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Ph.D. Thesis in Physics

Quantum Information transfer over Quantum Channels

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Contents

Introduction

The modern theory of quantum mechanics, developed since the early 1920s, is a cornerstone of theoretical physics and, in general, of science, and has been applied with enormous success everywhere, from the atomic structure to the structure of DNA. It has also enormously influenced the technological development of the 20th century. Indeed, the majority of the most important technological innovations, used nowadays, is substantially based on quantum effects, like laser devices, transistors, superconductors, digital clocks, barcode readers, computers, all digital electronics and many tools that we all use in everyday life. Recently, it has been understood that quantum physics can deliver very interesting advances also in information theory. It is now widely accepted that quantum mechanics allows for fundamentally new forms of communication and computation, more powerful and efficient than the traditional ones. In the last two decades, this has led to the development of a new exciting branch of physics in the overlap of quantum mechanics and classical information theory, i.e., quantum information theory [1, 2, 3, 4, 5, 6, 7, 8, 9].

Classical information theory is the mathematical theory that studies the transmission, the storage, and the processing of information [10]. The discoveries of Claude Shannon and Mathison Turing represent the starting point of the modern informatics. *The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point* (C. Shannon 1948) [11]. The latter statement perfectly summarizes one of the main concerns of information theory.

In 1959, Richard Feynman gave an after-dinner talk at the APS meeting in Pasadena, entitled *There's Plenty of Room at the Bottom*, in which he challenged, quite in advance, the realization of a quantum computer: *there is plenty of room to make them* [computers] *smaller. There is nothing that I can see in the physical laws that says the computer elements cannot be made enormously smaller than they are now. In fact, there may be certain* *advantages...* but *when we get to the very, very small world—say circuits of seven atoms—we have a lot of new things that would happen that represent completely new opportunities for design. Atoms on a small scale behave like nothing on a large scale, for they satisfy the laws of quantum mechanics. So, as we go down and fiddle around with the atoms down there, we are working with different laws, and we can expect to do different things. We can manufacture* [them] *in different ways. We can use, not just circuits, but some system involving the quantized energy levels, or the interactions of quantized spins, etc.*

Indeed, as predicted by Moore's Law, every 18 months the size of circuitry packed onto silicon chips is halved. Therefore, this continuous shrinking will eventually reach a point where individual elements will be no larger than a few atoms. At this scale, classical physics does not work and quantum theory comes into play. On one hand, this will be an obstacle to the further development of the information technology. On the other hand, recently another point of view has been adopted and the new idea is to exploit the strange behavior of the microscopic world in order to improve the traditional forms of computation and communication. In other words, the fact that *information is physical* (R. Landauer, 1991 [12]) means that the laws of quantum mechanics can be used to process and transmit it in ways that are not possible with existing traditional systems. In this context, the fundamental information unit is the quantum bit (or *qubit*), i.e. the physical state of a two-level quantum system, such as photon's polarization or electron spin. In the last twenty years, this new research area, known as quantum information science, has stimulated a lively interest both from the theoretical and the experimental point of view [1, 3].

From the theoretical point of view, this new rapidly developing field allows for a reconsideration of the foundations of quantum mechanics in an information theoretical context. Indeed, in many cases these studies have led to a deeper and more quantitative understanding of quantum theory, as it is the case for the entanglement, i.e., non-classical correlations between quantum systems violating Bell inequalities. This represents *not one but rather the characteristic trait of quantum mechanics* (E. Schrödinger, 1935) and drastically evidences the differences between quantum and classical physics [13]. Entanglement, which for a long time has been thought of only as an exotic property of quantum mechanics, has been discovered as a very important new resource in the quantum information theory context [14, 15, 16]. Nowadays, how to quantify entanglement is still an open problem and its solution could help to better understand the "strange" behavior of the quantum world.

Historically, the idea of a 'quantum computer' was devised by Feynman in 1982 in order to simulate quantum mechanical systems [17]. He showed that a classical Turing machine would experience an exponential slowdown when simulating quantum phenomena, because the size of the Hilbert space increases exponentially with the number of the constituents of the system. His hypothetical universal quantum simulator or quantum computer, instead, would allow to describe them efficiently since it is itself a quantum many-body system and the growth in memory resources would be only linear. This also explains why a quantum computer, operating with only a few thousands of qubits, could outperform classical computers. For example, a system of 500 qubits (i.e., 500 two-level quantum systems), which is impossible to simulate classically, can be in a quantum superposition of as many as 2^{500} states. Hence, for one tick of the computer clock, a quantum operation could operate not just on one machine state, as serial computers do, but on 2^{500} machine states at once. A similar operation could be performed on a classical supercomputer with about 10^{150} separate processors, that is impossible to be realized, of course. David Deutsch later, in 1985, took the ideas further and described the first true Universal quantum computer [18].

Today, from the technological point of view, these new forms of computation and communication are based on the ability to control quantum states of microsystems individually and to use them for information transmission and processing. In typical quantum information transfer protocols, information is encoded in delicate quantum states, like, for instance, the polarization states of single-photons, and subsequently it is manipulated and/or transported possibly without being destroyed. The promise of a new super quantum computer, capable of handing otherwise untractable problems, has so attracted the interest of researchers from many different fields like physics, mathematics and computer science. The basic idea is that, roughly speaking, a quantum computer can operate not only on one number per register but on superpositions of numbers and, besides, it can exploit the powerful resource of entanglement. This quantum parallelism leads to an exponential speedup for some computations making feasible problems that are intractable by any classical algorithm, like, for example, the number factoring problem. This last is one of the most important issues in the number theory and so far it seems to belong to the class of the hard problems, i.e. exponentially memory and time consuming. Besides, the factorization is not only a purely mathematical question but its computational complexity is fully used in modern cryptography (like RSA, after Rivest, Shamir, and Adleman) [19]. In order to transmit secret messages between two communicating parties, these crypto systems are essentially based on the computational complexity to factorize very large numbers. This means that it is computationally hard, but not impossible, for an adversary to get information on a secret message. Typically, for the most important private communications (e.g., bank transactions), numbers with more than 300 digits are used: actual computers, by using the algorithms known so far, would need thousands of years to factorize them and so to decipher the relative secret messages encoded with these key-numbers. Indeed, up to know, no classical algorithm is known that can factor these large numbers in a polynomial time.

In 1994, P. Shor discovered the first quantum algorithm (today well known as Shor algorithm) to solve the factoring problem in an efficient way, i.e. in polynomial time and memory resources [20]. Historically, it represents a crucial point in the development of quantum information science. Shor's algorithm could allow one to factorize a large number in a reasonable time and to attack modern cryptography, this violating our privacy and deciphering all private information, like credit card codes, bank and commercial transactions, etc. Even though quantum computers are far from being realized yet, the existence of such efficient factoring algorithm represents a huge challenge. In 2001, the principles of the Shor's algorithm were demonstrated by a group at IBM, using a small prototypical quantum computer with 7 qubits to factorize the number 15. The candidates for the realization of quantum bits for a future quantum computer basically are: photons and nonlinear optical media in quantum optics, cavity quantum electrodynamics devices, ion traps, nuclear and electron magnetic resonances (NMR) spectroscopy, atomic and quantum dot physics, Josephson junctions and superconducting electronics, and spin in semiconductors [1].

However, quantum information theory, does, on one hand, break current cryptography by using a quantum computer, on the other hand, it proposes also a new perfectly secure technology, i.e. quantum cryptography [21, 22, 23]. This is a new fully quantum technique of sending messages encoded in individual quantum states through a so-called quantum channel, such as the phase of photons transmitted through an optical fibre. The main concepts over which quantum cryptography, more correctly known as quantum key distribution, relies are two: 1) the Heisenberg uncertainty principle, i.e. any measurement on a quantum state will, in general, alter it; 2) the no-cloning theorem, i.e. the impossibility to clone generic unknown quantum states [24]. In other words, firstly a potential eavesdropper trying to intercept the message cannot avoid changing it and leaving own mark, enabling two communicating parties to detect his/her presence. Secondly, the eavesdropper is unable to acquire complete information about the private message. It is important to stress that the security of these protocols is not based on technological limits but it depends only on the postulates of quantum mechanics, that is the best experimentally tested of all physical theories. In this field, experiments are much further advanced than in other fields of quantum information processing, because they do not require many operations on many qubits (like in a quantum computer), but only preparation and manipulation of simple quantum states, and also because they are essentially based on well developed quantum optics technologies. This allows to overcome more easily the big problem of decoherence. For example, some quantum-cryptographic protocols are successfully implemented over tens of kilometers at rates of the order of thousand bits per second, by using single photons in optical fibers [25]. In the last years, many interesting new concepts in the field of quantum information [4, 5, 6, 7, 8, 9, 21, 22, 26, 27, 28, 29, 30], such as teleportation and superdense coding, have left the theoretical domain to be experimentally implemented and even to become commercial prototypes like quantum key distribution systems [31]. Indeed, quantum cryptography devices are realizable with today's technology and have attracted private investment in several start-up companies and major corporate players in the world. With several quantum-cryptography products already on the market, the quantum information industry has already arrived. Free-space quantum key distribution has been also proposed, sending single photons through open air even in daylight [32, 33]. In this project, a long-distance transmission system could be realized combining optical fibres and satellites. For example, in a near future the free-space quantum cryptography might help to protect the security of satellite television broadcasts. Recently, a quantum communication channel between space and Earth has been realized by sending single photons from a low-Earth orbit geodetic satellite to a ground-based receiver located in Matera (Italy) [34].

Another important and fascinating application of quantum information science is quantum teleportation. Charles H. Bennett and his co-workers have suggested that it is possible to transfer quantum states from one place to another, only by using classical communication and by exploiting an entanglement resource. Initially, teleportation of individual qubits has been limited only to laboratory distances. Later, after the first experiments of F.

De Martini and A. Zeilinger [28, 27], other techniques have allowed to reach larger distances by exploiting the possibility of using glass fiber optics in a channel underneath the river Danube in Vienna [35]. Efficient long-distance quantum teleportation is crucial for quantum communication and quantum networking schemes.

In this context, the term 'quantum communication' globally refers to this important research area of quantum information theory, including quantum cryptography, teleportation and other communication protocols, and has already found technological applications, as noted above. Loosely speaking, quantum communication studies the transfer of a quantum state from one place to another in the space [1, 3]. The main obstacle to the development of quantum information technology is the difficulty of transmitting quantum information (e.g., photons) over noisy quantum communication channels (such as an optical fiber), recovering and refreshing it at the receiver side, and then storing it in a reliable quantum memory. Indeed, the decoherence is one of the major obstacles to the realization of quantum information technologies. For these reasons, the theory of open quantum systems (i.e., systems interacting with a noisy environment), the study of quantum channels and the quantitative analysis of their capacity of transmitting quantum information are some of the main topics of quantum communication. In fact, the transition from the initial state to the final state of a quantum system can be described in terms of quantum channels [1]. At a mathematical level these are linear maps which operate on the set of bounded operators of the system, preserving the trace and (if any) the positivity of the operators on which they act. In order to represent a "physical" transformation, i.e. a transformation that could be implemented in an experimental laboratory, a quantum channel must also possess the property of complete positivity (i.e. the positivity of any initial joint operator acting on the system plus an external ancilla need to be preserved by the action of the map). An impressive effort has been devoted in the last decades to study the properties of quantum channels. They play a fundamental role in many different branch of physics, specifically in all those sectors where one is interested in studying the decoherence and noise effects.

Therefore, it is interesting to study the fundamental limits on quantum information transmission, that are due to the presence of noise in quantum channels. In other words, a goal of quantum information theory is to evaluate the information capacities of some important communication channels, i.e. optical fiber (Bosonic channels) or transmission of qubits in quantum systems (qubit channels). In general, the capacity of a communication channel

(Shannon, 1948) is the maximum rate, usually measured in bits per second, at which information can be transmitted reliably from the sender's side to the receiver's one with an asymptotically low probability of error.

The majority of the results obtained so far relate to two specific classes of channels, namely the qubit channels and the Bosonic Gaussian channels. The former are completely positive trace-preserving transformations which act on the state of a single two-level quantum system (qubit). Due to the small size of the Hilbert space a simple parametrization of these channels has been obtained [36, 37] while some additivity issues [38, 39, 40] and several classical and quantum capacities [38, 39, 41, 42, 43, 44] have been successfully solved (see also Ref. [3] for a review). Bosonic Gaussian channels [45, 46], on the contrary, are a specific subclass of maps acting on a continuous variable system that preserve certain symmetries. These channels include a variety of physical transformations that are of fundamental interest in optics, including thermalization, loss and squeezing. As in the qubit channel case, additivity issues [47, 48] and capacities [49, 50, 51, 52, 53, 54, 55] have been successfully solved for Bosonic Gaussian channels. Furthermore, they allow for a compact parametrization [48, 53, 56, 57] in terms of the characteristic function formalism [58, 59, 60, 61].

In this thesis we investigate the properties of both qubit and Bosonic Gaussian channels, using a phase space representation of a quantum state by means of the formalism of the characteristic function, extensively used in quantum optics. Particularly, we introduce a new property, called weak-degradability, that allows to characterize these noisy quantum communication channels, i.e. to evaluate their capacity of transmitting quantum information, encoded in quantum systems. Using this phase-space approach, we find that some quantum channels are not able at all to transmit quantum information (anti-degradable), while for the other ones (degradable) the quantum capacity can be greater than zero and can be calculated explicitly. In this last case, quantum information can be successfully transferred over noisy quantum channels. Therefore, this property represents a powerful tool to quantify the performance of noisy quantum channels and may be very useful for real quantum communication applications, in which the noise is the main obstacle to practical realizations.

This thesis is organized as follows.

In Chapter 1 we review some tools of quantum information science, useful in quantum communication theory. Particularly, we describe the idea of qubit with respect to the classical bit, the connection between entropy and information in both classical and quantum information theory, the quantum measurement problem and the concept of classical and quantum fidelity.

In Chapter 2, we focus on communication theory. We start describing the background of classical communication: the idea of communication channel, the encoding-decoding and error-correction procedures, and the notion of channel capacity. Moreover, we recall the definitions of Gaussian channel and degraded broadcast channel. Then we introduce quantum communication, based on the crucial idea to use quantum systems to transfer classical and quantum information between two communicating parties. The noisy quantum communication channels can be described as open quantum system by using the formalism of quantum operations. Therefore, we recall the notion of quantum capacity, Q, and we introduce a new property of quantum channels, that we call weak-degradability, simplifying the big issue of the calculation of Q [52, 53, 54, 44]. This property implies that, on one hand, for one subclass of quantum channels (degradable) the coherent information is additive; this fact allows one to express their quantum capacity, Q, in terms of a single-letter formula. On the other hand, for another subclass of quantum channels (anti-degradable) their quantum capacity Q is null (i.e., they cannot be used to transfer quantum information).

In Chapter 3, we examine Bosonic Gaussian channels, that describe most of the noise sources which are routinely encountered in optics, including those responsible for the attenuation (beam-splitter) and/or the amplification (amplifier) of signals along optical fibers. In the first part, we focus on beam-splitter/amplifier maps, their composition rules, weak-degradability properties and quantum capacity. We find that (almost) all one-mode Bosonic Gaussian channels are unitarily equivalent to beam-splitter/amplifier channels, up to squeezing transformations [52]. Therefore, we introduce a full classification of one-mode Bosonic Gaussian channels and, using a single-mode canonical representation, we study the weak-degradability properties [53]. Furthermore a new set of channels which have null quantum capacity is identified. This is done by exploiting the composition rules of one-mode

Gaussian maps and the fact that anti-degradable channels are not able to transfer quantum information. In the second part of this chapter, a complete analysis of multi-mode Bosonic Gaussian channels is proposed [54]. We clarify the structure of unitary dilations of general Gaussian channels involving any number of Bosonic modes and present a normal form, by proving the unitary dilation theorem. The minimal number of auxiliary modes that is needed is identified [62], including all rank deficient cases, and the specific role of additive classical noise is highlighted. It allows us to simplify, for instance, the weak-degradability classification. As an application of our theorem, we derive a canonical form of the noisy evolution of n system modes interacting unitarily with a Gaussian environment of n modes, based on a recent generalization of the normal mode decomposition for non-symmetric or locality constrained situations. Moreover, we investigate the structure of some singular multi-mode channels, like the additive classical noise channel. The latter can be used to decompose a noisy channel in terms of a less noisy one in order to find new sets of maps with zero quantum capacity. Particularly, for the two-mode case it is possible to follow out this analysis. In this case, apart from the simple situation of a noisy system-environment interaction not coupling the two Bosonic modes, we have found a (to some extent) counter-intuitive fact: increasing the level of the environmental noise, even if the coherence is progressively destroyed, nevertheless the recovering of the environment (system) output from the system (environment) output after the system-environment noisy evolution is an easier event. The latter property is called weak-degradability (anti-degradability) of the map. Finally, by exploiting the composition rules of two-mode maps and the fact that anti-degradable channels cannot be used to transfer quantum information, we identify sets of two-mode Bosonic channels with zero capacity.

In Chapter 4 we analyze the qubit channels along the same lines followed for Bosonic Gaussian channels [44]. Particularly, we introduce a characteristic function formalism for qubit maps in terms of generalized displacement operators and Grassmann variables. This new approach allows us to present a Green function representation of the quantum evolution and then to define the set of qubit Gaussian channels. It is shown clearly that the qubit Gaussian maps share analogous properties with their continuous variable counterpart, i.e. the Bosonic Gaussian channels. The weak-degradability properties are also analyzed. This approach could be generalized to d-level quantum systems (called *qudit*) in terms of generalized Grassmann variables [63, 64].

Finally, we introduce a class of the so-called memory quantum channels (in which the noise is correlated over different uses of the channel) with correlations acting on pairs of qubits, where the correlation takes the form of a shift operator onto a maximally entangled state [65]. In order to characterize the noise introduced by these maps, we optimize analytically and numerically all *purities* (measured using the p-norm, for any p) of the output states and show that, above a certain threshold of the "memory" parameter (i.e., a 'phase transition' behavior), the optimal value is achieved by the maximally entangled input state characterizing the shift, while below this threshold by partially entangled input states whose degree of entanglement depends on the characteristics of the quantum channel and increases monotonically with the correlation parameter.

The thesis, after the final remarks and outlook, includes also some appendices: in Appendix A we review a simple proof of Williamson theorem and a generalized normal mode decomposition (cited in Chapter 3), in Appendix B we present a brief excursus on Grassmann calculus, necessary to better understand the Chapter 4, and, finally, in Appendix C a quick look at some Fermionic channels is presented illustrating the physical difference between these maps and the qubit channels.

Chapter 1

Quantum Information Science

Quantum information theory is a rapidly developing branch of science, that joins quantum physics, mathematics and computer science. The basic idea is to exploit quantum mechanics in order to introduce new powerful techniques of communication and information processing, much more efficient than the traditional ones, working on the physical information that is held in the state of a quantum system.

The most popular unit of quantum information is the *quantum bit* or *qubit*, i.e. the state of a two-level quantum system. Indeed, unlike classical digital bit (which are discrete, i.e. 0 and 1), a qubit can actually be in a superposition of two states at any given time and can share the peculiar property of entanglement with another qubit. This essentially allows to realize quantum parallelism, that gives the possibility of an exponential speedup for some computations and the ability to manipulate quantum information performing tasks that would be unachievable in a classical context, such as unconditionally secure transmission of information. However, there is the problem of how to read the information, since quantum measurements destroy this parallelism for the Heisenberg uncertainty principle.

The research field of quantum information theory can be shared out in two main areas: quantum computation and quantum communication. The first one includes topics like the idea of a quantum computer/quantum simulator, quantum algorithms, quantum circuits, quantum gates and, more generally, quantum information processing. The quantum communication, instead, is based on the idea to transfer quantum states in space and, according to particular purposes, consists in quantum cryptography, teleportation, entanglement sharing and, more generally, the idea of quantum channel.

In this chapter we briefly introduce the basics of quantum information theory, starting from the description of a qubit. Then we recall the relation between entropy and information in both classical and quantum information theory and consider briefly the problem of quantum measurement in order to describe how to recover the message transmitted during a quantum communication. Finally, we recall another definition, useful in both classical and quantum communication, i.e. the fidelity.

1.1 Bit vs. Qubit

The classical information is stored in a logical state 0 and 1, called bit. All present information technology is based on the storing, the transmission and the processing of classical bits [10].

In quantum information theory, the information is, instead, typically stored in a two-level quantum state $|\psi\rangle$, e.g. a superposition of the states 0 and 1,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \ , \tag{1.1}$$

where α and β are complex numbers, and $|\psi\rangle$ is a generic quantum state (using Dirac's bra-ket notation). The *qubit*[1] is then the state of a two-level quantum system. It can be also represented as a vector in a bi-dimensional complex vectorial space, where the states $|0\rangle$ and $|1\rangle$ form an orthonormal basis in this space, known as computational basis.

In the von Neumann-Landau formalism[2], a quantum state can be described as a density matrix ρ [66]. For a *pure state* (as $|\psi\rangle$ above) the density matrix is defined as $\rho_{pure} = |\psi\rangle\langle\psi|$. In most realistic situations we have not a complete information about a quantum system and it can be defined only probabilistically. In other words, we are able only to assign given probabilities p_i with which the system is in a particular state $|\psi_i\rangle$. In these cases, the system is said to be in a mixture of quantum states or,

[1]The name *qubit* was born in conversations between Wootters and Schumacher, and first used in print by the latter in 1995 [69, 14].

[2]The density matrix was introduced, with different motivations, by von Neumann and by Landau. On one hand, Landau was inspired by the impossibility of describing a subsystem of a composite quantum system by a state vector. On the other hand, von Neumann introduced the density matrix in order to develop both quantum statistical mechanics and a theory of quantum measurements.

briefly, in a *mixed state*. They are described by density operators of the form $\rho_{mixed} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

In general, the density operator has to satisfy some mathematical properties in order to represent a physical state. First of all, it is a self-adjoint operator because the physical measured quantities are real numbers. Secondly, it is a trace-one operator ($\mathrm{Tr}[\rho] = 1$) because of the normalization of the quantum state, i.e., roughly speaking, all the probability distributions are normalized to unity. Thirdly, it is a positive operator, i.e. $\rho > 0$, because, ultimately, the probabilities calculated by ρ are always positive numbers. Finally, one can show that for pure state $\mathrm{Tr}[\rho^2] = 1$, while for mixed states $\mathrm{Tr}[\rho^2] < 1$. In particular, the density operators of a qubit can be represented by the following 2×2 matrix:

$$\rho \equiv \begin{pmatrix} p & \gamma \\ \gamma^* & 1-p \end{pmatrix} , \tag{1.2}$$

with p being real number in the range $[0, 1]$, γ complex and $|\gamma|^2 \leq p(1-p)$; for pure qubit states one has $|\gamma|^2 = p(1-p)$. We recall that the identity and Pauli matrices form a basis for $\mathbb{C}^{2\times 2}$ so that any 2×2 matrix ρ can be written as $r_0' \mathbb{1} + \vec{r}\,' \cdot \vec{\sigma}$ where $\vec{\sigma}$ denotes the vector of Pauli matrices, $r_0' \in \mathbb{C}$ and $\vec{r}\,' \in \mathbb{C}^3$. Then, for $\rho = r_0' \mathbb{1} + \vec{r}\,' \cdot \vec{\sigma}$ one has

a) ρ is self-adjoint $\iff$ $(r_0', \vec{r}\,')$ is real, i.e., $r_0' \in \mathbb{R}$ and $\vec{r}\,' \in \mathbb{R}^3$;

b) $\mathrm{Tr}[\rho] = 1 \iff r_0' = \frac{1}{2}$;

c) $\rho > 0 \iff |\vec{r}\,'| \leq r_0'$.

Thus, $\{\mathbb{1}, \vec{\sigma}\}$ also form a basis for the *real* vector space of self-adjoint matrices in $\mathbb{C}^{2\times 2}$ and every density matrix can be written in this basis as $\rho = \frac{1}{2}[\mathbb{1} + \vec{r} \cdot \vec{\sigma}]$ with $\vec{r} \in \mathbb{R}^3$ and $|\vec{r}| \leq 1$. Furthermore,

d) ρ is a one-dimensional projection (or pure state) $\iff$ $|\vec{r}| = 1$.

This elegant geometric method (see Fig. 1.1) for describing qubits is known as *Bloch representation* and $\vec{r}$ is called the *Bloch vector* of a three-dimensional sphere of radius 1, named *Bloch sphere* [1, 14, 67][3]. Note that the states $|0\rangle$ and $|1\rangle$ lie on the two poles of the Bloch sphere. This implies

[3]Actually, E. Majorana introduces for the first time the so-called *Majorana sphere* to represent spinors by a set of points on the surface of a sphere in 1932 [67]. These results

that a classical bit could only lie on one of the poles and, then, it is intuitive that one can, in principle, encode much more (quantum) information in a qubit by using the entire Bloch sphere.

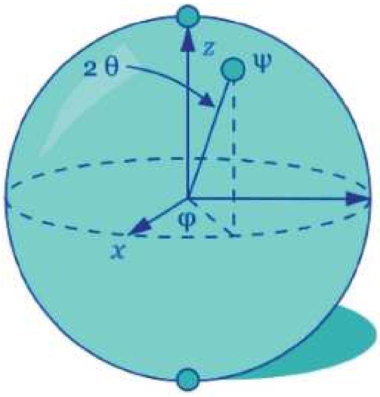

Figure 1.1: Every density matrix can be written in the basis $\{\mathbb{1}, \vec{\sigma}\}$ as $\rho = \frac{1}{2}[\mathbb{1} + \vec{r} \cdot \vec{\sigma}]$ with $\vec{r} \in \mathbb{R}^3$ and $|\vec{r}| \leq 1$: $\vec{r}$ is a real vector, called *Bloch vector*, of a three-dimensional sphere of radius 1, named *Bloch sphere*.

Usually, one is interested in a composite quantum system made up of two or more distinct physical subsystems (e.g., a system composed by many qubits). Its state space is built up from the tensorial product of the state spaces of each physical subsystem. This opens up the possibility to have one of the most peculiar features associated with composite quantum systems, i.e. *entanglement*[4] [13, 14] (see also Refs. [15, 16] for a review). An *entangled* state is defined as a state of a composite system which cannot be written as a tensorial product of states of individual subsystems. For example, the two-qubit state (known as *Bell state*),

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} , \tag{1.3}$$

have been extended later by Bloch and Rabi in 1945, who contributed to spread Majorana's results. Nowadays, this representation of a two-level quantum system is known mainly as Bloch sphere [1, 14]. The historical acknowledgement to E. Majorana for the introduction of this spinorial geometrical representation has been discussed extensively by R. Penrose and others (Penrose, 1987, 1993, and 1996) [67].

[4] In 1935, E. Schrödinger introduced the word *Verschränkung* to describe this phenomenon and, personally, translated it into English as *entanglement* [13, 14].

is an entangled state since there are no single qubit states $|a\rangle$ and $|b\rangle$ such that $|\psi\rangle = |a\rangle \otimes |b\rangle$. When, instead, such factorization is possible, the quantum state is called *separable*. Similarly, a mixed state ρ of a composite system AB is called separable if it can be written as a probability distribution over uncorrelated states, product states, i.e.

$$\rho = \sum_i p_i \; \rho_{Ai} \otimes \rho_{Bi} \; , \tag{1.4}$$

where p_i is a probability distribution, and ρ_{Ai} and ρ_{Bi} are density operators related to the subsystems A and B, respectively. Otherwise, the mixed state is called *entangled*. The entangled states play a crucial role in quantum computation and in quantum communication. They have not analogue in classical mechanics.

In the context of composite quantum systems, the density operator can be used also to describe each subsystem. Such a description is provided by the so-called reduced density operator. Indeed, suppose to have a composite system AB in the state ρ_{AB}, the reduced density operator, for instance, for the system A is defined as

$$\rho_A \equiv \mathrm{Tr}_B[\rho_{AB}] \; , \tag{1.5}$$

where Tr_B is the partial trace over system B. One can easily verify that ρ_A satisfies all the properties (see above) that a density operator has to satisfy. Moreover, it provides the correct statistics for the measurements made on system A. Rigorously, the partial trace respect to the subsystem B is defined in the following way

$$\mathrm{Tr}_B\left[|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right] \equiv c \; |a_1\rangle\langle a_2| \; , \tag{1.6}$$

where $|a_1\rangle$ and $|a_2\rangle$ are any two vectors in the Hilbert space of A and $|b_1\rangle$ and $|b_2\rangle$ are any two vectors in the Hilbert space of B and c is the scalar product $c = \langle b_2|b_1\rangle$. A similar definition is given for the partial trace over the system A and, more generally, the concept of reduced density operator does apply also for composite systems made up of more than two subsystems. We observe that, if a composite system AB is in a pure entangled state ρ_{AB}, then the subsystems A and B are always in a mixture of states, i.e. $\mathrm{Tr}[\rho_A^2] < 1$ and $\mathrm{Tr}[\rho_B^2] < 1$.

The inverse operation of the partial trace is the purification of a quantum state [1, 14]. Indeed, every mixed state ρ can be thought of as arising from a pure state $|\psi\rangle$ on a larger Hilbert space (system plus ancilla). In other words, given a generic (mixed) quantum state $\rho \in \mathcal{H}$ of a system A, one can always find a pure state $|\psi\rangle$ such that

$$\rho = \mathrm{Tr}_B[|\psi\rangle\langle\psi|] \; , \tag{1.7}$$

where $|\psi\rangle$ is the state of a composite system AB, with B being another system (ancilla). In particular, for a given mixed state with spectral decomposition $\rho_A = \sum_k p_k \, |k\rangle_A\langle k| \, \in \mathcal{H}$, such a purification is given by one of the following state in $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$,

$$|\psi\rangle_{AB} = \sum_k \sqrt{p_k} \, |k\rangle_A \otimes |k\rangle_B \; , \tag{1.8}$$

where $|k\rangle_B$ is an arbitrary orthonormal basis of $\mathcal{H}_\mathcal{B}$.

1.2 Entropy and information

In this section we start describing briefly the connection between entropy and information in classical information theory [10]. Then, we extend these ideas to the quantum case by using the notion of von Neumann entropy, that is the quantum version of the classical Shannon entropy.

Historically the notion of entropy appeared for the first time in the context of thermodynamics and statistical mechanics, but later its meaning assumed an information theory perspective because of the work of Shannon. Indeed, information can be associated to the ignorance level about some random variable. For instance, one can encode n letters of a message in the values $x_1, x_2, \ldots x_n$ assumed by a discrete random variable X (classical information source) with probabilities $p_1, p_2, \ldots p_n$. In this context, the Shannon (information) entropy is a measure of the uncertainty associated with the random variable X. Therefore, the entropy can be used to quantify the resources necessary for storing information or, also, to define the irreversibility of the computation because of information loss, according to the Landauer principle [68]. This principle states that, given a computer in an environment at temperature T, when an information bit is erased, the entropy of the environment increases at least by $k_B \ln 2$, where k_B is the Boltzmann constant.

At a mathematical level, the *Shannon entropy* of a discrete random variable X, assuming the values $x_1, x_2, \ldots x_n$ with probabilities $p_1, p_2, \ldots p_n$ (in the range $[0,1]$), respectively, is given by

$$H(X) \equiv H(p_1, p_2, \dots p_n) \equiv -\sum_x p_x \log_2 p_x \ . \tag{1.9}$$

Note that in this definition the logarithms are taken in base 2, as usual, in such a way that the information is measured in bits. Moreover, one makes the convention that $0 \log_2 0 \equiv 0$, because a never occurring event does not contribuite to the entropy; mathematically, it is justified by $\lim_{x\to 0} x \log_2 x = 0$. The Shannon entropy is also used to measure the complexity of an information source, i.e. to quantify the extent to which one can compress the information being produced by a classical information source [10, 1, 14] (Shannon's noiseless channel coding theorem, see Sec. 2.1.2). In this context, the data compression problem corresponds to the idea of determining which are the minimal physical requirements necessary to store an information source.

Now let us suppose to have two random variables, X and Y, with probability distribution $p(x)$ and $p(y)$, respectively. It is then interesting to relate the information content of X with the one of Y, by defining the conditional entropy and the mutual information. First of all, one has to introduce the joint entropy of X and Y as

$$H(X,Y) \equiv -\sum_{xy} p(x,y) \log_2 p(x,y) \ , \tag{1.10}$$

which measures the global uncertainty on both the random variables. Here $p(x,y)$ is the joint probability distribution, i.e. the probability of two simultaneously occurring events x and y. Now, assuming to know the value of Y, i.e. to have $H(Y)$ bit of information, the remain uncertainty about (X,Y) is due to our remaining ignorance of X. The conditional entropy is then defined as follows

$$H(X|Y) \equiv H(X,Y) - H(Y) \ , \tag{1.11}$$

and it measures the information content about X conditioned to the 'a priori' knowledge of Y. Finally, one is usually interested to measure the common information shared between X and Y, i.e. the so-called classical mutual information. It is given by

$$H(X:Y) \equiv H(X) + H(Y) - H(X,Y) \equiv H(X) - H(X|Y) \ . \tag{1.12}$$

This quantity plays an important role in quantifying the capability of transferring information over classical communication channels (see Sec. 2.1). Another way to write down the mutual information in terms of the probability distributions is the following [10]:

$$H(X:Y) \equiv \sum_{xy} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)} \; . \tag{1.13}$$

This quantity measures the dependence between the two random variables; it is symmetric in X and Y, always nonnegative, and equal to zero if and only if X and Y are independent. The entropy is sometimes referred to as the self-information of a random variable, because it corresponds to the mutual information of a random variable with itself, i.e. $H(X) \equiv H(X:X)$.

Von Neumann entropy

As the entropy plays a key role in classical information theory [10], it does the same in quantum information theory, since it measures *how much uncertainty* there is in a quantum state [1, 14]. As we have seen in Sec. 1.1, the quantum states are represented, in the von Neumann approach [66], by density operators. In quantum mechanics, they play a role which is similar to the probability distributions in classical mechanics. By using this analogy, the entropy of a quantum state ρ (quantum information source) is a simple generalization of the Shannon entropy. Particularly, the *von Neumann entropy* [66] is

$$S(\rho) \equiv -\mathrm{Tr}\left[\rho \log_2 \rho\right] \; , \tag{1.14}$$

that can be written in terms of the eigenvalues, λ_x, of ρ, i.e.,

$$S(\rho) \equiv -\sum_x \lambda_x \log_2 \lambda_x \; , \tag{1.15}$$

where $0 \log_2 0 \equiv 0$, like in the Shannon definition. This quantity satisfies the following properties:

a) It is non-negative and zero if and only if the state is pure;

b) It is at most $\log_2 d$ if ρ 'lives' in a d-dimensional Hilbert space. This upper bound is obtained if and only if the system is in the completely mixed state $\rho = \mathbb{1}_d/d$;

c) In a pure bipartite (composite system AB) state, one has $S(A) = S(B)$. The subsystem entropy, e.g. $S(A)$, is positive (mixed state) if the pure bipartite state is entangled;

d) The von Neumann entropy is a concave function of its arguments. Indeed, given the probabilities p_i (real non-negative numbers such that $\sum_i p_i = 1$) and the corresponding density operators ρ_i, the following inequality is satisfied:

$$S\Big(\sum_i p_i \rho_i\Big) \geq \sum_i p_i S(\rho_i) \ . \tag{1.16}$$

As for the Shannon entropy, the von Neumann entropy is also used to describe the complexity of a quantum information source, i.e. to quantify how much quantum information (quantum state), contained in a quantum information source, can be compressed. The quantum equivalent of the Shannon result is Schumacher's quantum noiseless channel coding theorem [69, 1, 14].

Besides, in analogy to the Shannon entropy, for quantum states it is possible to define the quantum joint entropy, the quantum conditional entropy and the quantum mutual information [1, 14]. For a quantum system, composed by the subsystems A and B, the quantum joint entropy is

$$S(A,B) \equiv -\mathrm{Tr}[\rho_{AB} \log_2(\rho_{AB})] \ , \tag{1.17}$$

where ρ_{AB} is the density operator for the composite system. Note that one can have $S(A,B) \leq S(A)$. Indeed, if the composite system AB is in a pure entangled state, then $S(A,B) = 0$ while $S(A) > 0$ because each subsystem is in a mixed state. This is a peculiar property of the von Neumann entropy, because of entanglement, and it is not true for classical composite systems.

In the same way, the quantum conditional entropy and mutual information are, respectively, given by

$$\begin{aligned} S(A|B) &\equiv S(A,B) - S(B) \ , \qquad (1.18) \\ S(A:B) &\equiv S(A) + S(B) - S(A,B) = S(A) - S(A|B) = S(B) - S(B|A) \ . \end{aligned}$$

1.3 Quantum measurements

In classical communication the transmitted messages are recovered by performing measurements on the received signals. Analogously, in a quantum communication scenario, at the end of the communication process, in order to recover the original message, the user has to perform quantum measurements on the physical systems (decoding).

The quantum measurement is still today a not yet well understood problem that has the historical roots in the Einstein-Bohr debates and in some paradoxes, i.e. the Einstein-Podolsky-Rosen (EPR) thought experiment [70] and the Schrödinger's cat state [71]. Briefly, the difficulties stemmed from an apparent conflict between the linear, reversible and deterministic dynamics of quantum mechanics and the nonlinear, irreversible and probabilistic process of quantum measurement. One of the postulates of the quantum mechanics states that during measurement a non-linear collapse of the wave packet occurs. Now we will describe briefly, first, the simple case of standard projective (von Neumann) measurements and then the most general formulation of a measurement in terms of positive operator-valued measure (POVM). The POVM formalism arises essentially from the fact that projective measurements on a larger system will act on a subsystem in such a way that they cannot be described by projective measurement on the subsystem alone [1, 14].

1.3.1 Von Neumann measurements

Let us consider a physical system in the state ρ of a finite-dimensional Hilbert space and an observable O we want to measure by a projective (von Neumann) measurement [66]. Particularly, the spectral decomposition of a non-degenerate[5] observable O is

$$O = \sum_n o_n |n\rangle\langle n| \equiv \sum_n o_n P_n \ , \tag{1.19}$$

with $\{P_n\}$ being a complete set of orthogonal projectors operators (i.e., $P_n P_m = P_n \delta_{nm}$ and $\sum_n P_n = \mathbb{1}$). According to the standard (von Neumann) measurement postulate [66], the possible outcomes of the measurement process correspond to the eigenvalues o_n of the observable O with probability

$$p_n = \langle n|\rho|n\rangle = \mathrm{Tr}[\rho P_n] \ , \tag{1.20}$$

and, immediately after the measurement, the state of the quantum system is 'projected' to $P_n \rho P_n^\dagger / p_n$.

[5]The projective measurement was originally formulated by von Neumann for non-degenerate observables [66] and, later, generalized by Lüders to the degeneracy case [72].

1.3.2 Positive operator-valued measurements

The von Neumann measurements, described above, give us two rules, describing the measurement statistics and the post-measurement state of the system, respectively. However, in a realistic scenario (i.e., in a experimental laboratory) one is interested only on the probabilities of the respective measurement outcomes, irrespective of the post-measurement state. This is the case, for instance, of a real experiment in which the system is measurement only once. In this context, the positive operator-valued measure (POVM) formalism is a mathematical tool well adapted to the analysis of more general measurements [1, 14, 2]. Suppose to have a system in the state $|\Psi\rangle$ and a measurement performed on the system described by measurement operators O_n. The probability of the outcome n is, then, given by

$$p_n = \langle\Psi|O_n^\dagger O_n|\Psi\rangle \ . \tag{1.21}$$

Now, we define

$$E_n \equiv O_n^\dagger O_n \ , \tag{1.22}$$

where E_n is a positive operator such that $\sum_n E_n = \mathbb{1}$ and $p_n = \langle\Psi|E_n|\Psi\rangle$. These operators E_n are called POVM elements of the measurement and the complete set $\{E_n\}$ is known as a POVM. For example, in a projective measurement (and only in this case) the POVM elements are the same as the measurement operators (projectors) P_n, i.e. $E_n \equiv P_n^\dagger P_n = P_n$.

Just to give a concrete example, an ideal photodetector implements the POVM $\{|n\rangle\langle n|\}_{n=0}^{\infty}$ on the electromagnetic field, with $|n\rangle$ being the number states. However, in real quantum optics experiments, the photodetector has the sensibility of a single photon but does not measure the photon number. In other words, it is described by the element POVM, corresponding to one 'click', and by the other one, corresponding to the result of 'no-click'.

According to the Naimark theorem (1940), any generalized measurement can be implemented by unitary dynamics and projective (von Neumann) measurements [73]. In other words, let us consider a system and environment interacting through an unitary operator, which simultaneously applies the operators O_n to the system and takes the environment from the initial state $|0\rangle$ to some state $|n\rangle$, i.e.

$$U|\Psi\rangle|0\rangle = \sum_n O_n|\Psi\rangle|n\rangle \ . \tag{1.23}$$

The normalization of the final state for any $|\Psi\rangle$,

$$\langle 0|\langle\Psi|U^\dagger U|\Psi\rangle|0\rangle = \sum_n \langle\Psi|O_n^\dagger O_n|\Psi\rangle = 1 \ , \tag{1.24}$$

is a consequence of the fact that $\sum_n O_n^\dagger O_n = \mathbb{1}$.

Now we perform a projective measurement on the environmental state (rather than the system one) using the operator $O = \sum_n o_n P_n$, as in Eq. (1.19), and the probability of outcome n becomes $p_n = \langle\Psi|O_n^\dagger O_n|\Psi\rangle$, as in Eq. (1.21). The final state of the whole system is

$$\frac{P_n U|\Psi\rangle|0\rangle}{\sqrt{p}_n} \equiv \frac{O_n|\Psi\rangle|n\rangle}{\sqrt{p}_n} \ . \tag{1.25}$$

1.4 Classical and Quantum Fidelity

During a communication protocol the message is always unavoidably perturbed by the presence of physical noise. It is so interesting to evaluate how this noise modifies the state encoding information, i.e. how the received message differs from the original one. The difference between the final state and the initial state of the communication protocol can be measured by using the concept of fidelity.

In classical information theory, for any two classical probability distributions, $\{p_i\}$ and $\{q_i\}$, the fidelity is defined as

$$F(p_i, q_i) \equiv \Big(\sum_x \sqrt{p_i q_i}\Big)^2 \ . \tag{1.26}$$

In other terms, the classical fidelity is the inner product of $(\sqrt{p_1}, \ldots \sqrt{p_n})$ and $(\sqrt{q_1}, \ldots \sqrt{q_n})$ viewed as vectors in Euclidean space. Note that $F(p_i, q_i) = 1$ when $\{p_i\} = \{q_i\}$ and, in general, $0 \leq F(p_i, q_i) \leq 1$.

The quantum extension of this notion from probability theory is the so-called quantum fidelity. It measures the "closeness"[6] of two quantum states, ρ and σ, i.e. it is measures how close two quantum states are in the Hilbert space [1, 14, 74, 75]. Uhlmann [74] defines it as follows

$$F(\rho, \sigma) \equiv \left(\mathrm{Tr}\left[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right]\right)^2 \ . \tag{1.27}$$

[6]Note that, however, it does not define a metric on the space of density operators; for instance, when two quantum states are equal (i.e., $\rho = \sigma$), $F(\rho, \rho) = \mathrm{Tr}[\rho] = 1$. The same fact holds for the classical fidelity.

It assumes values in the range $[0, 1]$ and it decreases as two states become more distinguishable while it increases as they become less distinguishable. Indeed, when the two quantum states are equal (i.e., $\rho = \sigma$), the fidelity is one, while, for example, the fidelity of two orthogonal pure states is exactly zero. Moreover, this quantity is symmetric in its arguments, i.e. $F(\rho, \sigma) = F(\sigma, \rho)$, and it is invariant under unitary transformations U, i.e. $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma)$.

Let us restrict to two special cases in which it is possible to give the fidelity a more explicit form. The first one is when ρ and σ commute, i.e. diagonal in the same basis,

$$\rho = \sum_i r_i \left|i\right\rangle \left\langle i\right| \qquad \sigma = \sum_i s_i \left|i\right\rangle \left\langle i\right| \ , \tag{1.28}$$

where $\{\left|i\right\rangle\}$ is an orthonormal basis in the Hilbert space associated to a particular quantum system. In this case, the fidelity is

$$F(\rho, \sigma) = \left(\mathrm{Tr}\left[\sum_i \sqrt{r_i s_i} \left|i\right\rangle \left\langle i\right|\right]\right)^2 = \left(\sum_i \sqrt{r_i s_i}\right)^2 = F(r_i, s_i) \ . \tag{1.29}$$

It is easy to realize that this quantity is the classical fidelity, $F(r_i, s_i)$, between the distributions of eigenvalues, r_i and s_i, respectively, of ρ and σ. The reason is that, when ρ and σ commute, the quantum states behave classically.

The second example, in which a more explicit form for the fidelity does exist, is represented by the fidelity between a pure state, $\left|\psi\right\rangle$, and a generic quantum state, ρ. In this circumstance we have

$$F(\left|\psi\right\rangle, \rho) = \left(\mathrm{Tr}\left[\sqrt{\left\langle\psi\right| \rho \left|\psi\right\rangle \left|\psi\right\rangle \left\langle\psi\right|}\right]\right)^2 = \left\langle\psi\right| \rho \left|\psi\right\rangle \ , \tag{1.30}$$

that is the mean value of ρ in the state $\left|\psi\right\rangle$.

These concepts will be useful in the next chapter, in which the idea of communication channel is explained. For instance, perfect (classical or quantum) fidelity (i.e., equal to one) in transmitting messages from the input to the output side means that the initial (classical or quantum) state can be reconstructed unambiguously from the output. This ideal situation refers to a noise-free channel with the (classical or quantum) capacity of transmitting (classical or quantum) information equal to one. In the real situations, there are unavoidable noise effects and errors during the communication and the main goal is essentially to maximize the fidelity between the output and input states. These considerations will be clearer in the following chapter.

Chapter 2

Communication Theory

The communication theory is the branch of information science which studies the transmission of information encoded in physical systems between two communicating parties. In this chapter, first of all we describe the background of classical communication and classical capacity in Sec. 2.1. Then, in Sec. 2.2, we consider the idea to use quantum systems (e.g., photons) to transfer classical and quantum information between two communicating parties (e.g., through an optical fiber): it leads to the so-called quantum communication theory. The main obstacle to the development of quantum communication technology is represented by the difficulty of transmitting quantum information over noisy quantum communication channels. Real quantum systems suffer from unwanted noisy interactions with the external environment (decoherence). At a mathematical level, they can be described by the formalism of open quantum systems, that we shall review in Sec. 2.2.1. The quantitative analysis of the capability of transmitting quantum states (quantum information) through a quantum channel leads to the notion of quantum capacity (Sec. 2.2.6). Finally, we characterize quantum channels with a new property, i.e. weak-degradability, that enables to considerably simplify the challenging open problem of the evaluation of the quantum capacity and to easily identify those channels that cannot be used to transfer quantum information (Sec. 2.3).

2.1 Classical communication

In the early 1940s the transmission of information at a positive rate with negligible error probability was considered impossible. Later, Shannon proved that this error probability can be made nearly zero for all communication rates below a certain threshold, called channel capacity [10]. The latter quantity can be computed by simply analyzing the noise characteristic of the channel. In particular, Shannon argued that the signal cannot be compressed below some irreducible complexity of the information source. In order to explain these results, we start by describing a generic communication channel (in Fig. 2.1) as composed by the following 4 parts:

1. An information source, which produces a message to be communicated from the sender to the receiver, e.g. symbols out of some finite alphabet.

2. A transmitter, which transforms the message into a physical signal suitable for the transmission over the channel (encoding).

3. A physical channel: this is the physical (noisy) medium used to transmit the signal between the two sides of the communication.

4. A receiver of the message. Her/his goal is to perform an inverse operation (from the signal to the message) with respect to the encoding (decoding). The resulting output signal sequence is usually random but has a probability distribution that depends on the input sequence. Therefore, he/she tries to recover the original message, despite the presence of noise in the channel.

The transfer of information is a physical process and therefore is unavoidably subjected to the uncontrollable environment noise and imperfections in the physical mechanism of signaling itself. The communication is successful when the two users agree on the message that was sent. A communication channel can be described as a system in which the output sequence depends probabilistically on the input sequence. Since two different input sequences may give rise to the same output sequence, the inputs are confusable. However, we can choose some particular sequences of input symbols, called codewords, that are distinguishable at the output side in a such way that the particular input can be reconstructed with a negligible error probability. Therefore, using this efficient encoding, we can transmit and reconstruct a

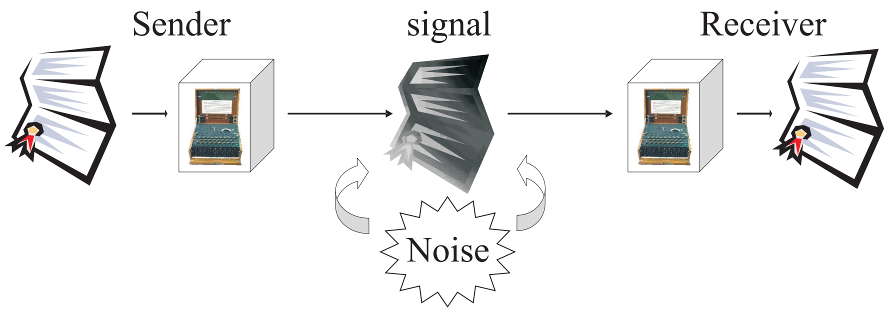

Figure 2.1: Scheme of a generic communication channel.

message at the receiver side with very low probability of error. The maximum rate at which this scheme can be implemented is essentially the capacity of the channel (see Sec. 2.1.2).

Mathematically, let us consider a classical communication channel Φ, in which a letter $x \in X$ is sent and $y \in Y$ is received, with probability $p(x)$ and $p(y)$, respectively [3, 10]. The channel is completely characterized by the probability transition matrix $p(y|x)$, i.e. the probability to get the information $y \in Y$ at the output side of the channel if $x \in X$ was sent at the input side. If $X = Y$ an ideal channel corresponds to have $p(y|x) = \delta_{xy}$, in which case the information is transmitted without errors. Actually, $p(y) = \sum_{xy} p(y|x)p(x)$ is the probability of getting y and $p(x, y) = p(y|x)p(x)$ is the probability that x is sent and y is received.

A channel is called *memoryless* if the probability distribution of the output, p, depends only on the input at that time and is conditionally independent of previous channel inputs and outputs. In the following we will often utilize the expression "use of the channel" referring specifically to a single transmission through the channel and many successive uses correspond to the situation in which, after receiving the output (relative to the first input) another input is sent and so on. Note that, some channels can exhibit memory effects, i.e. correlated noise affecting subsequent uses of the channel. In our communication scenario, sequences of input signals are sent through memoryless channels and, at the end side, because of the presence of noise, one receives corrupted output sequences, that, however, depend only on the input sequences at that time. In the next section we will show how to reduce the effect of noise during the transmission and how to decode the corrupted output sequence in order to recover the original message.

2.1.1 Encoding, decoding, and error-correction

As described above, the design of a communication system can be essentially reduced to two separate parts: (1) source coding, and (2) channel coding. This is a consequence of the source-channel separation theorem [10]. On one hand, the first part is a problem of data compression and consists in removing all redundancy in the data to form the most compressed and efficient representation of them. For example, most modern communication systems are digital and data are reduced to a binary representation; in this way different kinds of information (e.g., audio, video, etc.) are transmitted over the same classical channel. On the other hand, the channel coding, instead, is essentially a problem of data transmission, in which one tries to reduce the errors which unavoidably appear during the transmission through the classical channel, e.g. adding redundancy in a controlled way. The channel encoding can thus be designed independently of the source coding in order to achieve optimal performance. Sometimes, however, it may be better to send the uncompressed information over the noisy channel rather than the compressed version. Quantitatively, one requires that the classical fidelity between the input and the output is as close to one as possible (see Sec. 1.4).

In classical information theory the most obvious encoding procedure is to repeat information, in order to recover the original message even if some information is lost or corrupted (redundancy technique). In particular, it essentially consists in protecting a bit with many copies of itself; if one wants to send 0, he sends 000, and, to send 1, he sends 111. Note that, since this scheme, for instance, uses three symbols for each bit, it has a rate of 1/3 bit per symbol. On the other side, the optimal decoding procedure is realized taking the majority vote of each (output) block of three received bits. For example, if the noise has corrupted the string in such a way that 000 becomes 010, the decoding process is represented by measurement of the final state of the string and then by decoding the block as 0. This type of code is called repetition code. However, if (and only if) more than one bit is changed in one block, then that block is irremediably corrupted and an error occurs, despite the encoding-decoding scheme. For instance, if 000 becomes 011, one decodes the block as 1 and does a mistake. Using longer repetition codes, one can achieve an arbitrarily low error probability, but it is not convenient from a practical point of view because the rate of the code also goes to zero with the block length. In practice, more sophisticated error-correction techniques are implemented and, in effect, the presence of the noise can be efficiently controlled and overcome. A good example of application of these ideas of classical information theory is represented by the use of error-correcting codes on compact discs and DVDs. Nowadays, other more complex schemes, like turbo codes and low-density parity-check (LDPC) codes, are successfully applied to wireless and satellite communication channels.

2.1.2 Classical capacity

The maximum number of distinguishable output signals for n uses of a communication channel grows exponentially with n and the exponent is properly defined as the channel capacity. Of course, one can obtain perfect communication using the channel an infinite number of times but it is not an efficient way of communicating. For this reason, one is interested in the rate of reliably transmitted bits per channel use.

The most famous success of classical information theory is the characterization of the channel capacity as the maximum of the classical mutual information of the channel [11, 10]. In particular, if $p(x)$ and $p(y)$ are the probability distributions of the input and the output states, one can consider the mutual information of the channel, i.e. $H(X : Y)$ as in Eq. (1.13), with $p(x, y) = p(y|x)p(x)$ and $p(y|x)$ being the probability transition matrix (see Sec. 2.1). The mutual information describes, roughly speaking, the information shared by $p(x)$ and $p(y)$. For instance, if $p(x)$ and $p(y)$ are completely uncorrelated [i.e., $p(y|x) = p(y)$ and $p(x, y) = p(y)p(x)$] we get a null mutual information, that is $H(X : Y) = 0$ (see Sec. 1.2).

Thus, at a mathematical level, one can define the information channel capacity as follows.

Definition 1 (Shannon) *The information channel capacity, $C(\Phi)$, of a memoryless communication channel Φ is defined as*

$$C(\Phi) = \max_p H(X : Y) \tag{2.1}$$

where the maximum is taken over all the possible probability distributions of the input state.

Since $H(X : Y) \geq 0$, the channel capacity is always nonnegative. The presence of the maximum is justified by the fact that the mutual information $H(X : Y)$ is a concave function of p over a closed convex set and, therefore, a local maximum is also a global maximum. Usually, this maximum is obtained by using standard nonlinear optimization techniques, as the gradient search. However, in general, there is no closed-form solution for the classical capacity.

An operational definition of the channel capacity is the highest rate of transmitted bits per channel use at which information can be sent with a vanishingly low error probability. Shannon established that the information channel capacity in Eq. (2.1) is equal to the operational version in his second theorem (channel coding theorem, 1948) [11, 10]. It is a fundamental theorem of classical information theory and is rather counter-intuitive because, in order to correct the channel errors, one has to use correction processes that also contain errors and so on, at infinitum. An important idea is that using the same channel many times in succession, the law of large numbers comes into play. In fact, it states that it is always possible to transmit classical information in a classical communication channel in an error-free way up to a given maximum rate (capacity) through the channel. The Shannon coding theorem showed that this limit can be achieved by using codes with a long (enough) block length (i.e., particular typical input sequences, known as codewords, corresponding to distinguishable output states in such a way that the input can be recovered efficiently with an arbitrarily low error probability). In this theorem it is shown that, at least, one good code is represented by the randomly generated code. However, in practical communication systems, there are limitations on the code complexity that we can use and the real rate of communication of information over the channel is, usually, below this ultimate limit.

Now, let us show some examples of classical channels, trying to calculate their information capacities [10]. The simplest classical communication channel is the noiseless binary channel. It is a channel in which the binary input is reproduced exactly at the output, i.e. any bit is transmitted and received without error (ideal channel). Therefore, in each transmission we can send one bit reliably to the receiver and the capacity assumes the maximum possible value, i.e. $C = 1$. It can be obtained by using $\{p(x)\} = \{\frac{1}{2}, \frac{1}{2}\}$.

The basic example of a noisy communication system is the binary symmetric channel. The channel has again a binary input, while the output is equal to the input with probability $1 - p$. For example, if a user sends the input 0, at the output side one receives 0 with probability $1 - p$ and 1 with probability p. In this simple case, the channel capacity can be calculated explicitly and is given by $C = 1 + p \log p + (1 - p) \log(1 - p) = 1 - H(p)$ (with $H(p)$ being the so-called binary entropy function) and it is measured, of course, in bit per transmission. It is obtained with a uniform input distribution.

Another important classical channel is the binary erasure channel, in which some bits are lost rather than corrupted as in the binary symmetric one. In other words, the input bits are erased with probability α but, now, the receiver knows which bits are lost. Here, the input states are two, 0 and 1, while the output states are three, 0, 1, and e, where e is the event associated to the lost bit. The capacity of this channel is $C = 1 - \alpha$ and, intuitively, it corresponds to the fact that one can recover at most a fraction $1 - \alpha$ of bits.

2.1.3 Gaussian channels

Up to now, we have analyzed discrete channels, i.e. systems consisting of finite input and output alphabets. The most important continuous alphabet channel is the so-called Gaussian channel [10, 76]. It is characterized by an output $y_i \in Y$ at the time i, that is the sum of the input $x_i \in X$ and a Gaussian noise $z_i \in Z$, i.e.

$$y_i = x_i + z_i \tag{2.2}$$

where z_i is an independent and identically distributed (i.i.d.) variable from a Gaussian distribution Z with variance σ and independent from the input signal X. A characteristic property of Gaussian channels is that they map input Gaussian distributions into output Gaussian distributions.

If the noise variance is null, the transmission goes without errors. Furthermore, in this case X can assume any real value, so the channel can transmit an arbitrary real number with no error and the channel capacity is infinite. On the other hand, if $\sigma \neq 0$ and there are no further conditions on the input, one can choose an infinite subset of inputs such that the relative outputs are distinguishable with an arbitrary vanishingly low error probability, and the capacity becomes again infinite. In brief, the channel capacity is infinite if at least one of the two following conditions is verified: 1) the noise variance is zero, 2) no constraints are assumed on input[1]. Usually, the most common limitation on the input signal is represented by an energy or power constraint.

[1] A similar problem will be present for Gaussian channels of quantum Bosonic systems, analyzed in Chapter 3.

Suppose to transmit a codeword $(x_1, x_2, \ldots x_n)$ over the Gaussian channel, the following average power restriction can be required, i.e.

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P \ , \tag{2.3}$$

where P is a fixed real number.

Some widespread communication channels, like wired and wireless telephone channels, radio and satellite links, are well described by Gaussian channels. Indeed, in practical channels, since the cumulative effect of a large number of small random effects is approximately normal (central limit theorem), in a large number of situations the Gaussian assumption is valid and the noise can be considered additive, as in Eq. (2.2). In this context, the channel capacity is obtained as in Eq. (2.1) but maximizing the mutual information of the channel only over Gaussian distributions $p(x)$ with a power constraint P on the average of X^2, i.e. $\langle X^2 \rangle$ [10]. In this way, one can also find out that the capacity of a Gaussian channel Φ with power constraint P and noise variance σ is

$$C(\Phi) = \frac{1}{2}\log\left(1 + \frac{P}{\sigma}\right) \tag{2.4}$$

and it is measured in bit per transmission.

2.1.4 Degraded Broadcast channels

Suppose to have a channel with one input alphabet X and two output alphabets Y_1 and Y_2 and a probability transition function $p(y_1, y_2|x)$, i.e. the probability of getting the output symbols y_1 and y_2, if x is sent. This channel is called a broadcast channel and corresponds to the real situation in which there are one sender and two receivers [10]. Such a channel is said to be physically degraded if

$$p(y_1, y_2|x) = p(y_2|y_1)p(y_1|x) \ . \tag{2.5}$$

A broadcast channel is called stochastically degraded if its conditional marginal distributions are the same as that of a physically degraded broadcast channel, i.e. there exists a distribution $p'(y_2|y_1)$ such that

$$p(y_2|x) = \sum_{y_1} p'(y_2|y_1)p(y_1|x) \ . \tag{2.6}$$

In other terms, the receiver 1 can recover the output of the receiver 2 just applying another channel, i.e. $p'(y_2|y_1)$, to his/her output state y_1. Such property of classical broadcast channels will be extended to quantum communication in Sec. 2.3 and will be analyzed in detail for some important classes of quantum channels throughout this thesis.

2.2 Quantum communication

Quantum communication follows along the same principles of classical communication with the crucial difference that here the information is encoded not in a classical system, e.g. the state on or off of a capacitor, but in a quantum system, e.g. the polarization of a photon transmitted through an optical fiber. In this context, the quantum analog of a discrete information source is an ensemble of pure or mixed states $\rho_1, \rho_2, \dots \rho_n$ emitted with probabilities $p_1, p_2, \dots p_n$. Similarly, the quantum version of a classical noisy channel is a transformation that maps the input quantum states, on which the messages have been encoded, into the corresponding output quantum states which have been degraded by the interaction with an external (quantum) environment.

If the states ρ_i of a quantum source are all orthogonal and the channel preserves such orthogonality, the source can be treated as purely classical and quantum communication can be described as a classical communication line. Indeed, the sender measures the source, transmits the measurement results to the receiver, who can make arbitrarily many faithful copies of the source state. However, non-classical behaviors emerge when the source emits "non-orthogonal" or "entangled" states ρ_i: this leads to the notion of *classical capacity* of a quantum channel.

More generally, quantum communication gives us also the possibility to transmit on quantum systems not only classical information, that is bits encoded in quantum systems, but also "genuine" quantum information (e.g., qubit). This happens for instance when one tries to communicate a single copy of an "unknown" quantum system: in this case no classical measurement allows to extract complete information about the source state; then the sender can only transmit a unknown state ρ_i to the receiver, who tries to recover the original source state. This leads to the concept of *quantum capacity* of quantum channels. Interestingly enough, the transmission of unknown quantum states can be used to share entanglement among many parties, sending, for instance, one of two subsystems to another place. In fact, it is very useful for many quantum information protocols, like quantum teleportation. On the other hand, by using only classical communication it is impossible to share entanglement among different parties, because any measurement on any of two entangled subsystems will completely destroy it.

As we will see in Sec. 2.2.6, the classical capacity of a quantum channel "measures" the communication efficiency in transferring classical data (bits), while the quantum capacity "measures" the efficiency in transferring quantum data (qubits). In general, the transmission of quantum systems over quantum channels is, of course, influenced by the presence of physical noise. In this context, a quantum channel is well described treating the quantum system, in which the (classical or quantum) information is encoded, as an open quantum system, i.e. a system interacting with an external noisy environment (see Sec. 2.2.1). As in the classical scenario, a quantum channel is called memoryless when the effect of the noise on each use of the channel does not depend on the effect of the noise on the previous use, as well as for memoryless classical channels. In the following of this thesis we will consider basically memoryless quantum channels and we will analyze an example of memory channels only at the end of Chapter 4.

In real noisy quantum channels the quantum fidelity of transmitting quantum states is obviously less than one. In order to protect quantum information from the unavoidable noise the theory of quantum error-corrections was independently discovered by Shor in 1995 [77] and by Stean in 1996 [78]. The quantum error-correcting codes follow similar principles as the ones used for classical channels (see Sec. 2.1.1). However, there are some differences between classical and quantum information theory and, in the quantum case, we have to overcome three further obstacles. First of all, the no-cloning theorem[2] forbids to implement the repetition code, because one cannot clone quantum states [24]. Secondly, the quantum errors are continuous. The noise can change the value of the quantum bit, e.g. $|0\rangle \rightarrow |1\rangle$, but it can also introduce a continuous error in the relative phase of a quantum superposition, e.g. $|0\rangle + |1\rangle \rightarrow |0\rangle + e^{i\theta}|1\rangle$ with θ being a continuous real number in the range $[0, 2\pi)$. Thirdly, the simple measurement of the output from the

[2]The no-cloning theorem is one of the earlier results of the idea to apply quantum mechanics to information theory [24]. It makes impossible to clone an unknown quantum state, i.e. it is impossible to construct a device that copies unknown quantum states; nevertheless cloning only orthogonal states is possible. Moreover, if the cloning was possible, then it would be possible to communicate faster than light using quantum effects, violating the Einstein's theory of relativity.

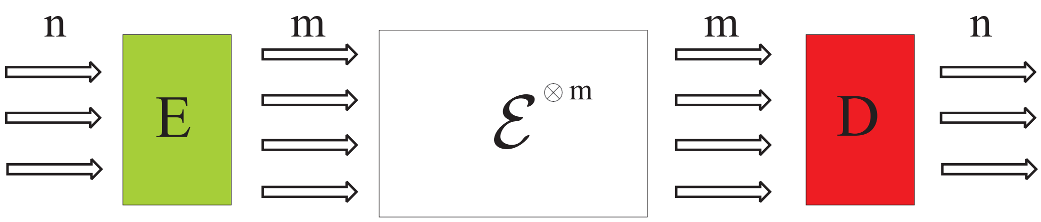

Figure 2.2: Block diagrams of the encoding and decoding procedures in the definition of classical and quantum capacity.

channel (decoding) destroys quantum information because of the Heisenberg principle.

However, all these limitations are not fatal and can be overcome (see Ref. [1] for more details). By adding extra qubits and carefully encoding the quantum state we wish to protect, a quantum system can be insulated against noise effects and errors. In other terms, error correcting codes allow one to reduce errors by a suitable encoding of logical qubits into larger systems. The basic idea is to construct the error-correcting codes, that allow, on one hand, to encode the quantum states in a special way making them robust against the effect of the noise, and, on the other hand, to decode them when the recovering of the original state is desired. As in the classical context, this approach is successful also in quantum communication. Indeed, both the quantum and the classical capacity of a channel are obtained maximizing, over all possible encoding and decoding procedures, the ratio among the number of bits or qubits transmitted and the "redundancy" employed into the code. Particularly, denote with E an encoding map from n qubits to m inputs of a quantum channel $\mathcal{E}$ and with D a decoding map from m channel outputs to n qubits, as in Fig. 2.2.

The classical capacity $C(\mathcal{E})$ of a noisy quantum channel $\mathcal{E}$ is thus defined as follows [79]

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\left\{\frac{n}{m}\,:\,\exists_{m,E,D}\ \ \forall_{|\psi\rangle\in\{|0\rangle,|1\rangle\}^{\otimes n}}\ \ F\left(|\psi\rangle, D\circ\mathcal{E}^{\otimes m}\circ E(|\psi\rangle\langle\psi|)\right) > 1-\epsilon\right\} \quad (2.7)$$

where $F(\ldots)$ is the quantum fidelity defined in Eq. (1.30) and $D\circ\mathcal{E}^{\otimes m}\circ E$ means that we first operate with the encoding map E on n qubits, then with $\mathcal{E}^{\otimes m}$ on m inputs and, finally, with the decoding map D on m channel outputs. The input state $|\psi\rangle$ is chosen to be a tensor product of states $|0\rangle$ and $|1\rangle$: since we are encoding classical information in quantum states, it is

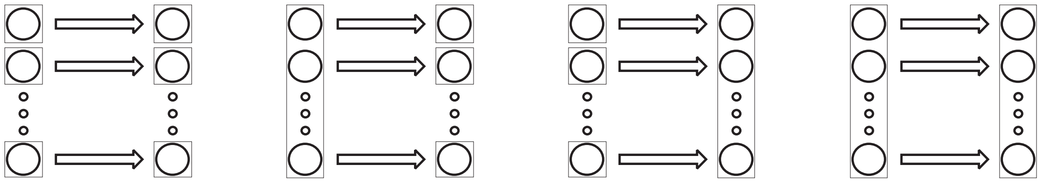

Figure 2.3: Possible schemes of encoding and decoding strategies on single and multi uses of the channel. Particularly, in the first one the sender (receiver) encodes (decodes) the message on individual systems (classical encoder/decoder). In the second and in the third one, only one of them uses a multi-use encoding/decoding procedure. In the last one the sender constructs an encoding on multi signals (quantum encoder) and the receiver realizes joint measurements of them (quantum decoder). The first three configurations are useful to described the classical capacity of the channel only. The quantum capacity, instead, can only be described in terms of the last configuration with optimal encodings being on entangled states and optimal decodings being joint quantum measurements.

not necessary to be able to transfer superpositions of the input messages. In other words, the classical capacity of a quantum channel is defined as the optimal rate (i.e., transmitted bits per channel use) at which the sender can send a tensor product state, i.e. $|\psi\rangle \in \{|0\rangle, |1\rangle\}^{\otimes n}$, of n qubits, for arbitrarily large n, to the receiver, who is able to recover it with fidelity greater than $1-\epsilon$, with arbitrarily small ϵ, after block-encoding, channel transmission and block decoding procedures.

