$
$\exists A_n(tA_{n+1} \in \mathsf{W}(\Delta_0\cup\{t\mathcal{K}_{j_1} A_n\})\wedge\forall\Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}(tA_n \in \mathsf{W}(\Delta_1)\vee$
$\vdots$
$\exists A_1(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n} A_1\}) \wedge \forall\Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}(tA_1 \in \mathsf{W}(\Delta_n)))\dots)$.
[5.1.i), 5.3) and 5.4)]

5.6) $sB = t\mathcal{K}_i C \in \mathsf{CSV}(Cnc_{m+n}(\mathsf{I}_i))$ [5.2), 5.5) and Def. 13]

5.7) $sB \in Cnc_{n+m+1}(\mathsf{I}_i)$ [5.6) and Def. 13]

6) $sB \in Cnc_{n+m+1}(\mathsf{I}_i)$ [4.2) and 5.7)]

Proposition 11.2 We split the proof into two parts.

1. For all $\Gamma \in \mathbb{A}$, $Cnu^0(\Gamma) = Cne^0(\Gamma)$.

We have already proved (Prop. 11.1) that $Cnu^0(\Gamma) \subseteq Cne^0(\Gamma)$. We need to prove that $Cne^0(\Gamma) \subseteq Cnu^0(\Gamma)$, that is to say, if $sA \in Cne^0(\Gamma)$, then $sA \in Cnu^0(\Gamma)$.

1) $sA \in Cne^0(\Gamma)$ [Hypothesis]

2) For any interpretation I_i in any model $\mathsf{M} \in \mathcal{M}_0$, if $\Gamma \subseteq Cne(\mathsf{I}_i)$, then $sA \in Cne(\mathsf{I}_i)$ [1) and Prop. 6]

3) Let I_i be any interpretation in any model $\mathsf{M} \in \mathcal{M}_0$ such that $\Gamma \subseteq Cnu(\mathsf{I}_i)$ and $\delta_i^{\mathsf{M}} = 0$ [Construction]

4) $\Gamma \subseteq Cne(\mathsf{I}_i)$ [3) and Lemma 3]

5) $sA \in Cne(\mathsf{I}_i)$ [2) and 4)]

6) $sA \in Cnu(\mathsf{I}_i)$ [5) and Lemma 7 (following)]

7) Let I'_i be any interpretation in any model $\mathsf{M}' \in \mathcal{M}_0$ such that $\Gamma \subseteq Cnu(\mathsf{I}'_i)$

8) $sA \in Cnu(\mathsf{I}'_i)$ [3), 6), 7) and Prop. 5.3]

9) $sA \in Cnu^0(\Gamma)$ [4), 8) and Prop. 6]

Lemma 7. Let I_i be any interpretation in any model $\mathsf{M} \in \mathcal{M}_0$ such that $\delta_i^{\mathsf{M}} = 0$. If $sA \in Cne(\mathsf{I}_i)$, then $sA \in Cnu(\mathsf{I}_i)$.

Proof. If $sA \in Cne(\mathsf{I}_i)$, then there exists some $n \in \mathbb{N}$ such that $sA \in Cne_n(\mathsf{I}_i)$. We prove by induction on n that if $sA \in Cne_n(\mathsf{I}_i)$, then there exists some $l_n \in \mathbb{N}$ such that $sA \in Cnu_{l_n}(\mathsf{I}_i)$.

- Base case: $n = 0$. We prove that if $sA \in Cne_0(\mathsf{I}_i)$, then there exists some $l_0 \in \mathbb{N}$ such that $sA \in Cnu_{l_0}(\mathsf{I}_i)$.
  1) $sA \in Cne_0(\mathsf{I}_i)$ [Hypothesis]
  2) $sA \in \varphi_i^{\mathsf{M}}$ [1) and Def. 12]
  3) $sA \in Cnu_0(\mathsf{I}_i)$ [2) and Def. 11]
- Inductive hypothesis: $n = m$. We assume that if $sA \in Cne_m(\mathsf{I}_i)$, then there exists some $l_m \in \mathbb{N}$ such that $sA \in Cnu_{l_m}(\mathsf{I}_i)$
- Inductive step: $n = m + 1$. We prove that if if $sA \in Cne_{m+1}(\mathsf{I}_i)$, then there exists some $l_{m+1} \in \mathbb{N}$ such that $sA \in Cnu_{l_{m+1}}(\mathsf{I}_i)$
  1) There are two cases to be considered: i. $sA \in \mathsf{W}(Cne_m(\mathsf{I}_i))$; ii. $sA = t\mathcal{K}_iC \in \mathsf{SV}(Cne_m(\mathsf{I}_i))$.
  2) Case i. $sA \in \mathsf{W}(Cne_m(\mathsf{I}_i))$.
     2.1) $Cne_m(\mathsf{I}_i) \subseteq Cnu_{l_m}(\mathsf{I}_i)$ [Inductive hypothesis]
     2.2) $sA \in \mathsf{W}(Cnu_{l_m}(\mathsf{I}_i))$ [2), 2.1) and Prop. 3.2]
     2.3) $sA \in Cnu_{l_m+1}$ [2.2) and Def. 11]
  3) Case ii. $sA = t\mathcal{K}_iC \in \mathsf{SV}(Cne_m(\mathsf{I}_i))$.

3.1) There exists some $\mathbf{J} = ((L, i), (J, j)) \in \mathsf{CAP}_i$ for which:

i. $\delta_i^{\mathsf{M}} \geq \delta_j^{\mathsf{M}}$; $\delta_i^{\mathsf{M}} \geq |L| + |J|$ and $\delta_j^{\mathsf{M}} \geq |J|$

ii. $\exists A_2(tC \in \mathsf{W}(Cne_m(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\}) \wedge \forall \Lambda \in \mathsf{R}^L_{Cne_m(\mathsf{I}_i)}(tA_2 \in \mathsf{W}(\Lambda) \vee$
$\exists A_1(tA_2 \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_j A_1\}) \wedge \forall \Delta \in \mathsf{R}^J_{\Sigma_j(\mathsf{W}(\Lambda))}(tA_1 \in \mathsf{W}(\Delta)))))$.

3.2) Since $\delta_j^{\mathsf{M}} = 0$ for all $j \in \mathcal{A}$, the only possible $\mathbf{J}$s here are of the kind $((\varnothing, i), (\varnothing, j))$ [3.1)]

3.3) The only minimal refinement of $Cne_m(\mathsf{I}_i)$ on $L = \varnothing$ is $Cne_m(\mathsf{I}_i)$; similarly, the only minimal refinement of $\Sigma_j(\mathsf{W}(Cne_m(\mathsf{I}_i))$ on $J = \varnothing$ is $\Sigma_j(\mathsf{W}(Cne_m(\mathsf{I}_i))$ [Def. 6]

3.4) for A_2 $(tC \in \mathsf{W}(Cne_m(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\}) \wedge (tA_2 \in \mathsf{W}(Cne_m(\mathsf{I}_i)) \vee$
for A_1 $(tA_2 \in \mathsf{W}(Cne_m(\mathsf{I}_i) \cup \{t\mathcal{K}_j A_1\}) \wedge (tA_1 \in \mathsf{W}(\Sigma_j(\mathsf{W}(Cne_m(\mathsf{I}_i))))))$.
[3.1.ii), 3.2), 3.3) and Prop. 4.8]

3.5) Consider the last disjunction of the expression in 3.4): $tA_2 \in \mathsf{W}(Cne_m(\mathsf{I}_i)) \vee$ for A_1 $\Big(tA_2 \in \mathsf{W}(Cne_m(\mathsf{I}_i) \cup \{t\mathcal{K}_j A_1\}) \wedge tA_1 \in \mathsf{W}(\Sigma_j (\mathsf{W}(Cne_m(\mathsf{I}_i))))\Big)$. Now, suppose that the second disjunct holds, namely, for A_1, it is the case that both the following hold:

i. $tA_2 \in \mathsf{W}(Cne_m(\mathsf{I}_i) \cup \{t\mathcal{K}_j A_1\})$ and

ii. $tA_1 \in \mathsf{W}(\Sigma_j(\mathsf{W}(Cne_m(\mathsf{I}_i))))$

3.6) $t\mathcal{K}_j A_1 \in \mathsf{W}(\mathsf{W}(Cne_m(\mathsf{I}_i)))$ [3.5.ii) and Prop. 4.7]

