Hopf ring structures on the cohomology of certain spaces

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Introduction

In this thesis, we are discussing a specific algebraic structure on the cohomology groups of certain spaces, mainly related to the symmetric groups and other finite reflection groups. The aforementioned algebraic structure is, in most cases, that of a Hopf ring. The formal definition of a Hopf ring is recalled in 32. This algebraic object should be thought as a (graded) coalgebra A with two bilinear products $A \times A \to A$ that behaves similarly to the sum and the multiplication in a commutative ring.

The existence of a Hopf ring structure on the cohomology of the disjoint union of the classifying spaces of the symmetric groups $\coprod_{n \geq 0} B(\Sigma_n)$ has first been observed by Strickland and Turner in 1997 [38]. They noted that, if E is a multiplicative cohomology theory, $E(\coprod_{n \leq 0} B(\Sigma_n))$ is a Hopf ring, where the two products are defined as the usual cup product in cohomology and the transfer map associated to the monomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, while the coproduct is the usual map induced in cohomology by these same monomorphisms.

A similar structure exists for the cohomology of other families of finite reflection groups. For example, the cohomology of $\coprod_{n \geq 0} B(W_{B_n})$, the classifying spaces of the Coxeter groups of Type B_n, is is a natural way a Hopf ring. Here, the structural morphisms are constructed in a way similar to the symmetric groups. It is only required to replace the inclusions $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ with other natural monomorphisms $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$.

An analogous reasoning can be done for the Coxeter groups of Type D_n. We can define two products and a coproduct via some natural maps $W_{D_n} \times W_{D_m} \to W_{D_{n+m}}$ also in this case. However, the reciprocal relations between these structural morphisms are more complicated than the previous families of groups: even if all the other axiomatic properties of Hopf rings are satisfied, the coproduct and the transfer product fail to form a bialgebra for some choices of the ring of coefficients. This leads to the introduction of a different algebraic object, that we call almost-Hopf ring. Fortunately, the failing of this bialgebra property is "controlled" in a certain sense, and can be fully described via the use of a canonical involution and a notion of "charge" on cohomology classes. However, this still makes a complete description of this almost-Hopf ring structure computationally more complicated.

Among others, we build our description of these (almost)-Hopf ring structures on the use of two techniques:

- some classical properties of the (co)homology of $\mathcal{C}$-spaces, where $\mathcal{C}$ is the little cubes operad, especially the Dyer–Lashof operations and their duals

- the existence of a CW model for the classifying spaces of Coxeter groups explained, for example, in [10], whose cells can be described in a geometric and combinatorial way from a Coxeter presentation of the group

A further application of these ideas is fruitful when applied to the study of the mod p cohomology of $Q(X)$, the free ∞-loop space over a topological space X. The study of these spaces has a very long tradition and many links between $Q(X)$ and the extended powers space $D(X)$ are classically known.

The cohomology of $D(X)$ has a Hopf ring structure that encompasses, as special cases, $\coprod_{n\geq 0} B(\Sigma_n)$ and $\coprod_{n\geq 0} B(W_{B_n})$ (for $X = S^0$ and $X = \mathbb{P}^\infty(\mathbb{R}) \cup \{*\}$). However, the almost-Hopf ring structure for $\coprod_{n\geq 0} B(W_{D_n})$ cannot be recovered this way. Although we cannot use the De Concini–Salvetti complex in this general case, we are still able to exploit the action of the Dyer–Lashof algebra and the existence of some divided powers operations that behave well with respect to the structural morphisms of our Hopf ring. This still allows us to obtain some results for $H^*(D(X); \mathbb{F}_p)$. This object can be obtained from $H^*(X; \mathbb{F}_p)$ by applying a suitable free functor.

Finally, the Hopf ring structure on the cohomology of $D(X)$ induces a similar structure on $H^*(Q(X); \mathbb{F}_p)$, that can be obtained from $H^*(D(X); \mathbb{F}_p)$ via a "stabilization" process.