Similarly, the quantum capacity $Q(\mathcal{E})$ is defined as [79]

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \left\{ \frac{n}{m} : \exists_{m,E,D} \ \forall_{|\psi\rangle\in\mathcal{H}^{\otimes n}} \ F\left(|\psi\rangle, D\circ\mathcal{E}^{\otimes m}\circ E(|\psi\rangle\langle\psi|)\right) > 1-\epsilon \right\} , \qquad (2.8)$$

where $\mathcal{H}$ is the two-dimensional Hilbert space of a two-level quantum system. Notice that Q assumes the same meaning as above for C with the only (but important) difference that now the encoding is realized on any possible superposition of n qubits, i.e. $|\psi\rangle \in \mathcal{H}^{\otimes n}$. Clearly, for any quantum channel $\mathcal{E}$, one has $Q(\mathcal{E}) \leq C(\mathcal{E})$, because the capability of transmitting quantum superpositions of n qubits includes trivially the ability of sending separable states. Besides, with respect to classical channels, in quantum information theory one can exploit a peculiar property of quantum mechanics, that is entanglement, to construct optimal encoding. Similarly, one can optimize the decoding strategies by considering joint quantum measurements of the received signals. Possible schemes of encoding and decoding procedures on single and multi uses of the channel are shown in Fig. 2.3.

Physical implementations

At present, the only suitable carrier for long-distance quantum communication is the photon [23]. However, other systems, e.g. atoms or ions, are

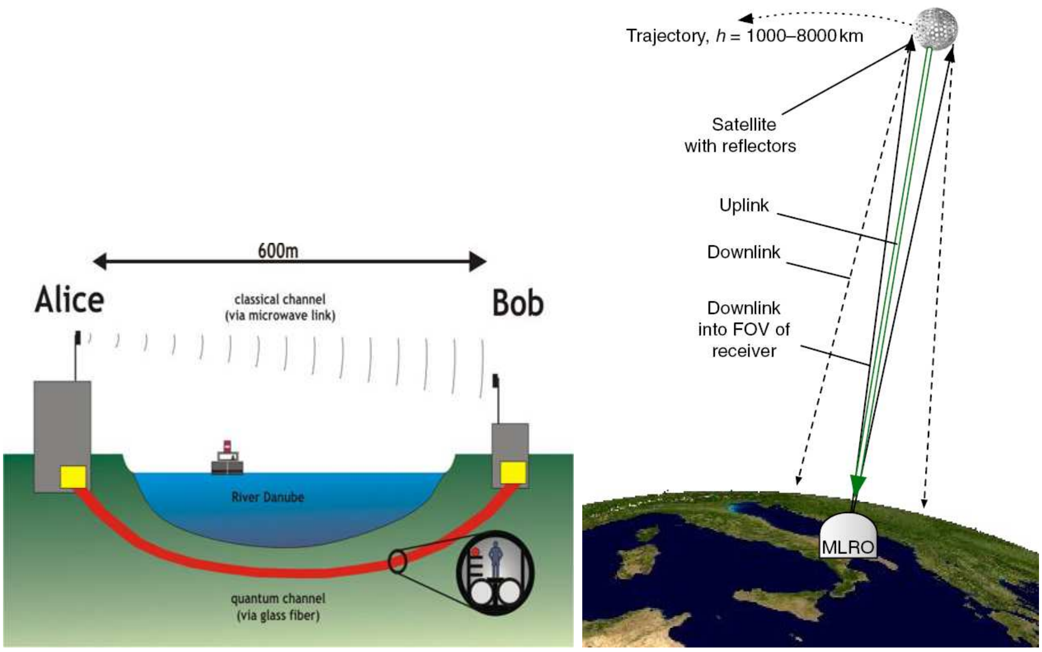

Figure 2.4: On the left, example of a real quantum channel, i.e. an 800-metre-long optical fibre installed in a public sewer system located in a tunnel underneath the River Danube, where it is exposed to temperature fluctuations and other environmental factors. It has been used to realize long-distance quantum teleportation across the River Danube [35]. On the right, scheme of a satellite single-photon link recently studied in Ref. [34]. They have implemented a quantum communication channel between Earth and space by simulating a single photon source on a low-Earth orbit geodetic satellite and by detecting the transmitted photons with the telescope at the Matera Laser Ranging Observatory of the Italian Space Agency.

studied but their applicability for quantum communication schemes is not feasible within the near future. Nevertheless, one of the problems of photon-based communication schemes is the loss of photons in the quantum channel. This limits the bridgeable distance for single photons to the order of 100 km with the present technology. Recent quantum communication experiments and some quantum cryptography applications already cover this distance [31]. In principle, this drawback can eventually be overcome by subdividing the larger distance to be bridged into smaller sections and applying periodically quantum purification through a full 'quantum repeater', in order to compensate the decoherence effects possibly induced by the physical channel.

Optical fibers are the most common type of channel for optical communications. The transmitters in optical fiber links are generally light-emitting diodes (LEDs) or laser diodes. Since optical fibers transmit infrared wavelengths with less attenuation and dispersion, the infrared light is commonly used; the classical telecom choices are 1300 and 1550 nm. The signal encoding is typically simple intensity modulation and the introduction of the erbium-doped fiber amplifier allows a periodic signal regeneration, at very low cost. Practically, a photon in some quantum state goes in the optical fiber, suffers noise (e.g., depolarization) and distortion in passing through it, and, if the photon is not absorbed and does not tunnel out, emerges in a transformed quantum state at the output side (see the left part of Fig. 2.4).

Free-space optical communication is also used today in a variety of applications. In this field, Free Space Optics (FSO) is a telecommunication technology, that uses light propagating in free space to transmit data between two points, and is useful, for example, in those cities where the laying of fibre optic cables is expensive. Moreover, since the air turbulence is not present outside the atmosphere, FSO is also used to communicate between space-crafts. For free space it is preferable to use either shorter wavelengths, around 800 nm, where efficient detectors exist, or much longer wavelengths, 4-10 μm, where the atmosphere is more transparent. Initially, free-space quantum key distribution has been proposed by sending single photons through open air even in daylight for a 10 km-long transmission [32, 33]. Another experiment of free-space secure quantum communication between La Palma and Tenerife Canary Islands has been shown in Ref. [30]. Recently, a very interesting experimental study of a single-photon exchange between a satellite and the telescope at the Matera (Italy) Laser Ranging Observatory (MLRO) of the Italian Space Agency (ASI) has been realized [34] (see the scheme on the right part of Fig. 2.4).

2.2.1 Open Quantum Systems

In this section we develop the mathematical tools that are commonly used to describe noise effects in quantum communication. A quantum system that does not interact with the outside world is said to be closed. It could be represented by an isolated atom, an electron in free space (isolated spin), or (at most) the universe as a whole. A closed quantum system can be described by a single wavefunction, Ψ, and by a time unitary evolution,

$$|\Psi(t)\rangle = U(t, t_0)|\Psi(t_0)\rangle \tag{2.9}$$

where $U(t, t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}$ is the unitary time evolution operator from the time t_0 to the time t, associated to a time-independent Hamiltonian H.

In the real world there are no perfectly closed quantum systems, except perhaps the universe as a whole. Real systems are “open” quantum systems that suffer from unwanted interactions with the external environment, whose dynamics we wish to neglect, or average over. Some examples are an atom in the presence of an external electromagnetic field, an electron interacting with other excitations in a solid or, in general, any physical systems (almost all) that suffer from the influence of an external environment.

The traditional tools used by physicists for the description of open quantum systems are represented by master equations, Langevin equations and stochastic differential equations [80]. These techniques provide us with differential equations, whose solution gives a continuous time description of the system dynamics. However, their application is limited to those cases where the environment is Markovian, i.e. a thermal bath in equilibrium and approximately unperturbed by the system.

The quantum operation (or *quantum channel*) formalism is a more general tool for describing the dynamical evolution of any open quantum system [1, 14, 3]. This approach is very powerful because it is useful to describe completely different physical scenarios. They describe, for instance, closed systems weakly or strongly coupled to the environment and a particular case of this situation is represented by quantum measurements. Moreover, they are well suited to describe any transition from the initial state to the final state of a system (discrete state change), without necessary explicit reference to time evolution [1, 14]. Indeed, the basic idea is to represent each processing step (free evolution, controlled time evolution, preparations, measurements, transmission of quantum information over long distances and the storing of it in some quantum memory) by a channel, which converts input states to the output states. The quantum channels play a fundamental role in many different branches of physics, specifically in all those situations where decoherence and noise effects come into play. Ideally, the quantum systems carrying the information do not interact with the environment; this kind of channels is called an ideal channel. In real situations, instead, interaction with an external noisy environment, i.e. additional, unobservable degrees of freedom, cannot be avoided. These real channels are called noisy quantum channels.

At a mathematical level, a *quantum operation* or *quantum channel* (sometimes called *superoperator*) is a linear map that transforms quantum states into quantum states as

$$\rho' = \mathcal{E}(\rho) \; . \tag{2.10}$$

In this formalism, we will use the notation $\mathcal{E}_1 \otimes \mathcal{E}_2$ to consider the composition in parallel of two quantum channels $\mathcal{E}_1$ and $\mathcal{E}_2$. For instance, the symbol $\mathcal{E}^{\otimes N}$ will indicate N uses of a memoryless channel. On the other hand, the composite channel $\mathcal{E}_2 \circ \mathcal{E}_1$ will mean a channel in which we first operate with $\mathcal{E}_1$ and then with $\mathcal{E}_2$ on the same system, i.e. $\mathcal{E}_2\left(\mathcal{E}_1(\rho)\right)$; this is a composition in series of two channels.

The dynamics of an open quantum system can be described basically in three different approaches, that turn out to be equivalent.

- The most natural and physical way of understanding quantum channels is as a unitary interaction between the principal system and an environment, that together form a closed quantum system (see Sec. 2.2.2) [1, 14, 3]. This approach is concrete and much related to the real world but it suffers from the drawback of not being mathematically convenient.

- A second approach, analyzed in Sec. 2.2.3, starts from a set of physically motivated axioms that a dynamical map in quantum mechanics is expected to satisfy. The advantage of this second approach is that it is very general and describes a wide range of physical situations.

- Finally, a third way of understanding open quantum systems is provided by a powerful mathematical representation, called operator-sum or Kraus representation (see Sec. 2.2.4). This procedure is more abstract but quite useful both for calculations and for theoretical work.

2.2.2 System-environment description

First of all, we start from the physical idea of studying open system dynamics as the result of an interaction with an environment. Indeed, one can always consider an "artificial" environment, E, starting in a (not necessarily pure) state ρ_E, and a model dynamics specified by a unitary operator U, describing the collective evolution of the system and the environment, such that

$$\mathcal{E}(\rho) = \mathrm{Tr}_E[U(\rho \otimes \rho_E)U^\dagger] , \tag{2.11}$$

where Tr_E is the partial trace over the environment. In this description of an open quantum system we assume that the initial state of the system and the environment is a product state. This is not true in general, but it is reasonable to assume it in many cases of practical interest. Indeed, the interaction between the system and the outside world is always on, constantly building up correlations, but, when the experimentalist prepares a quantum system in its initial state, all the correlations are completely destroyed.

Another important comment concerns the characterization of the environment. Usually, the outside world has nearly infinite degrees of freedom, therefore this system-environment representation is practically unfeasible. Actually, the Stinespring's dilation theorem [81] proves that there always exists an "artificial" environment with Hilbert space $\mathcal{H}_E$ and a unitary operation U such that Eq. (2.11) holds with ρ_E being a pure state. It can even be shown that the environment space $\mathcal{H}_E$ can be chosen such that $\dim \mathcal{H}_E \leq \dim^2 \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of the system. The artificiality of the environment is not a weak point of this formalism, because one wants to ignore the real environment dynamics and focus only on the principal system dynamics. Finally, note that this representation is unique up to unitary equivalence (strictly speaking, it is unique up to a partial isometry).

In the following, we call Eq. (2.11) with mixed ρ_E a "physical representation" of $\mathcal{E}$ to distinguish it from the Stinespring dilation, and to stress its connection with the physical picture of the noisy evolution represented by $\mathcal{E}$. In real physical experiments the environment initial state is usually a mixed state, e.g. a thermal state. Note that any Stinespring dilation gives rise to a physical representation. Moreover from any physical representation one can construct a Stinespring dilation by purifying ρ_E with an external ancillary system C, and by replacing U with the unitary coupling $U_C = U \otimes \mathbb{1}_C$. Of course, in a physical representation of $\mathcal{E}$ one has the advantage to treat with an (even more physical) environment of smaller size than the pure one.

2.2.3 Physically motivated axiomatic approach

An axiomatic approach to open quantum systems is based on a set of physical requirements which we expect to be satisfied by quantum channels. This method is more abstract respect to the previous one (system plus environment) but it is also extremely powerful for describing a wide range of circumstances. In order to represent a "physical" transformation, i.e. a transformation that could be implemented in a experimental laboratory, a quantum channel has to satisfy some reasonable physical constraints.

First of all, every map $\mathcal{E}$ representing noisy evolution in a quantum channel must preserve the trace of ρ, since $\mathcal{E}(\rho)$ is also a state, and so it must be *trace preserving*, i.e. $\mathrm{Tr}[\mathcal{E}(\rho)] = \mathrm{Tr}[\rho]$. Another property (stemming from physical requirements) is that $\mathcal{E}$ is a *convex-linear map* on the set of density matrices, i.e., for probabilities $\{p_i\}$,

$$\mathcal{E}\left(\sum p_i \rho_i\right) = \sum p_i \mathcal{E}(\rho_i) \ . \tag{2.12}$$

A quantum channel must also satisfy the physical property of *complete positiveness*. Indeed, if $\mathcal{E}$ maps density operators into density operators, then it must be positive for any positive operator A: the quantum map $\mathcal{E}$ must be *positive*. Furthermore, if $\mathcal{E}$ acts on a subsystem Q of a larger physical system $R+Q$ (R is an external ancilla), i.e. $(\mathcal{I} \otimes \mathcal{E})(\rho_{RQ})$, then it must be true that also $(\mathcal{I} \otimes \mathcal{E})$ is positive for any positive operator A on the combined system RQ, where $\mathcal{I}$ denotes the identity map on system R. In this case, the quantum map is said *completely positive*. Note that a completely positive map is also positive but the viceversa is not true. Indeed, an example of a positive map, which is not completely positive, is the operation of the mathematical transposition. It preserves positivity of operators on the principal system, but does not continue to preserve positivity when applied to systems which contain the principal system as a subsystem.

Therefore, a quantum channel is a *completely positive, trace-preserving* (CPT) map. Besides, a quantum map $\mathcal{E}$ is called *unital* if $\mathcal{E}(\mathbb{1}) = \mathbb{1}$, i.e. if $\mathcal{E}$ maps the identity operator to itself. Notice that, in general, the Hilbert space of the output can be also different from the input one.

2.2.4 Operator-sum or Kraus representation

Any quantum channel $\mathcal{E}$ can be also represented in an elegant form known as operator-sum (or Kraus) representation [1, 14, 2]. It is important because it allows us to describe the dynamics of the principal system in an intrinsic way, i.e. without considering explicitly the properties of the environment. The operator-sum representation is based on operators, A_k, acting on the principal system alone. In this way it is easy to show that many different environmental interactions may give rise to the same dynamics of the principal system. This representation is given by

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger \ , \tag{2.13}$$

where the operators A_k are known as *operation elements* or *Kraus operators* for the quantum channel $\mathcal{E}$ and satisfy the condition

$$\sum_k A_k^\dagger A_k = \mathbb{1} \ , \tag{2.14}$$

in such a way that the map is trace-preserving. Note that, if the output Hilbert space $\mathcal{H}_{out}$ is different from the input one $\mathcal{H}_{in}$, then A_k will be operators mapping $\mathcal{H}_{in}$ to $\mathcal{H}_{out}$. It is known that there always exists a representation with at most d^2 Kraus operators, if the principal system has a Hilbert space of dimension d. Moreover, the map is unital if

$$\sum_k A_k A_k^\dagger = \mathbb{1}. \tag{2.15}$$

Let us point out that this representation is not unique. In fact, there exist other Kraus operators that represent the same superoperator. In particular, for a given set of Kraus operators $\{A_k\}$, one can construct other infinite sets $\{B_h\}$, giving rise to the same quantum operation $\mathcal{E}$, according to the relation $B_h = \sum_k u_{hk} A_k$, where u_{hk} are the elements of a generic unitary matrix. This property is related to the important physical observation that the same system dynamics can characterize completely different physical processes. The only example, in which the Kraus representation is unique, is the unitary evolution of a physical system. It can be written in the form of a quantum channel, i.e. $\rho \to U\rho U^\dagger$, where the unitary evolution U is the unique Kraus operator. Viceversa, if in the Kraus representation there is only one Kraus operator, then the evolution is unitary because of the trace-preserving condition. More generally, let us suppose that our physical system is isolated but its Hamiltonian evolution is uncertain because of the presence of some (classical) random processes. The result is that the evolution is given by different Hamiltonians H_i applied with different probabilities p_i, i.e.

$$\rho \to \sum_i p_i U_i \rho U_i^\dagger \ , \tag{2.16}$$

with U_i being the unitary evolution associated with Hamiltonian H_i. Therefore, the relative Kraus operators have the form of $\sqrt{p_i}\ U_i$. Moreover, a projective measurement process can be regarded as a quantum channel in which the Kraus operators are the projection operators, while for a POVM process the Kraus operators are the generalized measurement operators (see Sec. 1.3).

Finally, let us stress that, given a system-environment unitary interaction U and an initial pure state of the environment $|e_0\rangle$, one can always provide a Kraus representation choosing a basis in the Hilbert space of the environment $\{|e_k\rangle\}$ and taking the Kraus operators as $A_k = \langle e_k|U|e_0\rangle$, i.e.

$$\mathcal{E}(\rho) = \mathrm{Tr}_E[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger] = \sum_k \langle e_k|[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger]|e_k\rangle = \sum_k A_k \rho A_k^\dagger \ .$$

Vice versa, supposing to have a CPT map with operation elements $\{A_k\}$, satisfying the completeness relation $\sum_k A_k^\dagger A_k = \mathbb{1}$, we can find an appropriate unitary operator U to model the evolution of the open quantum system as a system coupled unitarily to an environment in a larger Hilbert space. Indeed, let $|e_k\rangle$ be an orthonormal basis set for E, in one-to-one correspondence with the index k for the operation elements A_k, and let us define an operator U which acts on states of the form $|\psi\rangle|e_0\rangle$ as follows:

$$U|\psi\rangle|e_0\rangle = \sum_k A_k|\psi\rangle|e_k\rangle \ , \tag{2.17}$$

where $|e_0\rangle$ is just some standard pure state of the model environment. Note that for arbitrary states $|\psi\rangle$ and $|\phi\rangle$ of the principal system, one has

$$\langle\psi|\langle e_0|U^\dagger U|\phi\rangle|e_0\rangle = \sum_k \langle\psi|A_k^\dagger A_k|\phi\rangle = \langle\psi|\phi\rangle \ , \tag{2.18}$$

by the completeness relation. Thus the operator U can be extended to a unitary operator acting on the entire space of the joint system (that is, system plus environment). Requiring that

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger = \sum_k \langle e_k|[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger]|e_k\rangle \ , \tag{2.19}$$

we want U to satisfy

$$A_k = \langle e_k | U | e_0 \rangle \ . \tag{2.20}$$

Such U is conveniently represented as the following block matrix

$$U = \begin{pmatrix} [A_0] & \dots & \dots \\ [A_1] & \dots & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

in the basis $|e_k\rangle$. Note that the Kraus operators A_k only determine the first block column of this matrix and the determination of the rest can be done by us choosing the entries such that U is unitary.

2.2.5 Heisenberg picture: Dual channel

It is also useful to reformulate the quantum channels in the Heisenberg representation. Here we keep fixed the states of the system and the transformation induced on the system by the channel is described by means of a linear map $\mathcal{E}_H$ on the operators $\Theta \in \mathcal{B}(\mathcal{H})$. We represent with $\mathcal{B}(\mathcal{H})$ the algebra of all bounded[3] operators. This map on the operators is called dual channel [14] and is defined by the relation

$$\mathrm{Tr}[\mathcal{E}(\rho)\ \Theta] = \mathrm{Tr}[\rho\ \mathcal{E}_H(\Theta)] \ , \tag{2.21}$$

for all $\rho \in \mathcal{D}(\mathcal{H})$ ($\mathcal{D}(\mathcal{H})$ denotes the space of the density operators) and for all $\Theta \in \mathcal{B}(\mathcal{H})$. Therefore, starting from Eq. (2.13), the Kraus representation of the dual channel is

$$\mathcal{E}_H(\Theta) = \sum_k A_k^\dagger \Theta A_k \ , \tag{2.22}$$

where the operation elements satisfy Eq. (2.14). It is easy to note that the dual channel is always unital, since $\mathcal{E}_H(\mathbb{1}) = \sum_k A_k^\dagger A_k = \mathbb{1}$. Finally, we observe that the dual channel is a linear, completely positive map but, in general, is not trace preserving or, even, it is trace decreasing.

[3]In mathematics, a bounded operator is a linear transformation Θ mapping a normed vector space V to another one W for which the ratio of the norm of $\Theta(v)$ to that of v is bounded by the same number, over all non-zero vectors v in V.

2.2.6 Classical and quantum information transfer

In the context of quantum information theory, some emphasis is put on characterizing the properties of noisy quantum channels in terms of their information capacities [79, 3]. These figures of merit are the quantum counterpart of the Shannon capacity of a classical communication line [10], which "measures" the performances of the map in conveying classical or quantum information. More precisely, they coincide with the maximal rate, i.e. the maximum number of bits or qubits per channel use, at which we can transmit classical or quantum information asymptotically undisturbed through a noisy channel. Even though impressive achievements have been obtained in this field in recent years, several open questions are still under investigation — we refer the reader to Ref. [82] and references therein for details. For instance, for classical channels the capacity C is additive: the capacity associated with the use of two channels is equal to the sum of the capacity of each channel, individually considered, i.e. $C(\mathcal{E}_1 \otimes \mathcal{E}_2) = C(\mathcal{E}_1) + C(\mathcal{E}_2)$. For quantum channels, instead, the additivity issue is a difficult open problem and in the following chapters some additivity conditions are found for some class of quantum channels, i.e. Gaussian channels. Trivially, in an ideal channel the information is reliably transmitted through it without any loss, i.e. technically one says that the (classical or quantum) fidelity between the output and the input (classical or quantum) states is equal to one.

Consider now the transmission of classical data through a noisy quantum channel $\mathcal{E}$ (e.g., by encoding the message into the states of photons propagating through a lossy optical fiber). You can imagine to encode (E) the classical data (bit) in the polarization of photons, to transmit photonic quantum states through optical fibers, and to decode (D) them into classical information at the output side as in Fig. 2.2. In other words, the channel $E \circ \mathcal{E} \circ D$ has the form of a classical communication channel, where E and D are, respectively, the encoding and decoding maps. Therefore, in this situation, the one-shot classical capacity of a quantum channel $\mathcal{E}$ is given by

$$C_1(\mathcal{E}) = \sup_{E,D} C(E \circ \mathcal{E} \circ D) \tag{2.23}$$

where, now, the supremum is taken over all possible encodings and decodings of classical bits. Note that $C(E \circ \mathcal{E} \circ D)$ is the classical channel capacity in Eq. (2.1) of the classical channel $E \circ \mathcal{E} \circ D$. In this context, the adjective "one-shot" refers to the fact that we are encoding information only on tensor product states, as explained in Sec. 2.2. A computable expression for the one-shot classical capacity, thus making it an interesting quantity, is given in the Holevo-Schumacher-Westmoreland theorem [83, 84, 85, 86].

Theorem 1 (Holevo-Schumacher-Westmoreland) *: The one-shot classical capacity $C_1(\mathcal{E})$ of a quantum channel $\mathcal{E}$ is*

$$C_1(\mathcal{E}) = \sup_{p_j,\rho_j} \left[S\left(\sum_j p_j \mathcal{E}(\rho_j) \right) - \sum_j p_j S\left(\mathcal{E}(\rho_j)\right) \right] , \tag{2.24}$$

where the supremum is taken over all probability distributions p_j and collections of density operators ρ_j. The quantity in $[\ldots]$ is also called Holevo's information χ, i.e. $\chi(p_j, \rho_j) = S\left(\sum_j p_j \mathcal{E}(\rho_j)\right) - \sum_j p_j S\left(\mathcal{E}(\rho_j)\right)$.

This theorem quantifies the transmitted information in terms of quantum entropy and so it makes possible to apply quantum statistical physics to the issue of quantum capacity limits. As a consequence, at most n bits of classical information can be carried by a quantum system of n distinguishable qubits.

Moreover, one can consider many uses of the channel exploiting entanglement in the encoding and decoding procedures. The classical capacity of a quantum channel $\mathcal{E}$, defined in Eq. (2.7) is equivalent to

$$C(\mathcal{E}) = \lim_{N\to\infty} \frac{1}{N} C_1(\mathcal{E}^{\otimes N}) \tag{2.25}$$

where another optimization over the number of uses of the channel is considered. However, according to an additivity conjecture, it is believed that $C(\mathcal{E})$ coincides with $C_1(\mathcal{E})$.

In a similar way one can define the entanglement assisted classical capacity $C_e(\mathcal{E})$ of a quantum channel $\mathcal{E}$, by supposing that the sender and the receiver can share an arbitrary amount of maximally entangled states before the transmission [1, 3]. This quantity can be computed as

$$C_e(\mathcal{E}) = \sup_{\rho} I(\rho, \mathcal{E}) \ . \tag{2.26}$$

where the supremum is taken over all input states ρ and all possible entangled encoding and decoding procedures, and $I(\rho, \mathcal{E})$ is the quantum mutual information of the channel $\mathcal{E}$. This last quantity is defined as

$$I(\rho, \mathcal{E}) = S(\rho) + S(\mathcal{E}(\rho)) - S(\rho, \mathcal{E}) \ , \tag{2.27}$$

with $S(\rho, \mathcal{E})$ being the so-called entropy exchange of a channel $\mathcal{E}$ for a given input ρ, i.e.,

$$S(\rho, \mathcal{E}) = S[(\mathcal{E} \otimes \mathcal{I})(|\Psi\rangle\langle\Psi|)] , \tag{2.28}$$

where we are assuming that $|\Psi\rangle$ is a purification of $\rho \in \mathcal{H}$ in a larger Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $\rho = \mathrm{Tr}_1[|\Psi\rangle\langle\Psi|] = \mathrm{Tr}_2[|\Psi\rangle\langle\Psi|]$ (see Sec. 1.1). Note that $\mathcal{E} \otimes \mathcal{I}$ is a channel that applies $\mathcal{E}$ to the principal system input state and the identity map $\mathcal{I}$ to the ancilla state. The quantum mutual information $I(\rho, \mathcal{E})$ is positive, concave with respect to the input state and additive. Due to these properties of the quantum mutual information, the capacity $C_e(\mathcal{E})$ is additive and so coincides with the corresponding one-shot version. This is an important simplification with respect to what happens for the classical capacity $C(\mathcal{E})$.

The quantum capacity Q refers, instead, to the coherent transmission of quantum information (measured in number of qubits), i.e. quantum states, through a quantum channel. It is more difficult to treat than classical capacities discussed above and its explicit calculation is one of the basic issues in quantum communication. The quantum capacity, defined in Eq. (2.8), can be computed as [87, 88, 89, 90]

$$Q(\mathcal{E}) \;=\; \lim_{N\to\infty} \frac{1}{N} Q(\mathcal{E})^{(N)} = \lim_{N\to\infty} \frac{1}{N} \max_{\rho} J(\rho, \mathcal{E}^{\otimes N}) , \tag{2.29}$$

where the maximization is performed over all input states for N uses of the quantum cannel and $J(\rho, \mathcal{E}^{\otimes N})$ is the *coherent information* of $\mathcal{E}^{\otimes N}$, defined [91] for a generic channel $\mathcal{E}$ as follows

$$J(\rho, \mathcal{E}) = S(\mathcal{E}(\rho)) - S(\rho, \mathcal{E}) . \tag{2.30}$$

Note that the calculation of the quantum capacity is a daunting task essentially for two reasons. Firstly, since the coherent information is known not to be additive (unlike the classical and quantum mutual information), the regularization for N which tends to infinite (i.e., $\lim_{N\to\infty}$) is necessary. Secondly, due to lacking concavity properties, the coherent information may have local maxima which are not global ones. On top of this, for Bosonic channels this optimization is over an infinite dimensional Hilbert space.

In the following section we will study an important property of a quantum channel, i.e. degradability, that enables us to simplify the analysis of the quantum capacity issue and to identify channels (degradable) with additive coherent information (i.e., $Q(\mathcal{E}) = Q(\mathcal{E})^{(1)}$) and other channels (anti-degradable) with null quantum capacity (i.e., $Q(\mathcal{E}) = 0$). In this thesis we will study this property for Bosonic and qubit Gaussian channels.

2.3 Degradability of quantum channels

In Sec. 2.2.2 we have shown that any quantum channel $\mathcal{E}$ acting on a system A can be described as a unitary coupling between the system A in the input state ρ_a with an external ancillary system B (describing the *environment*) prepared in some fixed *pure* state $|e_0\rangle$, i.e.

$$\mathcal{E}(\rho_a) = \mathrm{Tr}_b[U_{ab}(\rho_a \otimes |e_0\rangle\langle e_0|)U_{ab}^\dagger] \ , \tag{2.31}$$

where $\mathrm{Tr}_b[\ldots]$ is the partial trace over the environment B, U_{ab} is a unitary operator in the composite Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$.

In Ref. [38], Devetak and Shor introduced the definition of degradability of a quantum channel. Loosely speaking, degradable are those channels where the modified state of the environment can be recovered from the output state of the channel through the action of a third channel (see Fig. 2.5).

Definition 2 *A quantum channel $\mathcal{E}$ is called degradable if there exists a CPT map $\mathcal{T}$ such that*

$$(\mathcal{T} \circ \mathcal{E}) = \mathcal{E}_{com} \ , \tag{2.32}$$

where $\mathcal{E}_{com}$ is the so-called complementary (conjugate) channel (see Refs. [38, 92, 93]), mapping the initial state of the system into the output state of the environment after the noisy evolution, i.e.

$$\mathcal{E}_{com}(\rho_a) = \mathrm{Tr}_a[U_{ab}(\rho_a \otimes |e_0\rangle\langle e_0|)U_{ab}^\dagger] \ . \tag{2.33}$$

Notice that we are considering a Stinespring representation of the channel, i.e. unitary interaction with an environment initially in a pure state. This property is a powerful tool for the challenging problem of channel quantum capacity's evaluation. Indeed, the degradable channels allow for a single letter formula expression for Q — i.e., the maximum of their output coherent information is additive (see below). The latter is much easier to handle than its regularized version and, in some cases, enables for a complete characterization of the Q (e.g., see the dephasing channel in Ref. [38] and amplitude damping channel in Ref. [41]). Besides, if a quantum channel is degradable, the coherent information is a concave function and, interestingly, the local maxima coincide with global ones. Another important feature of degradability is that it does not depend on physical operations applied locally on input and output states. Indeed, suppose to consider a degradable channel $\mathcal{E}$. On one hand, if one applies a unitary operation U_1 to the input state ρ, since the degradability is defined for a generic input state, it still holds also for the state $U_1 \rho U_1^\dagger$. On the other hand, any unitary operation U_2 applied to the output state of the channel, i.e. $U_2 \mathcal{E}(\rho) U_2^\dagger$, can be always absorbed in the CPT map $\mathcal{T}$, recovering trivially the degradability property in Eq. (2.32). This feature is very useful because, by using U_1 and U_2, one can analyze the degradability for simplified (canonical) forms of the quantum channel, as will be shown in Chapter 3 for Bosonic systems and in Chapter 4 for qubits.

Finally, let us point out that the definition of degradable quantum channels can be considered the quantum version of degraded broadcast classical channels in Sec. 2.1.4. Indeed, one can interpret a quantum channel as a two-user quantum broadcast channel connecting a single transmitter to two receivers, i.e. two output systems A and B, described by the noisy evolutions $\mathcal{E}(\rho_a)$ and $\mathcal{E}_{com}(\rho_a)$, respectively.

Degradability and additivity

Here we want to prove that the degradability implies the additivity of the coherent information and then simplifies the quantum capacity issue (i.e. single-letter formula for the quantum capacity), according to the following theorem [38].

Theorem 2 *For a degradable channel $\mathcal{E}$ the coherent information J is additive and the quantum capacity is given by the single-letter formula $Q(\mathcal{E}) = Q(\mathcal{E})^{(1)} := \max_\rho J(\rho, \mathcal{E})$.*

Proof: The coherent information is

$$J(\rho, \mathcal{E}) = S(\mathcal{E}(\rho)) - S(\rho, \mathcal{E}) \ , \tag{2.34}$$

where $S(\rho, \mathcal{E})$ is the entropy exchange of the channel $\mathcal{E}$ for a given input ρ, i.e.

$$S(\rho, \mathcal{E}) = S[(\mathcal{E} \otimes \mathcal{I})(|\Psi\rangle\langle\Psi|)] \tag{2.35}$$

where $|\Psi\rangle$ is a purification of ρ and $\mathcal{E} \otimes \mathcal{I}$ is a map applying $\mathcal{E}$ to the principal system input state and the identity map $\mathcal{I}$ to the ancilla state. Note that the coherent information for a process on a composite state is greater than or equal to the total of the 'marginal' coherent information for the reductions of the process and the initial state of the subsystems. Mathematically, one has

$$J(\rho, \otimes\mathcal{E}_i) \geq \sum_i J(\rho_i, \mathcal{E}_i) \ , \tag{2.36}$$

where ρ_i are the reduced density operators of ρ and $\{\mathcal{E}_i\}$ is a set of quantum channels. Therefore, one says that the coherent information is superadditive. Now we will prove that for a degradable channel it is also subadditivity and then, for this class of channels, the equality (i.e. the additivity of the coherent information) also holds. In particular, since the environment is initially in a pure state, one has that $S(\rho, \mathcal{E}) = S(\mathcal{E}_{com}(\rho))$ and the coherent information of the channel is

$$J(\rho, \mathcal{E}) = S(\mathcal{E}(\rho)) - S(\mathcal{E}_{com}(\rho)) \ , \tag{2.37}$$

where $S(\mathcal{E}_{com}(\rho))$ is the von Neumann entropy of the output state of the complementary channel, i.e. the von Neumann entropy of the final state of the environment. This will be show in the lemma proved below.

Now we use the degradability property of the channel $\mathcal{E}$, that is

$$\mathcal{T}(\mathcal{E}(\rho)) = \mathcal{E}_{com}(\rho) \ , \tag{2.38}$$

for any input state ρ. This intermediate map $\mathcal{T}$ can be described, of course, as a unitary interaction V between the final state of the system $\mathcal{E}(\rho)$ and another environment F, i.e.

$$\mathcal{T}(\mathcal{E}(\rho)) = \mathrm{Tr}_F \left[V \left(\mathcal{E}(\rho) \otimes |0\rangle_F\langle 0| \right) V^\dagger \right] \ , \tag{2.39}$$

hence,

$$S\left(\mathcal{E}_{com}(\rho)\right) = S\left(\mathrm{Tr}_F \left[V \left(\mathcal{E}(\rho) \otimes |0\rangle_F\langle 0| \right) V^\dagger \right] \right) \ . \tag{2.40}$$

Note that

$$S\left(V\left(\mathcal{E}(\rho)\otimes|0\rangle_F\langle 0|\right)V^\dagger\right)=S(\mathcal{E}(\rho))\ , \tag{2.41}$$

because the von Neumann entropy is invariant under unitary transformations and, moreover, one has

$$S\left(\left(\mathcal{E}(\rho)\otimes|0\rangle_F\langle 0|\right)\right)=S(\mathcal{E}(\rho))+S\left(|0\rangle_F\langle 0|\right)=S(\mathcal{E}(\rho))\ , \tag{2.42}$$

with the von Neumann entropy of a pure state being zero.

Combining Eqs. (2.40)-(2.41), the coherent information can be so written as the quantum conditional entropy between the environments F and E, i.e.

$$J(\rho,\mathcal{E})=S(F,E)-S(E)=S(F|E)\ . \tag{2.43}$$

It is know that the conditional entropy is subadditive [1, 14]. Therefore, the coherent information J is subadditive (and superadditive, by definition) and then additive. This fact implies that the quantum capacity is given by the single-letter formula, i.e.

$$Q(\mathcal{E})=Q(\mathcal{E})^{(1)}:=\max_\rho J(\rho,\mathcal{E})\ . \tag{2.44}$$

This concludes the proof of the theorem. Interestingly enough, apart from the additivity of the coherent information, since the quantum conditional entropy is also a concave function, the coherent information is too. Therefore, it implies an important useful simplification in the calculation of the quantum capacity Q, i.e. it is sufficient to search for local maxima in order to calculate Q. ■

Now we prove a lemma, used in the theorem above, that states the equivalence between the entropy exchange of $\mathcal{E}$ and the von Neumann entropy of the output state of the complementary channel, i.e. the von Neumann entropy of the final environmental state, when the initial environmental state is pure (Stinespring representation) [88, 94].

Lemma 1 *: If the initial state of the environment is pure, then the entropy exchange assume the more useful form*

$$S(\rho,\mathcal{E})=S(\mathcal{E}_{com}(\rho))\ , \tag{2.45}$$

where $S(\mathcal{E}_{com}(\rho))$ is the von Neumann entropy of the transformed state of the environment.

Proof: By definition, the entropy exchange of the channel $\mathcal{E}$ for a given input ρ of a system A has the expression

$$S(\rho, \mathcal{E}) = S[(\mathcal{E} \otimes \mathcal{I})(|\Psi\rangle\langle\Psi|)] \,, \tag{2.46}$$

where $|\Psi\rangle$ is a purification of ρ in a larger Hilbert space with an extra ancilla R. Recall that $\mathcal{E} \otimes \mathcal{I}$ is a composite channel in which $\mathcal{E}$ acts on the principal system input state and the identity map $\mathcal{I}$ on the ancilla state E. Besides,

$$(\mathcal{E} \otimes \mathcal{I})(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_E \left[(U \otimes \mathbb{1}_R)|\Psi\rangle\langle\Psi| \otimes |0\rangle_E\langle 0|(U \otimes \mathbb{1}_R)^\dagger\right] \,, \tag{2.47}$$

where U describes the unitary evolution of the map $\mathcal{E}$ and Tr_E is the partial trace over the environment. Then, one has

$$\begin{aligned} S\left((\mathcal{E} \otimes \mathcal{I})(|\Psi\rangle\langle\Psi|)\right) &= S\left(\mathrm{Tr}_E\left[(U \otimes \mathbb{1}_R)\left(|\Psi\rangle\langle\Psi| \otimes |0\rangle_E\langle 0|\right)(U \otimes \mathbb{1}_R)^\dagger\right]\right) \\ &= S\left(\mathrm{Tr}_{AR}\left[(U \otimes \mathbb{1}_R)\left(|\Psi\rangle\langle\Psi| \otimes |0\rangle_E\langle 0|\right)(U \otimes \mathbb{1}_R)^\dagger\right]\right) \\ &= S\left(\mathrm{Tr}_A\left[U\left(\rho \otimes |0\rangle_E\langle 0|\right)U^\dagger\right]\right) = S(\mathcal{E}_{com}(\rho)) \,. \end{aligned} \tag{2.48}$$

Let us stress that the second equality is due to the Schmidt decomposition [1, 14] of pure bipartite states[4] and there the purity of the environment comes into play. ■

2.3.1 Weakly complementary and degradable channels

Recently, a generalization of the notion of degradability has been proposed as an useful tool for studying the quantum capacity properties of one- and multi-mode Bosonic Gaussian and qubit channels [44, 52, 53, 54]. This suggested the possibility of classifying these maps in terms of simple canonical forms. Proceeding along similar lines, the exact solution of the quantum capacity of an important subset of those channels was obtained in Refs. [42, 55].

[4]Let us recall the Schmidt decomposition theorem [1, 14]. For a pure state $|\Psi\rangle$ of a composite system AB, there exist orthonormal bases $|i_A\rangle$ for the subsystem A and $|i_B\rangle$ for the subsystem B, such that $|\Psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$, with λ_i being non-negative numbers, called Schmidt coefficients, satisfying $\sum_i \lambda_i^2 = 1$. A corollary of this theorem is that the reduced density operators ρ_A and ρ_B have the same eigenvalues, i.e. $\rho_A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$ and $\rho_B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$, and then $S(\rho_A) = S(\rho_B)$. In other words, the von Neumann entropy of the two reduced states ρ_A and ρ_B of a bipartite system AB in a pure state are identical.

The definition of degradability in Eq. (2.32) can be generalized to the notion of *weak-degradability*, given by substituting the Stinespring representation of $\mathcal{E}$ with its physical representation of Eq. (2.11) with mixed ρ_E instead of the pure state $|e_0\rangle$ (see Sec. 2.2.2), i.e.

$$\mathcal{E}(\rho_a) = \mathrm{Tr}_b[U_{ab}(\rho_a \otimes \sigma_b)U_{ab}^\dagger] \;, \tag{2.49}$$

where U_{ab} is the unitary coupling of the system A with environment prepared in some *mixed* state σ_b. Equation (2.49) motivates the following [52, 53]

Definition 3 *For any physical representation* (2.49) *of the quantum channel* $\mathcal{E}$ *we define its* weakly complementary *as the map* $\tilde{\mathcal{E}} : \mathcal{D}(\mathcal{H}_a) \to \mathcal{D}(\mathcal{H}_b)$ *which takes the input state* ρ_a *into the state of the environment* B *after the interaction with* A*, i.e.*

$$\tilde{\mathcal{E}}(\rho_a) = \mathrm{Tr}_a[U_{ab}(\rho_a \otimes \sigma_b)U_{ab}^\dagger] \;. \tag{2.50}$$

The transformation (2.50) is CPT and describes a quantum channel connecting systems A and B. It is a generalization of the complementary (conjugate) channel $\mathcal{E}_{\mathrm{com}}$ defined above. In particular, if Eq. (2.49) arises from a Stinespring dilation (i.e. if σ_b of Eq. (2.50) is pure) the map $\tilde{\mathcal{E}}$ coincides with $\mathcal{E}_{\mathrm{com}}$. Hence the latter is a particular instance of a weakly complementary channel of $\mathcal{E}$. On the other hand, by using the purification procedure (see Sec. 1.1), we can always represent a weakly complementary map as a composition

$$\tilde{\mathcal{E}} = T \circ \mathcal{E}_{\mathrm{com}} \;, \tag{2.51}$$

where T is the partial trace over the purifying system (here " $\circ$ " denotes the composition of channels). As we will see, the properties of weakly complementary and complementary maps in general differ. Hence, we propose the following definition [52, 53] (see Fig. 2.5):

Definition 4 *Let* $\mathcal{E}, \tilde{\mathcal{E}}$ *be a pair of mutually weakly-complementary channels such that*

$$(\mathcal{T} \circ \mathcal{E})(\rho_a) = \tilde{\mathcal{E}}(\rho_a) \;, \tag{2.52}$$

for some CPT channel $\mathcal{T} : \mathcal{D}(\mathcal{H}_a) \to \mathcal{D}(\mathcal{H}_b)$ *and all density matrix* $\rho_a \in \mathcal{D}(\mathcal{H}_a)$*. Then* $\mathcal{E}$ *is called* weakly degradable *while* $\tilde{\mathcal{E}}$ – anti-degradable*. Similarly if*

$$(\overline{\mathcal{T}} \circ \tilde{\mathcal{E}})(\rho_a) = \mathcal{E}(\rho_a) \;, \tag{2.53}$$

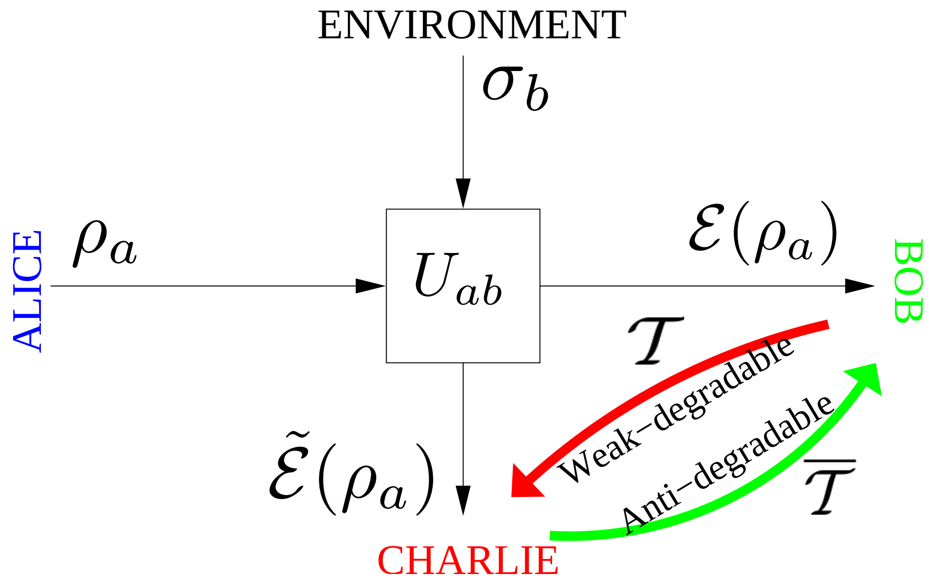

Figure 2.5: Weakly degradable vs. anti-degradable channels. A channel $\mathcal{E}$ is weakly degradable if there exists a CPT map $\mathcal{T}$ which, for all input ρ_a of Alice (the sender), allows Bob (the receiver) to recover the environment output $\tilde{\mathcal{E}}(\rho_a)$ from $\mathcal{E}(\rho_a)$ as in Eq. (2.52). A channel $\mathcal{E}$ is anti-degradable if, instead, a third party (Charlie) which is monitoring the channel environment can reconstruct Bob state, $\mathcal{E}(\rho_a)$, from $\tilde{\mathcal{E}}(\rho_a)$ via a CPT transformation $\overline{\mathcal{T}}$ as in Eq. (2.53). Weak-degradability reduces to the degradability notion of Ref. [38] if the environment σ_b is pure.

for some CPT channel $\overline{\mathcal{T}} : \mathcal{D}(\mathcal{H}_b) \to \mathcal{D}(\mathcal{H}_a)$ *and all density matrix* $\rho_a \in \mathcal{D}(\mathcal{H}_a)$*, then* $\mathcal{E}$ *is* anti-degradable *while* $\tilde{\mathcal{E}}$ *is* weakly degradable*.*

Weak-degradability and anti-degradability are not mutually exclusive properties – for instance, will see in Sec. 3.3.1 that a beam-splitter channel with transmissivity 1/2 satisfies both Eqs. (2.52) and (2.53).

Let us stress again that in Ref. [38] the channel $\mathcal{E}$ is called *degradable* if in Eq. (2.52) we replace $\tilde{\mathcal{E}}$ with the complementary map $\mathcal{E}_{\rm com}$ of $\mathcal{E}$. Clearly any degradable channel [38] is weakly degradable but the opposite is not necessarily true. Notice, however, that, due to Eq. (2.51), in the definition of anti-degradable channel we can always replace weakly complementary with complementary (for this reason there is no point in introducing the notion of weakly anti-degradable channel). This allows us to verify that if $\mathcal{E}$ is anti-degradable (2.53) then its complementary channel $\mathcal{E}_{\rm com}$ is degradable [38] and vice-versa. It is also worth pointing out that channels which are unitarily equivalent to a channel $\mathcal{E}$ which is weakly degradable (anti-degradable) are also weakly degradable (anti-degradable), as discussed for the degradability in Sec. 2.3.

One can verify that anti-degradable channels (where this property is defined irrespectively from the purity of σ_b associated with the physical representation) cannot be used to convey quantum messages in reliable fashion — i.e., their quantum capacity Q nullifies [41, 52, 53, 95, 96]. As discussed in [52] this is a consequence of the no-cloning theorem [24] (more precisely, of the impossibility of cloning with arbitrary high fidelity [97, 98, 99, 100]). Indeed, assume by contradiction $Q > 0$. This means that by employing sufficiently many times the map $\mathcal{E}$, Alice (the sender) will be able to transfer to Bob (the receiver) a generic unknown state $|\psi\rangle$. However, since the channel is anti-degradable, everything Bob gets from the channel can also be reconstructed by a third party (Charlie), which is monitoring the environment, by cascading $\tilde{\mathcal{E}}$ with the CPT map $\overline{\mathcal{T}}$ of Eq. (2.53). This implies that at the end of the day both Bob and Charlie will have a copy of $|\psi\rangle$, which is impossible. Therefore, the anti-degradable channels cannot be used to transfer quantum information.

Heisenberg picture

It is useful also to reformulate our definitions in the Heisenberg picture (see Sec. 2.2.5) [53]. Remind that the dual map $\mathcal{E}_H$ is defined on the algebra $\mathcal{B}(\mathcal{H}_a)$ of bounded operators of A in such a way that $\mathrm{Tr}_a[\mathcal{E}(\rho_a)\ \Theta_a] = \mathrm{Tr}_a[\rho_a\ \mathcal{E}_H(\Theta_a)]$ for all $\rho_a \in \mathcal{D}(\mathcal{H}_a)$ and for all $\Theta_a \in \mathcal{B}(\mathcal{H}_a)$. From this it follows that the Heisenberg picture counterpart of the physical representation (2.49) is given by the unital channel

$$\mathcal{E}_H(\Theta_a) \quad = \quad \mathrm{Tr}_b\big[\, U_{ab}^\dagger\, (\Theta_a \otimes \mathbb{1}_b)\, U_{ab}\, (\mathbb{1}_a \otimes \sigma_b)\, \big]\, . \tag{2.54}$$

Similarly, from (2.50) it follows that in the Heisenberg picture the weakly complementary of the channel is described by the completely positive unital map

$$\tilde{\mathcal{E}}_H(\Theta_b) = \mathrm{Tr}_b\ \big[\, U_{ab}^\dagger(\mathbb{1}_a \otimes \Theta_b)\, U_{ab}\, \big(\mathbb{1}_a \otimes \sigma_b\big)\, \big]\ , \tag{2.55}$$

which takes bounded operators in $\mathcal{H}_b$ into bounded operators in $\mathcal{H}_a$.

Within this framework the weak-degradability property (2.52) of the channel $\mathcal{E}_H$ requires the existence of a channel $\mathcal{T}_H$ taking bounded operators of $\mathcal{H}_b$ into bounded operators of $\mathcal{H}_a$, such that

$$(\mathcal{E}_H \circ \mathcal{T}_H)(\Theta_b) = \tilde{\mathcal{E}}_H(\Theta_b) \;, \tag{2.56}$$

for all $\Theta_b \in \mathcal{B}(\mathcal{H}_b)$. Notice that in the Heisenberg picture the maps are composed in the opposite order with respect to the Schrödinger representation. Similarly we say that a quantum channel $\mathcal{E}_H$ is anti-degradable, if there exists a channel $\overline{\mathcal{T}}_H$ from $\mathcal{B}(\mathcal{H}_a)$ to $\mathcal{B}(\mathcal{H}_b)$, such that

$$(\tilde{\mathcal{E}}_H \circ \overline{\mathcal{T}}_H)(\Theta_a) = \mathcal{E}_H(\Theta_a) \;, \tag{2.57}$$

for all $\Theta_a \in \mathcal{B}(\mathcal{H}_a)$.

Chapter 3

Bosonic Gaussian channels

Bosonic Gaussian channels (BGC) are ubiquitous in physics and, not very surprisingly, a lot of effort has been recently devoted to studying their properties [45, 46, 59, 60, 101, 102, 103, 104]. They arise whenever a harmonic system interacts linearly with a number of Bosonic modes which are inaccessible in principle or in practice. They provide realistic noise models for a variety of quantum optical and solid state systems. For instance, they describe the noise in transmission lines which employ photons as information carriers including optical fibers, wave guides, free-space electromagnetic communication, and account for all processes where the transmitted signals undergo loss, amplification, and/or squeezing transformations. Moreover, they also include noise models for Bosonic many-body systems, like quantum condensates.

Within the context of quantum information theory [1, 14, 3, 79] BGCs play a fundamental role and include all the physical transformations which preserve the "Gaussian character" of the transmitted signals, that can be seen as the quantum counterpart of Gaussian channels in classical information theory (see Sec. 2.1.3) [10, 76]. Due to their relatively simple structure, these channels provide also an ideal theoretical playground for the study of continuous variable quantum information processing [105], including quantum communication [49], teleportation and cryptography [106]. Specifically, from a quantum information perspective, a key question is whether or not a channel allows for the reliable transmission of classical or quantum information. It is relevant not only from a technological point of view but also from the point of view of quantum information theory where they pose some important open problems. Particularly, most of the efforts focused on the evaluation of the optimal transmission rates of these maps under the constraint on the input average energy both in the multi-mode scenario (where the channel acts on a collection of many input Bosonic modes) and in the one-mode scenario (where, instead, it operates on a single input Bosonic mode). As for classical Gaussian channels in Sec. 2.1.3, the unconstrained capacity would be infinite and the channel capacity becomes an interesting quantity when an input power constraint is taken. In particular, in order to avoid such nonphysical values, relevant physical constraints are necessary to be introduced, while maximizing over the classical or quantum information encoded in the physical system. For instance, one can consider only the subset of states

$$\mathcal{F} = \{\rho : \mathrm{Tr}[\rho H] < P\} \tag{3.1}$$

where H is the Hamiltonian of n harmonic oscillators $H = \sum_{i=1}^{n}(x_i^2 + p_i^2)/2$ and P is a constraint, for example, on the mean photon number $N = P - 1/2$. Analogously, for tensor products one considers $\mathcal{F}^{\otimes n} = \{\rho : \mathrm{Tr}[\rho H^{\otimes n}] < nP\}$.

In the calculation of the quantum capacity significant progress has been made in recent years [84, 85, 86, 87, 88, 89, 107], although for some important cases, like the thermal noise channel modelling a realistic fiber with offset noise, it is still not yet known. In few cases the exact values of the communication capacities of the channels have been computed [49, 50, 51, 96, 102]. In the general case, only certain bounds are available and, in the context of average input photon number constraint, it is believed that the optimal (classical or quantum) communication rates of such channels should be achieved by encoding messages into Gaussian input states [45, 96, 108, 109, 110, 111, 112]. However, apart from the noiseless case, the only nontrivial map for which such a conjecture has been proved is the purely lossy channel where photons, carrying information, couple through beam-splitters with an external vacuum state [51]. Recently, Gaussian encodings have been proved to be optimal for a class of Gaussian channels and some exact results for the quantum capacity have been shown in Ref. [55].

BGCs are generally believed to provide also a natural example of maps with additive properties [113]. For example, it has been conjectured that their maximum Holevo information and minimum Rényi entropies should be additive, although only preliminary results have been obtained so far [47, 48, 51]. In this context, the degradability properties represent a powerful tool to simplify the quantum capacity issue of such Gaussian channels. Indeed, in this chapter it will be shown that with some (important) exceptions, Gaussian channels which operate on a single Bosonic mode (i.e., one-mode Gaussian channels) can be classified as weakly degradable or anti-degradable [52, 53, 56]. This paved the way for the solution of the quantum capacity for a large class of these maps [55]. Therefore, a general construction of unitary dilations of multi-mode quantum channels is proved, allowing us, for instance, to characterize their weak-degradability/anti-degradability features [54].

In this chapter we analyze the Bosonic Gaussian channels. In Sec. 3.1 some basics of quantum optics are presented in order to describe the Bosonic systems. After a quick overview of Bosonic Gaussian maps in Sec. 3.2, we investigate the one-mode case in Sec. 3.3 [52, 53, 56]. In particular, we first analyze two basic examples, i.e. the beam-splitter and the linear amplifier. We discuss their composition rules, weak-degradability properties and quantum capacity. Then we extend these results to generic one-mode Gaussian channels by exploiting a unitary equivalence to beam-splitter/amplifier maps. Therefore, we consider a more general framework to characterize all one-mode BGCs in terms of a canonical representation. A full weak-degradability classification is discussed in more detail and we find that, apart from the class of channels which are unitarily equivalent to the channels with additive classical noise, all one-mode Bosonic Gaussian channels can be characterized in terms of weak- and/or anti-degradability. Finally, we determine a new set of channels with null quantum capacity.

In Sec. 3.4 we propose a complete analysis of multi-mode Bosonic Gaussian channels, clarifying the structure of unitary dilations of general Gaussian channels involving any number of Bosonic modes [54]. In this way, the weak-degradability properties of multi-mode channels are investigated. Moreover, we characterize the minimal number of environmental modes necessary to describe the unitary dilation of a generic multi-mode map [62]. The chapter ends with a detailed analysis of the two-mode case in Sec. 3.5. This is important since any n-mode channel can always be reduced to single-mode and two-mode parts [57]. We show their degradability features and investigate a useful decomposition of a generic map with the additive classical noise map that allows us to find new sets of channels with zero quantum capacity.

3.1 Bosonic systems

The quantization of the electromagnetic field is one of the most important cornerstones of the quantum optics theory [61]. The expansion of the vector potential of the electromagnetic field in terms of modes enables to reduce this problem to the quantization of the harmonic oscillator for each mode, i.e the following Hamiltonian for the electromagnetic field

$$H = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + 1/2 \right) \ , \tag{3.2}$$

where $\hbar$ is the Planck constant, ω_k are the mode frequencies, $\hbar\omega_k$ is the energy of a photon in the mode k, in which there is a number of photons n_k corresponding to the term $a_k^\dagger a_k$. The quantity $1/2\hbar\omega_k$ is the energy of the vacuum quantum fluctuations in each mode. Since photons are Bosons, the operators a_k and $a_k^\dagger$ satisfy the standard commutation relations, i.e.

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \ , \tag{3.3}$$

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \ . \tag{3.4}$$

Equivalently, each Bosonic mode is characterized by canonical observables Q_k and P_k, i.e.

$$Q_k = \frac{a_k + a_k^\dagger}{\sqrt{2}} \ , \tag{3.5}$$

$$P_k = \frac{a_k - a_k^\dagger}{i\sqrt{2}} \ , \tag{3.6}$$

obeying the canonical commutation relation $[Q_k, P_{k'}] = i\delta_{kk'}$ (from now on, $\hbar$ is, usually, set to one). The eigenvalues of the Hamiltonian H are $\hbar\omega_k (n_k + 1/2)$ and the relative eigenstates $|n_k\rangle$ are known as Fock or number states. These are the eigenstates of the number operator $N_k = a_k^\dagger a_k$, as follows

$$N_k|n_k\rangle = n_k|n_k\rangle \tag{3.7}$$

Particularly, the ground state of the k-th oscillator, corresponding to zero photons in that field mode, is given by

$$a_k|0\rangle = 0 \ , \tag{3.8}$$

and has an energy equal to $1/2\hbar\omega_k$ because of the quantum fluctuations of the vacuum. The total ground state energy is so $1/2\sum_k \hbar\omega_k$ and is known as zero-point energy. In this context, the raising and lowering operators of the harmonic oscillator eigenstate ladder, $a_k^\dagger$ and a_k, represent, respectively, the creation and the annihilation of a photon in the mode k (in the following, called Bosonic mode). For this reason, they are also called creation and annihilation operators, respectively. This behavior is mathematically expressed by the following relations,

$$a_k^\dagger|n_k\rangle = \sqrt{n_k+1}|n_k+1\rangle \ , \tag{3.9}$$

$$a_k|n_k\rangle = \sqrt{n_k}|n_k-1\rangle \ , \tag{3.10}$$

and, applying repeatedly the creation operator, the number or Fock states have the explicit expression

$$|n_k\rangle = \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle \ . \tag{3.11}$$

These states are pure states, corresponding to a definite number of photons in the field, and are extremely difficult to be created experimentally.

Displacement (Weyl) operators and characteristic functions

In order to describe more appropriately the state of each Bosonic mode k, we introduce the *displacement (Weyl) operators* as

$$D(\mu) \equiv \exp[\mu a^\dagger - \mu^* a] = \exp\left[i(Q,P)\cdot z\right] \equiv V(z) \tag{3.12}$$

with a being the annihilation operator of the mode k, $z = (x,y)^T$ being a column vector of R^2 and μ a complex variable. For simplicity, in the following we will consider a single mode k and we will neglect the subscripts k. The correspondence between the phase-space representation in terms of a and $a^\dagger$ and the Weyl representation in terms of P and Q is ruled out by setting[1] $z = \sqrt{2}(\Im[\mu], -\Re[\mu])^T$. In the Weyl framework the canonical commutation relation for each mode k is written as

$$V(z)\ V(z') = \exp\left[\frac{i}{2}\Delta(z,z')\right]\ V(z+z') \ ,$$

[1] $\Im[x]$ and $\Re[x]$ denote the imaginary and real parts of a complex number x, respectively.

where $\Delta(z,z')$ is the symplectic form

$$\Delta(z,z') = -i\, z^T \cdot \sigma_y \cdot z' = x'y - xy' \;, \tag{3.13}$$

with σ_y being the second Pauli matrix. In this chapter, in fact, we will use both of these frameworks because in some contexts it is better to use one in the place of the other one in order to simplify the calculations. Note that the displacement operators satisfy the following properties

$$\begin{aligned} D(\mu)^\dagger a D(\mu) &= a + \mu \;, & (3.14)\\ D(\mu)^\dagger a^\dagger D(\mu) &= a^\dagger + \mu^* \;, & (3.15)\\ D(\mu)^\dagger = D(-\mu) &= D(\mu)^{-1} \;. & (3.16) \end{aligned}$$

Moreover, applying the displacement operator to the vacuum state of the mode k, i.e.

$$|\mu\rangle \equiv D(\mu)|0\rangle \;, \tag{3.17}$$

one obtains a characterization of each Bosonic mode k in terms of its coherent states $|\mu\rangle$ [61]. By definition, they are eigenvectors of the annihilation operator a. Indeed, one has

$$a|\mu\rangle = \mu|\mu\rangle \;. \tag{3.18}$$

The coherent states have an indefinite number of photons but a definite phase (and are more accessible experimentally), while, on the contrary, the number states have a completely random phase. These vectors possess various appealing properties. Specifically, they minimize the uncertainty relations of any couple of conjugate quadratures, e.g. P and Q. In general, they have to satisfy the Heisenberg's uncertainty relation $\Delta P \Delta Q \geq \frac{1}{2}\hbar$, where $\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$ and $\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2}$ (here $\langle X\rangle$ stands for the expectation value of the observable X in a quantum state ρ, i.e. $\langle X\rangle = \mathrm{Tr}[\rho X]$). The condition $\Delta P \Delta Q = \frac{\hbar}{2}$ with $\Delta P = \Delta Q = \sqrt{\frac{\hbar}{2}}$ is satisfied by coherent states. An explicit expression of coherent states in terms of number states $|n\rangle$ is given by

$$|\mu\rangle = e^{-|\mu|^2/2} \sum_{n=0}^{\infty} {}_{,}\, \frac{\mu^n}{\sqrt{n!}} |n\rangle \tag{3.19}$$

and the probability of finding n photons in the state $|\mu\rangle$ follows a Poisson distribution, i.e.

$$p(n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!} , \tag{3.20}$$

where $\bar{n} = |\mu|^2$ is the mean number of photons in the coherent state $|\mu\rangle$. Besides, since the Fock states $|n\rangle$ form a complete orthonormal set, the following completeness relation for the coherent state holds, i.e.

$$\frac{1}{\pi}\int d^2\mu |\mu\rangle\langle\mu| = \mathbb{1} , \tag{3.21}$$

with $d^2\mu := d\Re[\mu]\ d\Im[\mu] = \frac{1}{2}\ dx\ dy$.

Using Eq. (3.21) it is possible to expand any other state of the system as a superposition of the $|\mu\rangle$s with coefficients which define quasi-probability density functions. Exploiting this and Eq. (3.17), one can also use displacement operators as an over-complete operator basis [46, 59, 60, 61]. In particular, given Θ any trace-class[2] operator of the system (e.g., a density matrix ρ), we can write

$$\Theta = \int \frac{d^2\mu}{\pi}\ \chi(\mu)\ D(-\mu) = \int \frac{d^2 z}{2\pi}\ \phi(z)\ V(-z) , \tag{3.22}$$

where

$$\chi(\mu) \equiv \mathrm{Tr}[\Theta\ D(\mu)] = \mathrm{Tr}[\Theta\ V(z)] \equiv \phi(z) . \tag{3.23}$$

with $dz := dx\ dy$

Equation (3.23) defines the *characteristic* function of the operator Θ. This is a complex function of the variables μ and μ^* which provides us with a faithful description of the original operator thanks to the "orthogonality" relation

$$\mathrm{Tr}[D(\mu)D(-\nu)] = \delta^{(2)}(\mu - \nu) , \tag{3.24}$$

with $\delta^{(2)}(\mu - \nu)$ being the Dirac delta in the complex plane. In particular, one can consider the characteristic function of a density operator ρ, and, in this case, the function (3.23) needs to possess certain properties [46, 59, 60]:

[2] Mathematically, a trace-class operator is a compact operator for which a trace may be defined.

1) $\phi(0) = 1$ and $\phi(z)$ is continuous at $z = 0$;

2) $\phi(z)$ is Δ-positive definite, i.e.

$$\sum_{j,\,k=1}^{n} c_j\, c_k^*\, \phi(z_j - z_k)\, \exp\left[\frac{i}{2}\Delta(z_j, z_k)\right] \geq 0 \;, \tag{3.25}$$

for any number n of $z_1, \ldots z_n$, and with $c_1, \ldots c_n$ being generic complex numbers.

Actually, they are sufficient and necessary conditions for $\phi(z)$ being a characteristic function of a quantum state ρ. Notice that the characteristic function is the Fourier transform of the Wigner function, i.e. a quasi-probability distribution that allows a pictorial representation of the abstract notion of a quantum state and also to calculate quantum mechanical expectation values simply using concepts of classical statistical mechanics [61].

Moreover, the phase space formalism gives the possibility of defining the so-called *Gaussian states*. Indeed, the Gaussian states of the mode a are density operators ρ whose characteristic function is Gaussian, i.e.,

$$\phi(z) = \exp\left[-\frac{1}{4} z^T \cdot \gamma \cdot z + i m^T z\right] \;, \tag{3.26}$$

with γ and m being the second and the first order moment of the state ρ [45, 46, 59, 60]. We will characterize them more explicitly for the generic multi-mode case in the following, as in Eqs. (3.40) and (3.41). Examples of Gaussian states are thermal, coherent and squeezed states, but a Fock state (except the vacuum) is not Gaussian.