3.7) $t\mathcal{K}_j A_1 \in \mathsf{W}(Cne_m(\mathsf{I}_i))$ [3.6) and Prop. 4.5]

3.8) $tA_2 \in \mathsf{W}(Cne_m(\mathsf{I}_i))$ [3.5.i), 3.7) and Prop. 3.3]

3.9) This means that if the second disjunct holds, then the first disjunct holds too. Of course, if the second disjunct does not hold, the first one holds. Thus, in both cases, the expression in 3.4) is logically equivalent to:
for A_2 $\Big(tC \in \mathsf{W}(Cne_m(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\}) \wedge tA_2 \in \mathsf{W}(Cne_m(\mathsf{I}_i))\Big)$

3.10) $Cne_m(\mathsf{I}_i) \subseteq Cnu_{l_m}(\mathsf{I}_i)$ [Inductive hypothesis]

3.11) $tA_2 \in \mathsf{W}(Cnu_{l_m}(\mathsf{I}_i))$ [3.9), 3.10) and Prop. 3.2]

3.12) $tA_2 \in \mathsf{W}(Cnu_{l_m+1}(\mathsf{I}_i))$ [3.11), Def. 11 and Prop. 3.2]

3.13) for all $\Delta \in \mathsf{R}^{\varnothing}_{Cnu_{l_m+1}(\mathsf{I}_i)}$, $tA_2 \in \mathsf{W}(\Delta)$ [3.12) and Prop. 4.8]

3.14) for $\varnothing \in \wp(\mathcal{P})$, $|\varnothing| \leq \delta_i^{\mathsf{M}}$ [Construction]

3.15) $t\mathcal{K}_i A_2 \in \mathsf{V}(Cnu_{l_m+1}(\mathsf{I}_i))$ [3.13), 3.14) and Def. 11]

3.16) $t\mathcal{K}_i A_2 \in Cnu_{l_m+2}(\mathsf{I}_i)$ [3.15) and Def. 11]

3.17) $t\mathcal{K}_i A_2 \in \mathsf{W}(Cnu_{l_m+2}(\mathsf{I}_i))$ [3.16) and Prop. 3.2]

3.18) $tC \in \mathsf{W}(Cnu_{l_m}(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\})$ [3.9), 3.10) and Prop. 3.2]

3.19) $tC \in \mathsf{W}(Cnu_{l_m+2}(\mathsf{I}_i) \cup \{t\mathcal{K}_i A_2\})$ [3.18) and Def. 11]

3.20) $tC \in \mathsf{W}(Cnu_{l_m+2}(\mathsf{I}_i))$ [3.17), 3.19) and Prop. 3.3]

3.21) for all $\Delta \in \mathsf{R}^{\varnothing}_{Cnu_{l_m+2}(\mathsf{I}_i)}$, $tC \in \mathsf{W}(\Delta)$ [3.20) and Prop. 4.8]

3.22) $t\mathcal{K}_i C \in \mathsf{V}(Cnu_{l_m+2}(\mathsf{I}_i))$ [3.14), 3.21) and Def. 11]

3.23) $t\mathcal{K}_i C \in Cnu_{l_m+3}(\mathsf{I}_i)$ [3.22) and Def. 11]

4) $sA \in Cnu_{l_m+3}(\mathsf{I}_i)$ [2.3) and 3.23)]

2. For all $\Gamma \in \mathbb{A}$, $Cne^0(\Gamma) = Cnc^0(\Gamma)$.

We have already proved (Prop. 11.1) that $Cne^0(\Gamma) \subseteq Cnc^0(\Gamma)$. We need to prove that $Cnc^0(\Gamma) \subseteq Cne^0(\Gamma)$, that is to say, if $sA \in Cnc^0(\Gamma)$, then $sA \in Cne^0(\Gamma)$.

1) $sA \in Cnc^0(\Gamma)$ [Hypothesis]

2) For any interpretation I_i in any model $\mathsf{M} \in \mathcal{M}_0$, if $\Gamma \subseteq Cnc(\mathsf{I}_i)$, then $sA \in Cnc(\mathsf{I}_i)$ [1) and Prop. 6]

3) Let I_i be any interpretation in any model $\mathsf{M} \in \mathcal{M}_0$ such that $\Gamma \subseteq Cne(\mathsf{I}_i)$ and $\delta_i^{\mathsf{M}} = 0$ [Construction]

4) $\Gamma \subseteq Cnc(\mathsf{I}_i)$ [3) and Lemma 5]

5) $sA \in Cnc(\mathsf{I}_i)$ [2) and 4)]

6) $sA \in Cne(\mathsf{I}_i)$ [5) and Lemma 8 (following)]

7) Let I'_i be any interpretation in any model $\mathsf{M}' \in \mathcal{M}_0$ such that $\Gamma \subseteq Cne(\mathsf{I}'_i)$

8) $sA \in Cne(\mathsf{I}'_i)$ [3), 6), 7) and Prop. 5.3]

9) $sA \in Cne^0(\Gamma)$ [4), 8) and Prop. 6]

Lemma 8. Let I_i be any interpretation in any $\mathsf{M} \in \mathcal{M}_0$ such that $\delta_i^{\mathsf{M}} = 0$. If $sA \in Cnc(\mathsf{I}_i)$, then $sA \in Cne(\mathsf{I}_i)$.

Proof. If $sA \in Cnc(\mathsf{I}_i)$, then there exists some $n \in \mathbb{N}$ such that $sA \in Cnc_n(\mathsf{I}_i)$. We prove by induction on n that if $sA \in Cnc_n(\mathsf{I}_i)$, then there exists some $l_n \in \mathbb{N}$ such that $sA \in Cne_{l_n}(\mathsf{I}_i)$.

- Base case: $n = 0$. We prove that if $sA \in Cnc_0(\mathsf{I}_i)$, then there exists some $l_0 \in \mathbb{N}$ such that $sA \in Cne_{l_0}(\mathsf{I}_i)$.

  1) $sA \in Cnc_0(\mathsf{I}_i)$ [Hypothesis]

2) $sA \in \varphi_i^{\mathsf{M}}$ [1) and Def. 13]

3) $sA \in Cne_0(\mathsf{I}_i)$ [2) and Def. 12]

- Inductive hypothesis: $n = m$. We assume that if $sA \in Cnc_m(\mathsf{I}_i)$, then there exists some $l_m \in \mathbb{N}$ such that $sA \in Cne_{l_m}(\mathsf{I}_i)$
- Inductive step: $n = m + 1$. We prove that if if $sA \in Cnc_{m+1}(\mathsf{I}_i)$, then there exists some $l_{m+1} \in \mathbb{N}$ such that $sA \in Cne_{l_{m+1}}(\mathsf{I}_i)$

1) There are two cases to be considered: i. $sA \in \mathsf{W}(Cnc_m(\mathsf{I}_i))$; ii. $sA = t\mathcal{K}_iC \in \mathsf{CSV}(Cnc_m(\mathsf{I}_i))$.

2) Case i. $sA \in \mathsf{W}(Cnc_m(\mathsf{I}_i))$.

2.1) $Cnc_m(\mathsf{I}_i) \subseteq Cne_{l_m}(\mathsf{I}_i)$ [Inductive hypothesis]

2.2) $sA \in \mathsf{W}(Cne_{l_m}(\mathsf{I}_i))$ [2), 2.1) and Prop. 3.2]

2.3) $sA \in Cne_{l_m+1}$ [2.2) and Def. 12]

3) Case ii. $sA = t\mathcal{K}_iC \in \mathsf{SV}(Cnc_m(\mathsf{I}_i))$.

