A variational treatment of hydrodynamic and magnetohydrodynamic flows

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Contents

Abstract

In order to describe stationary plasma flows in thrusters based on plasma propulsion, an ideal, axial symmetric, single-fluid motion is assumed. The conservation laws of conductive fluids and the Maxwell's equations lead to a second order differential equation for the magnetic flux function ψ, *i.e.* the generalized Grad Shafranov (GS) equation, and to two implicit constraints relating ψ to the plasma density and the azimuthal velocity. This set of three equations, one differential and two algebraic, is then expressed using a variational approach and the solution is obtained in a straightforward manner from the extremum of the appropriate Lagrangian functional. The numerical approach is based on Ritz's method, which has the advantage of producing analytic (though approximate) solutions. Both non-conductive fluids, where the acceleration can only be obtained exploiting the internal energy of the flow (thermodynamic process), and conductive fluids, where the electromagnetic forces play a fundamental role, are considered. In order to apply this approach to the acceleration processes in nozzle-like configurations, an open-boundary geometry is investigated and specific attention is paid to a physical definition of boundary conditions. Hydrodynamic shocks are taken into account and it is shown that the appropriate jump conditions follow implicitly from a natural extension of the Lagrangian variational principle. A comparison test with an explicit solution permits an estimate of the approximate results.

Introduction

Spacecraft propulsion outside the earth's atmosphere is normally obtained by ejecting a propellant fluid. The thrust is the product of the mass flow rate times a mean speed of the ejected flow. The amount of propellant, which greatly influences the cost, must be calculated based on a value estimated for this characteristic velocity. There are, however, chemical and thermodynamics constraints that limit the expansion of the combusted propellant beyond some speed levels. Electrically charged fluids, and in particular plasmas, have been considered over the last few decades, as they are subject to electromagnetic interactions. These interactions allow us to go beyond the limits of thermodynamics since other forces can be used in order to accelerate the propellant. This is why it has become important to understand the fundamentals of plasma dynamics in space propulsion.

In order to investigate thrust generation we exploit the fact that a non-charged fluid can be accelerated in a suitably shaped nozzle by letting it expand adiabatically. In this process a fraction of the internal energy of the fluid goes into kinetic energy, increasing the exit velocity. This fraction is related to the thermodynamic efficiency of the expansion and to the losses that prevent a full expansion. In spite of such a simple principle, the theory of de Laval nozzles (1890) is complex even for ideal fluids, where dissipative phenomena are neglected. The simplest way of studying this problem is that of assuming a steady state and a one-dimensional motion, whilst considering the section change in the conduit. Even with such simple assumptions, fluid dynamics equations are non-linear, and heavily influenced by a characteristic factor, the ratio of fluid speed to sound speed, the latter being the speed of propagation of changes in fluid properties. This ratio is named the Mach number, and the factor $\left(M^2 - 1\right)$ appearing in de Laval's model causes a change in flow behaviour when going from subsonic regimes ($M < 1$) to supersonic ones ($M > 1$). It can be shown that, to accelerate the fluid, the nozzle section must decrease along the axis in subsonic conditions, whilst it must increase in supersonic conditions. De Laval's theory, though very simplified, allowed a good understanding of the fundamental phenomena and has led to the first supersonic nozzles (convergent-divergent).

For a short historical background, the equations governing ideal hydrodynamics were written by Euler in 1775 and have not been solved in the general case so far. The achievements in hydrodynamics in the two centuries from its first formulation have been slower than in other fields of theoretical physics. Euler's theory allows us to decrease the number of unknown functions to deal with, but the equations remain quite complex. In the same period (in the years 1772-1788) J. L. Lagrange built the foundations of variational calculus, and showed its applications in classical mechanics and in fluid dynamics [17]. After the widespread introduction of variational calculus, in the early twentieth century, methods were introduced in order to solve variational problems directly. In particular, in 1907 Ritz proposed a fundamental method based on variational formulation [27].

Variational principles and their numerical applications have shown their power in handling hydrodynamics problems, in both the Eulerian and the Lagrangian descriptions. First attempts to define an Eulerian version of Hamilton's principle were only partly effective, as some over-simplifications were required for the equations of an ideal fluid. The starting point for these attempts was a transformation suggested by Clebsch [5] in 1859 of the hydrodynamics equations. In 1929 Bateman [2] showed a variational formulation of fluid dynamics that allowed the derivation of Clebsch's equations (Clebsch handled only an incompressible and isentropic fluid, but Lamb [18] included the simple extension of density change). This work, and the successive developments of Eckart [7], assumed the process to be isentropic. In addition, there was no connection between the functional used as a Lagrangian in [2][7] and the Hamilton's principle. The first general Eulerian formulations of Hamilton's principle were due to Lin [21] and Seliger & Whitham [31]. Starting from the Hamilton's principle for the Eulerian description and requiring the mass and entropy conservation by Lagrangian multipliers, Lin noticed that the solutions were limited to a subset of the general case and that additional constraints were necessary. To do so he added the conservation of the starting location of fluid particles (three components). These constraints, as shown by Bretherton [3], were forcing the equivalence between the Eulerian description of Hamilton's principle and the one based on Lagrangian description. In Selinger & Whitham's work the former achievements are clarified and settled down; also, the constraints implicit in the Eulerian formulation (with a single Lin binding coordinate required) become explicit and their correctness is proved (*i.e.* the equivalence of Clebsch's transformations to Euler's general equations). Moreover the same formulation is compared to the variational theory of electromagnetism and extended to plasma dynamics and to elasticity. Then, Broer & Kobussen [4] and Van Saarlos [35] showed that the formulations of Bateman and Lin may be obtained directly from the Lagrangian description by canonical transformations. After Lagrange's achievements, the Lagrangian version of Hamilton's principle has been extended to theoretical fluid dynamics by many authors. The work by Salmon [29] shows, as an example, how to model fluid behaviour at geophysical scales with this method.

In the case of plasma, things are more complex, partly because the theory is more recent (the first works are due to Thomson (1897) and Langmuir (1927)), but mainly because of the tight interaction between the dynamic and the electromagnetic effects in the fluid. A plasma may be described as a fluid containing a non-negligible amount of free charged particles, which allow for electromagnetic forces, capable of acting at non-contact distances, to influence the local behaviour of the fluid. As an example, such interactions allow for treating a rarefied gas as a fluid, even when that treatment would otherwise be incorrect. On the other hand, in spite of the highly ionized state, the fluid may macroscopically behave as a non charged medium, though the behaviour as a continuum is a consequence of the long range electromagnetic interactions, which keep the fluid in a globally non-charged state. The theory based on these assumptions is named ideal magnetohydrodynamics (MHD). It reduces to Euler's equations of classic hydrodynamics when electromagnetic phenomena are absent.

In order to deal with the ideal MHD, a first simplifying step has been made. When the plasma is stationary and there is axial symmetry, MHD equations may be transformed by the introduction of a scalar function ψ dependent on the flux of the poloidal magnetic field. This reduces the problem to a single partial derivatives differential equation, named after the work by Grad [11] and by Shafranov [33] (GS), who first derived it (sometimes referred to also as the equation of Grad-Shafranov\Lüst-Shlüter, from Lüst & Shlüter [24]). This equation was used for development work in the field of plasma-fusion and greatly contributed to the explanation of the first experiments. Many proposed configurations, the tokamak in the first place (Grad

& Rubin [11], Shafranov [33]), have been described in a first approximation using this model.

The most effective way to treat this problem mathematically is by calculus of variations. A variational formulation of hydromagnetic equilibrium conditions, including the velocity field, was extensively treated for the first time by Woltjer [37][36]. The procedure is to bound the energy of the system with a sufficient number of constraints, but the equations that describe these constraints are treatable only in the axisymmetric case. In the same line of action as GS and restraining the analysis to the axisymmetric case, Heinemann & Olbert [14] in 1978 derived a single equation describing the equilibrium of ideal MHD flows and proposed a variational principle useful to simplify the mathematical treatment.

Going from the stationary case to the conditions of steady state flow ($\frac{\partial}{\partial t} = 0, \mathbf{v} \neq 0$), it is found that the system is governed by two coupled equations, one a generalized GS equation (a partial differential equation), the other a generalized Bernoulli equation for fluid dynamics (a non linear algebraic equation). The same structure is found for uncharged fluids, although much simpler because of the absence of the electromagnetic terms (see Scott & Lovelace [30]). A detailed description of the GS equations with a non-stationary fluid is given by Lovelace et al. [23]. They deal with the difficulties of the generalized differential GS equation, due particularly to the coupling of this non-linear equation with an implicit algebraic Bernoulli's equation.

Because of these difficulties there is currently a wide interest in the variational description. Hameiri [13] demonstrated that, following the ideas present in the work by Woltjer, it is possible to derive a general (non-axisymmetric) variational principle for toroidal configurations. Also, an elegant variational description of MHD stationary flows is found in Goedbloed [8]. This simplified model is adopted in the problem treated here.

For simplicity, we first studied the behaviour of the non-charged fluids (dropping the Maxwell equations). The first part of each chapter is hence devoted to the hydrodynamics, whilst the second treats the plasma theory. The model that is developed here may be considered as quite limited, but, although it handles quasi-isentropic and axisymmetric stationary flows, it opens interesting views on astrophysical phenomena, fusion experiments and plasma propulsion.

The treatment for the model is explained in Chapter 1, whilst Chapter 2 is dedicated to the description of the numerical solution and to the analysis of the results. The numerical approach is systematically based on Rayleigh-Ritz's method [27][28], which has the additional advantage of including a simple and effective estimate of the solution.

Chapter 1

Models

In spacecraft propulsion a fundamental task is the efficient acceleration of the propellant fluid. By increasing the exhaust velocity and by generating a fast collimated jet it is possible to greatly improve the mission capabilities. Basically, the technical solution consists in expanding a highly compressed gas, typically the product of a chemical reaction, through a suitably shaped channel. To convert a considerable amount of internal energy into an effective speed growth, the shape of the acceleration channel (*nozzle*) has been deeply investigated both in theory and experimentally [22]. The basic geometry, as suggested by the ideal gasdynamics, consist in a convergent part that increases the fluid velocity up to the speed of sound and then in a divergent part in which the supersonic flow reaches the maximum speed. The optimum shape of this second part is designed in order to avoid a shock transition to subsonic flow and by minimizing the flux beam divergence, that is to obtain a velocity direction as uniform as possible at every point of the outlet surface.

To further increase the propellant velocity and maintain the flux control, the interaction between conductive fluids, or plasmas, and electromagnetic fields have been exploited. Although the plasma acceleration processes present many different possibilities with respect to the non-conductive gasdynamics, the use of a nozzle configuration like the one described here seems to improve the thrust performance. In this case, both an external coil and the plasma currents produce a magnetic field that is constrained by a conductive surface. This surface, typically shaped as a hyperboloid of revolution, defines the plasma acceleration channel. The plasma flow can be controlled and optimized by using the external magnetic coil. Due to the complex

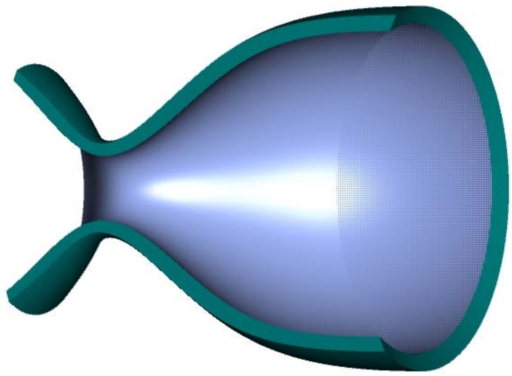

Figure 1-1: Hydrodynamic nozzle

combination of hydrodynamic motion and electromagnetic fields, the main features of this process have been described in its essential aspects only [15].

1.1 Governing Equations

In his fundamental paper on ionized gases Langmuir writes: *"Except near the electrodes, where there are sheaths containing very few electrons, the ionized gas contains ions and electrons in about equal numbers so that the resultant space charge is very small. We shall use the name plasma to describe this region containing balanced charges of ions and electrons"*[20]. Langmuir's model is important since, by considering spatial and time scales sufficiently large, the ensemble of different species of the plasma, ions and electrons, behave like that of a single, perfectly conducting, fluid.

The magnetohydrodynamic description of astrophysical plasma explains, for example, the behaviour of magnetosphere, the dynamo process inside the star nucleus, and the study of accretion disks and massive stars [23]. Many plasma phenomena have a technical importance for spacecraft propulsion [15], fused metal processing and MHD generators [16]. However, most of the technical applications of plasma dynamics are related to nuclear fusion research [9].

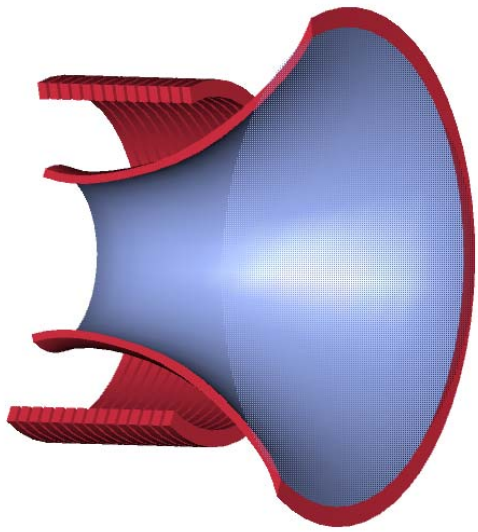

Figure 1-2: Magnetic nozzle

The MHD model, representing the starting assumption of this research, still permits a good approximation of the equilibrium state for fusion experiments.

Except for the magnetic force term in the momentum conservation law, Euler's equations can be used to describe the steady flow of both an ideal fluid or a hydromagnetic medium as they provide a convenient approximation of the equations obtained by averaging the kinetic plasma equations.

The first assumption, made in this model, is to neglect all the dissipative phenomena, such as plasma resistivity and viscosity. This is in general admissible for high energy plasmas and yields a great simplification. More arbitrary is the subsequent assumption of an isotropic pressure tensor, since the local magnetic field direction deeply influences the particles' motion. We leave the complexities of a non-isotropic model to a further development of this work.

In real applications these assumptions are not verified everywhere in the thruster but provide a very important first step in the investigation of the plasma dynamics.

Condition 1 *Ideal steady flows.*

We first consider the equations of ideal gas dynamics. In steady flow condition, the Eulerian time-derivative vanishes

$$\frac{\partial}{\partial t} = 0,$$

and *mass conservation law* becomes:

$$\nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.1}$$

where ρ and $\mathbf{v}$ represent the fluid density and velocity respectively. This equation holds for classical fluids, but also for other fluids employed in space propulsion, where there is no plasma generation or depletion. Considering the ionization process requires an additional source term in the right hand side of equation (1.1). However, since the plasma is assumed to be fully ionized before the inlet of the accelerating channel, equation (1.1) can be accepted.

The momentum equation for a fluid element

$$\underbrace{\rho(\mathbf{v} \cdot \nabla)\mathbf{v}}_{inertial\ force} = - \underbrace{\nabla p}_{pressure\ grad.} + \underbrace{\frac{1}{c}\mathbf{j} \times \mathbf{B}}_{Lorentz\ force}, \tag{1.2}$$

where p represents the isotropic pressure, expresses the fluid particle acceleration due to pressure gradient and, for conductive fluids, due to electromagnetic forces. This contribution, in non-relativistic regimes $|\mathbf{v}| \ll c$ (with c denoting the speed of light), reduces to the Lorentz force term, the vectorial product of the current density $\mathbf{j}$ and the magnetic induction $\mathbf{B}$. Compared with this term, the electrostatic acceleration can be neglected and space charge effects may be dropped. Thus the plasma quasi-neutrality is assumed to hold:

$$n_i \sim n_e \gg (n_i - n_e),$$

where n_i and n_e represent the ion and the electron numerical density, that is the number of particles for unit volume.

The last equation that determines the problem of non-conductive fluid motion, expresses the conservation of the entropy $S(\rho, p)$ of each fluid element

$$\mathbf{v} \cdot \nabla S = 0. \tag{1.3}$$

This equation is obtained by using the equation of state of an ideal gas (with adiabatic index γ) and by neglecting the dissipation of fluid energy in microscopic phenomena, otherwise described by a resistivity and a viscosity term. We also notice that equation (1.3) does not imply that the entropy is everywhere the same, since, in general, entropy variations are allowed for fluid elements belonging to different streamlines. These variations are determined by the initial non-equilibrium phase or, for flows passing through open domains, by the dissipative phenomena that may occur before the fluid entry. Assuming an isentropic motion for the fluid elements is a natural choice for the supersonic gas dynamics and is commonly adopted also for MHD.