A thermal state is defined by a density operator of the form

$$\rho = \frac{e^{-\frac{a^\dagger a}{k_B T}}}{\mathrm{Tr}[e^{-\frac{a^\dagger a}{k_B T}}]} \;, \tag{3.27}$$

where T is the temperature and k_B is the Boltzmann constant. Its characteristic function is as in Eq. (3.26) with $m = 0$ and covariance matrix

$$\gamma = (2N_0 + 1)\mathbb{1} \;. \tag{3.28}$$

Here $N_0 \equiv \langle a^\dagger a \rangle$ is the average photon number of the thermal state, which is related to T through the Planck distribution function

$$N_0 = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \;, \tag{3.29}$$

with $\hbar\omega$ being the photon energy. For $N_0 = 0$ one recovers the vacuum state. For a thermal state, one has $\langle Q\rangle = \langle P\rangle = 0$ and $\Delta P = \Delta Q = N_0 + \frac{1}{2}$.

A squeezed state can be defined as a state that may have less noise in one of quadratures (e.g., in P or in Q) than a coherent state. In other terms, it satisfies the condition $\Delta P\Delta Q = \frac{\hbar}{2}$ with $\Delta P \neq \Delta Q$. Therefore, they represent a more general class of minimum-uncertainty states than the particular case of coherent states. A squeezed state can be obtained applying the unitary *squeezing operator*,

$$S(r;\varphi) = \exp\left(1/2\ \epsilon^*\ a^2 - 1/2\ \epsilon\ {a^\dagger}^2\right)\ , \tag{3.30}$$

with $\epsilon = re^{i\varphi}$ being the squeezing factor, to a coherent state. Precisely, r represents the degree of the attenuation and amplification of two uncertainties, while φ describes a rotation in the phase space representation. When $r = 1$ one recovers a coherent state, in which the noise is equal in both quadratures. A squeezed state is so given applying first a squeezing and then a displacement to the vacuum, i.e.

$$|\mu,\epsilon\rangle = D(\mu)S(r;\varphi)|0\rangle\ . \tag{3.31}$$

These squeezing operators satisfy the relations

$$\begin{aligned} S(r;\varphi)\ a\ S^\dagger(r;\varphi) &= a\ \cosh r + e^{i\varphi}\ a^\dagger\ \sinh r\ , &(3.32)\\ S(r;\varphi)\ a^\dagger\ S^\dagger(r;\varphi) &= a^\dagger\ \cosh r + e^{-i\varphi}\ a\ \sinh r\ , &(3.33)\\ S(r;\varphi)^\dagger &= S(r;\varphi)^{-1} = S(-r;\varphi)\ . &(3.34)\end{aligned}$$

Multi-mode case and symplectic formalism

The previous definitions can be easily generalized to the multi mode case by using the Weyl operator formalism. Consider a system composed by n Bosonic modes having canonical coordinates $Q_1, P_1, \cdots, Q_n, P_n$. The canonical commutation relations of the canonical coordinates, $[R_j, R_{j'}] = i(\sigma_{2n})_{j,j'}$, where $R := (Q_1, \cdots, Q_n; P_1, \cdots, P_n)$, are grasped by the $2n \times 2n$ *commutation matrix*

$$\sigma_{2n} = \begin{bmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{bmatrix}\ , \tag{3.35}$$

when this order of canonical coordinates is chosen, (here $\mathbb{1}_n$ is the $n \times n$ identity matrix) [45, 59, 114]. Even though different reordering of the elements of R will not affect the definitions that follow, we find it useful to assume a specific form for σ_{2n}. One defines the group of real *symplectic matrices* $Sp(2n, \mathbb{R})$ as the set of $2n \times 2n$ real matrices S which satisfy the condition

$$S\sigma_{2n}S^T = \sigma_{2n} \,. \tag{3.36}$$

Since $\mathrm{Det}[\sigma_{2n}] = 1$, and $\sigma_{2n}^{-1} = -\sigma_{2n}$, any symplectic matrix S has $\mathrm{Det}[S] = 1$ and it is invertible with $S^{-1} \in Sp(2n, \mathbb{R})$. Similarly, one has $S^T \in Sp(2n, \mathbb{R})$. Symplectic matrices play a key role in the characterization of Bosonic systems. Indeed, the Weyl (displacement) operators, defined as

$$V(z) = V^\dagger(-z) := \exp[iRz] \,, \tag{3.37}$$

with $z := (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T$ being a column vector of $\mathbb{R}^{2n}$, generalize those for single mode above. Then it is possible to show [46] that for any $S \in Sp(2n, \mathbb{R})$ there exists a *canonical* unitary transformation U which maps the canonical observables of the system into a linear combination of the operators R_j, verifying the condition

$$U^\dagger \, V(z) \, U = V(Sz) \,, \tag{3.38}$$

for all z. This is often referred to as metaplectic representation. Conversely, one can show that any unitary U which transforms $V(z)$ as in Eq. (3.38) must correspond to an $S \in Sp(2n, \mathbb{R})$. As in the single-mode case, Weyl operators allow one to rewrite the canonical commutation relations as

$$V(z)V(z') = \exp[-\tfrac{i}{2} z^T \sigma_{2n} z'] V(z + z') \,, \tag{3.39}$$

and permit again a complete descriptions of the system in terms of (characteristic) complex functions as in Eq. (3.22-3.23) but replacing $x = (x, y)^T$ with $z = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T$ and $dz := dxdy$ with the differential $d^{2n}z := dx_1 \cdots dx_n dy_1 \cdots dy_n$. Within this framework, generalizing the single-mode definition above, a density operator ρ of n modes is said to represent a *Gaussian state* if its characteristic function $\phi(z)$ has a Gaussian form as in Eq. (3.26) with m being a real vector of mean values

$$m_j := \mathrm{Tr}[\, \rho \, R_j] \,, \tag{3.40}$$

and the $2n \times 2n$ real symmetric matrix γ being the *covariance matrix* [46, 59, 60] of ρ. For generic density operators ρ (not only the Gaussian ones) the latter is defined as the variance of the canonical coordinates R, i.e.,

$$\gamma_{j,j'} \;:=\; \mathrm{Tr}\Big[\rho\big\{(R_j - m_j), (R_{j'} - m_{j'})\big\}\Big] \,, \tag{3.41}$$

with $\{\cdot,\cdot\}$ being the anti-commutator, and it is bound to satisfy the uncertainty relations

$$\gamma \geqslant i\sigma_{2n}\,, \tag{3.42}$$

with σ_{2n} being the commutation matrix (3.35). Up to an arbitrary vector m (that can be nullified by displacement operators, without loss of generality), the uncertainty inequality presented above uniquely characterizes the set of Gaussian states, i.e. any γ satisfying (3.42) defines a Gaussian state. Let us first notice that if γ satisfies (3.42) then it must be (strictly) positive definite $\gamma > 0$, and have $\mathrm{Det}[\gamma] \geqslant 1$. From Williamson theorem [115] it follows that there exists a symplectic $S \in Sp(2n, \mathbb{R})$ such that (see Appendix A)

$$\gamma = S \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} S^T\,, \tag{3.43}$$

where $D := \mathrm{diag}(d_1, \cdots, d_n)$ is a diagonal matrix formed by the *symplectic eigenvalues* $d_j \geqslant 1$ of γ. For $S = \mathbb{1}_{2n}$ Eq. (3.43) gives the covariance matrix associated with thermal Bosonic states. This also shows that any covariance matrix γ satisfying (3.42) can be written as

$$\gamma = SS^T + \Delta\,, \tag{3.44}$$

with $\Delta \geqslant 0$. This is indeed the matrix

$$\Delta := S \begin{bmatrix} D - \mathbb{1}_n & 0 \\ 0 & D - \mathbb{1}_n \end{bmatrix} S^T\,,$$

with D as in Eq. (3.43) which is positive since $D \geqslant \mathbb{1}_n$. The extremal solutions of Eq. (3.44), i.e., $\gamma = SS^T$, are *minimal uncertainty solutions* and correspond to the *pure Gaussian states* of n modes (e.g., multi-mode squeezed vacuum states). They are uniquely determined by the condition $\mathrm{Det}[\gamma] = 1$ and satisfy the condition [104]

$$\gamma = -\sigma_{2n}(\gamma^{-1})\sigma_{2n}\,. \tag{3.45}$$

3.2 Bosonic Gaussian channels

Gaussian channels arise from linear dynamics of open Bosonic system interacting with a Gaussian environment via quadratic Hamiltonians [45, 54]. Loosely speaking, they can be characterized as CPT maps that transform Gaussian states into Gaussian states [114, 116, 117]. Recalling Sec. 2.2, in the Schrödinger picture the noise evolution is described by applying the transformation to the states (i.e., the density operators), $\rho \mapsto \mathcal{E}(\rho)$. In the Heisenberg picture the transformation is applied to the observables of the system, while leaving the states unchanged, $\Theta \mapsto \mathcal{E}_H(\Theta)$.

Due to the representation (3.22) and (3.23) any CPT transformation on the n-modes can be characterized by its action on the Weyl operators of the system in the Heisenberg picture (e.g., see Ref. [53]). In particular, a *Bosonic Gaussian channel* (BGC) is defined as a map which, for all z, operates on $V(z)$ according to [45]

$$V(z) \longmapsto \mathcal{E}_H(V(z)) := V(Xz)\, \exp\left[-\tfrac{1}{4} z^T Y z + i v^T z\right] , \tag{3.46}$$

with v being some fixed real vector of $\mathbb{R}^{2n}$, and with $Y, X \in \mathbb{R}^{2n\times 2n}$ being some fixed real $2n \times 2n$ matrices (the noise and the interaction term, respectively) satisfying the complete positivity condition

$$Y \geqslant i\Sigma \qquad \text{with} \quad \Sigma := \sigma_{2n} - X^T \sigma_{2n} X\,. \tag{3.47}$$

In the context of BGCs the above inequality is the quantum channel counterpart of the uncertainty relation (3.42). In fact, it is obtained simply imposing the condition in Eq. (3.25) to the output characteristic function. More generally, one has $\mathcal{E}_H(V(z)) = V(X \cdot z)\, f(z)$, where $f(z)$ has to satisfy some conditions in order to have a CPT map, e.g. for $n = 1$ the matrices $M_{jk} = f(z_j - z_k)\, \exp\left[-i/2\Delta(z_j, z_k) + i/2\Delta(X\ z_j, X\ z_k)\right]$ have to be positive, with $\Delta(z, z')$ defined in Eq. (3.13) [45]. When $f(z) = 1$ and X is a symplectic transformation, i.e. $\mathcal{E}$ is unitarily implemented, this map is better known as Bogoliubov transformation [3]. Up to a vector v, Eq. (3.47) uniquely determines the set of BGCs and bounds Y to be positive-semidefinite, $Y \geqslant 0$. However, differently from (3.42) in this case strict positivity is not a necessary prerequisite for Y. A completely positive map defined by Eqs. (3.46) and (3.47) will be referred to as Bosonic Gaussian channel. Notice that the action of BGC on a generic operator can be reduced to its action on the Weyl operators, because they represent an over-complete operator basis.

As mentioned before, such a map is a model for a wide class of physical situations, ranging from communication channels such as optical fibers, to open quantum systems, and to dynamics in harmonic lattice systems. Whenever one has only partial access to the dynamics of a system that can be well-described by a time evolution governed by a Hamiltonian that is a quadratic polynomial in the canonical coordinates, one will arrive at a model described by Eqs. (3.46) and (3.47).

An important subset of BGCs is given by set of *Gaussian unitary* transformations which have $Y = 0$, $X \in Sp(2n, \mathbb{R})$, and v arbitrary. They include the canonical transformations of Eq. (3.38) (characterized by $v = 0$), and the displacement transformations (characterized by having $X = \mathbb{1}_{2n}$ and v arbitrary). The latter simply adds a phase to the Weyl operators and correspond to unitary transformations of the form $\mathcal{E}_H(V(z)) := V(-v)V(z)V(v) = V(z)\,\exp[iv^T z]$.

In the Schrödinger picture the BGC transformation (3.46) induces a mapping of the characteristic functions of the form

$$\phi(z) \longmapsto \phi'(z) := \phi(Xz)\,\exp[-\tfrac{1}{4}z^T Y z + iv^T z]\,. \tag{3.48}$$

which in turn yields the following transformation of the mean and the covariance matrix

$$\begin{aligned} m &\longmapsto X^T m + v\,, \\ \gamma &\longmapsto X^T \gamma X + Y\,. \end{aligned} \tag{3.49}$$

Indeed, one has

$$\begin{aligned} \phi'(z) = \mathrm{Tr}[\rho\,\mathcal{E}_H(V(z))] &= \mathrm{Tr}[\rho\,V(Xz)]\,\exp\left[-\tfrac{1}{4}z^T Y z + iv^T z\right] \\ &= \phi(Xz)\,\exp[-\tfrac{1}{4}z^T Y z + iv^T z]\,. \end{aligned} \tag{3.50}$$

Clearly, BGCs always map Gaussian input states into Gaussian output states. Besides, one could derive a 'Green function' representation of these maps in the following way [44]. Consider the action of a linear superoperator $\mathcal{E}$ which transforms a generic trace-class operator Θ into $\Theta' = \mathcal{E}(\Theta)$. Equation (3.22) allows us to represent this mapping in terms of a linear transformation of the characteristic function $\phi(z)$ of Eq. (3.23). The characteristic function of the output operator Θ' is

$$\phi'(z) = \mathrm{Tr}[\Theta'\,V(z)] = \mathrm{Tr}[\Theta\,\mathcal{E}_H(V(z))] = \int \frac{d^{2n}z'}{(2\pi)^n}\,\phi(z')\,\mathrm{Tr}[V(-z')\mathcal{E}_H\,(V(z))]\,,$$

that can be written in a more compact form as

$$\phi'(z) = \int \frac{d^{2n}z'}{(2\pi)^n} \, \phi(z') \, G(z', z) \;, \tag{3.51}$$

with

$$G(z', z) \;\equiv\; \mathrm{Tr}[V(-z') \, \mathcal{E}_H(V(z))] = \mathrm{Tr}[\mathcal{E}(V(-z')) \, V(z)] \; . \tag{3.52}$$

In these expressions $\mathcal{E}_H$ is the dual of $\mathcal{E}$ which describes the channel in the Heisenberg picture (see Sec. 2.2.5). We call Eq. (3.52) the Green function of $\mathcal{E}$: according to previous definitions it provides us with a complete characterization of the channel. The Bosonic Gaussian channels are characterized by Green functions (3.52) of the form

$$G(z', z) \;=\; \delta^{(2n)}(z' - Xz) \; \exp\left[-\tfrac{1}{4} z^T Y z + i v^T z\right] \; . \tag{3.53}$$

Indeed, as can be directly verified from Eq. (3.51), we notice again that such maps have the peculiar property of transforming input Gaussian states into output Gaussian states.

Composition rules

For purposes of assessing quantum or classical information capacities, output entropies, or studying degradability or anti-degradability of a channel (as discussed in Sec. 2.3), the full knowledge of the channel is not required: transforming the input or the output with any unitary operation (say, Gaussian unitaries) will not alter any of these quantities [52, 53, 54, 55, 56]. It is then convenient to take advantage of this freedom to simplify the description of the BGCs. To do so we first notice that the set of Gaussian maps is closed under composition. Consider then $\mathcal{E}'$ and $\mathcal{E}''$ two BGCs described respectively by the elements X', Y', v' and X'', Y'', v''. The composition $\mathcal{E}'' \circ \mathcal{E}'$ where, in Schrödinger representation, we first operate with $\mathcal{E}'$ and then with $\mathcal{E}''$, is still a BGC and it is characterized by the parameters

$$\begin{aligned} v &= (X'')^T v' + v'' \,, \\ X &= X' X'' \,, \\ Y &= (X'')^T Y' X'' + Y'' \,. \end{aligned} \tag{3.54}$$

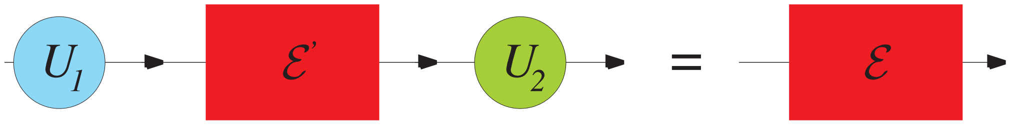

Figure 3.1: Simplification of $\mathcal{E}'$ by cascading it through proper encoding and decoding Gaussian unitary transformations U_1 and U_2. The resulting channel $\mathcal{E}$ is unitary equivalent to $\mathcal{E}'$ and it is characterized by the matrices of the form (3.55).

Exploiting these composition rules it is then easy to verify that the vector v can always be compensated by properly displacing either the input state or the output state (or both) of the channel. For instance by taking $X'' = \mathbb{1}_{2n}$, $Y'' = 0$ and $v'' = -v'$, Eq. (3.54) shows that $\mathcal{E}'$ is unitarily equivalent to the Gaussian channel $\mathcal{E}$ which has $v = 0$ and $X = X'$, $Y = Y'$. Therefore, without loss of generality, in the following we will focus on BGCs having $v = 0$.

More generally, consider the setup of Fig. 3.1 where we cascade a generic BGC $\mathcal{E}'$ described by matrices X', Y' as in Eq. (3.47) with a couple of canonical unitary transformation U_1 and U_2 described by the symplectic matrices S_1 and S_2 respectively. The resulting BGC $\mathcal{E}$ is then described by the matrices

$$\begin{aligned} X &= S_1(X')S_2, \\ Y &= S_2^T(Y')S_2\,. \end{aligned} \tag{3.55}$$

For single mode ($n = 1$) this procedure induces a simplified canonical form [48, 53, 56] which, up to a Gaussian unitarily equivalence, allows one to focus only on transformations characterized by X and Y which, apart from some special cases, are proportional to the identity (as we will show in Sec. 3.3). In Sec. 3.4 we will generalize some of those results to an arbitrary number of modes n. To achieve this goal, we first present an explicit dilation representation in which the mapping (3.46) is described as a (canonical) unitary coupling between the n modes of the system and some extra *environmental* modes which are initially prepared into a Gaussian state.

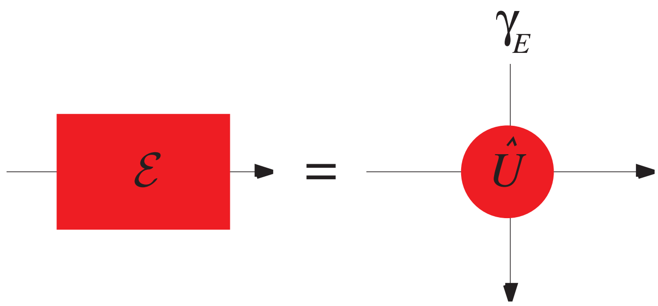

Figure 3.2: A general Gaussian channel of n modes, written as a unitary dilation by unitarily coupling them to a Gaussian state ρ_E of (at most) $2n$ environmental modes E as in Eq. (3.56). Here γ_E is the covariance matrix of ρ_E.

Unitary dilation problem

Let us briefly introduce the problem of unitary dilations of generic Bosonic Gaussian channels. Specifically, we ask if, given a BGC channel acting on n modes as in Eq. (3.46), it can be realized by invoking $\ell \leqslant 2n$ additional (environmental) modes E through the expression

$$\mathcal{E}(\rho) = \mathrm{Tr}_E[U(\rho \otimes \rho_E)U^\dagger]\,, \tag{3.56}$$

where ρ is the input n-mode state of the system, ρ_E is a Gaussian state of an environment, U is a canonical unitary transformation which couples the system with the environment, and Tr_E denotes the partial trace over E (see Fig. 3.2). In the case in which ρ_E is pure, Eq. (3.56) corresponds to a Stinespring dilation [81] of the channel $\mathcal{E}$, otherwise it is a physical representation, as in Sec. 2.2.2. It is worth to stress that this problem is not trivial because, although the Stinespring representation of the channel is always possible, however it is not necessarily true that one can find a physical representation with ρ_E being a (not necessarily pure) Gaussian state and U being a canonical unitary transformation. In the following, we will start to face directly this problem for one-mode channels [52, 53] and, later, we will generalize these results for multi-mode, proving a general unitary dilation theorem and analyzing also the two-mode case in detail [54].

3.3 One-mode Bosonic Gaussian channels

In this section, we focus on one-mode Bosonic Gaussian channels which act on the density matrices of a single Bosonic mode A. Loosely speaking, they can be characterized as a CPT transformation $\mathcal{E}$ operating on a Bosonic mode, described by the annihilation operator a, and producing output Gaussian states (see Sec. 3.1) when acting on Gaussian input states ρ_a [45, 46, 59, 60].

A complete description of these maps, specified by its action on the Weyl operators $V_a(z)$, is obtained as in Eq. (3.46), where $n = 1$ and $X, Y \in \mathbb{R}^{2\times 2}$ are real 2×2 matrices satisfying the CPT condition in Eq. (3.47). However, in this case the inequality in Eq. (3.47) reduces to the simpler condition

$$Y + i\,(\mathrm{Det}[X] - 1)\,\sigma_2 \geqslant 0 \tag{3.57}$$

(where σ_2 is defined in Eq. (3.35) for $n = 1$) that is trivially equivalent to the inequality

$$\mathrm{Det}[Y] \geqslant (\mathrm{Det}[X] - 1)^2 \ . \tag{3.58}$$

Within the limit imposed by Eq. (3.58) we can use Eq. (3.48) to describe the whole set of the one-mode Gaussian channels.

As shown in the following, an interesting fact about these channels is that, except for the additive classical noise channel [56], they admit a physical representation in terms of a single mode environment originally prepared in a Gaussian state [53]. Within such representation one can show that the Bosonic Gaussian channels (3.53) are either weakly degradable or anti-degradable, exploiting the unitary equivalence with the beam-splitter and amplifier channels [52]. A canonical classification of such maps obtained recently in the paper [56] enables us to simplify the analysis of the weak-degradability property and to study more easily the cases in which the environment is described by more than one Bosonic mode [52, 53] (cf. also Ref. [48]). Finally, we focus mostly on the anti-degradability property and we show that, by exploiting the composition rules of one-mode Bosonic Gaussian channels, one can extend the set of the maps with null quantum capacity well beyond the set of anti-degradable maps. In this way, we exhibit a new set of channels useless for the transfer of quantum information, extending a previous result in Ref. [45]. All these results will be shown in detail in this chapter [52, 53].

3.3.1 Beam-splitter and linear amplifier

Before taking generic Bosonic Gaussian channels into account, we start to study two simple particular cases, present in almost all real quantum optics experiments, i.e. the beam-splitter and the linear amplifier [61]. In particular, we will investigate their composition rules, weak-degradability features and quantum capacity. This will be propaedeutical for the characterization of the whole class of single-mode channels.

One of the simplest optical processes is attenuation by a beam-splitter (BS). It is an optical device (e.g., a semitransparent mirror) that splits an incident optical beam into two coherent parts by reflecting (with probability $1-k^2$) and transmitting (with probability k^2) some fraction of the incident beam. The parameter k^2 is called the transmissivity of the beam-splitter. It is made either of thin layers of metal or multi layer dielectric films deposited on glass. This device preserves all the mode properties, like the light frequency, the size of the beam, the curvature of the wavefront. However, the energy conservation requires a phase shift $\pi/2$ in the reflection of one of the two modes, while the direction of the propagation is given by geometrical considerations. From the photon point of view, the beam-splitter acts like a random selector with reflects the photons with probability $1-k^2$ and transmits them with a probability k^2, but its effect on any particular photon is unpredictable[3]. This model well describes, for instance, the losses inside an optical fiber and the optical attenuation over distance (but also what happens in a quantum memory). The beam-splitter is mathematically described by the following linear transformations of two single mode operators a and b

$$a' \equiv U_{ab}^\dagger \; a \; U_{ab} \;\; = \;\; k \; a - \sqrt{1-k^2} \; b \; , \tag{3.59}$$

$$b' \equiv U_{ab}^\dagger \; b \; U_{ab} \;\; = \;\; \sqrt{1-k^2} \; a + k \; b \; , \tag{3.60}$$

with $k^2 \in [0,1]$ being the transmissivity and U_{ab} is a unitary interaction between the two modes a and b, i.e.

$$U_{ab} = \exp\left[(ba^\dagger - b^\dagger a) \arctan\sqrt{\frac{1-k}{k}}\right] \tag{3.61}$$

If $k^2 = 1/2$, this device is known as balanced 50/50 beam-splitter. If $k = 1$, one obtains the identity map, i.e. $U_{ab}^\dagger a U_{ab} = a$ and $U_{ab}^\dagger b U_{ab} = b$. Moreover,

[3] This principle is used in a commercially available product, that is the Quantum Random Number Generator [31].

the beam-splitter can be seen as a Bosonic channel, where the mode a would correspond to the main system and the mode b to an external environment. If the initial state of the mode b is a Gaussian state, the relative beam-splitter is a Bosonic Gaussian channel.

An optical amplifier is any optical medium used to amplify the power of a laser beam. It is very useful for communication systems, because, for example, it enables to compensate the unavoidable losses in the transmission of optical beams over long distances and it is usually periodically placed in optical fiber links. Particularly, a linear amplifier is an amplifier whose output is linearly related to its input. This linear amplification process is described by the following operator transformations,

$$a' = k\,a - \sqrt{k^2-1}\,b\,, \tag{3.62}$$
$$b' = -\sqrt{k^2-1}\,a + k\,b\,, \tag{3.63}$$

where $k^2 \geq 1$ is the amplification factor. Again, for $k = 1$ this map corresponds to the identity, i.e. $a' = a$ and $b' = b$. A practical implementation of this model consists of a group of inverted two-level atoms; in this case the linearity is obtained by assuming that only one-photon processes take place. For example, it could correspond to a laser with the end mirrors removed, being perturbed by a weak external field. Notice that the linear amplifier can be considered a Bosonic channel and it is Gaussian if the input state of the mode b is Gaussian, e.g. an environmental thermal state.

According to the operator transformation above, it is easy to verify that the BS/amplifier map, defined as $\mathcal{E}[k, \sigma_b]$, (distinguished by $k < 1$ and $k > 1$, respectively) operates on a generic (not necessarily Gaussian) state ρ_a by transforming its characteristic function $\chi(\mu)$ as follows:

$$\chi(\mu) \to \chi'(\mu) = \begin{cases} \chi(k\mu)\,\xi(\sqrt{1-k^2}\mu) & k \in [0,1] \\ \chi(k\mu)\,\xi(-\sqrt{k^2-1}\mu^*) & k \geqslant 1\,, \end{cases} \tag{3.64}$$

with

$$\xi(\mu) = \mathrm{Tr}[\sigma_b \exp(\mu b^\dagger - \mu^* b)]\,, \tag{3.65}$$

being the Gaussian characteristic function of the environment state σ_b. Without loss of generality, in the following we will assume σ_b to have null first order momentum (it can always be compensated through a suitable unitary operator acting on the output of the channel). Particularly, from now on, σ_b is chosen to be a thermal state as in Eq. (3.27), whose characteristic function $\xi(\mu)$ is a Gaussian with γ given by Eq. (3.28) and with N_0 being the average photon number. The description of beam-splitter and amplifier channels, as in Eq. (3.46), corresponds to have $X = k\mathbb{1}$ and $Y = |k^2 - 1|(2N_0 + 1)\mathbb{1}$, satisfying trivially the inequality (3.58) because $\mathrm{Det}[Y] = (k^2 - 1)^2(2N_0 + 1)^2 \geqslant \left(k^2 - 1\right)^2 = \left(\mathrm{Det}[X] - 1\right)^2$.

Analogously the weakly complementary map $\tilde{\mathcal{E}}[k, \sigma_b]$ of Eq. (2.50) produces the transformation,

$$\chi(\mu) \to \chi'(\mu) = \begin{cases} \chi(-\sqrt{1-k^2}\mu)\,\xi(k\mu) & k \in [0,1] \\ \\ \chi(-\sqrt{k^2-1}\mu^*)\,\xi(k\mu) & k \geqslant 1\ . \end{cases} \tag{3.66}$$

It is worth noticing that the weakly complementary channel of a BS with transmissivity k^2 is another BS with transmissivity $1 - k^2$.

Examples of composite channels

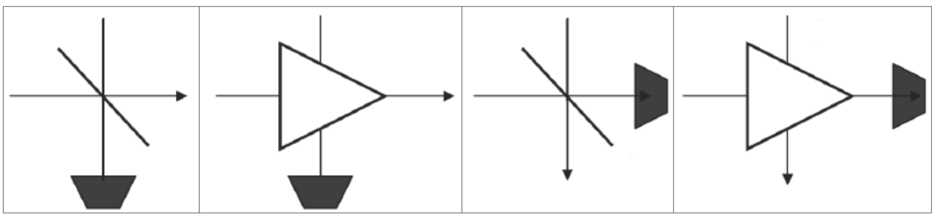

Figure 3.3: From left to right, pictorial representation of a BS, a linear amplifier, the weakly-complementary map of a BS and of a linear amplifier, respectively.

Now we try to compose beam-splitter and amplifier channels (see Fig. 3.3). As in Sec. 2.2.1, we use the word ‘composition’ in the sense that the channels are considered in succession, i.e. the second channel is applied to the output of the first channel and so on. We will find that these composite channels will be equivalent to a beam-splitter or an amplifier with “rescaled” environment (i.e., whose covariance matrix, γ, is a combination of two component ones, γ_1 and γ_2) and “rescaled” gain/attenuation coefficient (i.e., $k = k_1k_2$).

- BS + BS

  We combine two beam-splitter channels with attenuation parameter k_1^2 and k_2^2, respectively (see Fig. 3.4). The characteristic function evolution of these two maps is:

  - BS 1:
$$\chi(\mu) \rightarrow \chi'(\mu) = \chi(k_1\mu) \,, \xi(\sqrt{1-k_1^2}\mu) \tag{3.67}$$
  - BS 2
$$\chi(\mu)' \rightarrow \chi''(\mu) = \chi'(k_2\mu)\, \xi(\sqrt{1-k_2^2}\mu) \,, \tag{3.68}$$

  with $\xi(\mu)$ being a Gaussian characteristic function of the environment state σ_b as in Eq. (3.65). By defining a new attenuation coefficient

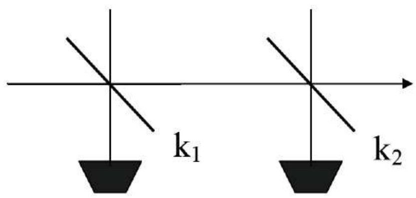

Figure 3.4: Graphical representation of the composition of two BS channels with attenuation coefficient k_1^2 and k_2^2, respectively.

  $k = k_1 k_2 < 1$, it is possible represent the channel BS+BS like another BS with transmissivity k^2. Indeed,
$$\chi(\mu) \rightarrow \chi''(\mu) = \chi(k\mu)\, \xi(\sqrt{1-k^2}\mu) \,, \tag{3.69}$$
  as in Eq. (3.64), where
$$\gamma = \frac{k_2^2(1-k_1^2)}{1-k^2}\gamma_1 + \frac{1-k_2^2}{1-k^2}\gamma_2 \,, \tag{3.70}$$
  with γ_1 and γ_2 being environment input state covariance matrices for the BS channels, respectively[4].

[4]Notice that the composition of two BS (amplifiers) with parameters k_1 and k_2 and environment, respectively, b_1 and b_2 (not necessarily Gaussian), is another BS (amplifier) with $k = k_1 k_2$ and with the environmental initial state being the output state of another BS (amplifier) with $\bar{k} = \frac{k_2\sqrt{|1-k_1^2|}}{\sqrt{|1-k_1^2 k_2^2|}}$ and input states b_1 and b_2.

- Amplifier + Amplifier

  Here we compose two amplifier channels with gain parameter k_1^2 and k_2^2, respectively (see Fig. 3.5).

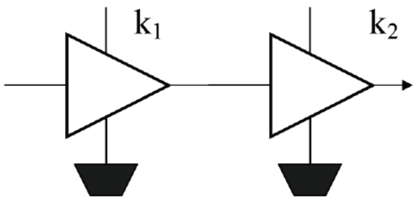

Figure 3.5: Graphical representation of the composition of two amplifier channels with gain parameter k_1^2 and k_2^2, respectively.

  As done for BS+BS case, by introducing a new amplification coefficient $k = k_1 k_2 > 1$, the composite map is another amplifier with efficiency k^2, where the Gaussian environmental state has the following covariance matrix

$$\gamma = \sigma_x \left[\frac{k_2^2(k_1^2 - 1)}{k^2 - 1}\gamma_1 + \frac{k_2^2 - 1}{k^2 - 1}\gamma_2 \right] \sigma_x \ , \tag{3.71}$$

  with σ_x being the first Pauli matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, γ_1 and γ_2 being the environment input state covariance matrices for the amplifier channels, respectively.

- BS + Amplifier

  Let us mix a beam-splitter with attenuation coefficient k_1^2 and an amplifier with gain parameter k_2^2 (see Fig. 3.6). We will show that the composite map (corresponding to the parameter $k = k_1 k_2$) is a BS or an amplifier, according to that $k < 1$ or $k > 1$, respectively.

  - $k = k_1 k_2 < 1$

    In this case, it is possible represent the channel BS+amplifier like a beam-splitter characterized by transmissivity k^2 and environmental covariance matrix

$$\gamma = \frac{k_2^2(1 - k_1^2)}{1 - k^2}\gamma_1 + \frac{(k_2^2 - 1)}{1 - k^2}\sigma_x \gamma_2 \sigma_x \ , \tag{3.72}$$

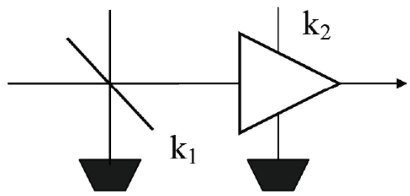

Figure 3.6: Graphical representation of the composition of a BS with attenuation coefficient k_1^2 and an amplifier with gain parameter k_2^2.

with γ_1 and γ_2 being the environment input state covariance matrices for the BS and the amplifier, respectively.

- $k = k_1 k_2 > 1$

  In this other case, instead, the map BS+amplifier is equivalent to a new amplifier with amplification coefficient k^2 and

$$\gamma = \sigma_x \Big[\frac{k_2^2(1-k_1^2)}{k^2-1} \sigma_x \gamma_1 \sigma_x + \frac{k_1^2(k_2^2-1)}{k^2-1} \gamma_2 \Big] \sigma_x \; . \tag{3.73}$$

- Amplifier + BS

  Let us consider an amplifier with gain parameter k_1^2 followed by a beam-splitter with attenuation coefficient k_2^2 (see Fig. 3.7), in an inverse order with respect to the results above.

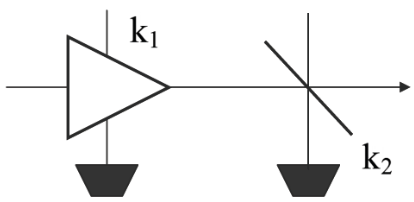

Figure 3.7: Graphical representation of the composition of an amplifier with gain parameter k_1^2 and a BS with attenuation coefficient k_2^2.

Analogously to the previous case, the composite map (corresponding to the parameter $k = k_1 k_2$) is a BS or an amplifier, according to that $k < 1$ or $k > 1$, respectively.

- $k = k_1 k_2 < 1$
  In this case, the channel amplifier+BS corresponds to a BS with attenuation coefficient k^2 and
$$\gamma = \frac{k_2^2(k_1^2-1)}{1-k^2}\sigma_x\gamma_1\sigma_x + \frac{1-k_2^2}{1-k^2}\gamma_2\ , \tag{3.74}$$
  with γ_1 and γ_2 being the environment input state covariance matrices for the amplifier and the BS, respectively.
- $k = k_1 k_2 > 1$
  Now, instead, the composite map is an amplifier with efficiency k^2, where the equivalent covariance matrix of the Gaussian environment for the composite map is
$$\gamma = \sigma_x\Big[\frac{k_2^2(k_1^2-1)}{k^2-1}\gamma_1 + \frac{1-k_2^2}{k^2-1}\sigma_x\gamma_2\sigma_x\Big]\sigma_x\ . \tag{3.75}$$

Weak-degradability

Now we will show that the BS/amplifier map, $\mathcal{E}[k,\sigma_b]$, whose characteristic function evolves as in Eq. (3.64), is weakly degradable for $k^2 \geqslant 1/2$ and anti-degradable for $k^2 \leqslant 1/2$ [52].

Consider first the amplifier case where $k \geqslant 1$. To show that $\mathcal{E}[k,\sigma_b]$ satisfies the weak-degradability condition in Eq. (2.52) we define the quantity $k'^2 \equiv (2k^2-1)/k^2$ and notice that this is always greater than or equal to 1. Our claim is that one can identify the map $\mathcal{T}$ of Eq. (2.52) with the weakly complementary map (2.50) of an amplifier of gain k'^2, i.e., $\mathcal{T} = \tilde{\mathcal{E}}[k',\sigma_b]$. This can be verified by studying how $\tilde{\mathcal{E}}[k',\sigma_b]\circ\mathcal{E}[k,\sigma_b]$ acts on a generic state ρ_a. Combining Eqs. (3.64), (3.66) and using similar calculations as above, it follows that the characteristic function $\chi(\mu)$ of ρ_a is transformed into
$$\begin{aligned}
&\chi(-\sqrt{k^2(k'^2-1)}\mu^*)\ \xi(\sqrt{(k'^2-1)(k-1)}\mu)\ \xi(k'\mu)\\
&\quad = \chi(-\sqrt{k^2(k'^2-1)}\mu^*)\ \xi(\sqrt{(k'^2-1)(k^2-1)+k'^2}\mu)\\
&\quad = \chi(-\sqrt{k^2-1}\mu^*)\ \xi(k\mu)\ ,
\end{aligned} \tag{3.76}$$
where we used the properties of the Gaussian function ξ and the identity $k^2(k'^2-1) = k^2-1$. By comparison with Eq. (3.66), we notice that $\tilde{\mathcal{E}}[k',\sigma_b]\circ\mathcal{E}[k,\sigma_b]$ operates on ρ_a as $\tilde{\mathcal{E}}[k,\sigma_b]$. Since this is true for all ρ_a we get
$$\tilde{\mathcal{E}}[k,\sigma_b] = \tilde{\mathcal{E}}[k',\sigma_b]\circ\mathcal{E}[k,\sigma_b]\ , \tag{3.77}$$

proving the thesis.

Consider now the BS case where $k \in [0,1]$. Here we distinguish two different regimes. For $k \in [1/\sqrt{2}, 1]$ the channel $\mathcal{E}[k, \sigma_b]$ is still weakly degradable and satisfies Eq. (3.77), the only difference being that now $\tilde{\mathcal{E}}[k', \sigma_b]$ represents the weakly complementary of a BS map of transmissivity $k'^2 = (2k^2-1)/k^2 \in [0,1]$. The formal proof goes as in Eq. (3.76), which now becomes

$$\begin{aligned}
&\chi(-\sqrt{k^2(1-k'^2)}\mu)\ \xi(-\sqrt{(1-k'^2)(1-k^2)}\mu)\ \xi(k'\mu) \\
&\quad = \chi(-\sqrt{k^2(1-k')^2}\mu)\ \xi(\sqrt{(1-k'^2)(1-k^2)+k'^2}\mu) \\
&\quad = \chi(-\sqrt{1-k^2}\mu)\ \xi(k\mu)\ . \qquad (3.78)
\end{aligned}$$

For $k \in [0, 1/\sqrt{2}]$ instead we can show that $\mathcal{E}[k, \sigma_b]$ is anti-degradable by observing that it satisfies the condition (2.53) with $\overline{\mathcal{T}}$ being the weakly complementary $\tilde{\mathcal{E}}[k'', \sigma_b]$ of a BS channel of transmissivity $k''^2 = (1-2k^2)/(1-k^2) \in [0,1]$, i.e.,

$$\mathcal{E}[k, \sigma_b] = \tilde{\mathcal{E}}[k'', \sigma_b] \circ \tilde{\mathcal{E}}[k, \sigma_b]\ . \qquad (3.79)$$

The proof is again obtained through Eqs. (3.64) and (3.66) by showing that the transformations on a generic $\chi(\mu)$ induced by $\tilde{\mathcal{E}}[k'', \sigma_b] \circ \tilde{\mathcal{E}}[k, \sigma_b]$ and by $\mathcal{E}[k, \sigma_b]$ coincide. Indeed, one has

$$\begin{aligned}
&\chi(\sqrt{(1-k^2)(1-k''^2)}\mu)\ \xi(\sqrt{(1-k''^2)k^2}\mu)\ \xi(k''\mu) \\
&\quad = \chi(\sqrt{(1-k^2)(1-k''^2)}\mu)\ \xi(\sqrt{(1-k''^2)k^2+k''^2}\mu) \\
&\quad = \chi(k\mu)\ \xi(\sqrt{1-k^2}\mu). \qquad (3.80)
\end{aligned}$$

Quantum Capacity

Here we analyze the quantum capacity of weakly degradable BS/amplifier channels, $\mathcal{E}[k, \sigma_b]$ [45, 55]. This will enable us to find BS/amplifier channels with null capacity. Let the input state ρ_a of the system A and σ_b of the environment be a thermal state with an average photon number N and N_0, respectively. Then the output state of $\mathcal{E}[k, \sigma_b](\rho_a)$ is again a thermal state with N replaced by

$$N' = k^2 N + \max\{0, (k^2-1)\} + |k^2-1| N_0\ , \qquad (3.81)$$

where k^2 is the attenuation/gain coefficient for the beam-splitter/amplifier channel. Since the coherent information (see Sec. 2.2.6) [45]

$$J(\rho, \mathcal{E}[k, \sigma_b]) = g(N') - g\left(\frac{D + N' - N - 1}{2}\right) - g\left(\frac{D - N' + N - 1}{2}\right)$$

increases with the input power N, we obtain a lower bound $Q_G(\mathcal{E}[k, \sigma_b])$ for the quantum capacity $Q(\mathcal{E}[k, \sigma_b])$ in the infinite power limit, i.e.

$$\begin{aligned} Q(\mathcal{E}[k, \sigma_b]) \geq Q_G(\mathcal{E}[k, \sigma_b]) &= \lim_{N \to \infty} J(\rho, \mathcal{E}[k, \sigma_b]) \\ &= \log k^2 - \log |k^2 - 1| - g(N_E) \end{aligned} \tag{3.82}$$

where $D = \sqrt{(N + N' + 1)^2 - 4k^2 N(N+1)}$ and $g(x) = (x+1)\log(x+1) - x \log x$ (see Fig. 3.8).

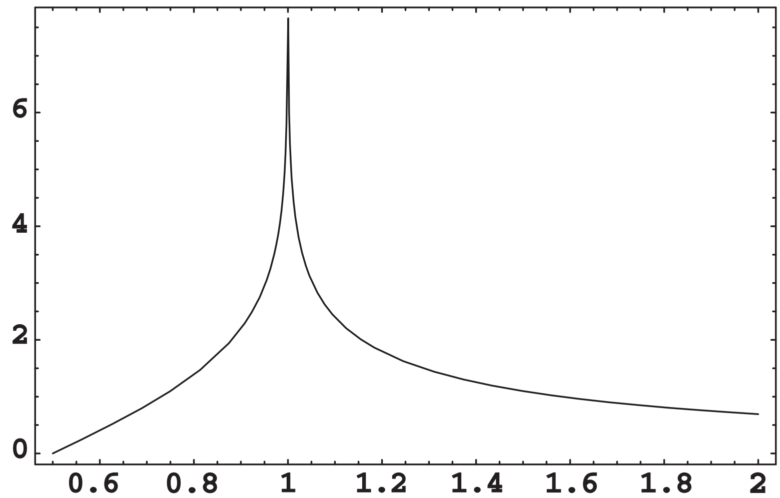

Figure 3.8: Quantum capacity of degradable BS/amplifier channels as a function of k^2, i.e. $Q = \log \frac{k^2}{|k^2-1|}$, with the environmental average photon number being initially null, i.e. $N_0 = 0$.

In Ref. [55] it was shown that the quantum capacity of degradable Gaussian Bosonic channels can be calculated explicitly by showing that Gaussian encodings (i.e., using Gaussian input states) are optimal. In other words, the lower bound in Eq. (3.82) coincides exactly with the quantum capacity of these degradable BS/amplifier channels, i.e. $Q(\mathcal{E}[k,\sigma_b]) = Q_G(\mathcal{E}[k,\sigma_b])$ with $N_0 = 0$ (Stinespring representation, see Sec. 2.2.2).

Besides, starting from Eq. (3.82), it is possible to find out how the input mean photon number of the environment N_0 depends on the factor k^2 for a generic BS/amplifier $\mathcal{E}[k,\sigma_b]$, that has $Q_G(\mathcal{E}[k,\sigma_b]) = 0$.

$$Q_G(\mathcal{E}[k,\sigma_b]) = 0 \quad \longrightarrow \quad k^2 = \frac{\exp g(N_0)}{\exp g(N_0) \mp 1} \tag{3.83}$$

where the sign in the denominator is $-$ or $+$ according to the fact that $k \geq 1$ or $k \leq 1$, respectively. Note in Fig. 3.9 that, when k becomes closer and closer to 1 (i.e., $\mathcal{E}[k,\sigma_b] \to$ identity map), one needs to put more and more noise in the channel (through N_0) in order to have $Q_G = 0$. For $k = 1$ one has, trivially, a noiseless quantum channel.

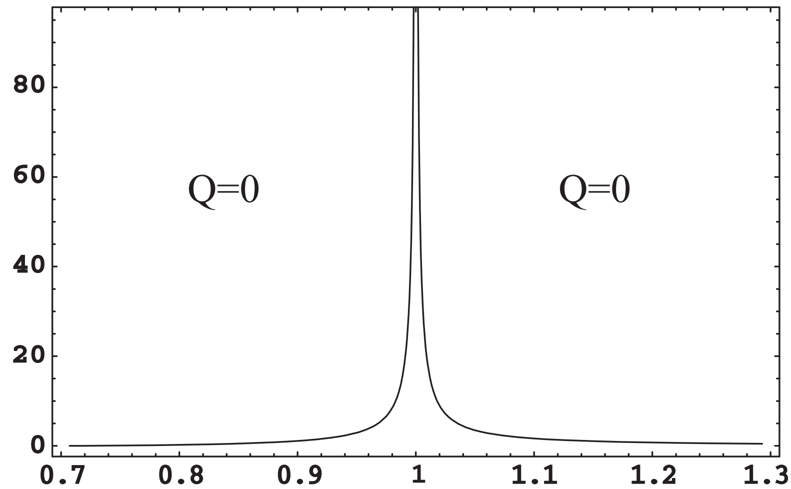

Figure 3.9: The input mean photon number of the environment N_0 as a function of the factor k, for a generic BS/amplifier with $Q(\mathcal{E}[k,\sigma_b]) = 0$. Above this lower bound, Q is always vanishing.

More generally, in the following we will use the composition rules examined above in order to study the relation between the noise (N_0) and the parameter k^2 for BS/amplifers with null quantum capacity, i.e. $Q = 0$. These results will be derived in a more powerful and compact way in Sec. 3.3.5.

Let us consider the composite amplifier + BS map and use the nice property that, if you compose a generic channel with an anti-degradable one (for which $Q = 0$), the global quantum capacity is always zero, of course. It is rigourously proved by using the so-called quantum data processing inequality [1]. Particularly, we consider a BS with transmissivity $k_2 = 1/\sqrt{2}$ (anti-degradable) and with a Gaussian environment in a pure state, i.e. $\gamma_2 = \mathbb{1}$. For simplicity, assume $\gamma_1 = \mathbb{1}$ also for the amplifier map.

- BS-like channels

  In this case, the equivalent BS has an environmental covariance matrix γ as in Eq. (3.74) and assuming $\gamma = (2N_0 + 1)\mathbb{1}$, it is possible to connect N_0 (noise parameter) and k for these zero capacity channels as

$$N_0 = \frac{2k^2 - 1}{2(1 - k^2)} \,. \tag{3.84}$$

  This relation is reported in Fig. 3.14, in which it is obtained in a more general approach, and it is a lower bound for the null quantum capacity region in the case of $k < 1$ (compare also with Eq. (3.154)).

  Now, let us suppose to know one point (BS-like) in the plane in Fig. 3.10, for example $\{\bar{k}^2 = 0.95, \bar{N}_0 = 10\}$, in which the quantum capacity Q is zero. Let us construct a family of the parametric curves (starting from $\{\bar{k}^2, \bar{N}_0\}$) such that their points correspond to BS/amplifier channels with null capacity. This result is important because it enables one to know 'a priori' if the quantum capacity of a BS/amplifier is zero, by knowing only k^2 and N_0, by assuming that the environmental input state is a thermal state. In order to achieve it, starting from a BS with parameters $\{\bar{k}^2, \bar{N}_0\}$, we add in 'series' an amplifier or a BS, in order to move in the 'direction' of channels with larger or smaller k'^2, i.e. more or less attenuating maps. However, the new noise parameter N_0' cannot change in an arbitrary way but following the composition rules above.

  First of all, let us consider how to obtain channels with smaller k'^2 and with null quantum capacity. Add another BS with transmissivity $\tilde{k}^2$ and $\gamma = (2\tilde{N}_0 + 1)\mathbb{1}$ (thermal state) and, as shown above, the resulting map will be another BS with the following relation between k' $(= \tilde{k}\bar{k} < \bar{k})$ and N_0':

$$N_0' = N_0 + \frac{1 - \tilde{k}^2}{1 - k'^2}(\tilde{N}_0 - N_0) \tag{3.85}$$

  This relation describes a family of zero-capacity curves as a function of $\tilde{k}^2$ and $\tilde{N}_0$ (reported on the right panel in Fig. 3.10). Equivalently, in order to 'move' towards channels with larger k' and with $Q = 0$, we add an amplifier with gain $\tilde{k}^2$ and $\gamma = (2\tilde{N}_0 + 1)\mathbb{1}$. In this way, k' ($= \tilde{k}\bar{k} > \bar{k}$) and N_0' are related as

$$N_0' = \bar{N}_0 + \frac{\tilde{k}^2 - 1}{1 - k'^2}(\tilde{N}_0 + \bar{N}_0 + 1) \tag{3.86}$$

and the relative family of zero-capacity curves, corresponding to different choices of $\tilde{k}^2$ and $\tilde{N}_0$, is shown on the left panel in Fig. 3.10.

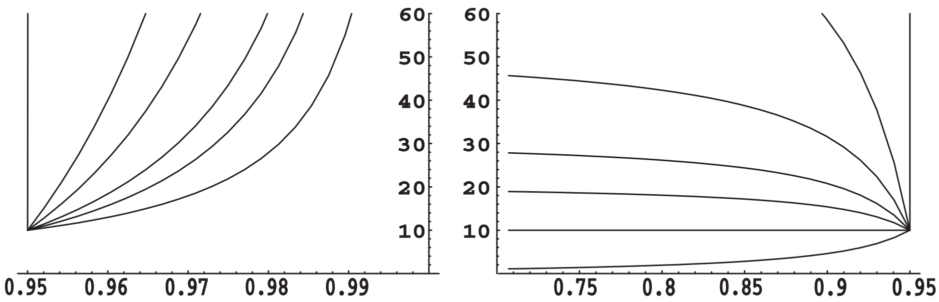

Figure 3.10: A family of curves N_0 vs. k^2, associated to BS-like channels with $Q = 0$ and corresponding to different $\tilde{k}^2$ and $\tilde{N}_0$, is shown in the cases of $k' < \bar{k}$ (right panel) and $k' > \bar{k}$ (left panel), as in Eqs. (3.85, 3.86), respectively. The starting point of these parametric curves is $\{\bar{k}^2 = 0.95, \bar{N}_0 = 10\}$.

- Amplifier-like channels

  If, instead, $k > 1$, from $\gamma = N_0 + \frac{1}{2}$ with γ as in Eq. (3.75), one obtains

$$N_0 = \frac{1}{2(k^2 - 1)} \tag{3.87}$$

  and this relation is reported again in Fig. 3.14 and it is associated to a null quantum capacity region for $k > 1$; compare also with Eq. (3.154). Again, suppose to know one point (amplifier-like) in Fig. 3.11, for example $\{\bar{k}^2 = 1.05, \bar{N}_0 = 10\}$, in which $Q = 0$. By using the same procedure as above, we find a similar family of the parametric curves with $Q = 0$ but, as in Eq. (3.85) for $k' > \bar{k}$, and as in Eq. (3.86) for $k' < \bar{k}$ (see Fig. 3.11).

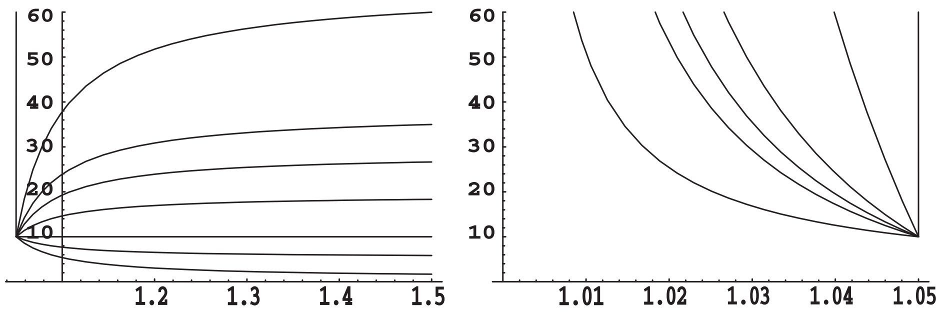

Figure 3.11: A family of curves N_0 vs. k^2, associated to amplifier-like channels with $Q = 0$ and corresponding to different $\tilde{k}^2$ and $\tilde{N}_0$, is shown in the cases of $k' < \bar{k}$ (right panel) and $k' > \bar{k}$ (left panel), as in Eqs. (3.86) and (3.85), respectively. The starting point is $\{\bar{k}^2 = 1.05, \bar{N}_0 = 10\}$.

3.3.2 Unitary equivalence

After analyzing the composition rules, the weak-degradability properties and the quantum capacity of BS and amplifier maps, we show that (almost) all one-mode Bosonic Gaussian channels are unitarily equivalent to beam-splitter/amplifier channels, up to squeezing transformations. Since the degradability properties are invariant under unitary transformations, applied individually to the input and the output states (see Sec. 2.3), proving these features for BS and amplifier channels turns out enough to characterize (almost) all one-mode Bosonic Gaussian channels [52].

As before, we focus on a generic one-mode Gaussian channel, which can be expressed as in Eq. (2.49) with σ_b being a (possibly mixed) Gaussian state of a *single* environmental Bosonic mode described by the annihilation operator b. We say that it admits a single-mode unitary representation. From now on, we will use the notation $\mathcal{E}(\rho_a) \equiv \mathcal{E}[U_{ab}, \sigma_b](\rho_a) = \mathrm{Tr}_b[U_{ab}(\rho_a \otimes \sigma_b)U_{ab}^\dagger]$ and $\tilde{\mathcal{E}}(\rho_a) \equiv \tilde{\mathcal{E}}[U_{ab}, \sigma_b](\rho_a) = \mathrm{Tr}_a[U_{ab}(\rho_a \otimes \sigma_b)U_{ab}^\dagger]$. In particular, U_{ab} describes a linear coupling which performs the transformation [46, 45],

$$U_{ab}\, \vec{v}\, U_{ab}^\dagger = A \cdot \vec{v}\ , \tag{3.88}$$

where $\vec{v}^T = (a, a^\dagger, b, b^\dagger)$ and A being a 4×4 complex symplectic matrix. In particular, to preserve the commutation relation among the operators a, $a^\dagger$,

b and $b^\dagger$, the matrix A satisfies the following constraints

$$\sum_{j=1}^{4} (-1)^{j+1} |A_{ij}|^2 = 1 \;, \tag{3.89}$$

for $i = 1, 3$ and

$$\begin{aligned} \sum_{j=1}^{4} (-1)^{j+1} A_{1j} A_{3\; j+(-1)^{j+1}} &= 0 \;, \\ \sum_{j=1}^{4} (-1)^{j+1} A_{1j} A_{3j}^{*} &= 0 \;. \end{aligned} \tag{3.90}$$

Almost all one-mode Gaussian channels can be expressed in this way. The only exception to this rule is represented by maps which are unitarily equivalent to additive classical noise channels (see Sec. 3.3.3) [53, 56]. Within the single-mode unitary representation of $\mathcal{E}$, the weakly complementary map in Eq. (2.50) of $\mathcal{E}$ is again a one-mode Gaussian channel [45, 93] which can be seen as a transformation which maps $\mathcal{D}(\mathcal{H}_a)$ into itself, by introducing an irrelevant isometry which exchanges a and b [41]. We will show that the weak-degradability of the Gaussian map $\mathcal{E}[U_{ab}, \sigma_b]$ with U_{ab} as in Eq. (3.88) depends only upon the real parameter,

$$q \equiv |A_{11}|^2 - |A_{12}|^2 \;, \tag{3.91}$$

with A_{11} and A_{12} being elements of the matrix A. The quantity (3.91) is an invariant of the unitary representation of the map, i.e., it depends on $\mathcal{E}$ but not on the choice of U_{ab} and σ_b. This property will be discussed in detail in the following when analyzing the canonical form of one-mode BGCs [52, 53, 56] (see also Ref. [48]). Notice that in the notation of generic one-mode BGCs in Eq. (3.46) q is directly related to X as $q = \mathrm{Det}[X]$. Without loss of generality, as previously discussed above, we assume σ_b to have null first order momentum. Another important simplification arises by considering the one-parameter family of unitaries $U_{ab}^{(k)}$ (3.88) associated with beam-splitter (BS) and amplifier transformations (examined in Sec. 3.3.1).

For $k \in [0,1]$ they are characterized by the matrix

$$A^{(k)} = \begin{pmatrix} k & 0 & -\sqrt{1-k^2} & 0 \\ 0 & k & 0 & -\sqrt{1-k^2} \\ \sqrt{1-k^2} & 0 & k & 0 \\ 0 & \sqrt{1-k^2} & 0 & k \end{pmatrix}, \tag{3.92}$$

which describes superposition of the modes a and b at the output of a beam-splitter of transmissivity k^2. For $k \geqslant 1$ instead the $U_{ab}^{(k)}$ are characterized by the matrix

$$A^{(k)} = \begin{pmatrix} k & 0 & 0 & -\sqrt{k^2-1} \\ 0 & k & -\sqrt{k^2-1} & 0 \\ 0 & -\sqrt{k^2-1} & k & 0 \\ -\sqrt{k^2-1} & 0 & 0 & k \end{pmatrix}, \tag{3.93}$$

which defines an amplification of a with gain parameter k^2. Notice that in both cases Eq. (3.91) yields

$$|A_{11}^{(k)}|^2 - |A_{12}^{(k)}|^2 = k^2 , \tag{3.94}$$

and $k^2 \equiv q$ according to Eq. (3.91).

Therefore, it is possible to demonstrate that the map $\mathcal{E}[U_{ab}, \sigma_b]$ is weakly degradable for $q \geqslant 1/2$ and anti-degradable for $q \leqslant 1/2$ (see Table 3.1). As discussed below, the BS/amplifier maps (examined in Sec. 3.3.1) $\mathcal{E}[k, \sigma_b] \equiv \mathcal{E}[U_{ab}^{(k)}, \sigma_b]$ and their weakly conjugates $\tilde{\mathcal{E}}[k, \sigma_b] \equiv \tilde{\mathcal{E}}[U_{ab}^{(k)}, \sigma_b]$ can be used to express a generic one-mode Gaussian channel via proper unitary transformations, with some remarkable exceptions in the case of $q = 0, 1$ [56]. We can, therefore, prove the weak-degradability or anti-degradability property of one-mode Gaussian maps by focusing only on the subset $\mathcal{E}[k, \sigma_b]$. Consider, in fact, a generic Gaussian map of the form $\mathcal{E}[U_{ab}, \sigma_b]$ with the real parameter q of Eq. (3.91) being positive and $\neq 1$. According to Eq. (3.106) we can write,

$$\mathcal{E}[U_{ab}, \sigma_b](\rho_a) = S_a \left(\mathcal{E}[k = q, \sigma_b'](\rho_a) \right) S_a^\dagger , \tag{3.95}$$

with $\sigma_b' \equiv S_b' \sigma_b S_b'^\dagger$ and S_a, S_b' being, respectively, unitary squeezing operators of a and b which depend on A but not on the input state ρ_a. Since squeezed

Value of q	Equivalent map	
$q < 0$	$\tilde{\mathcal{E}}[1-q, \sigma_b']$ conjugate amplifier	Anti-degradable ($Q = 0$)
$0 < q \leqslant 1/2$	$\mathcal{E}[q, \sigma_b']$ BS of transmissivity q	Anti-degradable ($Q = 0$)
$1/2 \leqslant q < 1$	BS of transmissivity q	
	$\mathcal{E}[q, \sigma_b']$	Weakly degradable
$1 < q$	amplifier	(degradable for σ_b' pure)

Table 3.1: Weak-degradability and anti-degradability conditions for the one-mode Bosonic channel $\mathcal{E}[U_{ab}, \sigma_b]$. In the first column we report the value of the characteristic parameter q of Eq. (3.91). In the second column we report the BS or amplifier map which, according to Eqs. (3.95) and (3.96), is unitarily equivalent to $\mathcal{E}$ (σ_b' are Gaussian states obtained by properly squeezing σ_b). Although for $q = 0$ and $q = 1$ the equivalent BS or amplifier map not always exists [56], still one can show that these maps are respectively anti-degradable and weakly degradable. Channels which are anti-degradable have null quantum capacity. Those which are weakly degradable with σ_b pure (i.e., degradable) have instead additive coherent information. The case $q = 1/2$ is an example of a channel which satisfies both the weak-degradability (2.52) and the anti-degradability (2.53) condition. This is a consequence of the symmetry of the fields emerging from the opposite output ports of a balanced 50/50 beam-splitter.

thermal states are Gaussian, the above expression shows that any Gaussian channel $\mathcal{E}[U_{ab}, \sigma_b]$ is unitarily equivalent to an amplifier channel for $q > 1$ and to a BS channel for $q \in]0, 1[$. This implies that $\mathcal{E}[U_{ab}, \sigma_b]$ is anti-degradable for $q \in]0, 1/2]$ and weakly degradable for $q \geqslant 1/2$ and $q \neq 1$. Consider now the case of maps with q of Eq. (3.91) being negative. Here Eq. (3.95) is replaced by

$$\mathcal{E}[U_{ab}, \sigma_b](\rho_a) = S_a \left(\tilde{\mathcal{E}}[1-q, \sigma_b'](\rho_a) \right) S_a^\dagger , \tag{3.96}$$

where, again, S_a and σ_b' are, respectively, a squeezing operator and a Gaussian state [in writing Eq. (3.96) an isometry $a \leftrightarrow b$ is implicitly assumed]. Since $1 - q > 1$, Eq. (3.96) shows that $\mathcal{E}$ is unitarily equivalent to the weakly conjugate map of the amplifier channel $\mathcal{E}[1 - q, \sigma_b']$. This is equivalent to say that $\mathcal{E}[U_{ab}, \sigma_b]$ with negative q are always anti-degradable. Finally, for $q = 0$ and $q = 1$ the channel is, respectively, anti-degradable and weakly degradable. The analysis of these maps is slightly more complex since it is not always possible to describe them in terms of BS/amplifier channels, as shown in Sec. 3.3.3 [56, 53].

Decomposition rules

Here we give an explicit derivation of the decomposition rules (3.95) and (3.96) which allow us to express any generic one-mode Gaussian map with $q \neq 0, 1$ in terms of BS or amplifier channels (studied in detail in Sec. 3.3.1). Remind that we are considering composition of channels in series, i.e. the output state of a quantum channel becomes the input of the next channel (see Sec. 2.2.1). For the sake of clarity we will analyze separately the cases $q \in]0, 1[$, $q > 1$ and $q < 0$ [52].

- Maps with $q \in]0, 1[$

  Consider first the case of one-mode Gaussian channel of the form $\mathcal{E}[U_{ab}, \sigma_b]$ with the real parameter q of Eq. (3.91) being positive and smaller than 1. Under this condition, apart from redefining the phases of a and b, the elements A_{1j} of the matrix (3.88) can be parametrized as follows

$$
\begin{aligned}
A_{11} &= \sqrt{q}\ \cosh r\ , \\
A_{12} &= \sqrt{q}\ e^{i\varphi} \sinh r\ , \\
A_{13} &= -\sqrt{1-q}\ \cosh s\ , \\
A_{14} &= -\sqrt{1-q}\ e^{i\psi} \sinh s\ ,
\end{aligned}
\tag{3.97}
$$

  where r, s, φ, and ψ are real quantities and where the last two expressions come from the constraint (3.89). Let us then introduce the (unitary) squeezing transformations [61]:

$$
\begin{aligned}
S_a(r;\varphi)\ a\ S_a^\dagger(r;\varphi) &= a\ \cosh r + e^{i\varphi}\ a^\dagger\ \sinh r\ , \\
S_b(s;\psi)\ b\ S_b^\dagger(s;\psi) &= b\ \cosh s + e^{i\psi}\ b^\dagger\ \sinh s\ .
\end{aligned}
\tag{3.98}
$$

On one hand, they allow us to write

$$(S_a^\dagger \otimes S_b^\dagger)\; a' \;(S_a \otimes S_b) \;=\; \sqrt{q}\, a - \sqrt{1-q}\, b = U_{ab}^{(q)}\; a\; \left[U_{ab}^{(q)}\right]^\dagger , \quad (3.99)$$

where $U_{ab}^{(q)}$ is the BS transformation defined as in Eq. (3.92) while $a' = U_{ab}\; a\; U_{ab}^\dagger$ represents the evolution of a under the unitary U_{ab} of $\mathcal{E}[U_{ab}, \sigma_b]$. On the other hand, we get

$$(S_a^\dagger \otimes S_b^\dagger)\; b' \;(S_a \otimes S_b) \;=\; \overline{A}_{21} a + \overline{A}_{22} a^\dagger + \overline{A}_{23} b + \overline{A}_{24} b^\dagger 1 , \quad (3.100)$$

with $b' = U_{ab}\; b\; U_{ab}^\dagger$ and with $\overline{A}_{2j}$ being complex parameters which satisfies the symplectic conditions analogous to those of Eqs. (3.89) and (3.90), i.e.,

$$\begin{aligned} |\overline{A}_{21}|^2 - |\overline{A}_{22}|^2 + |\overline{A}_{23}|^2 - |\overline{A}_{24}|^2 &= 1 , \\ \sqrt{q}\, \overline{A}_{21} - \sqrt{1-q}\, \overline{A}_{23} &= 0 , \\ \sqrt{q}\, \overline{A}_{22} - \sqrt{1-q}\, \overline{A}_{24} &= 0 . \end{aligned} \quad (3.101)$$

Equation (3.100) can be cast in a more compact form by properly parameterizing the $\overline{A}_{2j}$;

$$\begin{aligned} \overline{A}_{21} &= \sqrt{1-q}\, \cosh(t)\, e^{i\phi} , \\ \overline{A}_{22} &= \sqrt{1-q}\, \sinh(t)\, e^{i\phi'} , \\ \overline{A}_{23} &= \sqrt{q}\, \cosh(t)\, e^{i\phi} , \\ \overline{A}_{24} &= \sqrt{q}\, \sinh(t)\, e^{i\phi'} , \end{aligned} \quad (3.102)$$

with t, ϕ, and ϕ' real. This yields

$$(S_a^\dagger \otimes S_b^\dagger)\; b' \;(S_a \otimes S_b) = e^{i\phi}\, U_{ab}^{(q)} \left(S_b'\; b\; S_b'^\dagger \right) \left[U_{ab}^{(q)}\right]^\dagger , \quad (3.103)$$

where $S_b' \equiv S_b(t, \phi' - \phi)$ is a squeezing operator (3.100) acting on b and where $U_{ab}^{(q)}$ is the BS unitary coupling of Eq. (3.99). By absorbing the phase ϕ into the definition of b' and by noticing that S_b' does not affect a, Eqs. (3.99), (3.103), and (3.88) give

$$U_{ab}\, \vec{v}\, U_{ab}^\dagger = (S_a \otimes S_b)\, U_{ab}^{(q)} S_b'\, \vec{v}\, S_b'^\dagger\, [U_{ab}^{(q)}]^\dagger\, (S_a \otimes S_b)^\dagger , \quad (3.104)$$

which enables us to decompose U_{ab} as the following product:

$$U_{ab} = (S_a \otimes S_b)\, U_{ab}^{(q)} S_b' \;. \tag{3.105}$$

Replacing this into Eq. (2.49) we finally get

$$\begin{aligned}
\mathcal{E}[U_{ab},\sigma_b](\rho_a) &= S_a \mathrm{Tr}_b[S_b U_{ab}^{(q)}(\rho_a\otimes\sigma_b')[U_{ab}^{(q)}]^\dagger S_b^\dagger] S_a^\dagger \\
&= S_a \mathrm{Tr}_b[U_{ab}^{(q)}(\rho_a\otimes\sigma_b')[U_{ab}^{(q)}]^\dagger] S_a^\dagger \\
&= S_a \;\big(\; \mathcal{E}[k=q,\sigma_b'](\rho_a) \;\big)\; S_a^\dagger \;.
\end{aligned} \tag{3.106}$$

In this expression the S_a was brought out of the trace since it is acting on a. Vice versa, S_b has been simplified by exploiting the invariance of the trace under unitary transformation. Finally, the Gaussian state σ_b' is the squeezed version under S_b' of the environmental state σ_b, i.e.,

$$\sigma_b' \equiv S_b' \sigma_b {S_b'}^\dagger \;. \tag{3.107}$$

Equation (3.106) shows that, for $q \in]0,1[$ the map $\mathcal{E}[U_{ab},\sigma_b]$ is unitary equivalent to the BS channel $\mathcal{E}[k=q,\sigma_b']$.