3.1) There exists some $\mathbf{J} = ((L, i), (J_1, j_1), \ldots, (J_n, j_n)) \in \mathsf{LAP}_i$ for which:

i. $\delta_i^{\mathsf{M}} \geq \delta_{j_1}^{\mathsf{M}}$; $\delta_i^{\mathsf{M}} \geq |L|$ and $\forall n = 1, \ldots, n$, $\delta_n^{\mathsf{M}} \geq \sum\limits_{k=n}^{m} |J_k|$;

ii. and for $A_n \in \mathcal{L}$:
$\exists A_{n+1}(tC \in \mathsf{W}(Cnc_m(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge \forall \Delta_0 \in \mathsf{R}^{L}_{Cnc_m(\mathsf{I}_i)}(tA_{n+1} \in \mathsf{W}(\Delta_0) \vee$
$\exists A_n(tA_{n+1} \in \mathsf{W}(\Delta_0 \cup \{t\mathcal{K}_{j_1}A_n\}) \wedge \forall \Delta_1 \in \mathsf{R}^{J_1}_{\Sigma_{j_1}(\mathsf{W}(\Delta_0))}(tA_n \in \mathsf{W}(\Delta_1) \vee$
$\vdots$
$\exists A_1(tA_2 \in \mathsf{W}(\Delta_{n-1} \cup \{t\mathcal{K}_{j_n}A_1\}) \wedge \forall \Delta_n \in \mathsf{R}^{J_n}_{\Sigma_{j_n}(\mathsf{W}(\Delta_{n-1}))}(tA_1 \in \mathsf{W}(\Delta_n))) \ldots)$.

3.2) Since $\delta_j^{\mathsf{M}} = 0$ for all $j \in \mathcal{A}$, the only possible $\mathbf{J}$s here are of the kind $((\varnothing, i), (\varnothing, j_1), \ldots, (\varnothing, j_m))$ [3.1)]

3.3) The only minimal refinement Δ_0^m of $Cnc_m(\mathsf{I}_i)$ on $L = \varnothing$ is $Cnc_m(\mathsf{I}_i)$; similarly, the only minimal refinement Δ_1^m of $\Sigma_{j_1}(\mathsf{W}(Cnc_m(\mathsf{I}_i))$ on $J_1 = \varnothing$ is $\Sigma_{j_1}(\mathsf{W}(Cnc_m(\mathsf{I}_i))$; the only minimal refinement Δ_2^m of $\Sigma_{j_2}(\mathsf{W}(\Sigma_{j_1}(\mathsf{W}(Cnc_m(\mathsf{I}_i)))))$ on $J_1 = \varnothing$ is $\Sigma_{j_2}(\mathsf{W}(\Sigma_{j_1}(\mathsf{W}(Cnc_m(\mathsf{I}_i)))))$ and so on until we reach the only minimal refinement $\Delta_n^m = \Sigma_{j_n}(\mathsf{W}$ $(\Sigma_{j_{n-1}}(\mathsf{W}(\Sigma_{j_{n-2}}(\mathsf{W}(\ldots\ (\Sigma_{j_1}(\mathsf{W}(Cnc_m(\mathsf{I}_i))))))))$ [Def. 6]

3.4) for $A_{n+1}(tC \in \mathsf{W}(Cnc_m(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge (tA_{n+1} \in \mathsf{W}(\Delta_0^m) \vee$
for $A_n(tA_{n+1} \in \mathsf{W}(\Delta_0^m \cup \{t\mathcal{K}_{j_1}A_n\}) \wedge (tA_n \in \mathsf{W}(\Delta_1^m) \vee$
$\vdots$

for $A_2(tA_3 \in \mathsf{W}(\Delta^m_{n-2} \cup \{t\mathcal{K}_{j_{n-1}}A_2\}) \wedge (tA_2 \in \mathsf{W}(\Delta^m_{n-1})\vee$
for $A_1(tA_2 \in \mathsf{W}(\Delta^m_{n-1} \cup \{t\mathcal{K}_{j_n}A_1\}) \wedge (tA_1 \in \mathsf{W}(\Delta^m_n)))\ldots)$.
[3.1.ii), 3.2), 3.3) and Prop. 4.8]

3.5) Consider the last disjunction of the expression in 3.4): $tA_2 \in \mathsf{W}(\Delta^m_{n-1})\vee$ for $A_1 \left(tA_2 \in \mathsf{W}(\Delta^m_{n-1} \cup \{t\mathcal{K}_{j_n}A_1\}) \wedge tA_1 \in \mathsf{W}\ (\Delta^m_n)\right)$. Now, suppose that the second disjunct holds, namely, for A_1, it is the case that both of the following hold:
 i. $tA_2 \in \mathsf{W}(\Delta^m_{n-1} \cup \{t\mathcal{K}_{j_n}A_1\})$ and
 ii. $tA_1 \in \mathsf{W}(\Delta^m_n)$

3.6) $t\mathcal{K}_{j_n}A_1 \in \mathsf{W}(\Delta^m_{n-1})$ [3.5.ii), Prop 4.7 and Prop. 4.5]

3.7) $tA_2 \in \mathsf{W}(\Delta^m_{n-1})$ [3.5.i), 3.6) and Prop 3.3]

3.8) This means that if the second disjunct holds, then the first disjunct holds too. Of course, if the second disjunct does not hold, the first one holds. Thus, in both cases, the expression in 3.4) is logically equivalent to:
for $A_{n+1}(tC \in \mathsf{W}(Cnc_m(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge (tA_{n+1} \in \mathsf{W}(\Delta^m_0)\vee$
for $A_n(tA_{n+1} \in \mathsf{W}(\Delta^m_0 \cup \{t\mathcal{K}_{j_1}A_n\}) \wedge (tA_n \in \mathsf{W}(\Delta^m_1)\vee$
$\vdots$
for $A_3(tA_4 \in \mathsf{W}(\Delta^m_{n-3} \cup \{t\mathcal{K}_{j_{n-2}}A_3\}) \wedge (tA_3 \in \mathsf{W}(\Delta^m_{n-2})\vee$
for $A_2(tA_3 \in \mathsf{W}(\Delta^m_{n-2} \cup \{t\mathcal{K}_{j_{n-1}}A_2\}) \wedge (tA_2 \in \mathsf{W}(\Delta^m_{n-1})\ldots)$.

3.9) Consider the last disjunction of the expression in 3.8) and apply the same reasoning as before. This process has to be iterated until one gets that the expression in 3.4) is logically equivalent to:
for $A_{n+1}(tC \in \mathsf{W}(Cnc_m(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge tA_{n+1} \in \mathsf{W}(Cnc_m(\mathsf{I}_i))$

3.10) $Cnc_m(\mathsf{I}_i) \subseteq Cne_{l_m}(\mathsf{I}_i)$ [Inductive hypothesis]

3.11) $tC \in \mathsf{W}(Cne_{l_m}(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\})$ [3.5) and Prop. 3.2]

3.12) $tA_{n+1} \in \mathsf{W}(Cne_{l_m}(\mathsf{I}_i))$ [3.5) and Prop. 3.2]

3.13) $\forall\Lambda \in \mathsf{R}^{\varnothing}_{Cne_{l_m}(\mathsf{I}_i)}$, $tA_{n+1} \in \mathsf{W}(\Lambda)$ [3.12) and Prop. 4.8]

3.14) $tC \in \mathsf{W}(Cne_{l_m}(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge \forall\Lambda \in \mathsf{R}^{\varnothing}_{Cne_{l_m}(\mathsf{I}_i)}$, $tA_{n+1} \in \mathsf{W}(\Lambda)$ [3.11) and 3.13)]

3.15) $tC \in \mathsf{W}(Cne_{l_m}(\mathsf{I}_i) \cup \{t\mathcal{K}_iA_{n+1}\}) \wedge \forall\Lambda \in \mathsf{R}^{\varnothing}_{Cne_{l_m}(\mathsf{I}_i)}$ $(tA_{n+1} \in \mathsf{W}(\Lambda)\vee$ $(tA \in \mathsf{W}(\Lambda \cup \{t\mathcal{K}_jA\}) \wedge \forall\Delta \in \mathsf{R}^{\varnothing}_{\Sigma_j(\mathsf{W}(\Lambda))}$, $tA \in \mathsf{W}(\Delta))$ [3.14)]

3.16) For $\mathbf{J} = ((\varnothing, i), (\varnothing, j))$, $\delta^{\mathsf{M}}_i \geq \delta^{\mathsf{M}}_j$, $\delta^{\mathsf{M}}_i \geq |\varnothing| + |\varnothing|$, $\delta^{\mathsf{M}}_j \geq |\varnothing|$ [Construction]

3.17) $t\mathcal{K}_iC \in \mathsf{SV}(Cne_{l_m}(\mathsf{I}_i))$ [3.15), 3.16) and Def. 12]

3.18) $t\mathcal{K}_iC \in Cne_{l_m+1}(\mathsf{I}_i))$ [3.17) and Def. 12]

4) $sA \in Cne_{l_m+1}(\mathsf{I}_i)$ [2.3) and 3.18)]

Bibliography

Allison, Henry. (2004) *Kant's Transcendental Idealism: An Interpretation and Defense, Revised and Enlarged Edition*. New Haven, Conn.: Yale University Press.

