SCUOLA NORMALE SUPERIORE

On the K-theory of tame Artin stacks

Tesi di Perfezionamento in Matematica

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"Madame Bérenge, elle visait bas, elle visait juste"
Louis-Ferdinand Céline, *Mort à Crédit*

0.1 Abstract

This thesis pertains to the algebraic K-theory of tame Artin stacks. Building on earlier work of Vezzosi and Vistoli in equivariant K-theory, which we translate in stacky language, we give a description of the algebraic K-groups of tame quotient stacks. Using a strategy of Vistoli, we recover Grothendieck-Riemann-Roch-like formulae for tame quotient stacks that refine Toën's Grothendieck-Riemann-Roch formula for Deligne-Mumford stacks (as it was realized that the latter pertains to quotient stacks since it relies on the resolution property). Our formulae differ from Toën's in that, instead of using the standard inertia stack, we use the cyclotomic inertia stack introduced by Abramovich, Graber and Vistoli in the early 2000s. Our results involve the rational part of the K'-theory of the object considered. We establish a few conjectures, the main one (Conjecture 6.3) pertaining to the covariance of our Lefschetz-Riemann-Roch map for proper morphisms of tame stacks (not necessarily representable). Other future works might be dedicated to the study of torsion in K'-groups as well as more general Artin stacks.

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Contents

Chapter 1

Introduction

Equivariant algebraic K-theory is a common generalization of algebraic K-theory and the representation theory of algebraic groups. One can date its birth back to Thomason's paper [Th1] which was written in the early 1980s. Thomason's algebraic equivariant K-groups coincide with the Quillen K-groups of the category of locally free coherent sheaves on the quotient stack associated to the given group action, and likewise his equivariant K'-groups coincide with the Quillen K'-groups of the category of coherent sheaves on the quotient stack associated to the given group action (it is an obvious consequence of the definitions). As a consequence, equivariant algebraic K-theory can be seen as a subfield of algebraic K-theory of algebraic stacks. The latter, in fact, is a far-reaching generalization of the former, to the extent that not all algebraic stacks are isomorphic to global quotients of group scheme actions. It is to be noted at this point, however, that while many results have been obtained in equivariant K-theory over the past three decades, almost nothing is known about stacks that are not given by global quotients, and it is as yet not clear what the correct definitions should be. At the level of intersection theory, one should mention the work of Kresch [Ks], who defined a Chow homology functor on the category of Artin stacks of finite type over a field, and the first higher Chow groups thereof. Kresch's Chow groups are defined with integral coefficients. For Deligne-Mumford stacks and modulo torsions, they are isomorphic to the Vistoli's Chow groups defined in [V1], while for quotient stacks of actions of linear algebraic groups on algebraic spaces, they are isomorphic to Edidin and Graham's Chow groups defined in [EG1]. Kresch theory is best behaved with respect to the class of Artin stacks that admit a stratification by global quotients. It is not clear, however, how to pursue his efforts and obtain a satisfactory theory of higher Chow groups of arbitrary degrees.

Our purpose in this thesis is to focus on tame Artin stacks, and mostly to extend Toën's Riemann-Roch theorem ([T1,T2]) to this class of algebraic stacks. Tame stacks have been introduced around 2008 by Abramovich,

Olsson and Vistoli [AOV], as a class of algebraic stacks broader than that of tame Deligne-Mumford stacks and better behaved than general Deligne-Mumford stacks (in particular in positive and mixed characteristic).
In this thesis, we consider algebraic stacks as defined in [A] (also called Artin stacks). Recall that, for an algebraic stack $\mathcal{X} \longrightarrow S$ over an algebraic space S, we define its *inertia stack* as $I_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$. The natural morphism $I_{\mathcal{X}} \longrightarrow \mathcal{X}$ makes $I_{\mathcal{X}}$ a group stack over $\mathcal{X}$. $\mathcal{X}$ is said to have *finite inertia* is the morphism $I_{\mathcal{X}} \longrightarrow \mathcal{X}$ is finite. For any algebraic stack $\mathcal{X}$, we denote $QCoh(\mathcal{X})$ (resp. $Coh(\mathcal{X})$) the abelian category of quasi-coherent (resp. coherent) sheaves over it. The following definition, which is due to Gillet [G1], is fundamental.

Definition 1.1 : A *coarse moduli space* for an algebraic S-stack $\mathcal{X}$ is an algebraic S-space M together with a map $p : \mathcal{X} \longrightarrow M$ over S such that :
(i) p is initial among S-maps from $\mathcal{X}$ to algebraic spaces.
(ii) For every algebraically closed field K, the map $\bar{\mathcal{X}}(K) \longrightarrow M(K)$ is bijective, where $\bar{\mathcal{X}}(K)$ denotes the set of isomorphism classes of objects of $\mathcal{X}(K)$.

A celebrated result of Keel and Mori [KM] implies the existence of moduli spaces for algebraic stacks of finite type over a locally noetherian base algebraic space with finite inertia. It was originally formulated in terms of flat groupoids (recall that a consequence of the main result of [A] is that any groupoid algebraic space, locally of finite type over a reasonable[1] base algebraic space S, and quasi-separated, presents an algebraic stack of finite type over S - see [A, Theorem 7.1]).
The following slight refinement of the Keel-Mori theorem appears in [C] :

Theorem 1.2 (Keel-Mori-Conrad) : Let S be a scheme and let $\mathcal{X}$ be an Artin stack that is locally of finite presentation over S and has finite inertia. There exists a coarse moduli space $p : \mathcal{X} \longrightarrow M$, and it satisfies the following additional properties :
(i) The structure map $M \longrightarrow S$ is separated if $\mathcal{X} \longrightarrow S$ is separated, and it is locally of finite type if S is locally noetherian.
(ii) The map p is proper and quasi-finite.
Moreover, if $M' \longrightarrow M$ is a flat map of algebraic spaces then $p' : \mathcal{X}' = \mathcal{X} \times_M M' \longrightarrow M$ is a coarse moduli space. $\diamond$

Let S be a scheme. The following definition is made in [AOV] :

Definition 1.3 : A *tame* Artin stack over S is a locally finitely pre-

[1] In [A], S is assumed to be an algebraic space of finite type over an excellent Dedekind ring.

sented algebraic stack $\mathcal{X} \longrightarrow S$ with finite inertia, such that the natural map to its moduli space $p : \mathcal{X} \longrightarrow M$ induces an exact functor $p_* : QCoh(\mathcal{X}) \longrightarrow QCoh(M)$.

Let S be a scheme and let When G be a finite flat group scheme over S. Then $B_S G$ is tame if and only if G is linearly reductive : this results from the fact that, in this case, the moduli space of $B_S G \longrightarrow S$ is S itself. This is the simplest kind of tame stacks. The following theorem, which is the main theorem in the theory of tame stacks, gives a local description of tame stacks in the general case in terms of linearly reductive group actions (see [AOV, Theorem 3.2] for a proof).

Theorem 1.4 : Let S be a scheme and $\mathcal{X} \longrightarrow S$ a locally finitely presented algebraic stack over S with finite inertia. Let M denote the moduli space of $\mathcal{X}$. The following conditions are equivalent :
(i) $\mathcal{X}$ is tame.
(ii) If k is an algebraically closed field with a morphism $Spec(k) \longrightarrow S$ and ξ is an object of $\mathcal{X}(k)$, then the automorphism group scheme $Aut_k(\xi) \longrightarrow Spec(k)$ is linearly reductive.
(iii) There exists an fppf cover $M' \longrightarrow M$, a linearly reductive group scheme $G \longrightarrow M$ acting an a finite and finitely presented scheme $U \longrightarrow M$ together with an isomorphism $\mathcal{X} \times_M M \approx [U/G]$ of algebraic stacks over M
(iv) Same as (iii), but $M \longrightarrow M$ is assumed to be étale and surjective.$\diamond$

Remark 1.5 : A Deligne-Mumford stack is tame if and only if for evey algebraically closed field K and geometric point $s : Spec(K) \longrightarrow \mathcal{X}$, the (finite) group $Aut_K(s)$ has order prime to the characteristic of K.

Toën's Riemann-Roch theorem ([T1,T2]), which dates back to the late 1990s, is one of the major result known about the K-theory of Deligne-Mumford stacks. It applies to Deligne-Mumford stacks whose coarse moduli spaces are quasi-projective schemes, and which satisfy the resolution property (and hence are global quotients, thanks to the main result of [Tot1]). Toën's proof relies on the fact that Deligne-Mumford stack is, locally with respect to the étale site of its moduli space, given by the quotient stack of a finite group on a scheme, and a decomposition theorem for the equivariant K-theory of finite group actions on schemes proven by Vistoli in 1990 [V2]. By using methods of homotopical algebra, Toën reduces to Vistoli's decomposition theorem. The other fundamental ingredient in Toën's proof is the machinery introduced by Gillet in [G], which he uses to build Riemann-Roch maps with target a variety of cohomology theories (ie singular cohomology of the associated analytic stack for stacks over the complex numbers, De Rham comology, étale cohomology with torsion coefficients, and Chow groups).

Let us mention that, in the case of quotient Deligne-Mumford stacks over the field of compex numbers, Toën's Grothendieck-Riemann-Roch theorem has been considerably refined by Krishna and Sreedhar in 2017 (see [KrS]).

It is natural to wish to extend further this refined result to the class of tame quotient stacks, and this shall be one our main purposes. We obtain such an extension thanks to the work of Vezzosi and Vistoli in [VV]. While our approach differs a lot from Toën, to some extent one might say that the decomposition formula constituting the main result of [VV] (see Theorem 7.9 in the Appendix for the statement of it) will play a role similar to the main result of [V2] in [T1,T2]. Let us mention at this point a major difference between the decomposition formulae proven in [VV] and [V2] respectively : namely, the former involves a new variant of K-theory, coined geometric K-theory by Vezzosi and Vistoli, the definition of which we recall in Chapter 3, and which is further investigated in Chapters 4 to 7. For an algebraic stack $\mathcal{X}$, denoting $K'_*(\mathcal{X})$ the K-groups of the abelian category of coherent sheaves on $\mathcal{X}$, let us denote $K'_*(\mathcal{X})_{geom}$ the so called geometric K'-groups of $\mathcal{X}$ (cf Definition 3.3). In the case of tame quotient stacks, geometric K'-theory has a nice interpretation thanks to the following theorem, which is the main result of Chapter 4 :

Theorem 4.1 : Let $\mathcal{X}$ be a tame quotient stack. Let $p : \mathcal{X} \longrightarrow M$ be the projection to its coarse moduli space. Then the pushforward induces an isomorphism :

$$K'_*(\mathcal{X})_{geom} \xrightarrow{\approx} K'_*(M)$$

$\diamond$

Now, recall from [VV] that a *dual cyclic* group scheme is a group scheme isomorphic to the group scheme μ_n of n-th roots of 1, for some n. Theorem 7.9 essentially says two things. First, that given a well-behaved enough action of an algebraic group G on a regular algebraic space X of finite type over a field, the set of conjugacy classes of dual cyclic subgroup schemes of G is finite. And second, that the equivariant K-groups of X decompose as a product indexed by the latter set. This product involves geometric K-groups of subschemes of X fixed by the restricted action of dual cyclic subgroups of G. We call this second part of Theorem 7.9 the *Vezzosi-Vistoli decomposition formula.* It produces an isomorphism from which our Riemann-Roch morphisms are built.

We will here deal with algebraic spaces that are not perforce regular over the base, and therefore use K'-groups instead of K-groups in the following chapters. Furthermore, we will, unless mention to the contrary, only consider the torsion-free or rational part of K'-groups.

The two main ingredients of the Grothendieck-Riemann-Roch theorem obtained in Chapter 4 are the Vezzosi-Vistoli decomposition formula [VV, Theorem 5.4], and the notion of cyclic inertia stack of an Artin stack, introduced by Abramovich, Graber and Vistoli in their work on Gromov-Witten theory of (smooth) complex Deligne-Mumford stacks[2] [AGV].

Let us now review the main results presented in this work. We refer the reader to Chapter 2 for the definition of the cyclotomic inertia stack $I_\mu(\mathcal{X})$ of an Artin stack $\mathcal{X}$, and for the construction of the morphism $I_\mu(f) : I_\mu(\mathcal{X}) \longrightarrow I_\mu(\mathcal{Y})$ associated to a morphism of stacks $f : \mathcal{X} \longrightarrow \mathcal{Y}$. We also refer the reader to the first section of Chapter 7 for the definition of the Riemann-Roch map : for a tame stack $\mathcal{X}$, it is a morphism $\mathcal{L}_\mathcal{X} : K'_*(\mathcal{X}) \longrightarrow K'_*(I_\mu(\mathcal{X}))_{geom}$ of $\mathbf{Q}$-vector spaces that we furthermore prove to be an isomorphism.

Our main result is **Theorem 6.3.1**, and it is naturally followed by **Conjecture 6.3.2**. Modulo the latter, and some elementary Galois theory, it has the following consequence:

Theorem 1.6 : Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper representable morphism of quotient tame stacks. Then the following diagram is commutative:

$$\begin{tikzcd} K'_*(\mathcal{X}) \arrow[r, "\mathcal{L}_\mathcal{X}"] \arrow[d, "f_*"'] & K'_*(I_\mu(\mathcal{X}))_{geom} \arrow[d, "I_\mu(f)_*"'] \\ K'_*(\mathcal{Y}) \arrow[r, "\mathcal{L}_\mathcal{Y}"] & K'_*(I_\mu(\mathcal{Y}))_{geom} \end{tikzcd}$$

The morphism $\mathcal{L}_\mathcal{X}$ essentially comes from the combination of three morphisms that are studied separetely, and all proven to be isomorphism. The first one appears in Chapter 6 and is denoted $\aleph_\mathcal{X}$: it is closely related to the isomorphism of Theorem 1.6. The second morphism, denoted $\tilde{\alpha}_\mathcal{X}$, comes from an important comodule structure on the K-theory of coherent sheaves on cyclotomic inertia stacks. The third one is of a purely Galois theoretic nature.

[2]The work of Abramovich, Graber and Vistoli can be seen as an algebraic counterpart to Orbifold Gromov-Witten theory, which was introduced by Chen and Ruan in the symplectic setting in 2002 [CR]. The language of stacks is essential to computations in the former. Let us an extension of [CR] to tame Deligne-Mumford stacks in mixed characteristic has been recently studied by Poma [Po]. The cyclotomic inertia stack of a tame stack may be seen as a version of the inertia stack that is more appropriate than the latter in positive and mixed characteristic.

It is then shown that geometric K'-theory is closely related to higher Chow groups of algebraic stacks in our cases of interests, thanks to a result of Krishna in equivariant K-theory and equivariant intersection theory. This is illustrated by Proposition 6.9 below, using the comparison maps $\tau^K_{X,G,n}$ which Krishna associates to a linearly reductive group scheme G acting on an algebraic space X.

Proposition 6.9 : Let a linearly reductive group scheme G act on an algebraic space X such that the quotient stack $[X/G]$ associated to this action is tame. Then, for any $n \geq 0$

$$\tau^K_{X,G,n} : K'_n(X,G)_{geom} \xrightarrow{\approx} A^*_G(X,n) \otimes \mathbf{Q}$$

We give two definitions of Riemann-Roch maps, one relying on Theorem 4.1 and the other on Proposition 6.9. Arguably, the first definition is more natural, and it is also more general, but we use the second to state and prove our Grothendieck-Riemann-Roch theorem. It gives us a map

$$\tau_{\mathcal{X}} : K'_n(\mathcal{X})_{geom} \xrightarrow{\approx} A^*(\mathcal{X},n) \otimes \mathbf{Q}$$

which we call the Grothendieck-Riemann-Roch map associated to $\mathcal{X}$, and below is the statement of our Grothendieck-Riemann-Roch theorem.

Proposition 6.19 : Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper morphism of quotient tame Artin stacks. Then the following diagram commutes (for each $n \in \mathbf{N}$) :

$$\begin{tikzcd}
K'_n(\mathcal{X})_{geom} \arrow[r, "\tau_{\mathcal{X}}"] \arrow[d, "f_*"'] & A^*(\mathcal{X},n) \otimes \mathbf{Q} \arrow[d, "f_*"'] \\
K'_n(\mathcal{Y})_{geom} \arrow[r, "\tau_{\mathcal{Y}}"] & A^*(\mathcal{Y},n) \otimes \mathbf{Q}
\end{tikzcd}$$

In the following Chapters (2 to 5), we reformulate Theorem 1.6 as a comparison theorem between the K'-groups of a quotient tame Artin stack and the geometric K'-groups of its cyclotomic inertia stack. Chapter 4 is about cyclotomic inertia stacks of quotient tame stacks. Chapters 3 and 4 contain our general results on the K-theory of quotient stacks. In particular, Chapter 3 contains the proof of Theorem 3.1. Then, Chapter 5 is specifically about the K-theory of such cyclotomic inertia stacks. It is in

Chapter 5 that our work meets the main result of [VV]. In Chapter 6, we investigate the intrinsicness of our constructions, ie the extent to which our results and notions from equivariant K-theory can be formulated interms of K-theory of stacks. The definition of the Riemann-Roch map is the starting point of Chapter 6, building on the preceding chapters. Then in the remaining of Chapter 6 we obtain a Lefschetz-Riemann-Roch formula, and prove the contravariance thereof with respect to proper pushforwards. Next, we compare geometric K'-groups with Edidin and Grahams' higher equivariant Chow groups : in effect, we only need to check that we can apply a result of Kirshna proven in [Kr1]. We then obtain a Grothendieck-Riemann-Roch formula for tame quotient stacks. It would be especially interesting, in the future, to obtain results on non-quotient stacks, as well as to build a good theory of higher Chow groups of such, subsuming Kresch's theory. On the other hand, it would also be extremely interesting to obtain results capturing significant information about the torsion of equivariant K'-groups and K-groups. Specifically, an extension to equivariant contexts or algebraic stacks of the Friedlander-Suslin spectral sequence[3] [FS] would be a major breakthrough.[4]

We use throughout the next chapters a bunch of notations involving algebraic groups that are gathered in the Appendix (Section 7.1). The Appendix also contains some backgroud on algebraic groups (Section 7.2), and the statement of Thomason's slice theorem (Section 7.3).

[3]This spectral sequence pertains to any smooth scheme X. It has E_2-term all about the higher Chow groups (or equivalently motivic cohomology, in accordance with [Vo1]) of X and abuts to the Quillen K-groups of X. In the very special case of fields, it was constructed by Bloch and Lichtenbaum [BL] around 1995.

[4]In the case of actions of finite groups on schemes, such a spectral sequence has been constructed around 2005 by Levine and Serpé [LS]. In that paper, the authors express hope that their efforts can be pursued to generalize their construction to actions of more general algebraic groups and even non-quotient stacks. Incidentally, using the Levine-Serpé spectral sequence, Manh Toan Nguyen recovered the results of [V2] (see [Ng, Proposition 2.13]).

Chapter 2

Cyclotomic inertia stacks of quotient stacks

2.1 Definition and properties

The following definition was made in [AGV].

Definition 2.1 : Let $\mathcal{X}$ be an algebraic stack of finite type over a field k. For every integer $r \geq 1$, define a fibered category in groupoids $I_{\mu_r}(\mathcal{X})$ as follows.
(i) For every k-scheme T, $I_{\mu_r}(\mathcal{X})(T)$ has as objects pairs (ξ, α), where $\xi \in \mathcal{X}(T)$, and $\alpha : \mu_{r,T} \hookrightarrow Aut_T(\xi)$ is a monomorphism of group schemes.
(ii) An arrow from (ξ, α) over T to (ξ', α') over T' (T being over T') is an arrow $\xi \longrightarrow \xi'$ fitting in the following commutative diagram :

$$\begin{tikzcd}
\mu_{r,T} \arrow[r] \arrow[d, "\alpha"] & \mu_{r,T'} \arrow[d, "\alpha'"] \\
Aut_T(\xi) \arrow[r] \arrow[d] & Aut_{T'}(\xi') \arrow[d] \\
T \arrow[r] & T'
\end{tikzcd}$$

Thanks to [AGV, Proposition 3.2.3], the content of which is reproduced in the proposition below, there is another way to define $I_{\mu_r}(\mathcal{X})$.

Proposition 2.2 : Let $\mathcal{X}$ be an algebraic stack of finite type over a field

k. Then $I_{\mu_r}(\mathcal{X})$ is isomorphic to the category fibered in groupoids whose objects over a k-scheme T are representable morphisms $\phi : B\mu_{r,T} \longrightarrow \mathcal{X}$, and whose arrows over a morphism of k-schemes $f : T \longrightarrow T'$ are natural transformations $\rho : \phi \longrightarrow \phi' f_*$ such that the following diagram is commutative :

$$\begin{tikzcd}
B\mu_{r,T} \arrow[r, "f_*"] \arrow[dr, "\phi"'] & B\mu_{r,T'} \arrow[d, "\phi'"] \\
 & \mathcal{X}
\end{tikzcd}$$

$\diamond$

The following result, also from [AGV], is important in the Gromov-Witten theory of Deligne-Mumford stacks.

Proposition 2.3 : If $\mathcal{X}$ is a Deligne-Mumford stack, then $I_{\mu_r}(\mathcal{X})$ is a Deligne–Mumford stack, and the canonical projection morphism $\pi_{\mathcal{X},r} : I_{\mu_r}(\mathcal{X}) \longrightarrow \mathcal{X}$ is representable and finite.

Proof : This is [AGV, Proposition 3.1.2].$\diamond$

Now, for our purposes in this thesis, we need the following extension of the latter result.

Proposition 2.4 : If $\mathcal{X}$ is a tame quotient Artin stack, then so also is $I_{\mu_r}(\mathcal{X})$, and the canonical projection morphism $\pi_{\mathcal{X},r} : I_{\mu_r}(\mathcal{X}) \longrightarrow \mathcal{X}$ is representable and finite.

The proof of Proposition 2.4 requires a number of lemmas. Suppose that Δ is a finite diagonalizable group scheme over k, and G is an affine group scheme of finite type over k. Consider the contravariant functor $H_\Delta(G)$ from schemes over k to sets, sending a scheme S to the set of homomorphisms of group schemes $\Delta_S \longrightarrow G_S$. We will think of $H_\Delta(G)$ as a Zariski sheaf. Likewise, we can also consider the contravariant functor $H_\Delta 6^{in}(G)$ from schemes over $Spec(k)$ to sets, sending a scheme S to the set of monorphisms of group schemes $\Delta_S \longrightarrow G_S$. We denote by $H \in \Delta(G) \subseteq H_\Delta(G)$ the subfunctors consisting of the set of monomorphisms of group schemes $\Delta_S \longrightarrow G_S$. If $\phi : \Delta \longrightarrow \Delta_0$ is a homomorphism of diagonalizable group schemes, composing with ϕ gives a natural transformation $H_{\Delta_0}(G) \longrightarrow H_\Delta(G)$. Call $Q(\Delta)$ the set of quotients of Δ. For each $\Delta \in Q(\Delta)$, consider the composite $H^{in}_{\Delta_0}(G) \subseteq H_{\Delta_0}(G) \longrightarrow H_\Delta(G)$, which is immediately seen to be a monomorphism. This induces a morphism of Zariski sheaves a $\coprod_{\Delta_0 \in Q(\Delta)} H^{in}_{\Delta_0}(G) \longrightarrow H_\Delta(G)$.

Lemma 2.5 : (i) The functors $H_\Delta(G)$ and $H^{in}_\Delta(G)$ are represented by quasi-projective k-schemes.
(ii) Each $H^{in}_{\Delta_0}(G)$ is open and closed in $H_\Delta(G)$.
(iii) The morphism $\coprod_{\Delta_0 \in Q(\Delta)} H^{in}_{\Delta_0}(G) \longrightarrow H_\Delta(G)$ is an isomorphism.
(iv) If G is finite and linearly reductive, then $H_\Delta(G)$ and $H^{in}_\Delta(G)$ are finite over $Spec(k)$.

Proof : Choose an embedding $G \subseteq GL_n$ for some n; this gives an embedding of functors $H_\Delta(G) \subseteq H_\Delta(GL_n)$. It is a standard fact that the inclusion $H_\Delta(G) \subseteq H_\Delta(GL_n)$ is a closed embedding. Clearly we have $H^{in}_\Delta(G) = H_\Delta(G) \cap H^{in}_\Delta(GL_n)$. More generally, if $\Delta_0 \in Q(\Delta)$, the inverse image of $H_\Delta(G) \subseteq H^{in}_\Delta(GL_n)$ in $H^{in}_{\Delta_0}(GL)$ equals $H^{in}_{\Delta_0}(G)$. Hence, to prove (i), (ii) and (iii) we can assume that $G = GL_n$. So it is enough to prove that $H_\Delta(GL_n)$ is represented by a quasi-projective scheme over k. Let $\hat{\Delta}$ be the group of characters $\Delta \longrightarrow G_m$ of Δ. By the standard description of representations of diagonalizable groups, a representation $\Delta_S \longrightarrow GL_{n,S}$ corresponds to a decomposition $\mathcal{O}^n_S = \oplus\chi \in\hat{}\Delta V_\chi$ into eigenspaces. If $d : \hat{\Delta} \longrightarrow \mathbf{N}$ is a function, denote by $H^d_\Delta(GL_n) \subseteq H_\Delta(GL_n)$ the subfunctor of those representations of Δ such that the corresponding eigenspace V_χ has constant rank $d(\chi)$. We have a decomposition of Zariski sheaves $H_\Delta(GL_n) = \coprod_d H^d_\Delta(GL_n)$. If $0 \leq m \leq n$, denote by $G(m,n)$ the Grassmannian of m-dimensional subspaces of k^n. There is an obvious embedding of functors

$$H^d_\Delta(GL_n) \subseteq \textstyle\prod_d G(d(\chi), n)$$

which is easily seen to be an open embedding. This proves (i).