In order to describe the motion of a perfectly conductive fluid interacting with a magnetic field, we need to combine the previous equations with Maxwell's equations. This set of equations describes the evolution of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in response to the current density $\mathbf{j}$. It is expressed by the following equations

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} \qquad (Ampere's\ Law)\,, \tag{1.4}$$

$$\nabla \times \mathbf{E} = 0 \qquad (Faraday's\ Law)\,, \tag{1.5}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (Magnetic\ Divergence's\ Law) \tag{1.6}$$

Since we decided to neglect the effects of a space charge $\tau(\mathbf{x}) \propto (n_i - n_e)$, Poisson's law

$$\nabla \cdot \mathbf{E} = \tau$$

is no longer needed and may be dropped.

The last characteristic equation describing the electric field in a perfectly conducting moving medium is

$$\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} = 0, \tag{1.7}$$

which expresses that the electric field in a co-moving frame should vanish. In concrete terms, equation (1.7) represents Ohm's law for a perfectly conductive fluid.

If a plasma must be confined by a rigid conductive boundary or by a magnetic field, it is most likely that there will be a transition region where the plasma properties adapt themselves to the presence of boundaries or vacuum. The transition may be a sharp boundary or a gradual change in plasma quantities over some finite distance. In both cases it is necessary to match the variables for the solution of the model equations on adjacent sides of the transition region. Therefore the boundary conditions can be expressed in terms of the change of a plasma variable χ across the interface. The unit vector normal to the interface is denoted by $\mathbf{n}$ and the surface current in the boundary is denoted by $\mathbf{j}_s$.

If we define

$$\|\chi\| = \chi_i - \chi_o \tag{1.8}$$

the difference between the values assumed by χ on the two sides of the transition region, the boundary conditions result:

Plasma-plasma interface:

$$\left\|p + \frac{B^2}{8\pi}\right\| = 0, \qquad \mathbf{n}\cdot \|\rho\mathbf{v}\| = 0, \qquad \mathbf{n}\times \|\mathbf{E}\| = \mathbf{n}\times \frac{\|\mathbf{v}\times\mathbf{B}\|}{c} = 0,$$

$$\mathbf{n}\cdot \|\mathbf{B}\| = 0, \qquad \mathbf{n}\times \|\mathbf{B}\| = \frac{4\pi}{c}\mathbf{j}_s.$$

Plasma-perfectly conducting wall interface:

$$\mathbf{n}\cdot\mathbf{v} = 0, \qquad \mathbf{n}\times\mathbf{E} = 0, \qquad \mathbf{n}\cdot\mathbf{B} = 0.$$

The set of equations (1.1-1.7) and boundary conditions, or a part of them, can be used to describe different problems. We focus our attention on three models of increasing complexity:

1. the static plasma problem (equation 1.2 without the inertial term, and equations 1.4-1.7);

2. the hydrodynamic problem (equations 1.1-1.3, excluding the Lorentz force term in equation 1.2);

3. the general plasma flow problem (the whole set, equations 1.1-1.7).

The first of these problems leads to the well known Grad-Shafranov equation and will not be reviewed in the following, while the other two will be discussed in the next sections.

Condition 2 *The equilibrium is axisymmetric.*

Woltjer, in his papers on hydromagnetic equilibrium, writes: *"Even if in the dynamics the assumption of axial symmetry is, in general, impermissible, it may be expected that most equilibrium configurations of interest will be axisymmetric"* [36]. Thus, we study the axisymmetric dynamics of conductive and non-conductive fluids. This assumption is mainly due to the technical importance and the resulting simpler model. In fact, this kind of symmetry is typically assumed as a project's base not only in space propulsion research but also in fusion experiments. In what follows we use both spherical (σ, θ, ϕ) and cylindrical (r, ϕ, z) coordinates, on assuming that the solutions are independent of ϕ. This does not mean that the azimuthal velocity or magnetic field are zero.

Let us consider a vector field $\mathbf{C}$ that satisfies the divergence-free condition

$$\boldsymbol{\nabla} \cdot \mathbf{C} = 0$$

(in our case the mass flow $\rho\mathbf{v}$ and the magnetic field $\mathbf{B}$). A scalar function ψ independent on ϕ exists that defines the components of $\mathbf{C}$ in the symmetry plane (or *poloidal* plane $\phi = const$) as

$$\mathbf{C}_p = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi. \tag{1.9}$$

The function ψ is typically named a *stream-function* since it has a constant value along the streamlines of the vector field to which it belongs, or a *flux-function* as it labels each stream surface with the flux of the vector fields that is enclosed in it.

1.1.1 Axisymmetric hydrodynamics

First we consider a non-conductive fluid. In this case Maxwell's equations are not needed and the Lorentz force term in the momentum equation (1.2) can be dropped. The model reduces to the coupling of a partial differential equation and an algebraic constraint. In order to simplify these steps, we start assuming that the fluid entropy is uniform

$$S(\rho, p) = const.$$

Referring to flows passing through an open domain, this means that different fluid elements enter, and exit, the domain with equal entropy. The modifications necessary to deal with entropy variations will be discussed later, at the end of this section.

Hereafter we also explicitly separate the poloidal fluid velocity $\mathbf{v}_p$ from the azimuthal component $v_\phi = \frac{A}{r}$, where A indicates the specific angular momentum per unit mass about the symmetry axis.

Introducing the specific enthalpy $h = \int \frac{dp}{\rho}$ for the isentropic flow and the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ of the fluid, equation (1.2) can be written in the form

$$\mathbf{v} \times \boldsymbol{\omega} = \boldsymbol{\nabla} B, \tag{1.10}$$

where B represents the *Bernoulli 'constant'*

$$B = \frac{1}{2} v_P^2 + \frac{1}{2} \frac{A^2}{r^2} + h(\rho), \tag{1.11}$$

that is conserved along stream and vortex lines.

In order to satisfy the mass conservation law (equation. 1.1) and the axisymmetry condition (equation. 1.9) we can define a stream function ψ such as

$$\rho v_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \qquad \rho v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \tag{1.12}$$

or, in vectorial terms,

$$\rho \mathbf{v}_p = \boldsymbol{\nabla} \psi \times \boldsymbol{\nabla} \phi.$$

By using the definition (1.12), equation (1.10) leads to the following expressions

$$B = B(\psi) \qquad A = A(\psi),$$

where the second equality is obtained again from the axial symmetry condition. Taking the curl of equation (1.10) yields the vorticity equation:

$$\boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) = 0, \tag{1.13}$$

that is reduced by symmetry to:

$$\rho \mathbf{v} \cdot \boldsymbol{\nabla} \left(\frac{\omega_\phi}{r\rho} \right) = \frac{\partial}{\partial z} \left(\frac{A^2}{r^4} \right). \tag{1.14}$$

A first integral of equation (1.14) can be obtained from the momentum equation, substituting the definitions of ψ and ω:

$$\frac{\omega_\phi}{r\rho} = \frac{A}{r^2} \frac{dA}{d\psi} - \frac{dB}{d\psi}. \tag{1.15}$$

Using the expression of ω_ϕ in terms of ψ,

$$-r\omega_\phi = r \frac{\partial}{\partial r} \frac{1}{r} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial z} \right),$$

we obtain a second order partial differential equation

$$r \frac{\partial}{\partial r} \frac{1}{r} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial z} \right) = \rho r^2 \frac{dB}{d\psi} - \rho A \frac{dA}{d\psi}, \tag{1.16}$$

for the stream function ψ. Inspection of this equation shows that for incompressible fluids ($\rho = const$) the problem is linear only if $B(\psi) = b_2\psi^2 + b_1\psi + b_0$ and $A(\psi) = a_1\psi$ hold. For compressible fluids ($\rho \neq const$), the dependence of ρ on the stream function and its derivatives should be determined by using the Bernoulli equation

$$B(\psi) = \frac{1}{2} \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi}{\rho} \right)^2 + \frac{1}{2} \left[\frac{A(\psi)}{r} \right]^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho}, \tag{1.17}$$

where the last term is obtained from the adiabatic constitutive equation of an ideal gas:

$$p \propto \rho^\gamma.$$

This makes the differential problem non-linear.

In the case of non-uniform entropy propagation

$$\mathbf{v} \cdot \boldsymbol{\nabla} S = 0, \tag{1.18}$$

the model must be modified. By using the equation of state, S can be expressed in terms of ρ

and p as

$$S(\rho, p) = \frac{k_B}{\gamma - 1}\left[\ln\left(\frac{p}{\rho^\gamma}\right) - \ln\left(\frac{p_0}{\rho_0^\gamma}\right)\right], \tag{1.19}$$

where k_B represents the Boltzmann constant and $\{\rho_0, p_0\}$ are reference values for the gas density and pressure. Equation (1.18)

$$S(\rho, p) = S(\psi), \tag{1.20}$$

and equation (1.19) yields

$$p = C(\psi)\,\rho^\gamma = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma \exp\left[(\gamma - 1)\frac{S(\psi)}{k_B}\right].$$

The differential problem now becomes

$$r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial z}\right) = \rho r^2\frac{dB}{d\psi} - \rho A\frac{dA}{d\psi} - r^2\frac{p}{k_B}\frac{dS}{d\psi}, \tag{1.21}$$

and the modified Bernoulli equation results

$$B(\psi) = \frac{1}{2}\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi}{\rho}\right)^2 + \frac{1}{2}\left[\frac{A(\psi)}{r}\right]^2 + \frac{\gamma}{\gamma - 1}\rho^{\gamma-1}C(\psi). \tag{1.22}$$

The terms depending on $S(\psi)$ and $C(\psi)$ in the equations (1.21) and (1.22) add some further difficulties to the problem. However, this model permits us to find solutions for a wider class of inlet conditions.

1.1.2 Generalized Grad-Shafranov equations

The system of equations describing the plasma flow is now treated in the axisymmetric case. Starting from the equations (1.1-1.7) in the unknown fields ρ, $\mathbf{v}$, p, $\mathbf{j}$, $\mathbf{B}$, $\mathbf{E}$ it is possible to obtain a set of three equations depending on $\{v_\phi, \rho, \psi\}$, a partial differential equation and two algebraic equations.

From Faraday's law it follows that $E_\phi = 0$. By substituting this result into equation (1.7) we obtain

$$\mathbf{v}_p = \kappa(r, z)\mathbf{B}_p. \tag{1.23}$$

The continuity equation (1.1) reduces to $\nabla \cdot (\rho\mathbf{v}_p) = 0$ and, using equation (1.23), this can be rewritten as

$$\mathbf{B}_p \cdot \boldsymbol{\nabla}(\rho\kappa) = 0. \tag{1.24}$$

As explained in equation (1.9), the poloidal magnetic field and the poloidal plasma flux can be both expressed in terms of two different stream functions. However from equation (1.23) we obtain an easy way to express one of these functions in terms of the other and, for this reason, we only define one flux function ψ as:

$$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{1.25}$$

For plasma flows this is in general a good simplification but leads to some difficulties in the hydrodynamic limit since for $\mathbf{B}_p \to 0$ the coefficient κ becomes singular and the poloidal velocity cannot be expressed in terms of ψ. This difficulty can be avoided by using two different stream functions or by favouring the one related to the mass flow.

Thus, equation (1.24) reduces to

$$4\pi\rho\kappa = F(\psi), \tag{1.26}$$

where $F(\psi)$ is a generic function of ψ (*e.g.* a stream function) that needs to be specified in order to solve the problem. From equation (1.23) it follows that

$$\mathbf{v} \times \mathbf{B} = \frac{1}{r}\,(v_\phi - \kappa B_\phi)\boldsymbol{\nabla}\psi \tag{1.27}$$

and combining Faraday's law (1.5) with the perfect conductivity equation (1.7) yields to

$$\frac{1}{r}\,(v_\phi - \kappa B_\phi) = G(\psi), \tag{1.28}$$

where $G(\psi)$ is a stream function. The electric field can now be written as

$$\mathbf{E} = -\frac{1}{c}\,G(\psi)\boldsymbol{\nabla}\psi. \tag{1.29}$$

From equation (1.23-1.28) the plasma velocity can be expressed in the form

$$\mathbf{v} = \frac{F(\psi)}{4\pi\rho}\mathbf{B}_p + \left[\frac{F(\psi)}{4\pi\rho}B_\phi + rG(\psi)\right]\hat{\mathbf{e}}_\phi.$$

As for the hydrodynamic case, equation (1.3) and the first law of thermodynamics imply that the specific entropy depends on ψ only:

$$S(\rho, p) = S(\psi) \tag{1.30}$$

and, from the equation of state, it follows

$$p = I(\psi)\,\rho^\gamma = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma \exp\left[(\gamma-1)\,\frac{S(\psi)}{k_B}\right].$$

We are left with the three components of the momentum equation (1.2). The azimuthal component can be rewritten as

$$\mathbf{B}_p \cdot \boldsymbol{\nabla}(rB_\phi - rFv_\phi) = 0,$$

that implies:

$$rB_\phi - rFv_\phi = H(\psi). \tag{1.31}$$

A combination of this equation and equation (1.28) yields to an explicit form for the azimuthal plasma velocity v_ϕ and magnetic field B_ϕ respectively

$$rv_\phi = \left(r^2 G + \frac{FH}{4\pi\rho}\right)\left(1 - \frac{F^2}{4\pi\rho}\right)^{-1} \tag{1.32}$$

and

$$rB_\phi = \left(H + r^2 FG\right)\left(1 - \frac{F^2}{4\pi\rho}\right)^{-1}, \tag{1.33}$$

with the additional condition that $H = -r^2 FG$ on the surfaces where $F^2/(4\pi\rho) = 1$ (*Alfven surfaces*). However, this further rearrangement introduces an additional constraint in the solution. The component parallel to B_p of the Euler equation (1.2) in the symmetry plane results

$$\int (dp/\rho)\bigg|_{\psi=const} + v^2/2 - rv_\phi G = J(\psi), \tag{1.34}$$

where $J(\psi)$ represent a generalization of the Bernoulli constant (1.11). Once again a stream function, and equation (1.34) takes the name of a "generalized Bernoulli equation."

The poloidal component parallel to $\nabla\psi$ now gives the generalized Grad-Shafranov equation for $\psi(r,z)$ [23]:

$$\left(1 - \frac{F^2}{4\pi\rho}\right)\triangle^*\psi - F\boldsymbol{\nabla}\left(\frac{F}{4\pi\rho}\right)\cdot\boldsymbol{\nabla}\psi = -4\pi\rho r^2\left(J' + rv_\phi G'\right) - \tag{1.35}$$

$$(H + rv_\phi F)(H' + rv_\phi F') + 4\pi r^2 p\,(S'/k_B),$$

where $\triangle^*\psi$ is a differential operator defined as

$$\triangle^*\psi = r\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right).$$

The functional dependencies of $\{F, G, H, I, J\}$ on the stream function ψ must be prescribed. These are determined by arbitrary choice or via experimental measurements. With the appropriate boundary condition, the set of two algebraic equations, (1.31) and (1.34), and the partial differential equation (1.35) should, in principle, determine the three unknown fields $\{v_\phi, \rho, \psi\}$.

1.2 Variational Formulation

In the model described above, the main equation is the generalized differential GS equation (1.35) for the flux function ψ. In this differential formulation the plasma density that appears in the generalized GS equation is related to ψ by the Bernoulli equation, which is an algebraic equation that cannot in general be brought to the form $\rho = \rho(\psi)$. Similarly, the azimuthal velocity can be expressed in terms of ψ and ρ by equation (1.32), but this expression contains a potential singularity at the Alfven surface, where we need to impose the regularity condition $H = -r^2 FG$.

A solution of equation (1.35) with the implicit conditions stated by the Bernoulli equation (1.34) and the azimuthal momentum conservation (1.31) is found taking v_ϕ, ρ and ψ such that the Lagrangian of the system

$$L(v_\varphi, \rho, \psi) = \int_\Omega \mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \boldsymbol{\nabla}\psi\right) dV \tag{1.36}$$

is an extremum. In this expression $\mathcal{L}$ represents the Lagrangian density of the model. The Euler-Lagrange equations associated to the variational problem are the governing equations derived above.

To find a minimum of L we apply a variation $t_1 \cdot \delta v_\varphi$, $t_2 \cdot \delta\rho$ and $t_3 \cdot \delta\psi$ to the equilibrium state and we take the derivatives with respect to $t_{1,2,3}$. Hence we obtain

$$ext\left(L(v_\varphi, \rho, \psi)\right) \rightarrow \frac{\partial}{\partial t_i} L(v_\varphi + t_1 \cdot \delta v_\varphi, \rho + t_2 \cdot \delta\rho, \psi + t_3 \cdot \delta\psi) = 0,$$

with $i = 1, 2, 3$. With the classic procedure, taking into account the arbitrariness of the variation $(\delta v_\varphi, \delta\rho, \delta\psi)$, the above expressions reduce to the following

$$\frac{\partial}{\partial\psi}\left[\mathcal{L}\left(v_\varphi, \rho, \psi\right)\right] - \sum_i \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}\left(v_\varphi, \rho, \psi\right)}{\partial \psi_{,x_i}} = 0, \tag{1.37}$$

$$\frac{\partial}{\partial\rho}\left[\mathcal{L}\left(v_\varphi, \rho, \psi\right)\right] = 0, \tag{1.38}$$

$$\frac{\partial}{\partial v_\varphi}\left[\mathcal{L}\left(v_\varphi, \rho, \psi\right)\right] = 0. \tag{1.39}$$

The same procedure holds also for the purely hydrodynamic problem.