Allison, Henry. (1973) *The Kant-Eberhard Controversy.* Baltimore, Md.: The Johns Hopkins University Press.

Anderson, R. Lanier. (2015) *The Poverty of Conceptual Truth. Kant's Analytic/Synthetic Distinction and the Limits of Metaphysics.* Oxford: Oxford University Press.

Armstrong, Chloe, and Lapointe, Sandra. (2014) "Bolzano, Kant, Leibniz," In F. Přîhonský, *New Anti-Kant.* Hampshire-New York: Palgrave Macmillan, pp. 272-290.

Avron, Arnon, and Zamansky, Anna. (2001) "Non-Deterministic Semantics for Logical Systems," In D. M. Gabbay and F. Guenthner, eds., *Handbook of Philosophical Logic, volume 16.* Dordrecht: Springer, pp. 227-304.

Ayer, Alfred Jules. (1958) *Language, Truth and Logic.* London: V. Gollancs.

Bar-Hillel, Yehoshua. (1970) "Bolzano's Definition of Analytic Propositions," In Y. Bar-Hillel, *Aspects of language. Essays and lectures on Philosophy of Language, Linguistic Philosophy and Methodology of Linguistics.* Jerusalem: The Magnes Press-The Hebrew University, pp. 3-24.

Bar-Hillel, Yehoshua, and Carnap, Rudolf. (1952) "Semantic information," *The British Journal for the Philosophy of Science* 4 (14): pp. 147-157.

Bar-Elli, Gilead. (2017) "Conceptual Analysis and Analytical Definitions in Frege," *European Journal of Philosophy* 25(4): pp. 963-984.

Bar-Elli, Gilead. (2010) "Analyticity and Justification in Frege," *Erkenntnis* 73(2): pp. 165-184.

Beaney, Michael. (2016) "Analysis," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy(Summer 2016 Edition)*, URL = https://plato.stanford.edu/-archives/sum2016/entries/analysis/.

Beaney, Michael. (2005) "Sinn, Bedeutung and the Paradox of Analysis," In M. Beaney and E. H. Reck, eds., *Gottlob Frege. Critical Assessments of Leading Philosophers*. London-NewYork: Routledge, pp. 288-312.

Beaney, Michael. (2002) "Kant and Analytic Methodology,"*British Journal for the History of Philosophy* 10: pp. 455-466.

Beaney, Michael. (1996) *Frege: Making Sense*. London: Duckworth.

Boghossian, Paul Artin. (1999) "Analyticity," In B. Hale, C. Wright and A. Miller, eds., *A Companion to the Philosophy of Language*. Oxford: Blackwell, pp. 331-368.

Bolzano, Bernard. (TS) *Theory of Science*. P. Rusnock and R. George, trans. and eds. Oxford: Oxford University Press, 2014.

Bolzano, Bernard. (2014) *Dottrina fondamentale: §§1-45 della Dottrina della scienza*. G. Rigamonti and L. Fossati, trans. and eds. Milano: Bompiani.

Bolzano, Bernard. (2004) *On the Mathematical Method and the Correspondence with Exner*. Amsterdam: Rodopi.

Bolzano, Bernard. (1973) *Theory of Science*. J. Berg, ed. Dordrecht: Reidel.

Boolos, George. (1984) "Don't Eliminate Cut," *Journal of Philosophical Logic* 13: pp. 373-378.

Brittan, Gordon G. (2015) "Kant's Philosophy of Mathematics," In G. Brittan, *Kant's Theory of Science*. Princeton: Princeton University Press, pp. 43-67.

Brittan, Gordon G. (2006) "Kant's Philosophy of Mathematics," In B. Graham, ed., *A Companion to Kant*. Oxford: Blackwell, pp. 222-235.

Burge, Tyler. (2005) “Postscript to ‘Frege on Apriority’,” In T. Burge, *Truth, Thought, Reason. Essays on Frege.* Oxford: Oxford University Press, pp. 388-389.

Burge, Tyler. (2000) “Frege on Apriority,” In P. Boghossian and C. Peacocke, eds., *New Essays on the A Priori.* Oxford: Oxford University Press, pp. 11-42.

Capozzi, Mirella. (2002) *Kant e la logica.* Napoli: Bibliopolis.

Carapezza, Marco, and D’Agostino, Marcello. (2010) “Logic and the Myth of the Perfect Language,” *Logic and Philosophy of Science* 8 (1): pp. 1-29.

Carnap, Rudolf. (1962) *The Logical Foundation of Probability*, Chicago: University of Chicago Press.

Carnap, Rudolf. (1956) *Meaning and Necessity*, Chicago: University of Chicago Press.

Carnap, Rudolf. (1951) “Meaning Postulates,” In R. Carnap, *Meaning and Necessity. A Study in Semantics and Modal Logic.* Chicago-London: The University of Chicago Press, 1956, pp. 222-229.

Carnap, Rudolf. (1934) *The Logical Syntax of Language.* Peterson, New Jersey: Littlefield, Adams & Co, 1959.

Carnap, Rudolf, Hahn, Hans, and Neurath, Otto. (VC) “The Scientific Conception of the World: The Vienna Circle,” In O. Neurath, *Empiricism and Sociology*, M. Neurath and R. S. Cohen, eds. Dordrecht: D. Reidel, 1973, pp. 299-328.

Casari, Ettore. (2016) *Bolzano’s Logical System.* Oxford: Oxford University Press.

Chomsky, A. Noan. (1968) “Quine’s Empirical Assumptions,” *Synthese* 19 (1-2): pp. 53-68.

Coffa, J. Alberto. (1991) *The Semantic Tradition from Kant to Carnap. To the Vienna Station.* L. Wessels, ed. Cambridge: Cambridge University Press.

Cohen, Morris Raphael, and Nagel, Ernest. (1934) *An Introduction to Logic and Scientific Method.* New York: Harcourt.

Cook, Stephen A. (1971) “The Complexity of Theorem-Proving Procedures,” *Pro-*

ceedings of the Third Annual ACM Symposium on Theory of Computing: pp. 151-158.

Creath, Richard. (2004) “Quine on the intelligibility and relevance of analyticity,” In R. F. Gibson, ed., *The Cambridge Companion to Quine*. Cambridge: Cambridge University Press, pp. 47-64.

D’Agostino, Marcello. (Forthcoming) “Normality, Non-Contamination and Logical Depth in Classical Natural Deduction,” *Studia Logica*.

D’Agostino, Marcello. (2016) “The Philosophy of Mathematical Information,” In L. Floridi, ed., *Routledge Handbook of the Philosophy of Information.* London: Routledge, pp. 164-179.

D’Agostino, Marcello. (2015) “An Informational View of Classical Logic,” *Theoretical Computer Science* 606: pp. 79-97.

D’Agostino, Marcello. (2014a) “Informational Semantics, Non-Deterministic Matrices and Feasible Deduction,” *Electronic Notes in Theoretical Computer Science* 305: pp. 35-52.

D’Agostino, Marcello. (2014b) “Analytic Inference and the Informational Meaning of the Logical Operators,” *Logique et Analyse* 57 (227): pp. 407-437.

D’Agostino, Marcello. (2013a) “Semantic Information and the Trivialization of Logic: Floridi on the Scandal of Deduction,” *Information* 4 (1): pp. 33-59.

D’Agostino, Marcello. (2013b) “Depth-Bounded Logic for Realistic Agents,” *Logic and Philosophy of Science* 11: pp. 3-57.

D’Agostino, Marcello. (2010) “Tractable Depth-Bounded Logics and the Problem of Logical Omniscience,” In H. Hosni and F. Montagna, eds., *Probability, Uncertainty and Rationality.* Pisa: Edizioni della Normale, pp. 245-275.