Furthermore, if $d : \hat{\Delta} \longrightarrow \mathbf{N}$, denote by Δ'_d the quotient of Δ such that $\hat{\Delta}'_d \subseteq \hat{\Delta}$ is the group generated by the $\chi \in \hat{\Delta}$ with $d(\chi) > 0$. Then it is easy to see that $H^{in}_{\Delta'}(GL_n) \subseteq H_\Delta(GL_n)$ is the union of the components $H^d_\Delta(GL_n)$ with $\Delta'_d = \Delta'$. This proves (ii) and (iii). To prove (iv), assume that G is finite and linearly reductive. If $\Delta = \Delta' \times \Delta''$ is a decomposition into the product of two diagonalizable subgroups, and assume that $H_{\Delta'}(G)$ and $H_{\Delta''}(G)$ are finite over k; let us show that $H_\Delta(G)$ is also finite. We get an obvious morphism $H_\Delta(G) \longrightarrow H_{\Delta'}(G) \times H_{\Delta''}(G)$; let us show that this is a closed embedding. In fact, let $S \longrightarrow H_{\Delta'}(G) \times H_{\Delta''}(G)$ be a morphism, corresponding to an object (f', f'') of $(H(G) \times H_{\Delta''}(G))(S)$; here, $f' : \Delta'_S \longrightarrow G_S$ and $f' : \Delta''_S \longrightarrow G_S$ are homomorphisms of group schemes.

Then (f', f'') comes from a (unique) object of $H_\Delta(G)$ if and only if f' and f'' commute, that is, the morphism $(\Delta' \times \Delta'')_S \longrightarrow G_S$ that sends a pair (δ', δ'') into $f'(\delta')f''(\delta'')f'(\delta')^{-1}f''(\delta'')^{-1}$ factors through the identity section $S \longrightarrow G_S$. Then the result follows from the following standard fact.$\diamond$

Sublemma : Let $X \longrightarrow S$ and $Y \longrightarrow S$ be morphisms of schemes, $f, g : X \longrightarrow Y$ morphisms of S-schemes. Assume that $X \longrightarrow S$ is finitely presented, finite and flat, while $Y \longrightarrow S$ is separated. Then the functor from schemes to sets, sending a scheme T into the set of morphisms $T \longrightarrow S$ such that the pullbacks $f_T, g_T : X_T \longrightarrow Y_T$ coincide is represented by a closed subscheme of S.$\diamond$

Now let us prove the result in general. After extending the base field we may assume that it is well-split, that is, a semidirect product $G_1 |\times G_0$, where G_1 is constant, of order not divisible by $char(k)$, and G_0 is diagonalizable. We can split Δ into a finite product of group schemes of type μ_{p^r}, where p is a prime; because of the previous step, we can assume that $= \mu_{p^r}$. If $p = char k$, then $Hom_k(\mu_{p^r}, G_1) = Spec(k)$, so $H_{\mu_{p^r}}(G) = H_{\mu_{p^r}}(G_0)$; and, because of Cartier duality, $H_{\mu_{p^r}}(G_0)$ is a finite disjoint union of copies of $Spec(k)$. If $p \neq char(k)$, then μ_{p^r} is a constant cyclic group scheme of order p^r; hence $H_{\mu_{p^r}}(G)$ is represented by the inverse image of the identity $Spec(k) \subseteq G$ via the map $G \longrightarrow G$ sending x to x^{p^r}. This ends the proof of (iv).$\diamond$

Proof of Proposition 2.4 : Write $\mathcal{X} = [X/G]$. We will write the action of G on the right. There is a right action by conjugation of G on $H^{in}_{\mu_r}(G)$. Consider the closed subscheme $Y \subseteq X \times H^{in}_{\mu_r}(G)$ defined as follows. Let (x, ϕ) be a point of $(X \times H^{in}_{\mu_r}(G))(S)$; that is, x is a morphism of schemes $S \longrightarrow X$, and $\phi : \mu_{r,S} \longrightarrow G_S$ is a homomorphism of group schemes. Then ϕ induces an action of $\mu_{r,S}$ on the S-scheme $S \times X$; then (x, ϕ) is in $Y(S)$ if the section $S \longrightarrow S \times X$ defined by x is fixed by the action of $\mu_{r,S}$. The subscheme $Y \subseteq X \times H^{in}_{\mu_r}(G)$ is G-invariant; the projection $Y \longrightarrow X$ is G-equivariant, and defines a morphism of algebraic stacks $[Y/G] \longrightarrow\ = \mathcal{X}$. It is easy to see that $[Y/G]$ is equivalent to $I_{\mu_r}\mathcal{X}$. This proves that $I_{\mu_r}\mathcal{X}\mathcal{X}$ is representable. To show that it is finite, consider the moduli space M of $\mathcal{X}$. Formation of $I_{\mu_r}\mathcal{X}$ commutes with base change on M, so the question is fppf local on M. Locally on M the stack $\mathcal{X}$ is of the form $[X/\Gamma]$, where $X \longrightarrow M$ is a finite morphism and $\Gamma \longrightarrow M$ is a linearly reductive finite group scheme [AOV, Theorem 3.2]; by further refining in the fppf topology, we can assume that Γ is obtained by pullback from a finite linearly reductive group scheme G over k. From the construction above we have a factorization

$$I_{\mu_r}\mathcal{X} \subseteq [X \times H^{in}_{\mu_r}/G] \longrightarrow\ = \mathcal{X}$$

where the first homomorphism is a closed embedding, and the second is finite because of 2.1.(iv) Finally, the canonical projection morphism $\pi_{\mathcal{X},r}$:

$I_{\mu_r}(\mathcal{X}) \longrightarrow \mathcal{X}$ is representable and finite as the proof of the Deligne-Mumford case (Proposition 2.3) carries over to the case of tame Artin stacks. $\diamond$

Definition 2.6 : Let $\mathcal{X}$ be a tame Artin stack over k. The *cyclotomic inertia stack* $I_\mu(\mathcal{X})$ of $\mathcal{X}$ is the algebraic stack $\coprod_{r\geq 1} I_{\mu_r}(\mathcal{X})$. We denote $\pi_{\mathcal{X}} : I_\mu(\mathcal{X}) \longrightarrow \mathcal{X}$ the morphism $\coprod_r \pi_{\mathcal{X},r}$.

2.2 Computations

In this section, given a quotient stack $[X/GL_n]$, we determine the stacks $I_{\mu_r}(X/GL_n)$ associated to it. First, we determine $I_{\mu_r}(BGL_n)$. To begin with, as it is simpler, we introduce the stack $\tilde{I}_{\mu_r}(BGL_n)$ given, for any scheme T over k, by :

$$\tilde{I}_{\mu_r}(BGL_n)(T) = \{(E,\alpha) | E \in BGL_n(T), \alpha : \mu_{r,T} \longrightarrow Aut_T(E)\}$$

ie $\tilde{I}_{\mu_r}(BGL_n)$ differs from $I_{\mu_r}(BGL_n)$ in that α is no longer required to be injective. So an element of $\tilde{I}_{\mu_r}(BGL_n)(T)$ represents a vector bundle of rank n (still denoted) E, together with a μ_r-action (still denoted) α that is not (contrarily to as is required by the definition of the cylcotomic inertia) faithful. Now, such a pair (E,α) substantially amounts to a direct sum decompsition $\oplus_{\xi\in\hat{\mu_r}} E$ of E into subvector bundles on which μ_r acts through a single character $\xi \in \hat{\mu_r}$. Set Σ_n to be the set of maps (of sets) $\rho : \hat{\mu_r} \longrightarrow \mathbf{N}$ such that $\sum_{\xi\in\hat{\mu_r}} \rho(\xi) = n$. Set $I^\rho_{\mu_r}(BGL_n)(T)$ to be the category of μ_r-equivariant vector bundles $E \cong \oplus_{\xi\in\hat{\mu_r}} E_\xi$ such that, for any ρ in Σ_n, $rk(E_\xi) = \rho(\xi)$.

Remark 2.7 : We have :

$$\tilde{I}_{\mu_r}(BGL_n) = \coprod_{\rho\in\Sigma_n} I^\rho_{\mu_r}(BGL_n)$$

In the following sections, we will explicit $\tilde{I}_{\mu_r}(BGL_n)$ as a quotient of the form $[X/GL_n]$, where X is some algebraic space over k.

2.2.1 Grassmannians

We set $G(d,n)(T)$ to be the Grassmannian functor of subbundles of rank d of $\mathcal{O}^n_T$: according to [St, Lemma 27.22.1], this functor is representable by an algebraic space over k. Now let GL_n act on $G(d,n)$ in the following way : denoting $\alpha_T : G(d,n)(T) \times GL_n(T) \longrightarrow G(d,n)(T)$ the action, it sends a pair $\left(i : F \subset \mathcal{O}^n_T, a : \mathcal{O}^n_T \xrightarrow{\approx} \mathcal{O}^n_T\right)$ to $\alpha(i) = ai : F \hookrightarrow \mathcal{O}^n_T$.

We will eventually consider the following generalization : setting $\underline{d} = (d_1, ..., d_r)$ so that $d_1 + ... + d_r = n$, and $G(\underline{d}, n) = G(d_1, n) \times ... \times G(d_r, n)$ with the given product action of GL_n on it, let us consider the functor $\tilde{F}(\underline{d}, n) : Sch(k) \longrightarrow Grpds$ sending to a scheme T the groupoid of tuples $(E, F_1, ...F_r)$ where $E \in BGL_n(T)$, F_i is a subbundle of E, and $rk(F_i) = d_i$.

We have $I^{\rho}_{\mu_r}(BGL_n)(T) \subset \tilde{F}(\bar{\rho}, n)(T)$, where $\bar{\rho} = (\rho(\xi))_{\xi \in \hat{\mu}_r}$. We will prove the following :

Proposition 2.8 : $\tilde{F}(\underline{d}, n) \cong [G(\underline{d}, n)/GL_n]$

2.2.2 Proof of Proposition 2.8 :

To start with, we go back to the baby case where $r = 1$ and prove Proposition 1 in this case. In this case, $\tilde{F}(\underline{d}, n) : Sch(k) \longrightarrow Grpds$ is the functor sending T to the groupoid of pairs (E, F) consisting of a vector bundle of rank n over T and a vector subbundle F of E of rank d.

Lemma 2.9 : $\tilde{F}(\underline{d}, n) \cong [G(d, n)/GL_n]$ in this case.

Proof : First step : We express $[G(d, n)/GL_n]$ as a $BP(d, n)$ for some subgroup $P(d, n)$ of GL_n. Set $E = k^n$. $G(d, n)(k)$ is the set of d-dimensional subvector spaces F of E. Let $(e_1, ..., e_n)$ be a privileged basis of E, and $p \in G(d, n)(k)$ corresponding to F so that $(e_1, ..., e_d)$ is a basis of F. Any element of $GL_n(k)$ that stabilizes F is then a matrix of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} where$A $\in GLn(k)$. Let $P(d, n)(k)$ denote the group of such matrices : then $Stab_p(\alpha) = P(d, n)(k)$ and we have $G(d, n)(k) \cong GL_n(k)/P(d, n)(k)$. Hence $[G(d, n)/GL_n] = [(GL_n/P(d, n))/GL_n] = BP(d, n)$, the second equality holding because of [W p. 12].

Second step : Consider the morphism of groupoids $\tilde{F}(d, n) \xrightarrow{\phi_T} BP(d, n)(T)$ associating to a

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & E \arrow[d] \\ T \arrow[r, "Id"] & T \end{tikzcd}$$

representing an object in $\tilde{F}(d, n)(T)$ the morphism $Isom^F_T(\mathcal{O}^n_T, E) \longrightarrow T$.

where $Isom^F_T(\mathcal{O}^n_T, E) = \{\phi \in Isom_T(\mathcal{O}^n_T, E) | \phi|_F \in Isom_T(\mathcal{O}^n_T, F)\}$

Remark 2.10 : Recall that the functor $\begin{cases} Vect_r(T) \longrightarrow BGL_r(T) \\ E \longrightarrow Isom_T(\mathcal{O}_T^r, E) \end{cases}$ in [W, lemma 4.1.1], is precisely the one that is shown to be an equivalence between the categories of vector bundles of rank r over T and GL_r-torsors respectively. Here we abuse notation as we denote E both a vector bundle and the locally free sheaf associated to it, and it is the latter that is taken into account in $Isom_T(\mathcal{O}_T^r, E)$ of course. The same is assumed throughout this note.

Now we see that $\tilde{F}(d,n)$ is a stack, because $QCoh_T$ is one in the fpqc topology, and local freeness of fixed finite rank is respected by fpqc morphisms [W, p. 19]. We can straightforwardly adapt the proof of Wang's Lemma 4.1.1. to our case : taking a Zariski cover $\coprod_i T_i \longrightarrow T$ over which E is trivial, as well as F, we have $Isom^F_{T_i}(\mathcal{O}^n_{T_i}, \mathcal{O}^n_{T_i}) \cong T_i \times P(d,n)$, and from a descent datum $(\mathcal{O}^n_{T_i}, g_{ij})$ for E (with $g_{ij} \in P(d,n)(T_i \cap T_j)$, so that $\left(\mathcal{O}^d_{T_i}, g_{ij}|_{\mathcal{O}^d_{T_i}}\right)$ is a descent datum for F). We get likewise a descent datum $(T_i \times P(d,n), g_{ij})$ for $Isom^F_T(\mathcal{O}^n_T, E)$. In the other direction, we take $P \in BP(d,n)(T)$ and a descent datum $(T_i \times P(d,n), g_{ij})$ for P over an fpqc covering $\coprod_i T_i \longrightarrow T$. Set V the standard representation of $P(d,n)$, then ${}_PV$ is a locally free $\mathcal{O}_T$-module of rank n with an indicated rank d locally free submodule with a descent datum $\left(\mathcal{O}^d_{T_i}, g_{ij}|_{\mathcal{O}^d_{T_i}}\right)$ for the indicated submodule. Again, we proceed by comparing the descent data with one another to check that $P \xrightarrow[P]{} V$ is an inverse functor to $\tilde{F}(d,n) \xrightarrow{\phi_T} BP(d,n)(T)$, and thus ϕ is an isomorphism. $\diamond$

Lemma 2.11 : Let now $\underline{d} = (d_1, ..., d_r)$. Then :

$$\tilde{F}(\underline{d}, n) \cong [G(d_1,n) \times ... \times G(d_r,n)/GL_n]$$

Proof : We have :

$$[G(d_1,n) \times ... \times G(d_r,n)/GL_n] \cong \\ [G(d_1,n)/GL_n] \times_{BGL_n} ... \times_{BGL_n} [G(d_r,n)/GL_n]$$

by [W, Lemma 2.3.2]. So, by the previous lemma, we also have

$$[G(d_1,n) \times ... \times G(d_r,n)/GL_n](T) \cong \tilde{F}(d_1,n) \times_{BGL_n} ... \times_{BGL_n} \tilde{F}(d_r,n)(T)$$

Now, an element of the right hand side is an r-tuple of subvector bundles of ranks d_i of ambient rank n vector bundles that are isomorphic to one another : this amounts to an element of $\tilde{F}(\underline{d}, n)$. $\diamond$

2.2.3 Computation of $I^{\rho}_{\mu_r}(BGL_n)$

We now consider the following subfunctor of $\tilde{F}(\underline{d}, n)$, namely $\tilde{H}(\underline{d}, n)$: $Sch(k) \longrightarrow Grpd$ sending a scheme T to the groupoid of tuples $(E, F_1, ..., F_r)$ such that $E = F_1 \oplus ... \oplus F_r$, and $rank(F_i) = d_i$ for each i.

We want to prove that it is an open subfunctor. We first prove :

Proposition 2.12 : The functor $H''(\underline{d}, n) : Sch(k) \longrightarrow Set$ sending a scheme T to an r-tuple of submodules $(V_1, ..., V_r)$ of $\mathcal{O}_T^n$ such that $\mathcal{O}_T^n = V_1 \oplus ... \oplus V_r$ and $rank(V_i) = d_i$ is an open subfunctor of $G(\underline{d}, n) = G(d_1, n) \times ... \times G(d_r, n)$.

Proof : What we need to show is that, for any k-scheme T, there is an open subscheme $U \hookrightarrow T$ such that the following square is cartesian :

$$\begin{tikzcd} U \arrow[r] \arrow[d] & T \arrow[d, "t"] \\ H''(\underline{d}, n) \arrow[r, "\phi"] & G(\underline{d}, n) \end{tikzcd}$$

where ϕ is just the inclusion of functors. Let $t \in G(\underline{d}, n)(T)$ and let us form the following cartesian diagram in the category of functors :

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "y"'] & T \arrow[d, "t"] \\ H''(\underline{d}, n) \arrow[r, "\phi"] & G(\underline{d}, n) \end{tikzcd}$$

It is sufficient to show that U is representable by an open subscheme of T. By construction, we have for any scheme X :

$$U(X) = \left(T \times_{G(\underline{d},n)} H''(\underline{d}, n)\right)(X) =$$
$$\{(a : X \longrightarrow T, b : X \longrightarrow H''(\underline{d}, n)) \,|\phi(b) = ta$$

Now $ta : X \longrightarrow G(\underline{d}, n)$ corresponds to an r-tuple $(W_1, ..., W_r)$ of subsheaves locally free of rank $(d_1, ..., d_r)$ of $\mathcal{O}_X^n$ and $\phi(b)$ corresponds to a decomposition $\mathcal{O}_X^n = W_1' \oplus ... \oplus W_r'$ where $rk(W_i) = d_i$. $ta = \phi(b)$ really means that in fact $\mathcal{O}_X^n = W_1 \oplus ... \oplus W_r$ and (hence) $W_i = W_i'$ for any i. We can reformulate this by saying that :

$$U(X) = \left(T \times_{G(\underline{d},n)} H''(\underline{d},n)\right)(X) =$$
$$\left\{a : X \longrightarrow T | a^*V_1 \oplus ... \oplus a^*V_r \xrightarrow{\cong} \mathcal{O}_X^n\right\}$$

Let $a : X \longrightarrow T$ be any morphism of schemes. Then it is in $U(X)$, ie $a^*V_1 \oplus ... \oplus a^*V_r \longrightarrow \mathcal{O}_X^n$ is an isomorphism, if and only if it is surjective. (Indeed, it is an isomorphism of sheaves if it is a surjective and injective morphism of sheaves, and once we have proven it is surjective we can check injectivity on fibers and we can apply [M]). So a is in $U(X)$ if an only if the cokernel Q of the map $a^*V_1 \oplus ... \oplus a^*V_r \longrightarrow \mathcal{O}_X^n$ is zero, and what we have to verify is that it is an open condition. Recall that, writing $a^*V_i = W_i$, we have the support of Q which is $Supp(Q) = \{p \in T | W_1(p) \oplus ... \oplus W_r(p) \longrightarrow k(p)^n$ not surjective$\}$, which is a closed subscheme of T. And a is in U if and only if $a(X) \subset T$ does not meet Q, ie if and only if it is contained in the open complement $T - Supp(Q)$. So $U = T - Supp(Q)$ and is open. $\diamond$

We can thus represent $H''(\underline{d}, n)$ by an open subscheme $U(\underline{d}, n)$ of (the scheme representing) $G(d, n)$. We consider the restricted action of GL_n on $U(\underline{d}, n)$.

Proposition 2.13 : This action is transitive.

Proof : Le $a, b : T \longrightarrow U$ be two T-schemes in $U(\underline{d}, n)(T)$. We have a isomorphisms of sheaves $\phi : a^*V_1 \oplus ... \oplus a^*V_r \cong \mathcal{O}_T^n$ and $\psi : b^*V_1 \oplus ... \oplus b^*V_r \cong \mathcal{O}_T^n$. We can take local isomorphisms between a^*V_i and b^*V_i, ie a cover $V \longrightarrow U$ such that $(a^*V_i)_V \cong (b^*V_i)_V$. This enables us to dispose of an isomorphism of sheaves from $\mathcal{O}_T^n$ to itself : it corresponds to an element of $GL_{n,T}$. We check that it yields b by acting on a. $\diamond$

Now let us number the elements of $\hat{\mu}_r$, ie set $\hat{\mu}_r \{\xi_1, ..., \xi_r\}$. Set $\underline{d} = (\rho(\xi_1), ..., \rho(\xi_r))$.

Proposition 2.14 : $I^\rho_{\mu_r}(BGL_n) \cong [U(\underline{d}, n)/GL_n]$

Proof : We form the following cartesian square :

$$\begin{tikzcd}
\tilde{I}(\underline{d}, n) \arrow[r, "j"] \arrow[d, "h"'] & \tilde{F}(\underline{d}, n) \arrow[d, "f"] \\
{[U(\underline{d}, n)/GL_n]} \arrow[r, hook, "i"] & {[G(\underline{d}, n)/GL_n]}
\end{tikzcd}$$

This defines $\tilde{I}(\underline{d}, n)$. f is an isomorphism by Proposition 2.8 and i is an open embedding by Proposition 2.12, moreover for any k-scheme T, we have :

$$\tilde{I}(\underline{d}, n)(T) = \{((E, F_1, ..., F_r), (\mathcal{O}_T^n = V_1 \oplus ..._r))|(\forall \coprod_i T_i \longrightarrow T) F_j \times_T T_i \cong V_j\}$$

Thence $\tilde{I}(\underline{d}, n)(T) \cong I^{d}_{\mu_r}(BGL_n)$ which can be restated as $I^{\rho}_{\mu_r}(BGL_n) \cong [U(\underline{d}, n)/GL_n]$. $\diamond$

As a result, we have $\tilde{I}_{\mu_r}(BGL_n)(T) \cong \coprod_d [U(\underline{d}, n)/GL_n]$ Let $\eta = (V_1 \oplus ... \oplus V_r \cong \mathcal{O}_T^n)$ denote an element of $U(\underline{d}, n)(T)$. We have isomorphisms of sheaves $V_1 \cong \mathcal{O}_T^{d_1}$, ..., $V_r \cong \mathcal{O}_T^{d_r}$, so that the stabilizer of an element of $U(\underline{d}, n)(T)$ is the set of isomorphisms of $\mathcal{O}_T^n = \mathcal{O}_T^{d_1} \times .. \times \mathcal{O}_T^{d_r}$ that respect each factor $\mathcal{O}_T^{d_i}$; these can be represented by block diagonal matrices, with as blocks invertible matrices of rank $d_1, ..., d_r$. Whence $[U(\underline{d}, n)/GL_n] = BStab(\eta) = BGL_{d_1} \times ... \times BGL_{d_r}$. We have finally shown :

Proposition 2.15 : $\tilde{I}_{\mu_r}(BGL_n) \cong \coprod_{\underline{d}} BGL_{d_1} \times ... \times BGL_{d_r}$

2.2.4 Computation of $I_{\mu_r}(BGL_n)$

We have obviously an inclusion of functors $I_{\mu_r}(BGL_n) \subset \tilde{I}_{\mu_r}(BGL_n)$. We have to understand what being in the image of this inclusion means for an element of $\tilde{I}_{\mu_r}(BGL_n)(T)$ for all T. That is to say, we have to see how the faithfulness of the μ_r-action on vector bundles translates into a property of the eigenbundle decomposition. To this effect, let us consider $\pi : E \longrightarrow T$ a rank n vector bundle with a μ_r-action, that is $E \cong \oplus_\xi E_\xi$. Now on each factor E_ξ, μ_r acts through the character ξ and the action is free, if and only if $E_\xi \neq 0$ for a generator ξ of $\tilde{\mu}_r \cong \mathbf{Z}/(n)$. In the end, $I_{\mu_r}(BGL_n)$ turns out to be a coproduct of finite products of classifying spaces of general linear groups as was the case for $\tilde{I}_{\mu_r}(BGL_n)$, except that the indexing set is restricted.

2.2.5 Computation of $I_{\mu_r}([X/GL_n])$ for a GL_n-scheme X

We have a natural morphism $\nu_{X,n} : I_{\mu_r}([X/GL_n]) \longrightarrow I_{\mu_r}(BGL_n) \times_{BGL_n} [X/GL_n]$. (Indeed, we can form the latter fiber product in the category of stacks using the canonical morphisms $[X/GL_n] \longrightarrow BGL_n$ and $I_{\mu_r}(BGL_n) \longrightarrow BGL_n$, and it is obvious that $I_{\mu_r}([X/GL_n])$ maps naturally to both $I_{\mu_r}(BGL_n)$ and $[X/GL_n]$). The same holds for the 'tilded' versions of inertia stacks.

Proposition 2.16 : $\nu_{X,n}$ is fully faithful.

Proof : Indeed, let T be a k-scheme : an element of $\tilde{I}_{\mu_r}([X/GL_n])$ is a pair (ξ, α) where ξ denotes a diagram $\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd}$, and α denotes a morphism $\alpha : \mu_{r,T} \longrightarrow Aut_T(\xi)$. An element of the fiber product consists of a diagram $\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd}$, together with a morphism $\beta : \mu_{r,T} \longrightarrow Aut_T(E)$, ie a pair (ξ, β). The latter is in the image of the morphism if and only if β factors through α, ie if we have a commutative diaram :

$$\begin{tikzcd} \mu_{r,T} \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & Aut_T(E) \\ & Aut_T(\xi) \arrow[u, hook, "i"'] \end{tikzcd}$$

Where $i : Aut_T(\xi) \hookrightarrow Aut_T(E)$ is the natural inclusion. This clearly show that every α determines a unique β so as to fit the the above diagram, whence the fully faithfulness.$\diamond$

Remark 2.17 : We know X to be a GL_n-equivariant scheme, and it is natural to think of embedding μ_r into GL_n so that restricting the GL_n-action yields the same action as we are considering. In fact, we will consider the embedding of μ_r into $GL_{d_1} \times ... \times GL_{d_r}$ composed with the embedding of the latter in GL_n using block diagonal matrices. (We will denote this embedding $\mu_r^{(\underline{d})}$ when the value of $\underline{d}$ is to be specified).