- Maps with $q > 1$

  For q greater than one Eqs. (3.105) and (3.106) still hold: the only difference being that now $U_{ab}^{(q)}$ represents an amplifier map defined by the matrix of Eq. (3.93). This can be shown following the same derivation of the case $q \in]0,1[$ by replacing the parameterizations (3.97) and (3.102) with

$$\begin{aligned}
A_{11} &= \sqrt{q}\;\cosh r \;, \\
A_{12} &= \sqrt{q}\; e^{i\varphi} \sinh r \;, \\
A_{13} &= -\sqrt{q-1}\; e^{-i\psi} \sinh s \;, \\
A_{14} &= -\sqrt{q-1}\;\cosh s \;,
\end{aligned} \tag{3.108}$$

and

$$\begin{aligned}
\overline{A}_{21} &= -\sqrt{q-1}\;\sinh(t)\; e^{i\phi'} \;, \\
\overline{A}_{22} &= -\sqrt{q-1}\;\cosh(t)\; e^{i\phi} \;, \\
\overline{A}_{23} &= \sqrt{q}\;\cosh(t)\; e^{i\phi} \;, \\
\overline{A}_{24} &= \sqrt{q}\;\sinh(t)\; e^{i\phi'} \;.
\end{aligned} \tag{3.109}$$

- Maps with $q < 0$

  To analyze the channels $\mathcal{E}[U_{ab}, \sigma_b]$ with q negative it is useful to introduce a isometry $\Xi_{ab} = \Xi_{ab}^\dagger$ which transforms a in b and vice versa while leaving the vacuum state invariant, i.e., $\Xi_{ab}\ a\ \Xi_{ab} = b$, $\Xi_{ab}\ b\ \Xi_{ab} = a$, and $\Xi_{ab}|\emptyset\rangle = |\emptyset\rangle$. This is a unitary transformation which for any bounded operator Θ_{ab} on $\mathcal{H}_a \otimes \mathcal{H}_b$ satisfies the identity

$$\mathrm{Tr}_b[\Xi_{ab}\Theta_{ab}\Xi_{ab}] \otimes \mathbb{1}_b = \Xi_{ab}\left(\mathbb{1}_a \otimes \mathrm{Tr}_a[\Theta_{ab}]\right)\Xi_{ab}. \tag{3.110}$$

  Consider, then, the unitary transformation $\Xi_{ab}U_{ab}$ with U_{ab} being the unitary coupling associated with $\mathcal{E}[U_{ab}, \sigma_b]$. From Eq. (3.88) it follows

$$(\Xi_{ab}\ U_{ab})\ \vec{v}\ (U_{ab}^\dagger\ \Xi_{ab}) = \tilde{A} \cdot \vec{v} \tag{3.111}$$

  where $\tilde{A}$ is a 4×4 matrix which is obtained by shifting by 2 the columns of the matrix A which describes the unitary U_{ab}, i.e., $\tilde{A}_{ij} = A_{i,j\oplus2}$ where $\oplus$ represents the sum modulus 4. From the constraint (3.89) it then follows that the coefficient (3.91) of $\tilde{A}$ is greater than 1, i.e.,

$$\begin{aligned}\tilde{q} &= |\tilde{A}_{11}|^2 - |\tilde{A}_{12}|^2 = |A_{13}|^2 - |A_{14}|^2 \\ &= 1 - (|A_{11}|^2 - |A_{12}|^2) = 1 - q > 1\ .\end{aligned} \tag{3.112}$$

  We can then use the analysis above to show that there exist squeezing transformations S_a, S_b, and S'_b which allows us to write $\Xi_{ab}\ U_{ab} = (S_a \otimes S_b)\ U_{ab}^{(\tilde{q})} S'_b$ with $U_{ab}^{(\tilde{q})}$ being an amplifier coupling (3.93). Therefore, we get

$$U_{ab} = \Xi_{ab}\ (S_a \otimes S_b)\ U_{ab}^{(\tilde{q})} S'_b\ . \tag{3.113}$$

  Exploiting the identity (3.110) this yields

$$\begin{aligned}&\mathcal{E}[U_{ab}, \sigma_b](\rho_a) \otimes \mathbb{1}_b \qquad (3.114)\\ &= \mathrm{Tr}_b[\Xi_{ab}\ (S_a \otimes S_b)\ U_{ab}^{(\tilde{q})} S'_b(\rho_a \otimes \sigma_b) S_b'^\dagger \left[U_{ab}^{(\tilde{q})}\right]^\dagger (S_a \otimes S_b)^\dagger\ \Xi_{ab}] \otimes \mathbb{1}_b \\ &= \Xi_{ab}(\mathbb{1}_a \otimes \mathrm{Tr}_a[(S_a \otimes S_b) U_{ab}^{(\tilde{q})} \times (\rho_a \otimes \sigma'_b) \left[U_{ab}^{(\tilde{q})}\right]^\dagger (S_a \otimes S_b)^\dagger])\Xi_{ab} \\ &= \Xi_{ab}(\mathbb{1}_a \otimes S_b \mathrm{Tr}_a[U_{ab}^{(\tilde{q})}(\rho_a \otimes \sigma'_b) \left[U_{ab}^{(\tilde{q})}\right]^\dagger] S_b^\dagger)\Xi_{ab} \\ &= \Xi_{ab}(\mathbb{1}_a \otimes S_b\ \tilde{\mathcal{E}}[1-q, \sigma'_b](\rho_a)\ S_b^\dagger)\Xi_{ab}\ ,\end{aligned}$$

where we used the fact that $\mathrm{Tr}_a[U_{ab}^{(\tilde{q})}(\rho_a \otimes \sigma_b') \left[U_{ab}^{(\tilde{q})}\right]^\dagger]$ is the weakly complementary channel $\tilde{\mathcal{E}}[\tilde{q}, \sigma_b']$ of an amplifier with coupling $U_{ab}^{(\tilde{q})}$ and the identity $\tilde{q} = 1 - q$. Finally, the above expression can be cast in the less formal but certainly simpler form (3.96) where the isometry Ξ_{ab} is implicitly assumed.

3.3.3 Canonical representation

From now on, we will analyze one-mode BGCs in a more general framework in terms of Weyl operators, being able to study in more details also the 'singular' cases found above (e.g., $q = 0$ or $q = 1$) in terms of six canonical classes [53]. In fact, this formalism will allow us to generalize more easily all one-mode results to the multi-mode case in Sec. 3.4. As done in Sec. 3.3.2, let us focus first on an important subset of one-mode Gaussian channels, given by the maps $\mathcal{E}$ which possess a physical representation (2.49) with σ_b being a Gaussian state of a *single* external Bosonic mode B and with U_{ab} being a canonical transformation of Q_a, P_a, Q_b and P_b (the latter being the canonical observables of the mode B). In particular let σ_b be a thermal state of average photon number N_0 as in Sec. 3.3.1, and let U_{ab} be such that

$$U_{ab}^\dagger \, (Q_a, P_a, Q_b, P_b) \, U_{ab} = (Q_a, P_a, Q_b, P_b) \cdot M \; , \tag{3.115}$$

with M being a 4×4 symplectic matrix of block form

$$M \equiv \left(\begin{array}{c|c} m_{11} & m_{21} \\ \hline m_{12} & m_{22} \end{array} \right) \; . \tag{3.116}$$

This yields the following evolution for the characteristic function $\phi(z)$,

$$\begin{aligned} \phi'(z) &= \mathrm{Tr}_a[\mathcal{E}(\rho_a) \, V_a(z)] = \mathrm{Tr}_a[\rho_a \, \mathcal{E}_H(V_a(z))] \\ &= \mathrm{Tr}_{ab}\left[U_{ab}^\dagger \, (V_a(z) \otimes \mathbb{1}) U_{ab} \, (\rho_a \otimes \sigma_b)\right] \\ &= \mathrm{Tr}_{ab}\left[\big(V_a(m_{11} \cdot z) \otimes V_b(m_{12} \cdot z)\big) \, (\rho_a \otimes \sigma_b)\right] \\ &= \phi(m_{11} \cdot z) \; \exp[-(2N_0 + 1)|m_{12} \cdot z|^2/4] \; , \end{aligned} \tag{3.117}$$

which is of the form (3.48) by choosing $m = 0$, $X = m_{11}$ and $Y = (2N_0 + 1) \; m_{12}^T \cdot m_{12}$. It is worth stressing that in the case of Eq. (3.117) the CPT condition in Eq. (3.58) is guaranteed by the symplectic nature of the matrix

M, i.e. by the fact that Eq. (3.115) preserves the commutation relations among the canonical operators. Indeed, we have

$$\begin{aligned} \mathrm{Det}[Y] &= (2N_0+1)^2\ \mathrm{Det}[m_{12}]^2 = (2N_0+1)^2\ (\mathrm{Det}[m_{11}]-1)^2 \\ &\geqslant (\mathrm{Det}[m_{11}]-1)^2 = (\mathrm{Det}[X]-1)^2\ , \end{aligned} \tag{3.118}$$

where in the second identity the condition (3.126) was used.

As already shown in terms of beam-splitter and amplifier maps in Sec. 3.3.2, with certain important exception, one-mode Gaussian channels (3.46) are unitarily equivalent to transformations which admit physical representation with σ_b thermal state and U_{ab} as in Eq. (3.115). In the next sections, this unitary equivalence, in Eq. (3.95), will be described more generally in the symplectic formalism, where symplectic matrices take the place of the particular squeezing transformations used above, in order to obtain a canonical representation of all one-mode BGCs.

Canonical form

Following Ref. [56] any Gaussian channel (3.48) can be transformed (through unitarily equivalence as in Eq. (3.55)) into a simple canonical form. Namely, given a channel $\mathcal{E}$ characterized by the vector v and the matrices X, Y of Eq. (3.48), one can find unitary operators U_a and W_a such that the channel defined by the mapping

$$\rho_a \longrightarrow \mathcal{E}^{(\mathrm{can})}(\rho_a) = W_a\ \mathcal{E}(U_a\ \rho_a\ U_a^\dagger)\ W_a^\dagger \qquad \text{for all } \rho_a, \tag{3.119}$$

is still of the form (3.48) but with $v=0$ and with X, Y replaced, respectively, by the matrices X_{can}, Y_{can} of Table 3.2, i.e.

$$\phi'(z) = \phi(X_{\mathrm{can}}\cdot z)\ \exp\left[-\tfrac{1}{4}\ z^T\cdot Y_{\mathrm{can}}\cdot z\right]\ . \tag{3.120}$$

An important consequence of Eq. (3.120) is that, to analyze the weak-degradability properties of a one-mode Gaussian channel, it is sufficient to focus on the canonical map $\mathcal{E}^{(\mathrm{can})}$ which is unitarily equivalent to it (see Sec. 2.3).

The dependence on the matrix X_{can} of $\mathcal{E}^{(\mathrm{can})}$ upon the parameters of $\mathcal{E}$ can be summarized as follows,

$$X_{\mathrm{can}} = \begin{cases} \begin{cases} \sqrt{\mathrm{Det}[X]}\ \mathbb{1} & \mathrm{Det}[X]\geqslant 0 \\ \sqrt{|\mathrm{Det}[X]|}\ \sigma_z & \mathrm{Det}[X]<0 \end{cases} & \mathrm{rank}[X]\neq 1 \\ \\ \qquad (\mathbb{1}+\sigma_z)/2 & \mathrm{rank}[X]=1\ , \end{cases} \tag{3.121}$$

Channel $\mathcal{E}$ Det$[X]$		Class	$\mathcal{E}^{(\mathrm{can})}$ X_{can}	Y_{can}
0	$\mathrm{rank}[X]=0$	A_1	0	$(2N_0+1)\ \mathbb{1}$
0	$\mathrm{rank}[X]=1$	A_2	$(\mathbb{1}+\sigma_z)/2$	$(2N_0+1)\ \mathbb{1}$
1	$\mathrm{rank}[Y]=1$	B_1	$\mathbb{1}$	$(\mathbb{1}-\sigma_z)/2$
1	$\mathrm{rank}[Y]\neq 1$	B_2	$\mathbb{1}$	$N_0\ \mathbb{1}$
$k^2\quad (k\neq 0,1)$		C	$k\ \mathbb{1}$	$\lvert k^2-1\rvert(2N_0+1)\ \mathbb{1}$
$-k^2\quad (k\neq 0)$		D	$k\ \sigma_z$	$(k^2+1)(2N_0+1)\ \mathbb{1}$

Table 3.2: Canonical form for one-mode Gaussian Bosonic channels. In the first columns the properties of X and Y of the map $\mathcal{E}$ are reported. In last two columns instead we give the matrices X_{can} and Y_{can} of the canonical form $\mathcal{E}^{(\mathrm{can})}$ associated with $\mathcal{E}$ — see Eqs. (3.119) and (3.120). In these expressions σ_z is the third Pauli matrix, N_0 is a non-negative constant and k is a positive constant. Notice that the constraint (3.58) is always satisfied. In B_1 the free parameter N_c has been set equal to 1/2 — see discussion below Eq. (3.122).

with σ_z being the third Pauli matrix. Analogously for Y_{can} we have

$$Y_{\mathrm{can}} = \begin{cases} \sqrt{\mathrm{Det}[Y]}\ \mathbb{1} & \mathrm{rank}[Y]\neq 1 \\ \\ N_c\ (\mathbb{1}-\sigma_z) & \mathrm{rank}[Y]=1\ . \end{cases} \tag{3.122}$$

The quantity N_c is a free parameter which can set to any positive value upon properly calibrating the unitaries U_a and W_a of Eq. (3.119). Following Ref. [56] we will assume $N_c = 1/2$. Notice also that from Eq. (3.58), $\mathrm{rank}[Y]=1$ is only possible for $\mathrm{Det}[X]=1$.

Equations (3.121) and (3.122) show that only the determinant and the rank of X and Y are relevant for defining X_{can} and Y_{can}. One can verify that X_{can} and Y_{can} maintain the same determinant and rank of the original matrices X and Y, respectively. This is a consequence of the fact the $\mathcal{E}$ and $\mathcal{E}^{(\mathrm{can})}$ are connected through a symplectic transformation for which $\mathrm{Det}[X]$, $\mathrm{Det}[Y]$, $\mathrm{rank}[X]$, and $\mathrm{rank}[Y]$ are invariant quantities.

The six inequivalent canonical forms of Table 3.2 follow by parametrizing the value of $\sqrt{\mathrm{Det}[Y]}$ to account for the constraints imposed by the inequality (3.58). It should be noticed that to determine which class a certain channel belongs to, it is only necessary to know if $\mathrm{Det}[X]$ is null, equal to 1, negative or positive ($\neq 1$). If $\mathrm{Det}[X]=0$ the class is determined by the rank

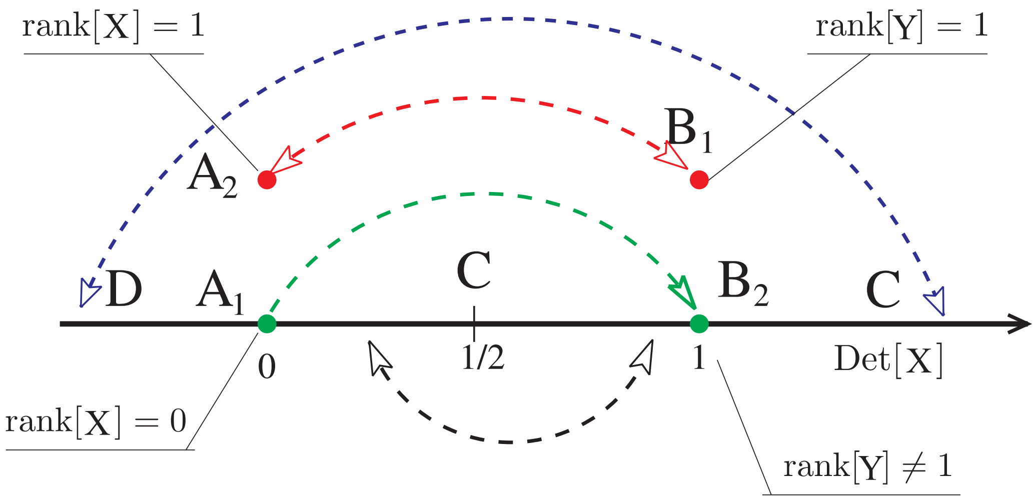

Figure 3.12: Pictorial representation of the classification in terms of canonical forms of Table 3.2. Depending on the values of Det$[X]$, rank$[X]$ and rank$[Y]$, any one-mode Gaussian channel can be transformed to one of the channels of the scheme through unitary transformations as in Eq. (3.119). The point on the thick oriented line for Det$[X] < 0$ represent the maps of D, those with Det$[X] > 0$ and Det$[X] \neq 1$ represent C. The classes $A_{1,2}$ and $B_{1,2}$ are represented by the four points of the graph. Notice that the channel B_2 and A_1 can be obtained as limiting cases of D and C. The dotted arrows connect channels which are weakly complementary (2.50) of each others with respect to the physical representations introduced in Sec. 3.3.3. For instance the weakly complementary of B_1 is channel of the class A_2 (and vice-versa) — see Sec. 3.3.4 and Table 3.3 for details. Note that the weakly complementary channel of A_1 belongs to B_2. However, not all the channels of B_2 have weakly complementary channels which are in A_1.

of the matrix. If Det$[X] = 1$ the class is determined by the rank of Y (see Fig. 3.12). Within the various classes, the specific expression of the canonical form depends then upon the effective values of Det$[X]$ and Det$[Y]$. We observe also that the class A_1 can be obtained as a limiting case (for $k \to 0$) of the maps of class C or D. Analogously the class B_2 can be obtained as a limiting case of the maps of class C. Indeed consider the channel with $X_{\rm can} = k\mathbb{1}$ and $Y_{\rm can} = |k^2-1|(2N_0'+1)\mathbb{1}$ with $N_0' = N_0/(|k^2-1|) - 1/2$, with N_0 and k positive ($k \neq 0, 1$). For k sufficiently close to 1, N_0' is positive and the maps belongs to the class C of Table 3.2. Moreover in the limit of $k \to 1$ this channel yields the map B_2. Let us notice that $\mathrm{Det}[X]$ is directly related to the invariant quantity q used above in the analysis in terms of BS/amplifiers, i.e. $\mathrm{Det}[X] \equiv q$, in Sec. 3.3.2.

Finally it is interesting to study how the canonical forms of Table 3.2 compose under the product (3.54), i.e.,

$$\begin{array}{c|cccccc} \circ & A_1 & A_2 & B_1 & B_2 & C & D \\ \hline A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 \\ A_2 & A_1 & A_2 & A_2 & A_2 & A_2 & A_2 \\ B_1 & A_1 & A_2 & B_1 & B_1/B_2 & C & D \\ B_2 & A_1 & A_2 & B_1/B_2 & B_2 & C & D \\ C & A_1 & A_2 & C & C & B_2/C & D \\ D & A_1 & A_2 & D & D & D & C \end{array} \tag{3.123}$$

In this table, for instance, the element on the row 2 and column 3 represents the class (i.e. A_2) associated to the product $\mathcal{E}'' \circ \mathcal{E}'$ between a channel $\mathcal{E}'$ of B_1 and a channel $\mathcal{E}''$ of A_2. Notice that the canonical form of the products $B_1 \circ B_2$, $B_2 \circ B_1$ and $C \circ C$ is not uniquely defined. In the first case in fact, even though the determinant of the matrix X of Eq. (3.54) is one, the rank of the corresponding Y might be one or different from one depending on the parameters of the two "factor" channels: consequently the $B_1 \circ B_2$ and $B_2 \circ B_1$ might belong either to B_1 or to B_2. In the case of $C \circ C$ instead it is possible that the resulting channel will have $\mathrm{Det}[X] = 1$ making it a B_2 map. Typically however $C \circ C$ will be a map of C. Composition rules analogous to those reported here have been extensively analyzed in Refs. [47, 52, 108].

Canonical single-mode physical representation

Apart from the case B_2 that will be treated separately (see below), all canonical transformations of Table 3.2 can be expressed as in Eq. (3.117), i.e. through a physical representation (2.49) with σ_b being a thermal state of a single external Bosonic mode B and U_{ab} being a linear transformation (3.115)[5]. To show this it is sufficient to verify that, for each of the classes of Table 3.2 but B_2, there exists a non-negative number N_0 and a

[5] The exceptional role of B_2 corresponds to the fact that any one-mode Bosonic Gaussian channel can be represented as a unitary coupling with a single-mode environment plus an additive classical noise (see below and Ref. [59]).

symplectic matrix M such that Eq. (3.117) gives the mapping (3.120). This yields the conditions

$$m_{11} = X_{\text{can}} \,, \tag{3.124}$$

$$m_{12} = O\sqrt{\frac{Y_{\text{can}}}{2N_0+1}} \,, \tag{3.125}$$

with $O^T = O^{-1}$ being an orthogonal 2×2 matrix to be determined through the symplectic condition

$$\text{Det}[m_{11}] + \text{Det}[m_{12}] = 1 \,, \tag{3.126}$$

which guarantees that $U_{ab}^\dagger Q_a U_{ab}$ and $U_{ab}^\dagger P_a U_{ab}$ satisfy canonical commutation relations. It is worth noticing that, once m_{11} and m_{12} are determined within the constraint (3.126), the remaining blocks (i.e. m_{21} and m_{22}) can always be found in order to satisfy the remaining symplectic conditions of M. An explicit example will be provided in few paragraphs. For the classes A_1, A_2, B_1, D, and C with $k<1$, Eqs. (3.125) and (3.126) can be solved by choosing $O = \mathbb{1}$. Note that for B_1 the latter setting is not necessary. Any non-negative number will do the job: thus we choose $N_0 = 0$ making the density matrix σ_b the vacuum of the B. For C with $k>1$ instead a solution is obtained by choosing $O = \sigma_z$. The corresponding transformations (3.115) for Q_a and P_a (together with the choice for the Gaussian environmental initial state) are summarized below and represent one solution of the unitary dilation problem discussed in Sec. 3.2.

Class	σ_b	$U_{ab}^\dagger\, Q_a\, U_{ab}$	$U_{ab}^\dagger\, P_a\, U_{ab}$
A_1	thermal (N_0)	Q_b	P_b
A_2	thermal (N_0)	$Q_a + Q_b$	P_b
B_1	vacuum ($N_0 = 0$)	Q_a	$P_a + P_b$
$C(k<1)$	thermal (N_0)	$k\, Q_a + \sqrt{1-k^2}\, Q_b$	$k\, P_a + \sqrt{1-k^2}\, P_b$
$C(k>1)$	thermal (N_0)	$k\, Q_a + \sqrt{k^2-1}\, Q_b$	$k\, P_a - \sqrt{k^2-1}\, P_b$
D	thermal (N_0)	$k\, Q_a + \sqrt{k^2+1}\, Q_b$	$-k\, P_a + \sqrt{k^2+1}\, P_b$.

To complete the definition of the unitary operators U_{ab} we need to provide also the transformations of Q_b and P_b. This corresponds to fixing the blocks m_{21} and m_{22} of M and cannot be done uniquely: one possible choice is presented in the following table

Class		$U_{ab}^\dagger\, Q_b\, U_{ab}$	$U_{ab}^\dagger\, P_b\, U_{ab}$
A_1		Q_a	P_a
A_2		Q_a	$P_a - P_b$
B_1		$Q_a - Q_b$	$-P_b$
C	$k<1$	$\sqrt{1-k^2}\, Q_a - k\, Q_b$	$\sqrt{1-k^2}\, P_a - k\, P_b$
C	$k>1$	$\sqrt{k^2-1}\, Q_a + k\, Q_b$	$-\sqrt{k^2-1}\, P_a + k\, P_b$
D		$\sqrt{k^2+1}\, Q_a + k\, Q_b$	$\sqrt{k^2+1}\, P_a - k\, P_b$.

The above definitions render explicit the fact that the canonical form C represents beam-splitter ($k<1$) and amplifier ($k>1$) channel (investigated in detail in Sec. 3.3.1) [45]. We will see in the following sections that the class D is formed by the weakly complementary of the amplifier channels of the class C and corresponds to the case $q<0$ in Sec. 3.3.2.

Finally it is important to note that the above physical representations are equivalent to Stinespring representations only when the average photon number N_0 of σ_b nullifies. In this case the environment B is represented by a pure input state (i.e. the vacuum). According to our definitions this is always the case for the canonical form B_1 while for the canonical forms A_1, A_2, C and D it happens for $N_0 = 0$.

For the sake of clarity, here we give the explicit expressions of the matrix M of Eq. (3.116) associated with the physical representations of the classes A_1, A_2, B_1, C and D, discussed above. They are

$$M_{A_1} \equiv \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) ,$$

$$M_{A_2} \equiv \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right) ,$$

$$M_{B_1} \equiv \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right) ,$$

$$M_C \equiv \left(\begin{array}{cc|cc} k & 0 & \sqrt{1-k^2} & 0 \\ 0 & k & 0 & \sqrt{1-k^2} \\ \hline \sqrt{1-k^2} & 0 & -k & 0 \\ 0 & \sqrt{1-k^2} & 0 & -k \end{array}\right) \qquad \text{(for } k<1\text{)},$$

$$M_C \equiv \left(\begin{array}{cc|cc} k & 0 & \sqrt{k^2-1} & 0 \\ 0 & k & 0 & -\sqrt{k^2-1} \\ \hline \sqrt{k^2-1} & 0 & k & 0 \\ 0 & -\sqrt{k^2-1} & 0 & k \end{array}\right) \qquad \text{(for } k>1\text{)},$$

$$M_D \equiv \left(\begin{array}{cc|cc} k & 0 & \sqrt{k^2+1} & 0 \\ 0 & -k & 0 & \sqrt{k^2+1} \\ \hline \sqrt{k^2+1} & 0 & k & 0 \\ 0 & \sqrt{k^2+1} & 0 & -k \end{array}\right) .$$

The class B_2: Additive classical noise channel

As mentioned above, the class B_2 of Table 3.2 must be treated separately. The map B_2 corresponds[6] to the additive classical noise channel [45], in which classical isotropic Gaussian noise is added to an otherwise lossless channel. In the framework of the Weyl operators it can be written as

$$\mathcal{E}(\rho_a) = \int d^2z \; p(z) \; V_a(z) \; \rho_a \; V_a(-z) \; , \tag{3.127}$$

with $p(z) = (2\pi N_v)^{-1} \; \exp[-|z|^2/(2N_v)]$ which, in Heisenberg picture, can be seen as a random shift of the annihilation operator a. Analogously, in terms of the complex phase space, it is defined by the one-mode CPT map acting on the Bosonic mode a, i.e. $\mathcal{E}(\rho_a) = \int d^2\mu \; p(\mu) \; D_a(\mu) \; \rho_a \; D_a^\dagger(\mu)$, where the integral is performed on the complex plain, $D_a(\mu)$ is the displacement operator of a and $p(\mu) \equiv (2\pi N_v)^{-1} \exp\left(-|\mu|^2/(2N_v)\right)$ is a Gaussian distribution with variance N_v.

These channels admit a natural physical representation which involve two environmental modes in a pure state (see Ref. [56] for details) but do not have a physical representations (2.49) involving a single environmental mode. This

[6]This can be seen for instance by evaluating the characteristic function of the state (3.127) and comparing it with Eq. (3.120).

can be verified by noticing that in this case, from Eqs. (3.124) and (3.125) we get

$$m_{11} = \mathbb{1} \,, \tag{3.128}$$
$$m_{12} = \sqrt{N_v/(N_0+1/2)}\; O \,, \tag{3.129}$$

which yields

$$\mathrm{Det}[m_{11}] + \mathrm{Det}[m_{12}] = 1 \pm N_v/(N_0+1/2) \,, \tag{3.130}$$

independently[7] of the choice of the orthogonal matrix O. Therefore, apart from the trivial case $N_v = 0$, the only solution to the constraint (3.126) is by taking the limit $N_0 \to \infty$. This would correspond to representing the channel B_2 in terms of a linear coupling with a single-mode thermal state σ_b of "infinite" temperature. Unfortunately this is not a well defined object. However, we can mention the "asymptotic" representation of B_2 as limiting case of C class maps, to claim at least that there exists a one-parameter family of one-mode Gaussian channels which admits single-mode physical representation and which converges to B_2. Indeed, the characteristic function of the output can be expressed as follows,

$$\chi'_a(\mu) \equiv \mathrm{Tr}[\mathcal{E}(\rho_a) D_a(\mu)] = \chi_a(\mu)\; \exp[-2N_v|\mu|^2] \,. \tag{3.131}$$

As discussed in [92] we can represent the transformation $\chi_a(\mu) \to \chi'_a(\mu)$ as the limit for $k \to 1$ of a BS channel $\mathcal{E}[U_{ab}^{(k)}, \sigma_b]$ with transmissivity k^2 with a thermal environment state σ_b having mean photon number $M_0 = 2N_v/(1-k^2)$, i.e.

$$\mathcal{E}(\rho_a) = \lim_{k^2 \to 1} \mathcal{E}[U_{ab}^{(k)}, \sigma_b](\rho_a) = \lim_{k^2\to 1} \mathrm{Tr}_b\Big[U_{ab}^{(k)} (\rho_a \otimes \sigma_b) U_{ab}^{(k)\dagger}\Big] \,, \tag{3.132}$$

where $U_{ab}^{(k)}$ is the unitary transformation

$$\begin{aligned} U_{ab}^{(k)\dagger}\; a\; U_{ab}^{(k)} &= k\; a + \sqrt{1-k^2}\; b \,, \\ U_{ab}^{(k)\dagger}\; b\; U_{ab}^{(k)} &= k\; b - \sqrt{1-k^2}\; a \,. \end{aligned} \tag{3.133}$$

Note that the quantity with the limit in the right hand side term of Eq. (3.132) is a physical representation of the map $\mathcal{E}[U_{ab}^{(k)}, \sigma_b]$. From it a Stinespring representation of $\mathcal{E}[U_{ab}^{(k)}, \sigma_b]$ can be constructed as follows

$$\mathcal{E}[U_{ab}^{(k)}, \sigma_b](\rho_a) = \mathrm{Tr}_{bc}\Big[U_{ab}^{(k)} \otimes \mathbb{1}_c \,\Big(\, \rho_a \otimes |\psi\rangle_{bc}\langle\psi| \,\Big)\, U_{ab}^{(k)\dagger} \otimes \mathbb{1}_c\Big] \,, \tag{3.134}$$

[7] This follows from the fact that $\mathrm{Det}[O] = \pm 1$ since $O^T = O^{-1}$.

where the trace is performed on b and on an ancillary system $\mathcal{H}_c$ and with $|\psi\rangle_{bc}$ being a purification of σ_b (see Sec. 1.1). Following Ref. [45] we choose $|\psi\rangle_{bc}$ to be a two mode Gaussian state having the following characteristic function

$$\chi_{bc}(\mu_1, \mu_2) = \mathrm{Tr}\Big[\; |\psi\rangle_{bc}\langle\psi| \; D_b(\mu_1) \; D_c(\mu_2) \; \Big]$$
$$= \exp\Big[-(M_0 + 1/2)(|\mu_1|^2 + |\mu_2|^2) - i\sqrt{M_0(M_0+1)}(\mu_1\mu_2^* - \mu_2\mu_1^*)\Big] \; .$$

Exploiting Eq. (3.133) we can compute the characteristic function of the output state: this is

$$\chi_a'(\mu) = \chi_a(k\mu) \; \chi_{bc}(\sqrt{1-k^2}\mu, 0) \; . \tag{3.135}$$

From Eq. (3.135) we have that the map $\mathcal{E}[U_{ab}^{(k)}, \sigma_b]$ yields an output state having characteristic functions of the form

$$\chi_a'(\mu) = \chi_a(k\mu) \; \exp[-(1-k^2)(M_0 + 1/2)|\mu|^2] \; , \tag{3.136}$$

where we used the fact that characteristic function of the thermal state σ_b is

$$\chi_b(\mu) = \exp[-(M_0 + 1/2)|\mu|^2] \; . \tag{3.137}$$

By replacing $M_0 = 2N_v/(1-k^2)$ into (3.136) and taking the limit $k \to 1$, it is easy to verify that this expression gives (3.131) for all input ρ_a. Let us remark that in the limit of $k \to 1$ the quantity $M_0 = 2N_v/(1-k^2)$ diverges and σ_b approaches a thermal state of infinity temperature. Even though this object and the corresponding purification (3.135) are not mathematically well defined, for all $k < 1$ the density matrix σ_b and its purification $|\psi\rangle_{bc}$ are legitimate states of the systems b and c [56].

As shown in Ref. [56], let us compute the coherent information J (see Sec. 2.2.6) at the output of the channel $\mathcal{E}[U_{ab}^{(k)}, \sigma_b]$ and its conjugate $\tilde{\mathcal{E}}[U_{ab}^{(k)}, \sigma_b]$ for a Gaussian input state ρ_a having characteristic function

$$\chi_a(\mu) = \exp[-(N + 1/2)|\mu|^2] \; . \tag{3.138}$$

The analysis simplifies since, for Stinespring conjugate channels, we have the following relation of the coherent information of complementary channels

$$J(\rho, \mathcal{E}[U_{ab}^{(k)}, \sigma_b]) = -J(\rho, \tilde{\mathcal{E}}[U_{ab}^{(k)}, \sigma_b]) \; . \tag{3.139}$$

Furthermore, if ρ is Gaussian, we can use the results of [45] (see also Sec. 3.3.1) to express the coherent information $J(\rho, \mathcal{E}[U_{ab}^{(k)}, \sigma_b])$ as follows

$$J(\rho, \mathcal{E}[U_{ab}^{(k)}, \sigma_b]) = g(N') - g\Big(\frac{D + N' - N - 1}{2}\Big) - g\Big(\frac{D - N' + N - 1}{2}\Big) \tag{3.140}$$

where $g(x) = (x+1)\log(x+1) - x\log x$ and

$$N' = k^2 N + (1-k^2)M_0 = k^2 N + 2N_v \;, \tag{3.141}$$

$$D = \sqrt{(N + N' + 1)^2 - 4k^2 N(N+1)} \,. \tag{3.142}$$

Notice that, even though for $k \to 1$ the quantity $M_0 = 2N_v/(1-k^2)$ diverges, Eq. (3.140) does not. In particular for $k = 1$ we get

$$N'|_{k=1} = N + 2N_v \;, \tag{3.143}$$

$$D|_{k=1} = D_1 \equiv \sqrt{(2N_v + 1)^2 + 8N_v N} \;, \tag{3.144}$$

and

$$J(\rho, \mathcal{E}[U_{ab}^{(k)}, \sigma_b])\Big|_{k=1} = g(N + 2N_v) - g\Big(\frac{D_1 + 2N_v - 1}{2}\Big) \tag{3.145}$$

$$-g\Big(\frac{D_1 - 2N_v - 1}{2}\Big) \,. \tag{3.146}$$

As pointed out in Ref. [56], in the limit of $N >> 1$, the function (3.146) becomes

$$\lim_{N\to\infty} J(\rho, \mathcal{E}[U_{ab}^{(k)}, \sigma_b])\Big|_{k=1} \simeq \log\left(\frac{1}{2eN_v}\right) \,, \tag{3.147}$$

which is positive for $N_v < 1/(2e)$. Indeed, by choosing $N_v = 0.99/(2e)$, the function $J(\rho, \mathcal{E}[U_{ab}^{(k)}, \sigma_b])\Big|_{k=1}$ becomes negative for N close to 0 and so it is positive for $N >> 1$. Note that analogous results apply for other elements of the family of maps $\mathcal{E}[U_{ab}^{(k)}, \sigma_b]$. Indeed, consider the limit $N >> 1$ of (3.140) for arbitrary $k < 1$. This yields

$$\lim_{N\to\infty} J(\rho, \mathcal{E}[U_{ab}^{(k)}, \sigma_b]) \simeq \log\left(\frac{k^2}{1-k^2}\right) - g\Big(\frac{2N_v}{1-k^2}\Big) \,. \tag{3.148}$$

Since for $x >> 1$ one has $g(x) \sim \log(ex)$, in the limit $k \to 1$ the above equation reduces to (3.147). As in the case $k = 1$, for $k^2 > 1/2$ there are values of $2N_v$ such that Eq. (3.148) is positive. On the other hand for the same values it is possible to have $J < 0$ for sufficiently small N (for instance $N_v = 1/10$ and $k^2 = 9/10$). In other words, using Eq. (3.139), both the channel (3.127) and its Stinespring conjugate have positive coherent information for some input states and, then, positive quantum capacities.

Therefore, the additive classical noise channel is a counterexample to the claim (shown in Sec. 3.3.2 and proved more generally in Sec. 3.3.4) that all one-mode Gaussian channel are either weakly degradable or anti-degradable, i.e. it is neither degradable nor anti-degradable.

3.3.4 Full weak-degradability classification

In the previous section we have shown that all one-mode Gaussian channels are unitarily equivalent to one of the canonical forms of Table 3.2. Moreover we have verified that, with the exception of the class B_2, all the canonical forms admits a physical representation (2.49) with σ_b being a thermal state of a single environmental mode and U_{ab} being a linear coupling. Here we will use such representations to construct the weakly complementary (2.50) of these channels and to study their weak-degradability properties, extending the results in Sec. 3.3.2 [53].

Weakly complementary channels

We construct the weakly complementary channels $\tilde{\mathcal{E}}$ of the class A_1, A_2, B_1, C and D starting from their single-mode physical representations (2.49) of Sec. 3.3.3. Because of the linearity of U_{ab} and the fact that σ_b is Gaussian, the channels $\tilde{\mathcal{E}}$ are Gaussian. This can be seen for instance by computing the characteristic function (3.23) of the output state $\tilde{\mathcal{E}}(\rho_a)$

$$\begin{aligned}\phi''(z) &= \mathrm{Tr}_b[\tilde{\mathcal{E}}(\rho_a)\; V_b(z)] = \mathrm{Tr}_b[\rho_a\; \tilde{\mathcal{E}}_H(V_b(z))] \\ &= \phi(m_{21}\cdot z)\; \exp[-\frac{1}{2}(N_0+1/2)\; |m_{22}\cdot z|^2]\;, \qquad (3.149)\end{aligned}$$

where m_{21}, m_{22} are the blocks elements of the matrix M of Eq. (3.116) associated with the transformations U_{ab}, and with N_0 being the average photon number of σ_b (the values of these quantities are given in the tables of Sec. 3.3.3). By setting $m = 0$, $X = m_{21}$ and $Y = (2N_0 + 1)\; m_{22}^T\; m_{22}$, Eq. (3.149) has the same structure (3.48) of one-mode Gaussian channels. Therefore, by cascading $\tilde{\mathcal{E}}$ with an isometry which exchanges A (system) with B (environment) (see Refs. [41, 52]), we can then treat $\tilde{\mathcal{E}}$ as an one-mode Gaussian channel operating on A (this is possible because both A and B are Bosonic one-mode systems). With the help of Table 3.2 we can then determine which classes can be associated with the transformation (3.149). This is summarized in Table 3.3.

Class of $\mathcal{E}$		$\tilde{\mathcal{E}}$		Class of $\tilde{\mathcal{E}}$
		X	Y	
A_1		$\mathbb{1}$	0	B_2
A_2		$\mathbb{1}$	$(2N_0+1)\,(\mathbb{1}-\sigma_z)$	B_1
B_1		$(\mathbb{1}+\sigma_z)/2$	$\mathbb{1}$	A_2
C	$k<1$	$\sqrt{1-k^2}\,\mathbb{1}$	$k^2(2N_0+1)\mathbb{1}$	$C\ (k<1)$
C	$k>1$	$\sqrt{k^2-1}\,\sigma_z$	$k^2(2N_0+1)\,\mathbb{1}$	D
D		$\sqrt{k^2+1}\,\mathbb{1}$	$k^2(2N_0+1)\,\mathbb{1}$	$C\ (k>1)$

Table 3.3: Description of the weakly complementary (2.50) of the canonical forms A_1, A_2, B_1, C and D of Table 3.2 constructed from the physical representations (2.49) given in Sec. 3.3.3. In the first column is indicated the class of $\mathcal{E}$. In the central columns instead is given a description of $\tilde{\mathcal{E}}$ in terms of the representation (3.48). Finally in the last column is reported the canonical form corresponding to the map $\tilde{\mathcal{E}}$. In all cases the identification is immediate: for instance the canonical form of the map $\tilde{\mathcal{E}}_{A_1}$ belongs to the class B_2, while the canonical form of the map $\tilde{\mathcal{E}}_D$ is the class C with $\mathrm{Det}[X_{\mathrm{can}}]>1$. In the case of $\tilde{\mathcal{E}}_{A_2}$ the identification with the class B_1 was done by exploiting the possibility freely varying N_c of Eq. (3.122) — see Ref. [56]. A pictorial representation of the above weak-degradability connections is given in Fig. 3.12.

Weak-degradability properties

Using the compositions rules of Eqs. (3.54) and (3.123) it is easy to verify that the canonical forms A_1, A_2, D and C with $k\leqslant\sqrt{1/2}$ are anti-degradable (2.53). Vice-versa one can verify that the canonical forms B_1 and C with $k\geqslant\sqrt{1/2}$ are weakly degradable (2.52) — for C, D and A_1 these

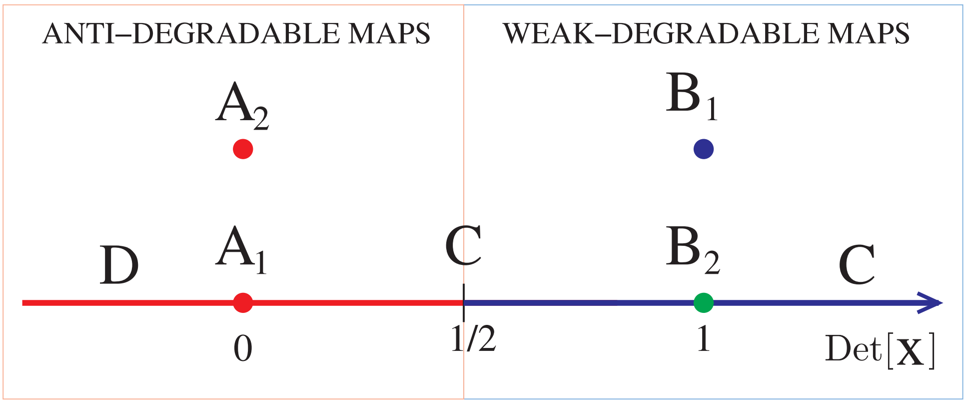

Figure 3.13: Pictorial representation of the weak-degradability regions for one-mode Gaussian channels. All canonical forms with $\mathrm{Det}[X] \leqslant 1/2$ are anti-degradable: this includes the classes A_1, A_2, D and part of the C. The remaining (with the exception of B_2) are instead weakly degradable. Moreover B_1 is also degradable in the sense of Ref. [38]. The same holds for channels of canonical form C with $N_0 = 0$: the exact expression for the quantum capacity of these channels was given in Sec. 3.3.1 [55].

results will be shown explicitly below [52]. Through unitary equivalence this can be summarized by saying that all one-mode Gaussian channels (3.48) having $\mathrm{Det}[X] \leqslant 1/2$ are anti-degradable, while the others (with the exception of the channels belonging to B_2) are weakly degradable (see Fig. 3.13). In particular, we recover all results obtained in terms of BS/amplifiers maps in Sec. 3.3.2, by recalling that $\mathrm{Det}[X] \equiv q$.

In the following we verify the above results by explicitly constructing the connecting channels $\mathcal{T}$ and $\overline{\mathcal{T}}$ of Eqs. (2.52) and (2.53) for each of the mentioned canonical forms:

- For a channel $\mathcal{E}$ of standard form A_1 or A_2, anti-degradability can be shown by simply taking $\overline{\mathcal{T}}$ of Eq. (2.53) coincident with the channel $\mathcal{E}$. The result immediately follows from the composition rule (3.54).

- For a channel $\mathcal{E}$ of B_1, weak-degradability comes by assuming the map $\mathcal{T}$ to be equal to the weakly complementary channel $\tilde{\mathcal{E}}$ of $\mathcal{E}$ (see Table 3.3). As pointed out in Ref. [56] this also implies the degradability of $\mathcal{E}$ in the sense of Ref. [38]. Let us remind that for B_1 the physical representation given in Sec. 3.3.3 was constructed with an environmental state σ_b initially prepared in the vacuum state, which is pure. Therefore in this case our representation gives rise to a Stinespring dilation.

- For a channel $\mathcal{E}$ of the class C with $X_{\text{can}} = k\ \mathbb{1}$ and $Y_{\text{can}} = |k^2 - 1|(2N_0 + 1)\mathbb{1}$ (see Sec. 3.3.1) we have the following three possibilities:
  - If $k \leqslant \sqrt{1/2}$ the channel is anti-degradable and the connecting map $\overline{\mathcal{T}}$ is a channel of C characterized by $X_{\text{can}} = k'\ \mathbb{1}$ and $Y_{\text{can}} = (1-(k')^2)(2N_0+1)\mathbb{1}$ with $k' = k/\sqrt{1-k^2} < 1$.
  - If $k \in [\sqrt{1/2}, 1[$ the channel is weakly degradable and the connecting map $\mathcal{T}$ is again a channel of C defined as in the previous case but with $k' = \sqrt{1-k^2}/k < 1$. For $N_0 = 0$ the channel is also degradable [38] since our physical representation is equivalent to a Stinespring representation.
  - If $k > 1$ the channel is weakly degradable and the connecting map $\mathcal{T}$ is a channel of D with $X_{\text{can}} = k'\ \sigma_z$ and $Y_{\text{can}} = ((k')^2+1)(2N_0+1)\mathbb{1}$ with $k' = \sqrt{k^2-1}/k$. As in the previous case, for $N_0 = 0$ the channel is also degradable [38].
- For a channel $\mathcal{E}$ of D with $X_{\text{can}} = k\ \sigma_z$ and $Y_{\text{can}} = (k^2+1)(2N_0+1)\mathbb{1}$ ($k > 0$ and $N_0 \geqslant 0$) we can prove anti-degradability by choosing $\overline{\mathcal{T}}$ of Eq. (2.53) to be yet another maps of D with $X_{\text{can}} = k'\ \sigma_z$ and $Y_{\text{can}} = ((k')^2+1)(2N_0+1)\mathbb{1}$ where $k' = k/\sqrt{k^2+1}$. From Eq. (3.54) and Table 3.3 it then follows that $\overline{\mathcal{T}} \circ \tilde{\mathcal{E}}$ is equal to $\mathcal{E}$.

Concerning the case B_2 it was shown in Sec. 3.3.3 [56] that the channel is neither anti-degradable nor degradable in the sense of [38] (apart from the trivial case $N_0 = 0$ which corresponds to the identity map). On the other hand one can use the continuity argument given there to claim that the channel B_2 can be arbitrarily approximated with maps which are weakly degradable (those belonging to C for instance).

3.3.5 Weakly degradable channels with zero capacity

In Sec. 3.3.4 we saw that all channels (3.48) with $\text{Det}[X] \leqslant 1/2$ are anti-degradable. Consequently these channels must have null quantum capacity [52, 53, 41]. Here we go a little further generalizing the results of Sec.

3.3.1 and showing that the set of the maps (3.48) which can be proved to have null quantum capacity include also some maps with $\mathrm{Det}[X] > 1/2$. To do this we will use the following simple fact (already used in Sec. 3.3.1): *Let be $\mathcal{E}_1$ a quantum channel with null quantum capacity and let be $\mathcal{E}_2$ some quantum channel. Then the composite channels $\mathcal{E}_1 \circ \mathcal{E}_2$ and $\mathcal{E}_2 \circ \mathcal{E}_1$ have null quantum capacity.* The proof of this property follows by interpreting $\mathcal{E}_2$ as a quantum operation performed either at the decoding or at encoding stage of the channel $\mathcal{E}_1$. This shows that the quantum capacities of $\mathcal{E}_1 \circ \mathcal{E}_2$ and $\mathcal{E}_2 \circ \mathcal{E}_1$ cannot be greater than the capacity of $\mathcal{E}_1$ (which is null). In the following we show two cases where the above property turns out to provide some nontrivial results [53].

Composition of two class D channels

We observe that according to composition rule (3.123) the combination of any two channels $\mathcal{E}_1$ and $\mathcal{E}_2$ of D produces a map $\mathcal{E}_{21} \equiv \mathcal{E}_2 \circ \mathcal{E}_1$ which is in the class C. Since the class D is anti-degradable the resulting channel must have null quantum capacity. Let then $k_j \sigma_z$ and $(k_j^2+1)(N_j+1/2)\mathbb{1}$ be the matrices X_{can} and Y_{can} of the channels $\mathcal{E}_j$, for $j = 1, 2$. From Eq. (3.54) one can then verify that $\mathcal{E}_{21}$ has the canonical form C with parameters

$$k = k_1 k_2 \,, \tag{3.150}$$

$$N_0 = \frac{(k_2^2+1)N_2 + k_2^2(k_1^2+1)N_1}{|k_1^2 k_2^2 - 1|} + \frac{1}{2}\left(\frac{k_1^2 k_2^2 + 2k_2^2 + 1}{|k_1^2 k_2^2 - 1|} - 1\right) \tag{3.151}$$

Equation (3.150) shows that by varying k_j, k can take any positive values: in particular it can be greater than $1/\sqrt{2}$ transforming $\mathcal{E}_{21}$ into a channel which does not belong to the anti-degradable area of Fig. 3.13. On the other hand, by varying the N_j and k_2, but keeping the product $k_1 k_2$ fixed, the parameter N_0 can assume any value satisfying the inequality

$$N_0 \geqslant \frac{1}{2}\left(\frac{k^2+1}{|k^2-1|} - 1\right) . \tag{3.152}$$

We can therefore conclude that all channels C with k and N_0 as in Eq. (3.152) have null quantum capacity — see Fig. 3.14. A similar bound was found in a completely different way in Ref. [45].

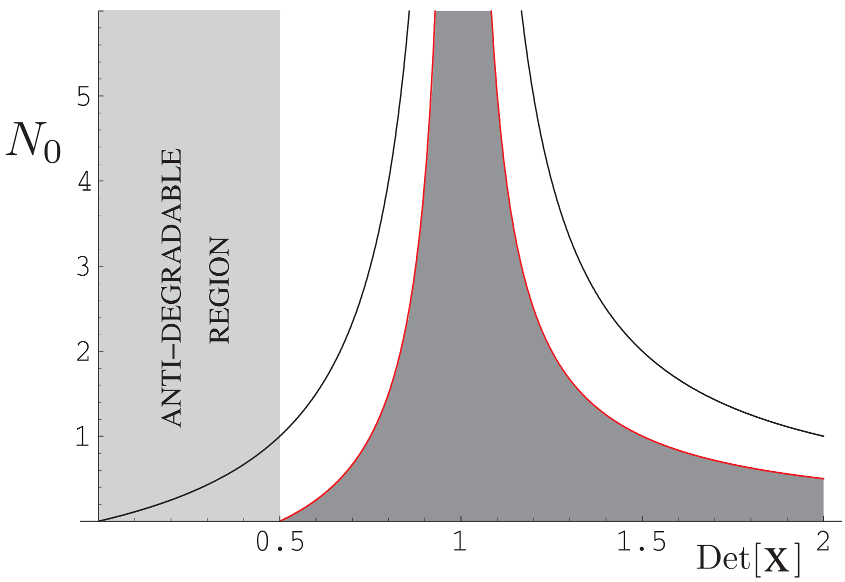

Figure 3.14: The dark-grey area of the plot is the region of the parameters N_0 and $\mathrm{Det}[X] = k^2$ where a channel with canonical form C can have not null quantum capacity. For $\mathrm{Det}[X] < 1/2$ the channel is anti-degradable. In the remaining white area the quantum capacity is null since these maps can be obtained by a composition of channels one of which being anti-degradable. The curve above refers to the bound of Eq. (3.152). The contour of the dark-grey area is instead given by Eq. (3.154).

Composition of two class C channels

Consider now the composition of two class C channels, i.e. $\mathcal{E}_1$ and $\mathcal{E}_2$, with one of them (say $\mathcal{E}_2$) being anti-degradable. The canonical form of $\mathcal{E}_1$ and $\mathcal{E}_2$ have matrices $X_{\rm can}$ and $Y_{\rm can}$ given by $X_i = k_j \mathbb{1}$ and $Y_j = |k_j^2 - 1|(2N_j + 1)\mathbb{1}$, where for $j = 1, 2$, N_j and k_j are positive numbers, with $k_1 \neq 0, 1$ and with $k_2 \in]0, 1/\sqrt{2}]$ (to ensure anti-degradability). From Eq. (3.54) follows then that the composite map $\mathcal{E}_{21} = \mathcal{E}_2 \circ \mathcal{E}_1$ has still a C canonical form with parameters

$$
\begin{aligned}
k &= k_1 k_2 \,, \qquad (3.153)\\
N_0 &= \frac{|k_2^2 - 1|N_2 + k_2^2|k_1^2 - 1|N_1}{|k_1^2 k_2^2 - 1|} + \frac{1}{2}\left(\frac{k_2^2|k_1^2 - 1| + |k_2^2 - 1|}{|k_1^2 k_2^2 - 1|} - 1\right) .
\end{aligned}
$$

As in the previous example, k can assume any positive value. Vice-versa keeping k fixed, and varying $k_1 > 1$ and $N_{1,2}$ it follows that N_0 can take any values which satisfy the inequality

$$N_0 \;\geqslant\; \frac{1}{2}\left(\frac{k^2}{|k^2-1|}-1\right) . \tag{3.154}$$

We can then conclude that all maps C with k and N_0 as above must possess null quantum capacity. The result has been plotted in Fig. 3.14. Notice that the constraint (3.154) is an improvement with respect to the constraint of Eq. (3.152). Finally let us point out that, by using a more general and powerful approach, we recover similar results of Sec. 3.3.1.

3.4 Multi-mode Bosonic Gaussian channels

In this section we propose a general construction of unitary dilations (see Sec. 3.2) of multi-mode quantum channels, including all rank-deficient cases [54]. We characterize the minimal noise maps involving only true quantum noise, showing a useful decomposition rule. Then, by using a generalized normal mode decomposition recently introduced in Ref. [57], we generalize the results of Refs. [52, 53] (analyzed above) concerning Gaussian weak complementary channels to the multi-mode case giving a simple weak-degradability/anti-degradability condition for such channels [54]. The minimal number of environmental modes of the multi-mode unitary dilations is also characterized [62]. The chapter ends with a detailed analysis of the two-mode case. This is important since any n-mode channel can always be reduced to single-mode and two-mode components [57]. Finally, we detalize the degradability analysis and investigate a useful decomposition of a channel with the additive classical noise map that allows us to find new sets of channels with zero quantum capacity. All results in this section include the one-mode BGCs results in Sec. 3.3 as a particular case for $n = 1$, of course.

3.4.1 Unitary dilation theorem

In relation to the problem examined in Sec. 3.2, we will construct Gaussian dilations of generic multi-mode BGCs, including an analysis of all rank-deficient cases, and later we will focus on dilations involving the minimal number of modes [54, 62]. To proceed, we establish some conventions and notation. To start with, we write the commutation matrix of our $n+\ell$ modes in the block structure

$$\sigma := \sigma_{2n} \oplus \sigma_{2\ell}^E = \left[\begin{array}{cc} \sigma_{2n} & 0 \\ 0 & \sigma_{2\ell}^E \end{array}\right] \begin{array}{l} \}\, 2n \\ \}\, 2\ell\,, \end{array} \tag{3.155}$$

where σ_{2n} and $\sigma_{2\ell}^E$ are $2n \times 2n$ and $2\ell \times 2\ell$ commutation matrices associated with the system and environmental modes, respectively. For σ_{2n} we assume the structure as defined in Eq. (3.35). For $\sigma_{2\ell}^E$, in contrast, we do not make any assumption at this point, leaving open the possibility of defining it later on[8]. Accordingly, the canonical unitary transformation U of Eq. (3.56) will be uniquely determined by a $2(n+\ell)\times 2(n+\ell)$ real matrix $S \in Sp(2(n+\ell), \mathbb{R})$ of block form

$$S := \left[\begin{array}{cc} s_1 & s_2 \\ s_3 & s_4 \end{array}\right], \tag{3.156}$$

which satisfies the condition

$$S\sigma S^T = \sigma\,, \qquad \Longleftrightarrow \qquad \left\{\begin{array}{l} s_1\, \sigma_{2n}\, s_1^T + s_2\, \sigma_{2\ell}^E\, s_2^T = \sigma_{2n}\,, \\ \\ s_1\, \sigma_{2n}\, s_3^T + s_2\, \sigma_{2\ell}^E\, s_4^T = 0\,, \\ \\ s_3\, \sigma_{2n}\, s_3^T + s_4\, \sigma_{2\ell}^E\, s_4^T = \sigma_{2\ell}^E\,. \end{array}\right. \tag{3.157}$$

In the above expressions, s_1 and s_4 are $2n \times 2n$ and $2\ell \times 2\ell$ real square matrices, while s_2 and s_3^T are $2n \times 2\ell$ real rectangular matrices. Introducing then the covariance matrices $\gamma \geqslant i\sigma_{2n}$ and $\gamma_E \geqslant i\sigma_{2\ell}^E$ of the states ρ and ρ_E, the identity (3.56) can be written as

$$S \left[\begin{array}{cc} \gamma & 0 \\ 0 & \gamma_E \end{array}\right] S^T \Bigg|_{2n} = s_1\, \gamma\, s_1^T + s_2\, \gamma_E\, s_2^T = X^T \gamma X + Y, \tag{3.158}$$

where $|_{2n}$ denotes the upper principle submatrix of degree $2n$, and where $X, Y \in \mathbb{R}^{2n\times 2n}$ satisfying the condition (3.47) are the matrices associated with the channel $\mathcal{E}$. In writing Eq. (3.158) we use the fact that due to the

[8] With this choice the canonical commutation relations of the $n+\ell$ mode read as $[R_j, R_{j'}] = i\sigma_{j,j'}$ where $R := (Q_1, \cdots, Q_n; P_1, \cdots, P_n; r_1, \cdots, r_{2\ell})$ with Q_j, P_j being the canonical coordinates of the j-th system mode and with and $r_1, \cdots, r_{2\ell}$ being some ordering of the canonical coordinates $Q_1^E, P_1^E; \cdots; Q_\ell^E, P_\ell^E$ of the environmental modes. For instance, taking $\sigma_{2\ell}^E = \sigma_{2\ell}$ corresponds to have $R := (Q_1, \cdots, Q_n; P_1, \cdots, P_n; Q_1^E, \cdots, Q_\ell^E; P_1^E, \cdots, P_\ell^E)$.

definition (3.155) the covariance matrix of the composite state $\rho \otimes \rho_E$ can be expressed as $\gamma \oplus \gamma_E$.

With these definitions, the first part of the unitary dilation property (3.56) can be written as follows:

Proposition 1 (Unitary dilations of Gaussian channels) *Let γ_E be the covariance matrix of a Gaussian state of ℓ modes and let $S \in Sp(2(n+\ell), \mathbb{R})$ be a symplectic transformation. Then there exists a symmetric $2n \times 2n$-matrix $Y \geqslant 0$ and a $2n \times 2n$-matrix X satisfying the condition (3.47), such that Eq. (3.158) holds for all γ.*

Proof: The proof is straightforward: we write S in the block form (3.156) and take $X = s_1^T$ and $Y = s_2 \gamma_E s_2^T$. Since γ_E is a covariance matrix of ℓ modes, $\gamma_E - i\sigma_{2\ell} \geqslant 0$ and therefore $s_2(\gamma_E - i\sigma_\ell)s_2^T \geqslant 0$. This leads to Eq. (3.47) through the symplectic condition $s_1 \sigma_{2n} s_1^T + s_2 \sigma_{2\ell} s_2^T = \sigma_{2n}$ in Eq. (3.157). ■

This proves that any CPT map obtained by coupling n modes with a Gaussian state of ℓ environmental Bosonic modes through a Gaussian unitary U is a BGC. The converse property is more demanding. In order to present it we find it useful to state first the following

Lemma 2 (Extensions of symplectic forms) *Let, for some skew symmetric $\sigma_{2\ell}^E$, s_1 and s_2 be $2n \times 2n$ and $2n \times 2\ell$ real matrices forming a symplectic system, i.e., $s_1\, \sigma_{2n}\, s_1^T + s_2\, \sigma_{2\ell}^E\, s_2^T = \sigma_{2n}$. Then we can always find real matrices s_3 and s_4 such that S of Eq. (3.156) is symplectic with respect to the commutation matrix (3.155).*

Proof: Since the rows of S form a symplectic basis, given s_1 and s_2 (an incomplete symplectic basis), it is always possible to find s_3 and s_4 as above. The proof easily follows from a skew-symmetric version of the Gram-Schmidt process to construct a symplectic basis [118]. Note that it does not restrict generality to take $\sigma_{2\ell}^E = \sigma_{2\ell}$, as this can always be accompanied by an appropriate similarity transform. Our problem at hand of extending a symplectic form is then equivalent to the following problem. Suppose we are given column vectors $e_1, \cdots, e_n$ and $f_1, \cdots, f_n$ from $\mathbb{R}^{2(n+\ell)}$ that satisfy

$$e_j^T \sigma_{2(n+\ell)} e_k = 0, \tag{3.159}$$

$$f_j^T \sigma_{2(n+\ell)} f_k = 0, \tag{3.160}$$

$$e_j^T \sigma_{2(n+\ell)} f_k = \delta_{j,k}, \tag{3.161}$$

for $j, k = 1, \cdots, n$. The procedure continues by identifying vectors e_{n+1} and f_{n+1} such that $e_{n+1}^T \sigma_{2(n+\ell)} f_{n+1} = 1$ and

$$e_{n+1}^T \sigma_{2(n+\ell)} w = f_{n+1}^T \sigma_{2(n+\ell)} w = 0 \tag{3.162}$$

for all

$$w \in W_n := \mathrm{span}(e_1, \cdots, e_n, f_1, \cdots, f_n). \tag{3.163}$$

Now define

$$W_n^\perp = \{w : w^T \sigma_{2(n+\ell)} v = 0 \,\forall v \in W_n\}. \tag{3.164}$$

It is now not difficult to see that $W_n \cap W_n^\perp = \{0\}$ and $\mathbb{R}^{2(n+\ell)} = W_n \oplus W_n^\perp$: Suppose that the vector v has $v^T \sigma_{2(n+\ell)} e_j =: \alpha_j$ and $v^T \sigma_{2(n+\ell)} f_j =: \beta_j$ for $j = 1, \cdots, n$. Then

$$v = \left[\sum_{j=1}^{n} (-\alpha_j f_j + \beta_j e_j)\right] + \left[v + \sum_{j=1}^{n} (\alpha_j f_j - \beta_j e_j)\right], \tag{3.165}$$

where the first term is element of W_n and the second of $W_n^\perp$. Following a symplectic Gram-Schmidt procedure, the symplectic basis can hence be completed, which is equivalent to extending the matrices s_1 and s_2 to a symplectic

$$S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \in Sp(2(n+\ell), \mathbb{R}). \tag{3.166}$$

For a special subset of BGCs, in Sec. 3.4.5 we will present an explicit expression for S based on a simplified (canonical) representation of the X matrix that defines $\mathcal{E}$. ■

Due to the above result, the possibility of realizing unitary dilation Eq. (3.56) for a generic BGC described by the matrices X and $Y \geqslant i\Sigma = i(\sigma_{2n} - X^T \sigma_{2n} X)$, can be proven by simply taking $s_1 = X^T$ and finding some $2n \times 2\ell$ real matrix s_2 and an ℓ-mode covariance matrix $\gamma_E \geqslant i\sigma_{2\ell}^E$ that solve the equations

$$\begin{aligned} s_2\, \sigma_{2\ell}^E\, s_2^T &= \sigma_{2n} - s_1\, \sigma_{2n}\, s_1^T = \Sigma\,, && (3.167) \\ s_2\, \gamma_E\, s_2^T &= Y\,. && (3.168) \end{aligned}$$

With this choice in fact Eq. (3.158) is trivially satisfied for all γ, while s_1 and s_2 can be completed to a symplectic matrix $S \in Sp(2(n+\ell), \mathbb{R})$. Note that $Sp(2(n+\ell), \mathbb{R})$ stands for the standard symplectic group here. The unitary dilation property (3.56) can hence be restated as follows:

Theorem 3 (Unitary dilations of Gaussian channels) *For any real $2n \times 2n$-matrices X and Y satisfying the condition (3.47), there exist ℓ smaller than or equal to $2n$, $S \in Sp(2(n+\ell), \mathbb{R})$, and a covariance matrix γ_E of ℓ modes, such that Eq. (3.158) is satisfied.*

Proof: As already noticed the whole problem can be solved by assuming $s_1 = X^T$ and finding s_2 and γ_E that satisfy Eqs. (3.167) and (3.168). We start by observing that the $2n \times 2n$ matrix Σ defined in Eq. (3.47) is skew-symmetric, i.e., $\Sigma = -\Sigma^T$. Moreover, according to Eq. (3.47), its support must be contained in the support of Y, i.e., $\mathrm{Supp}[\Sigma] \subseteq \mathrm{Supp}[Y]$. Consequently given $k := \mathrm{rank}[Y]$ and $r := \mathrm{rank}[\Sigma]$ as the ranks of Y and Σ, respectively, one has that $k \geqslant r$. We can hence identify three different regimes:

(i) $k = 2n$, $r = 2n$, i.e., both Y and Σ are full rank and hence invertible. Loosely speaking, this means that all the noise components in the channel are basically quantum (although may involve classical noise as well).

(ii) $k = 2n$ and $r < 2n$, i.e., Y is full rank and hence invertible, while Σ is singular. This means that the some of the noise components can be purely classical, but still nondegenerate.

(iii) $2n > k \geqslant r$, i.e., both Y and Σ are singular. There are noise components with zero variance.