D’Agostino, Marcello. (2005) “Classical Natural Deduction,” In S. Artemov, H. Barringer, A. d’Avila Garcez, L. C. Lamb and J. Woods, eds., *We will Show them! Essays in Honour of Dov Gabbay.* London: College Publications, pp. 429-468.

D’Agostino, Marcello, Finger, Marcelo, and Gabbay, Dov. (2013) “Semantics and Proof-Theory of Depth Bounded Boolean Logics,” *Theoretical Computer Science*

480: pp. 43-68.

D'Agostino, Marcello, and Floridi, Luciano. (2009) "The Enduring Scandal of Deduction. Is Propositional Logic Really Uninformative?," *Synthese* 167 (2): pp. 271-315.

De Jong, Willem R. (2010) "The Analytic-Synthetic Distinction and the Classical Model of Science: Kant, Bolzano and Frege," *Synthese* 174 (2): pp. 237-261.

De Jong, Willem R. (2001) "Bernard Bolzano, Analyticity and the Aristotelian Model of Science," *Kant Studien* 92 (3): pp. 328-349.

De Jong, Willem R. (1997) "Kant's Theory of Geometrical Reasoning and the Analytic-Synthetic Distinction. On Hintikka's Interpretation of Kant's Philosophy of Mathematics," *Studies in History and Philosophy of Science* 28 (1): pp. 141-166.

De Jong, William R. (1996) "Gottlob Frege and the Analytic-Synthetic Distinction within the Framework of the Aristotelian Model of Science," *Kant-Studien* 87 (3): pp. 290-324.

De Jong, Willem R. (1995) "Kant's Analytic Judgments and the Traditional Theory of Concepts," *Journal of the History of Philosophy* 33: pp. 631-641.

Dolev, Danny, Halpern, Joe and Moses, Yoram. (1986) "Cheating Husbands and Other Stories: A Case Study of Knowledge, Action, and Communication," *Distributed Computing* 1 (3): pp. 167-176.

Dummett, Michael. (1996) "Frege and the Paradox of Analysis," In M. Dummett, *Frege and Other Philosophers*. Oxford: Oxford University Press

Dummett, Michael. (1991a) "Analyticity," In M. Dummett, *Frege: Philosophy of Mathematics*. London: Duckworth, pp. 23-35.

Dummett, Michael. (1991b) "The Value of Analytic Propositions," In M. Dummett, *Frege: Philosophy of Mathematics*. London: Duckworth, pp. 36-46.

Dummett, Michael. (1991c) *The Logical Basis of Metaphysics*. London: Duckworth.

Dummett, Micheal. (1973) *Frege: Philosophy of Language*. London: Duckworth.

Ebbs, Gary. (Forthcoming) "Analyticity: the Carnap-Quine Debate and its Aftermath," In K. Becker and I. Thomson, eds., *Cambridge History of Philosophy: 1945-2015*. Cambridge: Cambridge University Press.

Eco, Umberto. (1997) *The Search for the Perfect Language*. Oxford: Blackwell.

Fagin, Ronald and Halpern, Joseph Y. (1987) "Belief, Awareness and Limited Reasoning," *Artificial Intelligence* 34 (1): pp. 39-76.

Fagin, Ronald, Halpern, Joseph Y., Moses, Yoram, and Vardi, Moshe Y. (1995) *Reasoning about Knowledge*. Cambridge (MA)-London: The MIT Press.

Fagin, Ronald, Halpern, Joseph Y., and Vardi, Moshe Y. (1995) "A Nonstandard Approach to the Logical Omniscience Problem," *Artificial Intelligence* 79 (2): pp. 203-240.

Flach, Peter A. (2002) "Modern Logic and its Role in the Study of Knowledge," In D. Jacquette, ed., *A Companion to Philosophical Logic*. Oxford: Blackwell Publishers, pp. 680-693.

Floridi, Luciano. (2010) *Information - A Very Short Introduction*. Oxford: Oxford University Press.

Friedman, Michael. (2010) "Synthetic History Reconsidered," In M. Domski and M. Dickinson, eds., *Discourse on a New Method: Reinvigorating the Marriage of History and Philosophy of Science*. Chicago: Open Court Publishing, pp. 573-813.

Friedman, Michael. (2000) "Geometry, Construction, and Intuition in Kant and his Successors," In G. Sher and R. L. Tieszen, eds., *Between Logic and Intuition: Essays in Honor of Charles Parsons*. Cambridge: Cambridge University Press, pp. 186-218.

Frege, Gottlob. (LM) "Logic in Mathematics," In G. Frege, *Posthumous Writings*, H. Hermes et al., eds. Oxford: Basil Blackwell, 1979, pp. 203-250.

Frege, Gottlob. (FG) "On the Foundations of Geometry: Second Series," In G. Frege, *Collected Papers on Mathematics, Logic, and Philosophy*, B. McGuinness, ed. Oxford: Basil Blackwell, 1984, pp. 293-340.

Frege, Gottlob. (RH) "Review of E. G. Husserl, *Philosophie der Arithmetik* I," In G. Frege, *Collected Papers on Mathematics, Logic, and Philosophy*, B. McGuinness, ed. Oxford: Basil Blackwell, 1984, pp. 195-209.

Frege, Gottlob. (GGA) *The Basic Laws of Arithmetic*, M. Furth, ed. Berkeley-Los Angeles: University of California Press, 1967.

Frege, Gottlob. (FTA) "On Formal Theories of Arithmetic," In G. Frege, *Collected Papers on Mathematics, Logic, and Philosophy*, B. McGuinness, ed. Oxford: Basil Blackwell, 1984, pp. 112-121.

Frege, Gottlob. (FA) *The Foundations of Arithmetic*, J. L. Austin, trans. Evanston, Illinois: Northwestern University Press, 1960^2.

Frege, Gottlob. (LTM) "Letter to Marty," In G. Frege, *The Frege Reader*, M. Beaney, ed. Oxford: Blackwell, 1997, pp. 79-83.

Frege, Gottlob. (BLC) "Boole's Logical Calculus and the Concept-script," In G. Frege, *Posthumous Writings*, H. Hermes et al., eds. Oxford: Basil Blackwell, 1979, pp. 9-46.

Gabbay, Dov, and Woods, John. (2003) "Normative Models of Rational Agency: the Theoretical Disutility of Certain Approaches," *Logic Journal of the IGPL* 11 (6): pp. 596-613.

Garey, Michael R., and Johnson, David S. (1979) *Computers and Intractability: A Guide to the Theory of NP-Completeness*, New York: W. H. Freeman and Company.

Gierasimczuk, Nina, and Szymanik, Jakub. (2011) "A Note on a Generalization of the Muddy Children Puzzle," In *Proceedings of the TARK XIII.* New York: ACM Press, pp. 257-264.

Goldfarb, Warren. (2010) "Frege's Conception of Logic," In J. Floyd and S. Shieh, eds., *Future Pasts: The Analytic Tradition in Twentieth Century Philosophy.* Oxford: Oxford University Press, pp.

Grice, H. Paul, and Strawson, Peter F. (1956) "In Defense of a Dogma," *The Philosophical Review* 65 (2): pp. 141-158.

Hahn, Hans. (1959) "Logic, Mathematics and Knowledge of Nature," In A. J. Ayer, ed., *Logical Positivism.* Glencoe, Illinois: The Free Press.

Hale, Bob, and Wright, Crispin. (2015) "Bolzano's Definition of Analytic Propositions," *Grazer Philosophische Studien* 91 (1): pp. 323-364.

Hanna, Robert. (2001) *Kant and the Foundations of Analytic Philosophy.* Oxford: Clarendon Press.

Harman, Gilbert. (1986) *Change in View.* Cambridge (MA)-London: The MIT Press.

Hazen, Allen P. (1999) "Logic and Analyticity," In A. C. Varzi, ed., *The Nature of Logic.* Stanford: Center for the Study of Language and Information, pp. 79-110.

Hempel, Carl Gustav. (2001) *The Philosophy of Carl G. Hempel. Studies in Science, Explanation, and Rationality*, J. H. Fetzer, ed. Oxford: Oxford University Press.