Remark 2.18 : Notice that the centralizer of μ_r in GL_n for the embedding we consider is precisely $GL_{d_1} \times ... \times GL_{d_r}$.

Now let us compute $I_{\mu_r}([X/GL_n])$. Proposition 2.14 implies, using the 'change of space formula' [W, Lemma 2.3.1], that we have :

Proposition 2.19 : $I_{\mu_r}(BGL_n) \times_{BGL_n} [X/GL_n] = \coprod_d [X \times U(d,n)/GL_n]$

On the other hand, we have an obvious section $s : Spec(k) \longrightarrow U(\underline{d}, n)$ of the canonical morphism $U(\underline{d}, n) \longrightarrow Spec(k)$, corresponding to the direct sum decomposition $k^n = k^{d_1} \oplus ... \oplus k^{d_r}$. This enables us to write $[X \times U(\underline{d}, n)/GL_n] \cong [X/GL_{d_1} \times ... \times GL_{d_r}]$. Indeed, any GL_n-equivariant morphism $E \longrightarrow X \times U(\underline{d}, n)$ now yields a morphism $E \longrightarrow X \times U(\underline{d}, n) \longrightarrow X$ (composing with $id_X \times s$), while the composition $E_1 \oplus ... \oplus E_r \cong E \longrightarrow T$ (obtained by keeping track of the morphism $E \longrightarrow U(\underline{d}, n)$) is a $GL_{d_1} \times ... \times GL_{d_r}$-bundle.

In the end we can write :

Proposition 2.20 :

$$I_\mu\left([X/GL_n]\right) = \coprod_{r,\underline{d}} [X^{\mu_r^{(\underline{d})}}/GL_{d_1} \times ... \times GL_{d_r}] = \coprod_{r,d} [X^{\mu_r^{(\underline{d})}}/C_{GL_n}(\mu_r^{(\underline{d})})]$$

2.3 Construction of $I_\mu(f)$

We now construct, for f not necessarily representable, the morphisms $I_\mu(f)$ functorially and so that their pushforwards is defined. This is where we use the local structure theory of tame Artin stacks. It will be crucially used in the fourth part of this work.

Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of Tame Artin stacks over a field k. Let $\alpha : B\mu_{r,T} \xrightarrow{rep} \mathcal{X}_T$ be a representable morphism in $I_{\mu_r}(\mathcal{X})(T)$ for some k-scheme T. We shall define $I_\mu(f)(\alpha) \in I_\mu(\mathcal{Y})(T)$. From $\alpha : B\mu_{r,T} \xrightarrow{rep} \mathcal{X}_T$ and $f_T : \mathcal{X}_T \longrightarrow \mathcal{Y}_T$ which is proper, we can naively form the composition $f_T\alpha : B\mu_{r,T} \longrightarrow \mathcal{Y}_T$, but in general it won't be representable. Hence, our task is to associate functorially to any morphism (not representable) $B_{\mu_r,T} \longrightarrow \mathcal{Y}_T$ a representable morphism $B_{\mu_s,T} \longrightarrow \mathcal{Y}_T$ for some s dividing r, Zariski locally on T. To this end, let now $\alpha : B\mu_{r,T} \longrightarrow \mathcal{Y}_T$ be a not necessarily representable morphism (it is understood that α comes from composing a representable morphism $B\mu_{r,T} \longrightarrow \mathcal{X}_T$ with f_T).

On the other hand, recall from [AOV, Proposition 3.6] that $\mathcal{Y}$ has a moduli space M over k and that $\mathcal{Y} \approx [Y/G]$ étale locally with respect to M, where $G \longrightarrow M$ is a linearly reductive group scheme and Y is some algebraic space. It is enough to work fppf locally.

Proposition 2.21 : $\forall \xi : T \longrightarrow \mathcal{Y} \in \mathcal{Y}(T)$, we have that $Aut_T(\xi) \subseteq G \times_M T = G_T$.

Proof : Indeed, for $\mathcal{Y} = [Y/G]$, we have $I_{\mathcal{Y}} = [Y'/G]$, where $Y' \subseteq G \times_M Y$ is the subscheme of pairs (g, y) such that $gy = y$. $\diamond$

Proposition 2.21 allows one to reduce to the case of a homomorphism $\mu_{r,T} \longrightarrow G_T$, which we can risklessly still denote α. From such a homomorphism, the aim is now to construct fppf-locally and functorially with respect to T a monomorphism of group schemes $\mu_{s,T} \hookrightarrow G_T$ through which α factors locally on T for some s dividing r.

Now, we can write a decomposition $\mu_r \approx \mu_{p^n} \times \mu_d$ with d an integer prime to the characteristic p of k. On the other hand, [AGV, Thm 2.19] gives us fppf-locally an exact sequence in the category of groups :

$$1 \longrightarrow \Delta \xrightarrow{c} G \xrightarrow{a} H \longrightarrow 1$$

where Δ is a diagonalizable group scheme and H is etale and tame.

Note that while μ_d is etale, μ_{p^n} is not. Set $\gamma = ker(\mu_r \longrightarrow G)$: it is a finite subgroup scheme of μ_r. What we want is γ to be flat so that one can form the quotient μ_r/γ which will be locally on T of the form μ_s for s dividing r and define $\tilde{\alpha} : \mu_s \longrightarrow G$ to be the monomorphism to be associated to α.

Now, we have that $a\alpha i(\mu_{p^n})$ is trivial because μ_{p^n} is infinitesimal and H is étale, denoting $i : \mu_{p^n} \hookrightarrow \mu_r$.

So, we have $Im(\alpha i) \subseteq ker\alpha = Imc$, whence there exists a morphism η fitting in the following commutative diagram with exact rows :

$$\begin{tikzcd}
1 \arrow[r] & \mu_{p^n,T} \arrow[r, "i"] \arrow[d, "\eta"] & \mu_{r,T} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mu_{d,T} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Delta \arrow[r, "c"] & G \arrow[r, "a"] & H \arrow[r] & 1
\end{tikzcd}$$

where furthermore the top exact sequence splits.

Zariski-locally on T, we have the following factorization :

$$\begin{tikzcd}
\mu_{p^n,T} \arrow[r] \arrow[dr] & \Delta \\
& \mu_{p^m,T} \arrow[u, hook]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mu_{d,T} \arrow[r] \arrow[dr] & H \\
& \mu_{d',T} \arrow[u, hook]
\end{tikzcd}$$

Set $\tilde{\Delta} = \mu_{p^m}$ and $\tilde{H} = \mu_{d'}$. We have :

$$Im\alpha \subseteq \tilde{H} \times' \Delta \subseteq H \times' \Delta$$

and a commutative diagram with exact rows :

$$\begin{tikzcd}
1 \arrow[r] & \mu_{p^n,T} \arrow[r] \arrow[d] & \mu_{r,T} \arrow[r] \arrow[d, "\alpha"'] & \mu_{d,T} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Delta_T \arrow[r] & \tilde{H}_T \times' \Delta_T \arrow[r] & \tilde{H}_T \arrow[r] & 1
\end{tikzcd}$$

Now, $\tilde{\Delta} \subseteq \tilde{H} \times' \Delta$ is left pointwise fixed by $\tilde{H}$ and :

$$\tilde{H} \times \Delta = \tilde{H} \times' \tilde{\Delta} \subseteq \tilde{H} \times \Delta$$

Moreover, since $Hom(\mu_{d,T}, \tilde{\Delta}) = 1$, we have that $Im\alpha \subseteq \tilde{H} \times \Delta$.

Summarizing, we have the following commutative diagram with exact rows, locally on T :

$$\begin{tikzcd}
1 \arrow[r] & \mu_{p^n,T} \arrow[r] \arrow[d, two heads] & \mu_{r,T} \arrow[r] \arrow[d, two heads] & \mu_{d,T} \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \Delta_T \arrow[r] \arrow[d, "="'] & \tilde{H}_T \times \Delta_T \arrow[r] \arrow[d, "="'] & \tilde{H}_T \arrow[r] \arrow[d, "="'] & 1 \\
1 \arrow[r] & \mu_{p^m} \arrow[r] & \mu_{p^m d'} \arrow[r] & \mu_{d'} \arrow[r] & 1
\end{tikzcd}$$

Hence, we get a surjective homomorphism $\mu_{r,T} \twoheadrightarrow \mu_{p^m d'}$ and we can choose $s = p^m d'$. This defines our morphism $I_\mu(f)$, functorially with respect to T and fppf-locally.

In other words, we have proven the following :

Proposition 2.22 : Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of tame Artin stacks. Then it induces a natural morphism $I_\mu(f) : I_\mu(\mathcal{X}) \longrightarrow I_\mu(\mathcal{Y})\diamond$

Chapter 3

Decomposition theorems in equivariant algebraic K-theory and algebraic K-theory of algebraic stacks

In this chapter, we use the equivariant higher algebraic K-theory of algebraic spaces introduced and developed in [Th1]. In the latter work, equivariant higher algebraic K-groups are defined using the exact category of equivariant coherent sheaves by applying to the latter Quillen's Q-construction [Q].

Our first step towards a Grothendieck-Riemann-Roch formula for tame quotient stacks involves a comparison of equivariant algebraic K-theory and a variant of it, introduced by Vezzosi and Vistoli in [VV], coined geometric K-theory. Let us fix a base field k for both X and G. The following definition of geometric K-groups of a noetherian tame quotient stack $[X/G]$ (Definition 0.3 below) is more intrinsic than the original one in [VV] and is due to Vistoli. (Using the terminology of [VV], we assume the G-action to be *sufficiently rational*).

Recall that $K_0(\mathcal{X})$ is the Grothendieck group of isomorphism classe of locally free sheaves of finite rank on $\mathcal{X}$, and that this construction is contravariant for maps of tame stacks.

Let, from now on, ζ_r denote a chosen primitive r-th root of 1, for $r \geq 1$.

Definition 3.1 : Let r be a positive integer. The representation ring $R\mu_r$ decomposes as $R\mu_r = \prod_{s|r} \mathbf{Q}(\zeta_s)$. Let $\mathcal{X} = [X/G]$ be a quotient stack. We let $\Sigma_r^{\mathcal{X}}$ be the subset of $K_0(\mathcal{X})$ consisting of differences of isomorphism classes α of locally free sheaves of finite rank such that, for all field extensions $k \hookrightarrow K$ and all representable morphisms $\phi : B\mu_{r,K} \longrightarrow \mathcal{X}$, the projection of $\phi^*\alpha \in R\mu_r$ to $\tilde{R}\mu_r = \mathbf{Q}(\zeta_r)$ is non-zero.

The latter are obviously multiplicative subsets.

Definition 3.2 : The μ_r-localization $K_*(\mathcal{X})_{(\mu_r)}$ of $K_*(\mathcal{X})$ is the $K_0(\mathcal{X})$-module $\left(\Sigma_r^{\mathcal{X}}\right)^{-1} K\left(\mathcal{X}\right)$.

The μ_r-localization $K'_*(\mathcal{X})_{(\mu_r)}$ of $K_*(\mathcal{X})$ is the $K_0(\mathcal{X})$-module $\left(\Sigma_r^{\mathcal{X}}\right)^{-1} K'\left(\mathcal{X}\right)$.

Definition 3.3 : The geometric K-theory of $\mathcal{X}$ is $K_*(\mathcal{X})_{geom} := K_*(\mathcal{X})_{(\mu_1)}$.
The geometric K'-theory of $\mathcal{X}$ is $K'_*(\mathcal{X})_{geom} := K'_*(\mathcal{X})_{(\mu_1)}$.

Remark 3.4 : The localizations of K-theory and K'-theory considered in the preceding definitions, and in particular the definition of geometric K-theory, are not a priori the ones considered in [VV]. However, in the case of a stack given by a global quotient of an algebraic space by a group scheme, they do coincide with the localizations considered in [VV], which is the content of the following proposition. We refer to the Appendix for the notations.

Proposition 3.5 : Let $\mathcal{X}$ be a quotient stack. The μ_r-localization of $K'_*(\mathcal{X})$ coincides with the product of localizations of $K'_*(X)$ at m_σ for every σ of order r. Note that for $r = 1$, this gives two approaches to geometric K'-theory.

Proof : Let $\mathcal{X} = [X/G]$. Let σ_1, ..., σ_m be the dual cyclic subgroups $\sigma \in C_r(G)$ such that $K'_*(X,G)_\sigma \neq 0$. Then

$$\prod_{\sigma\in C_r(G)} K'_*(X,G)_\sigma = \prod_{i=1}^m K'_*(X,G)_{\sigma_i}$$

Let S_r be the multiplicative system $RG-(m_{\sigma_1}\cup...\cup m_{\sigma_m})$. It follows easily from the fact that the support of $K'_*(X,G)$ in $Spec(RG)$ has a finite number of closed points, that the induced map $S_r^{-1}K'_*(X,G) \longrightarrow \prod_{i=1}^m K'_*(X,G)_{\sigma_i}$ is an isomorphism.

We claim that the image of $S_r \subseteq RG$ in $K_0(X,G)$ through the homomorphism $RG \longrightarrow K_0(X,G)$ is contained in Σ_r. In fact, let $\alpha \in S_r$ and let $B\sigma \longrightarrow \mathcal{X}$ be a representable morphism. After extending the base field we may assume that there is an embedding of σ into G_K (K denoting the new base field), and a rational point $p \in X(K)$ which is fixed under the action of σ. Hence $X_K^\sigma \neq \emptyset$, so σ is conjugate to some σ_i.The morphism $RG \longrightarrow \tilde{R}\sigma_i = \mathbf{Q}(\zeta_r)$ defined by the embedding $\sigma_i \subseteq G$ and the composite

$$RG \longrightarrow K_0(X,G) \longrightarrow K_0(B_K(\sigma)) = R\sigma \longrightarrow \tilde{R}\sigma = \mathbf{Q}(\zeta_r)$$

defined by the morphism $B_K(\sigma) \longrightarrow \mathcal{X}$ coincide; since by hypothesis α does not map to 0 in $\mathbf{Q}(\zeta_r)$, it follows that the image of α in $K_0(\mathcal{X})$ is in Σ_r as claimed.

Notice that if $\alpha \in K_0(\mathcal{X}) = K_0(X,G)$, the multiplication by α on $K'_*(X,G)$ gives a homomorphism of RG-modules, so that it preserves the multiplication above. We get a factorization

$$K'_*(X,G) \longrightarrow S_r^{-1}K'_*(X,G) = \prod_{\sigma\in C_r(G)} K'_*(X,G)_\sigma \longrightarrow \Sigma_r^{-1}K'_*(X,G) = K'_*(X,G)_{(\mu_r)}$$

and we need to show that the resulting homomorphism

$$\prod_{\sigma\in C_r(G)} K'_*(X,G)_\sigma \longrightarrow K'_*(X,G)_{(\mu_r)}$$

is an isomorphism.

This is equivalent to the following. First of all, notice that if $\alpha \in K_0(X,G)$, multiplication by α gives an endomorphism of the RG-module $K'_*(X,G)$, hence it descends to endomorphisms of $K'_*(X,G)_\sigma$ for each $\sigma \in C(G)$. We need to prove that if α is in Σ_r and the order of σ is r, then α induces an automorphism of $K'_*(X,G)_\sigma$. The proof of this fact is somewhat involved and requires several steps.

Step 1 : Assume that G is a finite diagonalizable group over k acting trivially on X. Then $\mathcal{X}$ is of the form $X \times BG$. Then we have $K'_*(\mathcal{X}) = K'_*(X) \otimes RG$ (as is proven in [Th1]) and $K_0(\mathcal{X}) = K_0(X) \otimes RG$. Then we have a decomposition $RG = \prod_{\sigma\in C(G)} \tilde{R}\sigma$. Then $K'_*(X,G) = K'_*(X) \otimes \tilde{R}\sigma$ and $K'_*(X,G)_{(r)} = \prod_{\sigma\in C_r(G)} K'_*(X) \otimes \tilde{R}\sigma$. The action of $K_0(X,G)$ on $K_0(X,G)_\sigma$ is induced by the action of $K_0(X,G) = K_0(X) \otimes R\sigma$ on $K'_*(X,G) = K'_*(X) \otimes \tilde{R}\sigma$, and this in turn is induced by the action of $K_0(X)$ on $K'_*(X)$. This factors through an action of $K_0(X) \otimes \tilde{R}\sigma$, thus it is enough to show that the image of α in $K_0(X) \otimes \tilde{R}\sigma$ is invertible. Let X_1, ..., X_m be the connected components of X. Then $K_0(X) = \prod_{i=1}^m K_0(X_i)$. There is a rank homomorphism $rk : K_0(X_i) \longrightarrow \mathbf{Q}$ whose kernel is nilpotent by a classical result of Grothendieck (see eg [SGA6]); from this we obtain a homomorphism

$$K'_*(X) \otimes \tilde{R}\sigma \longrightarrow (\tilde{R}\sigma)^m$$

with nilpotent kernel. Hence it is enough to show that the image of α in each copy of $\tilde{R}\sigma$ is non-zero. But α_i is obtained as follows : choose a point $Spec(K) \longrightarrow X_i$, this gives a morphism $BG_K \longrightarrow X_i \times BG$ where $G_K = G\times_{Spec(k)} Spec(K)$; by composing with the morphism $B\sigma_K \longrightarrow BG_K$ induced by the embedding $\sigma \subseteq G$, we get a morphism $B\sigma_K \longrightarrow X \times BG$ which induces a morphism $K_0(X,G) \longrightarrow \tilde{R}\sigma$ is immediately seen to coincide with the homomorphism $K'_*(X_i) \otimes \tilde{R}\sigma \longrightarrow \tilde{R}\sigma$. Since this is not zero, by hypothesis, this concludes the proof.

Step 2 : the case of a torus action. Here we assume that $G = \mathbf{G}_m^n$ is a split torus. If $Y \subseteq X$ is a G-invariant subscheme of X, we denote by α_Y the restriction of α to $K_0(Y,G)$. By noetherian induction, we can assume that

α_Y induces an automorphism of $K'_*(Y,G)_\sigma$ for all proper closed G-invariant subschemes $Y \longrightarrow X$. By [Th2] there exists an open G-invariant subscheme U that is G-equivariantly isomorphic to a scheme of the form $V \times (G/\Gamma)$ where $\Gamma \subseteq G$ is a finite diagonalizable subgroup scheme of G, V is some k-scheme, and the action of G over $V \times (G/\Gamma)$ is induced by the trivial action of G on V and the action of G/Γ by translation.

Assume that $X = U$. In this case $K'_*(X,G) = K'_*(U,\Gamma) = K'_*(U) \otimes R\Gamma$. If γ is not contained in Γ, then $X^\sigma = \emptyset$, so that $K'_*(X,G)_\sigma = 0$ and the result is obvious. If $\sigma \subseteq \Gamma$, then it is easy to convince oneself that $K'_*(X,G)_\sigma = K'_*(V,\Gamma)_\sigma$, and so the result follows from the preceding case.

If now $X \neq U$, call Y the complement of U with its reduced scheme structure. This is G-invariant. For each $i \geq 0$, we get a commutative diagram :

$$\begin{tikzcd}
K'_{i+1}(U,G) \arrow[r] \arrow[d, ".\alpha_U"'] & K'_i(Y,G) \arrow[r] \arrow[d, ".\alpha_Y"'] & K'_i(X,G) \arrow[r] \arrow[d, ".\alpha"'] & K'_i(U,G) \arrow[r] \arrow[d, ".\alpha_U"'] & K'_{i-1}(Y,G) \arrow[d, ".\alpha_Y"'] \\
K'_{i+1}(U,G) \arrow[r] & K'_i(Y,G) \arrow[r] & K'_i(X,G) \arrow[r] & K'_i(U,G) \arrow[r] & K'_{i-1}(Y,G)
\end{tikzcd}$$

with exact rows. Since the result is true for Y and for U,so that the first, second, fourth and fifth columns are isomorphisms, we see that the third column is also an isomorphism, as claimed.

Step 3 : the case $G = GL_n$. Consider the standard maximal torus $T \subseteq G$ of diagonal matrices, and its Weyl group S_n. The subgroup $\sigma \subseteq G$ is conjugate to a subgroup of T, well defined, up to conjugation by an element of S_n. In other words, $C(G)$ is in natural bijective correspondence with the set of orbits for the action of S_n on $C(T)$. We have $K'_*(X,G) = K'_*(X,T)^{S_n}$ (see [VV, Section 4]); hence

$$\prod_{\sigma \in C(G)} K'_*(X,G)_\sigma \approx \left(\prod_{\sigma \in C(T)} K'_*(X,T)_\sigma\right)^{S_n}$$

If $\Gamma \subseteq S_n$ is the stabilizer of $\sigma \subseteq T$ under the action of S_n, we deduce that

$$K'_*(X,G)_\sigma \approx K'_*(X,\Gamma)_\sigma^{S_n}$$

If β is the image in $K_0(X,T)$ of $\alpha \in K_0(X,G)$, then multiplication by β on $K'_*(X,T)_\sigma$ is Γ-equivariant, and its restriction to $K'_*(X,G)_\sigma = K'_*(X,T)_\sigma^\Gamma$ is multiplication by α. Since multiplication by β on $K'_*(X,T)_\sigma$ is an automorphism by the previous step, we deduced that multiplication by α is also an automorphism, as claimed.

Step 4 : in the most general case, choose an embedding $G \subseteq GL_n$, and set

$$Y = X \times^G GL_n = (X \times GL_n)/G$$

We have canonical isomorphisms

$$K'_*(X,G) \approx K'_*(Y,GL_n) \text{ and } K_0(X,G) \approx K_0(Y,GL_n)$$

The embedding $G \subseteq GL_n$ induces a homomorphism $RGL_n \longrightarrow RG$. If m'_σ is the inverse image of m_σ in RGL_n, then $K'_*(Y,GL_n)_\sigma = (RGL_n - m_\sigma)^{-1}K'_*(Y,GL_n)$ is a localization of $K'_*(X,G)_\sigma = (RG - m_\sigma)^{-1}K'_*(X,G)$. Since multiplication by α gives an automorphism of $K'_*(Y,GL_n)$ by the previous step, it also induces isomorphisms of all the localizations of $K'_*(Y,GL_n) \approx K'_*(X,G)$ as an RG-module. This concludes the proof of the proposition.$\diamond$

We will prove later that the geometric K'-theory of a tame Artin stack is isomorphic to the algebraic K'-theory of its moduli space. A direct consequence of Proposition 3.5 is that the algebraic K'-theory of a stack as a product of its localizations.

Chapter 4

On geometric K'-theory

In this chapter, we give more results pertaining to the K-theory of tame stacks, and then discuss possible extensions to a broader context than that of tame Artin stacks that are global quotients or that go beyond the study of K'-theory modulo torsion afforded by Grothendieck-Riemann-Roch formalisms. The first result we should like to disclose is the following theorem, which says that the geometric K'-groups of a tame quotient stack are isomorphic to the ordinary K'-groups of its moduli space [SV].

Theorem 4.1 : Let $\mathcal{X}$ be a quotient tame Artin stack. Let $p : \mathcal{X} \longrightarrow M$ be the projection to its coarse moduli space. Then the composition :

$$K'_*(\mathcal{X})_{geom} \subseteq K'_*(\mathcal{X}) \xrightarrow{p_*} K'_*(M)$$

is an isomorphism, where the first morphism is a (canonical) direct summand inclusion.

Remark : This result can be seen as an analogue of [T1, Corollaire 3.11]. What Toën calls "étale K-theory" in [T1,T2] is indeed very close to what is here called "geometric K-theory" (and should not be confused with Friedlander's étale K-theory [1] [F1,F2]). Toën's construction and Vistoli's coincide on tame Deligne-Mumford stacks that are global quotients. However, the proof of [T1, Corollaire 3.11] can only work for Deligne-Mumford stacks, because it relies on the fact that such stacks are locally quotients by finite group actions.

Theorem 4.1 results from a number of lemmas [SV] that we reproduce here together with their demonstration. Lemma 4.4 below is arguably the most important of them, and reveals a discrepancy between geometric K-theory and algebraic K-theory. We first need a deeper investigation of the

[1] The latter is closely related to *topological* K-theory, and fits in the context of étale homotopy theory.

functoriality of the decomposition theorem $K'_*(\mathcal{X}) \cong K'_*(\mathcal{X})_{geom} \oplus K'_*(\mathcal{X})_{extra}$ proven in Chapter 2, which we call the fundamental decomposition.

Proposition 4.2 : Let $f : \mathcal{Y} \longrightarrow \mathcal{X}$ be a representable morphism of tame Artin stacks. Then :
(i) If f has finite flat dimension, then f^* respects the fundamental decomposition.
(ii) If f is proper, then f_* respects the fundamental decomposition.