Even though (i) and (ii) admit similar solutions, it is instructive to analyze them separately. In the former case in fact there is a simple direct way of constructing a physical dilation of the channel with $\ell = n$ environmental modes.

(i) Since Σ is skew-symmetric and invertible there exists an $O \in O(2n, \mathbb{R})$ orthogonal such that

$$O\Sigma O^T = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}, \tag{3.169}$$

where $\mu = \mathrm{diag}(\mu_1, \cdots, \mu_n)$ and $\mu_i > 0$ for all $i = 1, \cdots, n$ (see page 107 in Ref. [119]). Hence $K := M^{-1/2} O$ with $M := \mu \oplus \mu$ satisfies

$$K\Sigma K^T = \sigma_{2n}\,. \tag{3.170}$$

Taking then $s_2 := K^{-1}$ we get[9]

$$s_2\,\sigma_{2n}\,s_2^T = K^{-1}\,\sigma_{2n}\,K^{-T} = \Sigma\,, \tag{3.171}$$

which corresponds to Eq. (3.167) for $\ell = n$. Since $s_1 = X^T$, Lemma 2 guarantees that this is sufficient to prove the existence of S. The condition (3.158) finally follows by taking $\gamma_E = KYK^T$ which is strictly positive (indeed K is invertible and $Y > 0$ because it has full rank) and which satisfies the uncertainty relation (3.42), i.e.,

$$Y \geqslant i\Sigma \quad \Longrightarrow \quad \gamma_E = KYK^T \geqslant iK\Sigma K^T = i\sigma_{2n}\,. \tag{3.172}$$

This shows that the channel admits a unitary dilation of the form as specified in Eq. (3.56) with $\ell = n$ environmental modes with commutation matrix, $\sigma_{2n}^E = \sigma_{2n}$ – see discussion after Eq. (3.155). Such a solution, however, will involve a pure state ρ_E only if $\mathrm{Det}[\gamma_E] = 1$, i.e.,

$$\mathrm{Det}[Y]\mathrm{Det}[K]^2 = 1 \quad \Longleftrightarrow \quad \mathrm{Det}[Y] = \mathrm{Det}[\Sigma]\,. \tag{3.173}$$

When $\mathrm{Det}[\gamma_E] > 1$, i.e., $\mathrm{Det}[Y] > \mathrm{Det}[\Sigma]$, we can still construct a pure dilation by simply adding further n modes which purify the state associated with the covariance matrix γ_E and by extending the unitary operator U associated with S as the identity operator on them. For details see the discussion of case (ii) given below. This corresponds to constructing a unitary dilation (3.56) with the pure state ρ_E being defined on $\ell = 2n$ modes.

(ii) In this case Y is still invertible, while Σ is not. Differently from the approach we adopted in solving case (i), we here derive directly a Stinespring unitary dilation, i.e., we construct a solution with a pure γ_E that involves $\ell = 2n$ environmental modes. In the next section, however, we will show that, dropping the purity requirement, one can construct unitary dilation that involves ρ_E with only $\ell = 2n - r/2$ modes.

To find s_2 and γ_E which solve Eqs. (3.167) and (3.168), it is useful to first transform Y into a simpler form by a congruent transformation, i.e.,

$$CYC^T = \mathbb{1}_{2n}\,, \tag{3.174}$$

[9] From now on, the symbol A^{-T} will be used to indicate the transpose of the inverse of the matrix A, i.e., $A^{-T} := (A^{-1})^T = (A^T)^{-1}$.

with $C \in Gl(2n, \mathbb{R})$ being not singular, e.g., $C := Y^{-1/2}$. From Eq. (3.47) it then follows that

$$\mathbb{1}_{2n} \geqslant i\Sigma' , \tag{3.175}$$

with $\Sigma' := Y^{-1/2}\,\Sigma\,Y^{-1/2}$ being skew-symmetric (i.e., $\Sigma' = -(\Sigma')^T$) and singular with rank$[\Sigma']$ = rank$[\Sigma] = r$ [119]. We then observe that introducing

$$s_2 = Y^{1/2}\, s_2' , \tag{3.176}$$

the conditions (3.167) and (3.168) can be written as

$$\begin{aligned} s_2'\, \sigma_{2\ell}^E\, (s_2')^T &= \Sigma' , &(3.177)\\ s_2'\, \gamma_E\, (s_2')^T &= \mathbb{1}_{2n} . &(3.178) \end{aligned}$$

Finding s_2' and γ_E which satisfy these expressions will provide us also a solution of Eqs. (3.167) and (3.168).

As in the case of Eq. (3.169), there exists an orthogonal matrix $O \in O(2n, \mathbb{R})$ which transforms the skew-symmetric matrix Σ' in a simplified block form. In this case however, since Σ' is singular, we find [119]

$$O\Sigma' O^T = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & \mu & 0 \\ & & 0 & 0 \\ \hline -\mu & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & & \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n - r/2 \\ \}\, r/2 \\ \}\, n - r/2, \end{array} \tag{3.179}$$

where now $\mu = \mathrm{diag}(\mu_1, \cdots, \mu_{r/2})$ is the $r/2 \times r/2$ diagonal matrix formed by the strictly positive eigenvalues of $|\Sigma'|$ which satisfy the conditions $1 \geqslant \mu_j > 0$, this being equivalent with

$$\mathbb{1}_{r/2} \geqslant \mu, \tag{3.180}$$

as a consequence of inequality (3.175). Define then $K := M^{-1/2}\, O$ with

$$M = \left[\begin{array}{cc|cc} \mu & 0 & \multicolumn{2}{c}{0} \\ 0 & \mathbb{1}_{n-r/2} & & \\ \hline \multicolumn{2}{c|}{0} & \mu & 0 \\ & & 0 & \mathbb{1}_{n-r/2} \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n - r/2 \\ \}\, r/2 \\ \}\, n - r/2. \end{array} \tag{3.181}$$

It satisfies the identity

$$K\Sigma' K^T = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & \mathbb{1}_{r/2} & 0 \\ & & 0 & 0 \\ \hline -\mathbb{1}_{r/2} & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & & \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n - r/2 \\ \}\, r/2 \\ \}\, n - r/2. \end{array} \tag{3.182}$$

To show that Eqs. (3.177) and (3.178) admit a solution we take $\ell = 2n$ and write $\sigma_{4n}^E = \sigma_{2n} \oplus \sigma_{2n} = \sigma_{4n}$ with σ_{2n} as in Eq. (3.35). With these definitions the $2n \times 4n$ rectangular matrix s_2' can be chosen to have the block structure

$$s_2' \;=\; \left[\; K^{-1} \;\middle|\; O^T A \;\right] , \tag{3.183}$$

with A being the following $2n \times 2n$ symmetric matrix

$$A = A^T \;=\; \begin{array}{c} \left[\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{n-r/2} \end{array} \\ \hline \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{n-r/2} \end{array} & 0 \end{array} \right] \end{array} \begin{array}{l} \}\, r/2 \\ \}\, n - r/2 \\ \}\, r/2 \\ \}\, n - r/2 . \end{array} \tag{3.184}$$

By direct substitution one can easily verify that Eq. (3.177) is indeed satisfied. In fact, assuming $\sigma_{4n}^E = \sigma_{2n} \oplus \sigma_{2n}$ with σ_{2n} as in Eq. (3.35), one has

$$\begin{aligned} s_2'\, \sigma_{4n}^E\, (s_2')^T - \Sigma' &= \left[\; K^{-1} \;\middle|\; O^T A \;\right] \left[\begin{array}{c|c} \sigma_{2n} & 0 \\ \hline 0 & \sigma_{2n} \end{array} \right] \left[\begin{array}{c} K^{-T} \\ \hline A^T O \end{array} \right] - \Sigma' \\ &= K^{-1} \sigma_{2n} K^{-T} + O^T A\, \sigma_{2n}\, A^T O - \Sigma' \\ &= K^{-1} \left(K \Sigma' K^T + B \right) K^{-T} + O^T A\, \sigma_{2n}\, A^T O - \Sigma' \\ &= K^{-1} B K^{-T} + O^T A\, \sigma_{2n}\, A^T O \\ &= O \left(M^{1/2} B M^{1/2} + A\, \sigma_{2n}\, A^T \right) O^T , \end{aligned} \tag{3.185}$$

where we used Eq. (3.182) to write $\sigma_{2n} = K \Sigma' K^T + B$, with B being the $2n \times 2n$ matrix

$$B \;:=\; \left[\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{n-r/2} \end{array} \\ \hline \begin{array}{c|c} 0 & 0 \\ \hline 0 & -\mathbb{1}_{n-r/2} \end{array} & 0 \end{array} \right] . \tag{3.186}$$

The identity (3.177) finally follows by noticing that the last term in Eq. (3.185) cancels since $M^{1/2} B = B M^{1/2} = B$ and $A\, \sigma_{2n}\, A^T = -B$.

Inserting Eq. (3.183) into Eq. (3.178) yields now the following equation

$$\alpha + A\, \delta^T + \delta\, A^T + A\, \beta\, A^T = M^{-1} , \tag{3.187}$$

for the $4n \times 4n$ covariance matrix

$$\gamma_E = \left[\begin{array}{cc} \alpha & \delta \\ \delta^T & \beta \end{array} \right] , \tag{3.188}$$

see below for details. A solution can be easily derived by taking

$$\alpha = \beta = \left[\begin{array}{cc|cc} \mu^{-1} & 0 & & \\ 0 & \xi \mathbb{1}_{n-r/2} & & 0 \\ \hline & & \mu^{-1} & 0 \\ 0 & & 0 & \xi \mathbb{1}_{n-r/2} \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2, \end{array} \tag{3.189}$$

with $\xi = 5/4$ and

$$\delta = \left[\begin{array}{cc|cc} & & f(\mu^{-1}) & 0 \\ & 0 & 0 & f(\xi \mathbb{1}_{n-r/2}) \\ \hline f(\mu^{-1}) & 0 & & \\ 0 & f(\xi \mathbb{1}_{n-r/2}) & & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, n-r/2 \\ \}\, r/2 \\ \}\, n-r/2, \end{array} \tag{3.190}$$

with $f(\theta) := -(\theta^2 - \mathbb{1})^{1/2}$. For all diagonal matrices μ compatible with the constraint (3.180) the resulting γ_E satisfies the uncertainty relation $\gamma_E \geqslant i\sigma_{4n}$. Moreover, since it has $\mathrm{Det}[\gamma_E] = 1$, this is also a minimal uncertainty state, i.e., a pure Gaussian state of $2n$ modes. It is worth stressing that for $r = 2n$, i.e., when also the rank of Σ is maximum, the above solution provides an alternative derivation of the unitary dilation discussed in the part (i) of the theorem. In this case the covariance matrix γ_E has block elements

$$\alpha = \beta = \left[\begin{array}{cc} \mu^{-1} & 0 \\ 0 & \mu^{-1} \end{array}\right] \begin{array}{l} \}\, n \\ \}\, n \end{array}, \qquad \delta = \left[\begin{array}{cc} 0 & f(\mu^{-1}) \\ f(\mu^{-1}) & 0 \end{array}\right] \begin{array}{l} \}\, n \\ \}\, n \end{array}, \tag{3.191}$$

where μ is now a strictly positive $n \times n$ matrix, while Eqs. (3.176) and (3.183) yield

$$s_2 := Y^{1/2} O^T \left[\begin{array}{cc|cc} \mu^{1/2} & 0 & 0 & 0 \\ \hline 0 & \mu^{1/2} & 0 & 0 \end{array}\right] \begin{array}{l} \}\, n \\ \}\, n. \end{array} \tag{3.192}$$

(iii) Here both Y and Σ are singular. This case is very similar to case (ii). Here, the dilation can be constructed by introducing a strictly positive matrix $\bar{Y} > 0$ which satisfies the condition

$$\Pi \, \bar{Y} \, \Pi = Y \,, \tag{3.193}$$

with Π being the projector onto the support of Y. Such a $\bar{Y}$ always exists ($\bar{Y} = Y + (\mathbb{1} - \Pi)$). By construction, it satisfies the inequality $\bar{Y} \geqslant Y \geqslant i\Sigma$. According to Sec. 3.2, $\bar{Y}$ and X define thus a BGC. Moreover, since $\bar{Y}$ is strictly positive, it has full rank. Therefore, we can use part (ii) of the proof to find a $2n \times 2\ell$ matrix $\bar{s}_2$ and $\bar{\gamma}_E \geqslant i\sigma_{2\ell}$ which satisfy the conditions (3.167) and (3.168), i.e.

$$\bar{s}_2\,\sigma_{2\ell}^E\,\bar{s}_2^T \;=\; \Sigma\,, \tag{3.194}$$

$$\bar{s}_2\,\bar{\gamma}_E\,\bar{s}_2^T \;=\; \bar{Y}\,. \tag{3.195}$$

A unitary dilation for the channel Y, X is then obtained by choosing $\gamma_E = \bar{\gamma}_E$ and $s_2 = \Pi\bar{s}_2$. In fact from Eq. (3.195) we get

$$s_2\,\gamma_E\,s_2^T = \Pi\,\bar{s}_2\,\bar{\gamma}_E\,\bar{s}_2^T\,\Pi = \Pi\,\bar{Y}\,\Pi = Y\,, \tag{3.196}$$

while from Eq. (3.194)

$$s_2\,\sigma_{2\ell}^E\,s_2^T = \Pi\,\bar{s}_2\,\sigma_{2\ell}^E\,\bar{s}_2^T\,\Pi = \Pi\,\Sigma\,\Pi = \Sigma\,, \tag{3.197}$$

where we have used the fact that $\mathrm{Supp}[\Sigma] \subseteq \mathrm{Supp}[Y]$. ■

In proving the second part of the unitary dilations theorem we provided explicit expressions for the environmental state ρ_E of Eq. (3.56). Specifically such a state is given by the pure $2n$ mode Gaussian state ρ_E characterized by the covariance matrix γ_E of elements (3.189) and (3.190). A trivial observation is that this can always be replaced by the $2n$ modes vacuum state $|\emptyset\rangle\langle\emptyset|$ having the covariance matrix $\gamma_E^{(0)} = \mathbb{1}_{2n}$. This is a consequence of the obvious property that according to Eq. (3.44) all pure Gaussian states are equivalent to $|\emptyset\rangle\langle\emptyset|$ up to a Gaussian unitary transformation. On the level of covariance matrices, Gaussian unitaries correspond to symplectic transformations. For a remark on unitarily equivalent dilations, see also Sec. 3.4.7. Hence, by means of a congruence with an appropriate symplectic transformation, we immediately arrive at the following corollary:

Corollary 1 (Gaussian channels with pure Gaussian dilations) *Any n-mode Gaussian channel $\mathcal{E}$ admits a Gaussian unitary dilation (3.56) with $\rho_E = |\emptyset\rangle\langle\emptyset|$ being the vacuum state on $2n$ modes.*

Proof: Let in fact s_2 and γ_E be a solution of Eq. (3.167) and (3.168) for the map $\mathcal{E}$, with γ_E being the covariance matrix of a pure Gaussian states of $2n$ modes (e.g., the solutions described in Theorem 3). Define now $S_\gamma \in Sp(4n;\mathbb{R})$ the symplectic transformation which realize the identity $\gamma_E =$

$S_\gamma\, \gamma_E^{(0)}\, S_\gamma^T$. Since by $S_\gamma\, \sigma_{2\ell}\, S_\gamma^T = \sigma_{2\ell}$ by replacing s_2 with $s_2^{(0)} = s_2\, S_\gamma$ it follows that $\gamma_E^{(0)}$, $s_2^{(0)}$ provide yet another solution of the Eqs. (3.167), (3.168), and thus a unitary dilation for the channel $\mathcal{E}$. ■

Now we first give an explicit derivation of Eq. (3.187). Then we analyze in detail the property of the state ρ_E associated with the covariance matrix γ_E defined be the Eqs. (3.189) and (3.190). Replacing Eq. (3.183) into Eq. (3.168), we get

$$\begin{aligned}
\mathbb{1}_{2n} = s_2'\, \gamma_E\, (s_2')^T \;&=\; \left[\begin{array}{c|c} K^{-1} & O^T A \end{array}\right] \left[\begin{array}{c|c} \alpha & \delta \\ \hline \delta^T & \beta \end{array}\right] \left[\begin{array}{c} K^{-T} \\ \hline A^T O \end{array}\right] \\
&=\; K^{-1}\, \alpha\, K^{-T} + O^T\, A\, \delta^T\, K^{-T} + K^{-1}\, \delta\, A^T\, O + O^T\, A\, \beta\, A^T\, O \\
&=\; O^T \left(M^{1/2}\, \alpha\, M^{1/2} + A\, \delta^T\, M^{1/2} + M^{1/2}\, \delta\, A^T + A\, \beta\, A^T \right) O\,,
\end{aligned}$$

which leads to

$$M^{-1} = \alpha + M^{-1/2} A\, \delta^T + \delta\, A^T M^{-1/2} + M^{-1/2}\, A\, \beta\, A^T M^{-1/2}\,, \tag{3.198}$$

and hence to Eq. (3.187) by the fact $M^{-1/2}A = A^T M^{-1/2} = A = A^T$. Such an equation admits the solution given in Eqs. (3.189) and (3.190). Explicitly this corresponds to the $4n \times 4n$ covariance matrix γ_E of the form

$$\left[\begin{array}{c|c|c|c}
\begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array} & 0 & 0 & \begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} \\ \hline
0 & \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array} & \begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} & 0 \\ \hline
0 & \begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} & \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array} & 0 \\ \hline
\begin{array}{c|c} f(\mu^{-1}) & 0 \\ \hline 0 & f(\xi\mathbb{1}) \end{array} & 0 & 0 & \begin{array}{c|c} \mu^{-1} & 0 \\ \hline 0 & \xi\mathbb{1} \end{array}
\end{array}\right]$$

where for easy of notation $\mathbb{1} := \mathbb{1}_{n-r/2}$. By looking at the structure of this covariance matrix, one realizes that it is composed by two independent sets formed by r and $2n - r$ modes, respectively. The first set describes $r/2$ thermal states characterized by the matrices μ^{-1} which have been purified adding further $r/2$ modes. The second set instead describes a collection of $2(n - r/2) = 2n - r$ modes prepared in a pure state formed by $n - r/2$ independent pairs of modes which are entangled. By reorganizing its rows and columns this can be cast into the simpler form

$$\gamma_E = \begin{array}{cl} \left[\begin{array}{cc|cc} \bar{\mu}^{-1} & f(\bar{\mu}^{-1}) & \multicolumn{2}{c}{0} \\ f(\bar{\mu}^{-1}) & \bar{\mu}^{-1} & & \\ \hline \multicolumn{2}{c|}{0} & \xi \mathbb{1}_{2n-r} & f(\xi \mathbb{1}_{2n-r}) \\ & & f(\xi \mathbb{1}_{2n-r}) & \xi \mathbb{1}_{2n-r} \end{array}\right] & \begin{array}{l} \}\, r \\ \}\, r \\ \}\, 2n-r \\ \}\, 2n-r\,, \end{array} \end{array} \tag{3.199}$$

where we used $\bar{\mu}$ to indicate the $r \times r$ matrix $\bar{\mu} = \mu \oplus \mu$.

3.4.2 Reducing the number of environmental modes

An interesting question is the characterization of the minimal number of environmental modes ℓ that need to be involved in the unitary dilation (3.56) [62]. From Theorem 3 we know that such number is certainly smaller than or equal to twice the number n of modes on which the BGC is operating: we have in fact explicitly constructed one of such representations that involves $\ell = 2n$ modes in a minimal uncertainty, i.e., pure Gaussian state. We also know, however, that there are situations[10] in which ℓ can be reduced to just n: This happens for instance for BGCs $\mathcal{E}$ with $\mathrm{rank}[Y] = \mathrm{rank}[\Sigma] = 2n$, i.e., case (i) of Theorem 3. In this case one can represent the channel $\mathcal{E}$ in terms of a Gaussian unitary coupling with $\ell = n$ environmental modes which are prepared into a Gaussian state with covariance matrix

$$\gamma_E = K Y K^T\,, \tag{3.200}$$

– see Eqs. (3.172). In general, this will not be of Stinespring form (not be a pure unitary dilation) since γ_E is not a minimal uncertainty covariance matrix. In fact, for $n = 1$ this corresponds to the physical representation of $\mathcal{E}$ in Sec. 3.3 [53]. However, if Y and X satisfy the condition (3.173), our analysis provides a unitary dilation involving merely $\ell = n$ modes in a pure Gaussian state.

We can then formulate a necessary and sufficient condition for the channels $\mathcal{E}$ of class (i) which can be described in terms of n environmental modes prepared into a pure state. It is given by (see also Sec. 3.4.3)

$$Y = \Sigma\, Y^{-1}\, \Sigma^T\,, \tag{3.201}$$

[10] Not mentioning the trivial case of Gaussian unitary transformation which does not require any environmental mode to construct a unitary dilation.

which follows by imposing the minimal uncertainty condition (3.45) to the n-mode covariance matrix (3.200) and by using (3.170). Similarly one can verify that given a pure n-modes Gaussian state ρ_E and an $S \in Sp(4n, \mathbb{R})$ (3.156) with an invertible subblock s_2, then the corresponding BGC satisfies condition (3.201). The above result can be strengthened by looking at the solutions for channels of class (ii) of which the channel of class (i) are a proper subset.

In order to reduce the number of the environmental modes, we prove the Theorem 3 in a slightly different way. We will analyze separately two different regimes of the rank of Y: i) k even and ii) k odd.

(i) First, let us consider the case in which k is an even number. To find s_2 and γ_E which solve Eqs. (3.167) and (3.168), it is useful to first transform Y into a simpler form by a congruent transformation. Particularly, it is always possible [119] to find $C \in Gl(2n, \mathbb{R})$ such that

$$Y' := C Y C^T = \left[\begin{array}{c|c} \mathbb{1}_k & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}\, k \\ \}\, 2n-k\,, \end{array} , \tag{3.202}$$

and

$$\Sigma' := C \Sigma C^T = \left[\begin{array}{c|c|c} 0 & \begin{array}{c|c} \mu & 0 \\ \hline 0 & 0 \end{array} & 0 \\ \hline \begin{array}{c|c} -\mu & 0 \\ \hline 0 & 0 \end{array} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, 2n-k \end{array} \tag{3.203}$$

where now $\mu = \mathrm{diag}(\mu_1, \cdots, \mu_{r/2})$ is the $r/2 \times r/2$ diagonal matrix formed by the strictly positive eigenvalues of $|\Sigma'|$ which satisfy the conditions $1 \geqslant \mu_j > 0$, this being equivalent with $\mathbb{1}_{r/2} \geqslant \mu$, as a consequence of inequality $Y' \geqslant i\Sigma'$. Note that $\mathrm{rank}[\Sigma'] = \mathrm{rank}[\Sigma] = r$ [119]. We then observe that, introducing $s_2 = C\, s_2'$, the conditions (3.167) and (3.168) can be written as

$$s_2'\, \sigma_{2\ell}^E\, (s_2')^T = \Sigma'\,, \tag{3.204}$$

$$s_2'\, \gamma_E\, (s_2')^T = Y'\,. \tag{3.205}$$

Finding s_2' and γ_E which satisfy these expressions will provide us also a solution of Eqs. (3.167) and (3.168). Hence, define the matrix $M \in \mathbb{R}^{2n\times 2n}$ as

$$K = \left[\begin{array}{c|c} \tilde{K} & 0 \\ \hline 0 & \mathbb{1}_{2n-k} \end{array}\right] \begin{array}{l} \}\, k \\ \}\, 2n-k\,, \end{array} , \tag{3.206}$$

where

$$\tilde{K} = \left[\begin{array}{cc|cc} \mu^{-1/2} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{(k-r)/2} & 0 & 0 \\ \hline 0 & 0 & \mu^{-1/2} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{(k-r)/2} \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2, \end{array}$$

satisfying

$$K\Sigma' K^T = \left[\begin{array}{cc|cc|c} 0 & 0 & \mathbb{1}_{r/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -\mathbb{1}_{r/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, 2n-k \end{array}$$

To show that Eqs. (3.204) and (3.205) admit a solution we take $\ell = k$ and write $\sigma_{2l}^E = \sigma_{2k} = \sigma_k \oplus \sigma_k$ with σ_k defined as above for σ_{2n}. With these definitions the $2n \times 2k$ rectangular matrix s_2' can be chosen to have the block structure

$$s_2' = \left[\begin{array}{c|c} \tilde{K}^{-1} & A \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}\, k \\ \}\, 2n-k. \end{array} \tag{3.207}$$

with A being the following $2n \times 2n$ symmetric matrix

$$A = A^T = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{(k-r)/2} \\ \hline 0 & 0 & 0 & 0 \\ 0 & \mathbb{1}_{(k-r)/2} & 0 & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2. \end{array}$$

By direct substitution one can easily verify that Eq. (3.204) is indeed satisfied. Inserting Eq. (3.207) into Eq. (3.205) yields now the following equation

$$\alpha + A\,\delta^T + \delta\,A^T + A\,\beta\,A^T = \tilde{K}^2\,, \tag{3.208}$$

for the $2k \times 2k$ covariance matrix γ_E as in Eq. (3.188). A solution can be easily derived by taking

$$\alpha = \beta = \left[\begin{array}{cc|cc} \mu^{-1} & 0 & 0 & 0 \\ 0 & \xi\mathbb{1}_{(k-r)/2} & 0 & 0 \\ \hline 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & 0 & \xi\mathbb{1}_{(k-r)/2} \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2, \end{array}$$

with $\xi = 5/4$ and

$$\delta = \begin{array}{c} \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & f(\mu^{-1}) & 0 \\ & & 0 & f(\xi \mathbb{1}_{(k-r)/2}) \\ \hline f(\mu^{-1}) & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & f(\xi \mathbb{1}_{(k-r)/2}) & & \end{array}\right] \end{array} \begin{array}{l} \} \, r/2 \\ \} \, (k-r)/2 \\ \} \, r/2 \\ \} \, (k-r)/2, \end{array}$$

with $f(\theta) := -(\theta^2 - \mathbb{1})^{1/2}$. For all diagonal matrices μ compatible with the constraint $\mathbb{1}_{r/2} \geqslant \mu$ the resulting γ_E satisfies the uncertainty relation $\gamma_E \geqslant i\sigma_{2k}$. It is also a pure Gaussian state of k modes, because of $\mathrm{Det}[\gamma_E] = 1$. By looking at the structure of this covariance matrix, one realizes that it is composed by two independent sets formed by r and $k-r$ modes, respectively. The first set describes $r/2$ thermal states characterized by the matrices μ^{-1} which have been purified adding further $r/2$ modes. The second set instead describes a collection of $2((k-r)/2) = k-r$ modes prepared in a pure state formed by $(k-r)/2$ independent pairs of modes which are entangled.

(ii) Now, suppose that k is odd. One can find $C \in Gl(2n, \mathbb{R})$ such that $Y' := CYC^T$ is as in Eq. (3.202) while

$$\Sigma' := C\Sigma C^T = \left[\begin{array}{cc|cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \mu & 0 & \multicolumn{2}{c}{\multirow{4}{*}{0}} \\ & & 0 & 0 & & \\ \cline{1-4} -\mu & 0 & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & & \\ 0 & 0 & & & & \\ \hline \multicolumn{4}{c|}{\multirow{2}{*}{0}} & 0 & \\ & & & & & 0 \end{array}\right] \begin{array}{l} \} \, r/2 \\ \} \, (k-1-r)/2 \\ \} \, r/2 \\ \} \, (k-1-r)/2 \\ \} \, 2 \\ \} \, 2n-k-1\,, \end{array}$$

where μ is defined as above and $1 \geqslant \mu_j > 0$. Again, we want to find s_2' and γ_E satisfying Eqs. (3.204-3.205). Hence, define the matrix $M \in \mathbb{R}^{2n\times 2n}$ as

$$K = \left[\begin{array}{c|c|c} \tilde{K} & 0 & 0 \\ \hline 0 & \mathbb{1}_2 & 0 \\ \hline 0 & 0 & \mathbb{1}_{2n-k-1} \end{array}\right] \begin{array}{l} \} \, k-1 \\ \} \, 2 \\ \} \, 2n-k-1\,, \end{array} \tag{3.209}$$

where

$$\tilde{K} = \left[\begin{array}{cc|cc} \mu^{-1/2} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \mathbb{1} & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \mu^{-1/2} & 0 \\ & & 0 & \mathbb{1} \end{array}\right] \begin{array}{l} \} \, r/2 \\ \} \, (k-1-r)/2 \\ \} \, r/2 \\ \} \, (k-1-r)/2, \end{array}$$

satisfying

$$
K\Sigma' K^T = \left[\begin{array}{c|c|c}
0 & \begin{array}{c|c} \mathbb{1}_{r/2} & 0 \\ \hline 0 & 0 \end{array} & \\ \cline{1-2}
\begin{array}{c|c} -\mathbb{1}_{r/2} & 0 \\ \hline 0 & 0 \end{array} & 0 & 0 \\ \hline
\multicolumn{2}{c|}{0} & \begin{array}{c|c} 0 & \\ \hline & 0 \end{array}
\end{array}\right]
\begin{array}{l} \}\, r/2 \\ \}\, (k-1-r)/2 \\ \}\, r/2 \\ \}\, (k-1-r)/2 \\ \}\, 2 \\ \}\, 2n-k-1\,, \end{array}
$$

Now, in order to find a solution for Eqs. (3.204) and (3.205), we take $\ell = k$ and write $\sigma_{2l}^E = \sigma_{2k} = \sigma_{k-1} \oplus \sigma_{k-1} \oplus \sigma_2$ with σ_{k-1} and σ_2 defined as above for σ_{2n}. With these definitions the $2n \times 2k$ rectangular matrix s_2' can be chosen to have the block structure

$$
s_2' = \left[\begin{array}{c|c|c}
\tilde{K}^{-1} & A & 0 \\ \cline{3-3}
0 & 0 & \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \\ \hline
0 & 0 & 0
\end{array}\right]
\begin{array}{l} \}\, k-1 \\ \}\, 1 \\ \}\, 1 \\ \}\, 2n-k-1. \end{array} \tag{3.210}
$$

with A being the following $2n \times 2n$ symmetric matrix

$$
A = \left[\begin{array}{c|c}
0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{(k-1-r)/2} \end{array} \\ \hline
\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{(k-1-r)/2} \end{array} & 0
\end{array}\right]
\begin{array}{l} \}\, r/2 \\ \}\, (k-1-r)/2 \\ \}\, r/2 \\ \}\, (k-1-r)/2. \end{array}
$$

and satisfies Eq. (3.204).

If one inserts Eq. (3.210) into Eq. (3.205), one obtains the same equation as in Eq. (3.208) but for the $2k \times 2k$ covariance matrix

$$
\gamma_E = \left[\begin{array}{c|c|c}
\alpha & \delta & 0 \\ \hline
\delta^T & \beta & 0 \\ \hline
0 & 0 & \mathbb{1}_2
\end{array}\right]
\begin{array}{l} \}\, k-1 \\ \}\, k-1 \\ \}\, 2\,, \end{array} \tag{3.211}
$$

where the matrices α, β and δ are $(k-1) \times (-1)k$ matrices defined as in the case 1) but replacing k with $k-1$. Again, the resulting γ_E satisfies the uncertainty relation $\gamma_E \geqslant i\sigma_{2k}$ and represents a pure Gaussian state of k modes (i.e., $\mathrm{Det}[\gamma_E] = 1$). By looking at the structure of this covariance matrix, one realizes that it is composed by three independent sets formed by r, $k-r$ and 1 modes, respectively. The first set describes $r/2$ thermal states characterized by the matrices μ^{-1} which have been purified adding further $r/2$ modes. The second set describes instead a collection of $2((k-1-r)/2) = k-1-r$ modes prepared in a pure state formed by $(k-1-r)/2$ independent pairs of modes which are entangled. The third set represents a single mode in the vacuum state.

In the following, however, we will show that one can construct a unitary dilation that involves ρ_E with only $\ell = k - r'/2$ modes and, dropping the purity requirement, with only $\ell = k - r/2$ modes. ■

Now, let us note that with the choice we made on the symplectic form $\sigma_{2\ell}^E = \sigma_{2k}$, the two matrices α and β are $k \times k$ covariance matrices for two sets of independent $k/2$ Bosonic modes satisfying the uncertainty relations with respect to the form σ_k. In turn, the matrices δ and δ^T represent cross-correlation terms among such sets. After all, the entire covariance matrix γ_E corresponds to a pure Gaussian state. We characterize the minimal number of the environmental modes in the unitary dilations of multi-mode maps according to the following theorem.

Theorem 4 (Minimal dilations of Gaussian channels) *Given a BGC Φ described by matrices X and Y satisfying the conditions (3.47) and characterized by the quantities $k = rank[Y]$ and*

$$r = rank[\Sigma]\,, \qquad r' = k - rank[Y - \Sigma Y^{\ominus 1}\Sigma^T]\,. \tag{3.212}$$

(where $Y^{\ominus 1}$ denotes the Moore-Penrose inverse [119]), then it is possible to construct a unitary dilation of Stinespring form (i.e., involving a pure Gaussian state ρ_E) with at most $\ell_{pure} = k - r'/2$ environmental modes. It is also always possible to construct a unitary dilation using $\ell = k - r/2$ environmental modes which are prepared in a Gaussian, but not necessarily pure state.

Proof: For simplicity we will consider the case in which k is even, but trivially it holds also for the odd case. The key point is now the observation that in Eq. (3.208), the matrix A couples only with those rows and columns of the matrices δ and β which contain elements $\xi \mathbb{1}_{n-r/2}$ or $f(\xi \mathbb{1}_{n-r/2})$: as far as A is concerned, one could indeed replace the element μ^{-1} and $f(\mu^{-1})$ of such matrices with zeros. The only reason why we keep these element is to render γ_E the covariance matrix of a minimal uncertainty state. In other words, the elements of δ and β proportional to μ^{-1} or $f(\mu^{-1})$ are only introduced to purify the corresponding element of the submatrix α, which is in itself hence a covariance matrix of a mixed Gaussian state.

Suppose then that μ has (say) the first $r'/2$ eigenvalues equal to 1, i.e., $\mu_1 = \mu_2 = \cdots = \mu_{r'/2} = 1$ while for $j \in \{r'/2+1, \cdots, r/2\}$ we have that $\mu_j \in (0,1)$. In this case the corresponding sub-matrix of α associated with those elements represent a pure Gaussian state, specifically the vacuum state. Accordingly, there is no need to add further modes to purify them. Taking this into account, one can hence reduce the number of environmental modes $\ell_{\rm pure}$ that allows one to represent Φ in term of a *pure state* ρ_E from k to

$$\ell_{\rm pure} = k/2 + (k/2 - r'/2) = k - r'/2. \tag{3.213}$$

Indeed, we need the $k/2$ modes of α plus $k/2 - r'/2$ additional modes of β to purify those of α which are not in a pure state yet. An easy way to characterize the parameter r' is to observe that, according to Eq. (3.203), it corresponds to the number of eigenvalues having modulus 1 of the matrix of $O\Sigma' O^T$, i.e.,

$$\begin{aligned} r' &= k - \mathrm{rank}[\mathbb{1}_{2n} - \Sigma'(\Sigma')^T] = k - \mathrm{rank}[\mathbb{1}_{2n} - \Sigma'(\Sigma')^T] \\ &= k - \mathrm{rank}[Y - \Sigma\, Y^{\ominus 1}\, \Sigma^T]\,, \end{aligned} \tag{3.214}$$

where $Y^{\ominus 1}$ denotes the Moore-Penrose inverse of Y [119]. Here we notice that for $r' = r = 2n$ we get $\ell_{\rm pure} = n$. This should correspond to the channels (3.201) of class (i) for which one can construct a unitary dilation with pure input states. Indeed, according to Eq. (3.214), when $r' = 2n$ the matrix $Y - \Sigma\, Y^{-1}\, \Sigma^T$ must be zero, leading to the identity (3.201). The explicit expressions for corresponding values of γ_E and s_2 are obtained in the following way. We choose the environmental symplectic form to be $\sigma_{2\ell}^E = \sigma_k \oplus \sigma_{k-r'}$ as above. A unitary dilation with $\ell_{\rm pure} = k - r'/2$ environmental modes in a pure state is obtained by having $s_2 = C s_2'$ with s_2' as in Eq. (3.210). In this case, however, A is a rectangular matrix $k \times 2(k/2 - r'/2)$ of the form

$$A = \left[\begin{array}{cc|cc} & & 0 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 0 \\ & & 0 & \mathbb{1}_{(k-r)/2} \\ \hline 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{0} \\ 0 & \mathbb{1}_{(k-r)/2} & & \end{array}\right] \begin{array}{l} \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, k/2 - r/2 \\ \}\, r'/2 \\ \}\, (r-r')/2 \\ \}\, k/2 - r/2 \end{array}. \tag{3.215}$$

Similarly, the covariance matrix γ_E can be still expressed as in Eq. (3.188). In this case, yet, α is a $k \times k$ matrix of block form

$$
\alpha = \left[\begin{array}{ccc|ccc}
\mathbb{1}_{r'/2} & 0 & 0 & & & \\
0 & \mu_o^{-1} & 0 & & 0 & \\
0 & 0 & \xi\mathbb{1}_{(k-r)/2} & & & \\
\hline
 & & & \mathbb{1}_{r'/2} & 0 & 0 \\
 & 0 & & 0 & \mu_o^{-1} & 0 \\
 & & & 0 & 0 & \xi\mathbb{1}_{(k-r)/2}
\end{array}\right]
\begin{array}{l}
\}\, r'/2 \\
\}\, (r-r')/2 \\
\}\, (k-r)/2 \\
\}\, r'/2 \\
\}\, (r-r')/2 \\
\}\, (k-r)/2,
\end{array}
$$

where $\xi = 5/4$ and μ_o is the $(r-r')/2 \times (r-r')/2$ diagonal matrix formed by the elements of μ which are strictly smaller than 1. β is the $(k-r') \times (k-r')$ matrix

$$
\beta = \left[\begin{array}{cc|cc}
\mu_o^{-1} & 0 & & \\
0 & \xi\mathbb{1}_{(k-r)/2} & & 0 \\
\hline
 & & \mu_o^{-1} & 0 \\
 & 0 & 0 & \xi\mathbb{1}_{(k-r)/2}
\end{array}\right]
\begin{array}{l}
\}\, (r-r')/2 \\
\}\, (k-r)/2 \\
\}\, (r-r')/2 \\
\}\, (k-r)/2,
\end{array}
$$

and

$$
\delta = \left[\begin{array}{cc|cc}
 & & 0 & 0 \\
 & 0 & f(\mu_o^{-1}) & 0 \\
 & & 0 & f(\xi\mathbb{1}_{(k-r)/2}) \\
\hline
0 & 0 & & \\
f(\mu_o^{-1}) & 0 & & 0 \\
0 & f(\xi\mathbb{1}_{(k-r)/2}) & &
\end{array}\right]
\begin{array}{l}
\}\, r'/2 \\
\}\, (r-r')/2 \\
\}\, (k-r)/2 \\
\}\, r'/2 \\
\}\, (r-r')/2 \\
\}\, (k-r)/2,
\end{array}
$$

with f defined above.

By looking at the structure of this covariance matrix, one realizes that it is composed by three independent pieces. The first one describes a collection of $r'/2$ vacuum states. The second one, in turn, describes $(r-r')/2$ thermal states characterized by the matrices μ_o^{-1} which have been purified by adding further $(r-r')/2$ modes. The third one, finally, reflects a collection of $2(k/2-r/2) = k-r$ modes prepared in a pure state formed by $k/2-r/2$ independent pairs of modes which are entangled.

Now, taking into account that $r' \leqslant r = \text{rank}[\Sigma]$, a further reduction in the number of modes ℓ can be obtained by dropping the requirement of γ_E being a minimal uncertainty covariance matrix. Indeed, an alternative unitary representation of Φ can be constructed with only

$$
\ell = k/2 + (k/2 - r/2) = k - r/2\,, \tag{3.216}
$$

environmental modes. In fact, choosing the symplectic form $\sigma_{2\ell}^E = k \oplus \sigma_{k-r}$, the matrix s_2' can be still expressed as in Eq. (3.210). but with A being a rectangular matrix $k \times (k-r)$ of the form

$$A \;=\; \left[\begin{array}{c|c} 0 & 0 \\ \cline{2-2} & \mathbb{1}_{(k-r)/2} \\ \hline 0 & \\ \cline{1-1} \mathbb{1}_{(k-r)/2} & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2, \end{array} \;. \tag{3.217}$$

Similarly, γ_E has the block form (3.188), where α is still the $k \times k$ matrix as above, while β and δ are, respectively, the following $(k-r) \times (k-r)$ and $k \times (k-r)$ real matrices:

$$\beta = \left[\begin{array}{c|c} \xi\mathbb{1}_{(k-r)/2} & 0 \\ \hline 0 & \xi\mathbb{1}_{(k-r)/2} \end{array}\right] \begin{array}{l} \}\, (k-r)/2 \\ \}\, (k-r)/2, \end{array} \tag{3.218}$$

$$\delta = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & f(\xi\mathbb{1}_{(k-r)/2}) \\ \hline 0 & 0 \\ \hline f(\xi\mathbb{1}_{(k-r)/2}) & 0 \end{array}\right] \begin{array}{l} \}\, r/2 \\ \}\, (k-r)/2 \\ \}\, r/2 \\ \}\, (k-r)/2, \end{array} \tag{3.219}$$

with ξ and f as above. This covariance matrix now consists of two independent parts: the first one describes a collection of $r/2$ thermal states described by the matrices μ^{-1}. The second instead reflects a collection of $2(k/2 - r/2) = k - r$ modes prepared in a pure state formed by $k/2 - r/2$ independent couples of modes which are entangled. ■

3.4.3 Minimal noise channels

In a very analogous fashion to the extremal covariance matrices corresponding to pure Gaussian states as in Eq. (3.44), one can introduce the concept of a minimal noise channel. In this section we review the concept of such minimal noise channels [104] and provide criteria to characterize them [54]. Given $X, Y \in \mathbb{R}^{2n\times 2n}$ satisfying the inequality (3.47), any other $Y' = Y + \Delta Y$ with $\Delta Y \geqslant 0$ will also satisfy such condition, i.e.,

$$Y' \geqslant Y \geqslant i(\sigma_{2n} - X^T \sigma_{2n} X)\,. \tag{3.220}$$

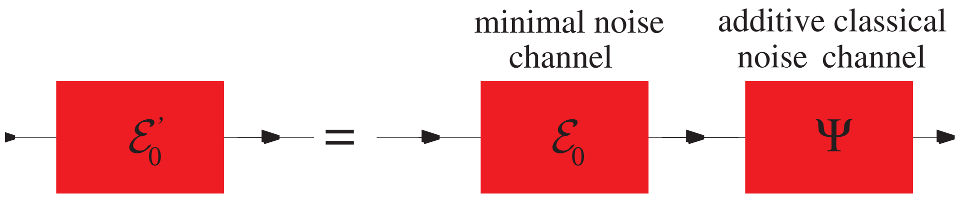

Figure 3.15: The noise associated with a generic BGC can be conceived of as originating from two sources. The first contribution may be regarded as *true quantum noise* associated with a *minimal noise channel* $\mathcal{E}_0$. The other one may be identified with an additive classical noise Ψ reflecting appropriate random displacements in phase space.

Furthermore, due to the compositions rules (3.54), the BGC $\mathcal{E}'$ associated with the matrices X, Y' can be described as the composition

$$\mathcal{E}' = \Psi \circ \mathcal{E} \,, \tag{3.221}$$

between the channel $\mathcal{E}$ associated with the matrices X, Y, and the channel Ψ described by the matrices $X = \mathbb{1}_n$ and $Y = \Delta Y$. The latter belongs to a special case of BGC that includes the so called *additive classical noise channels* [53, 45, 59] – see Fig. 3.15, Sec. 3.3.3 for $n = 1$ and Sec. 3.4.4 for $n > 1$.

For any $X \in \mathbb{R}^{2n \times 2n}$, one can then ask how much *noise* Y it is necessary to add in order to obtain a map satisfying the condition (3.47). This gives rise to the notion of *minimal noise* [104], as the extremal solutions Y of Eq. (3.47) for a given X. The corresponding *minimal noise channels* are the natural analogue of the Gaussian pure state and allows one to represent any other BGC as in Eq. (3.221) with a proper choice of the additive classical noise map Ψ.

Let us start considering the case of a generic channel $\mathcal{E}'$ of class (i) described by matrices X and Y'. According to Theorem 3 it admits unitary dilation with $\ell = n$ modes described by some covariance matrix γ_E' satisfying the condition

$$Y' = s_2 \, \gamma_E' \, s_2^T \,, \tag{3.222}$$

for some proper $2n \times 2n$ real matrix s_2. According to Eq. (3.44) γ_E can be written as

$$\gamma'_E = \gamma_E + \Delta\,, \tag{3.223}$$

with $\Delta \geqslant 0$ and γ_E minimal uncertainty state. Therefore writing $Y = s_2\gamma_E s_2^T$ and $\Delta Y = s_2 \Delta s_2^T$ we can express $\mathcal{E}'$ as in (3.221), where now $\mathcal{E}$ is the BGC associated with the minimal noise environmental state γ_E. Most importantly, since the decomposition (3.223) is optimal for γ'_E, the channel $\mathcal{E}$ is an extremal solution of Eq. (3.47). We stress that by construction $\mathcal{E}$ is still a channel of class (i): in fact it has the same Σ as $\mathcal{E}'$, while Y is still strictly positive since $\gamma_E > 0$ and s_2 is invertible – see Eq. (3.222). We can then use the results of Sec. 3.4.2 to claim that $\mathcal{E}$ must satisfy the equality (3.201). This leads us to establish three equivalent necessary and sufficient conditions for minimal noise channels of class (i):

$$(m_1) \qquad Y = \Sigma Y^{-1}\Sigma^T\,, \tag{3.224}$$

$$(m_2) \qquad \mathrm{Det}[Y] = \mathrm{Det}[\Sigma]\,, \tag{3.225}$$

$$(m_3) \qquad r = r'\,, \tag{3.226}$$

with r and r' as in Eq. (3.212). Since for class (i) we have that $r = 2n$, the minimal noise condition m_3 simply requires the eigenvalues of the matrix μ of Eq. (3.203) to be equal to unity. Similarly, minimal noise channels in case (ii) and (iii) can be characterized.

Theorem 5 (Minimal noise condition) *A Gaussian Bosonic channel characterized by the matrices Y and $X \in \mathbb{R}^{2n\times 2n}$ is a minimal noise channel if and only if, given $\Sigma = \sigma_{2n} - X^T\sigma_{2n}X$, one has*

$$Y = \Sigma Y^{\ominus 1}\Sigma^T\,. \tag{3.227}$$

Proof: The complete positivity condition (3.47) of a generic BGC is a positive semi-definite constraint for the symplectic form Σ, corresponding to the constraint $\gamma - i\sigma_{2n} \geqslant 0$ in case of covariance matrices of states of n modes. In general, $r = \mathrm{rank}[\Sigma]$ is not maximal, i.e., not equal to $2n$. When identifying the minimal solutions of the inequality (3.47), without loss of generality we can look for the minimal solutions of

$$Y' - i\Sigma' \geqslant 0, \tag{3.228}$$

where here

$$\Sigma' = \left[\begin{array}{cc|c} \begin{array}{c|c} 0 & \mu \\ \hline -\mu & 0 \end{array} & & \\ \hline & & 0 \end{array}\right], \tag{3.229}$$

with $\mu > 0$ being diagonal of rank $r/2$ (where $Y' = OYO^T$ and $\Sigma' = O\Sigma O^T$ with $O \in O(2n, \mathbb{R})$ orthogonal). The minimal solutions of inequality (3.228) are then given by $Y' = SS^T \oplus 0$, where S is a $r \times r$ matrix satisfying

$$S \left[\begin{array}{cc} 0 & \mu \\ -\mu & 0 \end{array}\right] S^T = \left[\begin{array}{cc} 0 & \mu \\ -\mu & 0 \end{array}\right], \tag{3.230}$$

so a symplectic matrix with respect to the modified symplectic form, so an element of $\{M \in Gl(r, \mathbb{R}) : M = (\mu^{1/2} \oplus \mu^{1/2}) S (\mu^{-1/2} \oplus \mu^{-1/2}), S \in Sp(r, \mathbb{R})\}$. From this, it follows that the minimal solutions of (3.228) are exactly given by the solutions of $Y' = \Sigma'(Y')^{\ominus 1}(\Sigma')^T$, from which the statement of the theorem follows. ■

3.4.4 Additive classical noise channel

In this section we focus on the maps Ψ which enter in the decomposition (3.221). They are characterized by having $X = \mathbb{1}_{2n}$ and $Y \geqslant 0$. Note that with this choice the condition (3.47) is trivially satisfied. This is the classical noise channel that has frequently been considered in the literature (for a review, see, e.g., Ref. [59]). For completeness of the presentation, we briefly discuss this class of multi-mode BGC [54].

If the matrix Y is strictly positive, the channel Ψ is the multi-mode generalization of the single mode additive classical noise channel, associated to the canonical form B_2 in Sec. 3.3.3 [53, 45, 59]. Indeed, one can show that these maps are the (Gaussian) unitary equivalent to a collection of n single mode additive classical noise maps. To see this, let us apply symplectic transformations (S_1 and S_2) before and after the channel Ψ. Following Eq. (3.55) this leads to $\{\mathbb{1}_n, Y\} \mapsto \{S_1 S_2, \;\; S_2^T Y S_2\}$. Now, since $Y > 0$, according to Williamson's theorem (see Appendix A) [115], we can find a $S_2 \in Sp(2n, \mathbb{R})$ such that $S_2^T Y S_2$ is diagonal $\mathrm{diag}(\lambda_1, \cdots, \lambda_n, \lambda_1, \cdots, \lambda_n)$ with $\lambda_i > 0$. We can then take $S_1 = S_2^{-1}$ to have $S_1 S_2 = \mathbb{1}_{2n}$. For $Y \geqslant 0$ but not $Y > 0$, the maps Ψ that enter the decomposition Eq. (3.221), however, include also channels which are not unitarily equivalent to a collection of B_2 maps. An explicit example of this situation is constructed in Sec. 3.4.8.

3.4.5 Canonical form

Analogously to the one-mode case in Sec. 3.3 [53, 48, 56], any BGC $\mathcal{E}$, described by the transformation Eq. (3.49), can be simplified through unitarily equivalence, by applying unitary canonical transformations before and after the action of the channel, which induce transformations of the form (3.55) [54]. Specifically, given a n-mode Gaussian channel $\mathcal{E}$ described by matrix X and Y, we can transform it into a new n-mode Gaussian channel $\mathcal{E}_c$ described by the matrices

$$X_c = S_1 X S_2 \,, \qquad Y_c = S_2^T Y S_2 \,, \tag{3.231}$$

with $S_{1,2} \in Sp(2n, \mathbb{R})$. As already discussed in the introductory Sec. 3.2, from an information theoretical perspective $\mathcal{E}$ and $\mathcal{E}_c$ are equivalent in the sense that, for instance, their unconstrained quantum capacities coincide. We can then simplify the analysis of n-mode Gaussian channels by properly choosing S_1 and S_2 to induce a parametrization of the interaction part (i.e., X) of the evolution. The resulting canonical form follows from the generalization of the Williamson theorem [115] presented in Ref. [57] (see Appendix A for more details). According to this result, for every non-singular matrix X there exist matrices $S_{1,2} \in Sp(2n, \mathbb{R})$ such that

$$X_c = S_1 X S_2 = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J^T \end{bmatrix} , \tag{3.232}$$

with J^T being a $n \times n$ block-diagonal matrix in the real Jordan form of $X \sigma_{2n} X^T \sigma_{2n}^T$ [119]. This can be developed a little further by constructing a canonical decomposition for the symplectic matrix S associated with the unitary dilation (3.56) of the channel.

For the sake of simplicity in the following we will focus on the case of generic quantum channels $\mathcal{E}$ which have non-singular $X \in Gl(2n, \mathbb{R})$ and belong to the class (i) of Theorem 3 (i.e., which have $r = \mathrm{rank}[\Sigma] = 2n$). Under these conditions X must admit a canonical decomposition of the form (3.232) in which all the eigenvalues of J are different from 1. In fact one has

$$\Sigma = \sigma_{2n} - X^T \sigma_{2n} X = S_2^{-T} \left[\sigma_{2n} - X_c^T \sigma_{2n} X_c \right] S_2^{-1} = S_2^{-T}\, \Sigma_c \, S_2^{-1} \,, \tag{3.233}$$

with Σ_c being the skew-symmetric matrix associated with the channel $\mathcal{E}_c$, i.e.,

$$\Sigma_c := \begin{bmatrix} 0 & \mathbb{1}_n - J \\ J^T - \mathbb{1}_n & 0 \end{bmatrix} . \tag{3.234}$$

Since $\mathrm{rank}[\Sigma_c] = \mathrm{rank}[\Sigma] = 2n$, it follows that J cannot have eigenvalues equal to 1. Similarly, it is not difficult to see that if X has a canonical form as in Eq. (3.232) with all the eigenvalues of J being different from 1, then $\mathcal{E}$ and $\mathcal{E}_c$ are of class (i). However, a special case in which $X = \mathbb{1}_{2n}$ is investigated in detail in Sec. 3.4.8.

Consider then a unitary dilation (3.56) of the channel $\mathcal{E}_c$ constructed with a not necessarily pure Gaussian state ρ_E of $\ell = n$ environmental modes. According to the above considerations, such a dilation always exists. Let $S \in Sp(4n, \mathbb{R})$ be the $4n \times 4n$ real symplectic transformation (3.156) associated with the corresponding unitary U. Assuming $s_1 = X_c^T$, an explicit expression for this dilation can be obtained by writing

$$s_4 = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J' \end{bmatrix}, \; s_j = \begin{bmatrix} F_j & 0 \\ 0 & G_j \end{bmatrix}, \tag{3.235}$$

where, for $j = 2, 3$, F_j, G_j are $n \times n$ real matrices. Imposing Eqs. (3.157), one obtains the following relations

$$\begin{aligned} &J^T + F_2 G_2^T = \mathbb{1}_n\,, \qquad && J'^T + F_3 G_3^T = \mathbb{1}_n\,, \\ &G_3^T + F_2 J'^T = 0\,, && G_2^T + F_3 J^T = 0\,, \end{aligned} \tag{3.236}$$

whose solution gives an $S \in Sp(4n, \mathbb{R})$ of the form

$$S = \left[\begin{array}{cc|cc} \mathbb{1}_n & 0 & (\mathbb{1}_n - J^T)G^{-T} & 0 \\ 0 & J & 0 & G \\ \hline -G^T J^{-T} & 0 & \mathbb{1}_n & 0 \\ 0 & G^{-1}J(J-\mathbb{1}_n) & 0 & G^{-1}JG \end{array}\right], \tag{3.237}$$

with G being an arbitrary matrix $G \in Gl(n, \mathbb{R})$. As a consequence of this fact, and because the eigenvalues of J are assumed to be different from 1, s_2, s_3 and s_4 are also non-singular. This is important because it shows that in choosing S as in the canonical form (3.237) we are not restricting generality: the value of s_2 can always be absorbed into the definition of the covariance matrix γ_E of ρ_E by writing (see also Sec. 3.4.7)

$$\gamma_E = s_2^{-1}\, Y_c\, s_2^{-T}\,. \tag{3.238}$$

Taking this into account, we can conclude that Eq. (3.237) provides an explicit demonstration of Lemma 2 for channels of class (i) with non-singular X.

Since $\mathcal{E}_c$ is fully determined by X_c and Y_c, the above expressions show that the action of $\mathcal{E}_c$ on the input state does not depend on the choice of G. As a matter of fact, the latter can be seen as a Gaussian unitary operation U_G characterized by the n-modes symplectic transformation $Sp(2n, \mathbb{R})$,

$$\Delta_G = \left[\begin{array}{c|c} G^T & 0 \\ \hline 0 & G^{-1} \end{array}\right], \tag{3.239}$$

applied to final state of the environment after the interaction with the input, i.e., $\tilde{\mathcal{E}}_G = U_G \tilde{\mathcal{E}} U_G^T$, where $\tilde{\mathcal{E}}$ is the weak complementary map for $G = \mathbb{1}_n$, and $\tilde{\mathcal{E}}_G$ is the weak complementary map in presence of $G \neq \mathbb{1}_n$ – see the next section for details. Since the relevant properties of a channel (e.g., weak degradability in Sec. 2.3 [52, 53]) do not depend on local unitary transformations to the input/output states, without loss of generality, we can consider $G = -J$ and the canonical form for $S \in Sp(4n, \mathbb{R})$ assumes the following simple expression

$$S = \left[\begin{array}{cc|cc} \mathbb{1}_n & 0 & \mathbb{1}_n - J^{-T} & 0 \\ 0 & J & 0 & -J \\ \hline \mathbb{1}_n & 0 & \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n - J & 0 & J \end{array}\right]. \tag{3.240}$$

The possibility of constructing different, but unitarily equivalent, canonical forms for S is discussed in Sec. 3.4.7.

3.4.6 Weak-degradability

Among other properties, the unitary dilations introduced so far are useful to define complementary or weak complementary channels of a given BGC $\mathcal{E}$, defined in Sec. 2.3, i.e. $\tilde{\mathcal{E}}(\rho) := \mathrm{Tr}_S[U(\rho \otimes \rho_E)U^\dagger]$, where ρ, ρ_E and U are defined as in Eq. (3.56), but the partial trace is now taken over the system modes. Specifically, let us recall that, if the state ρ_E we employed in constructing the unitary dilation of Eq. (3.56) is pure, then the map $\tilde{\mathcal{E}}$ is said to be the complementary of $\mathcal{E}$ and, up to partial isometry, it is unique [38, 93, 92, 80]. Otherwise it is called weak complementary [52, 53]. Since in Eq. (3.56) the state ρ_E is Gaussian and U is a unitary Gaussian transformation, one can verify that $\tilde{\mathcal{E}}$ is also BGC[11]. Expressing the Gaussian unitary transformation

[11] In general, however, it will not map the n input modes into n output modes. Instead, it will transform them into ℓ modes, with ℓ being the number of modes assumed in the unitary dilation (3.56).

U in terms of its symplectic matrix S of Eq. (3.156) the action of $\tilde{\mathcal{E}}$ is fully characterized by the following mapping of the covariance matrices γ of ρ, i.e.,

$$\tilde{\mathcal{E}}: \; \gamma \longmapsto \; s_3 \gamma s_3^T + s_4 \gamma_E s_4^T \,, \tag{3.241}$$

which is counterpart of the transformations (3.48) and (3.158) that characterize $\mathcal{E}$. The channel $\tilde{\mathcal{E}}$ is then described by the matrices $\tilde{X} = s_3^T$ and $\tilde{Y} = s_4 \gamma_E s_4^T$ which, according to the symplectic properties (3.157), satisfy the condition

$$\tilde{Y} \geqslant i\tilde{\Sigma} \qquad \text{with} \quad \tilde{\Sigma} := \sigma_{2\ell}^E - \tilde{X}^T \sigma_{2n} \tilde{X} \,. \tag{3.242}$$

As shown in Sec. 2.3, the relations between $\mathcal{E}$ and its weak complementary $\tilde{\mathcal{E}}$ contain useful information about the channel $\mathcal{E}$ itself. For instance, recall that we say that the channel $\mathcal{E}$ is weakly degradable (WD) while $\tilde{\mathcal{E}}$ is anti-degradable (AD), if Eq. (2.52) holds. In the following, we will use the compact notation WD/AD to refer to weakly and anti- degradable channels, respectively. A complete weak-degradability analysis of one-mode Bosonic Gaussian channels has been provided in Sec. 3.3 [52, 53]. Here we generalize those results to $n > 1$ [54].

A criterion for weak degradability

We review a general criterion for degradability of BGCs which was introduced in Ref. [55], adapting it to include also weak degradability [54]. Before entering the details of our derivation, let us point out that tensoring weakly degradable (anti-degradable) one-mode Gaussian channels with weakly degradable (anti-degradable) one-mode Gaussian channels yield multi-mode Gaussian channel which are weakly degradable (anti-degradable). For instance, $\mathcal{E}^{\otimes n}$ is weakly degradable (or anti-degradable) if $\mathcal{E}$ satisfies the same property. This can be easily verified. In the general case, however, it is worth noticing that generic multi-mode Gaussian channels are neither WD nor AD (e.g., the two-mode Gaussian channel which acts on the first mode as a beam-splitter transformation with transmissivity $k_1^2 < 1/2$ and on the second as a beam-splitter transformation with transmissivity $k_2^2 > 1/2$). In this respect the weak-degradability property of one-mode Gaussian maps (studied in Sec. 3.3) turns out to be rather remarkable. Consider in fact a WD single-mode Gaussian channel $\mathcal{E}$ having no zero quantum capacity $Q > 0$ (e.g., a beam-splitter channel with transmissivity $> 1/2$). Define then the two mode channel $\mathcal{E} \otimes \tilde{\mathcal{E}}$ with $\tilde{\mathcal{E}}$ being its weak complementary. This is Gaussian since both $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are Gaussian. The claim is that $\mathcal{E} \otimes \tilde{\mathcal{E}}$ is neither WD nor AD. Indeed, its weak complementary can be identified with the map $\tilde{\mathcal{E}} \otimes \mathcal{E}$. Consequently, since $\mathcal{E} \otimes \tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}} \otimes \mathcal{E}$ differ by a permutation, they must have the same quantum capacity Q'. Therefore if one of the two is WD than *both* of them must also be AD. In this case Q' should be zero which is clearly not possible given that $Q' \geqslant Q$. In fact, one can use $\mathcal{E} \otimes \tilde{\mathcal{E}}$ to reliably transfer quantum information by encoding it into the inputs of $\mathcal{E}$. In this respect the possibility (shown in Sec. 3.3) of classifying (almost) all single-mode Gaussian maps in terms of weak degradability property turns to be rather a remarkable property. We now turn to investigating the weak degradability properties of multi-mode Bosonic Gaussian channels deriving a criterion that will be applied in Sec. 3.5.1 for studying in detail the two-mode channel case [54].

Consider a n-mode Bosonic Gaussian channel $\mathcal{E}$ characterized the unitary dilation (3.56) and its weak complementary $\tilde{\mathcal{E}}$. Let $\{X, Y\}$, $\{\tilde{X}, \tilde{Y}\}$ be the matrices which define such channels. For the sake of simplicity we will assume X and $\tilde{X}$ to be non-singular, i.e. $X, \tilde{X} \in Gl(2n, \mathbb{R})$. Examples of such maps are for instance the channels of class (i) with X non-singular described in Sec. 3.4.5. Adopting in fact the canonical form (3.240) for S we have that

$$X = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J^T \end{bmatrix}, \qquad \tilde{X} = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n - J^T \end{bmatrix} \tag{3.243}$$

with all the eigenvalues of J being different from 1.

Suppose now that $\mathcal{E}$ is weakly degradable with $\mathcal{T}$ being the connecting CPT map which satisfies the weak degradability condition (2.52). As in Sec. 3.3 for single mode [52, 53], we will focus on the case in which $\mathcal{T}$ is BGC and described by matrices $\{X_{\mathcal{T}}, Y_{\mathcal{T}}\}$. Under these hypothesis the identity (2.52) can be simplified by using the composition rules for BGCs given in Eq. (3.54). Accordingly, one must have

$$\begin{aligned} X_{\mathcal{T}} &= X^{-1}\tilde{X}\,, \\ Y_{\mathcal{T}} &= \tilde{Y} - X_{\mathcal{T}}^T Y X_{\mathcal{T}}\,. \end{aligned} \tag{3.244}$$

These definitions must be compatible with the requirement that $\mathcal{T}$ should be a CPT map which transforms the n system modes into the ℓ environmental modes, i.e.,

$$Y_{\mathcal{T}} \geqslant i\left(\sigma_{2\ell}^E - X_{\mathcal{T}}^T \sigma_{2n} X_{\mathcal{T}}\right)\,. \tag{3.245}$$

Combining the expressions above, one finds the following weak-degradability condition for n-mode Bosonic Gaussian channels [55], i.e.

$$\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E \geqslant 0\,. \tag{3.246}$$

In order to obtain the anti-degradability condition (2.53), it is sufficient to swap $\{X, Y\}$ with $\{\tilde{X}, \tilde{Y}\}$ and the system commutation matrix σ_{2n} with $\sigma_{2\ell}^E$, in Eq. (3.246), i.e.,

$$Y - X^T \tilde{X}^{-T}(\tilde{Y} + i\sigma_{2\ell}^E)\tilde{X}^{-1}X + i\sigma_{2n} \geqslant 0\,. \tag{3.247}$$

Equations (3.246) and (3.247) are strictly related. Indeed, since

$$\begin{aligned} &Y - X^T \tilde{X}^{-T}(\tilde{Y} + i\sigma_{2\ell}^E)\tilde{X}^{-1}X + i\sigma_{2n} \qquad (3.248)\\ &\quad = -X^T\tilde{X}^{-T}\left(\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E\right)\tilde{X}^{-1}X\,, \end{aligned}$$

equation (3.247) corresponds to reverse the sign of the inequality (3.246), i.e.

$$\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E \leqslant 0\,. \tag{3.249}$$

Hence to determine if $\mathcal{E}$ is a weakly degradable or anti-degradable channel, it is then sufficient to study the positivity of the Hermitian matrix

$$W := \tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma_{2n})X^{-1}\tilde{X} + i\sigma_{2\ell}^E\,. \tag{3.250}$$

Note that for $n = 1$ the condition in Eq. (3.246) reduces more simply to $\mathrm{Det}(X) - 1/2 \geqslant 0$ (analogously, Eq. (3.247) reduces to $\mathrm{Det}(X) - 1/2 \leqslant 0$ for the anti-degradability), recovering the results for one-mode BGCs in Sec. 3.3.4.

In the case in which $\ell = n$ this can be simplified by reminding that an Hermitian $2n \times 2n$ matrix W partitioned as

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^\dagger & W_3 \end{bmatrix} \tag{3.251}$$

with W_i being $n \times n$ matrices is semi-positive definite if and only if

$$W_1 \geqslant 0 \;\text{ and }\; W_3 - W_2^\dagger W_1^{-1} W_2 \geqslant 0\,, \tag{3.252}$$

the right hand side being the Schur complement of W (see, e.g., page 472 in Ref. [119]). Using this result and the canonical form (3.240), Eq. (3.246) can be written as in Eq. (3.252) with

$$\begin{aligned} W_1 &= (\mathbb{1}_n - J^{-T})^{-1} Y_1 (\mathbb{1}_n - J^{-1})^{-1} - Y_1 \\ W_2 &= i(J^{-T} - 2\mathbb{1}_n) - Y_2(J^{-T} - \mathbb{1}_n) - (\mathbb{1}_n - J^{-T})^{-1} Y_2 \\ W_3 &= Y_3 - (J^{-1} - \mathbb{1}_n) Y_3 (J^{-T} - \mathbb{1}_n)\,, \end{aligned} \tag{3.253}$$

and

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}. \tag{3.254}$$

For the anti-degradability condition (3.247) simply replace [$\geqslant$] with [$\leqslant$] in Eq. (3.252).

3.4.7 Equivalent unitary dilations

Let

$$S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \tag{3.255}$$

and γ_E define a unitary dilation for a Bosonic Gaussian channel $\mathcal{E}$ characterized by matrices X and Y, as in Theorem 3. Then a full class of unitary dilations

$$S' = \begin{bmatrix} s_1' & s_2 \\ s_3' & s_4' \end{bmatrix} \tag{3.256}$$

can be obtained by taking $\gamma_E' = V\gamma_E V^T$ and

$$s_1' = s_1\,, \quad s_2' = s_2 V \quad s_3' = W s_3\,, \quad s_4' = W s_4 V\,, \tag{3.257}$$

with $V \in Sp(2\ell, \mathbb{R})$ and $W \in Sp(2n, \mathbb{R})$ being symplectic transformations of ℓ and n modes respectively [54]. With this choice in fact γ_E' is still a covariance matrix while the conditions (3.157) and (3.158) are automatically satisfied. From a physical point of view, the symplectic transformations V and W correspond to unitary local operations applied to the environmental input and output states, respectively, by virtue of the metaplectic representation. Consequently, the weak complementary channels $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}'$ associated with these two representations are unitarily equivalent and the weak-degradability properties one can determine for $\mathcal{E}$ will be the same when studied for $\mathcal{E}'$ (as shown in Sec. 2.3).