The structure of this work is as follows. In the first chapter, we recall the preliminary notions and results we will make use of, in particular, some facts regarding Coxeter groups and their classifying spaces, results for the homology of ∞-loop spaces and Hopf rings. The second chapter is devoted to the exposition of the results involving the cohomology of the symmetric groups. That chapter basically follows the treatment of two published papers, one by Giusti, Salvatore and Sinha, and the other by the author of this thesis. In the third chapter, we calculate the cohomology of the Coxeter groups of Type B_n and D_n as (almost-)Hopf rings. We also carry on the calculation of the restriction to elementary abelian subgroups and of the Steenrod algebra action. Finally, the fourth chapter deals with some results concerning the mod p cohomology of $D(X)$ and $Q(X)$ for a topological space X.

Contents

Chapter 1

Preliminary materials

This chapter is intended to provide some background material on Coxeter groups and infinite loop spaces. It is by no means an exhaustive exposition of these topics. We are only recalling some notions and standard notations that we will use in the following chapters, instead.

1.1 Finite reflection groups and hyperplane arrangements

We first recall the definition of reflection group. Let $\mathcal{O}_n$ be the group of linear isometries of the euclidean space $\mathbb{R}^n$, with the standard inner product.

Definition 1. An element $s \in \mathcal{O}_n$ is a *reflection* if s fixes a codimension-1 subspace and $s \neq \mathrm{id}$. A *(finite) reflection group* in $\mathbb{R}^n$ is a finite subgroup of $\mathcal{O}_n$ generated by reflections.

The finite reflection groups are completely classified up to isomorphisms. This is best explained with the notions of Coxeter graph and Coxeter group.

Definition 2. A *Coxeter graph* is a weighted simple finite graph Γ, with weights that are natural numbers $n \geq 3$ or ∞. The *Coxeter system* associated to a Coxeter graph Γ is a couple (G, S) where $S = \{s_v\}_{v \in V(\Gamma)}$ is a set indexed by the vertices of Γ and G is the group generated by S with the following relations:

- $s_v^2 = \mathrm{id}$ for all $v \in V(\Gamma)$
- $s_v s_{v'} = s_{v'} s_v$ if v and v' are not connected by an edge in Γ
- $(s_v s_{v'})^{w(v,v')} = \mathrm{id}$ if v and v' are connected by and edge with weight $w(v, v') < \infty$

The group G is called a *Coxeter group*.

When representing graphically the Coxeter graph of a reflection group, we follow in this paper a widely adopted convention and omit the labels of edges with weight 3. Note that if the Coxeter graph of a Coxeter system (G, S) is the disjoint union of two weighted graphs Γ_1 and Γ_2, then G is isomorphic to the direct product of G_1 and G_2, the Coxeter groups associated to Γ_1 and Γ_2.

Thus, every Coxeter group is isomorphic to a direct product of reflection groups arising from Coxeter systems (G, S) whose Coxeter graph is connected. We call these Coxeter systems, and the corresponding Coxeter groups, *irreducible*.

Finite reflection groups are Coxeter groups, as we explain below.

Definition 3. Let $G \leq \mathcal{O}_n(\mathbb{R})$ be a finite reflection group. A *root system* for G is a finite set $\Phi \subseteq \mathbb{R}^n \setminus \{0\}$ such that the two following conditions are satisfied:

- $\forall \alpha \in \Phi : \Phi \cap \mathrm{Span}_{\mathbb{R}}\{\alpha\} = \{\alpha, -\alpha\}$
- $G.\Phi = \Phi$

A *simple system* in Φ is a subset $\Delta \subseteq \Phi$ such that:

- Δ is linearly independent
- $\forall \alpha \in \Phi : \exists \{\lambda_x\}_{x\in\Delta} \subseteq \mathbb{R}^+ : \alpha = \sum_{x\in\Delta} \lambda_x x \ \vee \ -\alpha = \sum_{x\in\Delta} \lambda_x x$

Let $\langle \cdot, \cdot, \rangle$ be the standard euclidean inner product in $\mathbb{R}^n$. For any $v \in \Delta$, the reflections $s_v \colon x \mapsto \frac{\langle x,v\rangle}{\langle v,v\rangle} v$ are called *simple reflections*.

Note that every finite reflection group has a root system. For example, if , we can take $\Phi = \{v \in \mathbb{R}^n : (x \mapsto x - \langle x, v\rangle v) \in G\}$. Much less obvious is the existence of simple systems.