Proof : Write $\mathcal{X}$ as the global quotient stack $[X/G]$ of a sufficiently rational action of a linearly reductive group G on an algebraic space X. Then recall from Chapter 3 that :

$$K'_*(\mathcal{X}) \cong \prod_{\sigma \in C(G)} K_*(X,G)_\sigma$$

so that $K_*(X,G)_{geom} = K_*(X,G)_{\{1\}}$ and $K_*(X,G)_{extra} = \prod_{\sigma \neq 1} K_*(X,G)_\sigma$. Then it suffices to show that f^* (in the finite flat dimension case) and f_* (in the proper case) preserve the decomposition $K'_*(\mathcal{X}) \cong \prod_{\sigma \in C(G)} K(X,G)_\sigma$. Since in both cases f is representable, setting $Y = \mathcal{Y} \times_{\mathcal{X}} X$ defines an algebraic space such that $\mathcal{Y} \approx [Y/G]$ and f is induced from a G-equivariant map $Y \longrightarrow X$. Since K'-groups are then RG-modules and f^* (resp. f_*) induces an RG-linear homomorphism, the proposition follows.$\diamond$

Proposition 4.3 : Let $f : \mathcal{Y} \longrightarrow \mathcal{X}$ be a morphism of tame quotient Artin stacks. Then :
(i) If f is of finite flat dimension, then it induces a homomorphism

$$f^* : K'_*(\mathcal{X})_{geom} \longrightarrow K'_*(\mathcal{Y})_{geom}$$

such that the following diagram commutes :

$$\begin{tikzcd} K'_*(\mathcal{X}) \arrow[d, "f^*"'] & K'_*(\mathcal{X})_{geom} \arrow[l] \arrow[d, "f^*"'] \\ K'_*(\mathcal{Y}) & K'_*(\mathcal{Y})_{geom} \arrow[l] \end{tikzcd}$$

(ii) If f is proper, then it induces a homomorphism

$$f_* : K'_*(\mathcal{Y})_{geom} \longrightarrow K'_*(\mathcal{X})_{geom}$$

such that the following diagram commutes :

$$\begin{tikzcd}
K'_*(\mathcal{Y})_{geom} \arrow[r] \arrow[d, "f_*"'] & K'_*(\mathcal{Y}) \arrow[d, "f_*"'] \\
K'_*(M_{\mathcal{X}})_{geom} \arrow[r] & K'_*(\mathcal{X})
\end{tikzcd}$$

Proof : The homomorphism f_* and f^* defined above are clearly unique; they make K_{0geom} into a contravariant functor for maps of finite flat dimension, and a covariant functor for proper maps. The first part is a consequence of the fact that $f^* : K_0(\mathcal{X}) \longrightarrow K_0(\mathcal{Y})$ carries $\Sigma_1^{\mathcal{X}}$ into $\Sigma_1^{\mathcal{Y}}$ (the latter fact can be seen using the same reasoning we used in Section 2.6). For the second part, write $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$, where X and Y are algebraic spaces, and G and H are affine algebraic groups acting sufficiently rationally on X and Y. The projection $X \longrightarrow \mathcal{X}$ and the composite $Y \longrightarrow \mathcal{Y} \xrightarrow{f} \mathcal{X}$ are respectively G and H invariant. Set $Z = X \times_{\mathcal{X}} Y$; there is a natural action of $G \times H$ on Z, and it is easy to see that $[Z/G \times H] = \mathcal{Y}$. The projection $Z \longrightarrow \mathcal{X}$ is equivariant for the projection $pr_1 : G \times H \longrightarrow G$, and the induced morphism $\mathcal{Y} = [Z/G \times H] \longrightarrow \ = \mathcal{X}$ is isomorphic to f. The homomorphism $f_* : K'_*(Z, G \times H) \longrightarrow K'_*(X, G)$ is a homomorphism of RG-modules, where $K'_*(Z, G \times H)$ is considered as an RG-module through the homomorphism $pr_1 : RG \longrightarrow R(G \times H)$. Consider the decompositions

$$K'_*(Z, G \times H) = \prod_{\rho \in C(G \times H)} K'_*(Z, G \times H)_\rho$$

and

$$K'_*(X, G) = \prod_{\sigma \in C(G)} K'_*(X, G)_\sigma$$

If $\sigma \in C(G)$, we can also consider the σ-localization $K'_*(Z, G \times H)_\sigma = (RG_{m_\sigma})^{-1} K'_*(Z, G \times H)$ of $K'_*(Z, G \times H)$; it is immediate to see that it coincides with the quotient[2] $\prod_{\rho \in C'(G)} K'_*(Z, G \times H)_\rho$ of $K'_*(Z, G \times H)$. If

$$\eta \in K'_*(\mathcal{Y})_{geom} = K'_*(Z, G \times H)_1 \subseteq K'_*(Z, G \times H)$$

then, by definition the image of η in $K'_*(Z, G \times H)_\sigma$ is zero for every $\sigma \in C(G)$ with $\sigma \neq 1$; this implies that the image of $f_*\eta \in K'_*(\mathcal{X})$ in

$$K'_*(\mathcal{X})_{estra} = \prod_{\rho \in C(G \times H), \rho \neq 1} K'_*(X, G \times H)_\sigma$$

is zero, which implies $f_*(\xi) \in K'_*(X)_{geom}$, as claimed.$\diamond$

Lemma 4.4 : Let $\pi : \mathcal{X}' \longrightarrow \mathcal{X}$ be a finite flat morphism of tame Artin stacks. Then the following two sequences are exact :

[2] $C'(G)$ here stands for the subset of $C(G)$ consisting of ρs that surject onto σ

$$0 \longrightarrow K'_*(\mathcal{X})_{geom} \xrightarrow{\pi^*} K'_*(\mathcal{X}')_{geom} \xrightarrow{pr_1^*-pr_2^*} K'_*(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}')_{geom}$$

$$K'_*(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}')_{geom} \xrightarrow{pr_{1*}-pr_{2*}} K'_*(\mathcal{X}')_{geom} \xrightarrow{\pi_*} K'_*(\mathcal{X})_{geom} \longrightarrow 0$$

Proof : Consider the class $\alpha := [\pi_*\mathcal{O}_{\mathcal{X}'}] \in K_0(\mathcal{X})$; because of the hypotheses, we have $\alpha \in \Sigma_1^{\mathcal{X}}$. Hence, multiplication by α gives an automorphism of $K_0(\mathcal{X})_{geom}$. Let $\xi \in K'_*(\mathcal{X})$ be such that $\pi^*\xi = 0$. Then by the projection formula we have $0 = \pi_*\pi^*\xi = \alpha\xi$; hence $\xi = 0$. This proves that π^* is injective.
Now take $\xi' \in K'_*(\mathcal{X})$ such that $pr_1^*\xi' = pr_2^*\xi'$. Then :

$$\pi^*\pi_*\xi' = pr_{1*}pr_2^*\xi' = pr_{1*}pr_1^*\xi' = [pr_{1*}\mathcal{O}_{\mathcal{X}'\times_{\mathcal{X}}\mathcal{X}'}]\xi' = (\pi^*\alpha)\xi'$$

Call $\eta \in K'_*(\mathcal{X})_{geom}$ the element such that $\alpha\eta = \pi_*\xi'$. Then we have

$$(\pi^*\alpha)\pi^*\eta = \pi^*(\alpha\eta) = \pi^*\pi_*\xi' = (\pi^*\alpha)\xi'$$

But we have $\pi^*\alpha \in \Sigma_1^{\mathcal{X}'}$, hence $\xi' = \pi^*\eta$. This ends the proof of the first part. For the second part, consider a class $\xi \in K'_*(\mathcal{X})_{geom}$. Then $\alpha\xi = \pi_*\pi^*\xi$. If $\eta \in K'_*(\mathcal{X}')_{geom}$ is such that $(\pi^*\alpha)\eta = \pi^*\xi$, then $\alpha\xi = \pi_*(\pi^*\alpha)\eta = \alpha\pi_*\eta$, so $\xi = \pi_*\eta$, and π_* is surjective.

Now take $\xi' \in K'_*(\mathcal{X}')_{geom}$ such that $\pi_*\xi' = 0$. Then

$$pr_{2*}pr_1^*\xi' = \pi^*\pi_*\xi' = 0$$

On the other hand $pr_{1*}pr_1^*\xi' = (\pi^*\alpha)\xi'$. Denote by $\rho : \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \longrightarrow \mathcal{X}$ the morphism πpr_1. If $\eta \in K'_*(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}')_{geom}$ is such that $\rho^*\eta = pr_1^*\xi'$ then we have $pr_{1*}\eta = \xi'$, while $(\pi^*\alpha)pr_{2*}\eta = 0$, so $pr_{2*}\eta = 0$. Hence $(pr_{1*} - pr_{2*})\eta = \xi'$; this finishes the proof.$\diamond$

Corollary 4.5 : Let Γ be a finite group and let $\pi : \mathcal{X}' \longrightarrow \mathcal{X}$ be a Galois cover with Galois group Γ. Then the pullback and the pushforward induced by the latter yield isomorphisms :

$$\pi^* : K'_*(\mathcal{X})_{geom} \longrightarrow K'_*(\mathcal{X}')^{\Gamma}_{geom}$$

$$\pi_* : (K'_*(\mathcal{X}')_{geom})_{\Gamma} \longrightarrow K'_*(\mathcal{X})_{geom}$$

where $K'_*(\mathcal{X}')^{\Gamma}_{geom}$ (resp. $(K'_*(\mathcal{X}')_{geom})_{\Gamma}$) denotes the module of Γ-invariant elements of $K'_*(\mathcal{X}')_{geom}$ (resp. the module of Γ-coinvariant elements of $K'_*(\mathcal{X}')_{geom}$).$\diamond$

The next ingredient in the proof of Theorem 4.1 is the following result of Kresch and Vistoli (see [KV, Theorem 2.1]).

Theorem 4.6 : Let $\mathcal{X}$ be a separated quotient stack of finite type over a field k, with finite inertia, such that its coarse moduli space M is a quasi-projective k-scheme. Then there exists a finite faithfully flat morphism :

$$X \longrightarrow \mathcal{X}$$

where X is a quasi-projective k-scheme.$\diamond$

Proposition 4.7 : Let $\mathcal{X}$ be a tame Artin stack stack of finite type over a field k, such that its coarse moduli space M is a quasi-projective k-scheme. Then for every $\xi \in K'_*(\mathcal{X})$, we have that :
(i) $\xi \in K'_*(\mathcal{X})_{geom}$ if and only if there exists a proper morphism $p : Y \longrightarrow \mathcal{X}$ from a scheme Y, and an element $\eta \in K'_*(Y)$, such that $\xi = p_*\eta$.
(ii) $\xi \in K'_*(\mathcal{X})_{extra}$ if and only if for every morphism of finite flat dimension $p : V \longrightarrow \mathcal{X}$ from a scheme V, $f^*\xi = 0$.

Proof : In both cases, the "only if" part follows from the obvious fact that, for an algebraic space X, $K'_*(Y)_{extra} = 0$, together with Proposition 4.3. In both cases, the "if" part is a consequence of Theorem 4.6 together with Lemma 4.4.$\diamond$

Proof of Theorem 4.1 : It proceeds by noetherian induction. Let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed substack. Let N be the moduli space of $\mathcal{Y}$. Then, there is a canonical morphism $N \longrightarrow M$, and since $\mathcal{X}$ is tame, it is a closed embedding. The induction hypothesis is, that for any proper closed substack $\mathcal{Y} \hookrightarrow \mathcal{X}$, the proper pushforward $K'_*(\mathcal{Y})_{geom} \longrightarrow K'_*(N)$ is an isomorphism.
Recall that, the inclusion $i : \mathcal{X}_{red} \hookrightarrow \mathcal{X}$ induces an isomorphism on K'-theory, so it follows that $i_* : K'_*(\mathcal{X}_{red})_{geom} \longrightarrow K'_*(\mathcal{X})_{geom}$ is an isomorphism.
Now, the following diagram, the rows of which are isomorphisms, is commutative :

$$\begin{tikzcd}
K'_*(\mathcal{X}_{red})_{geom} \arrow[r] \arrow[d] & K'_*(\mathcal{X})_{geom} \arrow[d] \\
K'_*(N) \arrow[r] & K'_*(M)
\end{tikzcd}$$

If $\mathcal{X}$ is not reduced then $\mathcal{X}_{red} \neq \mathcal{X}$, and the left hand column is also an isomorphism, so that the thesis follows. Therefore we can assume that $\mathcal{X}$ is

reduced.
Let $N \subseteq M$ be a closed subscheme and set $U = M - N$. Set $\mathcal{Y} = \mathcal{X} \times_M N$ and $\mathcal{U} = \mathcal{X} \times_M U$. We have the following commutative diagram with exact rows, in which columns are given by proper pushforwards :

$$\begin{tikzcd}
K'_i(\mathcal{Y})_{geom} \arrow[r] \arrow[d] & K'_i(\mathcal{X})_{geom} \arrow[r] \arrow[d] & K'_i(\mathcal{U})_{geom} \arrow[r] \arrow[d] & K'_{i-1}(\mathcal{Y})_{geom} \arrow[r] \arrow[d] & K'_{i-1}(\mathcal{X})_{geom} \arrow[d] \\
K'_i(N) \arrow[r] & K'_i(M) \arrow[r] & K'_i(U) \arrow[r] & K'_{i-1}(N) \arrow[r] & K'_{i-1}(M)
\end{tikzcd}$$

If $N \neq M$, then $\mathcal{Y} \neq \mathcal{X}$, so the pushforward $K'_*(\mathcal{Y})_{geom} \longrightarrow K'_*(N)$ is an isomorphism. If $K'_*(\mathcal{U})_{geom} \longrightarrow K'_*(U)$ is an isomorphism, then the thesis follows. Therefore we can reduce M to a non-empty open subset.
Now, let $M' \longrightarrow M$ be an fppf cover. Set $M'' = M' \times_M M'$, $\mathcal{X}' = \mathcal{X} \times_M M'$, and $\mathcal{X}'' = \mathcal{X} \times_M M'' = \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'$. There is a commutative diagram :

$$\begin{tikzcd}
K'_*(\mathcal{X})_{geom} \arrow[r] \arrow[d] & K'_*(\mathcal{X}')_{geom} \arrow[r, shift left, "pr_1^*"] \arrow[r, shift right, "pr_2^*"'] \arrow[d] & K'_*(\mathcal{X}'')_{geom} \arrow[d] \\
K'_*(M) \arrow[r] & K'_*(M') \arrow[r, shift left, "pr_1^*"] \arrow[r, shift right, "pr_2^*"'] & K'_*(M'')
\end{tikzcd}$$

Lemma 4.4 implies that the rows of the above diagram are equalizers. Hence, if the result is true for $\mathcal{X}'$ and $\mathcal{X}''$, then it is true for $\mathcal{X}$.
Since M is reduced, by restricting to a non-empty subscheme of M we may assume that $\mathcal{X} \longrightarrow M$ is flat (this essentially follows from [EGA4, Theoreme 6.9.1]). By passing to an fppf cover, we can also assume that it has a section $\sigma : M \longrightarrow \mathcal{X}$, by faithfully flat descent. By restricting further, since $\mathcal{X}$ is reduced we can also assume that σ is flat. If $\Delta = M \times_{\mathcal{X}} M \longrightarrow M$ denotes the automorphism group scheme of σ, then $\Delta \longrightarrow M$ is a linearly reductive finite group scheme over M. Since $\mathcal{X}$ is the fppf quotient stack of the groupoid $M \times_{\mathcal{X}} M \Longrightarrow M$, it is equivalent to the classifying stack $B_M\Delta \longrightarrow M$. By taking a restriction and an fppf cover, we may assume that Δ is of the form $\Delta_1 \times_| \Delta_0$, where $\Delta_0 \longrightarrow M$ is finite and diagonalizable, and $\Delta_1 \longrightarrow M$ is a tame étale constant group scheme (see [AOV]); in the terminology of [AOV], such group schemes are called well-split. Such a group scheme is the pullback of a well split group scheme $\Gamma \longrightarrow Spec(k)$. So we are reduced to the case $\mathcal{X} = M \times B_k\Gamma$, where $\Gamma \longrightarrow Spec(k)$ is well-split. Then we have $K'_*(\mathcal{X})_{geom} = K'_*(M) \otimes (R\Gamma)_1$, and the result follows if we show that the homomorphism $(R\Gamma)_1 \longrightarrow Q$ induced by the rank map $R\Gamma \longrightarrow \mathbf{Q}$ is an isomorphism (notice that since Γ is well-split,

the trivial action of Γ is sufficiently rational). From Lemma 5.4 applied to the morphism $Spec(k) \longrightarrow B_k\Gamma$ we see that the pullback $(R\Gamma)_1 = K'_*(B_k\Gamma)_{geom} \longrightarrow K'_*(Spec(k))_{geom} = \mathbf{Q}$ is injective, which completes the proof.$\diamond$

Chapter 5

The K-theory of cyclotomic inertia stacks

5.1 On a generalization of the Vezzosi-Vistoli decomposition formula

The aim of this section is to investigate the connections between Proposition 2.20 and Theorem 7.9.

5.1.1 Original formulation (for regular algebraic spaces)

To begin with, let us recall how the map

$$K'_*(X,G) \longrightarrow \prod_{\sigma\in C(G)} \left(K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}(\sigma)\right)^{w_G(\sigma)}$$

occuring in Theorem 7.9 is constructed. First, there is a canonical homomorphism

$$K'_*(X,G) \longrightarrow \prod_\sigma K'_*(X, C_G(\sigma))_\sigma$$

which decomposes as

$$K'_*(X,G) \xrightarrow{\approx} \prod_\sigma K'_*(X,G)_\sigma \longrightarrow \prod_\sigma K'_*(X, C_G(\sigma))_\sigma$$

where the second arrow is the product of all morphisms induced by restriction with respect to subgroups $C_G(\sigma)$ of G for $\sigma \in C(G)$. Let X^σ be the maximal closed subscheme of X fixed under the action of σ. The product of pushforwards with respect to the regular[1] inclusions $j_\sigma : X^\sigma \hookrightarrow X$ for all σ's gives a moprhism

[1]Recall that, for σ of order prime to the characteristic of k, X^σ is the subscheme of fixed points of X under the action of σ is regular when X is regular ([VV, Section 1]).

$$\prod_\sigma K'_*(X^\sigma, C_G(\sigma))_\sigma \longrightarrow \prod_\sigma K'_*(X, C_G(\sigma))_\sigma$$

and composing the latter with the former yields a morphism

$$K'_*(X, G) \longrightarrow \prod_\sigma K'_*(X^\sigma, C_G(\sigma))_\sigma$$

By [VV, Proposition 4.5], the image of this morphism is contained in $\prod_\sigma (K'_*(X^\sigma, C_G(\sigma))_\sigma)^{w_G(\sigma)}$, and one has an isomorphism

$$K'_*(X, G) \xrightarrow{\approx} \prod_\sigma K'_*\,(X^\sigma, C_G(\sigma))_\sigma^{w_G(\sigma)}$$

Finally, there is the morhism

$$\theta_{C_G(\sigma),\sigma} : K'_*(X^\sigma, C_G(\sigma))_\sigma \longrightarrow K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma$$

which [VV, Proposition 4.6] introduces and proves to be an isomorphism. The construction of $\theta_{C_G(\sigma),\sigma}$ is given in the next subsection (it carries over to the general case). $w_G(\sigma)$ acts on both sides in such a way that it is an equivariant morphism. The map starred in Theorem 7.9 is the composition of the resulting isomorphism

$$\prod_\sigma K'_*(X^\sigma, C_G(\sigma))_\sigma^{w_G(\sigma)} \xrightarrow{\approx} \prod_\sigma \left(K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma\right)^{w_G(\sigma)}$$

with the preceding isomorphism

$$K'_*(X, G) \xrightarrow{\approx} \prod_\sigma (K'_*(X^\sigma, C_G(\sigma)))_\sigma^{w_G(\sigma)}$$

5.1.2 Generalization

We need to consider a slightly different map when X is not necessarily regular, to prove that it yields an isomorphism between the same source and target in this more general context. In doing so, we can assume that G is GL_n, using Morita isomorphisms, as is done in [VV, Section 5]. However, to construct the map we first restrict to the case when $G = T$ is a torus. In the latter case, for each essential dual cyclic subgroup σ of T, the group $w_G(\sigma)$ is trivial. Let $j_\sigma : X^\sigma \hookrightarrow X$ denote the inclusion of the fixed-point scheme. Recall from [VV, Proposition 3.4] that

$$(j_\sigma)_* : K'_*(X^\sigma, T)_\sigma \xrightarrow{\approx} K'_*(X, T)_\sigma$$

and

$$can : K'_*(X, T) \xrightarrow{\approx} \prod_{\sigma \in C(T)} K'_*(X, T)_\sigma$$

where the second morphism is the product of all localizations at ideals m_σ and the product on the left is finite. In [VV], this result is stated for X regular only, but it also holds when X fails to be regular, since the proof essentially relies on [Th3, Theorem 2.1] as to the first isomorphism, and Theorem 7.7 (Thomason's generic slice theorem for torus actions), as well as [Th2, Lemma 5.8] and [Th1, Proposition 7.2, Theorem 2.7] as to the second isomorphism, all of which results hold for non necessarily regular schemes. Moreover, [VV, proposition 3.5] which says that the morphisms $\theta_{T,\sigma} : K'_*(X^\sigma, T)_\sigma \longrightarrow K'_*(X^\sigma, T)_{geom} \otimes \tilde{R}\sigma$ induced from product morphism $\sigma \times T \longrightarrow T$ is an isomorphism, is also true in the more general case. In case $G = T$, we therefore have a map

$$\begin{gathered}\delta_{X,T} := \left(\prod_{\sigma\in C(T)} \theta_{T,\sigma}\right)\left(\prod_{\sigma\in C(T)} j_{\sigma *}\right)^{-1} can : \\ K'_*(X,T) \longrightarrow \prod_{\sigma\in C(T)} K'_*(X^\sigma, T)_{geom} \otimes \tilde{R}\sigma\end{gathered}$$

which is an isomorphism of $\mathbf{Q}$-vector spaces.

Let us now assume that $G = GL_n$. We have the canonical map can : $K'_*(X,G) \longrightarrow \prod_{\sigma\in C(G)} K'_*(X,G)_\sigma$ and canonical maps $K'_*(X,G) \longrightarrow K'_*(X, C_G(\sigma))$ induced by the group scheme inclusions $C_G(\sigma) \hookrightarrow G$. Again, let j_σ : $X^\sigma \hookrightarrow X$ denote the inclusion of the fixed-point scheme associated to a dual cyclic subgroup σ.

Claim : $(j_\sigma)_* : K'_*(X^\sigma, C_G(\sigma))_\sigma \longrightarrow K'_*(X, C_G(\sigma))_\sigma$ is an isomorphism for every σ.

Indeed, Lemma 3.10 implies that $K'_*(X, C_G(\sigma))_\sigma = 0$ if the fixed-point scheme X^σ is empty. Let Y denote the scheme $X - X^\sigma$. Let $n \geq 1$. The localization sequence in K'-theory around n and localized at m_σ reads as :

$$K'_n(X^\sigma, C_G(\sigma))_\sigma \longrightarrow K'_n(X, C_G(\sigma))_\sigma \longrightarrow K'_n(Y, C_G(\sigma))_\sigma \longrightarrow K'_{n+1}(X^\sigma, C_G(\sigma))_\sigma$$

and $Y^\sigma = 0$, whence the claim.$\diamond$

We can therefore consider the morphism

$$\left(\prod_{\sigma\in C(G)} j_{\sigma *}\right)^{-1} : \prod_{\sigma\in C(G)} K'_*(X, C_G(\sigma))_\sigma \longrightarrow \prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma$$

Finally, [VV, Proposition 4.6] is true for non necessarily regular schemes. This means the following. On the one hand, the multiplication morphism $m : C_G(\sigma) \times \sigma \longrightarrow C_G(\sigma)$ induces an morphism

$$m^* : K'_*(X^\sigma, C_G(\sigma)) \longrightarrow K'_*(X^\sigma, \sigma \times C_G(\sigma))$$

and on the other hand, as σ is contained in $C_G(\sigma)$, the classical lemma [VV, Lemma 2.7] gives a canonical isomorphism of $\mathbf{Q}$-vector spaces

$$K'_*(X^\sigma, \sigma \times C_G(\sigma)) \approx K'_*(X^\sigma, C_G(\sigma)) \otimes R\sigma$$

so that we can read m^* as a morphism

$$K'_*(X^\sigma, C_G(\sigma)) \longrightarrow K'_*(X^\sigma, C_G(\sigma)) \otimes R\sigma$$

Now, tensoring the geometric localization morphism $K'_*(X^\sigma, C_G(\sigma)) \longrightarrow K'_*(X^\sigma, C_G(\sigma))_{geom}$ with the projection $R\sigma \longrightarrow \tilde{R}\sigma$ yields a map

$$p : K'_*(X^\sigma, C_G(\sigma)) \otimes R\sigma \longrightarrow K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma$$

Consider

$$pm^* : K'_*(X^\sigma, C_G(\sigma)) \longrightarrow K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma$$

By [VV, Lemma 2.8] (which holds for noetherian separated algebraic spaces), this map factors through $K'_*(X^\sigma, C_G(\sigma)) \longrightarrow K'_*(X^\sigma, C_G(\sigma))_\sigma$. As a result, we get a map

$$\theta_{C_G(\sigma),\sigma} : K'_*(X^\sigma, C_G(\sigma))_\sigma \longrightarrow K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma$$

The content of [VV, Proposition 4.6] is that $\theta_{C_G(\sigma),\sigma}$ is an isomoprhism of $\mathbf{Q}$-vector spaces.

Consider the map $K'_*(X, G) \longrightarrow \prod_{\sigma \in C(G)} K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma$ defined as the composition :

$$\begin{tikzcd}
K'_*(X,G) \arrow[r, "can"] & \prod_{\sigma\in C(G)} K'_*(X,G)_\sigma \arrow[r, "res"] & \prod_{\sigma\in C(G)} K'_*(X,C_G(\sigma))_\sigma \arrow[r, "\prod_\sigma (j_{\sigma *})^{-1}"] & \prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma \arrow[d] \\
 & & & \prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma
\end{tikzcd}$$

The image of this map is contained in $\prod_\sigma (K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}\sigma)^{w_G(\sigma)}$ (with respect to the canonical action of each $w_G(\sigma)$ on $K'_*(X^\sigma, C_G(\sigma))_{geom}$ and $\tilde{R}\sigma$ ([VV, Corollary 2.5])). This gives us our morphism

$$\delta_{X,G} : K'_*(X,G) \longrightarrow \prod_{\sigma\in C(G)} \left(K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}(\sigma)\right)^{w_G(\sigma)}$$

Proposition 5.1 : The map $\delta_{X,G}$ defined above is an isomorphism of $\mathbf{Q}$-vector spaces.