Conversely, let us suppose to have two unitary dilations of $\mathcal{E}$, realized with $\ell = n$ environmental modes and characterized by the symplectic matrices S and S' as in Eq. (3.255) and (3.256), respectively, with s_i and s'_i being $2n \times 2n$ square matrices. Then it is possible to show that they must be related as in Eq. (3.257) under the hypothesis that s_2 and s_3 are non-singular. First of all, since Eq. (3.158) must be satisfied for all the input covariance matrices γ, we have $s_1 = X^T = s'_1$. Define then $V = s_2^{-1} s'_2$ and $W = s'_3 s_3^{-1}$. By using the first of Eq. (3.157) and exploiting the non-singularity of s_2 one has

$$s_2 \, V \, \sigma_{2\ell}^E \, V^T \, s_2^T = s_2 \, \sigma_{2n} \, s_2^T \quad \Longrightarrow \quad V \, \sigma_{2n} \, V^T = \sigma_{2n} \,, \tag{3.258}$$

which implies that V is a symplectic matrix (we are assuming $\sigma_{2\ell}^E = \sigma_{2n}$). Moreover, from the second condition in Eqs. (3.157) for S and S', we obtain

$$s_2 \sigma s_4^T W^T = s_2 V \sigma s_4'^T \,, \qquad \Longrightarrow \qquad s'_4 = W s_4 V \,, \tag{3.259}$$

because s_2 is non-singular and V is symplectic. By considering the third condition (3.157) one then has

$$W(s_3 \sigma_{2n} s_3^T + s_4 \sigma_{2n} s_4^T) W^T = W \sigma_{2n} W^T = \sigma_{2n} \tag{3.260}$$

which prove that W is a symplectic. Finally, let us observe that the proof above does not use the non-singularity of s_3. Indeed, one can relax this hypothesis and assume more simply that there exists a W such that $s'_3 = W s_3$; from Eqs. (3.157) W has to still be a symplectic matrix but s_3 and s'_3 may be singular.

As an application of these equivalent unitary dilation results, we can find an alternative canonical form to the one in Sec. 3.4.5 with the same s_1 and s_4 but with s_2 and s_3 of the following anti-diagonal block form

$$s_j = \left[\begin{array}{cc} 0 & F_j \\ G_j & 0 \end{array} \right] \tag{3.261}$$

where, for $j = 2, 3$, F_j, G_j are $n \times n$ real matrices. Imposing Eqs. (3.157), one obtains the following relations

$$\begin{aligned} J^T - F_2 G_2^T &= \mathbb{1}_n \quad , \quad J'^T - F_3 G_3^T = \mathbb{1}_n, \\ F_2 - F_3^T &= 0 \quad , \quad J G_3^T - G_2 J'^T = 0 \,, \end{aligned} \tag{3.262}$$

the solution of which provides the following unitary dilation,

$$S = \left[\begin{array}{cc|cc} \mathbb{1}_n & 0 & 0 & -(\mathbb{1}_n - J^T)G_2^{-T} \\ 0 & J & G_2 & 0 \\ \hline 0 & -G_2^{-1}(\mathbb{1}_n - J) & \mathbb{1}_n & 0 \\ G_2^T & 0 & 0 & G_2^T J^T G_2^{-T} \end{array}\right], \tag{3.263}$$

where again G_2 is an arbitrary (non-singular) matrix and the eigenvalues of J are assumed to be different from 1. This solution is unitarily equivalent to the one in Eq. (3.237) by applying $V = -\sigma_{2n}$ and

$$W = \left[\begin{array}{cc} 0 & G_2^{-1}J^{-1}G_2 \\ -G_2^T J^T G_2^{-T} & 0 \end{array}\right] \tag{3.264}$$

as above.

3.4.8 The ideal-like quantum channel

Here we consider a quantum channel with $X = \mathbb{1}_{2n}$ but $Y \geqslant 0$ with rank less than $2n$, which can be described in terms of only n additional (environmental) modes [54]. We call it ideal-like quantum channel. Accordingly, the canonical unitary transformation U of Eq. (3.56) will be uniquely determined by a $4n \times 4n$ real matrix $S \in Sp(4n, \mathbb{R})$ of block form in Eq. (3.156), where s_i are $2n \times 2n$ real matrices. Particularly, $s_1 = s_4 = \mathbb{1}_{2n}$,

$$s_3 = \left[\begin{array}{cc} F_3 & 0 \\ 0 & G_3 \end{array}\right], \qquad s_2 = \left[\begin{array}{cc} -G_3^T & 0 \\ 0 & -F_3^T \end{array}\right], \tag{3.265}$$

with F_3 and G_3 being $n \times n$ real matrices such that $F_3 G_3^T = G_3^T F_3 = 0$, in order to satisfy the symplectic conditions in Eqs. (3.157). Taking advantage of the freedom in the choice of the unitary dilation shown in Sec. 3.4.7, the matrix S can be put in the form of Eq. (3.156) in which $s_1' = s_4' = \mathbb{1}_{2n}$,

$$s_2' = \left[\begin{array}{cc} 0 & 0 \\ 0 & \mathbb{1}_n \end{array}\right], \qquad s_3' = \left[\begin{array}{cc} -\mathbb{1}_n & 0 \\ 0 & 0 \end{array}\right], \tag{3.266}$$

where F_3 is assumed non-singular. In this respect, one uses $V, W \in Sp(2n, \mathbb{R})$ (of Sec. 3.4.7) of the following form

$$V = \left[\begin{array}{cc} -F_3 & 0 \\ 0 & -F_3^{-T} \end{array}\right], \tag{3.267}$$

and $W = V^{-1}$. Similarly, one can proceed, if G_3 is non-singular, and obtains a similar structure for S as above. As concerns the weak-degradability properties, if one assumes the initial environmental input state in a thermal state, e.g. $\gamma_E = \mathrm{diag}(2N+1, 2M+1, 2N+1, 2M+1)$, the eigenvalues of $\tilde{Y} - \tilde{X}^T X^{-T}(Y + i\sigma)X^{-1}\tilde{X} + i\sigma$ are $\{2M, 2(M+1), 2N, 2(N+1)\}$, which are always positive for any $N \geqslant 0$ and $M \geqslant 0$; hence, this channel with γ_E as above is always weakly degradable.

Finally, one may consider another ideal-like channel with $X = \mathbb{1}_{2n}$ and $Y = [(1-\sigma_z)/2]^{\otimes n}$, i.e. $\mathcal{E}_{X,Y} = \bigotimes_{i=1}^n B_{1i}$, where the single-mode B_1 channel is defined in Sec. 3.3.3 [53] as $X = \mathbb{1}_2$ and $Y = (1-\sigma_z)/2$. Trivially, this multi-mode channel is always WD (like B_1) and is able to transfer a quantum state without decoherence with the maximum quantum capacity (like for the single-mode case).

3.5 Two-mode Bosonic Gaussian channels

Now we consider a particular case of n-mode Bosonic Gaussian channel analysis above, namely, the case of $n = 2$ [54]. This is by no means such a special case as one might at first be tempted to think since any n-mode channel can always be reduced to single-mode and two-mode parts [57]. For two-mode channels the interaction part and the noise term of a generic two-mode Bosonic Gaussian channel, X and Y, respectively, are 4×4 real matrices. Particularly, we will focus on two-mode channels $\mathcal{E}$ which have non-singular X and belong to the class (i) of Theorem 3 (i.e., which have $r = \mathrm{rank}[\Sigma] = 4$), like in Sec. 3.4.5. These maps can be grasped in terms of a unitary dilation of the form (3.240) coupling the two system Bosonic modes with two additional (environmental) modes, where J is a 2×2 real Jordan block. In order to characterize this large class of two-mode BGCs, one has to examine only three possible forms of J:

- Class A: this corresponds to taking a diagonalizable Jordan block, that is,

$$J := J_0 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} . \tag{3.268}$$

where a and b are real nonzero numbers. It represents the trivial case of a two-mode Bosonic Gaussian channel, whose interaction term does not couple the two modes. Actually, we call it of class A_1 if $a \neq b$ and of class A_2 otherwise.

- Class B: this is to take J as a non-diagonalizable matrix with a nonzero real eigenvalue a with double algebraic multiplicity (but with geometric multiplicity equal to one), i.e.

$$J := J_1 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} . \tag{3.269}$$

  In this case the Jordan block is called defective [119]. Here, a noisy interaction between the Bosonic system and the environment, coupling the two system modes, is switched on.

- Class C: the real Jordan block J has complex eigenvalues, i.e.

$$J := J_2 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} , \tag{3.270}$$

  with $b \neq 0$; the eigenvalues of J are $a \pm ib$. Again, the two system modes are coupled by the noisy interaction with the environment through the presence of the element b.

In order to explicit the form of $Y = s_2 \gamma_E s_2^T$, with s_2 being defined as in Eq. (3.240), we consider a generic two-mode covariance matrix in the so-called standard form [120] for the environmental initial state γ_E, i.e.

$$\gamma_E = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} , \tag{3.271}$$

where

$$\Gamma_{1,2} := \begin{bmatrix} x & z_{-,+} \\ z_{-,+} & y \end{bmatrix} , \tag{3.272}$$

and x, y, $z_{+,-}$ are real number satisfying $x + y \geqslant 0$, $xy - z_-^2 \geqslant 1$ and $x^2y^2 - y^2 - x^2 + (z_- z_+ - 1)^2 - xy(z_-^2 + z_+^2) \geqslant 0$ because of the uncertainty principle. More generally, one can apply a generic two-mode (symplectic) squeezing operator $V(\epsilon)$ to the environmental input state, i.e.,

$$\gamma_E' = V(\epsilon) \gamma_E V(\epsilon)^T , \tag{3.273}$$

where

$$V(\epsilon) = \begin{bmatrix} R^{-T} & 0 \\ 0 & R \end{bmatrix}, \; R = \begin{bmatrix} c+hs & -qs \\ -qs & c-hs \end{bmatrix}, \tag{3.274}$$

and $c = \cosh(2r)$, $s = \sinh(2r)$, $h = \cos(2\phi)$, $q = \sin(2\phi)$ and $\epsilon = re^{2i\phi}$ being the squeezing parameter [120]. Finally, it is interesting to study how the canonical forms of two-mode BGCs compose under the product. A simple calculation shows that the following rules apply

$\circ$	A	B	C
A	A	A_1/B	$A_1/B/C$
B	A_1/B	A_2/B	$A_1/B/C$
C	$A_1/B/C$	$A_1/B/C$	A/C

.

In this table, for instance, the element on row 1 and column 1 represents the class (i.e., A) associated to the composition of two channels of the same class A. Note that the canonical form of the products with a "coupled" channel (i.e., with B or C) is often not uniquely defined. For instance, composing two class B channels, with

$$(J_1)_i = \begin{bmatrix} a_i & 1 \\ 0 & a_i \end{bmatrix}, \tag{3.275}$$

for $i = 1, 2$, will give us either a class A_2 channel (if $a_1 + a_2 = 0$) or a class B channel (if $a_1 + a_2 \neq 0$). Composition rules analogous to those reported above have been analyzed in detail for the one-mode case in Sec. 3.3.3. In the following we will study the weak-degradability properties of these three classes of two-mode Gaussian channels.

3.5.1 Weak-degradability properties

The weak-degradability conditions in Eqs. (3.252) become

$$\Gamma_1 - (\mathbb{1}_2 - J^{-T})\Gamma_1(\mathbb{1}_2 - J^{-1}) \geqslant 0 \tag{3.276}$$

and

$$\begin{aligned} &J\Gamma_2 J^T - (\mathbb{1}_2 - J)\Gamma_2(\mathbb{1}_2 - J^T) \\ &\quad -(J^{-1} - 2\mathbb{1}_2)\left[\Gamma_1 - (\mathbb{1}_2 - J^{-T})\Gamma_1(\mathbb{1}_2 - J^{-1})\right]^{-1}(J^{-T} - 2\mathbb{1}_2) \geqslant 0\,. \end{aligned} \tag{3.277}$$

In the same way, the anti-degradability is obtained when both these quantities are non-positive. As concerns the environmental initial state of the unitary dilation, one can consider a generic two-mode state as in Eq. (3.273). On one hand, we find that, if $[J,R]=0$, this two-mode squeezing transformation $V(\epsilon)$ can be simply "absorbed" in local symplectic operations to the output states and then it does not affect the weak-degradability properties. On the other hand, if $[J,R]\neq 0$, we find numerically that the introduction of correlations between the two environmental modes contrasts with the presence of (anti-) weak-degradability features. Therefore, one can consider the particular case in which the environment is initially in a symmetric state γ_E as in Eq. (3.271) with $x=y=2N+1$ and $z_-=z_+=0$ where $N\geqslant 0$. In this case $\gamma_E=(2N+1)\mathbb{1}_2$ corresponds to a thermal state of two uncoupled environmental modes with the same photon average number N and it is possible to see the results above easily through analytical details. In fact, we study analytically the positivity condition in Eq. (3.246) in the three possible forms of the real Jordan block J_i [54].

In the uncoupled case J_0 as in Eq. (3.268), substituting in Eq. (3.246), we find that these two-mode Bosonic Gaussian channels are WD if $a,b\geqslant 1/2$ and AD for $a,b\leq 1/2$ (any $N\geqslant 0$). In other words, in the case of two uncoupled modes, the weak-degradability properties can be derived from the results for one-mode Bosonic Gaussian channels: tensoring two WD (AD) one-mode Gaussian channels with WD (AD) one-mode Gaussian channels yield two-mode Gaussian channels which are WD (AD).

In the case of defective J, i.e., J_1 as in Eq. (3.269), corresponding to noisy interaction coupling the two system modes, substituting in Eq. (3.246), we find that, on one hand, these two-mode Bosonic Gaussian channels are WD if $a>1$ and

$$N\geqslant N_1:=\frac{1}{2}\left[-1+\frac{1}{2}\frac{|2a-1|}{\sqrt{a(a-1)}}\right]. \tag{3.278}$$

On the other hand, it is AD if $a<0$ and $N\geqslant N_1$ (see Fig. 3.17). Note that the defective Jordan blocks are not usually stable with respect to perturbations [57]. Indeed, we find numerically that, applying proper two-mode squeezing transformations to the environmental input, these weak-degradability conditions reduce to the decoupled case ones. In Fig. 3.16 we consider, for simplicity, a symmetric environmental initial state γ_E' as in Eq. (3.273) with $x=y$, $z_-=0$ and $\epsilon=r$, and we plot the relation between x, z_+

and the minimum value of r such that $J := J_1$ reduces to $J := J_0$ corresponding to the decoupled case. One realizes that a squeezing parameter r close to 1 is enough to make the interaction not coupling the two system modes, carrying quantum information. Moreover, let us point out that this squeezing threshold (r) increases slightly with the presence of correlations (z_+) while decreases when increasing the level of noise (x) in the initial environmental state γ'_E.

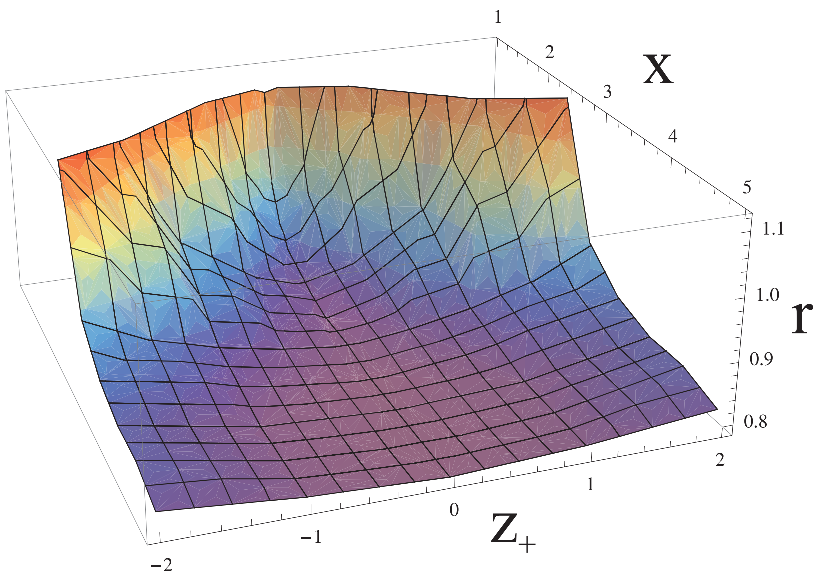

Figure 3.16: Relation between the parameters x, z_+ and the minimum value of r in the initial environmental state such that the two-mode channel with $X = \mathbb{1}_2 \oplus J_1$ reduces to the decoupled case $X' = \mathbb{1}_2 \oplus J_0$ with the same interaction parameter a for the two system modes.

Finally, in the case of real Jordan block with complex eigenvalues, i.e., J_2 as in Eq. (3.270), the corresponding two-mode Bosonic Gaussian channels are WD if $a > 1/2$ and

$$N \geqslant N_2 := \frac{1}{2}\left[-1 + \sqrt{1 + \frac{4b^2}{(1-2a)^2}}\,\right]. \tag{3.279}$$

while they are AD if $a < 1/2$ and $N \geqslant N_2$ (see Fig. 3.17). In both of these cases (real and complex eigenvalues), in which the interaction term couples the two Bosonic modes, there is the (apparently) counter-intuitive fact

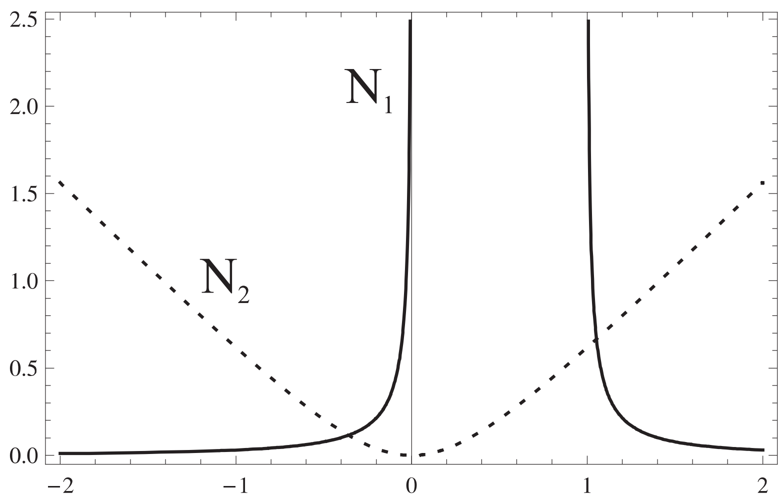

Figure 3.17: In continuous line we report N_1 as function of a in the case of J_1. For $N \geqslant N_1$ the map is WD if $a > 1$ and AD if $a < 0$. In dashed line we plot N_2 as function of b when $a = 0$ in the case of J_2. For $N \geq N_2$ the channel is (AD) WD if $a > 1/2$ ($a < 1/2$).

that above a certain environmental noise threshold the weak-degradability features appear, while for one-mode Bosonic Gaussian channels they do not depend on the initial state of the environment. Actually, one would expect at most that, when the level of the environmental noise increases, the coherence progressively decreases until to be destroyed. It would mean that it becomes more and more difficult to recover the environment (system) output from the system (environment) output after the noisy evolution. However, the things go the other way around when multi-mode Bosonic Gaussian channels are considered.

3.5.2 Channels with null quantum capacity

Analogously to Sec. 3.3.5 [53], where the one-mode case is investigated, one can enlarge (other than the AD maps) the class of two-mode BGCs with $Q = 0$, composing a generic channel with an AD one [54]. First of all, consider a channel $\mathcal{E}$ as in Sec. 3.4.3, but being AD (not necessarily minimal noise), then the maps $\mathcal{E}'$, defined in Eq. (3.221), have zero quantum capacity, i.e., they cannot be used to transfer quantum information. For instance, one can choose $\gamma_E = (2N_c + 1)\mathbb{1}_n$, i.e., the environmental initial state of the map $\mathcal{E}$ is a multi-mode thermal state with N_c being the average photon number for each mode, such that $\mathcal{E}$ is AD or simply with zero capacity. Therefore, for any $\gamma'_E \geqslant \gamma_E = (2N_c + 1)\mathbb{1}_n$, as in Eq. (3.223), the map $\mathcal{E}'$ of Eq. (3.221) has $Q = 0$. Particularly for $n = 2$, using these observations and choosing N_c equal to either N_1 (and $a < 0$) or N_2 (and $a < 1/2$) as in Eqs. (3.278) and (3.279), one obtains that for $X = \mathbb{1}_2 \oplus J_{1,2}$ and $Y' = s_2 \gamma'_E s_2^T$ [with s_2 as in Eq. (3.240)] the resulting channel $\mathcal{E}'$ has always zero capacity. In this way, one extends considerably the set of two-modes maps with zero capacity, other than the very particular cases of two-mode environmental thermal states studied above and shown in Fig. 3.17. For instance, two-mode squeezing can be applied to the thermal state γ_E including not only states with $N > N_c$ but also with not trivial two-mode correlations such that $\gamma'_E \geqslant (2N_c + 1)\mathbb{1}_2$. Therefore, just considering this last simple inequality one includes so a larger set of maps that have zero quantum capacity.

Moreover, we observe that, according to composition rules above, the combination $\mathcal{E} = \mathcal{E}_{II} \circ \mathcal{E}_I$ of two channels $\mathcal{E}_I$ and $\mathcal{E}_{II}$ of class A_2 and C, respectively, with Jordan blocks J_I as in Eq. (3.268) with $a_I = b_I$ and J_{II} as in Eq. (3.270) with a_{II} and $b_{II} \neq 0$, gives $J = a_I J_{II}$ which is in the class C. Now, since we have $N_1 \geqslant 0$, $N_2 \geqslant 0$ and assuming $a_I \leqslant 1/2$, the channel $\mathcal{E}_I$ is AD and the resulting channel $\mathcal{E}$ must have $Q = 0$. Varying the parameters but keeping the product $a_I a_{II} = a$ and $a_I b_{II} = b$ fixed, the parameter N can assume any value satisfying the inequality

$$N \geqslant \frac{1}{4}\left[\left(\frac{5(1 - 4a + 8a^2 + 8b^2)}{b^2 + (a-1)^2}\right)^{1/2} - 2\right]. \tag{3.280}$$

Notice that a_I has been chosen equal to $1/2$ and $\mathcal{E}_I$ corresponds to two uncoupled beam-splitter maps with transmissivity $1/2$ (AD). We can therefore conclude that all channels of the form C with N as in Eq. (3.280) have zero quantum capacity – see Fig. 3.18.

Consider now the composition $\mathcal{E} = \mathcal{E}_{II} \circ \mathcal{E}_I$ of two channels $\mathcal{E}_I$ and $\mathcal{E}_{II}$ of class C and A_2 (i.e., in the opposite order wrt above), respectively, with Jordan blocks J_I as in Eq. (3.270) with a_I and $b_I \neq 0$ and J_{II} as in Eq. (3.268) with $a_{II} = b_{II}$, giving $J = a_{II} J_I$ which is in the class C. As before, since we have $N_1 \geqslant 0$, $N_2 \geqslant 0$ and assuming again $a_{II} \leqslant 1/2$, the channel $\mathcal{E}_2$ is AD and the resulting channel has $Q = 0$. Varying the parameters but keeping the product $a_I a_{II} = a$ and $b_I a_{II} = b$ fixed, the parameter N can

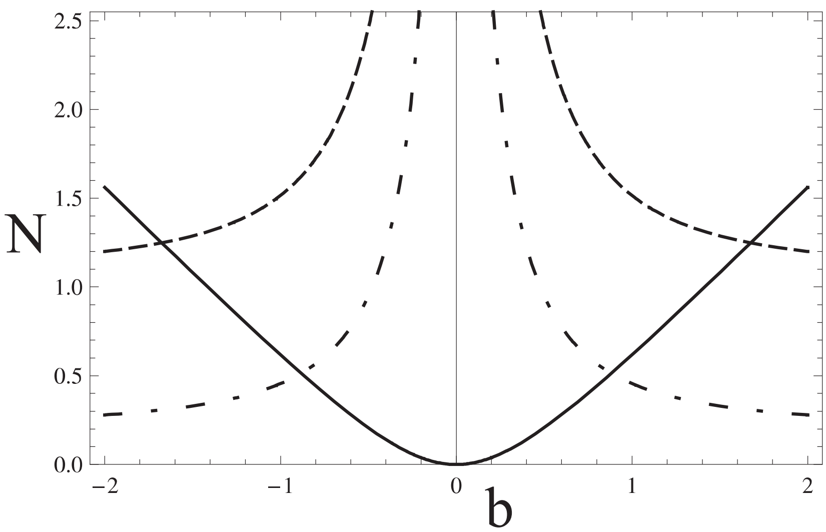

Figure 3.18: In continuous line we plot N_2 as in Eq. (3.279) versus b, with $a = 1$ in J_2 of Eq. (3.270). For $N \geq N_2$ the channel is WD (AD) if $a > 1/2$ ($a < 1/2$). The dashed line refers to the bound in Eq. (3.280), while the dashed-dot line to the one in Eq. (3.281); above these bounds the class C map is WD but with $Q = 0$. Note that Eq. (3.281) is an improvement with respect to the constraint of Eq. (3.280). Similar bounds can be obtained in the case $a < 1/2$, enlarging the group of AD maps with other channels with $Q = 0$.

assume any value satisfying the inequality

$$N \geqslant \frac{1}{4}\left[\left(\frac{(1+4a^2+4b^2)(1-4a+8a^2+8b^2)}{4(b^2+(a-1)^2)(a^2+b^2)}\right)^{1/2} - 2\right], \quad (3.281)$$

where again a_{II} is chosen equal to $1/2$. Again we can conclude that all class C channels with N as in Eq. (3.281) have zero quantum capacity. However, notice that the constraint in Eq. (3.281) is an improvement with respect to the constraint of Eq. (3.280) – see Fig. 3.18.

Chapter 4

Qubit channels

Many models and applications of quantum computing and communication are based on the processing and the transmission of individual qubits. The study of *qubit quantum channels*, i.e. channels in which the information carriers are single two-level systems (e.g., the polarization state of a single photon), thus plays a key role in quantum information theory.

At a mathematical level, the qubit channels are completely positive trace-preserving transformations which act on the state of a single two-level quantum system (qubit). Because of the small size of the Hilbert space, a simple (canonical) parametrization of these channels has been obtained in Ref. [36, 37] and some additivity issues [38, 39, 40] and several classical and quantum capacities [41, 38, 42, 39, 43, 44] have been successfully solved (see also the review in Ref. [3]). Here, we investigate the properties of qubit channels along the same lines followed for Bosonic Gaussian channels (described in Chapter 3) by introducing for the former a characteristic function representation [44]. To this aim we adapt the formalism introduced by Cahill and Glauber in Ref. [121] to represent the density operators of Fermions in the case of two-level systems. In this context, the channels are represented in terms of Green functions. Interestingly enough, this allows us to define a set of *Gaussian channels* for qubit that share analogous properties with their continuous variable counterpart [44].

The chapter is organized as follows. In Sec. 4.1 we introduce displacement operator and characteristic function for a qubit. We need to recall the notion of Grassmann variables and to use them to generalize the definition of coherent states for finite dimensional systems. We then present a Green function representation for qubit channels (Sec. 4.2) and some examples and canonical forms are described in Sec. 4.3. The set of qubit Gaussian channels is so defined in Sec. 4.4 and their degradability properties are discussed [44]. In Sec. 4.5 we will apply these results on some examples of qubit quantum channels, whose weak-degradability features are considered. Finally, from a different point of view, in Sec. 4.6 we briefly examine an example of memory (i.e., correlated noise) qubit quantum channels [65]. We find that the optimal (i.e., with maximum output purity) input states of the channel (roughly speaking, more robust against the decoherence) depend on the correlation parameter and show a sort of 'phase transition' behavior. Particularly, we optimize analytically and numerically all *purities* (measured using the p-norm, for any p) of the output states and show that, above a certain threshold of the correlation parameter, the optimal value is achieved by the maximally entangled input state, while below by partially entangled input states whose entanglement increases monotonically with the "memory" factor.

4.1 Representation of a qubit

Various proposals for defining a (discrete) phase space for finite dimensional systems have been discussed so far by introducing generalized position and momentum operators (see, for instance, Ref. [127] and references therein). Here we will not follow this line: instead we invoke the analogies between a qubit and a single Fermionic mode to adapt the results of Ref. [121]. A similar approach was developed in Ref. [128] to solve non-Markovian master equations of a two-level atom interacting with an external field.

The characteristic function formalism, presented in the previous chapters for Bosonic systems, can be generalized to describe Fermionic systems too [121] and, here, adapted to qubit. The main difference in this case is related to the fact that now the complex variables μ and μ^*, appearing as argument of the characteristic functions, are replaced by a couple of conjugate Grassmann variables ξ and ξ^* [122, 123, 124, 125] whose properties are reviewed in Appendix B. This is intrinsically related to the fact that the annihilation and creation operators of a Fermion obey anti-commutation rules instead of commutation rules [126] valid for Bosonic systems. We will not review the analysis of Ref. [121] since in the next section, when discussing the qubit case, we will rederive most of the results obtained in the Fermionic case.

The starting point of our analysis [44] is to observe that the lowering and raising operators of the qubit [i.e. $\sigma_+ \equiv |1\rangle\langle 0|$ and $\sigma_- \equiv (\sigma_+)^\dagger$] satisfy anti-commutation rules similar[1] to that of a Fermionic mode, i.e.

$$\begin{aligned}\{\sigma_-, \sigma_+\} &= |0\rangle\langle 0| + |1\rangle\langle 1| \equiv \mathbb{1} \;, \\ \{\sigma_-, \sigma_-\} &= \{\sigma_+, \sigma_+\} = 0 \;. \end{aligned} \tag{4.1}$$

Identifying the qubit state $|0\rangle$ with the Fermionic vacuum we can therefore treat σ_+ and σ_- as Fermionic creation and annihilation operators, respectively. Following [121] we introduce then a couple of conjugate Grassmann variables ξ and ξ^* (see Appendix B) and impose standard anti-correlation with the annihilation and creator operators of the system, i.e.

$$\{\xi, \sigma_\pm\} = \{\xi^*, \sigma_\pm\} = 0 \;. \tag{4.2}$$

It is worth noting that this implies that the projectors $|0\rangle\langle 0| = \sigma_-\sigma_+$ and $|1\rangle\langle 1| = \sigma_+\sigma_-$ as well as the Pauli matrix $\sigma_z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$ commute with ξ and ξ^*. Note that this parallelism with Fermions is valid only for one qubit and cannot be easily generalized to many qubits. In the following we will also require that

$$\begin{aligned}\xi \; |j\rangle &= (-1)^j |j\rangle \; \xi \;, \\ \xi^* \; |j\rangle &= (-1)^j |j\rangle \; \xi^* \;, \end{aligned} \tag{4.3}$$

for $j = 0, 1$. This is not strictly necessary but it is consistent with Eq. (4.2) and enables us to simplify the calculations. For instance, given any collection of qubit operators Θ_1, Θ_2, $\cdots$, Θ_{n+1} and the Grassmann numbers ξ_1, ξ_2, $\cdots$, ξ_n we can use Eq. (4.3) to verify that the following relation applies

$$\mathrm{Tr}[\Theta_1\xi_1\Theta_2\xi_2\cdots\Theta_n\xi_n\Theta_{n+1}] = \xi_1\xi_2\cdots\xi_n \; \mathrm{Tr}[\Theta_1\sigma_z\Theta_2\sigma_z\cdots\Theta_n\sigma_z\Theta_{n+1}] \tag{4.4}$$

(an analogous expression holds also when replacing all, or part of, the ξ_is with their complex conjugates).

The above definitions give us the possibility of operating with "hybrid" mathematical objects obtained by multiplying Grassmann variables and qubit operators. In this context we find it useful to define a generalized adjoint operation for these hybrid operators by arbitrarily imposing the conditions

$$(\Theta_1\xi_1\Theta_2\xi_2\cdots\Theta_n\xi_n\Theta_{n+1})^\dagger = \Theta_{n+1}^\dagger \; \xi_n^*\Theta_n^\dagger \cdots \; \xi_2^* \; \Theta_2^\dagger \; \xi_1^* \; \Theta_1^\dagger \;, \tag{4.5}$$

with ξ_i and Θ_i as in Eq. (4.4).

[1] This point is explained better in Sec. 4.1.2.

Equation (4.4) shows that the cyclicity of the trace needs to be modified when involving Grassmann terms. If we need to move only qubit operators, then the standard rule applies, i.e.

$$\mathrm{Tr}[\Theta_1\xi_1\cdots\Theta_n\xi_n\Theta_{n+1}] = \mathrm{Tr}[\Theta_{n+1}\Theta_1\xi_1\cdots\Theta_n\xi_n] = \mathrm{Tr}[\xi_1\cdots\Theta_n\xi_n\Theta_{n+1}\Theta_1] \ . \tag{4.6}$$

On the contrary, if we move also Grassmann variables, by exploiting the anti-commutation rules of the ξ_is, we get

$$\begin{aligned}\mathrm{Tr}[\Theta_1\xi_1\Theta_2\xi_2\cdots\Theta_n\xi_n\Theta_{n+1}] &= (-1)^{n-1}\mathrm{Tr}[\xi_n\Theta_{n+1}\Theta_1\xi_1\cdots\Theta_n] \\ &= (-1)^{n-1}\mathrm{Tr}[\Theta_2\xi_2\cdots\Theta_n\xi_n\Theta_{n+1}\Theta_1\xi_1] \ .\end{aligned}$$

Finally in conjunction with Eq. (4.5), Eq. (4.4) gives

$$\Big(\mathrm{Tr}[\Theta_1\xi_1\Theta_2\xi_2\cdots\Theta_n\xi_n\Theta_{n+1}]\Big)^* = \mathrm{Tr}[\Theta_{n+1}^\dagger\,\xi_n^*\Theta_n^\dagger\cdots\xi_2^*\,\Theta_2^\dagger\,\xi_1^*\,\Theta_1^\dagger] \ .$$

4.1.1 Qubit characteristic function

Qubit displacement operators can now be defined in analogy with [121] as

$$D(\xi) \ \equiv \ \exp\left(\sigma_+\xi - \xi^*\sigma_-\right) = \mathbb{1} + \sigma_+\xi - \xi^*\sigma_- - \sigma_z\xi^*\xi/2 \ ,$$

where in the second equality we used Eq. (B.34) in Appendix B. As in the Bosonic case, they satisfy the identity $D^\dagger(\xi) = D(-\xi)$. Moreover, the application of $D(\xi)$ to the vacuum originates eigenvectors of the annihilation operator of the system (i.e. σ_-). These are the coherent states of our qubit, i.e.

$$|\xi\rangle = D(\xi)|0\rangle = \left(1 - \frac{\xi^*\xi}{2}\right)|0\rangle - \xi|1\rangle \ , \tag{4.7}$$

whose norm is unity. These vectors are eigenvectors of σ_- in Grassmann sense (i.e., their eigenvalues are Grassmann variables; see Ref. [121] for more details). What is interesting for us is the fact that $D(\xi)$ can be used to define a characteristic function for the operators of the system as in Eq. (3.23), i.e.

$$\chi(\xi) \equiv \mathrm{Tr}[\Theta D(\xi)] \ . \tag{4.8}$$

In particular, consider an operator Θ which is characterized by the matrix

$$\Theta \equiv \begin{pmatrix} \theta_{00} & \theta_{01} \\ \theta_{10} & \theta_{11} \end{pmatrix} \ , \tag{4.9}$$

when expressed in the computational basis $\{|0\rangle, |1\rangle\}$. In this case, using the anti-commutation rules of Eq. (4.2) and the identity (4.4), we get

$$\chi(\xi) = \mathrm{Tr}[\Theta] + (\theta_{00} - \theta_{11})\frac{\xi\xi^*}{2} + \theta_{01}\xi - \theta_{10}\xi^*. \tag{4.10}$$

It is worth noticing that with respect to the analysis of Ref. [121] the characteristic functions analyzed here contain an extra term which is linear in ξ and ξ^*. In the Fermionic case analyzed by Cahill and Glauber the only allowed physical states are classical mixtures of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. Consequently the off-diagonal terms associated with θ_{01} and θ_{10} do not need to be considered. When analyzing qubit systems, instead, quantum superpositions among $|0\rangle$ and $|1\rangle$ are allowed and we need to include also the linear contributions. See Sec. 4.1.2 for more details about this subtle discussion.

As in the Bosonic case, Eq. (4.8) can be inverted. In this case, however, Eq. (3.22) is replaced by

$$\Theta = \int d^2\xi \; \chi(\xi) \; \tilde{E}(-\xi) \; , \tag{4.11}$$

with $\tilde{E}(\xi) \neq D(\xi)$ defined by

$$\tilde{E}(\xi) \;\; \equiv \;\; \sigma_z - \xi^*\xi/2 + \sigma_+\xi - \xi^*\sigma_- \; . \tag{4.12}$$

The easiest way to verify this is by direct substitution of Eqs. (4.10) and (4.12) into Eq. (4.11) and by employing the integration rules (B.16).

Let us recall that a density operators can be described by a 2×2 matrix as follows

$$\rho \equiv \begin{pmatrix} p & \gamma \\ \gamma^* & 1-p \end{pmatrix} \tag{4.13}$$

with p being a real number in the range $[0, 1]$ and γ complex (see Sec. 1.1). To represent a density operator the characteristic function, i.e. $\chi(\xi) \equiv \exp[\gamma\xi - \gamma^*\xi^* + (2p-1)\xi\xi^*/2]$, needs to satisfy certain physical requirements. First of all, the Hermitianity of ρ and the normalization condition $\mathrm{Tr}[\rho] = 1$ imply, respectively,

$$\chi(\mu) \;\; = \;\; [\chi(-\mu)]^* \; , \tag{4.14}$$

$$\chi(0) \;\; = \;\; 1 \; , \tag{4.15}$$

where complex conjugation is defined as in Eq. (B.8) in Appendix B [to verify this simply use Eq. (4.10) with $\Theta = \rho$]. The positivity of ρ imposes, instead, the following inequality to hold

$$\left| \int d^2\xi \ \chi(\xi)\xi \right|^2 + \left[\int d^2\xi \ \chi(\xi) \right]^2 \leqslant \frac{1}{4} \ . \tag{4.16}$$

This follows from the positivity condition $|\gamma|^2 \leqslant p(1-p)$ and by the identity

$$\begin{aligned} \gamma &= \int d^2\xi \ \chi(\xi) \ \xi^* \ , \\ p &= \int d^2\xi \ \chi(\xi) + 1/2 \ . \end{aligned}$$

Using similar arguments one can verify that Eqs. (4.14)-(4.16) are also sufficient conditions for $\chi(\xi)$ being a characteristic function of a density operator ρ, in a analogous way as for Bosonic systems in Sec. 3.1.

4.1.2 Qubit as Fermion?

The qubit characteristic function in Eq. (4.10) contains an extra term linear in ξ and ξ^*, which does not appear in the analysis of Ref. [121] for Fermions[2]. Basically, when considering the Fermionic case, the only allowed physical states are classical mixtures of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. Indeed, remind that any *physical* state $|\psi\rangle$ satisfies a global symmetry, called $U(1)$, i.e. it is invariant under transformations represented by the group of complex numbers with norm 1 under multiplication (circle group). In particular, a physical state needs to be invariant, apart from a global phase, when subjected to a rotation U of angle 2π about any axis $\hat{n}$, i.e.

$$U(\hat{n}, 2\pi)|\psi\rangle = e^{i\theta} \, |\psi\rangle. \tag{4.17}$$

Moreover, it is well known that Fermions carry half-odd-integer spin. This implies that, under such 2π rotations, a state of one Fermion or of any odd number of Fermions changes by the phase factor -1, while states with no Fermions or only even number of Fermions are, instead, invariant. Therefore, the physical states are represented only by linear combinations of states with

[2] A brief excursus on some Fermionic channels is shown in Appendix C.

either odd or even number of Fermions; otherwise, they do not exist in nature. For instance, the state

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{4.18}$$

is *nonphysical* because under a 2π rotation it changes into a different state, i.e.

$$U(\hat{n}, 2\pi)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \neq e^{i\theta}\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right). \tag{4.19}$$

In terms of density operator, this corresponds to the fact that the off-diagonal terms of the matrix ρ in Eq. (4.13) are not present (i.e., $\gamma = 0$) when considering *physical* Fermionic states.

This restriction does not hold, of course, when one considers qubits. In this case, quantum superpositions among $|0\rangle$ and $|1\rangle$ are naturally allowed and experimentally implemented and, therefore, the linear contributions (i.e., $\gamma \neq 0$) of the Grassmann characteristic function in Eq. (4.10) have to be included in the description of a generic qubit state.

4.2 Green function of a qubit channel

Let us now consider the effect of a qubit quantum channel $\mathcal{E}$ acting on a operator Θ of the system. We will derive a Green function representation of the qubit channels following the same lines as in Sec. 3.2 for Bosonic channels [44]. To do so we first evaluate the characteristic function $\chi'(\xi)$ associated with $A\Theta B$ with A and B being arbitrary qubit operators. This is

$$\chi'(\xi) = \mathrm{Tr}[A\Theta BD(\xi)] = \int d^2\zeta\ \mathrm{Tr}[A\Big(\chi(\zeta)\tilde{E}(-\zeta)\Big)BD(\xi)]\ , \tag{4.20}$$

where we used Eq. (4.11) with $\chi(\xi)$ being the characteristic function of Θ (from now on ζ and ξ should be considered entries of the same Grassmann set). Our goal is to find a function $G(\zeta, \xi)$ which gives

$$\chi'(\xi) \ = \ \int d^2\zeta\ \chi(\zeta)\ G(\zeta, \xi)\ , \tag{4.21}$$

for all $\chi(\xi)$. Notice that if ξ were a commuting variable (e.g., a complex variable) the problem could be solved by simply moving $\chi(\xi)$ out of the trace operation of Eq. (4.20) yielding $G(\zeta,\xi) = \mathrm{Tr}[A\tilde{E}(-\zeta)BD(\xi)]$. In the case under consideration, however, the situation is complicated by the fact that for moving out of trace the variables ξ or ξ^* we need to insert σ_zs as in Eq. (4.4). Taking into account this fact, the solution becomes

$$G(\zeta,\xi) \;=\; \mathrm{Tr}[A\sigma_z D(-\zeta)BD(\xi)]\;, \tag{4.22}$$

as can be easily verified by direct integration of the Eqs. (4.20) and (4.21) for the most general characteristic function (4.10).

The Green function (4.21) associated with a CPT map $\mathcal{E}$ can then be obtained [44] by using an operator sum representation [1, 14, 3] of such channel and exploiting the linearity of the trace. Indeed, writing $\mathcal{E}(\Theta) = \sum_k M_k \Theta M_k^\dagger$ with $\{M_k\}_k$ being Kraus operators of $\mathcal{E}$, we get

$$G(\zeta,\xi) = \sum_k \mathrm{Tr}[M_k\sigma_z D(-\zeta)M_k^\dagger D(\xi)] = \mathrm{Tr}\Big[\mathcal{E}\Big(\sigma_z D(-\zeta)\Big)D(\xi)\Big]\;. \tag{4.23}$$

Using Eq. (4.6) this can also be written as

$$G(\zeta,\xi) \;=\; \mathrm{Tr}\Big[\sigma_z D(-\zeta)\mathcal{E}_H\Big(D(\xi)\Big)\Big]\;, \tag{4.24}$$

with $\mathcal{E}_H$ being the Heisenberg representation of the map $\mathcal{E}$, defined in Sec. 2.2.5. Equation (4.24) shows that, as in the Bosonic case, a complete description of the channel is obtained by applying the dual map to the displacement operator — see Eq. (3.52). Exploiting the normalization condition $\sum_k M_k^\dagger M_k = \mathbb{1}$ we note that for $\xi = 0$ the above expression yields

$$G(\zeta,0) = \mathrm{Tr}[\sigma_z D(-\zeta)] = \zeta\zeta^*\;, \tag{4.25}$$

which corresponds to the Grassmann delta function $\delta^{(2)}(\zeta)$ defined in Eq. (B.19), in agreement with the requirement of channel being trace preserving — see Eqs. (4.10) and (4.21).

4.2.1 Composition rules

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two different qubit channels with Green functions $G_1(\zeta,\xi)$ and $G_2(\zeta,\xi)$, respectively, such that

$$\chi'(\xi) \;=\; \int d^2\zeta\; \chi(\zeta)\; G_1(\zeta,\xi)\;, \tag{4.26}$$

$$\chi'(\varsigma) \;=\; \int d^2\xi'\; \chi(\xi')\; G_2(\gamma,\varsigma)\;. \tag{4.27}$$

Now we want to analyze the Green function, $G_{12}(\zeta,\xi)$, of the composite map $\mathcal{E}_2 \circ \mathcal{E}_1$, in which we first operate with $\mathcal{E}_1$ and then with $\mathcal{E}_2$, i.e.

$$\chi'(\xi) = \int d^2\zeta\, \chi(\zeta)\, G_1(\zeta,\xi)\ , \tag{4.28}$$

$$\chi''(\varsigma) = \int d^2\xi'\, \chi'(\xi')\, G_2(\xi',\varsigma)\ . \tag{4.29}$$

Our goal is to find a function $G_{12}(\zeta,\xi)$ which gives

$$\chi''(\varsigma) = \int d^2\zeta\, \chi(\zeta)\, G_{12}(\zeta,\varsigma)\ . \tag{4.30}$$

Simply composing the two transformations of the characteristic function of the two maps, one has

$$\begin{aligned}\chi''(\varsigma) &= \int d^2\gamma \int d^2\zeta\, \chi(\zeta)\, G_1(\zeta,\gamma)\, G_2(\gamma,\varsigma)\\ &= \int d^2\zeta\, \chi(\zeta) \int d^2\gamma\, G_1(\zeta,\gamma)\, G_2(\gamma,\varsigma)\ ,\end{aligned} \tag{4.31}$$

and then the Green function $G_{12}(\zeta,\xi)$ can be expressed in terms of the following Grassmann convolution integral

$$G_{12}(\zeta,\xi) = \int d^2\xi'\, G_1(\zeta,\xi')\, G_2(\xi',\xi), \tag{4.32}$$

with ζ, ξ and ξ' Grassmann numbers.

4.3 Canonical representation

As a particular case of Green function consider the identity map $\mathcal{I}$ which leaves all operators invariant, i.e. $\mathcal{I}(\Theta) = \Theta$. According to our definition we get

$$G(\zeta,\xi) = \mathrm{Tr}[\sigma_z D(-\zeta) D(\xi)] = (\zeta-\xi)(\zeta^*-\xi^*)\ , \tag{4.33}$$

which, as expected, corresponds to the delta $\delta^{(2)}(\zeta-\xi)$ of Eq. (B.19).

The most generic qubit quantum channel $\mathcal{E} : \mathbb{C}^{2\times2} \to \mathbb{C}^{2\times2}$ can be represented [36] in the Bloch picture by a unique 4×4 matrix $\mathbb{T}$, i.e. $\mathbb{T} = \begin{pmatrix} 1 & \mathbf{0} \\ \vec{t} & \mathrm{T} \end{pmatrix}$ where T is a 3×3 matrix ($\mathbf{0}$ and $\vec{t}$ are row and column vectors, respectively), so that

$$\mathcal{E}(\rho) = \mathcal{E}\left(\frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2}\right) = \frac{\mathbb{1} + (\vec{t} + \mathrm{T}\vec{r})\cdot\vec{\sigma}}{2} \,, \tag{4.34}$$

where $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ is a vector containing the Pauli matrices and $\vec{r}$ is the Bloch vector describing the input state (see Sec. 1.1). The map $\mathcal{E}$ is unital if and only if $\vec{t} = \mathbf{0}$. Thus, any unital quantum map $\mathcal{E}$ acting on density matrices on $\mathbb{C}^{2\times2}$ can be written in the form

$$\mathcal{E}(\rho) = \mathcal{E}\left(\frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2}\right) = \frac{\mathbb{1} + (\mathrm{T}\vec{r})\cdot\vec{\sigma}}{2} \,, \tag{4.35}$$

where T is a real 3×3 matrix. Using the singular value decomposition, we can write [36]

$$\mathrm{T} = \mathrm{RS} \,, \tag{4.36}$$

where R is a rotation and S is self-adjoint, and define the map $\mathcal{E}_S$ by

$$\mathcal{E}_S(\rho) = \mathcal{E}\left(\frac{\mathbb{1} + \vec{r}\cdot\vec{\sigma}}{2}\right) = \frac{\mathbb{1} + (\mathrm{S}\vec{r})\cdot\vec{\sigma}}{2} \,. \tag{4.37}$$

The rotation R defines a unitary operator U such that for any state ρ

$$\mathcal{E}(\rho) = U\Big[\mathcal{E}_S(\rho)\Big]U^\dagger \,. \tag{4.38}$$

Since a unitary transformation leaves the spectrum unchanged, this last is the same for $\mathcal{E}$ and $\mathcal{E}_S$. Moreover, since S is self-adjoint it can be diagonalized by a change of basis and has eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$. The image of the set of pure state density matrices $\rho = \frac{1}{2}[\mathbb{1} + \vec{r}\cdot\vec{\sigma}]$ (with $|\vec{r}| = 1$) under the action of $\mathcal{E}_S$ is the ellipsoid

$$\left(\frac{x_1}{\lambda_1}\right)^2 + \left(\frac{x_2}{\lambda_2}\right)^2 + \left(\frac{x_3}{\lambda_3}\right)^2 = 1, \tag{4.39}$$

and the image under the action of $\mathcal{E}$ is obtained by a further rotation of the ellipsoid, corresponding to the operator U in (4.38) [36].

Similar reasoning applies when $\mathcal{E}$ is non-unital. The map $\mathcal{E}$ can be written in the form $\mathcal{E}(\rho) = U\mathcal{E}_D(V\rho V^\dagger)U^\dagger$ where U, V are unitary, D is diagonal and $\mathcal{E}_D$ is represented by the matrix

$$\mathbb{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t'_1 & \lambda_1 & 0 & 0 \\ t'_2 & 0 & \lambda_1 & 0 \\ t'_3 & 0 & 0 & \lambda_3 \end{pmatrix} \tag{4.40}$$

where $\vec{t'} = (t'_1, t'_2, t'_3)$ is a rotated vector of $\vec{t}$. In this case, the image of the set of pure state density matrices $\rho = \frac{1}{2}[\mathbb{1} + \vec{r}\cdot\vec{\sigma}]$ (with $|\vec{r}| = 1$) under the action of $\mathcal{E}_D$ is the translated ellipsoid

$$\left(\frac{x_1 - t'_1}{\lambda_1}\right)^2 + \left(\frac{x_2 - t'_2}{\lambda_2}\right)^2 + \left(\frac{x_3 - t'_3}{\lambda_3}\right)^2 = 1, \tag{4.41}$$

and again the image under $\mathcal{E}$ is a rotation of this (see Ref. [36] for more details).

Complete Positivity Conditions

The requirement for $\mathcal{E}$ to be a (CPT) quantum channel imposes a number of constraints on the matrix $\mathbb{T}$. In Ref. [36] the authors give explicit formulas for the matrix elements of $\mathbb{T}$, that imply constraints on the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ described in the previous section. Let T_{jk} denote the elements of $\mathbb{T}$, where $j, k = 0, 1, 2, 3$ and $T_{00} = 1$. Then, the points with coordinates (T_{11}, T_{22}, T_{33}) must lie inside a tetrahedron with corners at $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ (see Ref. [36]). These conditions are equivalent to four linear inequalities which can be written compactly as

$$|T_{11} \pm T_{22}| \quad \leq \quad |1 \pm T_{33}|. \tag{4.42}$$

In the special case where $\mathcal{E}$ is unital, (4.42) implies that the eigenvalues (which are necessarily real) satisfy

$$|\lambda_1 \pm \lambda_2| \quad \leq \quad |1 \pm \lambda_3|. \tag{4.43}$$

In fact, for unital $\mathcal{E}$ the condition (4.43) is a necessary and sufficient condition for the numbers $(\lambda_1, \lambda_2, \lambda_3)$ to arise as eigenvalues of the self-adjoint part of T [129].

Green function

Therefore, the canonical (diagonal) form of the qubit quantum operations is represented by $T = \mathrm{diag}(\lambda_1,\ \lambda_2,\ \lambda_3)$, with the real coefficients $\lambda_{1,2,3}$ and $t_{1,2,3}$ that need to satisfy certain conditions [36, 37] to guarantee the complete positivity of the map. The action of a generic linear map, represented by

$$\mathbb{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_1 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}, \tag{4.44}$$

on a single qubit density matrix ρ in Eq. (4.13), has then the form

$$\begin{aligned} \mathcal{E}(\rho) &= \mathcal{E}\begin{pmatrix} p & \gamma \\ \gamma^* & 1-p \end{pmatrix} \qquad\qquad (4.45) \\ &= \frac{1}{2}\begin{pmatrix} 1+t_3+\lambda_3(2p-1) & t_1 - it_2 + \lambda_1(\gamma+\gamma^*) + \lambda_2(\gamma-\gamma^*) \\ t_1+it_2+\lambda_1(\gamma+\gamma^*)-\lambda_2(\gamma-\gamma^*) & 1-t_3-\lambda_3(2p-1) \end{pmatrix}. \end{aligned}$$

In the Green function language such canonical form corresponds to have

$$\begin{aligned} G(\zeta,\xi) &= \delta^{(2)}\left(\zeta - \frac{\lambda_2+\lambda_1}{2}\xi - \frac{\lambda_2-\lambda_1}{2}\xi^*\right)\exp\left[-\frac{t_3}{2}\xi^*\xi\right] \\ &+ (\lambda_3-\lambda_1\lambda_2)\xi\xi^* + \frac{t_1-it_2}{2}\zeta\zeta^*\xi - \frac{t_1+it_2}{2}\zeta\zeta^*\xi^* . \end{aligned}$$

Hence, the output characteristic function is

$$\begin{aligned} \chi'(\xi) &= 1 - \frac{1}{2}[t_3+\lambda_3(|\alpha|^2-|\beta|^2)]\xi^*\xi + \frac{1}{2}[t_1 - it_2 + \lambda_1(\gamma+\gamma^*) \\ &+ \lambda_2(\gamma-\gamma^*)]\xi - \frac{1}{2}[t_1+it_2+\lambda_1(\gamma+\gamma^*)-\lambda_2(\gamma-\gamma^*)]\xi^* . \end{aligned}$$

4.4 Gaussian channels for qubits

In analogy with the Bosonic case, in this section we introduce the definition of qubit Gaussian channels [44]. We start noticing that in order to define these channels it does not make sense to focus on maps which transform Gaussian characteristic functions into Gaussian characteristic functions. Indeed, thanks to Eq. (B.34) in Appendix B, all characteristic functions of a qubit can be written in a Gaussian form. The characteristic function of the state (4.13) can be written as $\chi(\xi) \equiv \exp[\gamma\xi - \gamma^*\xi^* + (2p-1)\xi\xi^*/2]$.

Therefore, following Eq. (3.53) we say that a qubit map is Gaussian if its Green function has the form

$$G(\zeta,\xi) = \delta^{(2)}(\zeta - a\xi - b\xi^*)\ \exp[-c\xi^*\xi]\ , \tag{4.46}$$

with a and b complex and c real numbers, respectively, and with the exponential defined as in Eq. (B.34). The fact that c must be real can be derived by imposing the Hermitianity constraint in Eq. (4.14) to the output characteristic function (4.21). A trivial example is provided by the identity map $\mathcal{I}$ whose Green function (4.33) is of the form (4.46) for $b = c = 0$ and $a = 1$.

Generic mixtures of Gaussian channels do not necessarily have the form (4.46). Therefore the set of Gaussian channels is not convex. However, it has semi-group structure with respect to the channel composition rule $\circ$. Indeed, given two Gaussian channels $\mathcal{E}_1$ and $\mathcal{E}_2$ characterized by parameters (a_1, b_1, c_1) and (a_2, b_2, c_2), respectively, from Eq. (4.32) it is easy to verify that the Green function of $\mathcal{E}_2 \circ \mathcal{E}_1$ is again of the form (4.46) with

$$\begin{aligned} a &= a_1a_2 + b_1b_2^*\ , \\ b &= a_1b_2 + b_1a_2^*\ , \\ c &= c_1(|a_2|^2 - |b_2|^2) + c_2\ . \end{aligned} \tag{4.47}$$

Both the semi-group property and the non-convexity property hold also in the Bosonic case.

4.4.1 Canonical form for Gaussian channels

From Eq. (4.46) it is easy to verify that within the parametrization [36, 37] we can get Gaussian maps (4.46) by choosing

$$\lambda_3 = \lambda_1\lambda_2\ , \tag{4.48}$$

$$t_1 = t_2 = 0\ . \tag{4.49}$$

This in fact yields Gaussian Green functions with $a = (\lambda_2 + \lambda_1)/2$, $b = (\lambda_2 - \lambda_1)/2$ and $c = t_3/2$. We can then use [37] to show that the corresponding transformation is CPT if and only if the following inequalities hold,

$$\begin{cases} |\lambda_k| \leqslant 1 & \text{for } k = 1,\, 2; \\ |t_3| \leqslant \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)}\ . \end{cases} \tag{4.50}$$

This enables us to parametrize the whole set of Gaussian channels in terms of three real parameters only. First of all, as in Refs. [42, 37], we can use a trigonometric parametrization to express $\lambda_{1,2}$ in terms of the angles θ, ϕ in $[0, 2\pi[$ as follows

$$\lambda_1 = \cos(\theta - \phi) \; , \qquad \lambda_2 = \cos(\theta + \phi) \; . \tag{4.51}$$

Then we can parametrize t_3 by introducing the positive quantity $q \in [0, 1]$ to write

$$t_3 = (2q - 1) \frac{\cos(2\theta) - \cos(2\phi)}{2} \; . \tag{4.52}$$

Replacing all this into Eq. (4.46) yields the following canonical form for the Green function of a qubit Gaussian channel, i.e.

$$\begin{aligned} G(\zeta, \xi) &= \delta^{(2)} \left(\zeta - \xi \cos\theta \cos\phi + \xi^* \sin\theta \sin\phi \right) \\ &\times \exp\left[(2q - 1) \; \frac{\cos(2\theta) - \cos(2\phi)}{4} \xi\xi^* \right] \; . \end{aligned} \tag{4.53}$$

We will see that the maps of this form have the peculiar property that they can always be described in terms of a unitary interaction of the form (2.49) with a *single* (not necessarily pure) qubit environment. For this reason we call them "qubit-qubit" channels. Let us remark that so we have found that all qubit Gaussian channels are necessarily qubit-qubit [44]. It is worth stressing that once again a similar property holds for the Bosonic case: there (almost) all the one-mode Bosonic Gaussian maps are in fact describable in terms of a single mode environment (see Sec. 3.3.3) [52, 53].

4.4.2 Qubit-qubit maps: Pure environment case

An important subclass of qubit-qubit channels in Eq. (4.53) is obtained for $q = 1$ and θ and ϕ generic, i.e.

$$\begin{aligned} G(\zeta, \xi) = \delta^{(2)} \left(\zeta - \xi \cos\theta \cos\phi + \xi^* \sin\theta \sin\phi \right) \\ \times \exp\left[\frac{\cos(2\theta) - \cos(2\phi)}{4} \xi\xi^* \right] \; . \end{aligned} \tag{4.54}$$

According to Eq. (4.50) this corresponds to having $|t_3| = \sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)}$. As shown in Ref. [37] any CPT map which can be described in terms of an interaction with a single qubit environment originally prepared in a pure state can be expressed in this form by proper unitary rotation of the input and the output state. This implies that the maps (4.54) admit a Stinespring dilation with a two-dimensional (qubit) environment E (see Sec. 2.2.2). Without loss of generality, we can assume an initial state of the environment of the form $\rho_E \equiv |0\rangle_E\langle 0|$. Following Ref. [1], one can then choose the unitary coupling U to have the following block structure

$$U = \begin{pmatrix} [A_0] & [-\sigma_x A_1 \sigma_x] \\ [A_1] & [\sigma_x A_0 \sigma_x] \end{pmatrix} , \tag{4.55}$$

with

$$A_0 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\phi \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sin\phi \\ \sin\theta & 0 \end{pmatrix} , \tag{4.56}$$

being a Kraus set for the channel [the matrix (4.55) is expressed in the basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$ with $|jk\rangle \equiv |j\rangle \otimes |k\rangle_E$ for $j, k = 0, 1$].

The output density matrix is

$$\begin{aligned} \mathcal{E}_{qubit}(\rho) &= \mathcal{E}_{qubit}\begin{pmatrix} p & \gamma \\ \gamma^* & 1-p \end{pmatrix} \qquad (4.57) \\ = & \begin{pmatrix} p\cos^2\theta + (1-p)\sin^2\phi & \gamma\cos\theta\cos\phi + \gamma^*\sin\theta\sin\phi \\ \gamma^*\cos\theta\cos\phi + \gamma\sin\theta\sin\phi & p\sin^2\theta + (1-p)\cos^2\phi \end{pmatrix} , \end{aligned}$$

whose characteristic function is given by

$$\begin{aligned} \chi'(\xi) &= 1 - \frac{1}{2}[p\cos(2\theta) - (1-p)\cos(2\phi)]\xi^*\xi \qquad (4.58) \\ &+ [\gamma\cos\theta\cos\phi + \gamma^*\sin\theta\sin\phi]\xi - [\gamma^*\cos\theta\cos\phi + \gamma\sin\theta\sin\phi]\xi^* \\ &= \chi\left(\xi\cos\theta\cos\phi - \xi^*\sin\theta\sin\phi\right)\left[1 + \frac{\cos(2\theta) - \cos(2\phi)}{4}\xi\xi^*\right] . \end{aligned}$$

In the Bloch representation the Bloch vector is transformed as $(r_x, r_y, r_z) \rightarrow$

$$\left(\cos(\theta-\phi)\ r_x, \cos(\theta+\phi)\ r_y, \frac{\cos(2\theta) - \cos(2\phi)}{2} + \frac{\cos(2\theta) + \cos(2\phi)}{2}\ r_z\right) .$$

In general, the output states live in a deformed and shifted Bloch sphere. It reduces to a line if and only if $\cos(2\theta) = -\cos(2\phi)$. Indeed, $\cos(2\theta) = -\cos(2\phi)$ is satisfied by i) $\theta = \pi/2 - \phi$ and by ii) $\theta = \pi/2 + \phi$. In the former case, one has

$$(r_x, r_y, r_z) \rightarrow (\sin(2\phi)\ r_x, 0, -\cos(2\phi)) \tag{4.59}$$

and the output states are along a line in the plane $y = 0$. This line reduces to the two poles of Bloch sphere for $\phi = 0, \pi/2$. For $\phi = \pi/4$ the map projects the Bloch vector into the $\hat{x}$-axis, i.e. $(r_x, r_y, r_z) \to (r_x, 0, 0)$. For the other values, the output states belong to a segment (parallel to $\hat{x}$-axis) in the plane $y = 0$, contracted by a factor of $\sin(2\phi)$. In the latter case ii), one has

$$(r_x, r_y, r_z) \to (0, \sin(2\phi)\ r_y, -\cos(2\phi)) \tag{4.60}$$

and similar results as in the case i) but in the plane $x = 0$. Moreover, for $\theta = \phi$ this represents a *bit flip or dephasing* channel [38] and for $\theta = 0$ an *amplitude damping* channel [41] (see Sec. 4.5).

The complementary channel $\tilde{\mathcal{E}}$ [38, 93, 92] can now be computed as in Eq. (2.50). Since it represents a qubit channel — it connects two two-dimensional Hilbert spaces (the input Hilbert space with the environmental one) — we can use Eq. (4.23) to evaluate its Green function obtaining

$$\begin{aligned}\tilde{G}(\zeta, \xi) &= \delta^{(2)}\left(\zeta - \xi \cos\theta \sin\phi + \xi^* \sin\theta \cos\phi\right) \\ &\quad \times \exp\left[\frac{\cos(2\theta) + \cos(2\phi)}{4} \xi\xi^*\right] .\end{aligned} \tag{4.61}$$

It is still of the (pure-environment qubit-qubit) Gaussian form (4.54) and can be expressed in terms of the original Green function $G(\zeta, \xi)$ of $\mathcal{E}$ by simply shifting ϕ by $-\pi/2$ and by changing sign to θ, i.e.

$$\tilde{G}(\zeta, \xi) = G(\zeta, \xi)\Big|_{\substack{\theta \to -\theta \\ \phi \to \phi - \pi/2}} . \tag{4.62}$$

The output density matrix, $\tilde{\mathcal{E}}(\rho) = \mathrm{Tr}_S[U(\rho \otimes |0\rangle\langle 0|)U^\dagger]$, is

$$\begin{aligned}\tilde{\mathcal{E}}(\rho) &= \tilde{\mathcal{E}}\begin{pmatrix} p & \gamma \\ \gamma^* & (1-p) \end{pmatrix} \qquad (4.63) \\ &= \frac{1}{2}\begin{pmatrix} p\cos^2\theta + (1-p)\cos^2\phi & \gamma\cos\theta\sin\phi + \gamma^*\sin\theta\cos\phi \\ \gamma^*\cos\theta\sin\phi + \gamma\sin\theta\cos\phi & p\sin^2\theta + (1-p)\sin^2\phi \end{pmatrix} ,\end{aligned}$$

whose characteristic function is given by

$$\tilde{\chi}'(\xi) = \chi\left(\xi\cos\theta\sin\phi - \xi^*\sin\theta\cos\phi\right)\left[1 + \frac{\cos(2\theta) + \cos(2\phi)}{4}\xi\xi^*\right] . \tag{4.64}$$

The corresponding Kraus operators are

$$\tilde{A}_0 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \sin\phi \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} 0 & \cos\phi \\ \sin\theta & 0 \end{pmatrix} , \tag{4.65}$$

and $\tilde{\mathcal{E}}(\rho) = \sum_k \tilde{A}_k \rho \tilde{A}_k^\dagger$.

In the general framework of the qubit channels in Eq. (4.44), these qubit-qubit maps are obtained by using the following parameters:

$$\tilde{t}_1 = \tilde{t}_2 = 0 \, , \qquad \tilde{t}_3 = \frac{\cos(2\theta) + \cos(2\phi)}{2} \, , \tag{4.66}$$

$$\tilde{\lambda}_1 = \sin(\phi - \theta) \, , \qquad \tilde{\lambda}_2 = \sin(\theta + \phi) \, , \quad \tilde{\lambda}_3 = \frac{\cos(2\theta) - \cos(2\phi)}{2} \, . \tag{4.67}$$

The effect of a complementary qubit-qubit quantum channel in the Bloch representation is the Bloch vector transformation $(r_x, r_y, r_z) \rightarrow$

$$\left(\sin(\phi - \theta) \; r_x, \sin(\theta + \phi) \; r_y, \frac{\cos(2\theta) + \cos(2\phi)}{2} + \frac{\cos(2\theta) - \cos(2\phi)}{2} \; r_z \right) \, .$$

Degradability properties

In Ref. [42] it has been shown that qubit-qubit channels with pure environment are degradable for $\cos(2\theta)/\cos(2\phi) \geqslant 0$, and anti-degradable otherwise. Here we will rederive this same result in the Green function formalism as a consequence of the Gaussianity of these maps, pointing out an interesting parallelism with their Bosonic counterpart [44]. Moreover, this formalism allows us to find explicitly the intermediate map and to extend these results to environments initially in a mixed state.