Theorem 4. *[23] For every finite reflection group $G \leq \mathcal{O}_n$ and root system Φ for G, there exists a simple system Δ and (G, S_Δ) is a Coxeter system. Moreover, every finite Coxeter group arises this way.*

The finite irreducible reflection groups are classified by the following classical theorem.

Theorem 5. *[23] Let (G, S) be an irreducible Coxeter system. Then its Coxeter graph is isomorphic to an element of the infinite families A_n, B_n, D_n, and I_n ($n \in \mathbb{N}$) or to one of the sporadic cases E_6, E_7, E_8, F_4, G_2, H_2, H_3, and H_4 drawn below.*

In the following chapters, we will concentrate on the groups of Type A_n, B_n, and D_n, since these are the families that give rise to an almost-Hopf ring structure in cohomology. For this reason, we briefly recall, as examples, a geometric description of these groups:

- The reflection group W_{A_n} corresponding to the Coxeter graph A_n is the symmetric group Σ_n on $\{1, \ldots, n+1\}$. Here, we embed Σ_n in $\mathcal{O}_{n+1}$ by letting it act on $\mathbb{R}^{n+1}$ via permutation of the coordinates. This identifies Σ_{n+1} with W_{A_n} by letting, for every $1 \leq i \leq n$, s_i be the transposition $(i, i+1)$.
- W_{B_n} is the group of isometries $\mathrm{Isom}\left([-1, 1]^n\right)$ of a hypercube in $\mathbb{R}^n$. In order to understand this, note that a permutation of the n coordinates maps the hypercube onto itself, thus $W_{A_{n-1}} = \Sigma_n$ is a subgroup of $\mathrm{Isom}\left([-1, 1]^n\right)$. Moreover, the latter is generated by Σ_n and the element $s_0 \colon (x_1, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n)$. Since $(s_0 s_1)$ has order 4, and s_0 commutes with s_i for every $2 \leq i \leq n-1$, this determines an isomorphism with W_{B_n}.

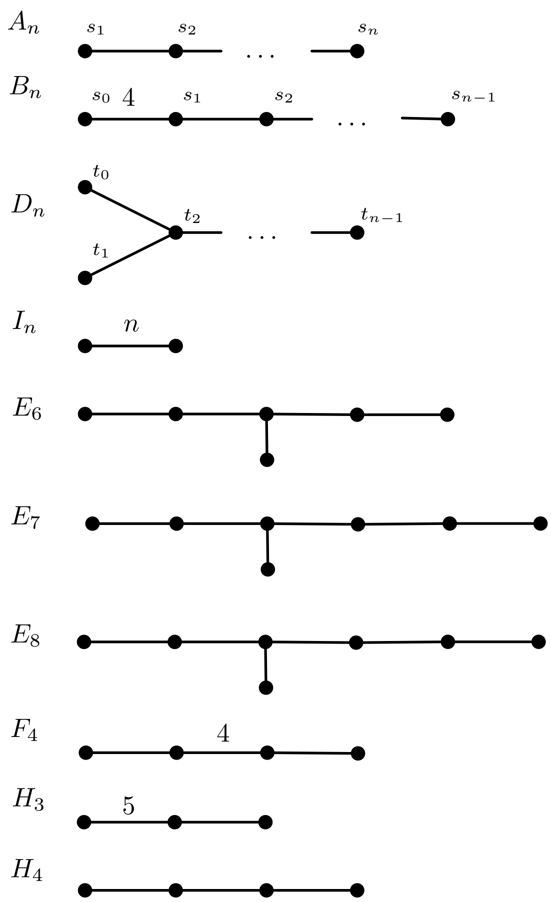


Figure 1.1: Coxeter graphs of irreductible finite reflection groups

- To understand W_{D_n}, define a vertex of the hypercube $[-1,1]^n$ to be "positive" if it has an even number of coordinates equal to -1, define it to be "negative" otherwise. W_{D_n} can be described as the index 2 subgroup of $W_{B_n} = \mathrm{Isom}\left([-1,1]^n\right) \cap \mathcal{SO}_n$ that maps positive (respectively negative) vertices into positive (respectively negative) vertices. It is generated by Σ_n together with the reflection $s_0' = s_0 s_1 s_0$.