Hendricks, Vincent, and Symons, John. (2006) "Where's the Bridge? Epistemology and Epistemic Logic," *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition* 128 (1): pp. 137-167.

Hintikka, Jaakko. (1997) *Lingua Universalis vs. Calculus Ratiocinator. An Ultimate Presupposition of Twentieth-Century Philosophy.* Dordrecht: Kluwer Academic Publishers.

Hintikka, Jaakko. (1987) "Self-profile," In R. J. Bogdan, ed., *Jaakko Hintikka: A Profile*, Dordrecht: D. Reidel Publishing Co., pp. 3-40.

Hintikka, Jaakko. (1984) "Kant's Transcendental Method and His Theory of Mathematics," *Topoi* 3 (2): pp. 99-108.

Hintikka Jaakko. (1982) "Kant's Theory of Mathematics Revisited," In J. N. Mohanty and R. W. Shehan, eds., *Essays on Kant's Critique of Pure Reason.* Oklahoma: University of Oklahoma, pp. 201-215.

Hintikka, Jaakko. (1975) "Impossible Possible Worlds Vindicated," *Journal of Philosophical Logic* 4 (4): pp. 475-484.

Hintikka, Jaakko. (1974, X) "'Dinge an sich' Revisited," In J. Hintikka, *Knowledge and the Known: Historical Perspective in Epistemology.* Dordrecht: D. Reidel Publishing Co., pp. 197-211.

Hintikka, Jaakko. (1974, VI) "Kant's 'New Method of Thought' and his Theories of Mathematics,"In J. Hintikka, *Knowledge and the Known: Historical Perspective in Epistemology.* Dordrecht: D. Reidel Publishing Co., pp. 126-134.

Hintikka, Jaakko. (1973) "Surface Semantics: Definitions and its Motivations," In H. Leblanc, ed., *Truths, Syntax and Modality*, Amsterdam: North Holland Publishing Company, pp. 128-147.

Hintikka, Jaakko. (1973, I) "Logic in Philosophy – Philosophy of Logic," In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 1-25.

Hintikka, Jaakko. (1973, V) "Quantifiers, Language-Games, and Transcendental Arguments," In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 98-122.

Hintikka, Jaakko. (1973, VI) "An Analysis of Analyticity," In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 123-149.

Hintikka, Jaakko. (1973, VII) "Are Logical Truths Tautologies?," In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 150-173.

Hintikka, Jaakko. (1973, VIII) "Kant vindicated," In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 174-198.

Hintikka, Jaakko. (1973, IX) "Kant and the Tradition of Analysis," In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 199-221.

Hintikka, Jaakko. (1973 X) "Information, Deduction and the *A Priori*," In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 222-241.

Hintikka, Jaakko. (1973 XI) “Distributive Normal Forms in First-Order Logic,” In J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press, pp. 242-286.

Hintikka, Jaakko. (1972) “Kantian Intuitions,” *Inquiry* 15: pp. 341-345.

Hintikka, Jaakko. (1970a) “On Semantic Information,” In J. Hintikka and P. Suppes, eds., *Information and Inference.* Amsterdam: North Holland Publishing Company, pp. 3-27.

Hintikka, Jaakko. (1970b) “Surface Information and Depth Information,” In J. Hintikka and P. Suppes, eds., *Information and Inference.* Amsterdam: North Holland Publishing Company, pp. 263-297.

Hintikka, Jaakko. (1969) “On Kant’s Notion of Intuition (*Anschauung*),” In T. Penelhum and J. J. MacIntosh, eds., *The First Critique. Reflections on Kant’s* Critique of Pure Reason. Belmont, CA: Wadsworth Publishing, pp. 38-54.

Hintikka, Jaakko. (1968) “Are Mathematical Truths Synthetic a Priori?,” *The Journal of Philosophy* 65 (20): pp. 640-651.

Hintikka, Jaakko. (1967) “Kant on the Mathematical Method,” *The Monist* 51 (3): pp. 352-375.

Hintikka, Jaakko. (1966) “Kant and the Tradition of Analysis,” In P. Weingartner, ed., *Deskription, Existenz, und Analytizität.* München: Pustet, pp. 254-272.

Hintikka, Jaakko. (1965a) “Are Logical Truths Analytic?,” *Philosophical Review* 74: pp. 178-203.

Hintikka, Jaakko. (1965b) “Distributive Normal Forms in First-Order Logic,” In J. N. Crossley and M. A. E. Dummett, eds., *Formal Systems and Recursive Functions, Proceedings of the Eighth Colloquium, Oxford, July 1963.* Amsterdam: North Holland Publishing Company, pp. 48-91.

Hintikka, Jaakko. (1962) *Knowledge and Belief. An Introduction to the Logic of the Two Notions.* Cornell: Cornell University Press.

Hintikka, Jaakko. (1953) *Distributive Normal Forms in the Calculus of Predicates.*

Acta Philosophica Fennica 6. Helsinki: Societas Philosophica Fennica.

Hintikka, Jaakko, and Remes, Unto. (1976) "Ancient Geometrical Analysis and Modern Logic," In R. S. Cohen, P. K. Feyerabend and M. W. Wartofsky, eds., *Essays in Memory of Imre Lakatos*, Dordrecht: D. Reidel Publishing Co., pp. 253-276.

Hintikka, Jaakko, and Remes, Unto. (1974) *The Method of Analysis. Its Geometrical Origin and Its General Significance,* Dordrecht: D. Reidel Publishing Co.

Horty, John. (1993) "Frege on the Psychological Significance of Definitions," *Philosophical Studies* 72 (2): pp. 223-263.

Hume, David. (1975) *An Enquiry concerning Human Understanding.* L. A. Selby-Bigge and P. H. Nidditch, eds. Oxford: Clarendon Press.

Ichikawa, Jonathan J., and Steup, Matthias. (2014) "The Analysis of Knowledge," *The Stanford Encyclopedia of Philosophy* URL = http://plato.stanford.-edu/archives/spr2014/entries/knowledge-analysis/.

Juhl, Cory, and Loomis, Eric. (2010) *Analyticity.* London-New York: Routledge.

Kant, Immanuel. (JL) *Immanuel Kant's Logic: A Manual for Lectures. Edited by Gottlob Benjamin Jäsche.* In J. M. Young, trans. and ed., *Lectures on Logic.* Cambridge: Cambridge University Press, 1992, pp. 517-640.

Kant, Immanuel. (PM) *What Real Progress has Metaphysics Made in Germany Since the Time of Leibniz and Wolff?* In G. Hatfield *et al.*, trans., and H. Allison and P. Heath, eds., *Theoretical Philosophy after 1781.* Cambridge: Cambridge University Press, 2002, pp. 337-424.

Kant, Immanuel. (Prol.) *Prolegomena to Any Future Metaphysics that May Be Able to Come forward as a Science.* G. Hartfield, trans. and ed. Cambridge: Cambridge University Press, 1997.

Kant, Immanuel. (CPR A/B) *Critique of Pure Reason.* P. Guyer and A. Wood. trans. and eds. Cambridge: Cambridge University Press, 1998.

Kasabova, Anita. (2002) "Is Logic a Theoretical or Practical Discipline? Kant and/or Bolzano," *Archiv für Geschichte der Philosophie* 84 (3): pp. 319-333.

Katz, Jerrold J. (1992) "Analyticity," In E. S. Jonathan Dancy, ed., *A Companion to Epistemology*. Oxford: Blackwell, pp. 11-17.

Kenny, Anthony. (2003) *Frege: un'introduzione*. Torino: Einaudi.

Kienzler, Wolfgang. (2011) "Wittgenstein and Frege," In O. Kuusela and M. McGinn, eds., *The Oxford Handbook of Wittgenstein*. Oxford: Oxford University Press, pp. 80-105.

Kluge, Eike-Henner W. (1980) "Frege, Leibniz and the Notion of an Ideal Language," *Studia Leibnitiana* 12 (1): pp. 140-154.

Kneale, William Calvert, and Kneale, Martha. (1962) *The Development of Logic*. Oxford: Clarendon Press.

Korte, Tapio. (2010) "Frege's *Begriffsschrift* as a *lingua characteristica*," *Synthese* 174: pp. 283-294.