In analogy with the previous chapters [52, 53, 54], we look for the intermediate map $\mathcal{T}$ that should connect $\mathcal{E}$ with $\tilde{\mathcal{E}}$, in the class of qubit-qubit channels (with pure environment). Rewriting the degradability condition (2.52) in terms of the compositions rules (4.32), we can then recast the problem as follows

$$\tilde{G}(\zeta, \xi) = \int d^2\xi' \; G(\zeta, \xi') \; G_x(\xi', \xi) \, , \tag{4.68}$$

where $G_x(\zeta, \xi)$ is the Green function (4.54) of the map $\mathcal{T}$ characterized by the parameters θ_x and ϕ_x. By using Eq. (4.47) we find that, for $\cos(2\theta)/\cos(2\phi) \geqslant 0$, θ_x, ϕ_x do exist such that Eq. (4.68) is satisfied. Specifically such parameters are defined by the relations

$$\begin{aligned} \cos(2\theta_x) &= \frac{\cos(2\theta) - \cos(2\phi) + 2\cos(2\theta)\cos(2\phi)}{\cos(2\theta) + \cos(2\phi)} \, , \\ \cos(2\phi_x) &= \frac{\cos(2\theta) - \cos(2\phi) - 2\cos(2\theta)\cos(2\phi)}{\cos(2\theta) + \cos(2\phi)} \, . \end{aligned} \tag{4.69}$$

The case $\cos(2\theta)/\cos(2\phi) \leqslant 0$ can be treated analogously to show that the corresponding channels are anti-degradable. In fact, in the Green function formalism the anti-degradability condition (2.53) becomes

$$G(\zeta, \xi) = \int d^2\xi' \, \tilde{G}(\zeta, \xi') \, \bar{G}_x(\xi', \xi) \,, \tag{4.70}$$

where $\bar{G}_x(\zeta, \xi)$ is the Green function of the connecting map $\overline{\mathcal{T}}$. We find that for $\cos(2\theta)/\cos(2\phi) \leqslant 0$, Eq. (4.70) is satisfied by choosing $\bar{G}_x(\zeta, \xi)$ in the subclass of qubit-qubit channels with pure environment – i.e. Eq. (4.54) – with θ_x and ϕ_x determined by the expressions (4.69) after replacing (θ, ϕ) with $(-\theta, \phi - \pi/2)$.

More directly this result can be established by using the correspondence in Eq. (4.62) and the fact that the complementary channels of degradable maps are anti-degradable. Consider in fact a (pure environment) qubit-qubit channel $\mathcal{E}$ with $\cos(2\theta)/\cos(2\phi) \leqslant 0$. According to Eq. (4.62) we know that its complementary $\tilde{\mathcal{E}}$ is still a (pure environment) qubit-qubit channel characterized by the parameters $(\theta', \phi') = (-\theta, \phi - \pi/2)$. Now it is easy to verify that $\cos(2\theta')/\cos(2\phi') = -\cos(2\theta)/\cos(2\phi) \geqslant 0$. Therefore from Eqs. (4.68) and (4.69) we can conclude that $\tilde{\mathcal{E}}$ is degradable while $\mathcal{E}$ is anti-degradable.

Note that, in the special case $\cos(2\theta) = \cos(2\phi) = 0$, the degradability relations are satisfied. Therefore, in this case the qubit-qubit channels with pure environment are both degradable and anti-degradable, with null quantum capacity.

Quantum capacity

Let us restrict ourselves to show only that the quantum capacity of degradable qubit-qubit channels with an environment initially in a pure state, calculated in Refs. [41, 42], is explicitly given by

$$Q(T) \;=\; \max_{p\in[0,1]} \, h\big(p\cos^2\theta + (1-p)\sin^2\phi\big) - h\big(p\sin^2\theta + (1-p)\sin^2\phi\big) \,,$$

where $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. In Figs. 4.1 and 4.2 the quantum capacity has been plotted in a two and three dimensional graphical representation, respectively.

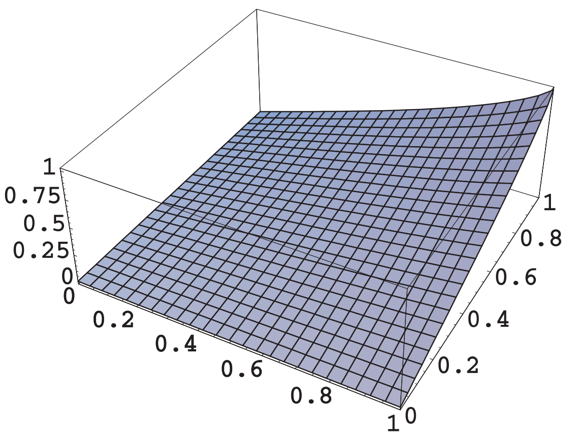

Figure 4.1: Quantum capacity Q in the region $\cos(2\theta)/\cos(2\phi) > 0$ as function of $\cos(2\theta)$ and $\cos(2\phi)$.

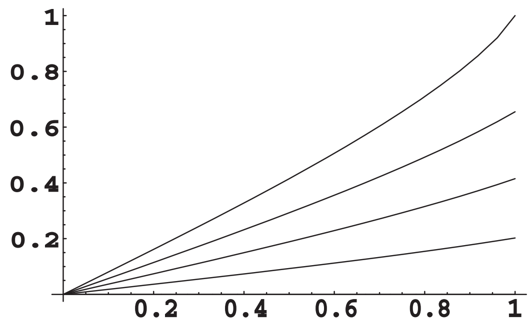

Figure 4.2: Quantum capacity Q in the region $\cos(2\theta)/\cos(2\phi) > 0$ as function of $\cos(2\phi)$ for $\cos(2\theta) = 0, 1/4, 1/2, 3/4, 1$, respectively from below. For $\cos(2\theta) = 0$ the channel is anti-degradable and Q is zero. For $\cos(2\theta) = \cos(2\phi) = 1$ the channel is degradable and, even, $Q = 1$.

4.4.3 Qubit-qubit maps: Mixed environment case

Now let us consider the Gaussian channels (4.53) for $q \neq 1$ [44]. They can be represented in terms of a physical representation (2.49) with U as in Eq. (4.55) and with E being a single qubit environment initially prepared in the mixed state,

$$\rho_E \equiv q|0\rangle_E\langle 0| + (1-q)|1\rangle_E\langle 1| \ . \tag{4.71}$$

To verify this, we observe that with the above prescriptions Eq. (2.49) gives

$$\begin{aligned}\mathcal{E}(\rho) &= \mathrm{Tr}_E[U\,(\rho\otimes(q|0\rangle_E\langle 0| + (1-q)|1\rangle_E\langle 1|))\,U^\dagger] \\ &= q\mathcal{E}_0(\rho) + (1-q)\mathcal{E}_1(\rho)\ ,\end{aligned} \tag{4.72}$$

with $\mathcal{E}_0 \equiv \mathrm{Tr}_E[U(\rho\otimes|0\rangle_E\langle 0|)U^\dagger]$ being the (pure environment) qubit-qubit channel of Sec. 4.4.2 associated with the operator U and with $\mathcal{E}_1(\rho) \equiv \sigma_x\mathcal{E}_0(\sigma_x\rho\sigma_x)\sigma_x$. From the properties of σ_x it follows that a Kraus set for $\mathcal{E}_1$ is given by the matrices (4.56) by exchanging θ and ϕ. Consequently the Green function of this channel is given by $G(\zeta,\xi)|_{\theta\leftrightarrow\phi}$ with $G(\zeta,\xi)$ as in Eq. (4.54). Using this fact and the linear dependence of Eq. (4.23) with respect to $\mathcal{E}$ we can now evaluate the Green function of the map (4.72) as follows

$$\begin{aligned}G(\zeta,\xi) &= q\,\delta^{(2)}\,(\zeta - \xi\cos\theta\cos\phi + \xi^*\sin\theta\sin\phi) \\ &\qquad \times\exp\left[\frac{\cos(2\theta)-\cos(2\phi)}{4}\xi\xi^*\right] \\ &+ (1-q)\,\delta^{(2)}\,(\zeta - \xi\cos\phi\cos\theta + \xi^*\sin\phi\sin\theta) \\ &\qquad \times\exp\left[\frac{\cos(2\phi)-\cos(2\theta)}{4}\xi\xi^*\right]\ .\end{aligned} \tag{4.73}$$

Equation (4.73) can finally be casted into the form (4.53) thanks to the identity

$$\begin{aligned}q\,e^{x\,\xi\xi^*} + (1-q)\,e^{-x\,\xi\xi^*} &= 1 + (2q-1)\,x\,\xi\xi^* \\ &= e^{(2q-1)\,x\,\xi\xi^*}\ ,\end{aligned} \tag{4.74}$$

which holds for all x complex — see Eq. (B.34). The above is an example of a convex combination of Gaussian channels (i.e. $\mathcal{E}_0$ and $\mathcal{E}_1$) which is still Gaussian.

Note that, in the Kraus representation, one has then

$$\mathcal{E}(\rho) = B_0\rho B_0^\dagger + B_1\rho B_2^\dagger + B_2\rho B_2^\dagger + B_3\rho B_3^\dagger \tag{4.75}$$

where

$$B_0 = \sqrt{p}A_0 = \sqrt{q}\begin{pmatrix}\cos\theta & 0\\ 0 & \cos\phi\end{pmatrix},\quad B_1 = \sqrt{p}A_1 = \sqrt{q}\begin{pmatrix}0 & \sin\phi\\ \sin\theta & 0\end{pmatrix}, \tag{4.76}$$

and

$$B_2 = \sqrt{1-q}\ \sigma_x A_0 \sigma_x\ , \qquad B_3 = \sqrt{1-q}\ \sigma_x A_1 \sigma_x\ . \tag{4.77}$$

Besides, it is easy to prove that $\sum_{k=0}^{3} B_k^\dagger B_k = \mathbb{1}$. The output density matrix is

$$\begin{aligned}
\mathcal{E}(\rho) &= \mathcal{E}\begin{pmatrix} p & \gamma \\ \gamma^* & (1-p) \end{pmatrix} \qquad (4.78)\\
&= \begin{pmatrix} C & \gamma\cos\theta\cos\phi + \gamma^*\sin\theta\sin\phi \\ \gamma^*\cos\theta\cos\phi + \gamma\sin\theta\sin\phi & 1-C \end{pmatrix} ,
\end{aligned}$$

with

$$C = p(q\cos^2\theta + (1-q)\cos^2\phi) + (1-p)(q\sin^2\phi + (1-q)\sin^2\theta).$$

Its characteristic function is given by

$$\chi'(\xi) = \chi\left(\xi\cos\theta\cos\phi - \xi^*\sin\theta\sin\phi\right)\left[1 + (2p-1)\frac{\cos(2\theta)-\cos(2\phi)}{4}\xi\xi^*\right] .$$

A natural question is then whether or not the weakly complementary channel (2.50) associated with Eq. (4.72) is also Gaussian. To see this we first use the linearity of trace to express the complementary $\tilde{\mathcal{E}}$ as a convex combination of the weakly complementaries of $\mathcal{E}_0$ and $\mathcal{E}_1$, i.e. $\tilde{\mathcal{E}} = q\,\tilde{\mathcal{E}}_0 + (1-q)\,\tilde{\mathcal{E}}_1$. Then we invoke the linearity of Eq. (4.23) and use Eq. (4.61) to write

$$\begin{aligned}
\tilde{G}(\zeta,\xi) &= q\,\delta^{(2)}\left(\zeta - \xi\cos\theta\sin\phi + \xi^*\sin\theta\cos\phi\right)\\
&\quad\times\exp\left[\frac{\cos(2\theta)+\cos(2\phi)}{4}\xi\xi^*\right]\\
&+ (1-q)\,\delta^{(2)}\left(\zeta + \xi\sin\phi\cos\theta - \xi^*\cos\phi\sin\theta\right)\\
&\quad\times\exp\left[-\frac{\cos(2\phi)+\cos(2\theta)}{4}\xi\xi^*\right] .
\end{aligned}\tag{4.79}$$

This is of the form (4.53) only for $q = 0, 1$. Therefore, in general the weakly complementaries of qubit-qubit maps with mixed environment are not Gaussian even though they can be expressed as a convex combination of Gaussian channels (i.e., $\tilde{\mathcal{E}}_0$ and $\tilde{\mathcal{E}}_1$). This can be pushed a little further by observing that for generic choices of θ, ϕ and q, the weakly complementaries (4.79) are not even unitarily equivalent to a Gaussian qubit channel.

It is worth observing that in the canonical form (4.34) the weakly complementaries (4.79) are characterized by $\lambda_1 = (2q-1)\sin(\theta+\phi)$, $\lambda_2 = (2q-1)\sin(\phi-\theta)$, $\lambda_3 = [\cos(2\theta) - \cos(2\phi)]/2$, $t_1 = t_2 = 0$ and $t_3 = (2q-1)/2[\cos(2\theta)+\cos(2\phi)]/2$. Since $\lambda_1\lambda_2 \neq \lambda_3$, this is an indirect way of verifying that these maps are not Gaussian (4.53). However, the canonical form [37] is uniquely determined only up to unitary transformations acting on the input and on the output of the map. Applying such unitary transformations one can permute the λs. After such permutations, one can have, for instance, $\lambda_1 = [\cos(2\theta)-\cos(2\phi)]/2$, $\lambda_2 = (2q-1)\sin(\theta+\phi)$, $\lambda_3 = (2q-1)\sin(\phi-\theta)$. Now $\lambda_1\lambda_2 = \lambda_3$ can be satisfied for some particular values of θ and ϕ, i.e. $\theta = \phi$, $\theta+\phi = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2$, and $\theta = \phi \pm \pi$. In these cases we can say that the weakly complementaries (4.79) are unitarily equivalent to a Gaussian channel.

Finally, we show that the output density matrix is given by

$$\mathcal{E}(\rho) = \mathcal{E}\begin{pmatrix} p & \gamma \\ \gamma^* & (1-p) \end{pmatrix} = \begin{pmatrix} \tilde{C} & \tilde{D} \\ \tilde{D}^* & 1-\tilde{C} \end{pmatrix}, \tag{4.80}$$

with

$$\tilde{C} = p(q\cos^2\theta + (1-q)\sin^2\phi) + (1-p)(q\cos^2\phi + (1-q)\sin^2\theta),$$

$$\tilde{D} = (2q-1)\left(\gamma\cos\theta\sin\phi + \gamma^*\sin\theta\cos\phi\right),$$

and the characteristic function of the output state is

$$\tilde{\chi}'(\xi) = [\chi_e(f) + (2p-1)\chi_o(f)]\left[1 + (2p-1)\frac{\cos(2\theta)+\cos(2\phi)}{4}\xi\xi^*\right], \tag{4.81}$$

where $\chi_e(\xi)$ and $\chi_o(\xi)$ are, respectively, the *even* and the *odd* characteristic functions of the input state (see Appendix B).

Weak-degradability properties

Let us analyze the weak-degradability properties of the qubit-qubit channels with mixed environment [44].

As in Sec. 4.4.2 we prove that the maps $\mathcal{E}$ of Eq. (4.53) are weakly degradable for $\cos(2\theta)/\cos(2\phi) \geqslant 0$. In this regime in fact one can easily check that Eq. (4.68) can still be solved with $G_x(\zeta,\xi)$ of the form (4.79) replacing θ and ϕ with $-\theta_x$ and $\phi_x + \pi/2$ where θ_x, ϕ_x satisfy the relations (4.69).

Proving anti-degradability for $\cos(2\theta)/\cos(2\phi) \leqslant 0$ is not simple because, in general, $\tilde{\mathcal{E}}$ is not in a Gaussian form — see Eq. (4.79). However, in this case we show that these channels cannot be used to transfer quantum information since their quantum capacity Q [87, 88, 89] is null. To see this we notice that for $\cos(2\theta)/\cos(2\phi) \leqslant 0$, $\mathcal{E}$ is a mixture (4.72) of two channels (i.e. $\mathcal{E}_0$ and $\mathcal{E}_1$) which are both anti-degradable and have hence null quantum capacity, i.e.

$$Q(\mathcal{E}_0) = Q(\mathcal{E}_1) = 0 \ . \tag{4.82}$$

Under these conditions it is easy to verify that also $\mathcal{E}$ must have a null Q. Indeed, let us consider a new CPT map,

$$\mathcal{E}'(\rho) = q \ \mathcal{E}_0(\rho) \otimes |0\rangle_B\langle 0| + (1-q) \ \mathcal{E}_1(\rho) \otimes |1\rangle_B\langle 1| \ ,$$

where B is an ancillary system. We can now verify that the $\mathcal{E}$ is isomorphic to $\mathcal{E} \circ \mathcal{E}'$ with $\mathcal{E}(\ldots) = \mathrm{Tr}_B[\ldots] \otimes |0\rangle_B\langle 0|$ being a CPT map which replaces all states of B with a fix given output $|0\rangle_B$. Expressing Q in terms of the output coherent information [91] of the channel and using the quantum data processing inequality [1] we can verify that

$$Q(\mathcal{E}) \leqslant Q(\mathcal{E}') \ . \tag{4.83}$$

Besides, by using the basic properties of von Neumann entropy [1, 14] we can express the coherent information of $\mathcal{E}'$ as

$$J(\rho, \mathcal{E}') = qJ(\rho, \mathcal{E}_0) + (1-q)J(\rho, \mathcal{E}_1) \ . \tag{4.84}$$

Putting all this together we get

$$Q(\mathcal{E}') = \lim_{N\to\infty} \max_{\rho} J(\rho, {\mathcal{E}'}^{\otimes N})/N \leqslant qQ(\mathcal{E}_0) + (1-q)Q(\mathcal{E}_1) = 0 \ , \tag{4.85}$$

and hence $Q(\mathcal{E}) = 0$.

4.5 Degradability of some qubit channels

In the following we will show some examples of qubit quantum channels [1, 14] and we will analyze their weak-degradability properties [130]. We recall that not all qubit channels are Gaussian but only those describable trough a noisy interaction between one qubit (for the system) and one qubit (for the environment).

4.5.1 Bit flip or dephasing channel

The *bit flip* or *dephasing* channel flips the state $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1-s$. Its Kraus operators [1, 14] are:

$$A_0 = \sqrt{s}\,\mathbb{1} = \sqrt{s}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.86}$$

$$A_1 = \sqrt{1-s}\,\sigma_x = \sqrt{1-s}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.87}$$

In Fig. 4.3 the effect of the bit flip channel is illustrated. It implies a contraction of the Bloch sphere and the Bloch vector can only ever decrease. Particularly, we note that $\mathrm{Tr}\rho^2 = \frac{1}{2}(1+|\vec{r}|^2)$, where $|\vec{r}|$ is the norm of the Bloch vector. The characteristic function of a generic input density operator in Eq. (4.13) is

$$\chi(\xi) = 1 - \frac{1}{2}(2p-1)\xi^*\xi + \gamma\xi - \gamma^*\xi^* \ . \tag{4.88}$$

After the noise evolution of the bit flip or dephasing channel, the density matrix of the output quantum state is:

$$\mathcal{E}(\rho) = \begin{pmatrix} sp+(1-s)(1-p) & q\gamma+(1-s)\gamma^* \\ s\gamma^*+(1-s)\gamma & s(1-p)+(1-s)p \end{pmatrix} , \tag{4.89}$$

whose characteristic function is

$$\begin{aligned} \chi'(\xi) &= 1 - \frac{1}{2}(2s-1)(2p-1)\xi^*\xi + [s\gamma+(1-s)\gamma^*]\xi - [s\gamma^*+(1-s)\gamma]\xi^* \\ &= \chi(s\,\xi - (1-s)\,\xi^*) \ . \end{aligned} \tag{4.90}$$

Note that this map can be obtained from the canonical form in Eq. (4.44) by putting

$$t_1 = t_2 = t_3 = 0 \quad \text{(unital map)} \tag{4.91}$$

$$\lambda_1 = 1 \ , \quad \lambda_2 = \lambda_3 = 2s-1 \ , \tag{4.92}$$

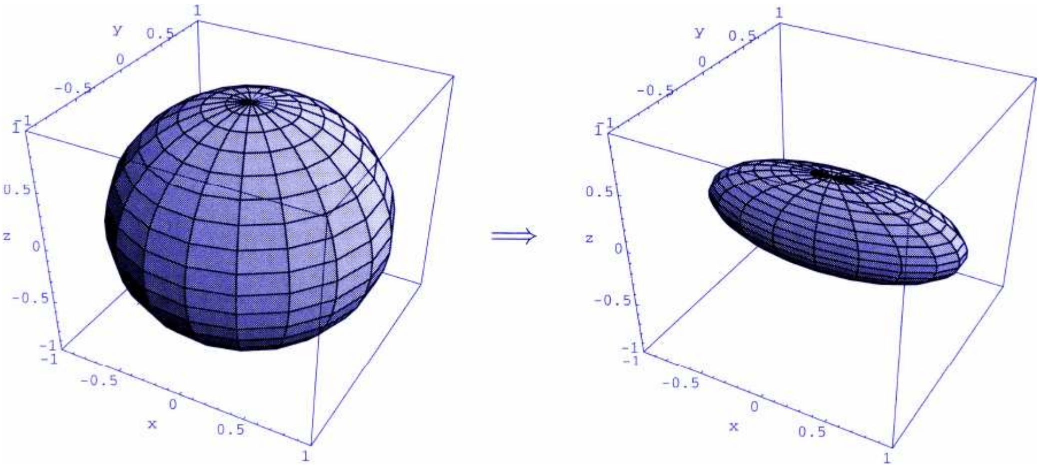

Figure 4.3: The effect of the bit flip channel on the Bloch sphere, for $s = 0.3$. The sphere on the left represents the set of all pure states, and the deformed sphere on the right represents the states after going through the channel. Note that the states on the $\hat{x}$ axis are left alone, while the $\hat{y} - \hat{z}$ plane is uniformly contracted by a factor of $1 - 2s$ [1].

and so it is a qubit-qubit map with pure environment (q=1). Indeed, $|t_3| = \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} = 0$. The relative Gaussian Green function is:

$$G(\zeta, \xi) \;\; = \;\; \delta^{(2)}\left(\zeta - s\xi - (s-1)\xi^*\right) \; . \tag{4.93}$$

Observing that $\cos(2\theta) = \frac{\lambda_3 + t_3}{2} = s - 1/2$ and $\cos(2\phi) = \frac{\lambda_3 - t_3}{2} = s - 1/2$, $\cos(2\theta)/\cos(2\phi) = 1 > 0$ and the bit flip channel is always degradable for any value of s.

4.5.2 Phase flip channel

The *phase flip* channel changes the phase of the state $|1\rangle$ with probability $1-s$; for instance, $\frac{1}{2}(|0\rangle + |1\rangle)$ is mapped to $\frac{1}{2}(|0\rangle - |1\rangle)$ with probability $1-s$ and with probability s it remains unchanged.

Its Kraus operators are [1, 14]:

$$A_0 \;\; = \;\; \sqrt{s}\, \mathbb{1} = \sqrt{s} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.94}$$

$$A_1 \;\; = \;\; \sqrt{1-s}\, \sigma_z = \sqrt{1-s} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.95}$$

In the special case of $s = \frac{1}{2}$ this channel corresponds to a measurement of the qubit in the basis $\{|0\rangle, |1\rangle\}$, with the result of the measurement unknown; indeed, $\mathcal{E}(\rho) = P_0\rho P_0 + P_1\rho P_1$, where $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are two projectors on the computational basis $\{|0\rangle, |1\rangle\}$.

In Fig. 4.4 the effect of the phase flip channel is illustrated. Geometrically, it implies that the Bloch vector is projected along the $\hat{z}$ axis and its x and y components are lost, i.e. $(r_x, r_y, r_z) \to (0, 0, r_z)$, where r_i are the components of the Bloch vector.

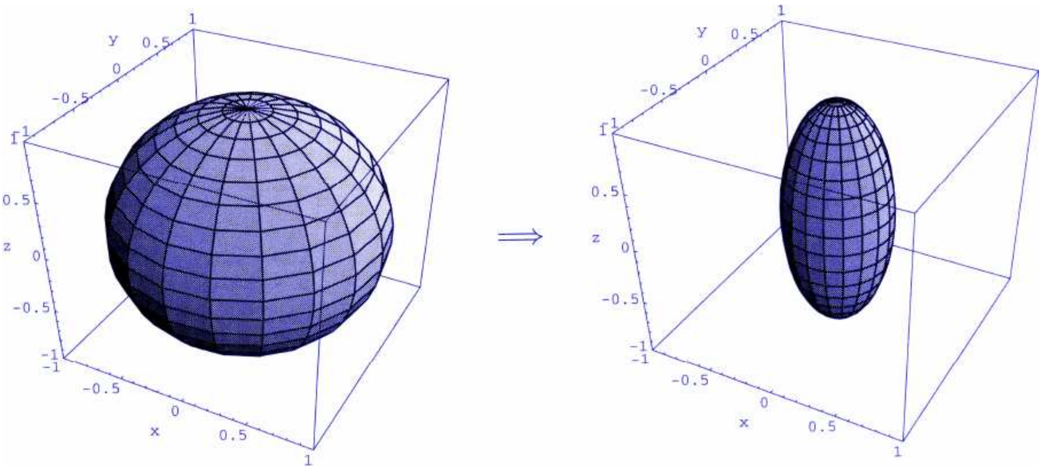

Figure 4.4: The effect of the phase flip channel on the Bloch sphere, for $s = 0.3$. Note that the states on the $\hat{z}$ axis are left alone, while the $\hat{x} - \hat{y}$ plane is uniformly contracted by a factor of $1 - 2s$ [1].

Starting from a generic input density operator, like in the previous section, after the noise evolution of the phase flip channel, the density matrix of the output state [with ρ as in Eq. (4.13)] is:

$$\mathcal{E}(\rho) = \begin{pmatrix} p & (2s-1)\gamma \\ (2s-1)\gamma^* & (1-p) \end{pmatrix} , \tag{4.96}$$

whose characteristic function is

$$\begin{aligned} \chi'(\xi) &= 1 - \frac{1}{2}(2p-1)\xi^*\xi + (2s-1)\gamma\xi - (2s-1)\gamma^*\xi^* \\ &= \chi_e(\xi) + \chi_o((2s-1)\xi) , \end{aligned} \tag{4.97}$$

where $\chi_e(\xi)$ and $\chi_o(\xi)$ are, respectively, the *even* and the *odd* characteristic functions of the input state (see Appendix B). This channel has the following (canonical) parameters in Eq. (4.44)

$$t_1 = t_2 = t_3 = 0 \quad \text{(unital map)} \tag{4.98}$$
$$\lambda_3 = 1\ , \quad \lambda_1 = \lambda_2 = 2s - 1\ , \tag{4.99}$$

and the relative Green function is not Gaussian, i.e.

$$G(\zeta, \xi) \ = \ \delta^{(2)}\left(\zeta - (2s-1)\xi\right) + 4s(1-s)\xi\xi^*\ .$$

Since the canonical form [37] is uniquely determined only up to unitary transformations, one can permute the λs and so the phase flip channel is unitarily equivalent to a bit-flip channel. Therefore, the phase flip channel is not a Gaussian channel but it is unitarily equivalent to a (degradable) Gaussian map.

4.5.3 Bit-phase flip channel

The *bit-phase flip* channel is a combination of a bit flip and a phase flip channels [1, 14]. Its operation elements are:

$$A_0 \ = \ \sqrt{s}\ \mathbb{1} = \sqrt{s}\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.100}$$
$$A_1 \ = \ \sqrt{1-s}\ \sigma_y = \sqrt{1-s}\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{4.101}$$

The geometric action of the bit-flip channel is shown in Fig. 4.5. Note that $\sigma_y = i\sigma_x\sigma_z$ and so the phase-flip channels includes both of the effects of the bit flip and the phase flip maps. The output density matrix has the following expression [with ρ as in Eq. (4.13)]

$$\mathcal{E}(\rho) = \begin{pmatrix} sp + (1-s)(1-p) & s\gamma - (1-s)\gamma^* \\ s\gamma^* - (1-s)\gamma & s(1-p) + (1-s)p \end{pmatrix}\ , \tag{4.102}$$

whose characteristic function is

$$\begin{aligned} \chi'(\xi) \ &= \ 1 - \frac{1}{2}(2s-1)(2p-1)\xi^*\xi + [s\gamma - (1-s)\gamma^*]\xi - [s\gamma^* - (1-s)\gamma]\xi^* \\ &= \ \chi(s\ \xi + (1-s)\ \xi^*). \end{aligned} \tag{4.103}$$

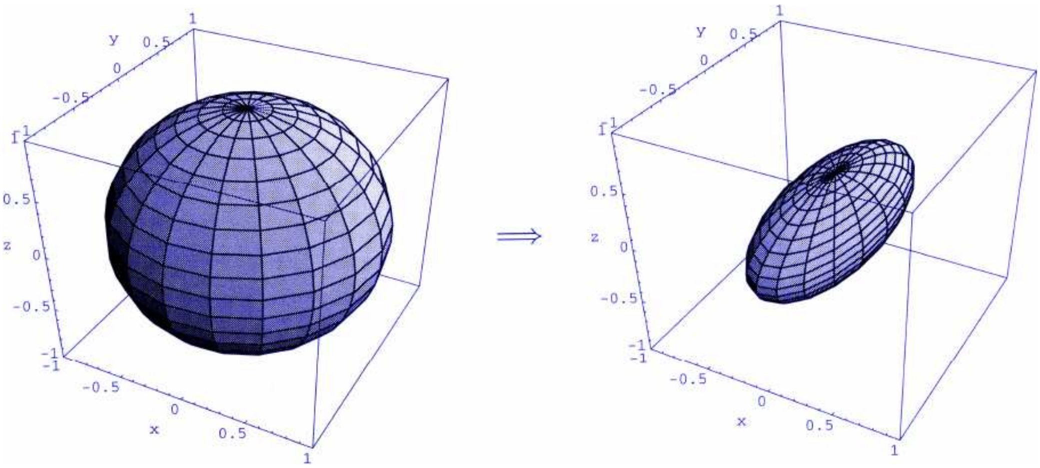

Figure 4.5: The effect of the bit-phase flip channel on the Bloch sphere, for $s = 0.3$. Note that the states on the $\hat{y}$ axis are left alone, while the $\hat{x} - \hat{z}$ plane is uniformly contracted by a factor of $1 - 2s$ [1].

The phase-flip channel is obtained with the following parameters in Eq. (4.44)

$$t_1 = t_2 = t_3 = 0 \ , \quad \text{(unital map)} \tag{4.104}$$

$$\lambda_2 = 1 \ , \quad \lambda_1 = \lambda_3 = 2s - 1 \ , \tag{4.105}$$

and so it is a qubit-qubit map with pure environment (q=1). Indeed, $|t_3| = \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} = 0$. The Green function is so Gaussian, i.e.

$$G(\zeta, \xi) \quad = \quad \delta^{(2)}\left(\zeta - s\xi - (1-s)\xi^*\right) \ .$$

Observing that $\cos(2\theta) = \frac{\lambda_3 + t_3}{2} = s - 1/2$ and $\cos(2\phi) = \frac{\lambda_3 - t_3}{2} = s - 1/2$, $\cos(2\theta)/\cos(2\phi) = 1 > 0$ and the bit-phase flip channel is always degradable for any value of s.

4.5.4 Depolarizing channel

The *depolarizing* channel represents an important kind of noise evolution, in which the qubit is *depolarized* (i.e. replaced by the completely mixed state, $\mathbb{1}/2$) with probability s and it is left untouched with probability $1 - s$. Therefore, the state of the quantum system after this noise evolution is

$$\mathcal{E}(\rho) = s\frac{\mathbb{1}}{2} + (1-s)\rho \tag{4.106}$$

where ρ is defined in Eq. (4.13). Observing that for an arbitrary density operator ρ the following relation holds

$$\frac{\mathbb{1}}{2} = \frac{\rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z}{4} , \tag{4.107}$$

one obtains the following four operation elements for the Kraus representation in Eq. (2.13) [1, 14]:

$$A_0 = \sqrt{1 - \frac{3}{4}s}\ \mathbb{1} = \sqrt{1 - \frac{3}{4}s} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.108}$$

$$A_1 = \frac{\sqrt{s}}{2}\ \sigma_x = \frac{\sqrt{s}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.109}$$

$$A_2 = \frac{\sqrt{s}}{2}\ \sigma_y = \frac{\sqrt{s}}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{4.110}$$

$$A_3 = \frac{\sqrt{s}}{2}\ \sigma_z = \frac{\sqrt{s}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.111}$$

Let us note that it is possible to generalize the depolarizing channel to quantum systems of any dimension d, according to the map $\mathcal{E}(\rho) = s\frac{\mathbb{1}}{d} + (1-s)\rho$.

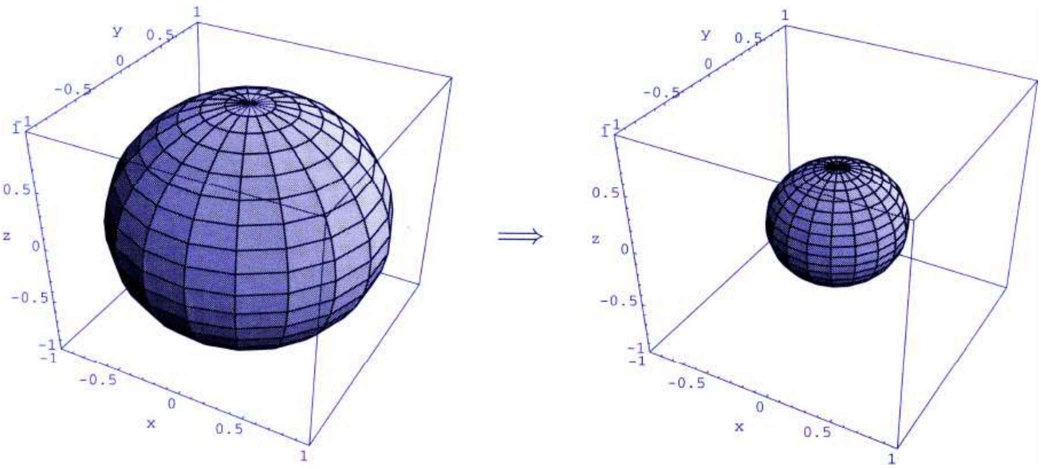

Figure 4.6: The effect of the depolarizing channel on the Bloch sphere, for $s = 0.5$. Notice how the entire sphere contracts uniformly as a function of s [1].

In Fig. 4.6 the geometric effect of the depolarizing channel is shown; increasing the probability s the Bloch sphere contracts uniformly. The output density matrix is

$$\mathcal{E}(\rho) = \begin{pmatrix} (1-s)p + s/2 & (1-s)\gamma \\ (1-s)\gamma^* & (1-s)(1-p) + s/2 \end{pmatrix} , \tag{4.112}$$

whose characteristic function is

$$\begin{aligned} \chi'(\xi) &= 1 - \frac{1}{2}(1-s)(2p-1)\xi^*\xi + (1-s)\gamma\xi - (1-s)\gamma^*\xi^* \\ &= \chi_e(\sqrt{1-s}\ \xi) + \chi_o((1-s)\ \xi). \end{aligned} \tag{4.113}$$

In the canonical representation in Eq. (4.44) the depolarizing channel is characterized by these parameters

$$t_1 = t_2 = t_3 = 0 \ , \quad \text{(unital map)} \tag{4.114}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1 - s \ , \tag{4.115}$$

and, since the Green function is not Gaussian, i.e.

$$G(\zeta, \xi) = \delta^{(2)}\left(\zeta - (1-s)\xi\right) + s(1-s)\xi\xi^* \ ,$$

we are not able to discuss its degradability properties in our formalism.

4.5.5 Amplitude damping or beam-splitter channel

Let us consider now a typical process of noise evolution in which a quantum system losses its energy. This physical scenario is well described by a quantum operation, known as *amplitude damping* or *beam-splitter* channel. Suppose to have the state of a photon in an interferometer or cavity when it is subject to scattering and attenuation. We can model the scattering of a photon from this mode by thinking a beam-splitter, e.g. a partially silvered mirror, in the path of the photon. This quantum channel can be described by the following Kraus operators [1, 14]:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{n} \end{pmatrix} , \tag{4.116}$$

$$A_1 = \begin{pmatrix} 0 & \sqrt{1-n} \\ 0 & 0 \end{pmatrix} , \tag{4.117}$$

where $1-n$ can be thought of as the probability of losing a photon. After the noise evolution of the amplitude damping channel, the density matrix of the output state is [with ρ as in Eq. (4.13)]:

$$\mathcal{E}(\rho) = \begin{pmatrix} p+(1-n)(1-p) & \sqrt{n}\gamma \\ \sqrt{n}\gamma^* & n(1-p) \end{pmatrix} , \tag{4.118}$$

whose characteristic function is

$$\begin{aligned} \chi'(\xi) &= 1-\frac{1}{2}[1-2n(1-p)]\xi^*\xi+\sqrt{n}\gamma\xi-\sqrt{n}\gamma^*\xi^* \\ &= \chi(\sqrt{n}\xi)\left(1+\frac{1-n}{2}\xi\xi^*\right), \end{aligned} \tag{4.119}$$

where $\chi(\xi)$ is the input characteristic function.

By considering the canonical form in Eq. (4.44) the amplitude damping or beam-splitter channel is given by

$$t_1 = t_2 = 0 \ , \quad t_3 = 1-n \quad \text{(not unital map)} \tag{4.120}$$

$$\lambda_1 = \lambda_2 = \sqrt{n} \ , \quad \lambda_3 = n \ . \tag{4.121}$$

Since $|t_3| = \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} = 1-n$, it is a qubit-qubit map with pure environment (q=1). The Gaussian Green function has the form

$$G(\zeta,\xi) = \delta^{(2)}\left(\zeta-\sqrt{n}\xi\right)\exp\left[-\frac{1-n}{2}\xi^*\xi\right] .$$

Since $\cos(2\theta) = \frac{\lambda_3+t_3}{2} = 1/2$, $\cos(2\phi) = \frac{2n-1}{2}$ and $\cos(2\theta)/\cos(2\phi) = \frac{1}{2n-1}$, the amplitude damping or beam-splitter channel is degradable for $n \geq 1/2$ and anti-degradable for $n \leq 1/2$. Let us stress that one obtains exactly the same results as the ones for the Bosonic beam-splitter, studied in Sec. 3.3.

In Fig. 4.7 we report the effect of amplitude damping in the Bloch representation as the Bloch vector transformation

$$(r_x, r_y, r_z) \ \to (\sqrt{n}\ r_x, \sqrt{n}\ r_y, (1-n)+nr_z) \ . \tag{4.122}$$

If one replaces $1-n$ with a time-varying function like $1-e^{-t/\tau}$ (where t is time and τ is a characteristic time scale that characterizes the speed of damping, e.g. in a quantum memory), the effect is a flow on the Bloch sphere, which moves every point in the unit ball towards a fixed point at the north pole, where there is $|0\rangle$.

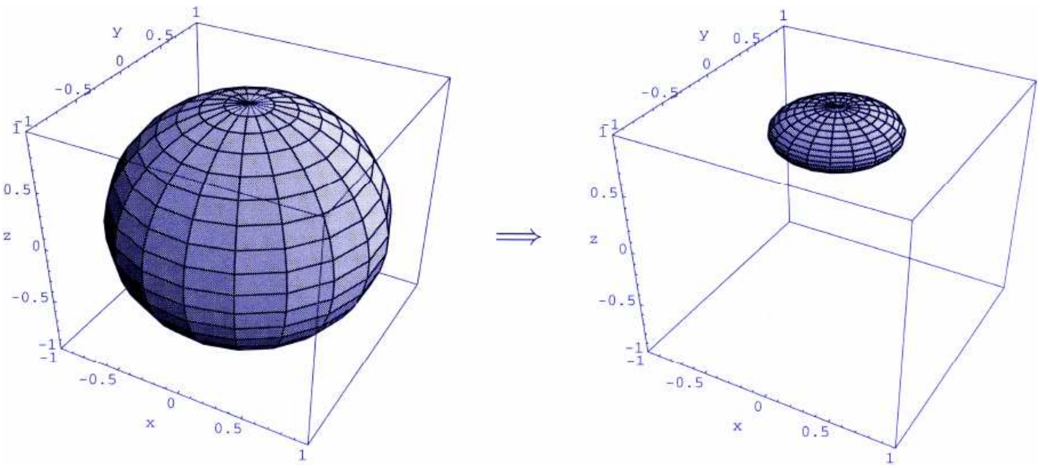

Figure 4.7: The effect of the amplitude damping channel on the Bloch sphere, for $n = 0.2$. Note how the entire sphere contracts shrinking towards the north pole, the $|0\rangle$ state [1].

The amplitude damping or beam-splitter channel can also be written [41] as a unitary transformation, U_{ab}, of the annihilation operators of the system (one qubit), a, and of the environment (one qubit) b,

$$U_{ab}^\dagger a U_{ab} = a' = \sqrt{n}a + \sqrt{1-n}[a, a^\dagger]b \ , \tag{4.123}$$

$$U_{ab}^\dagger b U_{ab} = b' = \sqrt{n}b - \sqrt{1-n}[b, b^\dagger]a \ , \tag{4.124}$$

where n is the damping coefficient. It is a canonical transformation and $\{a', a'^\dagger\} = 1$, $\{b', b'^\dagger\} = 1$, $[a', b'] = 0$, and $[a', b'^\dagger] = 0$, since $\{a, a^\dagger\} = 1$, $\{b, b^\dagger\} = 1$, $[a, b] = 0$, and $[a, b^\dagger] = 0$. Note that the commutation relations $[.,.] = 0$ are due to the fact that the two qubits are considered to be distinguishable. This transformation can be described as a coupling with environment prepared in an initial *mixed* state σ_b, that is

$$\mathcal{E}(\rho_a) = \mathrm{Tr}_b[U_{ab}(\rho_a \otimes \sigma_b)U_{ab}^\dagger] \ , \tag{4.125}$$

where $\mathrm{Tr}_b[\ldots]$ is the partial trace over the environment B. Therefore, the output characteristic function for the mode a is (like in the Bosonic case)

$$\chi_a'(\xi) = \chi_a(\sqrt{n}\xi)\chi_b(\sqrt{1-n}\xi) \ , \tag{4.126}$$

where $\chi_b(\xi)$ is the characteristic function of the environment in the initial state $|0\rangle$. The same transformation of the characteristic function is performed by the Fermionic beam-splitter channel (see Sec. C.1 in Appendix C).

An important characteristic of the amplitude damping channel is that only the ground state $|0\rangle$ is left invariant under the action of the map, since we are modelling the environment in the initial state $|0\rangle$, as if it were at zero temperature. In the following section we will consider the corresponding noise evolution at finite temperature.

4.5.6 Generalized amplitude damping channel

Here we describe the effect of dissipation due to the presence of an external environment at finite temperature. This quantum operation, called *generalized amplitude damping* channel, can be described by the following Kraus operators ($s \neq 1$) [1, 14]:

$$A_0 = \sqrt{s}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{n} \end{pmatrix}, \tag{4.127}$$

$$A_1 = \sqrt{s}\begin{pmatrix} 0 & \sqrt{1-n} \\ 0 & 0 \end{pmatrix}, \tag{4.128}$$

$$A_2 = \sqrt{1-s}\begin{pmatrix} \sqrt{n} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.129}$$

$$A_3 = \sqrt{1-s}\begin{pmatrix} 0 & 0 \\ \sqrt{1-n} & 0 \end{pmatrix}, \tag{4.130}$$

and the stationary state is

$$\rho_\infty = \begin{pmatrix} s & 0 \\ 0 & 1-s \end{pmatrix}. \tag{4.131}$$

After the noise evolution associated to the generalized amplitude damping channel, the output density matrix is

$$\mathcal{E}(\rho) = \begin{pmatrix} [n+s(1-n)]p + s(1-n)(1-p) & \sqrt{n}\gamma \\ \sqrt{n}\gamma^* & (1-s)(1-n)p + [1-s(1-n)](1-p) \end{pmatrix},$$

whose characteristic function is

$$\begin{aligned} \chi'(\xi) &= 1 - \frac{1}{2}\{[1-2(1-n)(1-s)]p - (1-2(1-n)s)(1-p)\}\xi^*\xi + \sqrt{n}\gamma\xi \\ &- \sqrt{n}\gamma^*\xi^* = \chi(\sqrt{n}\xi)\left[1 + (1-n)\left(s - \frac{1}{2}\right)\xi\xi^*\right], \end{aligned} \tag{4.132}$$

where $\chi(\xi)$ is the input characteristic function and ρ is defined in Eq. (4.13). The effect of this map in the Bloch representation is

$$(r_x, r_y, r_z) \; \rightarrow (\sqrt{n}\; r_x, \sqrt{n}\; r_y, (1-n)(2s-1) + n r_z) \; . \tag{4.133}$$

The only difference with the amplitude damping channel is the fix point of the flow, that is a mixed state along the $\hat{z}$ axis at the point $(2s-1)$.

The generalized amplitude damping channel corresponds to the following particular parameters in Eq. (4.44),

$$t_1 = t_2 = 0 \quad t_3 = (1-n)(2s-1) \quad \text{(not unital map)} \tag{4.134}$$

$$\lambda_1 = \lambda_2 = \sqrt{n} \; , \quad \lambda_3 = n \; . \tag{4.135}$$

Since $|t_3| = (2s-1)(1-n) \leq \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} = 1-n$, it is a qubit-qubit map with mixed environment ($q \neq 1$). The Gaussian Green function has the form

$$G(\zeta, \xi) \;\; = \;\; \delta^{(2)}\left(\zeta - \sqrt{n}\xi\right) \exp\left[-(2s-1)\frac{(1-n)}{2}\xi^* \xi\right] \; .$$

It is a qubit-qubit map with mixed environment ($q \equiv s \neq 1$) and is weakly degradable for $n \geq 1/2$ and with null quantum capacity for $n \leq 1/2$.

4.5.7 Phase damping channel

The *phase damping* channel describes the loss of quantum information without loss of energy. It is a typical quantum mechanical noise evolution regarding the loss of the coherence, when, for example, a photon scatters randomly as it travels through a waveguide, or how electronic states in an atom are perturbed upon interacting with distant electrical charge. Actually, the decoherence manifests itself with a loss of the information about the relative quantum phase between the energy eigenstates of a quantum system during the time evolution. This uniquely quantum mechanical process can be described by the following Kraus operators [1, 14]:

$$A_0 \;\; = \;\; \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-s} \end{pmatrix} \; , \tag{4.136}$$

$$A_1 \;\; = \;\; \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{s} \end{pmatrix} \; , \tag{4.137}$$

where s is the probability that, for instance, a photon or an electron from the system has been scattered (without loss of energy). It is exactly equivalent to the phase flip channel and so it can be visualized on the Bloch sphere as

$$(r_x, r_y, r_z) \rightarrow (\sqrt{1-s}\; r_x, \sqrt{1-s}\; r_y, r_z)\ , \tag{4.138}$$

with the effect of the shrinking the sphere into ellipsoids. As a function of time, i.e. $\sqrt{1-s} = e^{-t/\tau}$, the phase damping can be seen as a relaxation process (e.g., in a quantum memory), in which the damping increases and all points of the Bloch sphere flow towards the $\hat{z}$ axis, in which the states remain invariant.

4.6 Memory qubit channels

Throughout this thesis work, we have considered only the so-called standard memoryless channels, in which successive uses of the communication line are affected by the same noise [79]. Here, we will briefly analyze an example of correlated noise channels describing, instead, situations where consecutive channel uses suffer from the action of correlated noise source and which cannot be written as a simple tensor product of quantum channels. This phenomenon is quite common in physical situations, when the noise source exhibits, for instance, time-dependent statistical properties. Recently, there has been some interest in studying the behavior of these channels with correlations since such channels might be regarded as a small first step in studying the much more complex issue of channels with memory [131, 132, 133, 134, 135, 136, 137, 138]. They can describe multiple access channels and can be also related to critical quantum many-body physics [139]. In Ref. [132] the effect of correlated noise was analyzed showing specific examples for which there is a critical value μ_c below which the optimal input state (i.e., that suffers less noise) is a product state and above which the optimal input is maximally entangled.

In this section, we consider a very simple class of channels which exhibit quite different behavior [65]. We analyze the behaviour of a class of two-qubit correlated noise channels where, with probability $1-\mu$, the qubits suffer only from uncorrelated tensor product noise, while with probability μ they experience correlated noise. This situation can be modelled by a completely positive, trace preserving (CPT) map which transforms any bound operator

Θ of the joint system $\mathcal{H}_1 \otimes \mathcal{H}_2$ according to the transformation

$$\mathcal{E}(\Theta) = (1-\mu)(\Psi \otimes \Psi)(\Theta) + \mu\, \Gamma_{\rm corr}\, {\rm Tr}[\Theta] \tag{4.139}$$

with $\Gamma_{\rm corr}$ being a density matrix of the joint system $\mathcal{H}_1 \otimes \mathcal{H}_2$ and where Ψ represents a quantum channel acting on the a single qubit. In the following $\Gamma_{\rm corr}$ is simply a maximally entangled state. We find that the entanglement of the optimal input state increases with μ until it reaches a critical μ_c, after which the optimal input is always achieved with a maximally entangled state. Below μ_c, the optimal input is never maximally entangled.

Although one is ultimately interested in the effect of correlations on the capacities [79] of the channel, here we focus only on the maximal ℓ_p-norms achievable at the output of the channel. Given $p > 1$, the ℓ_p-norm of a state ρ is given by the expression

$$\|\rho\|_p \equiv \left({\rm Tr}[\rho^p]\right)^{1/p} \ . \tag{4.140}$$

These quantities are related to Rényi [140] and Tsallis entropies [141] (i.e., $S_p(\rho) \equiv \log[{\rm Tr}[\rho^p]]/(1-p)$ and $T_p(\rho) \equiv [1 - {\rm Tr}[\rho^p]]/(p-1)$ respectively), and provide us with a *measure* of the purity of ρ. Indeed values of $\|\rho\|_p$ close to one are indicative that ρ is close to a pure state. On the contrary small value of $\|\rho\|_p$ are associated with highly mixed states. The ℓ_p-norm $\|\mathcal{E}\|_p$ of a CPT map $\mathcal{E}$ is now defined as the maximum value of $\|\mathcal{E}(\rho)\|_p$ which can be achieved by varying the input states ρ. Exploiting the convexity of Eq. (4.140) and the linearity of $\mathcal{E}$ this can be formally written as

$$\|\mathcal{E}\|_p \equiv \max_{\psi}\ \|\mathcal{E}(|\psi\rangle\langle\psi|)\|_p \ , \tag{4.141}$$

where the maximization is performed over all possible pure input states $|\psi\rangle$. The quantities (4.141) have been extensively studied in the literature [39, 43, 47, 92, 142, 143, 144, 145, 146] to characterize the noise introduced by the map. The underlining idea is that inputs, whose outputs are close to pure states, are the least corrupted[3]. This allows one to introduce the concept of optimal inputs states of order p, as those vectors $|\psi^{(p)}_{opt}\rangle$ which saturate the maximization (4.141), i.e. $\|\mathcal{E}(|\psi^{(p)}_{opt}\rangle\langle\psi^{(p)}_{opt}|)\|_p = \|\mathcal{E}\|_p$. For channels with some covariance properties [147], one can make an explicit connection between the

[3]It should be notice, however, that a channel which produces pure outputs can still be extremely noisy.

classical capacity [84, 85] and the optimal output purity as measured by the minimal output von Neumann entropy.

The rest of this chapter is organized as follows. In Sec. 4.6.1 we introduce the family of two-qubit correlated channels and study a covariance property that will be useful to determine their optimal input states by allowing a convenient parametrization of the input states in Sec. 4.6.2 [65]. In Sec. 4.6.3 we study analytically the $p = 2$ purity of the output states for a specific class of correlated two-qubit channels (4.139), and we generalize these results for $p \neq 2$ by performing numerical optimizations of the corresponding norms. In Sec. 4.6.4 we discuss these results in relation to majorization and trumping concepts.

4.6.1 The model

Consider a two-qubit channel $\mathcal{E}_{\mu,\vec{\lambda}}$ which, with probability μ, maps the input density matrices ρ of the system into the maximally entangled state $|\beta_0\rangle \equiv \big(|00\rangle + |11\rangle\big)/\sqrt{2}$, while, with probability $1-\mu$, $\mathcal{E}_{\mu,\vec{\lambda}}$ operates on the two qubits independently, applying to each of them the unital transformation $\Psi_{\vec{\lambda}}$ defined by the relations

$$\Psi_{\vec{\lambda}}(\sigma_k) = \lambda_k \sigma_k \ , \tag{4.142}$$

where for $k = 1, 2, 3$, σ_k is the k-th Pauli[4] operator and λ_k are real coefficients. This is,

$$\mathcal{E}_{\mu,\vec{\lambda}}(\rho) = (1-\mu)\big(\Psi_{\vec{\lambda}} \otimes \Psi_{\vec{\lambda}}\big)(\rho) + \mu\, |\beta_0\rangle\, \langle\beta_0| \ , \tag{4.143}$$

which, by linearity and considering that $\mathrm{Tr}[\rho] = 1$, defines a correlated channel of the form (4.139). The remaining of this section will concentrate on the case where the λ_k are identical (i.e. $\lambda_k = \lambda$ for $k = 1, 2, 3$ and $\lambda \in [0, 1]$) and will use the symbols Ψ_λ and $\mathcal{E}_{\mu,\lambda}$ to represent the resulting channels (4.142) and (4.143). For such a choice Eq. (4.142) describes a depolarizing channel (as in Sec. 4.5.4) [43] which is known to be covariant under generic unitary transformations U, i.e. $\Psi_\lambda(U\rho U^\dagger) = U\Psi_\lambda(\rho)U^\dagger$. Exploiting this property and the identity

$$|\beta_0\rangle = (U \otimes \sigma_2 U \sigma_2)|\beta_0\rangle \ , \tag{4.144}$$

[4]From now on, for simplicity, we will use the notation $\{\sigma_1, \sigma_2, \sigma_3\}$ for the Pauli operators with respect to $\{\sigma_x, \sigma_y, \sigma_z\}$ used in the rest of this thesis.

with σ_2 being the second Pauli matrix, one can easily verify that the correlated channel $\mathcal{E}_{\mu,\lambda}$ is covariant with respect to local unitary transformation of the form

$$W_U \equiv U \otimes \sigma_2 U \sigma_2 \; . \tag{4.145}$$

Namely, for all two-qubits input ρ and for all single qubit unitary operator U one has

$$\mathcal{E}_{\mu,\lambda}\Big(W_U(\rho)W_U^\dagger\Big) = W_U\left(\mathcal{E}_{\mu,\lambda}(\rho)\right)W_U^\dagger \; . \tag{4.146}$$

This property is remarkable: it implies that, given a generic density matrix ρ, the set $\mathcal{C}_\rho$ composed by states of the form $W_U(\rho)W_U^\dagger$ are transformed by the channel (4.143) into output states which differ only by *local unitary* transformations. On one hand, the unitarity of W_U ensures that all the members of $\mathcal{C}_\rho$ have identical output ℓ_p-norms (4.140). On the other hand, the locality of W_U not only ensures that all states in $\mathcal{C}_\rho$ have the same entanglement structure of ρ, but also that all corresponding outputs will have the same entanglement structure of $\mathcal{E}_{\mu,\lambda}(\rho)$.

4.6.2 Canonical form

To fully exploit the covariance property (4.146) we first notice that the for $k = 1, 2, 3$ the vectors $|\beta_k\rangle \equiv (\mathbb{1} \otimes \sigma_k)|\beta_0\rangle$ together with $|\beta_0\rangle$ form an orthonormal basis for the two qubits system (indeed they form a Bell set). Hence, apart from a global phase, any pure state can be expressed as follows

$$|\psi\rangle = \sum_{k=0}^{3} a_k|\beta_k\rangle = a_0|\beta_0\rangle + (\mathbb{1} \otimes \sigma_{\vec{a}})|\beta_0\rangle \; , \tag{4.147}$$

with $a_0 \in [0, 1]$ and with $a_{1,2,3}$ being complex amplitudes satisfying the normalization condition $\sum_{k=1}^{3} |a_k|^2 = 1 - a_0^2$ (here we defined $\vec{a} = (a_1, a_2, a_3)$ and $\sigma_{\vec{a}} = \sum_{k=1}^{3} a_k\sigma_k$). We also notice that, apart from a global phase, the most generic unitary transformation of the form (4.145) can written as follows

$$W_U = \Big(\cos\theta\mathbb{1} + i\sin\theta\;\sigma_{\vec{n}}\Big) \otimes \Big(\cos\theta\mathbb{1} + i\sin\theta\sigma_{\vec{n}'}\Big), \tag{4.148}$$

with $\vec{n} = (n_1, n_2, n_3)$ being a normalized real vector and $\vec{n}' = (-n_1, n_2, -n_3)$. We then get[5]

$$W_U|\psi\rangle \;=\; a_0|\beta_0\rangle + (\mathbb{1}\otimes\sigma_{T(\vec{a})})|\beta_0\rangle \;, \tag{4.149}$$

with $T \equiv T_{\vec{n},\theta}$ being the linear transformation defined by the expression

$$\begin{aligned} T_{\vec{n},\theta}(\vec{a}) \;\equiv\; & \cos^2\theta\;\vec{a} - 2\sin\theta\cos\theta\;(\vec{n}'\times\vec{a}) \\ & +(\vec{n}'\cdot\vec{a})\;\sin^2\theta\;\vec{n}' - \sin^2\theta\;(\vec{n}'\times\vec{a})\times\vec{n}' \;. \end{aligned} \tag{4.150}$$

Equation (4.150) describes a rotation by 2θ along the axis $\vec{n}'$. This can be used to show that for any complex vector $\vec{a} = \vec{\alpha} + i\vec{\beta}$ there exists a suitable choice of $\vec{n}$ and θ which allows us to write

$$T_{\vec{n},\theta}(\vec{a}) = (|\vec{\alpha}|\cos\chi + i|\vec{\beta}|, 0, |\vec{\alpha}|\sin\chi) \;, \tag{4.151}$$

with $|\vec{\alpha}||\vec{\beta}|\cos\chi = \vec{\beta}\cdot\vec{\alpha}$ and $\sin\chi \geqslant 0$ — see below for details. Consequently from Eq. (4.149) it follows that for any input state $|\psi\rangle$ there exists a suitable choice of T (i.e. a suitable choice of U) which gives

$$W_U|\psi\rangle = |a_0,\varphi,\phi\rangle \equiv a_0|\beta_0\rangle + \sqrt{1-a_0^2}\,\Big(\cos\varphi|\beta_3\rangle + e^{i\phi}\sin\varphi|\beta_1\rangle\Big) \;, \tag{4.152}$$

where $\cos\varphi = |\vec{\alpha}|\sin\chi$, and $\phi = \arctan[|\vec{\beta}|/(|\vec{\alpha}|\cos\chi)]$. We call $|a_0,\varphi,\phi\rangle$ the canonical form associated with the input state $|\psi\rangle$[6]. As mentioned in the previous section, since all W_U are local transformations, the vectors $|\psi\rangle$ and $|a_0,\varphi,\phi\rangle$ share the same entanglement properties. Furthermore the covariance property of Eq. (4.146) ensures that the output ℓ_p-norms associated with $|\psi\rangle$ and with the vector on the right-hand-side of Eq. (4.152) coincide. Thus instead of maximizing $\|\mathcal{E}_{\mu,\lambda}(\psi)\|_p$ over the whole set of pure inputs states $|\psi\rangle$ as in Eq. (4.141), we can focus only on canonical inputs of the form (4.152), i.e.

$$\|\mathcal{E}_{\mu,\lambda}\|_p = \max_{a_0,\varphi,\phi}\|\mathcal{E}_{\mu,\lambda}(\,|a_0,\varphi,\phi\rangle\,\langle a_0,\varphi,\phi|\,)\|_p \;. \tag{4.153}$$

[5] Given any complex vector $\vec{m}$ and $\vec{n}$ one has $\sigma(\vec{n})\sigma(\vec{n}) = (\vec{n}\cdot\vec{m})\;\mathbb{1} + i\;\sigma(\vec{n}\times\vec{m})$ with $\cdot$ and $\times$ being, respectively, the scalar and the vector product. Moreover, from the invariance of $|\beta_0\rangle$ under the unitary W_U of Eq. (4.145) it follows that $\sigma(\vec{n})\otimes\sigma(\vec{n}')|\beta_0\rangle = -|\beta_0\rangle$ for all real normalized vectors $\vec{n} = (n_1, n_2, n_3)$ and $\vec{n}' = (-n_1, n_2, -n_3)$. This yields $\sigma(\vec{n})\otimes\mathbb{1}|\beta_0\rangle = -\mathbb{1}\otimes\sigma(\vec{n}')|\beta_0\rangle$ and $\sigma(\vec{n})\otimes\sigma(\vec{n}')|\beta_0\rangle = -(\vec{m}\cdot\vec{n}')\;|\beta_0\rangle - i\;\mathbb{1}\otimes\sigma(\vec{m}\times\vec{n}')|\beta_0\rangle$.

[6] It is worth noticing that the canonical decomposition of Eq. (4.152) is not uniquely defined. Indeed one can show that with a different choice of the transformation T of Eq. (4.151) it possible to express the pure vectors $|\psi\rangle$ as a superposition of $|\beta_0\rangle$ plus any other couples of orthogonal Bell states $|\beta_{1,2,3}\rangle$.

Derivation of Eq. (4.151)

Let us first verify that the transformations T of Eq. (4.150) represent a generic rotation of $\mathbb{R}^3$. To show this it is sufficient to introduce normalized vectors $\vec{n}''$ and $\vec{n}'''$ which, together with $\vec{n}'$, form an oriented orthonormal set $\{\vec{n}', \vec{n}'', \vec{n}'''\}$ of vectors of $\mathbb{R}^3$. Consider now a generic vector $\vec{m} \in \mathbb{R}^3$ and decompose it within such basis, i.e.

$$\vec{m} = c_1\, \vec{n}' + c_2\, \vec{n}'' + c_3\, \vec{n}''' \;,$$

with $c_{1,2,3}$ being real. Replacing the above expression in Eq. (4.150) gives

$$\begin{aligned} T_{\vec{n},\theta}(\vec{m}) \;=\; c_1\, \vec{n}' \;&+\; (\; \cos 2\theta\; c_2 + \sin 2\theta\; c_3 \;)\; \vec{n}'' \\ &+\; (\; \cos 2\theta\; c_3 - \sin 2\theta\; c_2 \;)\; \vec{n}''' \end{aligned} \tag{4.154}$$

which explicitly shows that $T_{\vec{n},\theta}$ is a rotation by 2θ along the axis $\vec{n}'$.

Equation (4.151) can now be derived by decomposing $\vec{a}$ into its real and imaginary part, i.e. $\vec{a} = \vec{\alpha} + i\vec{\beta}$ with $\vec{\alpha}$ and $\vec{\beta}$ being real 3-dim vectors. The transformation $T_{\vec{n},\theta}$ which satisfies Eq. (4.151) can then be constructed by employing the algebra of rotation to write $T_{\vec{n},\theta} = T_{\vec{n}_2,\theta_2} \circ T_{\vec{n}_1,\theta_1}$ with "$\circ$" being the composition rule associated with Eq. (4.154) and with $T_{\vec{n}_1,\theta_1}$ and $T_{\vec{n}_2,\theta_2}$ defined as follows. The transformation $T_{\vec{n}_1,\theta_1}$ is obtained by choosing $\vec{n}_1$ and θ_1 such that the map $T_{\vec{n}_1,\theta_1}$ rotates $\vec{\beta}$ into the $\hat{x}$ axis, i.e.

$$T_{\vec{n}_1,\theta_1}(\vec{\beta}) = (|\vec{\beta}|, 0, 0) \;. \tag{4.155}$$

This gives us

$$T_{\vec{n}_1,\theta_1}(\vec{a}) = T_{\vec{n}_1,\theta_1}(\vec{\alpha}) + i(|\vec{\beta}|, 0, 0) \;, \tag{4.156}$$

with $T_{\vec{n}_1,\theta_1}(\vec{\alpha})$ being a vector of $\mathbb{R}^3$. The transformation $T_{\vec{n}_2,\theta_2}$ is now defined by choosing $\vec{n}_2 = \hat{x}$ and θ_2 such that it rotates $T_{\vec{n}_1,\theta_1}(\vec{\alpha})$ into the $\hat{x}$, $\hat{z}$ place. With this choice the vector $(|\vec{\beta}|, 0, 0)$ is left invariant by $T_{\vec{n}_2,\theta_2}$ and $T_{\vec{n}_1,\theta_1}(\vec{a})$ is transformed as in Eq. (4.151).

4.6.3 Optimal output purity

Here we study the output ℓ_p-norm for the channel $\mathcal{E}_{\mu,\lambda}$ [65]. For the case $\ell = 2$ we solve analytically the optimization problem showing the existence of a threshold

$$\mu_c \equiv \frac{1-\lambda^2}{2-\lambda^2} \;, \tag{4.157}$$

for the probability μ. In the presence of a strong correlated noise (i.e. $\mu \geqslant \mu_c$) we see that the maximum of $\|\mathcal{E}_{\mu,\lambda}(\psi)\|_2$ is obtained by the maximally entangled state $|\beta_0\rangle$. Vice-versa below threshold, i.e. $\mu < \mu_c$, we show that the entanglement of the input states which achieve the maximum values of $\|\mathcal{E}_{\mu,\lambda}(\psi)\|_2$ decreases with μ. These results will then be generalized in the case of arbitrary p by showing (numerically) that the same threshold and the same optimal input states of the case $p = 2$ hold also for all $p \neq 2$.

Before presenting these results however we give a simple argument to show that, for all p, the state $|\beta_0\rangle$ is the one whose output achieves the maximum values for ℓ_p if one restrict the focus on the set of maximally entangled input states. To verify this let us recall that any maximally entangled state $|\beta\rangle$ can be written as $|\beta\rangle = (U \otimes V)|\beta_0\rangle$ where U and V are unitary transformations. Form the covariance property of Ψ_λ it then follows that

$$\begin{aligned}
&\mathcal{E}_{\mu,\lambda}(|\beta\rangle\langle\beta|) = \mathcal{E}_{\mu,\lambda}\Big((U \otimes V)\,|\beta_0\rangle\langle\beta_0|\,(U^\dagger \otimes V^\dagger)\Big) \qquad (4.158)\\
&= (U \otimes V)\Big[(1-\mu)(\Psi_\lambda \otimes \Psi_\lambda)(|\beta_0\rangle\langle\beta_0|) + \mu\,\Big|\widehat{\beta}\Big\rangle\Big\langle\widehat{\beta}\Big|\,\Big](U^\dagger \otimes V^\dagger)\\
&= (U \otimes V)\Big[\tfrac{1}{4}(1-\mu)(1-\lambda^2)\mathbb{1} + (1-\mu)\lambda^2\,|\beta_0\rangle\langle\beta_0| + \mu\,\Big|\widehat{\beta}\Big\rangle\Big\langle\widehat{\beta}\Big|\,\Big](U^\dagger \otimes V^\dagger)
\end{aligned}$$

where $|\widehat{\beta}\rangle = (U^\dagger \otimes V^\dagger)|\beta_0\rangle$ is again maximally entangled. If $|\widehat{\beta}\rangle \neq |\beta_0\rangle$ the operator $\mathcal{E}_{\mu,\lambda}(|\beta\rangle\langle\beta|)$ has three distinct eigenvalues (one double degenerate). Vice-versa if $|\widehat{\beta}\rangle = |\beta_0\rangle$ the operator $\mathcal{E}_{\mu,\lambda}(|\beta\rangle\langle\beta|)$ has only two distinct eigenvalues (one three times degenerate). One can easily verify that in this case the resulting density matrix majorizes all the other operators of Eq. (4.158). The claim then follows by noticing that such an output configuration can be achievable for instance by choosing $U = V = \mathbb{1}$, i.e. by having $|\beta\rangle = |\beta_0\rangle$.