This variational approach, where v_φ is considered as an independent variable, is convenient since the additional condition $H = -r^2 FG$ (at the Alfven surface) need not be imposed. In fact, when the extremum is found, this condition is automatically satisfied since the Euler-Lagrange equation (1.39) obtained by varying the Lagrangian with respect to v_φ is exactly the equation (1.32).

1.2.1 Axisymmetric hydrodynamics

In this case, the Lagrangian density $\mathcal{L}$, is defined by

$$\mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \boldsymbol{\nabla}\psi\right) = \frac{1}{2}\rho\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi}{\rho}\right)^2 + \frac{1}{2}\rho v_\phi^2 - \rho v_\phi \frac{A\left(\psi\right)}{r} + \rho B\left(\psi\right) - \frac{1}{\gamma - 1}\rho^\gamma C\left(\psi\right). \tag{1.40}$$

Since (1.40) can be regarded as

$$\mathcal{L} = \rho v_P^2 + p, \tag{1.41}$$

the dependence of $\mathcal{L}$ on the azimuthal velocity is relatively simple. Equation (1.39) yields

$$v_\phi = \frac{A(\psi)}{r}$$

and this expression has been directly substituted into the Lagrangian density

$$\mathcal{L}(\mathbf{x}, \rho, \psi, \boldsymbol{\nabla}\psi) = \frac{1}{2}\rho\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi}{\rho}\right)^2 - \frac{1}{2}\rho\left[\frac{A(\psi)}{r}\right]^2 + \rho B(\psi) - \frac{1}{\gamma - 1}\rho^\gamma C(\psi). \tag{1.42}$$

This reduces the independent variables of the variational problem and it simplifies the numerical calculation.

1.2.2 Generalized Grad-Shafranov equations

For the classic Grad-Shafranov problem ($\mathbf{v} = 0$) the Lagrangian density depends on ψ only:

$$\mathcal{L}(\mathbf{x}, \psi, \boldsymbol{\nabla}\psi) = \frac{1}{8\pi}\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 + \frac{1}{8\pi}\left(\frac{Q}{r}\right)^2 + P, \tag{1.43}$$

where $Q = rB_\phi$ and the plasma pressure P are two functions of ψ. Equation (1.43) can be rewritten as

$$\mathcal{L} = -\frac{B_P^2}{4\pi} + \frac{B^2}{8\pi} + p. \tag{1.44}$$

The form of $\mathcal{L}$ for the generalized problem is

$$\begin{aligned}\mathcal{L}(\mathbf{x}, v_\varphi, \rho, \psi, \boldsymbol{\nabla}\psi) = &\left(\frac{F^2}{4\pi\rho} - 1\right)\frac{1}{8\pi}\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 + \\ &\frac{1}{8\pi}\left(\frac{H + rv_\phi F}{r}\right)^2 - \frac{1}{2}\rho v_\phi^2 + \rho\left(J + rv_\phi G\right) - \frac{1}{\gamma - 1}\rho^\gamma I,\end{aligned} \tag{1.45}$$

where $\{F, G, H, I, J\}$ are the flux-functions described in the previous section. As can be deduced from equations (1.41) and (1.44), this last expression is equivalent to

$$\mathcal{L} = -\frac{B_P^2}{4\pi} + \frac{B^2}{8\pi} + p + \rho v_P^2. \tag{1.46}$$

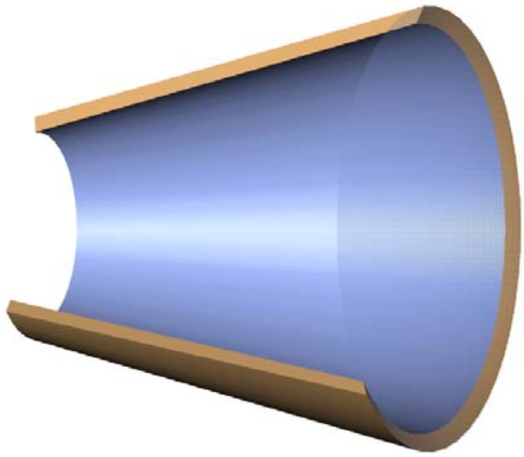

Figure 1-3: Conic nozzle

1.3 Boundary Conditions

In order to deal with open configurations, we consider the boundary to be divided into two regions: the inlet and outlet surfaces, where the fluid crosses the boundary; the nozzle's walls and the symmetry axis, assumed as flux surfaces.

A simple representation of the problem geometry is the conical nozzle illustrated in fig (1-3). This consists in a conical wall surface delimited by two spherical surfaces in the inlet and outlet region. The example is also a test case for the procedure. In particular, this geometry is easily described by using spherical coordinates (σ, θ, ϕ), as illustrated in figure (1-4), where the domain can be defined as $\sigma \in [\sigma_{inlet}, \sigma_{outlet}]$ and $\theta \in [0, \theta_{wall}]$.

In the variational formulation we consider the case of fixed boundary conditions where the values of the unknown function ψ is assigned on $\partial\Omega$. In another classic type of boundary condition, relevant to the problem of plasma propagation, we seek an extremum of the Lagrangian when the values assumed by ψ on a portion $\partial_1\Omega$ of the whole boundary $\partial\Omega$ are not specified. It can be shown that, due to the Lagrangian extremum condition, a specific boundary condition must be satisfied on $\partial_1\Omega$. In fact, taking the first variation of our functional, we obtain with

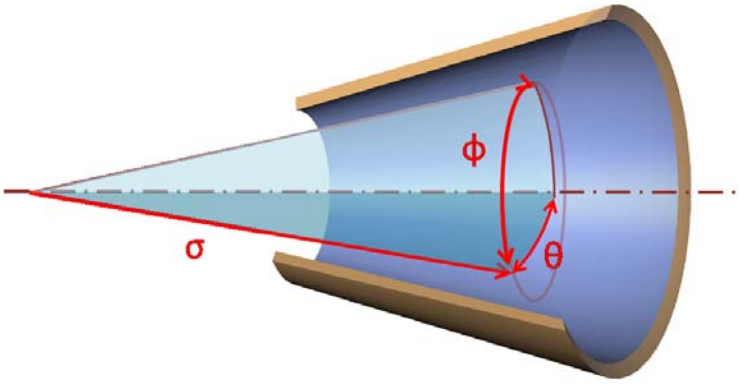

Figure 1-4: Spherical coordinate system used for the description of conic nozzle geometry

some manipulations:

$$\delta L = \int_\Omega [\mathcal{L}]_\psi \, \delta\psi dV + \int_\Omega [\mathcal{L}]_\rho \, \delta\rho dV + \int_\Omega [\mathcal{L}]_{v_\varphi} \, \delta v_\varphi dV + \int_{\partial_1\Omega} \left(\mathcal{L}_{\psi_z} \frac{dr}{ds} - \mathcal{L}_{\psi_r} \frac{dz}{ds} \right) \delta\psi dS,$$

where $[\mathcal{L}]_\psi$, $[\mathcal{L}]_\rho$, $[\mathcal{L}]_{v_\varphi}$ represent the Euler-Lagrange equations associated with the three variables of our problem. As before, this variation should be zero. It is easy to prove that, since the extremum solution for these boundary conditions is an extremum also for the classic boundary condition, the three volume integrals are zero. Thus, for the arbitrariness of the variations $\delta\psi$ the equation for the boundary $\partial_1\Omega$ results

$$\frac{\partial \mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \boldsymbol{\nabla}\psi\right)}{\partial\left(\partial\psi/\partial r\right)} \frac{dz}{ds} - \frac{\partial \mathcal{L}\left(\mathbf{x}, v_\varphi, \rho, \psi, \boldsymbol{\nabla}\psi\right)}{\partial\left(\partial\psi/\partial z\right)} \frac{dr}{ds} = 0, \tag{1.47}$$

where s is the boundary arc-length. This boundary condition is called the *natural boundary condition* of the problem.

As a consequence of the definition of the flux function ψ, we can now assume the Dirichlet condition on the nozzle's wall and on the symmetry axis:

$$\psi|_{axis} = \psi_0 \qquad \psi|_{wall} = \psi_1 \tag{1.48}$$

whereas in the open parts of the boundary we shall prescribe the natural conditions. The choice of natural boundary conditions fully determines the problem from a mathematical point of view and, since these conditions are naturally satisfy by the variational formulation, it permits a simple implementation of the open boundary geometry. However the solution still depends on the shape of the inlet and outlet surfaces and a study of this dependence is recommended in order to characterize the different solutions. In general we should consider the shape related to the plasma properties outside the domain of the nozzle and the analysis of different geometries can contribute to describe these relations. It is easy to see that the difference $\psi_1 - \psi_0$ represents the overall flow rate of the corresponding vector field passing through the domain. Since the flux function ψ is defined except for a constant additional value, it is a common choice to assume $\psi_0 = 0$. Thus, hereafter ψ represents the net flux enclosed in the axisymmetric surface that it labels.

1.3.1 Axisymmetric hydrodynamics

Substituting the Lagrangian density $\mathcal{L}$ in (1.47) and carrying out the calculation, the natural boundary condition can be rewritten in the form:

$$\frac{1}{r^2}\frac{\boldsymbol{\nabla}\psi}{\rho}\cdot\mathbf{n} = 0. \tag{1.49}$$

Considering the relation between the mass flow and ψ (1.12), in the part of $\partial\Omega$ where we impose no condition on ψ, we are asking for a solution with a poloidal velocity field flowing perpendicularly to the boundary. This assumption is the same as prescribing the tangent component of the crossing fluid velocity in each point of the open-boundaries and from equation (1.49) we obtain:

$$\boldsymbol{\nabla}\psi\cdot\mathbf{n}\big|_{i,o} = 0 \rightarrow \left.\frac{\partial\psi}{\partial n}\right|_{i,o} = \left.\frac{\partial\psi}{\partial\sigma}\right|_{i,o} = 0, \tag{1.50}$$

where the indices i and o distinguish respectively the inlet and outlet boundary properties. Integrating the normal flux

$$\rho_i V_i^{\sigma} = \left.\frac{1}{\sigma^2\sin(\theta)}\frac{\partial\psi}{\partial\theta}\right|_i$$

along the inlet surface yields

$$\sigma_i^2\int_0^{\theta}\sin(\theta)\,\rho_i V_i^{\sigma}d\theta = \psi_i(\theta) - \psi_0 \tag{1.51}$$

and for $\theta = \theta_{wall}$ we obtain the difference between the two boundary values.

From equation (1.51), giving an estimate of ρ_i, p_i, V_i and V_ϕ in the inlet region, we obtain the explicit expressions of A, B and C in terms of stream function ψ. Recalling the definition, we have

$$A\left(\theta\right)|_i = V_\phi\left(\theta\right) \cdot \sigma_i \sin\left(\theta\right), \tag{1.52}$$

$$B\left(\theta\right)|_i = \frac{V_i^2\left(\theta\right)}{2} + \frac{V_\phi^2\left(\theta\right)}{2} + \frac{\gamma}{\gamma-1}\frac{p_i\left(\theta\right)}{\rho_i\left(\theta\right)}, \tag{1.53}$$

$$C\left(\theta\right)|_i = \frac{p_i}{\rho_i^\gamma}, \tag{1.54}$$

while

$$\theta|_i = \theta\left(\psi\right) \tag{1.55}$$

is obtained by inverting equation (1.51).

Given these conditions (1.48-1.50 and 1.52-1.55) the problem is fully formulated and is possibly solvable, provided that a solution exists.

1.3.2 Generalized Grad-Shafranov equations

By substituting the Lagrangian density (1.45) into equation (1.47), we obtain the equivalent expression of the natural condition for the MHD problem:

$$\frac{1}{r^2}\left(\frac{F^2}{4\pi\rho} - 1\right)\nabla\psi \cdot n = 0.$$

In the open parts of the boundary, where we assume the natural condition to hold, this equation yields

$$\boldsymbol{\nabla}\psi \cdot \mathbf{n}|_{i,o} = 0 \rightarrow \left.\frac{\partial\psi}{\partial n}\right|_{i,o} = \left.\frac{\partial\psi}{\partial\sigma}\right|_{i,o} = 0, \tag{1.56}$$

where the indices i and o distinguish respectively the inlet and outlet boundary. It is easy to see that the difference $\psi_1 - \psi_0$ represents the overall magnetic induction passing through the domain. In fact, by integrating the normal flux

$$B_i^\sigma = \frac{1}{\sigma^2 \sin\left(\theta\right)}\left.\frac{\partial\psi}{\partial\theta}\right|_i$$

in the inlet surface, we derive

$$\sigma_i^2 \int_0^\theta \sin(\theta) B_i^\sigma d\theta = \psi_i(\theta) - \psi_0 \tag{1.57}$$

and for $\theta = \theta_{wall}$ we obtain exactly the difference between the two boundary values.

From equation (1.57), giving an estimate of ρ_i, p_i, V_i, V_ϕ, B_i, B_ϕ, in the inlet region, we deduce the explicit expressions of F, G, H, I and J in terms of stream function ψ. Recalling the definition, we have

$$F(\theta)|_i = 4\pi\rho_i(\theta) \frac{V_i(\theta)}{B_i(\theta)} \tag{1.58}$$

$$G(\theta)|_i = \frac{1}{\sigma_i \sin(\theta)} \left[V_\phi(\theta) - \frac{V_i(\theta)}{B_i(\theta)} B_\phi(\theta) \right], \tag{1.59}$$

$$H(\theta)|_i = \sigma_i \sin(\theta) \left[B_\phi(\theta) - 4\pi\rho_i(\theta) V_\phi(\theta) \frac{V_i(\theta)}{B_i(\theta)} \right] \tag{1.60}$$

$$I(\theta)|_i = \frac{p_i(\theta)}{\rho_i^\gamma(\theta)} \tag{1.61}$$

$$J(\theta)|_i = \frac{V_i^2(\theta)}{2} + \frac{V_\phi^2(\theta)}{2} + \frac{\gamma}{\gamma - 1} \frac{p_i(\theta)}{\rho_i(\theta)} - V_\phi(\theta) \left[V_\phi(\theta) - \frac{V_i(\theta)}{B_i(\theta)} B_\phi(\theta) \right], \tag{1.62}$$

while

$$\theta|_i = \theta(\psi) \tag{1.63}$$

is obtained by inverting equation (1.57). The substitution of equation (1.63) into equation. (1.58-1.62) yields the explicit expressions of the five functions of ψ.

1.4 Discontinuous Solutions

The variational formulation allows us to find the solution in a more general class of functions. This yields a simple method to include the *shocks* in our analysis. It is well known that hydrodynamic equations are locally elliptic or hyperbolic depending on the ratio between the velocity and the speed of sound (Mach number). We can see that in some cases, for the same mass flow rate, two different solutions can be found, one with a subsonic flux and the other with a supersonic one. If we relax the hypothesis of isentropic flow considering that somewhere in the domain the entropy function passes from S to $S + \Delta S$, it is possible to determine a new solution with both subsonic and supersonic regimes. This solution and the position of the entropy's transition surface depend on the chosen value of the entropy jump ΔS. For the physically meaningful cases ($\Delta S > 0$) the transition is from a supersonic flow to a subsonic one and is called a "*shock*" because the fluid properties change discontinuously through it.

An important feature of the variational principle (1.36) is that the solution of the general problem of hydrodynamic and magnetohydrodynamic flow with *shocks* can be implicitly carried out assuming that the derivatives of the flux function ψ and the density ρ are piecewise continuous.

Let us assume that a discontinuity surface λ (a curve in the poloidal plane) exists which divides the domain into two parts: Ω_1 and Ω_2. Thus, given a general function $\chi\,(x, y)$, we define as

$$\|\chi\| = \chi_{\lambda^2} - \chi_{\lambda^1} \tag{1.64}$$

the difference between the two limits

$$\chi_{\lambda^2} = \lim_{x \to \lambda} \chi \qquad x \in \Omega_2,$$

$$\chi_{\lambda^1} = \lim_{x \to \lambda} \chi \qquad x \in \Omega_1$$

at a generic point of λ.