Kripke, Saul A. (1972) *Naming and Necessity*. Cambridge, MA: Harvard University Press.

Künne, Wolfgang. (2006) "Analyticity and Logical Truth. From Bolzano to Quine," In M. Textor, ed., *The Austrian Contribution to Analytic Philosophy*. London: Routledge, pp. 184-249.

Künne, Wolfgang. (1997) "Propositions in Bolzano and Frege," *Grazer Philosophische Studien* 53: pp. 203-240.

Langford, Cooper Harold. (1992) "The Notion of Analysis in Moore's Philosophy," In P. A. Schilpp, ed., *The Philosophy of G. E. Moore*. La Salle: Open Court, pp. 321-342.

Lapointe, Sandra. (2014a) "Bolzano and the Analytical Tradition," *Philosophy Compass* 9 (2): pp. 96-111.

Lapointe, Sandra. (2014b) "Bolzano, Quine, and Logical Truth," In G. Harman and E. Lepore, eds., *A Companion to W. V. O. Quine*. Malden: Blackwell, pp. 296-312.

Lapointe, Sandra. (2012) "Is Logic Formal? Bolzano, Kant and the Kantian Logicians," *Grazer Philosophische Studien* 85 (1): pp. 11-32.

Lapointe, Sandra. (2011) *Bolzano's Theoretical Philosophy. An Introduction.* Hampshire-New York: Palgrave Macmillan.

Lapointe, Sandra. (2010) "Bolzano *a Priori* Knowledge, and the Classical Model of Science," *Synthese* 174 (2): pp. 263-281.

Lapointe, Sandra. (2007) "Bolzano's Semantics and his Critique of the Decompositional Conception of Analysis," In M. Beaney, ed., *The Analytic Turn: Analysis in Early Analytic Philosophy and Phenomenology.* New-York: Routledge.

Leibniz, Gottfried Wilhelm. (1989) *Philosophical Essays.* R. Ariew and D. Garber, trans. and eds. Indianapolis, Ind.: Hackett.

Leibniz, Gottfried Wilhelm. (1966) *Logical Papers.* G. H. R. Parkinson, trans. and ed. Oxford: Clarendon Press.

Levine, James. (2002) "Analysis and Decomposition in Frege and Russell," *Philosophical Quarterly* 52 (207): pp. 195-216.

Lewis, David. (1972) "How to Define Theoretical Terms," *Journal of Philosophy* 67: pp. 427-446.

Locke, John. (1975) *An Essay Concerning Human Understanding.* P. H. Nidditch, ed. Oxford: Oxford University Press.

Lu-Adler, Huaping. (2018) *Kant and the Science of Logic: A Historical and Philosophical Reconstruction.* Oxford: Oxford University Press.

MacFarlane, John. (2004) "In What Sense (If Any) Is Logic Normative for Thought?," unpublished, URL = https://johnmacfarlane.net/normativity_of_logic.pdf.

MacFarlane, John. (2002) "Frege, Kant and the Logic in Logicism," *The Philosophical Review* 111 (1): pp. 25-65.

Mayer, Verena (2003) "Implicit Thoughts: Quine, Frege and Kant on Analytic Proposition," *Grazer Philosophische Studien* 66: pp. 61-90.

McGinn, Colin. (2011) "The Paradox of Analysis," In C. McGinn, *Truth by Analysis: Games, Names, and Philosophy.* Oxford: Oxford University Press, pp. 48-62.

Meyer, John-Jules C. (2001) "Epistemic logic," In L. Goble, ed., *The Blackwell Guide to Philosophical Logic.* Oxford: Blackwell Publishers, pp. 183-202.

Moore, George Edward. (1992) "A Reply to My Critics," In P. A. Schilpp, ed., *The Philosophy of G. E. Moore.* La Salle: Open Court, pp. 533-687.

Morscher, Edgar. (2013) "Bernard Bolzano," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy (Fall 2014 Edition)*, URL = https://plato.stanford.-edu/archives/fall2014/entries/bolzano/.

Morscher, Edgar. (2006) "The Great Divide within Austrian Philosophy," In M. Textor, ed., *The Austrian Contribution to Analytic Philosophy.* London: Routledge, pp. 250-263.

Morscher, Edgar. (1997) "Bolzano's Method of Variation: Three Puzzles," *Grazer Philosophische Studien* 53: pp. 139-165.

Mugnai, Massimo. (2016) "Ontology and Logic: The Case of Scholastic and Late-Scholastic Theory of Relations," *British Journal for the History of Philosophy* 24 (3): pp. 532-553.

Mugnai, Massimo. (2001) *Introduzione alla filosofia di Leibniz.* Torino, Einaudi.

Mugnai, Massimo. (1976) *Astrazione e realtà: saggio su Leibniz.* Milano, Feltrinelli.

Neemann, Ursula. (1970) "Analytic and Synthetic Propositions in Kant and Bolzano," *Ratio* 12: pp. 1-25.

Nelson, Michael. (2006) "Frege and the Paradox of Analysis," *Philosophical Studies* 137 (2): pp. 159-181.

Otte, Michael. (2014) "The Analytic/Synthetic Distinction in Kant, Bolzano and Peirce," *Logique et Analyse* 225: pp. 83-112.

Parikh, Rohit Jivanlal. (2008) "Sentences, Belief and Logical Omniscience, or What Does Deduction Tell Us?," *The Review of Symbolic Logic* 1 (4): pp. 1-18.

Parikh, Rohit Jivanlal. (2005) "Logical Omniscience and Common Knowledge; WHAT Do We Know and What Do WE Know," In R. van der Meyden, ed., *Proceedings of the Tenth Conference on Theoretical Aspects of Rationality and Knowledge, TARK '05.* Singapore: National University of Singapore, pp. 62-77.

Parikh, Rohit Jivanlal. (1994) "Logical Omniscience," In D. Leivant, ed., *Logic and Computational Complexity. International Workshop LCC '94.* New York: Springer, pp. 22-29.

Parikh, Rohit Jivanlal. (1987) "Knowledge and the Problem of Logical Omniscience," In Z. W. Ras and M. Zemankova, eds., *Methodologies of Intelligent Systems. Proceedings of the Second International Symposium, ISMIS 1987.* Amsterdam: Elsevier, pp. 432-439.

Parsons, Charles. (1983) "Kant's Philosophy of Arithmetic," In *Philosophy, Science and Method: Essays in Honor of Ernest Nagel*, S. Morgenbesser *et al.*, eds., New York: St Martins Press, pp. 568-594.

Parsons, Charles. (1980) "Mathematical Intuition," *Proceedings of the Aristotelian Society* 80: pp. 145-168.

Parsons, Charles. (1969) "Kant's Philosophy of Arithmetic," In S. Morgenbesser, P. Suppes and M. White, eds., *Philosophy and Method: Essays in Honor of Ernst Nagel.* New York: St. Martin's Press, pp. 568-594.

Peckhaus, Volker. (2004) "Calculus Ratiocinator versus Characteristica Universalis? The Two Traditions in Logic, Revisited," *History and Philosophy of Logic* 25: pp. 3-14.

Perissinotto, Luigi. (2010) *Wittgenstein. Una guida.* Milano: Feltrinelli.

Plato. (2002) "Meno," In Plato, *Five Dialogues*, G.M.A. Grube, trans. Indianapolis: Hackett Publishing Company.

Priest, Graham. (2008) *An Introduction to Non-Classical Logic. From If to Is.* Cambridge: Cambridge University Press.

Přîhonský, František. (2014) *New Anti-Kant.* Hampshire-New York: Palgrave Macmillan.

Proops, Ian. (2005) "Kant's Conception of Analytic Judgment," *Philosophy and Phenomenological Research* 70 (3): pp. 588-612.

Proust, Jöelle. (1989) *Questions of Form. Logic and the Analytic Proposition from Kant to Carnap.* A. A. Brenner, trans. Minneapolis: University of Minnesota Press.

Proust, Jöelle. (1981) "Bolzano's Analytic Revisited," *The Monist* 64 (2): pp. 214-230.

Putnam, Hilary. (1975) "The Analytic and the Synthetic," In H. Putnam, *Mind, Language and Reality: Philosophical Papers.* Cambridge: Cambridge University Press, pp. 33-69.