Proof : Since $RG \longrightarrow RT$ is faithfully flat, it suffices to show that $\delta_{X,G} \otimes id_{RT}$ is an isomorphism. This can be done as in the proof of [VV, Proposition 4.5].$\diamond$

Remark 5.2 : Proposition 2.20 immediately implies that

$$\prod_r K'_* \left(I_{\mu_r}([X/GL_n])\right)_{geom} \otimes \tilde{R}\mu_r \approx \prod_{r,\underline{d}} K'_* \left(X^{\mu_r^{(\underline{d})}}, C_G(\mu_r^{(\underline{d})})\right)_{geom} \otimes \tilde{R}\mu_r$$

We have a canonical action of $Aut(\mu_r)$ on each r-term of the right hand side : $Aut(\mu_r)$ acts on $\tilde{R}\mu_r$, as $\tilde{R}\mu_r \approx \mathbf{Q}(\zeta_r)$ and $Aut(\mu_r) \approx Gal\left(\mathbf{Q}(\zeta_r)/\mathbf{Q}\right)$, and on each $K'\left([X^{\mu_r^{(\underline{d})}}/C_G(\mu_r^{(\underline{d})})]\right)$ via the pullbacks

$$K_0\left(Spec(k), C_G(\mu_r^{(\underline{d})})\right) \longrightarrow K_0\left(X^{\mu_r^{(\underline{d})}}, C_G(\mu_r^{(\underline{d})})\right)$$

It is time at this point to recall the following lemma from [VV, Lemma 2.10] :

Lemma 5.3 : Let W be a finite group acting on the left on a set A, and let $B \subset A$ be a set of representatives for the orbits. Assume that W acts on the left on a product of abelian groups of the type $\prod_{\alpha\in A} M_\alpha$ in such a way that $sM_a = M_{s\alpha}$ for any $s \in W$. For each $\alpha \in B$, let us denote by W_α the stabilizer of α in W. Then the canonical projection $\prod_{\alpha\in A} M_\alpha \longrightarrow \prod_{\alpha\in B} M_\alpha$ induces an isomorphism :

$$\left(\prod_{\alpha\in A} M_\alpha\right)^W \xrightarrow{\approx} \prod_{\alpha\in B}(M_\alpha)^{W_\alpha}$$

Let us apply this lemma to rewrite Proposition 5.1, denoting by A the set of indices d and by B the set of their conjugacy classes.

Proposition 5.4 : Let G be GL_n. The canonical projection induces an isomorphism of $\mathbf{Q}$-vector spaces :

$$\left(K'_*(I_{\mu_r}([X/G]))_{geom} \otimes \tilde{R}(\mu_r)\right)^{Aut(\mu_r)} \xrightarrow{\approx} \prod_{\underline{d}\in B}\left(K'_*\left(X^{\mu_r^{(\underline{d})}}, C_G(\mu_r^{(\underline{d})})\right)_{geom} \otimes \tilde{R}(\mu_r)\right)^{Aut(\mu_r)_{\underline{d}}}$$

This suggests a connection between the K'-theory of the cyclotomic inertia of order r and the geometric K'-theory of the fixed spaces (for the action of a rank r dual cyclic subgroup).

5.2 The tautological part of the K-theory of cyclotomic inertia stacks

The main result of this section is Proposition 5.8, which requires two things : first, the use of an interesting comodule structure on the K-theory of cyclotomic inertia stacks, and second, a localization of these modules that we may call tautological because of its definition.[2]

5.2.1 $R\mu_r$-comodule structure on $K'_*(I_{\mu_r}\mathcal{X})$

Let $r \geq 1$. The map $R\mu_r \longrightarrow R\mu_r \otimes R\mu_r$ sending a μ_r-representation V to $V \otimes V$ turns $R\mu_r$ into a coalgebra.

Let $\mathcal{F}$ be a coherent sheaf on $I_{\mu_r}(\mathcal{X})$ (we could as well work with quasi-coherent sheaves instead of coherent sheaves). By [TV, Propositio 4.13.(d)], $\mathcal{F}$ amounts to the data of all sheaves of the form $\chi^*\mathcal{F}$ for every morphism $\chi : U \longrightarrow I_{\mu_r}(\mathcal{X})$ from a scheme U. Let $\chi : U \longrightarrow I_{\mu_r}(\mathcal{X})$ be a morphism of stacks with U a scheme : this amounts to a morphism of stacks $U \longrightarrow \mathcal{X}$ together with a monomorphism of group schemes $\mu_{r,U} \longrightarrow Aut_U(\chi)$, and the latter gives an action of $\mu_{r,U}$ on the sheaf $\chi^*\mathcal{F}$. Now, since μ_r is diagonalizable, we can canonically decompose $\chi^*\mathcal{F}$ into a direct sum of eigensheaves indexed by the element of the group $\hat{\mu}_{r,U}$ of characters of $\mu_{r,U}$. As a result, we get an $R\mu_r$-comodule structure on $K'_*(I_{\mu_r}(\mathcal{X}))$, namely a map

$$\alpha_{\mathcal{X},r} : K'_*(I_{\mu_r}(\mathcal{X})) \longrightarrow K'_*(I_{\mu_r}(\mathcal{X})) \otimes R\mu_r$$

subjected to the comodule axioms.

5.2.2 Tautological localization of $K'_*(I_{\mu_r}\mathcal{X})$

Now, let us introduce the new localization. Suppose that K is an extension of k, and let (ξ, a) be an object of $I_\mu\mathcal{X}(K)$. The homomorphism $\phi : \mu_{\infty,K} \longrightarrow Aut_K(\xi)$ induces an action of $\mu_{\infty,K}$ on ξ, which commutes with itself, since $\mu_{\infty,K}$ is abelian; thus ϕ can be considered as a homomorphism $\mu_{\infty,K} \longrightarrow Aut_K(\xi, a)$. We can think of this as follows : a morphism $B_K\mu_{\infty,K} \xrightarrow{a} \mathcal{X}$ has a canonical lifting to a morphism $B_K\mu_{\infty,K} \xrightarrow{a'} I_\mu\mathcal{X}$, which we call its *tautological lifting* (over a K-scheme T, we have $a'(T) = (a(T), a'_T))$. It sends Now, a morphism $B_K\mu_{\infty,K} \longrightarrow I_\mu\mathcal{X}$ corresponds to an object (ξ, a) of $I_\mu\mathcal{X}(K)$, together with a homomorphism $b : \mu_{\infty,K} \longrightarrow Aut_K(\xi, a)$. This gives two action of $\mu_{\infty,K}$ on ξ: one given by $a : \mu_{\infty,K} \longrightarrow Aut_K(\xi)$, and the other by the composite

[2] The localized modules are then called the tautological part of the K-theory of cyclotomic inertia stacks.

$$\mu_{\infty,K} \xrightarrow{b} Aut_K(\xi, a) \subseteq Aut_K\xi$$

The morphism $B_K\mu_{\infty,K} \longrightarrow I_\mu\mathcal{X}$ is said to be *tautological* if and only if the two actions coincide. As we saw, there is an equivalence between morphisms $B_K\mu_{\infty,K} \longrightarrow \mathcal{X}$ and representable morphisms $B_K\mu_{r,K} \longrightarrow \mathcal{X}$ for some r; therefore we can also talk about tautological representable morphisms $B_K\mu_{r,K} \longrightarrow \mathcal{X}$. Since there is an equivalence between morphisms $B_K\mu_{\infty,K} \longrightarrow \mathcal{X}$ and representable morphisms $B_K\mu_{r,K} \longrightarrow \mathcal{X}$ for some r, we can talk about tautological representable morphisms $B_K\mu_{r,K} \longrightarrow \mathcal{X}$.

Tautological morphisms are very special among morphisms from $B_K\mu_{\infty,K}$, as they are so easily defined. This motivates the following definition.

Definition 5.5 : The *tautological part* of $K'_*(I_{\mu_r}(\mathcal{X}))$ is the localization of $K'_*(I_{\mu_r}(\mathcal{X}))$ as a $K_0(\mathcal{X})$-module, with respect to the multiplicative system consisting of elements $\alpha \in K_0(\mathcal{X})$ such that for every field K extending k and for every tautological representable morphism $\phi : B_K\mu_{r,K} \longrightarrow \mathcal{X}$, the image of $\phi^*\alpha \in R\mu_r$ in $\mathbf{Q}(\zeta_r)$ is non-zero.

It is denoted $K'_*(I_{\mu_r}(\mathcal{X}))_{taut}$

The $K_0(\mathcal{X})$-module $K'_*(I_{\mu_r}(\mathcal{X}))$ can also be localized with respect to the multiplicative system defined using non-tautological morphisms instead of tautological ones. The module thus obtained is called the non-tautological part of $K'_*(I_{\mu_r}(\mathcal{X}))$, and $K'_*(I_{\mu_r}(\mathcal{X}))$ is isomorphic to the product of its tautological part with its non-tautological part.

We will from now on consider $K'_*(I_{\mu_r}(\mathcal{X}))_{taut}$ as a submodule of $K'_*(I_{\mu_r}(\mathcal{X}))$. There is furthermore a natural action of $Aut(\mu_r)$ on $K'_*(I_{\mu_r}(\mathcal{X}))_{taut}$.

5.2.3 The case of quotient stacks

The following proposition explains what the tautological K'-theory of cyclotomic inertia stacks of tame quotient stacks boils down to.

Let $\mathcal{X} \approx [X/G]$ be a tame quotient stack. We can reduce the general case to the case when $G = GL_{n,k}$ for some $n \geq 1$, as in [VV, Section 5]. Adopting the notations set in the Appendix, Section 7.1, we can rephrase Proposition 2.20 as :

$$I_{\mu_r}\mathcal{X} \approx \textstyle\coprod_{\alpha\in\tilde{C}_r(G)} \left[X^{\alpha(\mu_r)}/C_G(\alpha)\right]$$

which gives in particular isomorphisms of $\mathbf{Q}$-vector spaces :

$$K'_*\left(I_{\mu_r}\mathcal{X}\right) \xrightarrow[\gamma'_{X,G,r}]{\approx} \textstyle\prod_{\alpha\in\tilde{C}_r(G)} K'_*\left(X^{\alpha(\mu_r)}, C_G(\alpha)\right)$$

Also, recall from Chapter 3 that, for each $\sigma \in C(G)$, we have :

$$K'_*\left(X^\sigma, C_G(\sigma)\right) \approx K'_*\left(X^\sigma, C_G(\sigma)\right)_\sigma \times \prod_{\tau \neq \sigma} K'_*\left(X^\sigma, C_G(\sigma)\right)_\tau$$

Proposition 5.6 : Let $\mathcal{X} \approx [X/G]$ be a presentation of a tame quotient stack and let $r \geq 1$. Then there is a natural isomoprhism of $\mathbf{Q}$-vector spaces of the form

$$K'_*\left(I_{\mu_r}\mathcal{X}\right)_{taut}^{Aut(\mu_r)} \xrightarrow[\gamma_{X,G,r}]{\approx} \prod_{\sigma \in C_r(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma^{w_G(\sigma)}$$

Setting $\gamma_{X,G} := \prod_{r\geq 1} \gamma_{X,G,r}$, we have :

$$\prod_{r\geq 1} K'_*\left(I_{\mu_r}\mathcal{X}\right)_{taut}^{Aut(\mu_r)} \xrightarrow[\gamma_{X,G}]{\approx} \prod_{\sigma \in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma^{w_G(\sigma)}$$

Proof : We have a natural map $\tilde{C}_r(G) \longrightarrow C_r(G)$ that assigns to any $\alpha : \mu_r \hookrightarrow G$ its image $\alpha(\mu_r)$ in G which we denote σ_α. This map is surjective. Indeed, for all $\sigma \in C(G)$, we can precompose with an isomorphism $c : \mu_r \xrightarrow{\approx} \sigma$ so that $\sigma_{\alpha_\sigma} = \sigma$.

On the other hand, let c and c' be two such isomorphisms. Then $(c')^{-1}c =: g \in Aut(\mu_r)$. Therefore, denoting A' a set of representatives of $Aut(\mu_r)$-orbits of elements of $\tilde{C}(G)$, we have a map $A' \twoheadrightarrow C(G)$. It is in fact a bijection, and from now on we identify these two sets. Now, for any $\sigma \in C(G)$, we have that $w_G(\sigma) \subseteq Aut(\mu_r)$. It is immediate to see that $w_G(\sigma)$ is the stabilizer of σ.

Combining Definition 5.5 with Proposition 3.5, we immediately get

$$K'_*(I_{\mu_r}\mathcal{X})_{taut} \approx \prod_{\alpha \in \tilde{C}_r(G)} K'_*\left(X^{\sigma_\alpha}, C_G(\alpha)\right)_{\sigma_\alpha}$$

Now, we have a projection map :

$$\prod_{\alpha \in \tilde{C}(G)} K'_*(X^{\sigma_\alpha}, C_G(\sigma_\alpha))_{\sigma_\alpha} \longrightarrow \prod_{\alpha \in A'} K'_*(X^{\sigma_\alpha}, C_G(\sigma_\alpha))_{\sigma_\alpha} \approx \prod_{\sigma \in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma$$

Lemma 5.3 implies that this projection map induces an isomoprhism

$$\left(\prod_{\alpha \in \tilde{C}(G)} K'_*(X^{\alpha(\mu_r)}, C_G(\alpha))_{\sigma_\alpha}\right)^{Aut(\mu_r)} \xrightarrow{\approx} \prod_{\sigma \in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma^{w_G(\sigma)}$$

Combining this isomorphism with $\gamma'_{X,G,r}$ yields the isomorphisms $\gamma_{X,G,r}$.$\diamond$

Proposition 5.7 : Let $\mathcal{X} \approx [X/G]$ be a fixed presentation of a tame quotient stack. For each $\alpha \in \tilde{C}_r(G)$, there exists a unique connected component $\mathcal{U}$ of $I_{\mu_r}(\mathcal{X})$ such that $\alpha \in \mathcal{U}$. Setting $m_{\mathcal{U}} := m_\alpha$, we have :

$$K'_*(I_\mu\mathcal{X})_{taut} = \prod_{\mathcal{U}} K'_*(\mathcal{U})_{m_{\mathcal{U}}}$$

where the product is indexed by the connected components of $I_\mu\mathcal{X}$.

Proof : We adopt the notations of Theorem 7.5 and Theorem 7.6 with respect to G. Let C be the set of connected components of $\tilde{S}_r(G)$. We can identify C with $\tilde{C}_r(G)$: indeed, since the G-action on X is sufficiently rational, we can reduce to the case where the base field k is algebraically closed, so that the closed points of $\tilde{S}_r(G)$ are its k-points. Now, we can invoke Theorem 7.6 and [SGA3, II, XII, §5]. Let B be the set of connected components of $I_{\mu_r}(\mathcal{X})$, ie $I_{\mu_r}(\mathcal{X}) = \coprod_{\mathcal{U}\in B}\mathcal{U}$. We have :

$$I_{\mu_r}(\mathcal{X}) \approx \coprod_{\alpha\in C}\left[X^{\alpha(\mu_r)} \times U_\alpha/G\right]$$

Now, for anu $\mathcal{U} \in B$, there exists exactly one $\alpha \in C$ such that

$$\mathcal{U} \subseteq \left[X^{\alpha(\mu_r)} \times U_\alpha/G\right],$$

by connectedness as a result of Theorem 7.6.$\diamond$

5.2.4 From tautological localization to geometric K-theory

Let β_r be defined by the following diagram :

$$\begin{array}{ccc}
K'_*(I_{\mu_r}(\mathcal{X})) & \xrightarrow{\alpha_{\mathcal{X},r}} & K'_*(I_{\mu_r}(\mathcal{X})) \otimes R\mu_r \\
\cup & & \downarrow \\
K'_*(I_{\mu_r}(\mathcal{X}))_{taut} & \xrightarrow{\beta_{\mathcal{X},r}} & K'_*(I_{\mu_r}(\mathcal{X}))_{geom} \otimes \tilde{R}\mu_r
\end{array}$$

Proposition 5.8 : $\beta_{\mathcal{X},r}$ is an isomorphism for every $r \geq 1$, so that

$$K'_*(I_{\mu_r}\mathcal{X})_{taut} \xrightarrow[\beta_{\mathcal{X},r}]{\approx} K'_*(I_{\mu_r}\mathcal{X})_{geom} \otimes \tilde{R}\mu_r$$

Proof : Fix a presentation $\mathcal{X} \approx [X/G]$, so that $I_\mu(\mathcal{X}) \approx \coprod_d \left[X^{\mu_r^{(d)}}/C_G\left(\mu_r^{(d)}\right)\right]$. Then, $\alpha_{\mathcal{X},r}$ corresponds to the morphism induced from the product morphism $C_G\left(\mu_r^{(d)}\right) \times \mu_r \longrightarrow C_G\left(\mu_r^{(d)}\right)$ considered in Section 5.1, and the proof outlined in the latter section implies that $\beta_{\mathcal{X},r}$ is an isomorphism.$\diamond$

Now, there is a canonical isomorphism :

$$\prod_{r\geq 1}\left(K'_*(I_{\mu_r}(\mathcal{X})_{geom} \otimes \tilde{R}\mu_r\right)^{Aut(\mu_r)} \xrightarrow[can]{\approx} \left(K'_*(I_\mu(\mathcal{X}))_{geom} \otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)}$$

where $\tilde{R}\mu_\infty \approx \mathbf{Q}(\zeta_\infty) := \bigcup_n \mathbf{Q}(\zeta_n)$, $\mu_\infty = lim_r \mu_r$, and $Aut(\mu_\infty) \approx Gal\left(\mathbf{Q}(\zeta_\infty)/\mathbf{Q}\right)$.

Letting $\beta_{\mathcal{X},*} = \coprod_{r\geq 1}\beta_{\mathcal{X},r}$ we have :

$$\prod_{r\geq 1}\left(K'_*(I_{\mu_r}(\mathcal{X}))_{taut}\right)^{Aut(\mu_r)} \xrightarrow[\approx]{\beta_{\mathcal{X},*}} \prod_{r\geq 1}\left(K'_*(I_{\mu_r}(\mathcal{X}))_{geom} \otimes \tilde{R}\mu_r\right)^{Aut(\mu_r)}$$

Let us set $\beta_{\mathcal{X}} := can\beta_{\mathcal{X},*}$. It is a first isomorphism of the form

$$\prod_{r\geq 1}\left(K'_*(I_{\mu_r}(\mathcal{X}))_{taut}\right)^{Aut(\mu_r)} \xrightarrow{\approx} \left(K'_*(I_\mu(\mathcal{X}))_{geom} \otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)}$$

5.2.5 Covariance

The isomorphism $\beta_{\mathcal{X}}$ exhibited above, however, won't be suitable in the next chapter, because it fails to be covariant with respect to proper morphisms of tame stacks. This can be readily seen, for instance, by testing the map $B\mu_3 \longrightarrow Spec(k)$. In this case, $I_\mu B_{\mu_3} = B_{\mu_3} \coprod I_{\mu_2}B_{\mu_3} \coprod I_{\mu_3}B_{\mu_3}$ where :

- $I_{\mu_2}B_{\mu_3} = \emptyset$
- $I_{\mu_3}B_{\mu_3} \approx B\mu_3 \coprod B\mu_3$

and we have (assuming first that the ground field is a splitting field) :

- $R\mu_3 = \mathbf{Q1} \oplus \mathbf{Q}\xi \oplus \mathbf{Q}\bar{\xi} \approx \mathbf{Q} \oplus \mathbf{Q}(\zeta_3)$
- $K(I_\mu B_{\mu_3}) \approx (\mathbf{Q} \oplus \mathbf{Q}(\zeta_3)) \oplus ((\mathbf{Q} \oplus \mathbf{Q}(\zeta'_3)) \oplus (\mathbf{Q} \oplus \mathbf{Q}(\zeta''_3)))$
- $K(I_{\mu_3}B_{\mu_3})_{taut} \approx \mathbf{Q}(\zeta'_3) \oplus \mathbf{Q}(\zeta''_3)$
- $K(I_{\mu_3}B_{\mu_3})_{geom} \otimes \tilde{R}\mu_3 \approx \left(\mathbf{Q} \otimes \mathbf{Q}\left(\zeta^!_3\right)\right) \oplus \left(\mathbf{Q} \otimes \mathbf{Q}\left(\zeta^!_3\right)\right)$

The first isomorphism sends $\mathbf{1}$ to $(1,1)$, ξ to $(1,\zeta_3)$, and $\bar{\xi}$ to $(1,\bar{\zeta}_3)$. Let us identify these entities. Let us denote $(1,0)$ by 1 and $(0,\zeta_3)$ by ζ_3. Then $\mathbf{1}+\xi+\bar{\xi}=3$, $\mathbf{1}-2\xi+\bar{\xi}=-3\zeta_3$ and $\mathbf{1}+\xi-2\bar{\xi}=-3\bar{\zeta}_3$. As a result, the projection to $\mathbf{Q}$ sends 1 to $1/3$. Furthermore, both ζ_3 and $\bar{\zeta}_3$ are sent to $-1/3$.

$(1,\zeta_3')$ and $(1,\zeta_3'')\in K(I_\mu B_{\mu_3})$ correspond to representations ξ' and ξ'' in $R\mu_3'$ and $R\mu_3''$ respectively, and as before

$$\mathbf{1}'+\xi'+\bar{\xi}'=3,\ \mathbf{1}'-2\xi'+\bar{\xi}'=-3\zeta_3' \text{ and } \mathbf{1}'+\xi'-2\bar{\xi}'=-3\bar{\zeta}_3'$$

$$\mathbf{1}''+\xi''+\bar{\xi}''=3,\ \mathbf{1}''-2\xi''+\bar{\xi}''=-3\zeta_3'' \text{ and } \mathbf{1}''+\xi''-2\bar{\xi}''=-3\bar{\zeta}_3''$$

$\left(1,\zeta_3^!\right)\in R\mu_3$ and $\left(1,\bar{\zeta}_3^!\right)\in R\mu_3$, correspond to the representations $\xi^!$ and $\bar{\xi}^!$ in the ring $R\mu_3^!$. The map $\beta_{B\mu_3}:K(I_{\mu_3}B\mu_3)_{taut}\to K(I_{\mu_3}B\mu_3)_{geom}\otimes\tilde{R}\mu_3$ sends $\mathbf{1}'$ to $1\otimes\mathbf{1}'$, to $\mathbf{1}''$ to $1\otimes\mathbf{1}''$, ξ' to $1\otimes\xi^!$ and ξ'' to $1\otimes\bar{\xi}^!$.

Let us see whether it is covariant with respect to the map $f:B\mu_3\to *$. $\beta_{B\mu_3}$ is essentially the map :

$$\mathbf{Q}\left(\zeta_3'\right)\oplus\mathbf{Q}\left(\zeta_3''\right)\to\left(\mathbf{Q}\otimes\mathbf{Q}\left(\zeta_3^!\right)\right)\oplus\left(\mathbf{Q}\otimes\mathbf{Q}\left(\zeta_3^!\right)\right)$$

sending ζ_3' to $1\otimes\zeta_3^!$ and ζ_3'' to $1\otimes\bar{\zeta}_3^!$. $\zeta_3'\in\mathbf{Q}(\zeta_3')\subseteq K(I_{\mu_3}B\mu_3)_{taut}$ is sent to $-1/3$ in $\mathbf{Q}$ by $\tilde{I}_\mu f_*:K(I_\mu B\mu_3)_{taut}\to K(*)_{taut}=\mathbf{Q}$. By contrast, $1\otimes\zeta_3^!$ is sent to $1\otimes\zeta_3$ by $I_\mu f_*:K(I_\mu B\mu_3)_{geom}\otimes\tilde{R}\mu_3\to K(*)_{geom}\otimes\tilde{R}\mu_3=\mathbf{Q}\otimes\mathbf{Q}(\zeta_3)$.

Let us unravel the map $(R\mu_3'\otimes R\mu_3^!)\oplus(R\mu_3''\otimes R\mu_3^!)\to\mathbf{Q}\otimes R\mu_3$. Recall that the following diagram commutes :

$$\begin{tikzcd} B\mu_3'\times B\mu_3 \arrow[r, "\alpha"] \arrow[d, "g"] & B\mu_3' \arrow[d, "f"] \\ *\times B\mu_3 \arrow[r, "\alpha"] & * \end{tikzcd}$$

where $g:=I_\mu f\otimes id$. This implies that :

$g_*\left(\left(\xi'\otimes\xi^!\right)\oplus\left(\xi''\otimes\bar{\xi}^!\right)\right)=g_*\left(\xi'\otimes\xi^!\right)+g_*\left(\xi''\otimes\bar{\xi}^!\right)=I_\mu f_*\left(\xi'\right)\otimes\xi^!+I_\mu f_*\left(\xi''\right)\otimes\bar{\xi}^!=1\otimes\xi^!+1\otimes\bar{\xi}^!=1\otimes(-1)$.