Consider a frame of reference locally aligned with the discontinuity surface. In this frame the velocity and the magnetic induction vectors can be written as

$$\mathbf{V} = v_n \mathbf{n} + v_t \mathbf{t} + v_\phi \boldsymbol{\phi},$$

$$\mathbf{B} = B_n \mathbf{n} + B_t \mathbf{t} + B_\phi \boldsymbol{\phi},$$

where $\mathbf{n}$ and $\mathbf{t}$ represent respectively the normal and the tangential unit vector to λ in the symmetry plane.

A first way to determine the jump conditions across the discontinuity is to use the integral form of the model's equations (1.1)-(1.7). Consider an infinitesimal cylindrical volume element, illustrated in figure (1-5), where the bases of the cylinder are parallel to the shock surface and the bases' typical dimension is bigger than the distance between them. The mass conservation

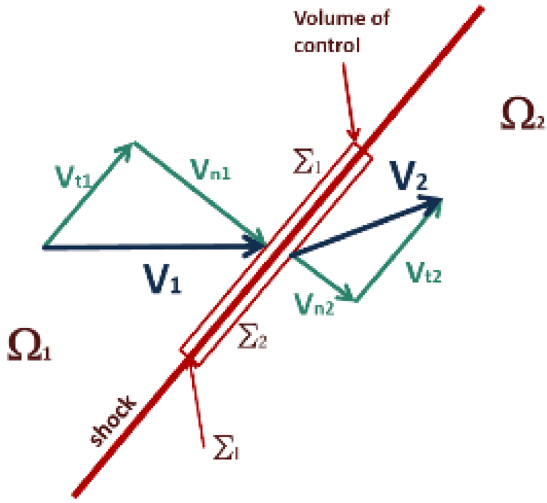

Figure 1-5: Schematic description of a shock transition

equation (1.1), which is the same for the two models we are investigating, in its integral form can be written as

$$\int_{\Sigma_1} \rho_1 \mathbf{V}_1 \cdot \mathbf{n}_1 dS_1 + \int_{\Sigma_l} (\rho \mathbf{V} \cdot \mathbf{n})|_{\Sigma_l} dS_l + \int_{\Sigma_2} \rho_2 \mathbf{V}_2 \cdot \mathbf{n}_2 dS_2 = 0,$$

where the flux term through the surface Σ_l can be neglected, for $h \ll r$, thus obtaining

$$\int_{\Sigma_1} \rho_1 \mathbf{V}_1 \cdot \mathbf{n}_1 dS_1 + \int_{\Sigma_2} \rho_2 \mathbf{V}_2 \cdot \mathbf{n}_2 dS_2 = 0.$$

This expression can be rewritten, by using the definition (1.64), as

$$\|\rho v_n\| = 0. \tag{1.65}$$

These same considerations hold for the remaining model's equations (1.2)-(1.7) but specific care must be taken for energy conservation. Our attempt is to model entropy discontinuity in an adiabatic flow. Thus, instead of equation (1.3), we shall use the energy equation

$$\int (\rho \mathbf{V} \cdot \mathbf{n}) \left(e + \frac{V^2}{2} \right) dS + \int p\mathbf{n} \cdot \mathbf{V} dS + \int \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \cdot \mathbf{n} dS = 0. \tag{1.66}$$

In this way we obtain a number of jump constraints equal to the number of conservation equations.

The same constraints can be found through the extremization of the Lagrangian functional (1.43) on the set of functions

$$\mathcal{U} \equiv \left\{ \bar{\rho} \;\; \textit{piecewise continuous}, \, \bar{\psi} \in \mathbb{C}^0 : \boldsymbol{\nabla} \bar{\psi} \;\; \textit{piecewise continuous} \right\} \tag{1.67}$$

Hence equation (1.43) can be written as:

$$L(\rho, \psi) = \int_{\Omega_1} \mathcal{L}^1 \left(\mathbf{x}, \rho, \psi, \boldsymbol{\nabla}\psi \right) dV + \int_{\Omega_2} \mathcal{L}^2 \left(\mathbf{x}, \rho, \psi, \boldsymbol{\nabla}\psi \right) dV \tag{1.68}$$

where the only difference between $\mathcal{L}^1$ and $\mathcal{L}^2$ is in the entropy term. Let us suppose that two functions $\{\rho, \psi\} \in \mathcal{U}$ give an extremum of the functional (1.68). By assuming that the discontinuity surface λ is fixed according to the solution's one, if we search for an extremum in the subset $\mathcal{U}$ it obviously follows that the solution will be the same as in the general case (no fixed λ). However, since this problem is equivalent to the traditional one, we can conclude that in the two sub-domains the solution satisfies the classical Euler-Lagrange equations associated with the Lagrangian principle.

Considering a double integral of a general function $u = u\,(x, y)$

$$J = \iint\limits_{\Omega} F\left(x, y, u, u_x, u_y\right) dxdy, \tag{1.69}$$

the first variation became:

$$\delta J = \int_{\partial\Omega} \left(F_{u_x} \frac{dy}{ds} - F_{u_y} \frac{dx}{ds} \right) \delta u \cdot ds + \iint\limits_{\Omega} [F]\, dxdy.$$

Observing that $\frac{dy}{ds}$ and $-\frac{dx}{ds}$ are the components of a normal vector n pointing outside the domain, we can write the first term on the right side in the form

$$\int_{\partial\Omega} \left(F_{u_x} n_x + F_{u_y} n_y\right) \delta u \cdot ds. \tag{1.70}$$

Now, using the expression (1.64) for the integral (1.70), it follows that

$$\delta J = \int_{\lambda} \left(\|F_{u_x}\| \, n_x + \left\|F_{u_y}\right\| n_y\right) \delta u \cdot ds = 0,$$

where n is the normal vector on λ, pointing outside the domain Ω_1. For the arbitrariness of δu, we conclude that

$$\|F_{u_x}\| \, n_x + \left\|F_{u_y}\right\| n_y = 0, \tag{1.71}$$

for each point on λ.

In the derivation of this condition we have assumed that λ is fixed. If we take the variation induced by the arbitrariness of the shock surface we obtain one more condition in the form

$$\|F\| = \left. F_{u_x}\right|_1 \|u_x\| + \left. F_{u_y}\right|_1 \|u_y\|\,, \tag{1.72}$$

where the index 1 means that the expression is evaluated on the Ω_1 side (details on this derivation can be found in Smirnov [34]).

So, the problem can be equivalently solved using the differential approach, with the boundary conditions and the shock conditions like (1.65), or finding an extremum of the Lagrangian principle with the condition (1.67). From a numerical point of view it is thus possible to model the hydrodynamic and magnetohydrodynamic flows with shock surfaces within the variational theory by including the position of the shock among the unknowns.

1.4.1 Axisymmetric hydrodynamics

In the hydrodynamic case the equations necessary to obtain the jump condition across a discontinuity of the flux entropy are the mass conservation equation, previously described, the momentum and the energy conservation equations. The integral formulation of the momentum balance can be written as

$$\oint (\rho \mathbf{V} \cdot \mathbf{n}) \mathbf{V} dS + \oint p \mathbf{n} dS = 0,$$

while the energy conservation yields

$$\oint (\rho \mathbf{V} \cdot \mathbf{n}) \left(e + \frac{V^2}{2} \right) dS + \oint p \mathbf{n} \cdot \mathbf{V} dS = 0,$$

where $e(\mathbf{x})$ represents the specific internal energy of the fluid. By exploiting the notation introduced before, due to the arbitrariness of the integration surfaces we obtain, from the first equation,

$$\| \rho v_n \mathbf{V} + p \mathbf{n} \| = 0, \tag{1.73}$$

and, from the second equation,

$$\left\| \rho v_n \left(e + \frac{V^2}{2} \right) + p v_n \right\| = 0. \tag{1.74}$$

The condition (1.65) can now be used to simplify the term ρv_n in the above equations. Eventually we have the following system of equations that describes univocally the discontinuous changes of the flow properties

$$\| \rho v_n \| = 0, \tag{1.75a}$$

$$\left\| \rho v_n^2 + p \right\| = 0, \tag{1.75b}$$

$$\| v_t \| = 0, \tag{1.75c}$$

$$\| v_\phi \| = 0, \tag{1.75d}$$

$$\left\| \frac{V^2}{2} + h \right\| = 0, \tag{1.75e}$$

where $h = \frac{p}{\rho} + e$ represents the specific fluid's enthalpy. The arrangement of the conditions (1.75), called the *Rankine-Hugoniot's* conditions, yields the well-known shock equation

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2}{2 + (\gamma - 1) M_1^2}, \tag{1.76}$$

where $M_1 = \sqrt{\frac{\rho_1 v_1^2}{\gamma p_1}}$ is the Mach number before the shock surface.

From the foregoing, a different way to determine the Rankine-Hugoniot conditions is by exploiting the variational principle of the model. The continuity of the stream functions still holds and yields to equations (1.75d) and (1.75e)

$$\|v_\phi\| = \left\|\frac{A(\psi)}{r}\right\| = 0,$$

$$\left\|\frac{V^2}{2} + h\right\| = \|B(\psi)\| = 0.$$

By using the definition of ψ, equation (1.12), we obtain

$$\|\rho v_n\| = \frac{1}{r}\|\boldsymbol{\nabla}\psi \cdot \mathbf{t}\| = 0,$$

where the last equality can be deduced from the continuity properties of the stream function in the direction tangential to the shock. This directly implies the condition (1.75a).

Two conditions remain that can be derived from the equations (1.71) and (1.72), in the hypothesis of a discontinuous adiabatic flow. In fact, substituting the expression (1.42), equation (1.71) yields

$$\left\|\frac{1}{r^2}\frac{\boldsymbol{\nabla}\psi}{\rho}\right\| \cdot \mathbf{n} = 0 \tag{1.77}$$

that represent the tangential velocity conservation across the shock surface, equation (1.75c). Due to expression (1.42), the equation (1.72) reduces to

$$|\mathcal{L}| = \frac{1}{r^2\rho}\frac{\partial\psi}{\partial z}\bigg|_1 \left\|\frac{\partial\psi}{\partial z}\right\| + \frac{1}{r^2\rho}\frac{\partial\psi}{\partial r}\bigg|_1 \left\|\frac{\partial\psi}{\partial r}\right\| \tag{1.78}$$

Combining this jump condition with equation (1.77) and with the continuity of the flux function ψ through the shock yields the requested equation (1.76) (the details are worked out in Appendix A).

1.4.2 Generalized Grad-Shafranov equations

Following the arguments of the hydrodynamic model, we now describe the changes in the jump condition for MHD flows with shock transitions. In the momentum and energy equations are added terms related to the magnetic field,

$$\oint (\rho\mathbf{V}\cdot\mathbf{n})\,\mathbf{V}dS + \oint p\mathbf{n}dS + \oint \frac{B^2}{8\pi}\mathbf{n}dS - \oint \left(\frac{\mathbf{B}}{\mathbf{4\pi}}\cdot\mathbf{n}\right)\mathbf{B}dS = 0, \tag{1.79}$$

$$\oint (\rho\mathbf{V}\cdot\mathbf{n})\left(e + \frac{V^2}{2}\right)dS + \oint p\mathbf{n}\cdot\mathbf{V}dS + \oint \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\cdot\mathbf{n}dS = 0. \tag{1.80}$$

Moreover new conditions arise from Maxwell equations (1.5) - (1.7).

The equation (1.79) yields

$$\left\|\rho v_n\mathbf{V} + p\mathbf{n} + \frac{B^2}{8\pi}\mathbf{n} - \frac{B_n}{4\pi}\mathbf{B}\right\| = 0, \tag{1.81}$$

which expresses shows the conservation of momentum in the three components, normal to the discontinuity surface

$$\left\|p + \rho v_n^2 + \frac{B^2}{8\pi} - \frac{B_n^2}{4\pi}\right\| = 0, \tag{1.82}$$

and in the tangent plane

$$\left\|\rho v_n v_t - \frac{B_n B_t}{4\pi}\right\| = 0, \qquad \left\|\rho v_n v_\phi - \frac{B_n B_\phi}{4\pi}\right\| = 0. \tag{1.83}$$

The analogy between the Lagrangian density and the equation (1.82) shows a possible meaning of the variational principle: the solution is an extremum of the total momentum in the flow direction.

From the energy equation it follows that

$$\left\|\rho v_n\left(e + \frac{V^2}{2}\right) + p v_n + \frac{B^2}{4\pi}v_n - (\mathbf{v}\cdot\mathbf{B})\frac{B_n}{4\pi}\right\| = 0,$$

that can be further manipulated using the definition of specific enthalpy and the mass flow condition (1.65) as

$$\left\|h + \frac{V^2}{2} + \frac{B^2}{4\pi\rho} - (\mathbf{v}\cdot\mathbf{B})\frac{B_n}{4\pi\rho v_n}\right\| = 0. \tag{1.84}$$

The Maxwell equations for the magnetic induction yield three more condition. From the divergence equation we obtain

$$\|\mathbf{B}\cdot\mathbf{n}\| = 0, \tag{1.85}$$

and from Faraday's equation

$$\mathbf{n}\times \|\mathbf{E}\| = \mathbf{n}\times \left\| \frac{\mathbf{v}\times\mathbf{B}}{c} \right\| = \mathbf{0}, \tag{1.86}$$

where the second equality derived from the perfect conductivity equation. This last vectorial condition can be expressed in terms of each single component, in the reference system defined by the discontinuity position. It is easily shown that the component parallel to the normal unit vector is nothing. In the azimuthal direction the equation (1.86) becomes

$$\|-v_n B_\phi + v_\phi B_n\| = 0,$$

and the other component tangent to the shock results

$$\|v_n B_t - v_t B_n\| = 0.$$

The system of conditions formed by the equations (1.65), (1.81), (1.85), (1.86) defines the jump conditions for a plasma flow through an entropy discontinuity. Like the hydrodynamic case, this system can be derived from the variational formulation of the problem. The continuity of the stream function ψ yields

$$\|B_n\| = \frac{1}{r}\|\boldsymbol{\nabla}\psi\cdot\mathbf{t}\| = 0 \tag{1.87}$$

and the four functions that depend on ψ give four more of the jump condition,

$$\|F(\psi)\| = \left\|4\pi\frac{\rho v_n}{B_n}\right\| = \|\rho v_n\| = 0,$$

$$\|G(\psi)\| = \left\|\frac{1}{r}\left(v_\phi - \frac{v_n B_\phi}{B_n}\right)\right\| = \|-v_n B_\phi + v_\phi B_n\| = 0,$$

$$\|H(\psi)\| = \left\|4\pi r\frac{\rho v_n}{B_n}v_\phi - rB_\phi\right\| = \left\|\rho v_n v_\phi - \frac{B_n B_\phi}{4\pi}\right\| = 0,$$

$$\|J(\psi)\| = \left\|h + \frac{V^2}{2} + \frac{B^2}{4\pi\rho} - (\mathbf{v}\cdot\mathbf{B})\frac{B_n}{4\pi\rho v_n}\right\| = 0,$$

where equation (1.87) has been used to simplify the term B_n.

By assuming the discontinuity of the plasma density ρ and of the gradient of ψ in the direction normal to the shocks, two more conditions follow. In fact, substituting the expression (1.45) in (1.71) we obtain

$$\left\| \frac{1}{r^2}\left(\frac{F^2}{4\pi\rho} - 1\right) \boldsymbol{\nabla}\psi \right\| \cdot \mathbf{n} = 0, \tag{1.88}$$

that represent the tangential momentum conservation across the shock surface, equation (1.83). Due to expression (1.45), the equation (1.72) reduces to

$$|\mathcal{L}| = \left(\frac{F^2}{4\pi\rho} - 1\right)_1 \left[\frac{1}{4\pi r^2} \frac{\partial\psi}{\partial z}\bigg|_1 \left\| \frac{\partial\psi}{\partial z} \right\| + \frac{1}{4\pi r^2} \frac{\partial\psi}{\partial r}\bigg|_1 \left\| \frac{\partial\psi}{\partial r} \right\| \right]. \tag{1.89}$$

Combining this jump condition with equations (1.88)-(1.89) yields the required remaining equation.

Chapter 2

Numerical Procedure

After the discussion of the theoretical model, we apply a numerical procedure that permits a complete description of fluid and plasma flow and the achievement of a reasonable understanding of the acceleration processes. The variational approach has a fundamental role in our attempts. This permits the use of a simple approximation method, the Ritz method. By scaling the Lagrangian functional it is first possible to render all the equations dimensionless. The extremum is confined in a finite-dimensional functions subspace and the solution is obtained through a system of non-linear equations. This system is solved by using the Newton-Raphson algorithm. A simple semi-analytic solution is finally presented to be utilized as a validation test of the procedure.

2.1 Non-Dimensionalization

In order to obtain a dimensionless model we start scaling the Lagrangian density variables with typical values, values of the size we expect to see or dictated by the geometry. Instead of a large number of physical parameters and variables, all with dimensional units, we are left with equations written in dimensionless variables. All the physical parameters and typical values are collected together into a smaller number of dimensionless parameters (or dimensionless groups) which, when suitably interpreted, should tell us the relative importance of the various mechanisms.