Quine, Willard Van Orman. (1991) "Two Dogmas in Retrospect," *Canadian Journal of Philosophy* 3: pp. 265-274.

Quine, Willard Van Orman. (1974) *The Roots of Reference.* La Salle-Chicago: Open Court Publishing Co.

Quine, Willard Van Orman. (1951) "Two Dogmas of Empiricism," *The Philosophical Review* 60: pp. 20-43.

Rantala, Veikko. (1987) "Constituents," In R. J. Bogdan, ed., *Jaakko Hintikka: A Profile.* Dordrecht: D. Reidel Publishing Co., pp. 43-76.

Rantala, Veikko. (1982) "Impossible World Semantics and Logical Omniscience," *Acta Philosophica Fennica* 35: pp. 106-115.

Rantala, Veikko, and Tselishchev, Vitali. (1987) "Surface Information and Analyticity," In R. J. Bogdan, ed., *Jaakko Hintikka: A Profile.* Dordrecht: D. Reidel Publishing Co., pp. 77-90.

Rescher, Nicholas. (2002) "Epistemic Logic," In D. Jacquette, ed., *A Companion to Philosophical Logic.* Oxford, Blackwell Publishers, pp. 478-490.

Rey, Georges. (2018) "The Analytic/Synthetic Distinction," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy (Spring 2018 Edition)*, URL = https://-plato.stanford.edu/archives/spr2018/entries/analytic-synthetic/.

Roski, Stefan. (2010) "*A Priori* Knowledge in Bolzano, Conceptual Truths, and Judgements," In M. van der Schaar, ed., *Judgement and the Epistemic Foundation of Logic. Logic, Epistemology, and the Unity of Science, vol 31.* Dordrecht: Springer, pp. 101-130.

Rusnock, Paul. (2013) "Kant and Bolzano on Analyticity," *Archiv für Geschichte der Philosophie* 95 (3): pp. 298-335.

Rusnock, Paul. (2012) "Remarks on Bolzano's Conception of Necessary Truth," *British Journal for the History of Philosophy* 20 (4): pp. 817-837.

Rusnock, Paul. (2011) "Kant and Bolzano on Logical Form," *Kant-Studien* 102 (4): pp. 477-491.

Rusnock, Paul. (1997) "Bolzano and the Traditions of Analysis," *Grazer Philosophische Studien* 53: pp. 61-85.

Russell, Bertrand. (1967) *Introduction to Mathematical Philosophy.* London: George Allen and Unwin.

Russell, Christopher. (1990) "Hintikka on Kant and Logic", *Erkenntnis* 33: pp. 23-38.

Russell, Gillian. (2008) *Truth in Virtue of Meaning. A Defense of the Analytic/Synthetic Distinction.* Oxford: Oxford University Press.

Sanchez, Francisco. (2018) *Nulla si sa.* C. Buccolini, ed. Pisa: Edizioni della Normale.

Sanchez, Francisco. (1988) *That Nothing is Known.* E. Limbrick and D. F. Limbrick, eds. Cambridge: Cambridge University Press.

Schirn, Matthias. (1989) "Frege on the Purpose and Fruitfulness of Definitions," *Logique et Analyse* 32 (125/6): pp. 61-80.

Scholz, Heinrich. (1952) "Review: Y. Bar-Hillel, Bolzano's Definition of Analytic Propositions," *Journal of Symbolic Logic* 17 (2): pp. 119-122.

Šebestik, Jan. (2018) "Bolzano's Logical System," *History and Philosophy of Logic*

39 (2): pp. 164-186.

Šebestik, Jan. (2017) "Bolzano's Logic," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy (Fall 2017 Edition)*, URL = https://plato.stanford.edu-/archives/fall2017/entries/bolzano-logic/.

Šebestik, Jan. (1997) "Bolzano, Exner and the Origin of Analytical Philosophy," *Grazer Philosophische Studien* 53 (1): pp. 33-59.

Sequoiah-Grayson, Sebastian. (2008) "The Scandal of Deduction. Hintikka on the Information Yield of Deductive Inferences," *The Journal of Philosophical Logic*, 37 (1): pp. 67-94.

Shabel, Lisa. (2006) "Kant's Philosophy of Mathematics," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy(Spring 2016 Edition)*, URL = https://-plato.stanford.edu/archives/spr2016/entries/kant-mathematics/.

Shieh, Sanford. (2008) "Frege on Definitions," *Philosophy Compass* 3 (5): pp. 992-1012.

Siebel, Mark. (2011) "'It Falls Somewhat Short of Logical Precision.' Bolzano on Kant's Definition of Analyticity," *Grazer Philosophische Studien* 82 (1): pp. 91-127.

Sim, Kwang Mong. (1997) "Epistemic Logic and Logical Omniscience. A Survey," *International Journal of Intelligent Systems* 12 (1): pp. 57-81.

Simmons, Peter. (1987) "Bolzano, Tarski, and the Limits of Logic," *Philosophia Naturalis* 24 (4): pp. 378-405.

Smullyan, Raymond S. (1995) *First-Order Logic.* New York: Dover Publications.

Sorensen, Roy. (2003) *A Brief History of the Paradox. Philosophy and the Labyrinths of the Mind.* Oxford: Oxford University Press.

Stalnaker, Robert C. (2006) "On Logics of Knowledge and Belief," *Philosophical Studies* 128 (1): pp. 169-199.

Stalnaker, Robert C. (1999) "The Problem of Logical Omniscience, II," In R. C. Stalnaker, *Context and Content. Essays on Intentionality in Speech and Thought.*

Oxford, Oxford University Press, pp. 255-273.

Stalnaker, Robert C. (1991) "The Problem of Logical Omniscience I," *Synthese* 89 (3): pp. 425-440.

Stang, Nicholas F. (2014) "Kant, Bolzano, and the Formality of Logic," In F. Přîhonský, *New Anti-Kant*. Hampshire-New York: Palgrave Macmillan, pp. 192-234.

Stang, Nicholas F. (2012) "A Kantian Reply to Bolzano's Critique of Kant's Analytic-Synthetic Distinction," *Grazer Philosophische Studien* 85: pp. 33-61.

Tennant, Neil. (2017) "Logicism and Neologicism," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy Winter 2017 Edition*, URL = https://plato.stanford.edu/entries/logicism/.

Textor, Mark. (2013) "Bolzano's Anti-Kantism. From A Priori Cognitions to Conceptual Truth," In M. Beaney, ed., *The Oxford Handbook of The History of Analytic Philosophy*. Oxford: Oxford University Press.

Textor, Mark. (2010) "Frege on Conceptual and Propositional Analysis," *Grazer Philosophische Studien* 81 (1): pp. 235-257.

Tolley, Clinton. (2013) "Kant on the Generality of Logic," In *Proceedings of the 11th International Kant Congress. Vol. 2*, Margit Ruffing *et al.*, eds., Berlin: De Gruyter, pp. 431-442.

Tolley, Clinton. (2012) "Bolzano and Kant on the Nature of Logic," *History and Philosophy of Logic* 33 (4): pp. 307-27.

Tolley, Clinton. (2006) "Kant on the Nature of Logical Laws," *Philosophical Topics* 34 (1-2): pp. 371-407.

Uebel, Thomas. (2019) "Vienna Circle," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy Spring 2019 Edition*, URL = https://plato.stanford.edu/entries /vienna-circle/.

Van Ditmarsch, Hans, van der Hoek, Wiebe, and Kooi, Barteld. (2007) *Dynamic Epistemic Logic*. New York: Springer.

Van Harmelen, Frank, Lifschitz, Vladimir, and Porter, Bruce. (2008) *Handbook of Knowledge Representation.* Amsterdam: Elsevier.

Voltaggio, Franco. (1974) *Bernard Bolzano e la Dottrina della Scienza.* Milano: Edizione di Comunità.

Wittgenstein, Ludwig. (1921) *Tractatus Logico-Philosophicus.* London-New York: Routledge, 2001.

Zalta, Edward N. (2019) "Gottlob Frege," In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy Summer 2019 Edition*, URL = https://plato.stanford.edu/cgi-bin/encyclopedia/archinfo.cgi?entry=frege.