Now, the projection of the latter in $\mathbf{Q}(\zeta_3)$, which is -1, coincides with the image of $1\otimes\zeta_3+1\otimes\bar{\zeta}_3=1\otimes(-1)$ in $\mathbf{Q}$ in $\mathbf{Q}(\zeta_3)$. This projects to $-1/3$ in $\mathbf{Q}$, which is $((-1/3)+(-1/3))/\phi(3)$. Therefore,

$$f_*\beta_{B\mu_3}\left(\zeta_3'+\zeta_3''\right)\neq\beta_{Spec(k)}\left(f_*\left(\zeta_3'+\zeta_3''\right)\right)$$

To remedy this, we shall consider an appropriate readjustment of the map $\beta_{\mathcal{X}}$ in the general case, as follows. Let's fix presentations $\mathcal{X}\approx[X/G]$ and $\mathcal{Y}\approx[Y/H]$. We can suppose that G surjects onto H, so that each $\sigma\in C_r(G)$ surjects onto one $\tau\in C_s(H)$ for $s|r$. The morphism $I_\mu f_*$: $K'_*(I_\mu\mathcal{X})\longrightarrow K'_*(I_\mu\mathcal{Y})$ boils down to

$$\prod_r\prod_{\sigma\in C_r(G)}\prod_{\sigma\approx\mu_r}K'_*(X^\sigma,C_G(\sigma))\xrightarrow{I_\mu f_*}\prod_s\prod_{\sigma\in C_s(H)}\prod_{\sigma\approx\mu_s}K'_*(Y^\tau,C_H(\tau))$$

Let us fix some $\sigma\in C(G)$ and $\tau\in C(H)$ so that $I_\mu f_*$ maps $K'_*(X^\sigma,C_G(\sigma))$ to $K'_*(Y^\tau,C_H(\tau))$. To keep our notations light, we can suppose that $C_G(\sigma)=G$ and $C_H(\tau)=H$. We have $\tilde{R}\sigma\approx\mathbf{Q}(t)/(\Phi_r(t))$.

Let $[\mathcal{E}]\in K'_0(X^\sigma,G)$ be the class of a G-equivariant coherent sheaf on X^σ. Since the σ-action on $\mathcal{E}$ is diagonalizable, we have a decomposition into eigensheaves, namely $\mathcal{E}=\bigoplus_{j=0}^{r-1}\mathcal{E}^{(j)}$. This gives the following decomposition of K'-theory (see eg [KrS, Lemma 4.1]) :

$$K'(X^\sigma,G)=\bigoplus_{j=0}^{r-1}K'(X^\sigma,G)^{(j)}$$

Let t^{r_1}, ..., $t^{r_{\phi(r)}}$ be the $\phi(r)$ primitive r-th roots of 1, so that $1=r_1\leq\ldots\leq r_{\phi(r)}$ (in other words, the r_is are the integers prime to r and less than r). σ acts on $\mathcal{E}_i$ as the group $\langle t^{r_i}\rangle$. Note that if $\mathcal{P}_r$ is the set of primitive r-th roots of 1, then $\sigma=\bigcup_{d|r}\mathcal{P}_d$. Thanks to these notations, we can now write the preceding decomposition as :

$$K'(X^\sigma,G)=\bigoplus_{d|r}K'(X^\sigma,G)^{\{d\}}$$

where $K'(X^\sigma,G)^{\{d\}}=\bigoplus_{t^j\in\mathcal{P}_d}K'(X^\sigma,G)^{(j)}$.

We also have a decomposition of the form $[\mathcal{E}]=\sum_{\tau\in C(G)}[\mathcal{E}]_\tau=[\mathcal{E}]_g+[\mathcal{E}]_\sigma+[\mathcal{F}]$ as a result of the decomposition

$$K'(X^\sigma,G)\approx\bigoplus_{\tau\in C(G)}K'(X^\sigma,G)_\tau=K'(X^\sigma,G)_1\oplus K'(X^\sigma,G)_\sigma\oplus\bigoplus_{\tau\notin\{1,\sigma\}}K'(X^\sigma,G)_\tau$$

Bearing these decompositions in mind, we can make the following definitions, which do not depend upon the presentation of the quotient stack $\mathcal{X}$.

Definition 5.9 : We set:

(*i*)

$$\alpha^{\dagger}_{\mathcal{X},r} := \sum_{d|r} a_0(d)\phi\left(\frac{r}{d}\right)\alpha_{\mathcal{X},r}|_{K'_*(X^\sigma,G)^{\{d\}}}$$

(*ii*)

$$\tilde{\alpha}_{\mathcal{X},r} := (pr_{geom} \otimes pr_\sigma) \circ \alpha^{\dagger}_{\mathcal{X},r} \circ inc_\sigma$$

where $a_0(d)$ is the constant term of the d-th cyclotomic polynomial[3], and $\gamma : \tilde{R}\tau \to \tilde{R}\sigma$ is the map sending the generator u of $\tilde{R}\tau \approx \mathbf{Q}(u)/(\Phi_s(u))$ to $t^d \in \tilde{R}\sigma \approx \mathbf{Q}(t)/(\Phi_r(t))$, where $r = ds$. Moreover, the injective morphism

$$inc_\sigma : K'_*(X^\sigma, G)^{w_G(\sigma)}_\sigma \hookrightarrow \prod_{\sigma\approx\mu_r} K'_*(X^\sigma, G)$$

is the composition

$$K'_*(X^\sigma, G)^{w_G(\sigma)}_\sigma \xrightarrow[a]{\approx} \left(\prod_{\sigma\approx\mu_r} K'_*(X^\sigma, G)_\sigma\right)^{Aut(\mu_r)} \subseteq \prod_{\sigma\approx\mu_r} K'_*(X^\sigma, G)$$

These morphisms are the ones that will be relevant in the next chapter.

5.3 Intrinsicness

Let $\mathcal{X} \approx [X/G]$ be a tame quotient stack over k. In this section, we prove that the isomorphism

$$\delta_{X,G} : K'_*(X, G) \xrightarrow{\approx} \prod_{\sigma\in C(G)} \left(K'_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}(\sigma)\right)^{w_G(\sigma)}$$

exhibited by Proposition 5.1 essentially only depends on the quotient stack $[X/G]$, namely that $\delta_{X,G} \approx \delta_{Y,H}$ when $[X/G] \approx [Y/H]$. It is essentially what remains to do build our Riemann-Roch morphisms of stacks (which has to be intrinsic, ie not depend on the presentation of the stacks). Now, thanks to Proposition 5.6 and Proposition 5.8, it suffices to prove that for the isomorphism

$$\gamma^{-1}_{X,G}\beta^{-1}_{[X/G]}\delta_{X,G} : K'_*(X, G) \xrightarrow{\approx} \prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma))^{w_G(\sigma)}_\sigma$$

This is achieved in Corollary 5.15, which is the main result of this chapter.

In order to prove Corollary 5.15, we relate the latter isomoprhism to the morphism $\pi_{\mathcal{X}*} : K'_*(I_\mu(\mathcal{X})) \longrightarrow K'_*(\mathcal{X})$ induced from the projection map $\pi_{\mathcal{X}} : I_\mu\mathcal{X} \longrightarrow \mathcal{X}$ (cf Definition 2.6).

[3]Recall that $a_0(1) = -1$, and it is known that $a_0(n) = 1$ for every $n > 1$ (this can actually be readily seen by induction on n).

Definition 5.10 : Let $x : X \longrightarrow \mathcal{X}$ be the smooth cover of $\mathcal{X}$ asociated to the presentation $\mathcal{X} \approx [X/G]$. We define the G-equivariant algebraic space $I_\mu X$ by forming the following cartesian square :

$$\begin{tikzcd} I_\mu X \arrow[r] \arrow[d] & X \arrow[d, "x"] \\ I_\mu \mathcal{X} \arrow[r, "\pi_{\mathcal{X}}"] & \mathcal{X} \end{tikzcd}$$

In accordance with this diagram, the morphism of algebraic spaces $I_\mu X \longrightarrow X$ is G-equivariant. We call it $\pi_{X,G}$. Moreover, $I_{\mu_r} X \longrightarrow I_{\mu_r}\mathcal{X}$ which we call $I_r x$. We have $I_\mu \mathcal{X} \approx [I_\mu X/G]$, and we can legitimately write

$$I_\mu X = \coprod_r I_{\mu_r} X$$

so as to have $I_{\mu_r}\mathcal{X} \approx [I_{\mu_r} X/G]$ for all $r \geq 1$, or alternatively

$$I_\mu X = \coprod_{\alpha \in \tilde{C}(G)} I_\mu X_\alpha$$

so as to have $I_{\mu_r}\mathcal{X} \approx \coprod_{\alpha \in \tilde{C}_r(G)} [I_\mu X_\alpha / G] \approx: \coprod_{\alpha \in \tilde{C}_r(G)} I_\mu \mathcal{X}_\alpha$

Remark 5.11 : (i) For our purpose in this chapter, we can suppose that $G = GL_{n,k}$ for some $n \geq 1$. For every $\sigma \in C_r(G)$, $C_G(\sigma)$ acts on $G \times X$ by $n \cdot (g', x) = (gn^{-1}, nx)$, and this action is free. Let $G \times^{C_G(\sigma)} X$ denote the associated quotient : it is a G-equivariant algebraic space. There is a natural morphism of algebraic spaces $X = C_G \times^{C_G(\sigma)} X \longrightarrow G \times^{C_G(\sigma)} X$, which we denote c_σ, and which is induced from the closed subgroup inclusion $C_G(\sigma) \hookrightarrow G$. It is $C_G(\sigma)$-equivariant.

(ii) Morita equivalence of equivariant K-theory gives isomorphisms of $\mathbf{Q}$-vector spaces

$$M_{X,G,\sigma} : K'_*\left(X^\sigma, C_G(\sigma)\right) \xrightarrow{\approx} K'_*\left(G \times^{C_G(\sigma)} X^\sigma, G\right)$$

Setting $M_{X,G,r} := \prod_{\sigma \in C_r(G)} M_{X,G,\sigma}$ and $M_{X,G} := \prod_r M_{X,G,r}$, we have

$$\prod_{\sigma \in C(G)} K'_*\left(X^\sigma, C_G(\sigma)\right) \xrightarrow[M_{X,G}]{\approx} \prod_{\sigma \in C(G)} K'_*\left(G \times^{C_G(\sigma)} X^\sigma, G\right)$$

(iii) By Chapter 2, we have

$$I_{\mu_r} X = \coprod_{\alpha \in \tilde{C}_r(G)} I_{\mu_r} X_\alpha := \coprod_{\alpha \in \tilde{C}_r(G)} G \times^{C_G(\sigma)} X^{\alpha(\mu_r)}$$

so that

$$K'_*\left(I_{\mu_r}X,G\right)=\prod_{\alpha\in\tilde{C}_r(G)}K'_*\left(G\times^{C_G(\sigma)}X^{\alpha(\mu_r)},G\right)$$

Henceforth, we will denote the fixed-point spaces $X^{\alpha(\mu_r)}$ by X^α for each $\alpha\in\tilde{C}_r(G)$. And we will denote $M_{X,G}$ the Morita equivalence isomorphisms

$$\prod_{\alpha\in\tilde{C}(G)}K'_*\left(X^\alpha,C_G\left(\alpha\right)\right)\xrightarrow{\approx}\prod_{\alpha\in\tilde{C}(G)}K'_*\left(G\times^{C_G(\alpha)}X^\alpha,G\right)$$

Definition 5.12 : We define

$$\aleph_{X,G}:\prod_{\sigma\in C(G)}K'_*\left(X^\sigma,C_G\left(\sigma\right)\right)_\sigma^{w_G(\sigma)}\longrightarrow K'_*\left(X,G\right)$$

to be the map $\delta^{-1}_{X,G}\beta_{[X/G]}\gamma^{-1}_{X,G}$

Definition 5.13 : We define

$$\aleph_{\mathcal{X}}:\prod_{r\geq1}K'_*\left(I_{\mu_r}\mathcal{X}\right)^{Aut(\mu_r)}_{taut}\longrightarrow K'_*\left(\mathcal{X}\right)$$

to be the morphism $(x^*)^{-1}\aleph_{X,G}\gamma_{X,G}$, where $x^*:K'_*(\mathcal{X})\xrightarrow{\approx}K'_*(X,G)$ is the isomorphism induced by x. A priori, it depends on the considered presentation of $\mathcal{X}$.

Proposition 5.14 : (i) Let $\iota_{X,G}$ denote the natural inclusion

$$\prod_{\sigma\in C(G)}K'_*(X^\sigma,C_G\left(\sigma\right)^{w_G(\sigma)}_\sigma\hookrightarrow\prod_{\alpha\in\tilde{C}(G)}K'_*\left(X^\alpha,C_G\left(\alpha\right)\right)$$

Then $\aleph_{X,G}=(\pi_{X,G})_*M_{X,G}\iota_{X,G}$
(ii) Let $\iota_{\mathcal{X}}$ denote the natural inclusion

$$\prod_{r\geq1}K'_*\left(I_{\mu_r}\mathcal{X}\right)^{Aut(\mu_r)}_{taut}\hookrightarrow K'_*\left(I_\mu\mathcal{X}\right)$$

Then $\aleph_{\mathcal{X}}=\pi_{\mathcal{X}*}\iota_{\mathcal{X}}$

Proof : (i) $\aleph_{X,G}$ fits in the following commutative diagram :

$$\begin{tikzcd}
K'_*\left(X,G\right) \arrow[r, shift left, "can"] & \prod_{\sigma\in C(G)}K'_*\left(X,G\right)_\sigma \arrow[l, shift left] \arrow[r, "res"] & \prod_{\sigma\in C(G)}K'_*\left(X,C_G\left(\sigma\right)\right)_\sigma \arrow[r, "\prod_\sigma(j_{\sigma*})^{-1}"] & \prod_{\sigma\in C(G)}K'_*\left(X^\sigma,C_G\left(\sigma\right)\right)_\sigma \\
 & & & \prod_{\sigma\in C(G)}K'_*\left(X^\sigma,C_G\left(\sigma\right)\right)^{w_G(\sigma)}_\sigma \arrow[u, hook] \arrow[ulll, "\aleph_{X,G}"]
\end{tikzcd}$$

and what we shall prove is that the following diagram is commutative :

$$\begin{tikzcd}
K'_*(X,G) & K'_*(I_\mu X,G) \arrow[l, "(\pi_{X,G})_*"'] & \prod_{\alpha\in\tilde{C}(G)} K'_*(X^\alpha, C_G(\alpha)) \arrow[l, "M_{X,G}"'] \\
 & & \prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma^{w_G(\sigma)} \arrow[u, hook, "\iota_{X,G}"] \arrow[ull, "\aleph_{X,G}"]
\end{tikzcd}$$

which amounts to proving that the following diagram is commutative :

$$\begin{tikzcd}
\prod_{\alpha\in\tilde{C}(G)} K'_*(X^\alpha, C_G(\alpha)) \arrow[r, "M_{X,G}"] \arrow[d, "can", "\approx"'] & K'_*(I_\mu X,G) \arrow[r, "(\pi_{X,G})_*"] & & K'_*(X,G) \arrow[d, "can", "\approx"'] \\
\prod_{\alpha,\sigma} K'_*(X^\alpha, C_G(\alpha))_\sigma & \prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma \arrow[l, hook', "inc"'] & \prod_{\sigma\in C(G)} K'_*(X, C_G(\sigma))_\sigma \arrow[l, "\prod_\sigma (j_{\sigma*})^{-1}"'] & \prod_{\sigma\in C(G)} K'_*(X, \arrow[l, "res"']
\end{tikzcd}$$

The morphism $M_{X,G}can^{-1}inc$ coincides with the composition :

$$\prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma))_\sigma \hookrightarrow \prod_{\sigma\in C(G)} K'_*(X^\sigma, C_G(\sigma)) \xrightarrow{\prod_{\sigma\in C(G)} M_{X,G,\sigma}} \prod_{\sigma\in C(G)} K'_*(I_\mu X_\sigma, G)$$

so it suffices to prove that the diagram

$$\begin{tikzcd}
K'_*(X^\sigma, C_G(\sigma)) \arrow[r, "M_{X,G,\sigma}"] & K'_*(I_\mu X_\sigma, G) \arrow[r, "(\pi_{X,G,\sigma})_*"] & K'_*(X,G) \\
K'_*(X^\sigma, C_G(\sigma))_\sigma \arrow[u, hook] & K'_*(X, C_G(\sigma))_\sigma \arrow[l, "j_{\sigma*}^{-1}"'] & K'_*(X,G)_\sigma \arrow[l, "res"'] \arrow[u, hook]
\end{tikzcd}$$

commutes, where $\pi_{X,G,\sigma} := \pi_{X,G}|_{I_\mu X_\sigma}$, in other words that the square

$$\begin{tikzcd}
K'_*(X^\sigma, C_G(\sigma))_\sigma \arrow[r, "M_{X,G,\sigma}"] \arrow[d, "j_{\sigma*}"'] & K'_*(I_\mu X_\sigma, G)_\sigma \arrow[d, "(\pi_{X,G,\sigma})_*"'] \\
K'_*(X, C_G(\sigma))_\sigma & K'_*(X,G)_\sigma \arrow[l, "res"']
\end{tikzcd}$$

is commutative for every dual cyclic subgroup σ. We claim that this is true without σ-localizing.

By [Th1, Proposition 6.2], $M^{-1}_{X,G,\sigma}$ is induced from restriction along the $C_G(\sigma)$-equivariant map c_σ, ie

$$\begin{tikzcd}
K'_*\left(G \times^{C_G(\sigma)} X^\sigma, C_G(\sigma)\right) \arrow[r, "c_\sigma^*"] & K'_*(X^\sigma, C_G(\sigma)) \\
K'_*\left(G \times^{C_G(\sigma)} X^\sigma, G\right) \arrow[u, "res"] \arrow[ur, "M^{-1}_{X,G,\sigma}"'] &
\end{tikzcd}$$

commutes.

Let $\mathcal{F}$ be a G-equivariant coherent sheaf on $G \times^{C_G(\sigma)} X^\sigma$. Since $\pi_{X,G,\sigma}$ is a finite morphism of noetherian schemes, $(\pi_{X,G,\sigma})_*\mathcal{F}$ is coherent. Our claims follows from the fact that

$$(\pi_{X,G,\sigma})_*\mathcal{F} \approx j_{\sigma *} c_\sigma^* \mathcal{F}$$

as G-equivariant coherent sheaves on X, since for any morphism of k-schemes $x : T \longrightarrow X^\sigma$, $\pi_{X,G,\sigma} c_\sigma x$ and $j_\sigma x$ are in the same $G(T)$-orbit.

(ii) We have :

$$\begin{aligned}\aleph_{\mathcal{X}} &= (x^*)^{-1}\aleph_{X,G}\gamma_{X,G} = (x^*)^{-1}\pi_{X,G}M_{X,G}\iota_{X,G}\gamma_{X,G} \\ &= ((x^*)^{-1}(\pi_{X,G})_* I_\mu x^*)((I_\mu x^*)^{-1}M_{X,G}\iota_{X,G}\gamma_{X,G}) = \pi_{\mathcal{X}*}\iota_{\mathcal{X}}\end{aligned}$$

where the second equality follows from (i) and the others follow by definition.$\diamond$

As an immediate consequence of Proposition 5.14.(ii), we have :

Corollary 5.15 : $\aleph_{\mathcal{X}}$ is an intrinsic isomorphism of $\mathbf{Q}$-vector spaces.$\diamond$

Chapter 6

The Grothendieck-Riemann-Roch theorem

6.1 Definition of the Riemann-Roch maps

In this chapter, we introduce the pre-Riemann-Roch map and the Lefschetz-Riemann-Roch map, which we denote $\tilde{\tau}_{\mathcal{X}}$ and $\mathcal{L}_{\mathcal{X}}$ respectively, associated to an algebraic stack $\mathcal{X}$ over a field k, and state and prove the Toën-Riemann-Roch theorem for tame Artin stacks. These two maps are very closely related.

Definition 6.1 : Let $\mathcal{X}$ be a quotient tame Artin stack.

The pre-Riemann-Roch map $\tilde{\tau}_{\mathcal{X}} : K'_*(\mathcal{X}) \longrightarrow \left(K'_*(I_\mu(\mathcal{X}))_{geom} \otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)}$ is defined to be the map $\tilde{\alpha}_{\mathcal{X}}\aleph_{\mathcal{X}}^{-1}$, where $\tilde{\alpha}_{\mathcal{X}}$ is as in Definition 5.9, and $\aleph_{\mathcal{X}}$ is as in Definition 5.13.

The Lefschetz-Riemann-Roch map is a map $\mathcal{L}_{\mathcal{X}} : K'_*(\mathcal{X}) \longrightarrow K'_*\left(I_\mu(\mathcal{X})\right)_{geom}$ which is obtained from the former in Section 6.3.

Remark 6.2 : It follows from Corollary 5.15 and Proposition 5.8 that $\tilde{\tau}_{\mathcal{X}}$ is an isomorphism for every quotient tame Artin stack.

As to the *Grothendieck-Riemann-Roch map*, which appears in the Grothendieck-Riemann theorem proven in Section 6.4, we shall postpone its definition to that section, as it involves some higher intersection theory.

6.2 Covariance of the pre-Riemann-Roch map

Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of tame Artin stacks. Proposition 2.22 gives us a natural morphism $I_\mu(f) : I_\mu(\mathcal{X}) \longrightarrow I_\mu(\mathcal{Y})$. The latter in turn induces a morphism

$$I_\mu(f)_* : K'_*\left(I_\mu(\mathcal{X})\right) \longrightarrow K'_*\left(I_\mu(\mathcal{X})\right)$$

and hence, thanks to Proposition 3.3, a morphism

$$\left(K'_*(I_\mu(\mathcal{X}))_{geom} \otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)} \xrightarrow{I_\mu(f)_*} \left(K'_*(I_\mu(\mathcal{Y}))_{geom} \otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)}$$

The main result of this section is Theorem 6.3 below. Ideally, we should have the following theorem, but we were only able to prove one half of it (namely, Theorem 6.3.1). By contrast, the second half (so to speak) is Conjecture 6.3.2, and we indicate below its statement how far we have been to proving it, thereby suggesting an approach that we believe might be appropriate.

Conjecture 6.3 : Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper morphism of quotient tame Artin stacks. Then the following diagram is commutative :

$$\begin{tikzcd}
K'_*(\mathcal{X}) \arrow[r, "\tilde{\tau}_{\mathcal{X}}"] \arrow[d, "f_*"'] & \left(K'_*(I_\mu(\mathcal{X}))_{geom} \otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)} \arrow[d, "I_\mu(f)_*"'] \\
K'_*(\mathcal{Y}) \arrow[r, "\tilde{\tau}_{\mathcal{Y}}"] & \left(K'_*(I_\mu(\mathcal{Y}))_{geom} \otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)}
\end{tikzcd}$$

By Proposition 5.7, we can see $\aleph_{\mathcal{X}}$ as a morphism of the form

$$\textstyle\prod_{r\geq 1}\left(\prod_{\mathcal{U}\subseteq I_{\mu_r}\mathcal{X}} K'_*(\mathcal{U})_{m_{\mathcal{U}}}\right)^{Aut(\mu_r)} \longrightarrow K'_*(\mathcal{X})$$

By definition of the pre-Riemann-Roch map, we have to prove that the outer square in the diagram below is commutative. To do so, we construct the dotted arrow, which is subject to the condition that the left inner square must commute. Once this map, which we denote $\tilde{I}_\mu f_*$ is constructed, it suffices to prove that the right inner square is commutative. This is essentially the content of Theorem 6.3.1.

$$\begin{tikzcd} K'_*(\mathcal{X}) \arrow[r, "\aleph_{\mathcal{X}}^{-1}"] \arrow[d, "f_*"] & \prod_{r\geq 1}\left(\prod_{\mathcal{U}} K'_*(\mathcal{U})_{m_{\mathcal{U}}}\right)^{Aut(\mu_r)} \arrow[r, "\tilde{\alpha}_{\mathcal{X}}"] \arrow[d, dotted] & \prod_{r\geq 1}\left(\prod_{\mathcal{U}} K'_*(\mathcal{U})_{geom}\otimes \tilde{R}\mu_r\right)^{Aut(\mu_r)} \approx \left(K'_*(I_\mu(\mathcal{X}))_{geom}\otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)} \arrow[d, "I_\mu(f)_*"] \\ K'_*(\mathcal{Y}) \arrow[r, "\aleph_{\mathcal{Y}}^{-1}"] & \prod_{s\geq 1}\left(\prod_{\mathcal{V}} K'_*(\mathcal{V})_{m_{\mathcal{V}}}\right)^{Aut(\mu_s)} \arrow[r, "\tilde{\alpha}_{\mathcal{Y}}"] & \prod_{s\geq 1}\left(\prod_{\mathcal{V}} K'_*(\mathcal{V})_{geom}\otimes \tilde{R}\mu_s\right)^{Aut(\mu_s)} \approx \left(K'_*(I_\mu(\mathcal{Y}))_{geom}\otimes \tilde{R}\mu_\infty\right)^{Aut(\mu_\infty)} \end{tikzcd}$$

Theorem 6.3.1 : Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper morphism of quotient tame Artin stacks. Then the following diagram is commutative :

$$\begin{tikzcd} K'_*(\mathcal{X}) \arrow[r, "\aleph_{\mathcal{X}}^{-1}"] \arrow[d, "f_*"] & \prod_{r\geq 1} K'_*\left(I_{\mu_r}\mathcal{X}\right)_{taut}^{Aut(\mu_r)} \arrow[d, "I_\mu(f)_*"] \\ K'_*(\mathcal{Y}) \arrow[r, "\aleph_{\mathcal{Y}}^{-1}"] & \prod_{s\geq 1} K'_*\left(I_{\mu_s}\mathcal{Y}\right)_{taut}^{Aut(\mu_s)} \end{tikzcd}$$

Proof : Let us first unravel the map

$$\tilde{I}_\mu f_* : \prod_r \prod_{\mathcal{U}} K'_*(\mathcal{U})_{m_{\mathcal{U}}} \longrightarrow \prod_s \prod_{\mathcal{V}} K'_*(\mathcal{V})_{m_{\mathcal{V}}}$$

corresponding to $I_\mu f_* : K'_*\left(I_\mu\mathcal{X}\right)_{taut} \longrightarrow K'_*\left(I_\mu\mathcal{Y}\right)_{taut}$.