Another advantage of the scaling is the improvement in the numerical precision. This is because the numerical operations are carried out between numbers of the order of the unity.

All of this is much easier to see by working through an example, so we start directly with the hydrodynamic model where we can see the advantages of this technique.

2.1.1 Axisymmetric Hydrodynamics

In both the hydrodynamic and magnetohydrodynamic cases, the natural candidate for the length scale is the inlet spherical radius $r_0 = \sigma_{in}$ of the conical nozzle. Then, for a non-conductive fluid we can scale the fluid density and velocity with the inlet typical values ρ_0 and V_0 and the flux function with $\psi_0 = r_0^2\rho_0 V_0$. Since the azimuthal behavior is governed by a different equation we shall use a different value for the scaling of v_ϕ. We introduce the value $V_{0\phi}$, again related to the inlet condition, and consequently A_0 results

$$A_0 = V_{0\phi} r_0. \tag{2.1}$$

The remaining stream functions can be scaled in different ways but here we adopt these typical values

$$B_0 = \frac{V_0^2}{2}, \tag{2.2}$$

$$C_0 = \frac{p_0}{\rho_0^\gamma}, \tag{2.3}$$

where p_0 is the reference value for the fluid pressure.

Rewriting the Lagrangian density (1.42) in dimensionless units, hereafter designated by the hat sign (*e.g.* $\hat{a}$), we obtain

$$\mathcal{L}\left(\hat{\mathbf{x}}, \hat{\rho}, \hat{\psi}, \hat{\boldsymbol{\nabla}}\hat{\psi}\right) = \frac{1}{2}\rho_0 \left(\frac{\psi_0}{\rho_0 r_0^2}\right)^2 \hat{\rho}\left(\frac{1}{\hat{r}}\frac{\hat{\boldsymbol{\nabla}}\hat{\psi}}{\hat{\rho}}\right)^2 - \frac{1}{2}\rho_0\left(\frac{A_0}{r_0}\right)^2 \hat{\rho}\left[\frac{\hat{A}}{\hat{r}}\right]^2 + \rho_0 B_0 \hat{\rho}\hat{B} - \frac{1}{\gamma - 1}\rho_0^\gamma C_0 \hat{\rho}^\gamma \hat{C}.$$

The dimensionless gradient can be expressed as:

$$\boldsymbol{\nabla} = \frac{1}{r_0}\hat{\boldsymbol{\nabla}}.$$

For the test geometry, the conical nozzle illustrated in Sec. 1.3, we first consider the gradient as expressed in spherical coordinates

$$\boldsymbol{\nabla} = \mathbf{e}_\sigma \frac{\partial}{\partial \sigma} + \mathbf{e}_\theta \frac{1}{\sigma} \frac{\partial}{\partial \theta}.$$

Then we resize the coordinates to obtain a unit square domain:

$$\hat{\boldsymbol{\nabla}} = \mathbf{e}_x \frac{1}{h_x} \frac{\partial}{\partial x} + \mathbf{e}_y \frac{1}{h_y} \frac{\partial}{\partial y},$$

where the elements $h_{x,y}$ defines the new metric adopted. The following equations hold

$$\frac{\partial}{\partial \sigma} = \frac{1}{\Delta\sigma} \frac{\partial}{\partial y} \qquad \frac{1}{\sigma} \frac{\partial}{\partial \theta} = \frac{1}{\theta_{wall}} \left(\frac{1}{\sigma_{in} + \Delta\sigma s} \right) \frac{\partial}{\partial x}.$$

These last expressions can be simplified by putting

$$X_{\rm dim} = \theta_{wall}, \qquad Y_{\rm dim} = \frac{\Delta\sigma}{\sigma_{in}}, \tag{2.4}$$

which imply

$$h_x = X_{\rm dim} \left(1 + Y_{\rm dim} \cdot y \right),$$
$$h_y = Y_{\rm dim}.$$

From equations (2.2-2.3) it follows that

$$\frac{\mathcal{L}}{\frac{1}{2}\rho_0 V_0^2} = \rho \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi}{\rho} \right)^2 - R_0^2 \rho \left[\frac{A}{r} \right]^2 + \rho B - \frac{2M_0^{-2}}{(\gamma - 1)\gamma} \rho^\gamma C, \tag{2.5}$$

where the hats have been dropped and we have introduced the two dimensionless numbers

$$M_0 = \frac{V_0}{a_0} = \frac{V_0}{\sqrt{\gamma \frac{p_0}{\rho_0}}} \qquad \textit{Mach number}, \tag{2.6}$$

$$R_0 = \frac{V_{0\phi}}{V_0} \qquad \textit{Azimuthal velocity ratio}. \tag{2.7}$$

The problem resolution now depends only on the two geometrical factors (2.4) and the two dimensionless numbers (2.6-2.7) in order to characterize different solutions.

2.1.2 Generalized Grad-Shafranov equations

As for the hydrodynamic case, a general simplification for the numerical solution and for the results analysis is obtained through the non-dimensionalization of the model equations. We can begin scaling all the variables with typical values and rewriting the Lagrangian density (1.45)

$$\mathcal{L} = \left(\frac{\psi_0}{r_0^2}\right)^2 \left(\frac{F_0^2}{4\pi\rho_0}\frac{\hat{F}^2}{\hat{\rho}} - 1\right) \frac{1}{8\pi}\left(\frac{\hat{\boldsymbol{\nabla}}\hat{\psi}}{\hat{r}}\right)^2 + \frac{1}{8\pi r_0^2}\left(\frac{H_0\hat{H} + r_0 V_0^{\phi} F_0 \hat{r}\hat{v}_{\phi}\hat{F}}{r}\right)^2 - \frac{1}{2}\rho_0\left(V_0^{\phi}\right)^2 \hat{\rho}\hat{v}_{\phi}^2 + \rho_0\hat{\rho}\left(J_0\hat{J} + r_0 V_0^{\phi} G_0 \hat{r}\hat{v}_{\phi}\hat{G}\right) - \frac{1}{\gamma - 1}\rho_0^{\gamma} I_0 \hat{\rho}^{\gamma}\hat{I} \tag{2.8}$$

In this expression we have again denoted the *dimensionless* quantities by the hat sign '$\hat{a}$' and the characteristic scale-lengths by the subscript zero 'a_0'. The length scale is the inlet spherical radius $r_0 = \sigma_{in}$ of the conical nozzle, B_0 is the typical value of the magnetic field and we can scale the flux function with $\psi_0 = r_0^2 B_0$. The dimensionless gradient can be expressed as

$$\boldsymbol{\nabla} = \frac{1}{r_0}\hat{\boldsymbol{\nabla}}$$

where the definition of $\hat{\boldsymbol{\nabla}}$ and the two geometrical parameters $(X_{\text{dim}}, Y_{\text{dim}})$ are the same defined in the previous subsection.

Then we take some more assumptions on the non-dimensional quantities and we scale $F ... J$ respectively with

$$F_0 = 4\pi\frac{\rho_0 V_0}{B_0},$$
$$G_0 = \frac{V_0}{2r_0},$$
$$H_0 = r_0 B_0,$$
$$I_0 = \frac{p_0}{\rho_0^{\gamma}},$$
$$J_0 = \frac{V_0^2}{2}.$$

By using these definitions it follows that

$$\frac{\mathcal{L}}{\frac{1}{8\pi}\left(\frac{\psi_0}{r_0^2}\right)^2} = \left(M_1^2\frac{F^2}{\rho} - 1\right)\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 + \left(\frac{H + M_1 M_2 r v_{\phi} F}{r}\right)^2 - ... \tag{2.9}$$
$$-M_2^2\rho v_{\phi}^2 + M_1^2\rho J + M_1 M_2 r\rho v_{\phi} G - \frac{M_1^2}{M_0^2}\frac{2}{(\gamma - 1)\gamma}\rho^{\gamma} I,$$

where the hats have been dropped and we have introduced the three dimensionless numbers

$$M_0 = \frac{V_0}{a_0} = \frac{V_0}{\sqrt{\gamma \frac{p_0}{\rho_0}}} \qquad \textit{Mach number,} \tag{2.10}$$

$$M_1 = \frac{V_0}{V_{A0}} = \sqrt{\frac{F_0^2}{4\pi\rho_0}} \qquad \textit{Mach-Alfven number,} \tag{2.11}$$

$$M_2 = \frac{V_{0\phi}}{V_{A0}} = \sqrt{\frac{4\pi\rho_0 V_{0\phi}^2}{B_0^2}} \qquad \textit{Mach-Alfven azimuthal number.} \tag{2.12}$$

We are left with an equation written in dimensionless variables and all the physical parameters and typical values are collected together into a smaller number of dimensionless parameters. Again the solution depends on the two geometrical factors (2.4) and on the three dimensionless numbers (2.10-2.12).

2.2 The Ritz Method

In giving an existence proof for a solution to a variational problem one requires the existence of minimizing sequences, with suitable convergence and functional properties. In practical applications there still remains the problem of actually constructing a minimizing sequence, and, furthermore, one which converges with a fair degree of rapidity. The method described below was first introduced by W. Ritz [27][28], who applied it to problems concerning elastic plates.

We consider a variational integral $I(\Phi)$ defined over a compact set $\mathcal{R}$ of admissible functions. A numerable sequence of functions $w_1, \ldots, w_n$ contained in the class $\mathcal{R}$ is said to be *complete* if every function Φ in $\mathcal{R}$ can be approximated by a finite linear combination

$$W_n = a_1 w_1 + a_2 w_2 + \ldots + a_n w_n$$

of functions belonging to the sequence $\{w_n\}$ with pre-assigned accuracy. The approximation can be understood in several senses. Given any Φ in $\mathcal{R}$ and any ε, we want a W_n such that

$$
\begin{aligned}
&(a) \qquad |I(\Phi) - I(W_n)| < \varepsilon, \\
&(b) \qquad \int_\Omega (\Phi - W_n)^2 \, d\mathbf{x} < \varepsilon, \\
&(c) \qquad |\Phi - W_n| < \varepsilon.
\end{aligned}
$$

In the following we shall take the approximation in the sense (a).

For example, we know from the theory of Fourier series that the sequence of functions

$$\sin(n\pi x) \qquad (n = 1, 2, \ldots)$$

forms a complete system for all functions $\Phi(x)$ which are continuous, have a piecewise continuous derivative, and vanish at 0 and 1.

Except for the trigonometric functions, the most important and most useful complete system is given by the integer powers of x, or, in two dimensions, $x^n y^m$. The linear combination of such functions are polynomials. Weierstrass proved the following important theorem:

If $f(x)$ is an arbitrary continuous function in a closed interval, then it may be approximated in this interval to any desired degree of accuracy by a polynomial $P_n(x)$, provided that n is taken sufficiently large. This theorem is valid for higher dimensions as well.

Returning to the given variational integral $I(\Phi)$, we suppose, in order that the problem make sense, that the integral has a greatest lower bound d. It immediately follows the existence of minimizing sequences Φ_n such that $I(\Phi_n) \to d$. The Ritz method consists in replacing the minimizing sequence by means of a sequence of auxiliary minimum problems.

We consider, for a fixed n, the integral

$$I(W_n) = I(a_1 w_1 + a_2 w_2 + \ldots + a_n w_n),$$

where $w_1, \ldots, w_n$ are the first n numbers of a complete system $\{w_n\}$ of the admissible functions of the set $\mathcal{R}$. Then, the integral becomes a function of the n coefficients $a_1, \ldots, a_n$ varying independently. We next consider the problem of finding the set of coefficients $a_1, \ldots, a_n$ which makes $I(W_n)$ a minimum. Since I has a lower bound and depends continuously on the n

parameters $a_1, \ldots, a_n$, it must attain a minimum; according to the ordinary theory of maxima and minima, the system of n equations

$$\frac{\partial}{\partial a_i} I\left(W_n\right) = 0$$

permits to determine the particular values $a_i = c_i$ which give the minimum. We denote the minimizing function by $u_n = c_1 w_1 + \ldots + c_n w_n$. The essence of the Ritz method is then contained in the following theorem:

The sequence of functions $u_1, \ldots, u_n$, which are the solutions to the successive minimum problems $I\left(W_n\right)$ formed for each n, are a minimizing sequence to the original variational problem.

First, it is seen that $I\left(u_n\right)$ is a monotonically decreasing function of n, since we may regard every function W_{n-1} admissible in the $(n-1)^{th}$ minimum problem as an admissible function for the n^{th} minimum problem with the additional side condition $a_n = 0$. Therefore

$$I\left(u_n\right) \to \delta \geq d.$$

Next, the existence of a minimizing sequence $\{\Phi_n\}$ to the variational problem implies that, for some sufficiently large k,

$$I\left(\Phi_k\right) < d + \frac{\varepsilon}{2}.$$

Since the system $w_1, \ldots, w_n$ is complete, there exists a suitable function $W_n = a_1 w_1 + \ldots + a_n w_n$ such that

$$I\left(W_n\right) < I\left(\Phi_k\right) + \frac{\varepsilon}{2}.$$

But, by definition of u_n,

$$I\left(u_n\right) \leq I\left(W_n\right),$$

hence

$$I\left(u_n\right) < d + \varepsilon,$$

which establishes the convergence of $I\left(u_n\right)$ to d.

The process of constructing the minimizing sequence $\{u_n\}$ depends on solving the system of n equations:

$$\frac{\partial}{\partial a_i} I\left(W_n\right) = 0.$$

The process is considerably simplified if the given functional is quadratic, since in that case we have a system of linear equations in the a's.

As an example of the Ritz method, let us consider the case where on the boundary $\Phi = g$ is a polynomial, the boundary being given by $B\left(x,y\right) = 0$. If we take for functions Φ the functions

$$\Phi = g + B\left(x,y\right)\left(a + bx + cy + \ldots\right),$$

this sequence of functions Φ is a minimizing sequence, and $I\left(\Phi\right)$ is a function $I\left(a,b,c,\ldots\right)$ of the coefficients $a,b,c,\ldots$, and the problem is reduced to finding the minimum of I with respect to $a,b,c,\ldots$.

2.2.1 Axisymmetric Hydrodynamics

By exploiting the Rayleigh-Ritz Method, we search for a simple approximation of ψ and ρ through the extremization of the functional (1.42). We start assuming that the approximation functions belong to a finite dimensional subspace of the solution space, thus considering ψ and ρ sums of base-function. These can be written in the form

$$\psi\left(x,y\right) = \sum_{n=0}^{n_T} \psi_n F_n\left(x,y\right) = \sum_{i=0}^{n_i}\sum_{j=0}^{n_j} \psi_{ij} F_i^x\left(x\right) F_j^y\left(y\right), \tag{2.13}$$

$$\rho\left(x,y\right) = \sum_{m=0}^{m_T} \rho_m G_m\left(x,y\right) = \sum_{i=0}^{m_i}\sum_{j=0}^{m_j} \rho_{ij} G_i^x\left(x\right) G_j^y\left(y\right), \tag{2.14}$$

where $F^{x,y}$ and $G^{x,y}$ are two families of base-functions. The two series are truncated and n_T and m_T are the numbers of functions used. Some common choices are:

$$F_i^{x,y}\left(t\right), G_i^{x,y}\left(t\right) \begin{cases} t^i \\ \cos\left(it\right) \\ \ldots \end{cases}.$$

It is usually preferred to approximate the solution with smooth functions, choosing the base that best represents the solution as we suppose it will be. It is also important to avoid singularities in the integrals of these functions. Substitution of this expression into (1.36) implies

$$L\left(\psi_1,...\psi_{n_T},\rho_1,...,\rho_{m_T}\right)=\int_\Omega \mathcal{L}\left(\mathbf{x},\rho\left(\rho_m\right),\psi\left(\psi_n\right),\boldsymbol{\nabla}\psi\left(\psi_n\right)\right)dV, \tag{2.15}$$

and the extremization process can be developed differentiating (2.15) with respect to ψ_n and ρ_m

$$\frac{\partial L}{\partial \psi_n}=\int_\Omega \frac{\partial \mathcal{L}}{\partial \psi_n}dV \qquad \frac{\partial L}{\partial \rho_m}=\int_\Omega \frac{\partial \mathcal{L}}{\partial \rho_m}dV.$$

From (2.5) we have

$$\frac{\partial \mathcal{L}}{\partial \psi_n}=P_n\left(\psi_1,...\psi_{n_T},\rho_1,...,\rho_{m_T}\right), \tag{2.16}$$

where $P_n\left(\psi_1,...\psi_{n_T},\rho_1,...,\rho_{m_T}\right)$ has the expression

$$P_n=\frac{2}{r^2}\frac{\boldsymbol{\nabla}\psi}{\rho}\cdot\boldsymbol{\nabla}f_n+\rho\left[-2R_0^2\frac{A\dot{A}}{r^2}+\dot{B}-\frac{2M_0^{-2}}{\left(\gamma-1\right)\gamma}\rho^{\gamma-1}\dot{C}\right]\cdot f_n,$$

and

$$\frac{\partial \mathcal{L}}{\partial \rho_m}=R_m\left(\psi_1,...\psi_{n_T},\rho_1,...,\rho_{m_T}\right), \tag{2.17}$$

with $R_m\left(\psi_1,...\psi_{n_T},\rho_1,...,\rho_{m_T}\right)$ given by

$$R_m=\left[-\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi}{\rho}\right)^2-R_0^2\left(\frac{A}{r}\right)^2+B-\frac{2M_0^{-2}}{\left(\gamma-1\right)}\rho^{\gamma-1}C\right]\cdot g_m.$$

Hence the approximation coefficients are a solution of the non-linear system of equations

$$\begin{bmatrix}P_1 & \cdots & P_{n_T} & R_1 & \cdots & R_{m_T}\end{bmatrix}=0. \tag{2.18}$$

Observe that in the incompressible case the problem can be linear if the condition $A\propto\psi$ and $B\propto\psi^2$ hold.