Optimizing $\|\mathcal{E}_{\mu,\lambda}(\psi)\|_2$

In the following we will determine the optimal input states $|\psi^{(2)}_{opt}\rangle$ which solve the maximization in Eq. (4.153) for the ℓ_p-norm of order $p = 2$. To do so we observe that, for any input state $|\psi\rangle$, the first contribution of (4.143) is proportional to

$$\begin{aligned}
\big(\Psi_\lambda \otimes \Psi_\lambda\big)(|\psi\rangle\langle\psi|) = \lambda^2|\psi\rangle\langle\psi| \;&+\; (1-\lambda)^2\mathbb{1}/4 \qquad (4.159)\\
&+\; (1-\lambda)\lambda(\mathbb{1}_1 \otimes \rho_2 + \rho_1 \otimes \mathbb{1}_2)/2\,,
\end{aligned}$$

where $\rho_1 \equiv \mathrm{Tr}_2[|\psi\rangle\langle\psi|]$ and $\rho_2 \equiv \mathrm{Tr}_1[|\psi\rangle\langle\psi|]$ are the reduced density matrices of $|\psi\rangle$ associated with the qubits 1 and 2, respectively (see Sec. 1.1). Replacing this in Eq. (4.143) the output $p = 2$ purity of $|\psi\rangle$ can be computed by exploiting the following identities:

$$\begin{aligned} \langle\psi|(\rho_1 \otimes \mathbb{1}_2)|\psi\rangle &= \mathrm{Tr}_1[\rho_1^2] \, , \\ \langle\psi|(\mathbb{1}_1 \otimes \rho_2)|\psi\rangle &= \mathrm{Tr}_2[\rho_2^2] \, , \\ \mathrm{Tr}_1[\rho_1^2] &= \mathrm{Tr}_2[\rho_2^2] \, , \\ \langle\beta_0|(\rho_1 \otimes \mathbb{1}_2)|\beta_0\rangle &= \langle\beta_0|(\mathbb{1}_1 \otimes \rho_2)|\beta_0\rangle = 1/2 \, . \end{aligned}$$

This yields

$$\|\mathcal{E}_{\mu,\lambda}(|\psi\rangle\langle\psi|)\|_2 = \sqrt{\mathrm{Tr}[\mathcal{E}_{\mu,\lambda}(|\psi\rangle\langle\psi|)^2]} = \sqrt{A\ |\langle\psi|\beta_0\rangle|^2 + B\ \mathrm{Tr}_1[\rho_1^2] + C} \quad (4.160)$$

with A, B and C being positive quantities defined as

$$\begin{aligned} A &= 2\mu(1-\mu)\lambda^2 \, , \\ B &= (1-\mu)^2(1-\lambda^2)\lambda^2 \, , \\ C &= \mu^2 + (1-\mu)^2[(\lambda^2 + (1-\lambda)^2/2)^2 + (1-\lambda)^3\lambda] + \mu(1-\mu)(1-\lambda^2)/2 \, . \end{aligned} \quad (4.161)$$

To maximize Eq. (4.160) one would be tempted to have both $|\langle\psi|\beta_0\rangle|^2$ and $\mathrm{Tr}_1[\rho_1]^2$ as bigger as possible. This is, however, is impossible since the first quantity measures how "close" $|\psi\rangle$ is to the maximally entangled state $|\beta_0\rangle$ which, in turns, has minimum purity for the reduced density matrix components. Consequently high values of $|\langle\psi|\beta_0\rangle|^2$ corresponds to low value of $\mathrm{Tr}_1[\rho_1]^2$ and vice-versa. To find the maximum of (4.160), it is thus useful write the input state as in Eq. (4.147). In this case we get $\rho_1 = (\mathbb{1}_1 + \vec{\omega}\cdot\vec{\sigma})/2$ and $\mathrm{Tr}_1[\rho_1^2] = (1 + |\vec{\omega}|^2)/2$, where $\vec{\omega}$ is a three dimensional real vector of components

$$\begin{aligned} \omega_1 &= 2\,\Re[a_0 a_2^* - a_1 a_3^*] \, , \\ \omega_2 &= 2\,\Im[a_0 a_3^* - a_1 a_2^*] \, , \\ \omega_3 &= 2\,\Re[a_0 a_1^* + a_2 a_3^*] \, . \end{aligned} \quad (4.162)$$

Observing that one has also $\langle\beta_0|\psi\rangle = a_0$ this allows us to write

$$\|\mathcal{E}_{\mu,\lambda}(|\psi\rangle\langle\psi|)\|_2 = \sqrt{A\ a_0^2 + B\ (|\vec{\omega}|^2 + 1)/2 + C} \, . \quad (4.163)$$

Thanks to the covariance properties analyzed in the previous section the maximization of this expression can be performed by focusing on the inputs of the form (4.152) which have $a_3 = 0$, $a_1 = \sqrt{1-a_0^2}\cos\varphi$, and $a_2 = \sqrt{1-a_0^2}\sin\varphi e^{i\phi}$. Under these conditions we have

$$|\vec{\omega}|^2 \;=\; 4(1-a_0^2)\left\{ a_0^2 + \sin^2\varphi\sin^2\phi\big[(1-2\,a_0^2) - (1-a_0^2)\,\sin^2\varphi\,\big]\right\},$$

which for any given a_0 can be maximized with respect to $\varphi \in [0,\pi]$ and $\phi \in [0, 2\pi)$.

For $a_0 > 1/\sqrt{2}$, the quantity $\big[(1-2\,a_0^2) - (1-a_0^2)\,\sin^2\varphi\big]$ is always negative. Therefore $|\vec{\omega}|^2$ is always smaller than $4(1-a_0^2)a_0^2$. By simply looking at Eq. (4.164) we see that such maximum is achievable by setting either $\varphi = 0$ and ϕ generic, or $\phi = 0$ and φ generic, yielding $|\vec{\omega}|^2_{\max} = 4a_0^2(1-a_0^2)$.

On the contrary, for $a_0 \leqslant 1/\sqrt{2}$ the maximum value of $|\vec{\omega}|^2$ can be determined by studying the function

$$F(x,y) \equiv (1-2a_0^2)yx - (1-a_0^2)yx^2\,, \tag{4.164}$$

for $x, y \in [0,1]$. A simple analysis reveals that it achieves its maximum for $x = \frac{1-2a_0^2}{2(1-a_0^2)}$ and $y = 1$. Therefore, if $a_0 < 1/\sqrt{2}$ the maximum is 1 and it is obtained for $\phi = \pi/2$ and $\sin^2\varphi = \frac{1-2a_0^2}{2(1-a_0^2)}$. Replacing this into Eq. (4.164) one can easily verify that this corresponds to have $|\vec{\omega}|^2_{\max} = 1$.

Replacing all these results in Eq. (4.163) gives

$$\max_{\varphi,\phi} \|\mathcal{E}_{\mu,\lambda}(|a_0,\varphi,\phi\rangle\langle a_0,\varphi,\phi|\|_2 = \sqrt{A\;a_0^2 + B\;[h(a_0^2)+1]/2 + C}\;, \tag{4.165}$$

where

$$h(x) \equiv \begin{cases} 1 & \text{for } x \in [0,1/2] \\ 4x(1-x) & \text{for } x \in]1/2, 1]. \end{cases} \tag{4.166}$$

Equation (4.165) can then easily optimized with respect to a_0. In particular one verifies that there are two independent regimes identified by the parameter

$$\frac{A}{2B} = \frac{\mu}{1-\mu}\frac{1}{1-\lambda^2} \quad\longrightarrow\quad \mu = \mu_c = \frac{1-\lambda^2}{2-\lambda^2}\,. \tag{4.167}$$

Indeed if $\mu > \mu_c$ (i.e. $A/(2B) > 1$) then Eq. (4.165) has maximum value $\sqrt{A + B/2 + C}$ which is achieved for $a_0 = 1$. On the contrary for $\mu \leqslant \mu_c$

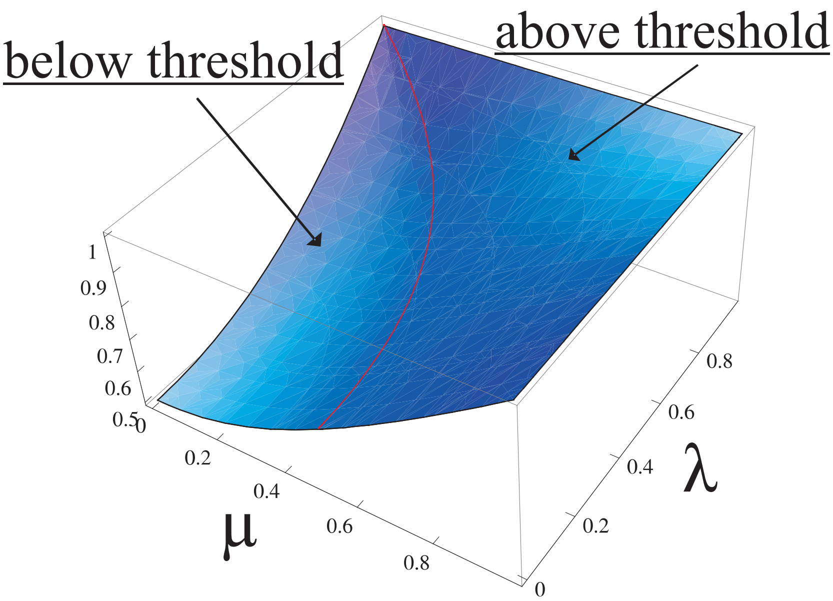

Figure 4.8: Plot of the ℓ_p-norm of order $p = 2$ for the channel $\mathcal{E}_{\mu,\lambda}$. The continuous curve represents the threshold condition (4.157). For $\mu = 1$ and $\lambda = 1$ the norm is maximal: in the former case the channel sends every input state into $|\beta_0\rangle$; in the latter case instead it transforms the inputs $|\psi\rangle$ into a mixture of $|\psi\rangle$ and $|\beta_0\rangle$ so that choosing $|\psi\rangle = |\beta_0\rangle$ one gets a pure output.

(i.e. $A/(2B) \leqslant 1$) the maximum is $\{\frac{B}{2}\left[1 + \left(1 + \frac{A}{2B}\right)^2\right] + C\}^{1/2}$, and it is achieved for

$$a_0 = \sqrt{\frac{1 + A/(2B)}{2}} = \sqrt{\frac{1-(1-\mu)\lambda^2}{2(1-\mu)(1-\lambda^2)}} \,. \tag{4.168}$$

Putting all these together we then arrive at the following result – see also Fig. 4.8:

$$\|\mathcal{E}_{\mu,\lambda}\|_2 = \begin{cases} \sqrt{A + B/2 + C} & \text{for } \mu > \mu_c \\ \\ \sqrt{\frac{B}{2}\left[1 + \left(1 + \frac{A}{2B}\right)^2\right] + C} & \text{for } \mu \leqslant \mu_c. \end{cases} \tag{4.169}$$

The corresponding optimal input states can similarly be determined. In particular, since above threshold (i.e. $\mu > \mu_c$) the optimization of Eq. (4.165)

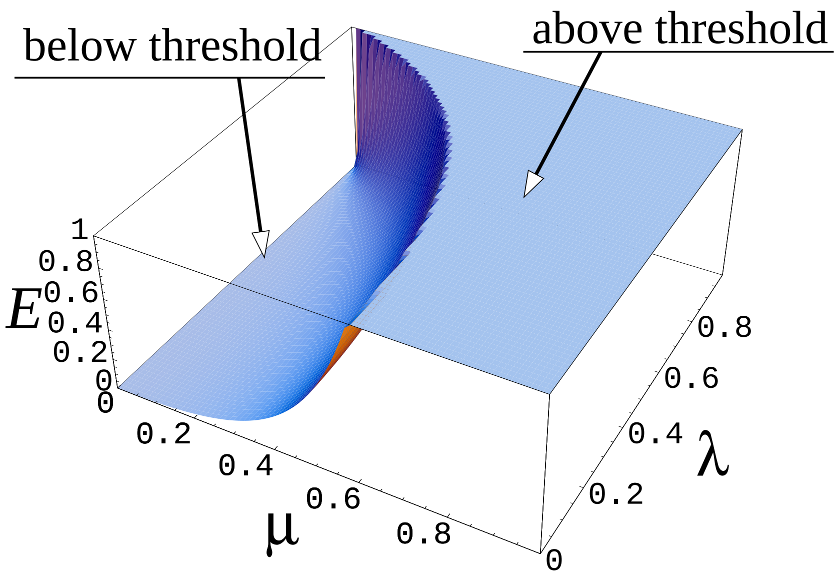

Figure 4.9: Entanglement (4.171) of the optimal input states $|\psi^{(2)}_{opt}\rangle$ which maximize the ℓ_2-norm for the map $\mathcal{E}_{\mu,\lambda}$. For values of μ above the threshold (4.157), $|\psi^{(2)}_{opt}\rangle$ is maximally entangled, while for μ below the threshold the entanglement of $|\psi^{(2)}_{opt}\rangle$ continuously decreases.

requires $a_0 = 1$, from Eq. (4.152) it follows that the optimal input state $|\psi^{(2)}_{opt}\rangle$ is unique and equal to the maximally entangled state $|\beta_0\rangle$. On the contrary below threshold (i.e. $\mu \leqslant \mu_c$) the optimal state $|\psi^{(2)}_{opt}\rangle$ can be determined by noticing that the quantity in the left hand side of Eq. (4.168) is always greater than $1/\sqrt{2}$. Consequently the optimal input state must have the following canonical form

$$|\psi^{(2)}_{opt}\rangle = a_0|\beta_0\rangle + \sqrt{1-a_0^2}\Big[\cos\varphi|\beta_3\rangle + \sin\varphi|\beta_1\rangle\Big] , \qquad (4.170)$$

with a_0 as in Eq. (4.168) and φ generic. These vectors become increasingly entangled as the correlation parameter μ increases from 0 to μ_c. To verify this for instance we can evaluate the linear entropy E of their reduced density matrix. Considering also the case for $\mu > \mu_c$ we get,

$$E = 2(1 - \mathrm{Tr}[\rho_1^2]) = \begin{cases} 1 & \text{for } \mu > \mu_c \\ \left[\frac{\mu}{(1-\mu)(1-\lambda^2)}\right]^2 & \text{for } \mu \leqslant \mu_c\ . \end{cases} \tag{4.171}$$

In Fig. 4.9 we report this quantity as a function of μ and λ. The peculiar behavior of the channel $\mathcal{E}_{\mu,\lambda}$ can be contrasted with that of the shifted depolarizing channel

$$\mathcal{E}(\rho) \equiv (1-\mu)\big[(1-\tau)\rho + \tau\, \mathbb{1}/d\big] + \mu|\phi\rangle\langle\phi|\ , \tag{4.172}$$

with $0 < \tau < 1$. It is always optimized by the input $|\phi\rangle\langle\phi|$ regardless of whether $|\phi\rangle$ is a product or maximally entangled [146]. Introducing a correlation into a product of depolarizing channels by shifting with a maximally entangled state yields quite different behavior than introducing it into a depolarizing channel in higher dimensions.

Optimal ℓ_p-norm for $p \neq 2$

In general, it is not easy to perform an exact analytical analysis of the output p-norms for $p \neq 2$. However, extensive numerical studies of optimization were carried out using the parametrization in Eq. (4.152) and the equivalent Rényi entropy [148], for over 2000 pairs of randomly chosen value of μ and λ. In all cases, we found that the input states which are optimal for $p = 2$ are also optimal for $p > 1$, i.e. $|\psi^{(p)}_{opt}\rangle = |\psi^{(2)}_{opt}\rangle$. In Figs. 4.10-4.11, we show typical numerical results of our findings by comparing the minimal Rényi output entropies for randomly chosen inputs with that of optimal inputs for $p = 2$, which lie on the bottom curve. In all cases the output Rényi entropy was larger than the expected minimum. Indeed, the minimal Rényi output entropy is monotonically related to the ℓ_p-norm of the channel through the identity,

$$S_p(\mathcal{E}_{\mu,\lambda}) \equiv \min_{\psi} \frac{\log \mathrm{Tr}[(\mathcal{E}_{\mu,\lambda}(|\psi\rangle\langle\psi|)^p]}{1-p} = \frac{\log \|\mathcal{E}_{\mu,\lambda}\|_p^p}{1-p}.$$

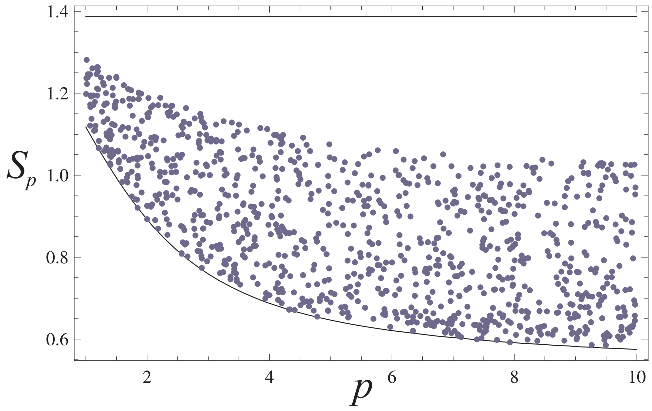

Figure 4.10: Scatter plots of the output Rényi entropy $S_p[\mathcal{E}_{\beta_0,\mu,\lambda}(|\psi\rangle\langle\psi|)]$ as a function of p for randomly chosen inputs compared to that for the conjectured optimal input, in the case of $\lambda = \frac{1}{2}, \mu = \frac{1}{4} < \mu_c = \frac{3}{7}$ below threshold. The horizontal line corresponds to the maximum possible Rényi entropy of $\log 4$.

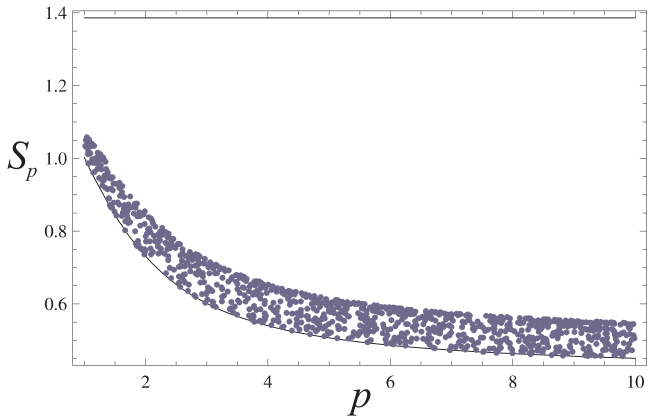

Figure 4.11: Scatter plots of the output Rényi entropy $S_p[\mathcal{E}_{\beta_0,\mu,\lambda}(|\psi\rangle\langle\psi|)]$ as a function of p for randomly chosen inputs compared to that for the conjectured optimal input, in the case of $\lambda = \frac{1}{3}, \mu = \frac{1}{2} > \mu_c = \frac{8}{17}$ above threshold. The horizontal line corresponds to the maximum possible Rényi entropy of $\log 4$.

4.6.4 Majorization

Finally, we conclude this section about memory qubit channels with some observations on majorization and trumping properties applied to the optimal input states above [65]. First to proceed, let us define the notion of majorization in a general context [149]. There are three alternative equivalent useful definitions of majorization. The first one is given in terms of a set of inequalities between partial sums of the two distributions. In other terms, given two normalized probability distributions, described by the vectors $\vec{x}$, $\vec{y} \in \mathbb{R}^{+n}$, the distribution $\vec{y}$ is said to majorize distribution $\vec{x}$, i.e. $\vec{x} \prec \vec{y}$, if and only if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \qquad k = 1, 2, \ldots, n-1 \quad , \tag{4.173}$$

where the components of the two probability vectors have been sorted in decreasing order. The second one is that $\vec{x} \prec \vec{y}$ if and only if there exist a set of permutation matrices P_k and probabilities $p_k \geq 0$, with $\sum_k p_k = 1$, such that

$$\vec{x} = \sum_k p_k P_k \vec{y} \, . \tag{4.174}$$

So, loosely speaking, it means that the probability distribution $\vec{x}$ is more disordered than probability distribution $\vec{y}$. Indeed, it defines a partial order in the space of probability distributions. The third equivalent definition of majorization is that $\vec{y}$ majorizes $\vec{x}$ if and only if there is a doubly stochastic matrix D such that

$$\vec{x} = D\vec{y} \, , \tag{4.175}$$

where a double stochastic matrix is defined as a matrix with nonnegative entries and in which each row and column adds up to unity. A powerful relation between majorization and any convex function f over the set of probability distributions states that

$$\vec{x} \prec \vec{y} \Rightarrow f(\vec{x}) \leq f(\vec{y}) \quad . \tag{4.176}$$

It implies that, for instance, if $\vec{x} \prec \vec{y}$, one has $H(\vec{x}) \geq H(\vec{y})$, where $H(\vec{t})$ is the Shannon entropy $H(\vec{t}) \equiv -\sum_{i=1}^{N} t_i \log_2 t_i$ of a probability distribution $\vec{t} \in \mathbb{R}^n$. Therefore, majorization is a stronger notion of order for probability distributions with respect to the one imposed by the entropy $H(\vec{t})$. In quantum information science, the notion of majorization is easily extended to quantum states considering that the spectrum of a density matrix represents a normalized probability distribution. In other words, for two density operators ρ and σ with spectrums $\vec{\rho}$ and $\vec{\sigma}$, one says that $\rho \prec \sigma$ if and only if $\vec{\rho} \prec \vec{\sigma}$; it implies that $S(\rho) > S(\sigma)$ with S being the von Neumann entropy.

After this brief introduction, since the family of states (4.170) optimizes the output purity for any value of p (i.e., minimizes the output von Neumann entropy), one could expect that the corresponding output family majorizes all other possible output states. Actually this is not true. As an illustration of this we report two counterexamples of output states that are not majorized by the output states corresponding to the family (4.170). As a first example consider the two-qubit channel $\mathcal{E}_{\mu,\lambda}$ with parameters $\lambda = 1/3$ and $\mu = 1/2$. In this case $\mu > \mu_c$ and therefore the optimal input state is $|\beta_0\rangle$, whose output eigenvalues take in this case the explicit values $(0, 667, 0.111, 0.111, 0.111)$. By numerical analysis it turns out that the input product state of the form (4.152) with $a_0 = 0$, $\varphi = \pi/4$ and $\phi = \pi/2$ yields an output state with eigenvalues $(0.611, 0.222, 0.111, 0.056)$ which trivially is not majorized by $(0, 667, 0.111, 0.111, 0.111)$. As a second example, in the regime $\mu \leq \mu_c$, consider $\lambda = 1/2$ and $\mu = 1/4$. In this case the eigenvalue of output state associated with the optimal input state (4.170) is given by the set of values $(0.596, 0.141, 0.141, 0.123)$. On the other hand, the same product state introduced above has an output state with eigenvalues $(0.422, 0.391, 0.141, 0.047)$ which, again, is not majorized by $(0.596, 0.141, 0.141, 0.123)$. Therefore we can conclude that there is no family of input states that gives output states majorizing all other output states of the qubit quantum channel with small correlations, analyzed in this section.

However, one can suppose that there exist a weak majorization between the optimal output states and the other output [119, 150], e.g. the so-called "trumping" relation [151, 152]. This concept, introduced in connection with the phenomenon known as entanglement catalysis [153], plays an important role in quantum information theory. If there is a vector $\vec{z}$ such that $\vec{x} \otimes \vec{z} \prec \vec{y} \otimes \vec{z}$, then one says that the vector $\vec{y}$ trumps $\vec{x}$ and writes $\vec{x} \prec_T \vec{y}$. Moreover, it follows that $\|\vec{x}\|_p \leq \|\vec{y}\|_p$ for all $p > 1$. Even if some analytical and numerical arguments seem to show that this conjecture holds, however, it is not easy in general to prove it between the optimal output states and the other outputs.

Conclusions and Outlook

Quantum communication is one of the most prominent applications of the new rapidly developing field of quantum information theory. It consists in the transmission of (classical or quantum) information between two communicating parties, encoding it in a quantum system (e.g., photons) and sending it through a quantum channel (e.g., an optical fiber). In this context, an important research area is represented by the analysis of quantum channels, described as open quantum systems, and then by the quantitative evaluations of the fundamental limits of their capacity of transmitting quantum information from one place to another in the space [1, 3].

The results in this thesis regard two specific classes of channels, namely the Bosonic Gaussian channels and the qubit channels. We introduce a new property of quantum channels (i.e., weak-degradability) by exploiting a more "physical" picture of the noisy evolution (i.e., interaction with a thermal-like environment) and by generalizing the degradability definition given by Devetak and Shor [38]. In particular, it implies that their quantum capacity Q is null (i.e. they cannot be used to transfer quantum information) for anti-degradable channels, while it is allows us to establish the additivity of the coherent information (and the single-letter formula for Q) for those weakly degradable maps which admit unitary representation with pure environment.

Regarding the Bosonic channels, investigated in Chapter 3, we prove that with the exception of the additive classical noise channel [56], all one-mode Gaussian maps are either weakly degradable or anti-degradable. First of all, we use the unitary equivalence of these maps with beam-splitter/amplifier channels [52] and, then, we provide a full weak-degradability classification of one-mode Gaussian channels [53] by exploiting the canonical form decomposition of Ref. [56]. Within this context we identify those channels which are anti-degradable. Besides, we explicitly compute the quantum channel capacity of degradable Gaussian maps. By exploiting composition rules of

Gaussian maps and the fact that anti-degradable channels cannot be used to transfer quantum information, we are able to strengthen the bound for one-mode Gaussian channels which have nonvanishing quantum capacity [53].

Therefore, we have presented a complete analysis of generic multi-mode Bosonic Gaussian channels by proving a unitary dilation theorem and by finding their canonical form [54]. This is a simple form that can be achieved for any Gaussian quantum channel, as a convenient starting point for various considerations. For instance, it allows us to simplify the analysis of their weak-degradability properties. Minimal output entropies, or quantum and classical information capacities and other difficult questions might be tackled using the canonical form shown in this thesis. Moreover, we characterize the minimal noise channel, involving only true quantum noise, and we show a useful decomposition in terms of it and of the classical noise channel. Then, we show an interesting characterization of the minimal required number of environmental modes to be involved in the unitary dilation describing the multi-mode channel [62]. Finally, we investigate in detail the particular case of the two-mode scenario [54] that is relevant since any n-mode channel can always be reduced to single-mode and two-mode parts [57]. In this case, apart from the simple situation of a noisy system-environment interaction which does not couple the two Bosonic modes, we have found the (maybe) surprising fact that increasing the level of the environmental noise the coherence is progressively destroyed but it becomes easier to recover the environment (system) output from the system (environment) output after the noisy evolution, i.e. to recover the weak-degradability (anti-degradability) property. These results could play a basic role in characterizing the efficiency of continuous-variables quantum information processing, quantum communication and quantum key distribution protocols.

Regarding the qubit channels, examined in Chapter 4, we introduce a characteristic function formalism in terms of generalized displacement operators and Grassmann variables, along the same lines followed for Bosonic Gaussian channels. We then present a Green function representation of the quantum evolution that allows us to define the set of qubit Gaussian maps [44]. In this context, we find that all Gaussian channels are qubit-qubit, i.e. they can always be described in terms of a unitary interaction of a qubit system with a single (not necessarily pure) qubit environment. Similarly, it is known that in the Bosonic case (almost) all the one-mode Bosonic Gaussian maps can be described in terms of a single mode environment. This formalism turns out to be elegant and powerful and, in particular, it can be used to study the weak-degradability properties of qubit-qubit maps, for both pure and mixed qubit environments, in terms of Green functions [44]. On one hand, in the case of pure environment, the qubit-qubit maps are either degradable (i.e., single-letter formula for Q) or anti-degradable (i.e., $Q = 0$). Besides, the complementary maps are still qubit-qubit channels and so Gaussian. It is interesting to note that an equivalent property holds for one-mode Bosonic Gaussian channels. On the other hand, in the case of mixed environment, we show that the qubit-qubit maps are either weakly degradable or they are not able to transfer quantum information (i.e., Q=0). However, in this case the weakly complementary maps do not belong to the set of qubit-qubit channels and are not Gaussian. It is important to stress that this Green function formalism shows clearly that the qubit Gaussian maps share analogous properties with their continuous variable counterpart, i.e. the Bosonic Gaussian channels.

Let us remark that the characteristic function approach, introduced in this thesis, for qubit systems, can be generalized to d-level quantum systems (qudit) in terms of generalized Grassmann variables [63, 64]. In other words, one can think to describe d-level quantum systems using para-Grassmann variables and generalized displacement operators and coherent states. In this way, one can try to define a characteristic function, a Green representation of qudit channels and to introduce the set of qudit Gaussian maps. One could study the weak-degradability property for these channels and understand how general the results in this thesis are. For example, one could answer to the interesting questions if all qudit Gaussian channels are describable in terms of unitary interaction between two qudits (like for Bosonic and qubit channels) and if Gaussianity implies weak-degradability also for these maps.

Finally, we have introduced a class of memory (i.e., correlated noise over many channel uses) two-qubit quantum channels with a peculiar property [65]. Namely, for certain values of the channel parameters, above a ‘critical’ threshold, the optimal input states (defined as those inputs which maximize the channel output *purities* of order p, measured using the p-norm) are always maximally entangled. Vice-versa, in the remaining parameter region, the optimal inputs do have decreasing entanglement properties (i.e., there is a kind of ‘phase transition’ behavior). Such optimal input states have been analytically derived in the case of $p = 2$. For $p \neq 2$ we have, instead, performed a numerical analysis which confirms the same results. They are interesting in order to characterize the noise properties of memory channels and could be generalized to continuous variable systems.

Recently, there is a continuously growing interest in studying the behavior of memory quantum channels and, even, in relating them to the physics of critical many-body quantum systems [139]. For instance, there it has been shown that the non-analytic behavior of correlated noise channels can be treated as an indicator of the presence of critical points in many-body systems, playing the role of environment in the correlated noise evolution. It is developing a very promising connection between quantum communication and quantum many-body theory. Besides, it would enable us to use techniques from many-body physics to better characterize the memory effects in such channels and vice versa. Interestingly, the unitary dilation theorem for multi-mode Bosonic Gaussian channels, proved in this thesis, may pave the way to a full analysis of Bosonic Gaussian memory channels and, for instance, to an interesting connection of their features (e.g., channel capacity) with the appearance of long-range correlations in the ground state of a quantum many-body system undergoing a quantum phase transition at zero temperature.

Appendix A

Normal mode decomposition

In Chapter 3 the Williamson theorem has been referred to in relation to the ordinary normal mode decomposition, as starting point for the generalization for non-symmetric or locality constrained situations and, particularly, for multi-mode Bosonic Gaussian channels. Here, first, we show a simple proof of Williamson's theorem presented in Ref. [154], and, then, we describe the generalized normal mode decomposition investigated in Ref. [57].

A.1 Williamson theorem

Consider n quantum mechanical oscillators, that are characterized by a set of momentum and position operators $(P_1, \ldots, P_n, Q_1, \ldots, Q_n) =: R$ which obey the canonical commutation relations $[R_k, R_l] = i\sigma_{kl}$, where

$$\sigma = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{A.1}$$

is the symplectic matrix. The symplectic group $Sp(2n, \mathcal{R})$ is formed by all real matrices S satisfying the condition

$$S\sigma S^T = \sigma \ . \tag{A.2}$$

This condition allows to preserve the commutation relations in a canonical/symplectic transformation $R_k \mapsto \sum_l S_{kl} R_k$. It also implies that, for any symplectic matrix S, one has $\mathrm{Det}[S] = 1$.

The ordinary normal mode decomposition [115], i.e. the Williamson theorem (proved below), is basically based on the fact that for any positive definite matrix $X \in \mathbb{R}^{2n\times 2n}$, there is an $S \in Sp(2n, \mathcal{R})$ such that

$$SXS^T = \mathrm{diag}(\nu_1, \ldots, \nu_n, \nu_1, \ldots, \nu_n). \tag{A.3}$$

For instance, if X represents a Hamiltonian $H = \sum_{kl} X_{kl} R_k R_l$, the ν_k are the normal mode frequencies. If, instead, $X_{kl} = \langle\{R_k - \langle R_k\rangle, R_l - \langle R_l\rangle\}_+\rangle$ is a covariance matrix, then $(\nu_k - 1)/2$ is the mean occupation number (phonons/photons) in the k'th normal mode.

Theorem 6 (Williamson theorem) *Let X be a $2n$-dimensional real symmetric positive definite matrix. Then there exists an $S \in Sp(2n, \mathcal{R})$ such that*

$$\begin{aligned} SXS^T &= D^2 > 0\,, \\ D^2 &= \mathrm{diag}(\nu_1, \nu_2, \cdots, \nu_n, \nu_1, \nu_2, \cdots, \nu_n). \end{aligned} \tag{A.4}$$

Proof: The most general $S \in GL(2n, \mathcal{R})$ which solves $SXS^T = D^2$ is $S^T = X^{-1/2}RD$, where $R \in O(2n)$. Now, we show that a X-dependent choice of D and R can be made in a such way that the product $X^{-1/2}RD$ is an element of $Sp(2n, \mathcal{R})$, even if none of the factors D, R or $X^{-1/2}$ is an element of $Sp(2n, \mathcal{R})$. Indeed, since $\sigma^T = -\sigma$, one has that $\mathcal{M} = X^{-1/2}\sigma X^{-1/2}$ is antisymmetric. Hence, there exists [119] an $R \in SO(2n)$ such that

$$R^T X^{-1/2}\sigma X^{-1/2} R = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}, \quad \Omega = \text{diagonal} > 0\,. \tag{A.5}$$

Now, we define a diagonal positive definite matrix

$$D = \begin{pmatrix} \Omega^{-1/2} & 0 \\ 0 & \Omega^{-1/2} \end{pmatrix}. \tag{A.6}$$

Hence,

$$DR^T X^{-1/2}\sigma X^{-1/2} RD = \sigma\,, \tag{A.7}$$

and S is given by $S^T = X^{-1/2}RD$. Indeed, S satisfy the following properties:

$$\begin{aligned} S\sigma S^T &= \sigma\,, \\ SXS^T &= D^2 = \text{diagonal}. \end{aligned} \tag{A.8}$$

The first equation says that $S \in Sp(2n, \mathcal{R})$ and the second one says that X is diagonalized by congruence through S. This completes the proof of the Williamson theorem. ■

A.2 Generalized canonical form

In Ref. [57] a generalized canonical form is proved. At a mathematical level, this result is a canonical matrix form with respect to real symplectic equivalence transformations, i.e., a symplectic analogue of the singular value decomposition. The statement is that, for any nonsingular matrix $X \in \mathbb{R}^{2n\times 2n}$ there exist real symplectic transformations S_1, S_2 such that

$$S_1 X S_2 = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & J \end{pmatrix} \tag{A.9}$$

where

$$J = \left(\begin{array}{ccc|ccc} \bar{J}_{n_1}(\lambda_1) & & 0 & & & \\ & \ddots & & & \mathbf{0} & \\ 0 & & \bar{J}_{n_p}(\lambda_p) & & & \\ \hline & & & J_{n_{p+1}}(\lambda_{p+1}) & & 0 \\ & \mathbf{0} & & & \ddots & \\ & & & 0 & & J_{n_k}(\lambda_k) \end{array}\right),$$

is a $n \times n$ block-diagonal matrix in the real Jordan canonical form [119] and λ_i are the eigenvalues of $X\sigma X^T\sigma^T$. In the most general case, we can have not necessarily distinct p complex conjugate pairs of the eigenvalues $\{\lambda_{1\ldots p}, \bar{\lambda}_{1\ldots p}\}$ and $k-p$ real eigenvalues $\lambda_{p+1\ldots k}$. Note that $2(n_1+\ldots+n_p)+n_{p+1}+\ldots+n_k = n$ and the factor 2 counts the multiplicity of the complex conjugate pairs of eigenvalues. Moreover, each block is a real Jordan block corresponding to either the complex conjugate pair of eigenvalues ($\bar{J}_{n_i}(\lambda_i)$, $i = 1, \ldots p$) or to one of its real eigenvalues ($J_{n_j}(\lambda_j)$, $j = p+1, \ldots k$) [119].

In the former case $\bar{J}_{n_i}(\lambda_i) \in \mathbb{R}^{2n_i\times 2n_i}$ has the form

$$\bar{J}_{n_i}(\lambda_i) = \begin{pmatrix} \Lambda & \mathbb{1}_2 & & 0 \\ & \Lambda & \ddots & \\ & & \ddots & \mathbb{1}_2 \\ 0 & & & \Lambda \end{pmatrix}, \quad \Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \lambda_i = a + ib,$$

and it is called non-defective if $\bar{J}_{n_i}(\lambda_i) = \Lambda$.

In the case of real eigenvalues $J_{n_j}(\lambda_j) \in \mathbb{R}^{n_j \times n_j}$ has, instead, the form

$$J_{n_j}(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

Let us show some basic properties of the matrix J [119]:

1) $k + p$ is the number of linearly independent eigenvectors of J.

2) J is diagonalizable if and only if $k + p = n$.

3) The number of Jordan blocks corresponding to a given eigenvalue is the geometric multiplicity of the eigenvalue, i.e. the dimension of the associated eigenspace. Moreover, for each complex eigenvalue, there is another eigenspace associated to the complex conjugate eigenvalue and with the same dimension.

4) The sum of the orders of all the Jordan blocks corresponding to a given eigenvalue is the algebraic multiplicity of the eigenvalue, i.e. the multiplicity in the characteristic polynomial. For the complex eigenvalues, this number includes also the algebraic multiplicity of the complex conjugate. They must occur in conjugate pair with the same (algebraic and geometric) multiplicity because the matrix is real.

Note that an entirely diagonal normal form, analogous to the usual normal mode decomposition, cannot exist in general because real diagonal matrices have only real invariants. Indeed, the remaining 2×2 blocks are not diagonalizable because they correspond to complex λ's. Let us point out that for positive definite matrices X the canonical form in Eq. (A.9) and the usual normal mode decomposition in Eq. (A.3) coincide up to a simple squeezing transformation and the invariants are related via $\lambda_k = \nu_k^2$.

Appendix B

Grassmann algebra and Fermionic coherent states

Here we review some properties of Grassmann calculus [121], used in the Chapter 4 to describe qubit quantum channels. A system of Fermions can be described by the creation $a_n^\dagger$ and annihilation a_m operators which satisfy the anti-commutations relations

$$
\begin{aligned}
\{a_n, a_m^\dagger\} &= \delta_{nm} , && \text{(B.1)} \\
\{a_n, a_m\} &= 0 , && \text{(B.2)} \\
\{a_n^\dagger, a_m^\dagger\} &= 0 , && \text{(B.3)} \\
a_n|0\rangle &= 0 , && \text{(B.4)}
\end{aligned}
$$

in which $|0\rangle$ is the vacuum state.

Now we look at the problem to find the Fermionic coherent states, as eigenstates of the annihilation operator, like in the Bosonic case (see Sec. 3.1). Actually, for Fermions the vacuum state is the only physically realizable eigenstate of the annihilation operator. However, it is possible to define such eigenstates in a formal way and to use the obtained coherent states in a similar fashion and with similar properties like for Bosonic coherent states. Moreover, differently from the Bosonic case in which the eigenvalues are complex numbers, here the eigenvalues of the Fermionic annihilation operator must be 'strange' anti-commuting numbers, since Fermionic variables anti-commute, as noted by Schwinger [126]. Such numbers are called Grassmann variables and they generate an algebra, called Grassmann algebra [122, 123, 124, 125].

A Grassmann variable ξ spans over a set of objects (the Grassmann numbers) ξ_1, ξ_2, $\cdots$, which anti-commute. Given any ξ_i and ξ_j elements of the set, they satisfy the relation

$$\xi_i \xi_j = -\xi_j \xi_i \ , \tag{B.5}$$

while obeying ordinary commutation relations with respect to the multiplication by a complex number. In particular Eq. (B.5) implies that a Grassmann variable is 2-nilpotent, i.e. $\xi^2 = 0$ (note that 0 is trivially included in the Grassmann variable set). At a mathematical level, the above conditions can be rigorously formalized by saying that Grassmann numbers are the generators of an algebra over the complex field which obey anti-commutation relations.

Complex conjugation of ξ can be defined by introducing an extra Grassmann variable ξ^* whose elements ξ_1^*, ξ_2^*, $\cdots$, obey the same relation (B.5) and anti-commute with all the ξ_is, i.e.

$$\xi_i^* \xi_j^* = -\xi_j^* \xi_i^* \ , \tag{B.6}$$

$$\xi_i^* \xi_j = -\xi_j \xi_i^* \ . \tag{B.7}$$

To identify ξ_i^* with the complex conjugate of ξ_i we finally require the relations

$$\begin{aligned} (\xi_i^*)^* &= \xi_i \\ (\xi_i x)^* &= x^* \xi_i^* \ , \end{aligned} \tag{B.8}$$

to be satisfied for any x complex number or product of the $\xi_1, \xi_2, \cdots$ and $\xi_1^*, \xi_2^*, \cdots$. From now on, we will use lower-case Greek letters to denote Grassmann variables. One assumes that Grassmann variables anti-commute with Fermionic operators

$$\{\xi_i, a_j\} = 0 \ , \tag{B.9}$$

and commute with Bosonic operators. Moreover, one makes the arbitrary choice that hermitian conjugation reverses the order of all Fermionic quantities, both the operators and the Grassmann numbers; for instance,

$$\left(a_1 \beta_2 a_3^\dagger \xi_4^* \right)^\dagger = \xi_4 a_3 \beta_2^* a_1^\dagger. \tag{B.10}$$

In the following we show some rules of the Grassmann algebra and then we present a formalism, introduced in Ref. [121], similar to what is well known in quantum optics.

Let us recall that the Bosonic coherent states, i.e. eigenstates of the Bosonic annihilation operator, contain an intrinsically indefinite number of quanta but, notwithstanding, they can be used as a basis for describing all the electromagnetic field states. In other words, pure coherent states are not physically attainable in Bosonic systems with fixed numbers of particles, but they are useful to describe Boson fields in terms of suitably weighted superpositions and mixtures of coherent states. The weight functions associated with these combinations may be treated as quasi-probability densities in the spaces of coherent-state amplitudes. Particularly, the Wigner function [58, 155] and its Fourier transform, i.e. the characteristic function, play an important role in representing the quantum states. So we will define similar quasi-probability distributions for Fermions by using the Grassmann algebra and some results in Ref. [121].

B.1 Functions, derivatives and integrals

First of all, we define the functions of Grassmann numbers. Since the square of any Grassmann variable vanishes, the most general function $f(\xi)$ of a single anti-commuting variable ξ is linear in ξ and so can be written as follows

$$f(\xi) = u + \xi t. \tag{B.11}$$

Now we can define the derivative of a Grassmann function. Let us point out that, since the anti-commutation relations between the Grassmann variables, one can define a left derivative and a right derivative of a Grassmann function. For instance, the left derivative of the function $f(\xi)$ in Eq. (B.11) with respect to the Grassmann variable ξ is

$$\frac{df(\xi)}{d\xi} = t. \tag{B.12}$$

Moreover, if the variable t is anti-commuting, then we may also write the function $f(\xi)$ as

$$f(\xi) = u - t\xi. \tag{B.13}$$

In this last case, the left derivative is obtained following this procedure: we first move ξ past t, picking up a minus sign, and then we obtain the form (B.11) and its derivative gives the result (B.12). In a similar way we can define the right derivative, that in this case is $-t$.

Integration over ξ and ξ^* can be defined by introducing the "differential" $d\xi$ and $d\xi^*$. These are assumed to obey the same anti-commutation relations obeyed by the variables ξ and ξ^*, including Eq. (B.5) and Eqs. (4.2) and (4.3) of Chapter 4. The integrals are then defined according to the Berezin rules

$$\int d\xi_n = \int d\xi_n^* = 0 \,, \tag{B.14}$$

$$\int d\xi_n \, \xi_m = \delta_{nm} \,, \tag{B.15}$$

$$\int d\xi_n^* \, \xi_m^* = \delta_{nm}. \tag{B.16}$$

This integration due to Berezin [122, 123, 124, 125] is exactly equivalent to left differentiation. Indeed,

$$\int d\xi f(\xi) = \int d\xi (u + \xi t) = t = \frac{df(\xi)}{d\xi}. \tag{B.17}$$

We shall typically be concerned with pairs of anti-commuting variables ξ_i and ξ_i^*, and for such pairs joint integration with respect to ξ and ξ^* is finally defined by identifying the double differential $d^2\xi$ as follows,

$$d^2\xi \equiv d\xi^* \, d\xi = -d\xi \, d\xi^* \,. \tag{B.18}$$

In this context one can identify an analogous of the Dirac delta function $\delta^{(2)}(\mu - \nu)$ in the complex plane. Such Grassmann delta is defined as

$$\delta^{(2)}(\xi - \zeta) \equiv \int d^2\kappa \, \exp\left[\kappa \, (\xi^* - \zeta^*) - (\xi - \zeta) \, \kappa^*\right] = (\xi - \zeta) \, (\xi^* - \zeta^*) \,, \tag{B.19}$$

with ξ, ζ and κ Grassmann variables. Indeed, from Eq. (B.16) and from Eq. (B.11) we have

$$\int d^2\xi \, \delta^{(2)}(\xi - \zeta) \, f(\xi, \xi^*) = f(\zeta, \zeta^*) \,, \tag{B.20}$$

for all $f(\xi, \xi^*)$. Notice that the delta function (B.19) commutes with any Grassmann numbers and satisfies the relation $\delta^{(2)}(\xi - \zeta) = \delta^{(2)}(\zeta - \xi) = -\delta^{(2)}(\xi^* - \zeta^*)$.

A useful property is the following. Given the function $f(\xi,\xi^*)$ one can define its even and odd parts, i.e.

$$f_\pm(\xi,\xi^*) \equiv \frac{f(\xi,\xi^*) \pm f(-\xi,-\xi^*)}{2} \ . \tag{B.21}$$

According to Eq. (B.11) they are of the form $f_+(\xi,\xi^*) = A + C\xi^*\xi$ and $f_-(\xi,\xi^*) = B_1\ \xi + B_2\ \xi^*$, respectively. Now given $g(\xi,\xi^*)$ another function we can write

$$\int d^2\xi\ f_\pm(\xi,\xi^*) g_\mp(\xi,\xi^*) = 0$$

and thus

$$\int d^2\xi\ f(\xi,\xi^*) g(\xi,\xi^*) = \int d^2\xi\ f_+(\xi,\xi^*) g_+(\xi,\xi^*) + \int d^2\xi\ f_-(\xi,\xi^*) g_-(\xi,\xi^*) \ .$$

Therefore, it is always possible to write down a generic function of Grassmann variables as the sum of its even part and its odd one, that is

$$f(\xi) = f_+(\xi) + f_-(\xi) \ . \tag{B.22}$$

Note that the even part of a Grassmann function commutes with Grassmann variables while the odd one anti-commutes.

Suppose, for simplicity, to have a single Fermionic mode and to consider the quantum states $|0\rangle$ and $|1\rangle$. We assume that

$$\langle 0|\xi|0\rangle = \xi \ , \tag{B.23}$$
$$\langle 1|\xi|1\rangle = -\xi \ , \tag{B.24}$$
$$\langle 0|\xi|1\rangle = 0 \ , \tag{B.25}$$
$$\langle 1|\xi|0\rangle = 0 \ . \tag{B.26}$$

In this way, one has consistently

$$\langle 1|\xi a^\dagger|0\rangle = \langle 1| - a^\dagger\xi|0\rangle = \langle 1|\xi|1\rangle = -\langle 0|\xi|0\rangle = -\xi \ . \tag{B.27}$$

So, it is easy to prove that

$$\mathrm{Tr}\left[f_-(\xi)\right] = 0 \ , \tag{B.28}$$
$$\mathrm{Tr}\left[f_+(\xi)\right] = 2\ f_+(\xi) \ , \tag{B.29}$$
$$\mathrm{Tr}\left[f_-(\xi)A\right] = f_-(\xi)\mathrm{Tr}\left[\sigma_z A\right] \ , \tag{B.30}$$
$$\mathrm{Tr}\left[f_+(\xi)A\right] = f_+(\xi)\mathrm{Tr}\left[A\right] \ , \tag{B.31}$$

where A is a generic operator and σ_z being the third Pauli matrix, i.e.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.32}$$

in the *computational* basis $|0\rangle$, $|1\rangle$ (eigenstates of σ_z). In general, we can write

$$\mathrm{Tr}\left[f(\xi)A\right] = \mathrm{Tr}\left\{\left[f_+(\xi) + f_-(\xi)\right]A\right\} = f_+(\xi)\mathrm{Tr}\left[A\right] + f_-(\xi)\mathrm{Tr}\left[\sigma_z A\right]. \tag{B.33}$$

Finally, note that the exponentials of Grassmann variables become

$$\begin{aligned} \exp(B_1\xi + B_2\xi^* + C\xi^*\xi) &\equiv \sum_{n=0}^{\infty} \frac{(B_1\xi + B_2\xi^* + C\xi^*\xi)^n}{n!} \\ &= 1 + B_1\xi + B_2\xi^* + C\xi^*\xi + B_1\xi B_2\xi^*/2 + B_2\xi^* B_1\xi/2 \,. \end{aligned} \tag{B.34}$$

This expression can be used to verify that (apart from a global multiplicative term) any function (B.11) can be written as an exponential.

B.2 Displacement and coherent states

In analogy to the Bosonic formalism in Sec. 3.1, let us define (see Ref. [121]) the unitary displacement operator $D(\boldsymbol{\xi})$ as the exponential

$$D(\boldsymbol{\xi}) = \exp\left(\sum_i \left(a_i^\dagger \xi_i - \xi_i^* a_i\right)\right). \tag{B.35}$$

where $\boldsymbol{\xi} = \{\xi_i\}$ is a generic set of Grassmann variables.

We may rewrite the displacement operator as the product

$$\begin{aligned} D(\boldsymbol{\xi}) &= \prod_i \exp\left(a_i^\dagger \xi_i - \xi_i^* a_i\right) && \text{(B.36)} \\ &= \prod_i \left[1 + a_i^\dagger \xi_i - \xi_i^* a_i + \left(a_i^\dagger a_i - \frac{1}{2}\right)\xi_i^*\xi_i\right], && \text{(B.37)} \end{aligned}$$

where we use the useful property of Grassmann numbers, such that, if they multiply Fermionic annihilation or creation operators, their anti-commutativity cancels that of the operators; for instance the operators $a_i^\dagger \xi_i$ and $\xi_j^* a_j$ simply commute for $i \neq j$. Using the fact that the annihilation operator a_n

commutes with all the operators $a_i^\dagger \xi_i$ and $\xi_j^* a_j$ when $n \neq i$, the displaced annihilation operator can be calculated by ignoring all modes but the nth, i.e.

$$\begin{aligned}
D^\dagger(\boldsymbol{\xi}) a_n D(\boldsymbol{\xi}) &= \prod_i \exp\left(\xi_i^* a_i - a_i^\dagger \xi_i\right) a_n \prod_j \exp\left(a_j^\dagger \xi_j - \xi_j^* a_j\right) \\
&= \exp\left(\xi_n^* a_n - a_n^\dagger \xi_n\right) a_n \exp\left(a_n^\dagger \xi_n - \xi_n^* a_n\right) \\
&= \left(1 - a_n^\dagger \xi_n - \frac{1}{2}\xi_n^* a_n a_n^\dagger \xi_n\right) a_n \left(1 + a_n^\dagger \xi_n - \frac{1}{2} a_n^\dagger \xi_n \xi_n^* a_n\right) \\
&= \left(1 - a_n^\dagger \xi_n - \frac{1}{2}\xi_n^* \xi_n\right) a_n \left(1 + a_n^\dagger \xi_n + \frac{1}{2}\xi_n^* \xi_n\right) \\
&= a_n - a_n^\dagger \xi_n a_n + a_n a_n^\dagger \xi_n = a_n + \xi_n. \qquad \text{(B.38)}
\end{aligned}$$

Similarly

$$D^\dagger(\boldsymbol{\xi}) a_n^\dagger D(\boldsymbol{\xi}) = a_n^\dagger + \xi_n^*. \qquad \text{(B.39)}$$

For any set $\boldsymbol{\xi} = \{\xi_i\}$ of Grassmann numbers, we so naturally define [121] the normalized coherent state $|\boldsymbol{\xi}\rangle$ as the displaced vacuum state, i.e.

$$|\boldsymbol{\xi}\rangle = D(\boldsymbol{\xi})|0\rangle. \qquad \text{(B.40)}$$

By using the displacement relation (B.38), the coherent state is verified to be an eigenstate of every annihilation operator a_n, i.e.

$$\begin{aligned}
a_n|\boldsymbol{\xi}\rangle &= a_n D(\boldsymbol{\xi})|0\rangle = D(\boldsymbol{\xi})\, D^\dagger(\boldsymbol{\xi}) a_n D(\boldsymbol{\xi})|0\rangle = D(\boldsymbol{\xi})\, (a_n + \xi_n)\,|0\rangle \\
&= D(\boldsymbol{\xi})\, \xi_n |0\rangle = \xi_n\, D(\boldsymbol{\xi})|0\rangle = \xi_n |\boldsymbol{\xi}\rangle. \qquad \text{(B.41)}
\end{aligned}$$

Finally, the coherent state has the following expression

$$\begin{aligned}
|\boldsymbol{\xi}\rangle &= D(\boldsymbol{\xi})|0\rangle = \exp\left(\sum_i \left(a_i^\dagger \xi_i - \frac{1}{2}\xi_i^* \xi_i\right)\right)|0\rangle \\
&= \prod_i \left[1 + a_i^\dagger \xi_i - \xi_i^* a_i + \left(a_i^\dagger a_i - \frac{1}{2}\right)\xi_i^* \xi_i\right]|0\rangle \\
&= \prod_i \left(1 + a_i^\dagger \xi_i - \frac{1}{2}\xi_i^* \xi_i\right)|0\rangle\,, \qquad \text{(B.42)}
\end{aligned}$$

where the product formula in Eq. (B.37) for the displacement operator has been used. This formula takes a form closely analogous to the one for Bosonic coherent states. Let us stress that the only difference is that, in this formula for Fermions, the creation operator $a_i^\dagger$ stands to the *left* of the Grassmann number ξ_i. Besides, one can define the adjoint of the coherent state $|\boldsymbol{\xi}\rangle$ as

$$\langle\boldsymbol{\xi}| = \langle 0|D^\dagger(\boldsymbol{\xi}) = \langle 0|\exp\left(\sum_i \left(\xi_i^* a_i - \frac{1}{2}\xi_i^*\xi_i\right)\right), \tag{B.43}$$

obeying the relation

$$\langle\boldsymbol{\xi}|a_n^\dagger = \langle\boldsymbol{\xi}|\xi_n^*. \tag{B.44}$$

Therefore, the inner product of two coherent states is the following

$$\langle\boldsymbol{\xi}|\boldsymbol{\beta}\rangle = \exp\left(\sum_i \left(\xi_i^*\beta_i - \frac{1}{2}(\xi_i^*\xi_i + \beta_i^*\beta_i)\right)\right), \tag{B.45}$$

hence

$$\langle\boldsymbol{\beta}|\boldsymbol{\xi}\rangle\,\langle\boldsymbol{\xi}|\boldsymbol{\beta}\rangle = \exp\left[-\sum_i(\beta_i^* - \xi_i^*)(\beta_i - \xi_i)\right] = \prod_i\left[1 - (\beta_i^* - \xi_i^*)(\beta_i - \xi_i)\right].$$

For a single Fermionic mode, the identity operator $\mathbb{1}$ and the traceless operators $a, a^\dagger$, and $\frac{1}{2} - a^\dagger a$ form a complete set of operators. By using the expression (B.37) and the Grassmann calculus, each of these operators can be written as an integral over the displacement operators [121]

$$\begin{aligned}
\mathbb{1} &= \int d^2\xi\,\xi\xi^*\,D(\xi)\,, && \text{(B.46)}\\
a &= \int d^2\xi\,(-\xi)\,D(\xi)\,, && \text{(B.47)}\\
a^\dagger &= \int d^2\xi\,\xi^*\,D(\xi)\,, && \text{(B.48)}\\
\frac{1}{2} - a^\dagger a &= \int d^2\xi\,D(\xi)\,. && \text{(B.49)}
\end{aligned}$$

It turns out that the displacement operators form a complete set of operators for that mode. Actually, it is over-complete and this integral decomposition can be easily generalized to the multi-mode case.

Finally, let us point out that some operators can be written as sums of products of even or odd numbers of creation and annihilation operators. In the former case, they are called even, while in the latter they are said to be odd. For instance, the number operator $a^\dagger a$ is even, while the creation and annihilation operators, $a^\dagger$ and a, are odd. Even if most operators are neither even nor odd, the operators of physical interest are either even or odd. Particularly, one may define the even part of the displacement operator of a single Fermionic mode, $D_+(\xi)$, as

$$D_+(\xi) = 1 + \left(a^\dagger a - \frac{1}{2} \right) \xi^* \xi \, , \tag{B.50}$$

and $D_+(\xi) = D_+(-\xi)$. In the same way, the odd part of the displacement operator, $D_-(\xi)$, is defined as

$$D_-(\xi) = a^\dagger \xi - \xi^* a \tag{B.51}$$

and $D_-(\xi) = -D_-(-\xi)$.

Appendix C

Fermionic channels

Here we present a brief excursus about some Fermionic channels. The only difference with respect to the qubit channels (analyzed in Chapter 4) is that, now, the system and the environment are true Fermions and then anti-commute. Remember that, in the case of qubit channels, the system and the environment are single qubits describable as Fermionic systems by using the Grassmann algebra, but the system and environmental qubits are considered to be distinguishable, i.e. they commute. In the following we will analyze the beam-splitter and the amplifier channels for Fermions.

C.1 Fermionic beam-splitter

The Fermionic beam-splitter channel is given by the following transformation

$$\begin{aligned} U_{ab}^{\dagger} a U_{ab} &= a' = \sqrt{n} a + \sqrt{1-n} b \ , \\ U_{ab}^{\dagger} b U_{ab} &= b' = \sqrt{n} b - \sqrt{1-n} a \ , \end{aligned} \tag{C.1}$$

where a and b are, respectively, the annihilation operators of the one Fermionic mode of the system and another one of the environment, $n \in [0,1]$ is the *attenuation coefficient* or the *transmissivity*. Eq. (C.1) represents a canonical transformation: indeed, $\{a', a'^{\dagger}\} = 1$, $\{b', b'^{\dagger}\} = 1$, $\{a', b'\} = 0$, and $\{a', b'^{\dagger}\} = 0$, since $\{a, a^{\dagger}\} = 1$, $\{b, b^{\dagger}\} = 1$, $\{a, b\} = 0$, and $\{a, b^{\dagger}\} = 0$. Note that here a and b are completely Fermionic operators, while in the amplitude damping channel a and b (and also a' and b') satisfy independently the anti-commutation relations but they commute between each other.

This map $\mathcal{E}$ can be described as a coupling with environment prepared in some *mixed* state σ_b, i.e.

$$\mathcal{E}(\rho_a) = \mathrm{Tr}_b[U_{ab}(\rho_a \otimes \sigma_b)U_{ab}^\dagger] \ , \tag{C.2}$$

where $\mathrm{Tr}_b[\ldots]$ is the partial trace over the environment B, U_{ab} is the unitary operator in the composite Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$. In the Heisenberg picture (see Sec. 2.2.5) the Fermionic displacement operator for the mode a evolves as

$$\mathcal{E}_H(D_a(\xi)) = D_a(\sqrt{n}\xi)\chi_b(\sqrt{1-n}\xi) \ , \tag{C.3}$$

where $\chi_b(\xi)$ is the characteristic function of the input density operator, σ_b, of the environment (e.g., in a thermal state; see below) and ξ is the Grassmann variable (see Appendix B). Therefore, the output characteristic function for the mode a is

$$\chi_a'(\xi) = \chi_a(\sqrt{n}\xi)\chi_b(\sqrt{1-n}\xi) \ . \tag{C.4}$$

Note that this transformation has exactly the same expression like in the Bosonic case in Sec. 3.3.1. In particular, it implies the simple composition rule for which two beam-splitter channels, with transmissivity n_1 and n_2, are equivalent to a new beam-splitter with transmissivity $n_1 n_2$.

The Fermionic beam-splitter is equivalent to the amplitude damping map (Sec. 4.5), when the initial state of the environment is the vacuum state, i.e. the thermal state with $T = 0$ (see below and also Ref. [41]). Indeed, if $\sigma_b = |0\rangle\langle 0|$, Eq. (C.4) provides the same transformation of the amplitude damping channel. In the same way, if the environment initial state is a thermal state at finite temperature, as in Eq. (C.7) with

$$\tanh\left(\frac{1}{2k_B T}\right) = \frac{n(q-1/2)}{1-n} \tag{C.5}$$

or

$$T = \frac{1}{2k_B}\left\{\mathrm{arctanh}\left[\frac{n(q-1/2)}{1-n}\right]\right\}^{-1} , \tag{C.6}$$

with k_B being the Boltzmann constant, we recover the same characteristic functions transformation of the generalized amplitude damping channel.

The Fermionic thermal state can be described [similarly to the Bosonic one in Eq. (3.27)] from the density operator τ_b, as follows

$$\tau_b = \frac{e^{-\frac{a^\dagger a}{k_B T}}}{\mathrm{Tr}[e^{-\frac{a^\dagger a}{k_B T}}]} \ , \tag{C.7}$$

which, in the 'computational basis' (i.e., eigenstates of σ_z), can be written as

$$\tau_b = \left(\begin{array}{cc} \frac{1}{1+e^{-\frac{1}{k_B T}}} & 0 \\ 0 & \frac{e^{-\frac{1}{k_B T}}}{1+e^{-\frac{1}{k_B T}}} \end{array} \right) . \tag{C.8}$$

In the limit of zero temperature, i.e. $T \to 0$,

$$\tau_b = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) , \tag{C.9}$$

which naturally corresponds to the ground state, $|0\rangle\langle 0|$. In the limit of the infinite temperature (i.e., maximal noise)

$$\tau_b = \frac{1}{2}\mathbb{1} = \frac{1}{2} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) , \tag{C.10}$$

which is the completely mixed state, of course. The characteristic function of the thermal state, i.e. $\chi(\xi) = \mathrm{Tr}[\tau_b D(\xi)]$, is

$$\chi(\xi) = 1 - \frac{1}{2} \, \frac{1-e^{-\frac{1}{k_B T}}}{1+e^{-\frac{1}{k_B T}}} \, \xi^* \xi = 1 + \frac{1}{2} \tanh \left(\frac{1}{2 k_B T} \right) \xi \xi^* , \tag{C.11}$$

and

$$\chi^{T \to 0}(\xi) = 1 + \frac{1}{2} \xi \xi^* , \tag{C.12}$$

$$\chi^{T \to \infty}(\xi) = 1 . \tag{C.13}$$

C.2 Fermionic amplifier

The Fermionic amplifier channel can be obtained mathematically (i.e. physically one cannot create Fermions) as in the Bosonic case in Sec. 3.3.1, according to the following canonical transformations

$$U_{ab}^\dagger a U_{ab} = a' = \sqrt{k} a + \sqrt{k-1} b^\dagger , \tag{C.14}$$

$$U_{ab}^\dagger b U_{ab} = b' = \sqrt{k+1} a^\dagger + \sqrt{k} b , \tag{C.15}$$

where a and b are, respectively, the annihilation operators of the one Fermionic mode of the system and another one of the environment, and k is the *gain parameter*. Again U is a canonical transformation and $\{a', a'^\dagger\} = 1$, $\{b', b'^\dagger\} = 1$, $\{a', b'\} = 0$, and $\{a', b'^\dagger\} = 0$, since $\{a, a^\dagger\} = 1$, $\{b, b^\dagger\} = 1$, $\{a, b\} = 0$, and $\{a, b^\dagger\} = 0$. The quantum channel is, again, described through a coupling with an environment as in Eq. (C.2). The action of the dual channel on the Fermionic displacement operator for the mode a, $D_a(\xi)$, is

$$\mathcal{E}_H(D_a(\xi)) = D_a(\sqrt{k}\xi)\chi_b(\sqrt{k-1}\xi^*)[1 + (k-1)\xi^*\xi] \ , \tag{C.16}$$

where $\chi_b(\xi)$ is the characteristic function of the input density operator, σ_b, of the environment. Therefore, the output characteristic function for the mode a is

$$\chi'_a(\xi) = \chi_a(\sqrt{k}\xi)\chi_b(\sqrt{k-1}\xi^*)[1 + (k-1)\xi^*\xi] \ . \tag{C.17}$$

Note that the Fermionic amplifier channel has a similar transformation of the Bosonic amplifier, apart from the last coefficient. However, again one has that two amplifier channels, with amplification coefficient k_1 and k_2, are equivalent to a new amplifier with gain parameter k_1k_2. Indeed, one has

$$\begin{aligned}
\chi'_a(\xi) &= \chi_a(\sqrt{k_1}\xi)\chi_b(\sqrt{k_1-1}\xi^*)[1 + (k_1-1)\xi^*\xi] \\
\chi''_a(\xi) &= \chi'_a(\sqrt{k_2}\xi)\chi_b(\sqrt{k_2-1}\xi^*)[1 + (k_2-1)\xi^*\xi] \\
&= \chi_a(\sqrt{k_2k_1}\xi)\chi_b(\sqrt{k_1-1}\sqrt{k_2}\xi^*)\chi_b(\sqrt{k_2-1}\xi^*)[1 + k_2(k_1-1)\xi^*\xi] \\
&\quad [1 + (k_2-1)\xi^*\xi] = \chi_a(\sqrt{k_2k_1}\xi)\chi_b(\sqrt{k_1k_2-1}\xi^*)[1 + (k_2k_1-1)\xi^*\xi],
\end{aligned}$$

which is an amplifier with gain k_1k_2.

Related publications

This thesis is basically based on the following papers:

1. F. Caruso and V. Giovannetti,
   "*Degradability of Bosonic Gaussian Channels*",
   Phys. Rev. A **74**, 062307 (2006).

2. F. Caruso, V. Giovannetti, and A. S. Holevo,
   "*One-mode Bosonic Gaussian channels: A full weak-degradability classification*",
   New J. Phys. **8**, 310 (2006).

3. F. Caruso and V. Giovannetti,
   "*Qubit quantum channel: A characteristic function approach*",
   Phys. Rev. A **76**, 042331 (2007).

4. F. Caruso, J. Eisert, V. Giovannetti, and A. S. Holevo,
   "*Multi-mode Bosonic Gaussian channels*",
   e-print arXiv:0804.0511 (2008).

5. F. Caruso, V. Giovannetti, C. Macchiavello, and M.B. Ruskai,
   "*Qubit channels with small correlations*",
   Phys. Rev. A **77**, 052323 (2008).

Other publications:

6. F. Caruso and V. Giovannetti,
"*A new approach to characterize qubit channels*",
accepted for publication in the special issue of IJQI "Noise, Information and Complexity at Quantum Scale" (2008), eds. S. Mancini and F. Marchesoni, e-print arXiv:0802.2822.

7. F. Caruso and C. Tsallis,
"*Extensive nonadditive entropy in quantum spin chains*",
invited paper in *Complexity, Metastability and Nonextensivity*, eds. S. Abe, H.J. Herrmann, P. Quarati, A. Rapisarda, C. Tsallis, American Institute of Physics Conference Proceedings 965 (New York, 2007), pp. 51, e-print arXiv:0711.2641.

8. F. Caruso and C. Tsallis,
"*Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics*",
e-print arXiv:cond-mat/0612032 (2006).

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