We now need to recall a geometric construction introduced by De Concini and Salvetti in [10], that we will require in order to describe our (almost)-Hopf ring structures. It must be noted that this is a generalization of a previous idea by Salvetti [37]. Given a reflection $s \in \mathcal{O}_n$, its fixed point locus $H_s = \ker(s - \mathrm{id}_{\mathbb{R}^n})$ is a codimension 1 subspace of $\mathbb{R}^n$. Note that the hyperplane H_s univocally determines s. Thus, a finite reflection group $G \leq \mathcal{O}_n$ is uniquely identified by the family $\mathcal{A}_G = \{H_s \colon s \in G \text{ reflection}\}$. $\mathcal{A}_G$ is a hyperplane arrangement, as defined below.

Definition 6. Let k be a field. Let V be a finite-dimensional vector space over k. A *(central) hyperplane* in V is a codimension-1 k-linear subspace $H \leq V$. A *(central) hyperplane arrangement* is a finite family of hyperplanes in V.

We now give some classical geometric definitions, well-known to experts. In what follows, we will assume that k is a field, V is a finite-dimensional vector space over k and $\mathcal{A}$ is a central hyperplane arrangement in V. More details can be found in Orlik and Terao's book [34].

Definition 7. The *intersection poset* of $\mathcal{A}$ is the family $L(\mathcal{A})$ of subspaces $X \leq V$ of the form $X = H_1 \cap \cdots \cap H_r$, for some $H_1, \ldots, H_r \in \mathcal{A}$, ordered by reverse inclusion. We consider, by convention, V itself as an element of $L(\mathcal{A})$ (as the intersection of the empty family of hyperplanes).

Although we do not need it, it is worth recalling that $L(\mathcal{A})$ is a geometric lattice, as defined on page 24 of Orlik and Terao's book.

Definition 8. Let $X \in L(\mathcal{A})$. We define two new hyperplane arrangements:

- $\mathcal{A}_X = \{H \in \mathcal{A} : X \subseteq H\}$
- $\mathcal{A}^X = \{X \cap H : H \in \mathcal{A} \setminus \mathcal{A}^X\}$

We call $\mathcal{A}^X$ the restriction of $\mathcal{A}$ to X.

Note that $L(\mathcal{A}^X)$ and $L(\mathcal{A}_X)$ can be canonically identified with the sub-posets of $L(\mathcal{A})$ $[X, \{0\}] = \{Y \in L(\mathcal{A}) : X \leq Y\}$ and $[V, X] = \{Y \in L(\mathcal{A}) : X \geq Y\}$ respectively.

Definition 9. The *complement* of $\mathcal{A}$ is the set $\mathcal{M}(\mathcal{A}) = V \setminus \bigcup \mathcal{A}$.

If $k = \mathbb{R}$ there are further meaningful geometric notions.

Definition 10. Assume that $k = \mathbb{R}$. A *chamber* of $\mathcal{A}$ is a connected component of $\mathcal{M}(\mathcal{A})$. The set of the chambers of $\mathcal{A}$ is denoted by $\mathcal{C}(\mathcal{A})$. The *face poset* of $\mathcal{A}$ is the family $\mathcal{L}(\mathcal{A}) = \bigcup_{X \in L(\mathcal{A})} \mathcal{C}(\mathcal{A}^X)$. The ordering of $\mathcal{L}(\mathcal{A})$ is given by the following condition:

$$X \leq Y \;\Leftrightarrow\; Y \subseteq \overline{X}$$

An element of $\mathcal{L}(\mathcal{A})$ is called a *face* of $\mathcal{A}$.

Note that $\mathcal{L}(\mathcal{A})$ gives a topological stratification of V, where each stratum is homeomorphic to an open disk.

We now consider the special case when $\mathcal{A} = \mathcal{A}_G$ is the hyperplane arrangement associated to a finite reflection group G. In this context, the action of G permutes the strata of $\mathcal{L}(\mathcal{A})$, and this action restricts to a transitive and faithful action on $\mathcal{C}(\mathcal{A})$. In particular, the set of the chambers of $\mathcal{A}_G$ is in bijective correspondence with G itself, via the choice of a "fundamental chamber" $C_0 \in \mathcal{C}(\mathcal{A})$.