2.2.2 Generalized Grad-Shafranov equations

Again we start assuming that the approximation functions belong to a finite dimensional subspace of the solution space. Considering the expansion of ψ, ρ and v_ϕ as sums of base-functions, the three unknown functions can be written in the form:

$$\psi\left(x,y\right)=\sum_{n=0}^{n_T}\psi_n F_n\left(x,y\right)=\sum_{i=0}^{n_i}\sum_{j=0}^{n_j}\psi_{ij}F_i^x\left(x\right)F_j^y\left(y\right), \tag{2.19}$$

$$\rho\left(x,y\right)=\sum_{m=0}^{m_T}\rho_m G_m\left(x,y\right)=\sum_{i=0}^{m_i}\sum_{j=0}^{m_j}\rho_{ij}G_i^x\left(x\right)G_j^y\left(y\right), \tag{2.20}$$

$$v_\phi\left(x,y\right)=\sum_{l=0}^{l_T}v_{\phi l} H_l\left(x,y\right)=\sum_{i=0}^{l_i}\sum_{j=0}^{l_j}v_{\phi ij}H_i^x\left(x\right)H_j^y\left(y\right), \tag{2.21}$$

where $F^{x,y}, G^{x,y}$ and $H^{x,y}$ are three families of base-functions and n_T, m_T and l_T are the number of functions used.

Following the same arguments as before, we obtain

$$L=\int_\Omega \mathcal{L}\left(\mathbf{x}, v_\phi\left(v_{\phi l}\right), \rho\left(\rho_m\right), \psi\left(\psi_n\right), \boldsymbol{\nabla}\psi\left(\psi_n\right)\right)dV, \tag{2.22}$$

and the extremization process can be developed differentiating (2.15) with respect to three sets of coefficients $\{\psi_n,\ \rho_m, v_{\phi l}\}$

$$\frac{\partial L}{\partial \psi_n}=\int_\Omega \frac{\partial \mathcal{L}}{\partial \psi_n}dV \qquad \frac{\partial L}{\partial \rho_m}=\int_\Omega \frac{\partial \mathcal{L}}{\partial \rho_m}dV \qquad \frac{\partial L}{\partial v_{\phi l}}=\int_\Omega \frac{\partial \mathcal{L}}{\partial v_{\phi l}}dV.$$

The arguments of the integrals, deduced from the dimensionless expression (2.9) of the Lagrangian density, result:

$$\frac{\partial \mathcal{L}}{\partial \psi_n}=P_n\left(\psi_1,\ldots,\psi_{n_T},\rho_1,\ldots,\rho_{m_T},v_{\phi 1},\ldots,v_{\phi l_T}\right), \tag{2.23}$$

with

$$P_n = \left(M_1^2\frac{F^2}{\rho} - 1\right)\frac{2}{r^2}\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}f_n +$$
$$\left[2M_1^2\frac{F\dot{F}}{\rho}\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 + 2\left(\frac{H + M_1M_2rv_\phi F}{r}\right)\left(\frac{\dot{H} + M_1M_2rv_\phi\dot{F}}{r}\right) +\right.$$
$$\left. M_1^2\rho\dot{J} + M_1M_2r\rho v_\phi\dot{G} - \frac{M_1^2}{M_3^2}\frac{2}{(\gamma-1)\gamma}\rho^\gamma\dot{I}\right]\cdot f_n, \tag{2.24}$$

and

$$\frac{\partial\mathcal{L}}{\partial\rho_m} = R_m\left(\psi_1, ...\psi_{n_T}, \rho_1, ..., \rho_{m_T}, v_{\phi 1}, \ldots, v_{\phi l_T}\right), \tag{2.25}$$

$$R_m = \left[-M_1^2\frac{F^2}{\rho^2}\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 - M_2^2v_\phi^2 + M_1^2J + M_1M_2rv_\phi G - \frac{M_1^2}{M_3^2}\frac{2}{(\gamma-1)}\rho^{\gamma-1}I\right]\cdot g_m, \tag{2.26}$$

$$\frac{\partial\mathcal{L}}{\partial v_{\phi l}} = V_l\left(\psi_1, ...\psi_{n_T}, \rho_1, ..., \rho_{m_T}, v_{\phi 1}, \ldots, v_{\phi l_T}\right), \tag{2.27}$$

$$V_l = \left[2\left(\frac{H + M_1M_2rv_\phi F}{r}\right)M_1M_2F - 2M_2^2\rho v_\phi + M_1M_2r\rho G\right]\cdot h_l. \tag{2.28}$$

The approximation coefficients are a solution of the non-linear equation system

$$\begin{bmatrix}P_1 & \cdots & P_{n_T} & R_1 & \cdots & R_{m_T} & V_1 & \cdots & V_{l_T}\end{bmatrix} = 0. \tag{2.29}$$

We should notice that in both cases it is in general not possible to assert that the solution is a minimum (or a maximum) of the Lagrangian of the system. Thus the convergence theorem for the Ritz method applies only if some more assumptions hold. In particular, if it is possible to determine an *a priori* bound of the term $\nabla\psi$, this permits a satisfactory condition for the theorem to hold. A more complete analysis of this issue which is related bot to the structure of the equations and to the regimes considered will be the addressed in future works.

2.3 Newton-Raphson Algorithm

Since each equation of the non-linear equation's system (2.18) is a smooth function of unknown coefficients and can be easily differentiated with respect to these coefficients, we decide to exploit the Newton-Raphson algorithm [26]. Generally, given a vectorial equation like

$$\mathbf{f}\left(\bar{\mathbf{x}}\right) = 0, \tag{2.30}$$

the search for its solutions is equivalent to asking for a fixed point of the mapping

$$\mathbf{x} = \mathbf{x} - \mathbf{H}\left(\mathbf{x}\right) \cdot \mathbf{f}\left(\mathbf{x}\right), \tag{2.31}$$

where $\mathbf{H}\left(\mathbf{x}\right)$ is an arbitrary non-degenerate matrix. The Newton-Raphson algorithm is obtained choosing

$$\mathbf{H}\left(\mathbf{x}\right) = \mathbf{J}^{-1}\left(\mathbf{x}\right),$$

with $\mathbf{J}\left(\mathbf{x}\right)$ representing the Jacobian matrix of the system (2.18). Thus, we can rewrite equation (2.31) in iterative terms as

$$\mathbf{x}^{i+1} = \mathbf{x}^{i} - \mathbf{J}^{-1}\left(\mathbf{x}^{i}\right) \mathbf{f}\left(\mathbf{x}^{i}\right), \tag{2.32}$$

and solve at each step the linear system

$$\mathbf{J}\left(\mathbf{x}^{i}\right) \cdot \left(\mathbf{x}^{i+1} - \mathbf{x}^{i}\right) = -\mathbf{f}\left(\mathbf{x}^{i}\right).$$

A simple convergence estimate holds for the Newton-Raphson method. Let $\mathbf{f}\left(\mathbf{x}\right) \in \mathbf{C}^{2}\left(\mathbf{D}\right)$ and $\bar{\mathbf{x}} \in \mathbf{D}$ the solution of (2.30), where $\mathbf{D}$ is the domain of the problem. If the Jacobian matrix of $\mathbf{f}\left(\mathbf{x}\right)$ is non-degenerate in $\mathbf{D}$, therefore there exists in $\mathbf{D}$ a neighborhood $\mathbf{I}$ of $\bar{\mathbf{x}}$ where the sequence (2.32) converges to $\bar{\mathbf{x}}$ for each $\mathbf{x}^{0} \in \mathbf{I}$.

Furthermore, defined as a generic vectorial norm $\left\|\cdot\right\|$, there exists a β such as

$$\left\|\mathbf{x}^{i+1} - \bar{\mathbf{x}}\right\| \leq \beta \cdot \left\|\mathbf{x}^{i} - \bar{\mathbf{x}}\right\|^{2},$$

i.e. the convergence is almost quadratic.

2.3.1 Axisymmetric Hydrodynamics

In this case the vectorial non-linear function is given by equations (2.16-2.17) with

$$\mathbf{x} = [\psi_1, \ldots, \psi_{n_l}, \rho_{1,\ldots,}\rho_{m_l}],$$

while the elements of $\mathbf{J}(\mathbf{x})$ are defined as

$$J_{ij} = \frac{\partial f_i}{\partial x_j} = \int_\Omega \frac{\partial}{\partial x_j}\left[\frac{\partial \mathcal{L}}{\partial x_i}\right] dV = J_{ji}. \tag{2.33}$$

Details on the analithical expressions obtained substituting the Lagrangian density (1.42) into equation (51) can be found in Appendix B.

2.3.2 Generalized Grad-Shafranov equations

In this case the vectorial non-linear function is given by equations (2.23-2.27) with

$$\mathbf{x} = [\psi_1, \ldots, \psi_{n_l}, \rho_{1,\ldots,}\rho_{m_l}, v_{\phi 1,\ldots,}v_{\phi l_l}],$$

while the elements of $\mathbf{J}(\mathbf{x})$ are

$$J_{ij} = \frac{\partial f_i}{\partial x_j} = \int_\Omega \frac{\partial}{\partial x_j}\left[\frac{\partial \mathcal{L}}{\partial x_i}\right] dV = J_{ji}.$$

Details on the analithical expressions obtained substituting the Lagrangian density (1.42) into equation (51) can be found in Appendix B.

2.4 Test Cases

Before using the numerical algorithm on the experimental configurations, to prove the validity of our assumptions some simple cases have been considered. These are particular solutions of the hydrodynamic and MHD models that can be expressed in analytic form or easily interpolated. More important, for the hydrodynamic model a solution with an entropy discontinuity has been determined. Thus the variational procedure described for shock capture can be tested.

In the following subsections these solutions are briefly presented and discussed.

2.4.1 Axisymmetric Hydrodynamics

For the very simple case of constant boundary condition ($\mathbf{v} \cdot \mathbf{n}|_i = V_0,\ \rho|_i = \rho_0$ and $p|_i = p_0$) and zero azimuthal velocity ($v_\phi = 0$) it is possible to determine an analytic solution of the conical nozzle problem.

We start deducing the three stream functions $[A, B, C]$ from the inlet conditions: the zero azimuthal velocity implies

$$A(\psi) = 0.$$

and the Bernoulli equation written on the inlet surface leads to

$$B(\psi) = \frac{V_0^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} = B_0,$$

where the flux entropy is uniform and the related function results

$$C(\psi) = \frac{p_0}{\rho_0^\gamma}.$$

By simplifying the terms on the right hand side of equation (1.21), we obtain, in spherical coordinates,

$$\frac{\partial}{\partial \sigma}\left[\frac{1}{\rho}\frac{\partial \psi}{\partial \sigma}\right] + \frac{\sin(\theta)}{\sigma^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin(\theta)}\frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\right] = 0,$$

where the density derived from the algebraic Bernoulli relation (1.17)

$$B_0 = \frac{1}{2}\left(\frac{1}{\sigma \sin(\theta)}\frac{\boldsymbol{\nabla}\psi}{\rho}\right)^2 + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0^\gamma}\rho^{\gamma-1}. \tag{2.34}$$

Let us consider a stream function like

$$\psi = \sigma_0^2 \rho_0 V_0 \left[1 - \cos(\theta)\right] \tag{2.35}$$

and substitute this in the equation (2.34). We obtain a polynomial equation for the density field

$$\left(\frac{\rho}{\rho_0}\right)^{\gamma+1} - (K+1)\left(\frac{\rho}{\rho_0}\right)^2 + K\left(\frac{\sigma}{\sigma_0}\right)^{-4} = 0, \tag{2.36}$$

with K defined by

$$K = \frac{V_0^2}{2}\frac{\gamma - 1}{\gamma}\frac{\rho_0}{p_0} = \frac{\gamma - 1}{2}M_0^2. \tag{2.37}$$

From equation (2.36) it follows that the density field depends on the spherical radius only

$$\rho = \rho(\sigma). \tag{2.38}$$

Expressions (2.35) and (2.38),as can be checked by substitution in (1.21), provide a solution of the hydrodynamic problem.

The equation (2.36), although it is not in general explicitly solvable with respect to ρ, can be numerically interpolated and the results can be compared with that obtained from the variational algorithm. Solutions of equation (2.36) are presented in fig (2-1).

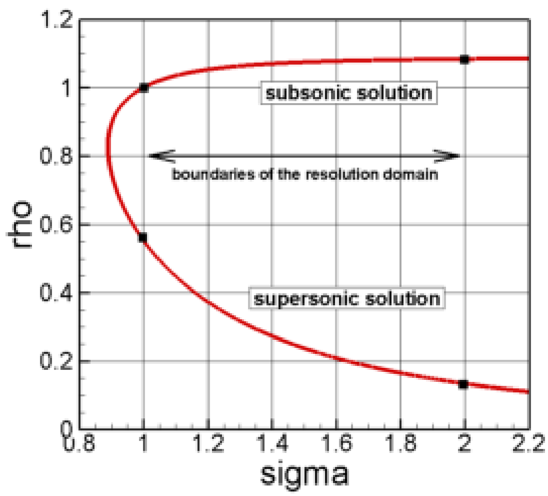

Figure 2-1: Behaviour of a particular solution for the density ρ

It is easy to see that the density profile presents two solutions depending on the flow regime in the inlet region, in fig (2-1) the upper branch shows a typical subsonic flow ($M < 1$) while the lower decreasing branch characterizes the supersonic ($M > 1$) regimes. We also notice that for a smaller inlet radius the solutions still exist if and only if $\sigma_i \leq \sigma_{\min}$, where $\sigma_{\min}$ is the point of inlet transonic condition.

Considering an entropy jump ΔS inside the domain, the flux function ψ defined by equation

(2.35) is still a solution of the problem while in figure (2-2) we have shown both the density profiles associated with S and $S + \Delta S$. The jump condition (1.76) is plotted over the two solutions and the crossing point gives the shock position.

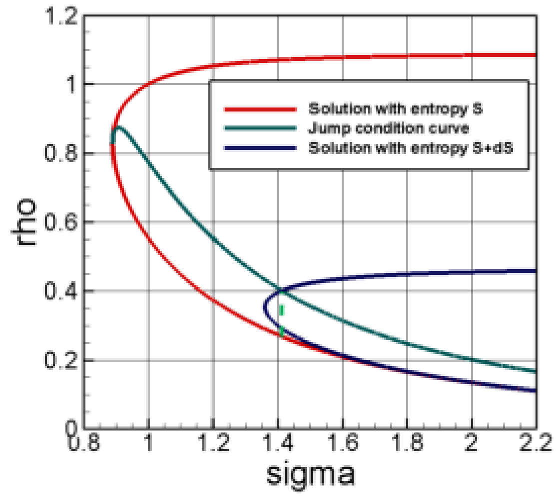

Figure 2-2: Behaviour of two density solutions with different entropy (the jump conditions are also shown that lead to a discontinuous solution)

2.4.2 Generalized Grad-Shafranov equations

Despite of the greater complexity of the conductive fluid equations, a solution exists close to the hydrodynamic one already presented. Both the equations and the numerical algorithm only slightly differ from the previous case and the solution consists of a plasma flow parallel to the magnetic field, with no further interactions.