Let us first work using fixed presentations of our quotient stacks. Suppose that $\mathcal{X} \approx [X/G]$ and $\mathcal{Y} \approx [Y/H]$. Then $I_\mu\mathcal{X} \approx [I_\mu X/G]$ and $I_\mu\mathcal{Y} \approx [I_\mu Y/H]$ as in the preceding section, and thanks to Proposition 5.7, we have $(\forall \mathcal{U})$ $m_{\mathcal{U}} = m_\sigma$ for some $\sigma \in C(G)$, and $(\forall \mathcal{V})$ $m_{\mathcal{V}} = m_\tau$ for some $\tau \in C(H)$. There is a natural map $I_\mu f : I_\mu\mathcal{X} \longrightarrow I_\mu\mathcal{Y}$, and functoriality of taking pushforwards yields a commutative diagram :

$$\begin{tikzcd} K'_*(\mathcal{X}) \arrow[d, "f_*"] & K'_*(I_\mu\mathcal{X}) \arrow[l, "\pi_{\mathcal{X}*}"'] \arrow[d, "I_\mu f_*"] \\ K'_*(\mathcal{Y}) & K'_*(I_\mu\mathcal{Y}) \arrow[l, "\pi_{\mathcal{Y}*}"'] \end{tikzcd}$$

Now, $K'_*(I_\mu\mathcal{X})_{m_\mathcal{U}} \subseteq K'_*(I_\mu\mathcal{X})$ and $K'_*(I_\mu\mathcal{Y})_{m_\mathcal{V}} \subseteq K'(I_\mu\mathcal{Y})$ both as direct summands. What we now need to check to verify that $\tilde{I}_\mu f$ is well defined is that to each $\mathcal{U}$ corresponds a $\mathcal{V}$ such that $I_\mu f$ localizes to a map $I_\mu f_{m_\mathcal{U}} =: \tilde{I}_\mu f|_\mathcal{U} : K'_*(I_\mu\mathcal{X})_{m_\mathcal{U}} \longrightarrow K'_*(I_\mu\mathcal{Y})_{m_\mathcal{V}}$. This would yield the following commutative rectangle with commutative cells :

$$\begin{tikzcd}
K'_*(I_\mu\mathcal{X})_{m_\mathcal{U}} \arrow[r] \arrow[d] & K'_*(I_\mu\mathcal{X}) \arrow[r] \arrow[d] & K'_*(\mathcal{X}) \arrow[d] \\
K'_*(I_\mu\mathcal{Y})_{m_\mathcal{V}} \arrow[r] & K'_*(I_\mu\mathcal{Y}) \arrow[r] & K'_*(\mathcal{Y})
\end{tikzcd}$$

We have $K'_*(I_\mu\mathcal{X})_{m_\mathcal{U}} = K'_*(I_\mu X, G)_\sigma$ and $K'_*(I_\mu\mathcal{Y})_{m_\mathcal{V}} = K'_*(I_\mu Y, H)_\tau$.

At this point, we shall work with a $(G \times H)$-equivariant cover Z of $\mathcal{X}$, instead of the G-equivariant cover we initially considered. Z is obtained as follows : as we have a commutative rectangle with cartesian cells :

$$\begin{tikzcd}
Y \times_\mathcal{Y} X \arrow[r] \arrow[d] & X \arrow[d] \\
Y \times_\mathcal{Y} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
Y \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

exhibiting the fibre product $Y \times_\mathcal{X} X$ as a $(G \times H)$-torsor over $\mathcal{X}$, we can take $Z = Y \times_\mathcal{X} X$, so that $\mathcal{X} \approx [Z/G \times H]$.

Now, consider the following two compositions :

$$m_\sigma \subseteq RG \otimes RH \longrightarrow R\sigma \twoheadrightarrow \tilde{R\sigma}$$

$$m_\tau \subseteq RH \longrightarrow R\tau \twoheadrightarrow \tilde{R\tau}$$

We have $\tau = Im(\sigma \longrightarrow H)$ and hence we have a commutative square :

$$\begin{tikzcd}
\sigma \arrow[r, hook] \arrow[d] & G \times H \arrow[d] \\
\tau \arrow[r, hook] & H
\end{tikzcd}$$

inducing another one :

$$\begin{tikzcd}
R\sigma & RG \otimes RH \arrow[l] \\
R\tau \arrow[u, "1\otimes id"] & RH \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

Note that $1 \otimes id$ is a section of pr_2, so that the latter also fits in the last diagram. Considering $K_0(I_\mu \mathcal{X})$ and $K_0(\mathcal{X})$ as $RG \otimes RH$-modules, and $K_0(I_\mu \mathcal{Y})$ and $K_0(\mathcal{Y})$ as RH-modules, it now suffices to prove that the following diagram commutes :

$$\begin{tikzcd}
m_\sigma & \subseteq RG \otimes RH \arrow[r] & R\sigma \arrow[r] & \tilde{R\sigma} \\
m_\tau \arrow[u] & \subseteq \quad RH \arrow[u] \arrow[r] & R\tau \arrow[u] \arrow[r] & \tilde{R\tau} \arrow[u]
\end{tikzcd}$$

which is what we checked when we proved the intrinsicness of the decomposition $K_*(\mathcal{X}) \approx K_*(\mathcal{X})_{geom} \times K_*(\mathcal{X})_{extra}$. Therefore, $\tilde{I}_\mu f$ is defined and has the desired properties.

As a result of this construction, and by virtue of Proposition 5.14.(ii), the following diagram is commutative :

$$\begin{tikzcd}
\prod_{r\geq 1} K'_*(I_{\mu_r}\mathcal{X})_{taut} \arrow[r, hook] \arrow[d, "\tilde{I}_\mu f_*"] & \prod_{r\geq 1} K'_*(I_{\mu_r}\mathcal{X}) \arrow[r, "\pi_{\mathcal{X}*}"] \arrow[d, "I_\mu f_*"] & K'_*(\mathcal{X}) \arrow[d, "f_*"] \\
\prod_{s\geq 1} K'_*(I_{\mu_s}\mathcal{Y})_{taut} \arrow[r, hook] & \prod_{s\geq 1} K'_*(I_{\mu_s}\mathcal{Y}) \arrow[r, "\pi_{\mathcal{Y}*}"] & K'_*(\mathcal{Y})
\end{tikzcd}$$

Now, the restriction of $\tilde{I}_\mu f_*$ to $\prod_{r\geq 1} K'_*\,(I_{\mu_r}\mathcal{X})^{Aut(\mu_r)}_{taut}$ has image contained in $\prod_{s\geq 1} K'_*\,(I_{\mu_s}\mathcal{Y})^{Aut(\mu_s)}_{taut}$, so that it yields a map

$$\prod_{r\geq 1} K'_*(I_{\mu_r}\mathcal{X})^{Aut(\mu_r)}_{taut} \longrightarrow \prod_{s\geq 1} K'_*(I_{\mu_s}\mathcal{Y})^{Aut(\mu_s)}_{taut}$$

fitting in the following commutative diagram :

$$\begin{tikzcd}
\prod_{r\geq 1} K'_*(I_{\mu_r}\mathcal{X})^{Aut(\mu_r)}_{taut} \arrow[r, hook] \arrow[d, "\tilde{I}_\mu f_*"'] & \prod_{r\geq 1} K'_*(I_{\mu_r}\mathcal{X})_{taut} \arrow[d, "\tilde{I}_\mu f_*"'] \\
\prod_{s\geq 1} K'_*(I_{\mu_s}\mathcal{Y})^{Aut(\mu_s)}_{taut} \arrow[r, hook] & \prod_{s\geq 1} K'_*(I_{\mu_s}\mathcal{Y})_{taut}
\end{tikzcd}$$

Conjecture 6.3.2 : The right inner square is commutative. namely, the following diagram is commutative :

$$\begin{tikzcd}
 & \prod_{\sigma\approx\mu_r} K'_*(X^\sigma,G) \arrow[rr, "\alpha^\dagger_{\mathcal{X},r}"] & & \prod_{\sigma\approx\mu_r}\left(K'_*(X^\sigma,G)\otimes R\sigma\right) \arrow[dl, "pr_{geom}\otimes pr_\sigma"] \\
K'_*(X^\sigma,G)^{w_G(\sigma)}_\sigma \arrow[ur, "inc_\sigma"] \arrow[rr, "\tilde{\alpha}_{\mathcal{X},r}"] \arrow[dd, "\tilde{I}_\mu f_*"] & & \left(K'_*(X^\sigma,G)_{geom}\otimes\tilde{R}\sigma\right) \arrow[dd] & \\
 & \prod_{\tau\approx\mu_s} K'_*(Y^\tau,H) \arrow[rr] & & \prod_{\tau\approx\mu_s}\left(K'_*(Y^\tau,H)\otimes R\sigma\right) \arrow[dl, "pr_{geom}\otimes pr_\sigma"] \\
K'_*(Y^\tau,H)^{w_H(\tau)}_\tau \arrow[ur] \arrow[rr, "(1\otimes\gamma)\tilde{\alpha}_{\mathcal{Y},s}"] & & \left(K'_*(Y^\tau,H)_{geom}\otimes\tilde{R}\sigma\right) &
\end{tikzcd}$$

Let us indicate how we think Conjecture 6.3.2 could be proved.

Let $[\mathcal{E}]$ be the image in $K'_*(X^\sigma,G)^{w_G(\sigma)}_\sigma$ of a G-equivariant sheaf $\mathcal{E}$ on X^σ left invariant by $w_G(\sigma)$. We have

$$a([\mathcal{E}]) = ([\mathcal{E}_1],...,[\mathcal{E}_{\phi(r)}])$$

As is shown by Krishna and Sreedhar in [KrS, Lemma 2.8(4)], the morphism f factors as $f = gf'$ where g is representable and f' is a stacky moduli space map (while Kirshna and Sreedhar use this result only to deal with complex Deligne-Mumford stacks, their proof works for arbitrary tame stacks). In detail, this implies that there is an H-equivariant stack $\mathcal{X}'$ with moduli space M' such that the following diagram is commutative

$$\begin{tikzcd}
{[X/G] = [\mathcal{X}'/H]} \arrow[r, "f"] \arrow[d, "f'"'] & {[Y/H]} \\
{[M'/H]} \arrow[ur, "g"'] &
\end{tikzcd}$$

and, since the case of a representable map is well-known, we are reduced to prove the proposition when f is as in [KrS, Lemma 2.8(4)] (namely, what the authors call a *coarse moduli stack map*). Now, $[X/G]$ and $[M'/H]$ have isomorphic coarse moduli spaces. We denote M one of them.

Let us consider the following diagram

$$\begin{tikzcd}
K'(X^\sigma, G)_\sigma^{w_G(\sigma)} \arrow[r, "I_\mu f_*"] \arrow[d] & K'(M'^\tau, H)_\tau^{w_H(\tau)} \arrow[r] \arrow[d] & K'(M_\sigma) \arrow[d, "id\otimes 1"] \\
K'(X^\sigma, G)_{geom} \otimes \tilde{R}\sigma \arrow[r, "I_\mu f_*\otimes id"] & K'(M'^\tau, H)_{geom} \otimes \tilde{R}\sigma \arrow[r, "\approx"] & K'(M_\sigma) \otimes \tilde{R}\sigma
\end{tikzcd}$$

where M_σ is the moduli space of the quotient stack $[X^\sigma/G]$. Now, since the right bottom horizontal arrow is an isomorphism, it suffices to prove that the right inner square and the outer square commute to prove that the left inner square is commutative (as can be easily verified, and as is done in [KrS, Lemma 8.5]).

As a result, we need to check that the following square is commutative

$$\begin{tikzcd}
K'_*(X^\sigma, G)_\sigma^{w_G(\sigma)} \arrow[r, "\tilde{\alpha}"] \arrow[d, "I_\mu f_*"'] & K'_*(X^\sigma, G)_{geom} \otimes \tilde{R}\sigma \arrow[d, "I_\mu f_*\otimes id"'] \\
K'_*(M_\sigma) \arrow[r, "id\otimes 1"] & K'_*(M_\sigma) \otimes \tilde{R}\sigma
\end{tikzcd}$$

Let $[\mathcal{E}] \in K'(X^\sigma, G)$ be the class of a G-equivariant coherent sheaf on X^σ. We have

$$I_\mu f_*[\mathcal{E}] = \sum_i [\mathcal{E}_i]^G = \phi(r)\,[\mathcal{E}_1]^G$$

and, in turn, one has $[\mathcal{E}_1] = [\mathcal{E}_{1,1}] + ... + \left[\mathcal{E}_{1,|w_G(\sigma)|}\right]$ in $K'_*(X^\sigma, G)_\sigma$, so that in the end

$$I_\mu f_*[\mathcal{E}] = \phi(r)|w_G(\sigma)|\,[\mathcal{E}_{1,1}]^G$$

On the other hand,

$$\begin{aligned}
\alpha^*([\mathcal{E}]) &= \alpha^*\left(\sum_{i,j}\left[\mathcal{E}_i^{(j)}\right]\right) = \sum_{i,j}\left[\mathcal{E}_i^{(j)}\right]\otimes t^{r_i j} \\
&= \sum_i \left[\mathcal{E}_i^{(0)}\right]\otimes 1 + \sum_i\sum_{j\geq 1}\left[\mathcal{E}_i^{(j)}\right]\otimes t^{r_i j} \\
&= \phi(r)\left[\mathcal{E}_1^{(0)}\right]\otimes 1 + \sum_i\sum_{j\geq 1}\left[\mathcal{E}_i^{(j)}\right]\otimes t^{r_i j} \\
&= \phi(r)\left[\mathcal{E}_1^{(0)}\right]\otimes 1 + \sum_{j\geq 1}\left[\mathcal{E}_1^{(j)}\right]\otimes\left(\sum_i t^{r_i j}\right) \\
&= \phi(r)\left[\mathcal{E}_1^{(0)}\right]\otimes 1 + \sum_{j\geq 1}\left[\mathcal{E}_1^{(j)}\right]\otimes \mu_j(r)
\end{aligned}$$

Where we have set

$$\mu_j(r) = \sum_{i=1}^{\phi(r)} (t^{r_i})^j$$

for $j \leq r$. We can call it the *j-th twisted Möbius function*, since $\mu_1(r) = \mu(r)$, where μ is the Möbius function[1]. If $gcd(r,j) = d$, then

$$\mu_j(r) = \mu_d(r) = \mu\left(\frac{r}{d}\right)\frac{\phi(r)}{\phi(r/d)}$$

since t^j is then a primitive r/d-th root of 1.

Now,

$$\begin{aligned}
\alpha^\dagger([\mathcal{E}]) &= -\phi(r)\left[\mathcal{E}_1^{(0)}\right]\otimes 1 + \sum_{j\geq 1}\left[\mathcal{E}_1^{(j)}\right]\otimes\mu\left(\frac{r}{gcd(j,r)}\right)\frac{\phi(r)}{\phi(r/gcd(j,r))}\phi\left(\frac{r}{gcd(j,r)}\right) \\
&= -\phi(r)\left[\mathcal{E}_1^{(0)}\right]\otimes 1 + \sum_{j\geq 1}\left[\mathcal{E}_1^{(j)}\right]\otimes\mu\left(\frac{r}{gcd(j,r)}\right)\phi(r)
\end{aligned}$$

[1] $\mu : \mathbf{N} \to \mathbf{Z}$ is the function defined by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is an integer divisible by a square} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \end{cases}$$

Note that the function μ is multiplicative, ie $\mu(ab) = \mu(a)\mu(b)$ when a and b are coprime.

so that

$$\tilde{\alpha}\left([\mathcal{E}]_\sigma\right) = -\phi(r)\left[\mathcal{E}_1^{(0)}\right]_g \otimes 1 + \sum_{j\geq 1}\left[\mathcal{E}_1^{(j)}\right]_g \otimes \mu\left(\frac{r}{gcd(j,r)}\right)\phi(r)$$

We thus have

$$(I_\mu f_* \otimes id)\tilde{\alpha}\left([\mathcal{E}]_\sigma\right) = \phi(r)\left(-\left[\mathcal{E}_1^{(0)}\right]_g^G \otimes 1 + \sum_{j\geq 1}\left[\mathcal{E}_1^{(j)}\right]_g^G \otimes \mu\left(\frac{r}{gcd(j,r)}\right)\right)$$

It remains to check that

$$[\mathcal{E}]_\sigma^G = \phi(r)\left(-\left[\mathcal{E}_1^{(0)}\right]_g^G + \sum_{j\geq 1}\mu\left(\frac{r}{gcd(j,r)}\right)\left[\mathcal{E}_1^{(j)}\right]_g^G\right)$$

$a_n(r) \in \mathbf{Z}$ for all $r, n \in \mathbf{N}$.

A proof of the latter identity would finish the proof of Conjecture 6.3.2, and hence of Conjecture 6.3. It could be the subject of future investigations.

6.3 A Lefschetz-Riemann-Roch isomorphism

The goal of this section is to prove Proposition 6.4, which enables us to give an equivalent and simpler reformulation of Theorem 6.3, namely Theorem 1.6 which was announced in the Introduction. This is essentially a result in Galois theory.

Let $k \subseteq L$ be a Galois extension of fields. Let M be a k-vector space. Consider the action of $G = Gal(L/k)$ on M induced from the natural action of G on L. By the normal basis theorem (which holds for all Galois extensions), L has a normal basis $(\sigma(\alpha))_{\sigma\in G}$, where α denotes some element of L. As a result, considering the action of G on $\oplus_{\sigma\in G} k$ by permutations, the map

$$L \longrightarrow \oplus_{\sigma\in G}\, k$$

sending $x \in L$ to its decomposition over this normal basis, is an equivariant isomorphism. And then

$$M \otimes L \xrightarrow{\approx} \oplus_{\sigma\in G}\, M$$

equivariantly for every k-vector space M, G acting on the right side by permuting elements. This gives the following isomoprhism of $\mathbf{Q}$-vetcor spaces :

$$(M \otimes_k L)^G \xrightarrow[\phi_\alpha]{\approx} M$$

When $k = \mathbf{Q}$, $L = \mathbf{Q}(\zeta_r)$, $M = K'_*\left(I_{\mu_r}(\mathcal{X})\right)_{geom}$, picking the normal basis consisting of i-th powers ζ_r^i, for every i prime to r, we get in particular the following isomoprhisms :

$$\left(K'_*\left(I_{\mu_r}\left(\mathcal{X}\right)\right)_{geom} \otimes \mathbf{Q}\left(\zeta_\infty\right)\right)^{Gal(\mathbf{Q}(\zeta_r)/\mathbf{Q})} \xrightarrow[\phi_{\zeta_r}]{\approx} K'_*\left(I_{\mu_r}\left(\mathcal{X}\right)\right)_{geom}$$

which are the inverses of the morphisms sending $u \in K'_*\left(I_{\mu_r}\left(\mathcal{X}\right)\right)_{geom}$ to $\sum_i u \otimes \zeta_r^i$.

Combining all of these isomorphisms yields the isomorphism

$$\prod_r \left(K'_*\left(I_{\mu_r}\left(\mathcal{X}\right)\right)_{geom} \otimes \mathbf{Q}\left(\zeta_\infty\right)\right)^{Gal(\mathbf{Q}(\zeta_r)/\mathbf{Q})} \xrightarrow[\prod_r \phi_{\zeta_r}]{\approx} \prod_r K'_*\left(I_{\mu_r}\left(\mathcal{X}\right)\right)_{geom}$$

which we abusively call ϕ_ζ and can be written as

$$\left(K'_*\left(I_{\mu}\left(\mathcal{X}\right)\right)_{geom} \otimes \mathbf{Q}\left(\zeta_\infty\right)\right)^{Gal(\mathbf{Q}(\zeta_\infty)/\mathbf{Q})} \xrightarrow[\phi_{\zeta}]{\approx} K'_*\left(I_{\mu}\left(\mathcal{X}\right)\right)_{geom}$$

Now, remarking that $Aut(\mu_\infty) \approx Gal\left(\mathbf{Q}\left(\zeta_\infty\right)/\mathbf{Q}\right)$ and combining this with Remark 6.2, we get the following proposition.

Proposition 6.4 : Let $\mathcal{X}$ be a quotient tame Artin stack. Then $\mathcal{L}_\mathcal{X} := \phi_\zeta \tilde{\tau}_\mathcal{X}$ is an isomorphism :

$$K'_*(\mathcal{X}) \xrightarrow[\mathcal{L}_\mathcal{X}]{\approx} K'_*(I_\mu(\mathcal{X}))_{geom}$$

It is the *Lefschetz-Riemann-Roch isomorphism.*

We can now state and prove Theorem 1.6 (modulo Conjecture 6.3.2).

Theorem 1.6 : Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper representable morphism of quotient tame stacks. Then the following diagram is commutative :

$$\begin{tikzcd}
K'_*(\mathcal{X}) \arrow[r, "\mathcal{L}_\mathcal{X}"] \arrow[d, "f_*"'] & K'_*(I_\mu(\mathcal{X}))_{geom} \arrow[d, "I_\mu(f)_*"'] \\
K'_*(\mathcal{Y}) \arrow[r, "\mathcal{L}_\mathcal{Y}"] & K'_*(I_\mu(\mathcal{Y}))_{geom}
\end{tikzcd}$$

Proof : This follows from Theorem 6.3 and Proposition 6.4.◇

6.4 Geometric K-theory and Bloch's higher Chow groups

In 1986, Spencer Bloch [B] introduced higher Chow groups $A^*(X,n)$ for algebraic schemes X. His main motivations seem to have been to reach a deeper understanding of the relationship between the category of coherent sheaves on a scheme and algebraic cycles. Modulo torsion, and restricting one's attention to K_0, this relationship is completely determined, in the case of algebraic schemes and proper morphisms thereof, by the Grothendieck-Riemann-Roch theorem of Baum, Fulton and MacPherson [BFM]. In 1981, Gillet [G] made first steps towards a generalization of this to higher K-theory. He introduced a very general theory of Chern classes for elements of all higher K-groups with values in some cohomology theories (eg étale cohomology and De Rham cohomology), constructed a Riemann-Roch map and proved its covariance with respect to proper pushforwards. This led to applications in studying regulators and some Euler characteristics. However, Gillet did not prove that this Riemann-Roch map was an isomorphism modulo torsion. By specializing the target map of Gillet's Chern classes to higher Chow groups, this is what Bloch achieved in the seminal paper [2] [B].

In 1998, Edidin and Graham [EG1] generalized Bloch's higher Chow groups to quotient Artin stacks, by providing an alternative definition directly inspired from Totaro's definition of the Chow ring of the classifying stack of an algebraic group [Tot]. On the other hand, it is clear that, modulo the idea of introducing inertia stacks in a stacky generalization of the aforementioned context, Toën's work on the Grothendieck-Riemann-Roch theory of quotient Deligne-Mumford stacks is based on Gillet's preceding constructions. As a result, Toën could not obtain that his Riemann-Roch map induces an isomorphism modulo torsion. In this section, using results obtained by Krishna in 2009 [Kr1] to relate geometric K-theory to higher Chow groups, and the results from the preceding subsections, we address the problem of identifying the higher K-theory of coherent sheaves of quotient tame Artin stacks with its higher intersection theory modulo torsion. We first prove Proposition 6.10. Then, we investigate the intrinsicness thereof, and restate it in terms of proper representable maps of algebraic stacks. We should like to point out that the proof of the Grothendieck-Riemann-Roch theorem for

[2] Most of the results of [B] depends on a moving lemma given in the first section of the latter paper. It turned out that, while all results in [B] are correct, the proof of the aforementioned lemma had a flaw. This inconsistency was resolved by Bloch in 1994, who gave in [B1] a correct (much longer) proof of the moving lemma for higher Chow groups.

proper representable maps of algebraic stacks (ie equivariant morphisms), follows from Gillet's method, as in [T1,T2].

Let us first review a number of definitions required to state Proposition 6.10.

We refer the reader to [G, Definition 2.34], for the definition of Chern characters for quasi-projective k-schemes valued in "generalized cohomology theories" in the sense of [G, Definition 1.1], and to [G, Definition 2.1] for the notion of generalized Chern classes (defined for higher algebraic K-theory and valued in the same cohomology theories) from which the Chern characters are built. We denote $A^*(X,n)$ the n-th graded higher Chow groups of a quasi-projective k-scheme X defined in [B]. After pointing out that these higher Chow groups are a particular instance of Gillet's "generalized cohomology theories", Bloch defined, for X regular, higher Chern characters $ch_n : K'_n(X) \longrightarrow A^*(X,n) \otimes \mathbf{Q}$ and Riemann-Roch maps $\tau^B_{X,n} : K'_n(X) \longrightarrow A^*(X,n) \otimes \mathbf{Q}$, in such a way that $ch_0 = ch$ and $\tau^B_{X,0} = \tau_X$, where ch and τ_X denote the Chern character and Riemann-Roch map defined in [BS]. Moreover, by construction, $\tau^B_{X,n} = (td(X))ch_n$, where $td(X)$ is the Todd class of X as in [BS]. We refer to [B,§7] for these.
Let us now suppose that X is endowed with an action of a linear algebraic group G. We denote $A^*_G(X,n)$ the n-th graded equivariant higher Chow group of X defined in [EG1]. The following definition is due to Edidin and Graham.

Definition 6.5 : Let G be a linear algebraic group acting on a quasi-projective k-scheme X. Let V be a representation of G, let j be a positive integer, and let U be a G-invariant open subset of V, such that :
(i) $V - U$ has codimension greater than j in V.
(ii) G acts freely on U.
(iii) U/G is a quasi-projective k-scheme.
Then the pair (V,U) is called a *good pair* with respect to X, G and j.

The results of [EG1] imply that for any action of the kind we shall consider, there exists a good pair. The following definition is due to Krishna.