The action of G on positive-codimensional strata is known too and can be described with the use of simple systems.

Theorem 11. *(from [9] and [23]) Let $G \leq \mathcal{O}_n(\mathbb{R})$ be a finite reflection group and Φ be a root system for G. Let $\langle \cdot, \cdot \rangle$ denote the euclidean inner product of $\mathbb{R}^n$. For any simple system $\Delta \subseteq \Phi$, the set $C_\Delta = \{y \in \mathbb{R}^n : \forall x \in \Delta : \langle x, y \rangle > 0\}$ is a chamber for the hyperplane arrangement $\mathcal{A}_G$. Moreover, this defines a bijection between the set of simple systems in Φ and $\mathcal{C}(\mathcal{A}_G)$.*

For any $A \subseteq \Delta$, let X_A be the set of points in the closure of C_Δ orthogonal to all $x \in A$. $\{X_A\}_{A \subseteq \Delta}$ is the set of faces of the polyhedral cone $\overline{C_\Delta}$. Furthermore,

$\mathrm{Stab}_G(X_A)$ *is the subgroup generated by* $\{s_v : v \in A\}$ *and every face* $F \in \mathcal{L}(\mathcal{A}_G)$ *of the arrangement* $\mathcal{A}_G$ *is conjugate, via the action of* G*, to the interior of* X_A *for exactly one* $A \subseteq \Delta$*. In particular, every reflection in* G *is* G*-conjugate to* s_x *for some* $x \in \Delta$*.*

Given G and Φ, the choice of a fundamental chamber C_0 corresponds to the choice of a simple system Δ, and thus to a set $S = S_\Delta$ of simple reflections. In this context, S consists of the reflections that fix a codimension-1 face F contained in $\overline{C_0}$.

The set $\mathcal{L}(\mathcal{A}_G)$ can be used to construct an explicit model for a $K(G;1)$ space. We begin with the following definition.

Definition 12. Let $\mathcal{A}$ be a hyperplane arrangement in a finite-dimensional vector space V over $\mathbb{R}$. Let $d \in \mathbb{N}$. The *d-complexification* of $\mathcal{A}$ is the subspace arrangement $\mathcal{A}^{(d)}$ in $V \otimes \mathbb{R}^d$ given by $\mathcal{A}^{(d)} = \{H \otimes \mathbb{R}^d\}_{H \in \mathbb{A}}$. We also define $\mathcal{M}(\mathcal{A}^{(d)}) = (V \otimes \mathbb{R}^d) \setminus \bigcup \mathcal{A}^{(d)}$.

Note that, when $d = 2$, the previous definition coincides with the classical complexification of $\mathcal{A}$.

With reference to the notation in the previous definition, for any subset $F \subseteq \mathbb{R}^n$, we can define, as before,

$$\mathcal{A}_F = \{H \in \mathcal{A} : F \subseteq H\}.$$

This gives rise to a stratification $\mathcal{L}(\mathcal{A}_F)$ of V. For all $d \in \mathbb{N}$, we can define a stratification $\mathcal{L}^{(d)}(\mathcal{A})$ of $V \otimes \mathbb{R}^d$ whose strata are cartesian products of the form $F_1 \times \cdots \times F_k \times \ldots$, with $F_k \in \mathcal{L}(\mathcal{A}_{F_{k-1}})$ for $k \geq 1$ (where we identify $V \otimes \mathbb{R}^d$ with V^d). Here we put, by convention, $F_0 = \{0\}$. This gives rise, by an obvious limiting process, to a stratification $\mathcal{L}^{(\infty)}(\mathcal{A})$ of $V \otimes \mathbb{R}^\infty$. It should be observed that $\mathcal{L}^{(d)}(\mathcal{A})$ differs from the product stratification. Since $\bigcup \mathcal{A}^{(d)}$ is a union of strata in $\mathcal{L}^{(d)}(\mathcal{A})$, this restricts to a stratification of $\mathcal{M}(\mathcal{A}^{(d)})$.