For the very simple case of constant inlet conditions ($\mathbf{B} \cdot \mathbf{n}|_i = B_0$, $\mathbf{v} \cdot \mathbf{n}|_i = V_0, \rho|_i = \rho_0$ and $p|_i = p_0$), zero azimuthal velocity ($v_\phi = 0$) and zero azimuthal magnetic field ($B_\phi = 0$), it is possible to determine an analytic solution of the conical nozzle problem. First, we deduce the five stream functions $[F, G, H, I, J]$ from the inlet conditions: the ratio between the mass flow rate and the magnetic field in the poloidal plane gives

$$F(\psi) = \frac{4\pi\rho_0 V_0}{B_0} = F_0,$$

the zero azimuthal velocity and azimuthal magnetic field imply

$$G(\psi) = 0, \qquad H(\psi) = 0,$$

the Bernoulli equation written at the inlet surface leads to

$$J(\psi) = \frac{V_0^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0} = J_0,$$

and the flux entropy is uniform and the related function is

$$I(\psi) = \frac{p_0}{\rho_0^\gamma}.$$

By simplifying the terms on the right hand side of equation (1.35), we obtain in spherical coordinates,

$$\frac{\partial}{\partial\sigma}\left[\frac{1}{\rho}\frac{\partial\psi}{\partial\sigma}\right] + \frac{\sin\theta}{\sigma^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{1}{\rho}\frac{\partial\psi}{\partial\theta}\right] = 0,$$

where the density ρ is given by the algebraic Bernoulli equation

$$J_0 = \frac{1}{2}\left(\frac{F_0}{\sigma\sin\theta}\frac{\boldsymbol{\nabla}\psi}{4\pi\rho}\right)^2 + \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0^\gamma}\rho^{\gamma-1}. \tag{2.39}$$

Considering a stream function of the form $\psi = \sigma_0^2 B_0\left[1-\cos\theta\right]$ and inserting it into equation (2.39), we find that the density depends only on the spherical radius, $\rho = \rho(\sigma)$, and obeys the polynomial equation

$$(\rho/\rho_0)^{\gamma+1} - (K+1)(\rho/\rho_0)^2 + K(\sigma/\sigma_0)^{-4} = 0, \tag{2.40}$$

with K defined as in equation (2.37).

The equation (2.40) can be interpolated numerically and the result can be compared with that obtained from the variational algorithm. The flow velocity in the inlet region is taken to be equal to the speed of sound and the flow after it is supposed to be supersonic. Both subsonic and supersonic solutions of equation (2.40) are shown in figure (2-3).

In figure (2-3) the upper branch shows a typical subsonic flow ($M < 1$) while the lower (decreasing) branch is characteristic of supersonic ($M > 1$) regimes.

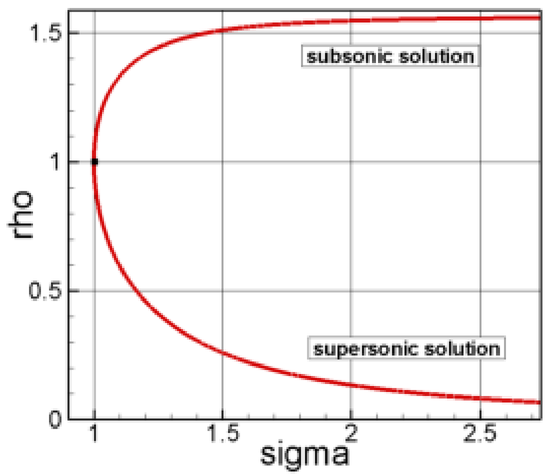

Figure 2-3: Subsonic and supersonic solutions for the density ρ as a function of the spherical radius σ for inlet sonic conditions. Both curves are plotted in dimensionless units.

An example of thruster operation

For better testing our procedure, we explore a configuration closer to the condition of thruster operation, that is a case where the electric field influences the flow behaviour. As before, we consider a conical nozzle. The function $G(\psi)$, that defines the electric field in the plasma, and the function $H(\psi)$ are both related to the azimuthal inlet velocity and azimuthal magnetic field. We choose these two functions in the inlet region so as to give

$$G(\psi) = G_0 \qquad H(\psi) = 0,$$

and leave all other quantities unchanged. Thus, the values of $F(\psi)$ and $I(\psi)$ remain the same as in the previous case. For the azimuthal velocity we obtain

$$v_\phi\big|_i = V_{\phi 0} \sin\theta$$

and $G_0 = (V_{\phi 0}/\sigma_0)\left(1 - M_1^2\right)$. The generalized Bernoulli equation becomes

$$J(\psi) = [V_0^2 + V_{\phi 0}^2 \sin^2\theta]/2 + [\gamma/(\gamma - 1)]p_0/\rho_0 - \sigma_0 V_{\phi 0} G_0 \sin^2\theta, \tag{2.41}$$

where $\theta|_i = \theta(\psi)$ can be obtained by integrating $\mathbf{B}\cdot\mathbf{n}|_i$ which gives

$$\psi|_i = \sigma_0^2 B_0 \left[1 - \cos(\theta)\right].$$

Thus, equation (2.41) can be rewritten as the quadratic form

$$J(\psi) = J_0 + J_1(\psi/\psi_0) + J_2(\psi/\psi_0)^2$$

where $\psi_0 = \sigma_0^2 B_0$ and

$$J_0 = (V_0^2/2)\,\left[1 + 1/[(\gamma - 1)M_0^2]\right],$$

$$J_1 = V_{\phi 0}^2\left(2M_1^2 - 1\right), \qquad J_2 = -J_1/2.$$

Chapter 3

Summary and Conclusions

The theoretical model described in the first part has been discussed in its fundamental features: the governing equations; a variational principle; boundary and jump conditions. In the second part, a numerical approach based on the variational formulation has been described in detail. Then, a test for the validation of the resolution algorithm is presented.

We have implemented a numerical code by using the C++ language. The procedure we followed has a great advantage in determining the code's complexity, which appears to be acceptable. The algorithm performance has been evaluated only for the test cases, which are relatively simpler with respect to the simulation we are interested in. In the next sections the results are finally presented.

3.1 Validation Results

This thesis describes the theoretical model and numerical simulation of a typical thruster, based on both gas or plasma acceleration through a shaped nozzle. The results obtained in the testing phase are presented in the following sections.

3.1.1 Axisymmetric Hydrodynamics

Exploiting the known results previously described (cfr. Sec. (2.4)), the first tests carried out are intended to verify the procedure. A first validation was performed on the incompressible model, in the limit of gamma that tends to infinity. The fundamental fields we are interested in are the flux function ψ and its derivatives

$$v_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \qquad v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z},$$

that give us the velocity profile. The comparison between the exact solution and our approximated result is very satisfying, as can be seen in fig. (3-1).

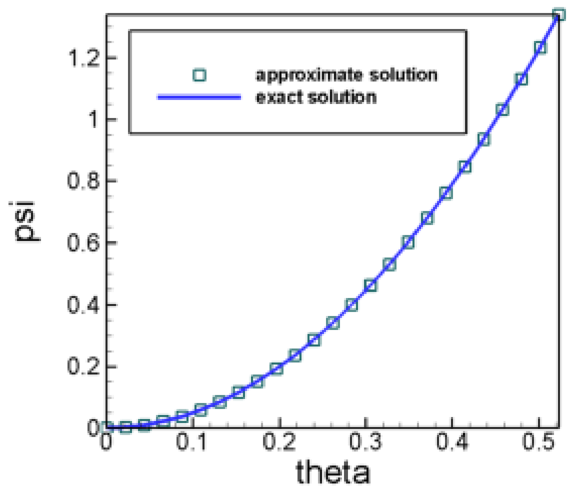

Figure 3-1: The behavior of the fluid flux function ψ as a function of the polar angle θ is shown (squares) together with the exact solution (straight line) for the case of an incompressible hydrodynamic flow.

Then, a second validation was carried out on the full model, including the compressibility effects, and investigation was concerned with both the density field and the flux functions. Since we have decided to use an iterative algorithm, in order to reach the solution the choice of a specific initial condition (in particular the density field) depends on whether the inlet condition is supersonic or subsonic. Although this is a crucial limitation of the adopted procedure, it can be easily seen that a wide class of simple initial conditions leads the approximation to converge to the expected values. As illustrated in figures (3-3-3-2), the same satisfying results can be obtained for the subsonic flows, figure (3-2), as well as for the supersonic ones, figure (3-3).

We conclude by recording some results obtained with an entropy jump condition. Figure (3-4) shows the behavior of the Lagrangian function with respect to the shock's position and

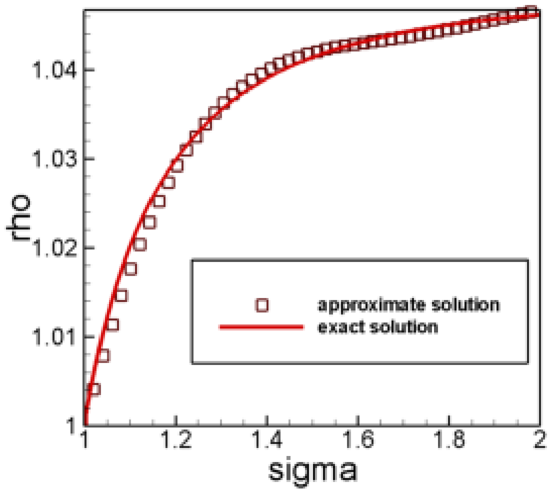

Figure 3-2: The behavior of the fluid density ρ as a function of spherical radius σ is shown (squares) together with the exact solution (straight line) for the case of a compressible subsonic hydrodynamic flow.

in figure (3-5) the solution is shown. The extremum found interpolating the points defines the shock position with a relative error of less than 10^{-3}. Moreover, we should notice that to approximate the solution with a relatively low error ($\varepsilon_{\max} \sim 10^{-3}$) we only need few base functions ($m_T = 4$). Even for flow regimes with discontinuities (*i.e.* shocks), the proposed method has turned out to be accurate in describing general flow properties, proving itself to be a reliable instrument for the study of axial symmetric hydrodynamic flows.

3.1.2 Generalized Grad-Shafranov equations

The results for the simple problems described before appear to be extremely satisfactory. In the first case described, the fields we are interested in are the two unknown functions (ψ, ρ). The comparison between the exact solution and our approximated results is good, as can be seen in figures (3-6)-(3-7)

The numerical solution, obtained with the variational method, of the example of thruster operation, is illustrated in figures (3-8)-(3-10).

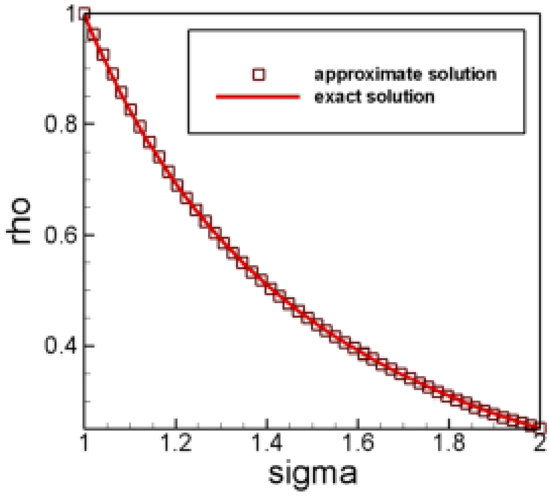

Figure 3-3: The behavior of the fluid density ρ as a function of the spherical radius σ is shown (squares) together with the exact solution (straight line) for the case of a compressible supersonic hydrodynamic flow.

3.2 Further Developments

Some important questions that we intend to analyze remain. A part of this work has been carried out while writing down the present thesis. However, results are not yet available and these additions will thus be only briefly presented here. Two problems with the numerical procedure have been so far identified:

Ritz Method Continuous base functions defined for the whole domain failed to describe equilibria within regions of relatively high plasma concentration. In particular, a solution with a sharp density profile increases, due to the iterative algorithm, the chance of having regions with negative density values.

Domain's Geometry The main assumption of the numerical procedure is related to the domain. To give a simple form to the boundary condition only simple geometries can be used. A more complex domain, closer to the experimental one, can be used, but a suitable system of coordinates should be chosen in order to transform the integration domain into the unit square.

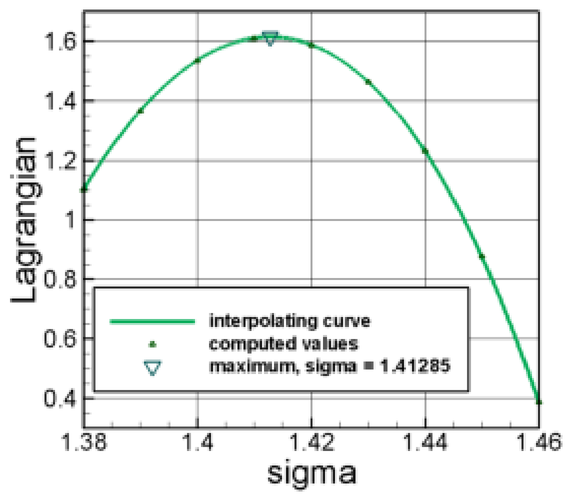

Figure 3-4: Lagrangian values evaluated at different shock positions for a hydrodynamic flow. The extremum indicates the correct position of the flow discontinuity.

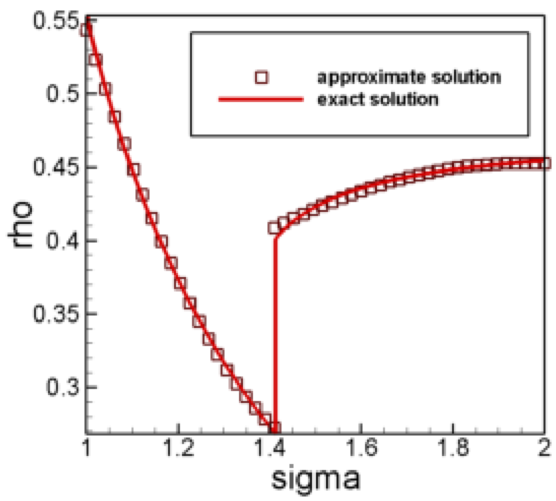

Figure 3-5: The behavior of the fluid density ρ as a function of the spherical radius σ is shown (squares) together with the exact solution (straight line) for the case of a compressible hydrodynamic flow with shock.

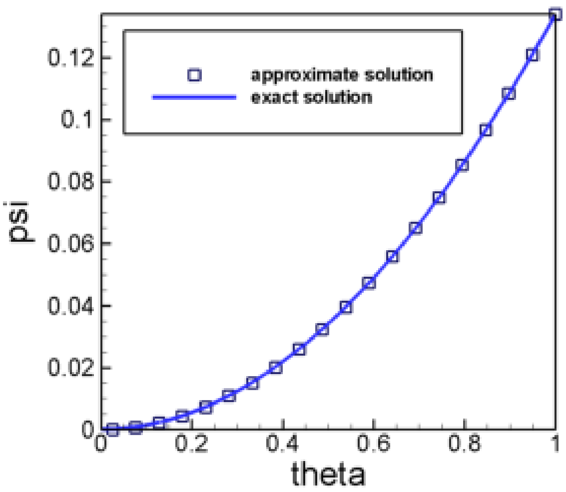

Figure 3-6: The behavior of the plasma flux function ψ as a function of the polar angle θ is shown (squares) together with the exact solution (straight line) for the case of a magnetohydrodynamic flow.

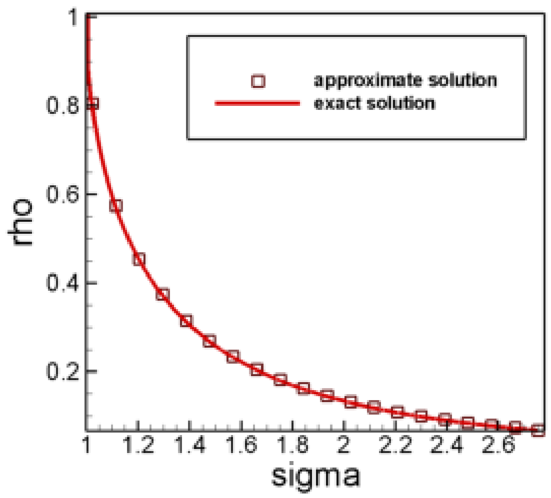

Figure 3-7: The behavior of the plasma density ρ as a function of the spherical radius σ is shown (squares) together with the exact solution (straight line) for the case of a supersonic magnetohydrodynamic flow.

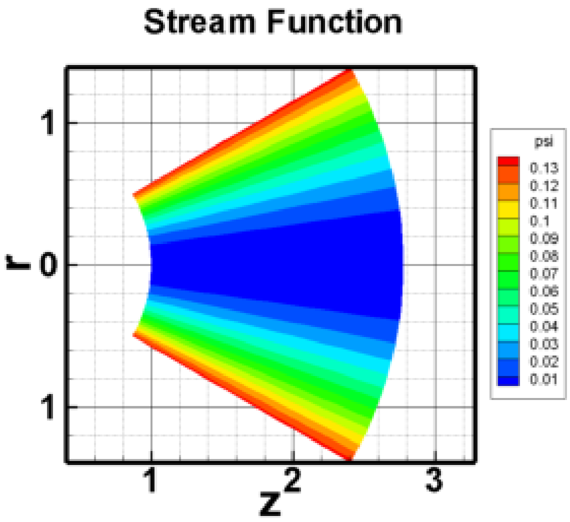

Figure 3-8: Contour plot of the magnetic flux function ψ for the example of thruster operation, plotted in dimensionless units.