Definition 6.6 : Let G be a linear algebraic group acting on a quasi-projective k-scheme X. Let (V,U) be a good pair associated to this action and some fixed integer. Set $X_G = X \times^G U$. Let $E_V \longrightarrow X_G$ be the vector bundle corresponding to the map $X \times^G (U \times V) \longrightarrow X \times^G U$. Let $pr_1 : X \times U \longrightarrow X$ denote the first projection. The n-th equivariant Riemann-Roch map

$$\tau^K_{X,G,n} : K'_n(X,G) \longrightarrow A^*_G(X,n) \otimes \mathbf{Q}$$

is given by $\tau^K_{X,G,n}(x) = (td(E_V))^{-1}\tau_{X_G} pr_2^*(x)$ for every x in $K'_n(X,G)$.

Proposition 6.7 : Let a linearly reductive group scheme G act on an algebraic space X such that the quotient stack $[X/G]$ associated to this action is tame. Then, for any $n \geq 0$, the composition :

$$\tau^K_{X,G,n} : K'_n(X,G)_{geom} \subseteq K'_n(X,G) \longrightarrow A^*_G(X,n) \otimes \mathbf{Q}$$

is an isomorphism of $\mathbf{Q}$-vector spaces.

Remark 6.8 : For $n = 0$, this was first proven by Edidin and Graham in [EG2].

Proof : We first recall the following result obtained by Krishna in [Kr1] :

Theorem 6.9 (Krishna) : Let G be a linear algebraic group over a field k, acting on an algebraic k-scheme X (that is, X is a reduced connected separated noetherian scheme of finite type over k with an ample family of line bundles). Let $\mathcal{X} = [X/G]$ be the associated quotient stack. If $\mathcal{X}$ satisfies the hypothesis of the Keel-Mori theorem, then :

(i) $\tau^K_{X,G,n} : K'_n(X,G) \twoheadrightarrow A^*_G(X,n) \otimes \mathbf{Q}$

(ii) $\left(\alpha \in ker\left(\tau^K_{X,G,n}\right)\right) \iff ((\exists \epsilon \in RG - m)$ such that $\epsilon\alpha = 0)$ $\diamond$

Now, a tame Artin stack always has a coarse moduli scheme by Theorem 1.2. Therefore, by Kirshna's theorem (i) yields an epimorphism $\tilde{\tau}^K_{\mathcal{X}} : K'_n(X,G) \twoheadrightarrow A^*_G(X,n) \otimes \mathbf{Q}$, (ii) implies that it is in fact an isomorphism, so that $\tau^K_{X,G,n} : K'_n(X,G) \xrightarrow{\approx} A^*_G(X,n) \otimes \mathbf{Q}$, whence the proposition. $\diamond$

Proposition 6.10 : Let $f : X \longrightarrow Y$ be a proper G-equivariant morphism of quasi-projective k-schemes. Then the following diagram is commutative for all n :

$$\begin{tikzcd}
K'_n(X,G)_{geom} \arrow[r, "\tau^G_X"] \arrow[d, "f_*"'] & A^*_G(X,n) \otimes \mathbf{Q} \arrow[d, "f_*"'] \\
K'_n(Y,G)_{geom} \arrow[r, "\tau^G_Y"] & A^*_G(Y,n) \otimes \mathbf{Q}
\end{tikzcd}$$

Proof : The proof is a rather straightforward extension of the proof of [G, Theorem 4.1] to the G-equivariant case. There exists a factorization of f as :

$$X \hookrightarrow \mathbf{P}^n_Y \longrightarrow Y$$

where $\mathbf{P}^n_Y$ is endowed with a G-action, in such a way that both i and p are G-equivariant. Let us first prove the covariance with respect to i. First, we reduce to the case when Y is a smooth k-variety. This is possible, given that there exist composable closed G-immersions $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ with Z a smooth or regular G-variety, if the following equivariant version of Quillen's purity theorem holds :

Lemma 6.11 : j induces an isomorphism $j_* : K'_n(Y,G) \xrightarrow{\approx} K'_n(Z, Z-Y, G)$, and moreover ji induces an isomorphism $(ji)_* : K'_n(X,G) \xrightarrow{\approx} K'_n(Z, Z-X, G)$, where $K_n(Z, Z-Y)$ is the π_{n+1} of the homotopy fiber of the morphism of classifying spaces $BQ\mathcal{P}(Y,G) \longrightarrow BQ\mathcal{P}(Z,G)$.

Let us assume the lemma, a proof of which is supplied immediately the proof of the proposition. We need to prove that the following is commutative :

$$\begin{array}{ccc}
K'_n(X,G) & \xrightarrow{\tau^G_X} & A^*_G(X,n) \otimes \mathbf{Q} \\
\Big\downarrow{\scriptstyle i_*} & & \Big\downarrow{\scriptstyle i_*} \\
K'_n(Y,G) & \xrightarrow{\tau^G_Y} & A^*_G(Y,n) \otimes \mathbf{Q}
\end{array}$$

Now, τ^G_X is, like in the non-equivariant case, the map given by the composition :

$$K'_n(X,G) \xrightarrow{\approx} K_n(Z, Z-X, G) \longrightarrow A^*_G(Z,X,n) \longrightarrow A^*_G(X,n) \otimes \mathbf{Q}$$

Where the first morphism is the purity isomorphism, the second is the higher Chern character with support in X, ch^X_n of Gillet and Bloch, and the third assigns to $ch^X_n(\alpha)$ the Bloch cycle $\eta_Z \cap \left(Td(Z) \cup ch^X_n(\alpha)\right)$. Thanks to the lemma, it suffices to check the commutativity of :

$$\begin{tikzcd}
K_n(Z, Z-X, G) \arrow[r, "ch_n^X"] \arrow[d, "i_*"'] & A_G^*(Z, X, n) \otimes \mathbf{Q} \arrow[d, "i_*"] \\
K_n(Z, Z-Y, G) \arrow[r, "ch_n^Y"] & A_G^*(Z, Y, n) \otimes \mathbf{Q}
\end{tikzcd}$$

The latter results from the fact that ch_n^X and ch_n^Y come from the same map of simplicial presheaves on Z (see [G, paragraph 2]). It now remains to prove the covariance with respect to p, ie the commutativity of :

$$\begin{tikzcd}
K'_n\left(\mathbf{P}_Y^m, G\right) \arrow[r, "\tau^G_{\mathbf{P}_Y^m}"] \arrow[d, "p_*"'] & A_G^*\left(\mathbf{P}_Y^m, n\right) \otimes \mathbf{Q} \arrow[d, "p_*"] \\
K'_n\left(Y, G\right) \arrow[r, "\tau_Y^G"] & A_G^*\left(Y, n\right) \otimes \mathbf{Q}
\end{tikzcd}$$

At this point, using the equivariant projective bundle theorem proven in [Th1], we have surjections :

$$\odot : K'_n\left(Y, G\right) \otimes K_0\left(\mathbf{P}_k^m\right) \twoheadrightarrow K'_n\left(\mathbf{P}_Y^m\right)$$

and the same at the level of equivariant higher Chow groups. Now

$$\tau_n^{\mathbf{P}_Y^m}(\alpha \odot \beta) = \Big((\alpha \odot \beta) \cup Td^G\left(\mathbf{P}_Y^m\right)\Big) \cap \eta_{\mathbf{P}_Y^m}$$

Denote $\pi : \mathbf{P}_Y^m \longrightarrow \mathbf{P}_k^m$ the canonical projection. Then $\alpha \odot \beta = p^*\alpha \cup \pi_*\beta$ and

$$\tau_n^{\mathbf{P}_Y^m}(\alpha \odot \beta) = \Big(p^! \Big(ch_n^Y(\alpha) \cup \pi^! ch(\beta)\Big) \cup \left(\pi^* Td\left(Y\right) \cup Td\left(\mathbf{P}_Y^m\right)\right)\Big) \cap \eta_{\mathbf{P}_Y^m}$$

Hence $\tau_n^{\mathbf{P}_Y^m}(\alpha \odot \beta) = \tau_n^Y(\alpha) \odot \tau_0^{\mathbf{P}_k^m}$, as the equivariant Todd class is multiplicative. The projection formula and the equivariant Riemann-Roch theorem for K_0 now imply that $\tau_n^Y p_* = p_* \tau_n^{\mathbf{P}_Y^m}$, whence finally the proposition. $\diamond$

We now turn to the proof of the purity theorem in equivariant K-theory.

Lemma 6.12 (Purity) : Let X be a G-equivariant smooth k-scheme. Let $i : Z \hookrightarrow X$ be a G-equivariant closed immersion. Then i induces an isomorphism of $\mathbf{Q}$-vector spaces

$$i_* : K'_n(Z,G) \xrightarrow{\approx} K'_n(X, X-Z, G)$$

Proof : We have by definition $K'_n(Z,G) = K_n^{Quillen}(Coh_G(Z))$ and $K_n(X, X-Z, G) = K_n^{Quillen}\left(Coh_G^Z(X)\right)$, where $Coh_G^Z(X)$ is the exact category of coherent G-modules on X supported on Z. Obviously, there is an exact inclusion of exact categories as follows :

$$j : Coh_G(Z) \hookrightarrow Coh_G^Z(X)$$

Now, let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{O}_X$ defining Z. Clearly, for any $\mathcal{M} \in Coh_G^Z(X)$, one has $\mathcal{I}^n\mathcal{M} = 0$ for some n. Therefore, Quillen's devissage theorem [Q, Theorem 4] implies that $K(Coh_G(Z)) \xrightarrow{\approx} K\left(Coh_G^Z(X)\right)$. $\diamond$

Remark 6.13 : With the same notations as above, let $X-Z = U \hookrightarrow X$ be the complementary open immersion. U is G-invariant and moreover, as in the non-equivariant case, we have that every coherent G-module on U extends to a coherent G-module on X, ie the restriction functor res induces an essentially surjective exact functor $res : Coh_G(X) \longrightarrow Coh_G(U)$. This was proven by Thomason as [Th1, Corollary 2.4]. $Coh_G^Z(X)$ is a Serre subcategory of the abelian category $Coh_G(X)$, and the general theory of Serre categories gives the existence of a quotient abelian category $Coh_G(X) \longrightarrow Coh_G(X)/Coh_G^Z(X)$ where $Coh_G(X)/Coh_G^Z(X)$ turns out to be equivalent to $Coh_G(U)$.

Proposition 6.10 proves the equivariance of the Krishna-Riemann-Roch map with respect to equivariant proper maps of equivariant quasi-projective schemes. It is now time to introduce higher Chow groups of algebraic stacks, following the standard approach ([EG,KrS]). This point of view forces us from the very beginning to restrict our attention to algebraic stacks admitting quasi-projective coarse moduli spaces.

Definition 6.14 : Let $\mathcal{X}$ be a separated algebraic stack with a quasi-projective coarse moduli space $M_{\mathcal{X}}$. The n-th higher Chow group of $\mathcal{X}$ is defined to be :

$$A^*(\mathcal{X}, n) := A^*(M_{\mathcal{X}}, n)$$

Our main interest is in proper maps of stacks and pushforwards theorof induced on higher Chow groups. Following [KrS], we need to further restrict ourselves to quotient stacks. The following definition was made in [KrS].

Definition 6.15 : Let $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$ be separated quotient stacks with quasi-projective coarse moduli spaces $M_{\mathcal{X}}$ and $M_{\mathcal{Y}}$ respectively, and let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper morphism. Then the proper pushforward $f_* : A^*(\mathcal{X}, n) \longrightarrow A^*(\mathcal{Y}, n)$ induced by f is defined to be equal to $(Mf)_* : A^*(M_{\mathcal{X}}, n) \longrightarrow A^*(M_{\mathcal{Y}}, n)$.

The above two definitions call for the following one to be made.

Definition 6.16 : Let $\mathcal{X}$ be a separated quotient stack with quasi-projective coarse moduli space $M_{\mathcal{X}}$. Let $[X/G]$ be a presentation of $\mathcal{X}$, and let $x : X \longrightarrow \mathcal{X}$ be the smooth cover associated to it. Let n be a positive integer. The *Krishna-Riemann-Roch map*[3] $\tau^K_{\mathcal{X}} : K'_n(\mathcal{X})_{geom} \longrightarrow A^*(\mathcal{X}) \otimes \mathbf{Q}$ is defined to be the composition :

$$K'_n(\mathcal{X})_{geom} \xrightarrow{x^*} K'_n(X, G)_{geom} \xrightarrow{\tau^K_{X,G,n}} A^*_G(X, n) \otimes \mathbf{Q} \xrightarrow{\approx} A^*(\mathcal{X}, n) \otimes \mathbf{Q}$$

where the third isomorphism is given by [EG1, Proposition 14].

By contrast, Theorem 4.1 calls for the following alternative definition.

Definition 6.17 : Let $\mathcal{X}$ be a quotient tame stack with a quasi-projective moduli space $M_{\mathcal{X}}$. Let n be a positive integer. The *Grothendieck-Riemann-Roch map* (or the *Bloch-Riemann-Roch map*) $\tau_{\mathcal{X}} : K'_n(\mathcal{X})_{geom} \longrightarrow A^*(\mathcal{X}, n) \otimes \mathbf{Q}$ is defined to be the composition :

$$K'_n(\mathcal{X})_{geom} \xrightarrow{x^*} K'_n(M_{\mathcal{X}})_{geom} \xrightarrow{\tau^B_{M_{\mathcal{X}},n}} A^*(M_{\mathcal{X}}, n) \otimes \mathbf{Q} = A^*(\mathcal{X}, n) \otimes \mathbf{Q}$$

While Definition 6.16 is in our opinion the most natural one to make, it is Definition 6.17 that we shall use in our main result. It is not clear that these two definitions coincide when they both make sense. This leads to the conjecture below.

[3] Thanks to [KrS, Section 10.3], it seems reasonable to expect that $\tau^K_{\mathcal{X}}$ does not depend on the presentation of $\mathcal{X}$ as a quotient. Since we don't use this map later, we include this expectation in Conjecture 7.18.

Conjecture 6.18 : Let $\mathcal{X}$ be a quotient tame stack with a quasi-projective moduli space. Then $\tau_{\mathcal{X}} = \tau_{\mathcal{X}}^{K}$ (in particular, $\tau_{\mathcal{X}}^{K}$ is intrinsic).

Proposition 6.19 : Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper morphism of quotient tame Artin stacks. Then the following diagram commutes (for each $n \in \mathbf{N}$) :

$$\begin{tikzcd}
K'_n(\mathcal{X})_{geom} \arrow[r, "\tilde{\tau}^K_{\mathcal{X}}"] \arrow[d, "f_*"'] & A^*(\mathcal{X}, n) \otimes \mathbf{Q} \arrow[d, "f_*"'] \\
K'_n(\mathcal{Y})_{geom} \arrow[r, "\tilde{\tau}^K_{\mathcal{Y}}"] & A^*(\mathcal{Y}, n) \otimes \mathbf{Q}
\end{tikzcd}$$

Proof : Let $M_{\mathcal{X}}$ (respectively $M_{\mathcal{Y}}$) denote the coarse moduli space of $\mathcal{X}$ (respectively $\mathcal{Y}$). The morphism $Mf_* : M_{\mathcal{X}} \longrightarrow M_{\mathcal{Y}}$ induced by f is a proper morphism of algebraic spaces, and the Bloch-Grothendieck-Riemann-Roch theorem proven in [B] yields the following commutative diagrams :

$$\begin{tikzcd}
K'_n(M_{\mathcal{X}}) \arrow[r] \arrow[d, "Mf_*"'] & A^*(M_{\mathcal{X}}, n) \otimes \mathbf{Q} \arrow[d, "Mf_*"'] \\
K'_n(M_{\mathcal{Y}}) \arrow[r] & A^*(M_{\mathcal{Y}}, n) \otimes \mathbf{Q}
\end{tikzcd}$$

On the other hand, thanks to the proposition above, we have the following isomorphisms :

$$p_{\mathcal{X}*} : K'_n(\mathcal{X})_{geom} \xrightarrow{\approx} K'_n(M_{\mathcal{X}})$$

and

$$p_{\mathcal{Y}*} : K'_n(\mathcal{Y})_{geom} \xrightarrow{\approx} K'_n(M_{\mathcal{Y}})$$

given by the projections $p_{\mathcal{X}} : \mathcal{X} \longrightarrow M_{\mathcal{X}}$ and $p_{\mathcal{Y}} : \mathcal{Y} \longrightarrow M_{\mathcal{Y}}$ respectively. By functoriality of geometric K'-groups, we have a commutative diagram as follows (recall that geometric K'-theory conincides with K'-theory in the non-equivariant case of algebraic spaces) :

$$\begin{tikzcd}
K'_n(\mathcal{X})_{geom} \arrow[r, "\approx"] \arrow[d, "f_*"'] & K'_n(M_{\mathcal{X}}) \arrow[d, "Mf_*"'] \\
K'_n(\mathcal{Y})_{geom} \arrow[r, "\approx"] & K'_n(M_{\mathcal{Y}})
\end{tikzcd}$$

Moreover, by definition :

$$\begin{tikzcd}
A^*(\mathcal{X}, n) \otimes \mathbf{Q} \arrow[r] \arrow[d, "f_*"'] & A^*(M_{\mathcal{X}}, n) \otimes \mathbf{Q} \arrow[d, "Mf_*"'] \\
A^*(\mathcal{Y}, n) \otimes \mathbf{Q} \arrow[r] & A^*(M_{\mathcal{Y}}, n) \otimes \mathbf{Q}
\end{tikzcd}$$

To prove the proposition, it now suffices (for $\mathcal{X}$, hence also for $\mathcal{Y}$) to prove that the following diagram is commutative :

$$\begin{tikzcd}
K'_n(\mathcal{X})_{geom} \arrow[r] \arrow[d, "f_*"'] & A^*(\mathcal{X}, n) \otimes \mathbf{Q} \arrow[d, "f_*"'] \\
K'_n(M_{\mathcal{X}}) \arrow[r] & A^*(M_{\mathcal{X}}, n) \otimes \mathbf{Q}
\end{tikzcd}$$

Chapter 7

Appendix

7.1 Notations pertaining to algebraic groups

Recall that an affine group scheme G over k is said to be *of multiplicative type* if it is of finite type and its base change to a separable closure k^{sep} of k is diagonalizable fppf locally. We refer the reader to [Mi, Chapter XIV] for the basic theory surrounding this notion.

Definition 7.1 : A subgroup scheme σ of an algebraic group G is called *dual cyclic* if it is isomorphic to the group scheme $\mu_{n,k}$ for some n.

Let G be an algebraic group.

Notation 7.2.1 : We denote $C(G)$ the set of conjugacy classes of dual cyclic subgroups of G.

Notation 7.2.2 : $C_r(G)$ denotes the set of conjugacy classes of dual cyclic subgroups σ of G of order r fixed, so that $C(G) = \coprod_{r\geq 1} C_r(G)$.

Let σ be a dual cyclic subrgoup of G. Let on the other hand $\alpha : \mu_r \longrightarrow G$ be a monomorphism of group schemes.

Notation 7.2.3 : (i) $N_G(\sigma)$ stands for the normalizer of σ in G.
(ii) $N_G(\alpha)$ stands for the normalizer of $\alpha(\mu_r)$ in G.

Notation 7.2.4 : (i) $C_G(\sigma)$ stands for the centralizer of σ in G.
(ii) $C_G(\alpha)$ stands for the centralizer of $\alpha(\mu_r)$ in G.

Notation 7.2.5 : (i) Finally, we set $w_G(\sigma) = N_G(\sigma)/C_G(\sigma)$ for every dual cyclic subgroup σ of G.
(ii) We set $w_G(\alpha) = N_G(\alpha)/C_G(\alpha)$ for every monomorphism $\alpha \longrightarrow G$.

Definition 7.3 : Let G be an algebraic group. For each σ, we let $\tilde{C}_r(G)$ denote the set of conjugacy classes of closed points $\alpha : \mu_{r,k} \hookrightarrow G$ of $\tilde{S}_r(G)$. fixed, and $\tilde{C}(G) := \coprod_{r\geq 1} \tilde{C}_r(G)$.

Definition 7.4 : (i) Let $\sigma \in C_r(G)$. Let I be the ideal of $R\sigma$ generated by the r-th cyclotomic polynomial $\Phi_r(t)$. We denote $\tilde{R}\sigma := R\sigma/I$. Each dual cyclic subgroup σ of G defines a maximal ideal m_σ of RG as follows :

$$m_\sigma := ker \left(RG \longrightarrow \tilde{R}\sigma \right)$$

(ii) Let $\alpha \in \tilde{C}_r(G)$. We set $\tilde{R}\mu_r := R\mu_r/I$ and

$$m_\alpha := ker \left(RG \longrightarrow R\mu_r \twoheadrightarrow \tilde{R}\mu_r \right)$$

7.2 Results on algebraic groups

The following two theorems were proven by Grothendieck, Demazure et al. in [SGA3, II, XII, §5.8] and [SGA3, II, XI and XII §5.8] respectively.

Theorem 7.5 : Let G be a product of general linear groups over k. There exists a k-scheme $S_r(G)$ representing the moduli of subgroup schemes of G that are étale-locally isomorphic to μ_r. Namely, the functor $Sch/k \longrightarrow Sets$ sending a k-scheme T to the set of subgroup schemes of G étale-locally isomorphic to μ_r is represented by a scheme $S_r(G)$ of finite type over k. Furthermore one has :

$$S_r(G) = \textstyle\coprod_{\sigma\in C_r(G)} V_\sigma$$

where each V_σ is an orbit under the G-action.$\diamond$

Theorem 7.6 : Under the notations of the preceding theorem, there exists a k-scheme $\tilde{S}_r(G)$ representing the moduli of monomorphisms $\alpha : \mu_r \hookrightarrow G$. Furthermore :

$$\tilde{S}_r(G) = \textstyle\coprod_{\alpha\in \tilde{C}_r(G)} U_\alpha$$

where U_α is a connected scheme.$\diamond$

7.3 Thomason's generic slice theorem for torus actions

The next theorem is Thomason's so called generic slice theorem for torus actions, namely [Th2, Proposition 4.10]. It is arguably the most crucial ingredient in the proofs of the main results of [VV].

Theorem 7.7 : Let T be a diagonalizable torus of finite type over an excellent noetherian base scheme S. Let T act on a reduced separated algebraic space X of finite type over S. Then there exists a non-empty T-invariant open subspace U of X with all the following properties :
(i) U is an affine scheme which is regular.
(ii) The geometric quotient[1]$_Y \approx \pi_* \left(\mathcal{O}_X^G\right)$
The scheme Y is denoted X/G. Our reference for this notion is [MFK]. U/T exists, and $U - U/T$ exists, U/T is affine, of finite type over S, regular and $U - U/T$ is smooth.
(iii) There is a diagonalizable subgroup T' of T, with quotient torus $T'' = T/T'$, and an action of T'' on U such that T acts on U via $T \longrightarrow T''$. Further T'' acts freely on U and U is a trivial principal homogeneous space for T'' over $U/T'' = U/T$. Thus there is an isomorphism of schemes with T-action :

$$U \xrightarrow{\approx} T'' \times_S U/T \xrightarrow{\approx} T/T' \times_S U/T$$

$\diamond$

Remark 7.8 : We are only interested in the special case of Theorem 7.7 where S is the spectrum of a field. Furthermore, we don't need the fact that U can be taken to be affine or regular. It is (iii) which is the point of the theorem both in [VV] and here.

7.4 The Vezzosi-Vistoli decomposition formula

Below is the statement of the Vezzosi-Vistoli decomposition formula [VV, Main Theorem].

[1] Recall that the *geometric quotient* of a scheme X by an action of an algebraic group G is a morphism of schemes $\pi : X \to Y$ such that
(i) For each $y \in Y$, the fiber $\pi^{-1}(y)$ is an orbit of G.
(ii) The topology of Y is the quotient topology : a subset $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open.
(iii) There is an isomorphism of sheaves $\mathcal{O}$

Theorem 7.9 : Let G be an affine group scheme of finite type over a field k, acting on a noetherian regular separated algebraic k-space X. Assume that :
(1) The action has finite geometric stabilizers.
(2) The action is sufficiently rational[2].
(3) For any essential dual cyclic subgroup σ of G, the quotient $G/C_G(\sigma)$ is smooth.
Then $C(G)$ is a finite set. Moreover, there is a canonical isomorphism of RG-algebras :

$$K_*(X, G) \otimes \mathbf{Z}[1/N] \xrightarrow{\approx} \prod_{\sigma \in C(G)} \left(K_*(X^\sigma, C_G(\sigma))_{geom} \otimes \tilde{R}(\sigma)\right)^{w_G(\sigma)}$$

where N denotes the least common multiple of the orders of all the essential dual cyclic subgroups of G.$\diamond$

In effect, in [VV] Theorem 7.9 is proven first in the case where $G = T$ is a split torus, then in the case $G = GL_n$ and then in the general case where G is a linearly reductive group, following an Atiyah-Segal-like strategy. In the case where $G = T$, two isomorphisms are constructed, namely $\Psi_{X,T} : K_*(X, T) \xrightarrow{\approx} \prod_{\sigma \in C(T)} K_*(X^\sigma, T)_{geom} \otimes \tilde{R}\sigma$, which is a ring isomorphism, and $\Phi_{X,T} : \prod_{\sigma in C(T)} K'_*(X^\sigma, T)_{geom} \otimes \tilde{R}\sigma \xrightarrow{\approx} K'_*(X, T)$, which is a module isomorphism. While Theorem 7.9 displays the former, the Lefschetz-Riemann-Roch and Grothendieck-Riemann-Roch formulae use the latter.

[2]This means the following two conditions are met :
(i) Each essential dual cyclic subgroup $\sigma \subseteq G_{\bar{k}}$ is conjugate by an element of $G(\bar{k})$ to a dual cyclic subgroup of G.
(ii) If two essential dual cyclic subgroups of G are conjugate by an element of $G(\bar{k})$, they are also conjugate by an element of $G(k)$.
($\bar{k}$ denotes an algebraic closure of k).

Chapter 8

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