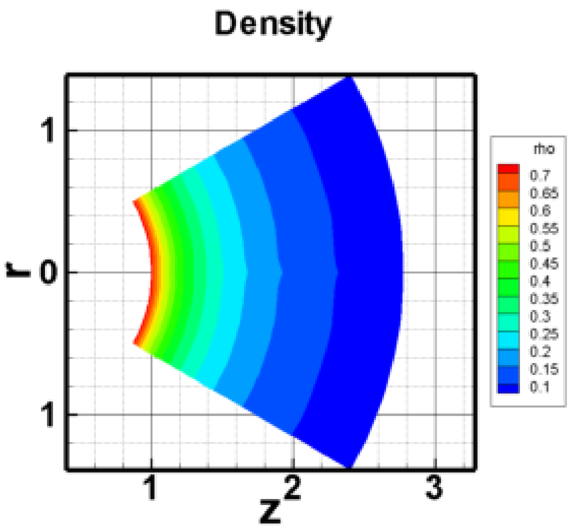

Figure 3-9: Contour plot of the plasma density function ρ for the example of thruster operation, plotted in dimensionless units.

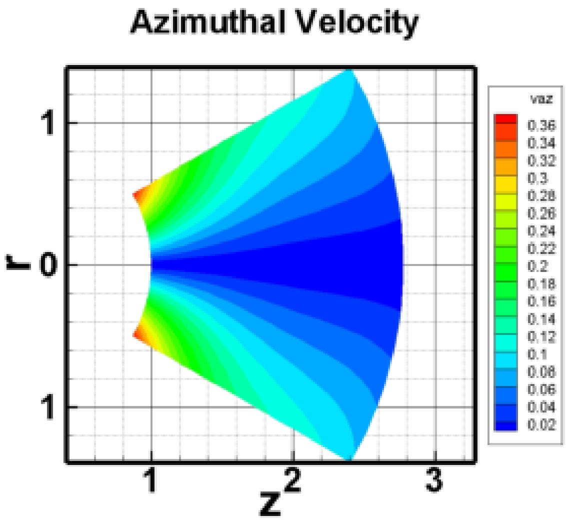

Figure 3-10: Contour plot of the azimuthal velocity function v_ϕ for the example of thruster operation, plotted in dimensionless units.

An attempt to overcome these problems has been made by changing the kind of base functions. In more detail, a set of functions has been defined similarly to finite elements' bases, thus introducing a grid and a set of functions with a small support.

The main aim of the forthcoming work is to make the code fit for experiment behaviour predictions. The code performances can be improved by adopting a more efficient routine of integration and by parallel-computing the integrals for the Newton-Raphson algorithm.

The theory beyond this thesis is rich in interesting and unexplored features, the development of a more sophisticated analysis of flow transitions is a promising field of research.

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Appendix A

Appendix: Jump conditions

With the variational approach (described in Sec. 1.4), we have obtained three jump conditions that characterize the solution on the entropy discontinuity surface λ:

$$\|\psi\| = 0, \tag{A.1}$$

$$\left\| \frac{1}{r^2} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{n}}{\rho} \right\| = 0, \tag{A.2}$$

with the normal versor $\mathbf{n}$ on λ pointing in the direction of the increasing entropy region, and

$$\|\mathcal{L}\| = \frac{1}{r^2 \rho} \frac{\partial \psi}{\partial z} \left\| \frac{\partial \psi}{\partial z} \right\| + \frac{1}{r^2 \rho} \frac{\partial \psi}{\partial r} \left\| \frac{\partial \psi}{\partial r} \right\|. \tag{A.3}$$

In these expressions and in the following derivations, the $\|\cdot\|$ represents the difference between the value assumed after the shock and that assumed before while all the expressions without the $\|\cdot\|$ operator are evaluated in the region before the entropy jump.

The aim of this section is to show that the conditions (A.1-A.3) are equivalent to those obtained in the classic shock theory (1.65-1.76). It is quite easy to demonstrate that the first two equations represent the mass flow and the tangential velocity conservation respectively. Hence, we concentrate our attention on the density ratio equation across the discontinuity (1.76)

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\, M_1^2}{2 + (\gamma - 1)\, M_1^2}, \tag{A.4}$$

proving that it follows from a combination of all the variational shock constraints.

We start by recalling some expressions that we will use in the next steps: the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi}{\rho}\right)^2 - \frac{1}{2}\rho\left(\frac{A}{r}\right)^2 + \rho B - \frac{1}{\gamma-1}p \tag{A.5}$$

and the Bernoulli constraint

$$B = \frac{1}{2}\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi}{\rho}\right)^2 + \frac{1}{2}\left(\frac{A}{r}\right)^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} \tag{A.6}$$

where the pressure p is defined in the two regions as

$$\rho^\gamma C_2 = p|_2 \qquad \rho^\gamma C_1 = p|_1\,. \tag{A.7}$$

As a simplification, given a generic function f we define as

$$f_x = \frac{f_2}{f_1} \tag{A.8}$$

the ratio of the function's values across the shock. Exploiting this definition and the definition of $\|\cdot\|$ it easily follows that:

$$\|f+g\| = \|f\| + \|g\|\,, \tag{A.9}$$

$$\left\|\frac{1}{f}\right\| = -\frac{\|f\|}{f_x f^2}, \tag{A.10}$$

$$\frac{\|f\|}{f} = (f_x - 1)\,. \tag{A.11}$$

Using the expression (A.8), the equation (A.2) reduces to

$$(\boldsymbol{\nabla}\psi\cdot\mathbf{n})_x = \rho_x$$

and, from equation (A.1), we obtain

$$\|\boldsymbol{\nabla}\psi\cdot\mathbf{t}\| = 0$$

where $\mathbf{t}$ is the tangential unit vector on λ in the azimuthal plane.

The term on the left side of the condition (A.3) can be written in the form

$$\|\mathcal{L}\| = \left\| \frac{1}{2}\rho \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi}{\rho} \right)^2 \right\| - \|\rho\| \left[\frac{1}{2} \left(\frac{A}{r} \right)^2 - B \right] - \left\| \frac{1}{\gamma - 1} \frac{p}{\rho} \right\|$$

while the right hand side expression reduces to

$$\frac{1}{r^2\rho} \frac{\partial \psi}{\partial z} \left\| \frac{\partial \psi}{\partial z} \right\| + \frac{1}{r^2\rho} \frac{\partial \psi}{\partial r} \left\| \frac{\partial \psi}{\partial r} \right\| = \rho \left[(\boldsymbol{\nabla}\psi \cdot \mathbf{n})_x - 1 \right] \left(\frac{\boldsymbol{\nabla}\psi \cdot \mathbf{n}}{r\rho} \right)^2 .$$

Considering that the expression

$$\left\| \frac{1}{2}\rho \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi}{\rho} \right)^2 \right\| = \left\| \frac{1}{2} \frac{1}{r^2} \frac{(\boldsymbol{\nabla}\psi \cdot \mathbf{n})^2}{\rho} \right\| + \left\| \frac{1}{2} \frac{1}{r^2} \frac{(\boldsymbol{\nabla}\psi \cdot \mathbf{t})^2}{\rho} \right\|$$

can be written as

$$\left\| \frac{1}{2} \frac{1}{r^2} \frac{(\boldsymbol{\nabla}\psi \cdot \mathbf{n})^2}{\rho} \right\| = \left\| \frac{1}{2} \left(\frac{1}{r^2} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{n}}{\rho} \right) \boldsymbol{\nabla}\psi \cdot \mathbf{n} \right\| = \frac{1}{2} \left(\frac{1}{r^2} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{n}}{\rho} \right) \|\boldsymbol{\nabla}\psi \cdot \mathbf{n}\| = \frac{1}{2} \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{n}}{\rho} \right)^2 \|\rho\| ,$$

$$\left\| \frac{1}{2} \frac{1}{r^2} \frac{(\boldsymbol{\nabla}\psi \cdot \mathbf{t})^2}{\rho} \right\| = \frac{1}{2} \frac{1}{r^2} (\boldsymbol{\nabla}\psi \cdot \mathbf{t})^2 \left\| \frac{1}{\rho} \right\| = -\frac{1}{2} \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{t}}{\rho} \right)^2 \frac{\|\rho\|}{\rho_x}$$

and that the subsequent relations hold

$$\left\| \frac{1}{\gamma - 1} p \right\| = \frac{1}{\gamma - 1} \left(\|\rho\| \frac{p}{\rho} + \rho_x \rho \left\| \frac{p}{\rho} \right\| \right) ,$$

$$\left[\frac{1}{2} \left(\frac{A}{r} \right)^2 - B \right] = -\frac{1}{2} \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi}{\rho} \right)^2 - \frac{\gamma}{\gamma - 1} \frac{p}{\rho},$$

we obtain

$$-\frac{1}{2} \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{t}}{\rho} \right)^2 \frac{\|\rho\|}{\rho_x} + \frac{1}{2} \|\rho\| \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{t}}{\rho} \right)^2 + \|\rho\| \frac{p}{\rho} - \frac{1}{\gamma - 1} \rho_x \rho \left\| \frac{p}{\rho} \right\| = 0. \tag{A.12}$$

Hence we exploit the Bernoulli constraint in the form

$$\frac{\gamma}{\gamma - 1} \left\| \frac{p}{\rho} \right\| = \left\| B - \frac{1}{2} \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi}{\rho} \right)^2 - \frac{1}{2} \left(\frac{A}{r} \right)^2 \right\| = -\frac{1}{2} \left\| \left(\frac{1}{r} \frac{\boldsymbol{\nabla}\psi \cdot \mathbf{t}}{\rho} \right)^2 \right\|$$

and we substitute this equation into equation (A.12)

$$-\frac{1}{2}\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi\cdot\mathbf{t}}{\rho}\right)^2+\frac{1}{2}\rho_x\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi\cdot\mathbf{t}}{\rho}\right)^2+\rho_x\frac{p}{\rho}+\frac{1}{2\gamma}\frac{\rho_x^2\rho}{\|\rho\|}\left\|\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi\cdot\mathbf{t}}{\rho}\right)^2\right\|=0. \tag{A.13}$$

Using the property (A.10) we obtain

$$(\rho_x-1)\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi\cdot\mathbf{t}}{\rho}\right)^2+2\rho_x\frac{p}{\rho}-\frac{1}{\gamma}\frac{\rho}{\|\rho\|}\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi\cdot\mathbf{t}}{\rho}\right)^2\left(\rho_x^2-1\right)=0$$

and, recalling the definitions

$$V_n^2=\left(\frac{1}{r}\frac{\boldsymbol{\nabla}\psi\cdot\mathbf{t}}{\rho}\right)^2, \tag{A.14}$$

$$M^2=\frac{\rho V_n^2}{\gamma p}, \tag{A.15}$$

it follows that

$$\rho_x\gamma-\gamma+\rho_x 2M^{-2}-1-\rho_x=0.$$

(note: since $C_2\neq C_1$ we can exclude the trivial solution $\rho_x=1$) Finally, rearranging this last equation we obtain

$$\rho_x=\frac{(\gamma+1)\,M^2}{(\gamma-1)\,M^2+2}, \tag{A.16}$$

i.e. the expression (A.4) derived from the classic theory.

Appendix B

Appendix: Newton-Raphson Equations

Axisymmetric Hydrodynamics

Developing the integral we obtain, for $j \leq i \leq n_l$

$$\frac{\partial}{\partial \psi_j} \frac{\partial \mathcal{L}}{\partial \psi_i} = \frac{2}{r^2} \frac{\boldsymbol{\nabla} f_j \cdot \boldsymbol{\nabla} f_i}{\rho} + \rho \left[-2R_0^2 \left(\frac{A\ddot{A} + \dot{A}^2}{r^2} \right) + \ddot{B} - \frac{2M_0^{-2}}{(\gamma - 1)\,\gamma} \rho^{\gamma - 1} \ddot{C} \right] f_j f_i,$$

for $j \leq n_l < i$

$$\frac{\partial}{\partial \rho_i} \frac{\partial \mathcal{L}}{\partial \psi_j} = -\frac{2}{r^2} \frac{\boldsymbol{\nabla} \psi}{\rho^2} \cdot \boldsymbol{\nabla} f_j \cdot g_i + \left[-2R_0^2 \frac{AA}{r^2} + \dot{B} - \frac{2M_0^{-2}}{(\gamma - 1)} \rho^{\gamma - 1} \dot{C} \right] g_i f_j,$$

and for $n_l < j \leq i$

$$\frac{\partial}{\partial \rho_j} \frac{\partial \mathcal{L}}{\partial \rho_i} = \left[\frac{2}{\rho} \left(\frac{1}{r} \frac{\boldsymbol{\nabla} \psi}{\rho} \right)^2 - 2M_0^{-2} \rho^{\gamma - 2} C \right] g_j g_i.$$

Generalized Grad-Shafranov equations

Developing the integral we obtain, for $j \leq i \leq n_T$

$$\frac{\partial}{\partial \psi_j}\frac{\partial \mathcal{L}}{\partial \psi_i} = \left(M_1^2\frac{F^2}{\rho}-1\right)\frac{2}{r^2}\boldsymbol{\nabla} f_i\cdot\boldsymbol{\nabla} f_j + 4M_1^2\frac{F\dot{F}}{\rho}\frac{1}{r^2}\boldsymbol{\nabla}\psi\cdot\left[f_j\boldsymbol{\nabla} f_i + f_i\boldsymbol{\nabla} f_j\right] +$$
$$\left[\frac{2M_1^2}{\rho}\left(F\ddot{F}+\dot{F}^2\right)\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 + 2\left(\frac{H+M_1M_2rv_\phi F}{r}\right)\left(\frac{\ddot{H}+M_1M_2rv_\phi\ddot{F}}{r}\right)+\right.$$
$$\left. 2\left(\frac{\dot{H}+M_1M_2rv_\phi\dot{F}}{r}\right)^2 + M_1^2\rho\ddot{J} + M_1M_2r\rho v_\phi\ddot{G} - \frac{M_1^2}{M_3^2}\frac{2}{(\gamma-1)\gamma}\rho^\gamma\ddot{I}\right]\cdot f_jf_i,$$

for $j \leq n_T < i < n_T + m_T$

$$\begin{aligned}\frac{\partial}{\partial \rho_i}\frac{\partial \mathcal{L}}{\partial \psi_j} &= -M_1^2\frac{F^2}{\rho^2}\frac{2}{r^2}\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla} f_j\cdot g_i + \\ &\left[-2M_1^2\frac{F\dot{F}}{\rho^2}\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 + M_1^2\dot{J} + M_1M_2rv_\phi\dot{G} - \frac{M_1^2}{M_3^2}\frac{2}{(\gamma-1)}\rho^{\gamma-1}\dot{I}\right]\cdot g_if_j,\end{aligned}$$

for $j \leq n_l$ and $n_I + m_T < i < n_T + m_T + l_T$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial v_{\phi i}}\frac{\partial \mathcal{L}}{\partial \psi_j} &= \left[2M_1M_2\left(\frac{H+M_1M_2rv_\phi F}{r}\right)\dot{F}+\right.\\ &\left. 2M_1M_2\left(\frac{\dot{H}+M_1M_2rv_\phi\dot{F}}{r}\right)F + M_1M_2r\rho\dot{G}\right]\cdot f_jh_i,\end{aligned}$$

for $n_T < j \leq i < n_T + m_T$

$$\frac{\partial}{\partial \rho_j}\frac{\partial \mathcal{L}}{\partial \rho_i} = \left[M_1^2\frac{2}{\rho}\frac{F^2}{\rho^2}\left(\frac{\boldsymbol{\nabla}\psi}{r}\right)^2 - 2\frac{M_1^2}{M_3^2}\rho^{\gamma-2}I\right]\cdot g_jg_i,$$

for $n_T < j \leq n_T + m_T < i \leq n_T + m_T + l_T$

$$\frac{\partial}{\partial \rho_i}\frac{\partial \mathcal{L}}{\partial v_{\phi j}} = \left[-2M_2^2v_\phi + M_1M_2rG\right]\cdot g_ih_j$$

and for $n_T + m_T < j \leq i \leq n_T + m_T + l_T$

$$\frac{\partial}{\partial v_{\phi i}}\frac{\partial \mathcal{L}}{\partial v_{\phi j}} = \left[2\left(M_1M_2F\right)^2 - 2M_2^2\rho\right]\cdot h_ih_j = 2M_2^2\rho\left[M_1^2\frac{F^2}{\rho}-1\right]\cdot h_ih_j.$$