If $\mathcal{A} = \mathcal{A}_G$ is the hyperplane arrangement associated to a finite reflection group G, then $\mathcal{L}^{(d)}(\mathcal{A})$ is G-equivariant. Moreover, a straightforward homotopical argument proves that, for all $d \in \mathbb{N}$, $\mathcal{M}(\mathcal{A}^{(d)})$ is $(d-1)$-connected. By passing to the limit we obtain a contractible space $\mathcal{M}(\mathcal{A}_G^{(\infty)}) = \varinjlim_d \mathcal{M}(\mathcal{A}_G^{(d)})$ with a free and discrete action of G. Thus $\mathcal{M}(\mathcal{A}_G^{(\infty)})$ is a model for the total space $E(G)$, and its quotient $\overline{\mathcal{M}}(\mathcal{A}^{(\infty)})$ by the action of G is a model for the classifying space $B(G)$.

In [10], De Concini and Salvetti use this fact to construct a CW model for $B(G)$. More explicitly, they construct, for all $1 \leq d \leq \infty$, a regular G-equivariant CW-complex $X^{(d)} \subseteq \mathcal{M}(\mathcal{A}^{(d)})$ that is “dual” with respect to the stratification $\mathcal{L}^{(d)}(\mathcal{A}_G)$. The duality here means that the following three conditions are satisfied:

- the intersection of a cell of $X^{(d)}$ and a stratum of $\mathcal{L}^{(d)}(\mathcal{A}_G)$ is always transverse
- $\forall F \in \mathcal{L}^{(d)}(\mathcal{A}_G) : \exists! e_F \subseteq X \text{ cell} : \dim(e_F) = \mathrm{codim}(F) \wedge e_F \cap F \neq \varnothing$
- $|e_F \cap F| = 1$

Then they prove that $X^{(d)}$ is a G-equivariant strong deformation retract of $\mathcal{M}(\mathcal{A}^{(d)})$. In particular, the quotient $\overline{X}_G$ of $X^{(\infty)}$ with respect to the action of G is a CW model for $B(G)$.

Note that the cells of $B(G)$ are indexed by the set of strata $F \in \mathcal{L}^{(\infty)}(\mathcal{A}_G)$ contained in $\mathcal{M}(\mathcal{A}_G^{(\infty)})$. Suppose we fixed a root system Φ for G. For example, we can take the (unique) root system composed by vectors of length 1. As described before, the choice of a fundamental chamber $C_0 \in \mathcal{C}(\mathcal{A})$ determines a simple system $\Delta \subseteq \Phi$, and thus a Coxeter presentation (G, S), where S is the set of simple reflections with respect to Δ. Due to Theorem 11, the relevant strata (and thus the cells of $X^{(\infty)}$) are indexed by the set of couples $(\underline{\Gamma}, \gamma)$, where $\underline{\Gamma} = (S \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k \supseteq \varnothing)$ is a "flag" of finite length in S and $\gamma \in G$. The action of G on the cells is understood in this setting as the diagonal G-action that fixes the $\underline{\Gamma}$ component and acts as left multiplication on γ.

Moreover, by construction, $e_{F_1} \subseteq \overline{e_{F_2}} \Leftrightarrow F_2 \subseteq \overline{F_1}$. It turns out that this can be translated to an algebraic-combinatorial condition on the corresponding $(\underline{\Gamma}, \gamma)$. Also the incidence number $[e_{F_1}, e_{F_2}]$ and, as a consequence, the whole cellular chain complex of $X^{(\infty)}$, can be described combinatorially. In order to state the relevant result, we require the notion of minimal length coset representatives for parabolic subgroups, that we briefly recall below.

Definition 13. Let (G, S) be a Coxeter system. The *length* of an element $\gamma \in G$ is the minimum of the length of a word in S representing γ. The length of γ is denoted $\ell(\gamma)$.

Theorem 14. *[23] Let (G, S) be a Coxeter system. For any $T \subseteq S$, let G_T be the subgroup generated by T. Then:*

- *every left coset γG_T of G_T in G possesses a unique element β such that*

$$\forall \gamma' \in G_T : \ell(\beta\gamma') = \ell(\beta)\ell(\gamma')$$

- *β is the unique element of minimal length among those belonging to the coset γG_T*

Definition 15. With reference to the previous theorem, the subgroups of the form G_T for some $T \subseteq S$ are called *parabolic subgroups* of the Coxeter system (G, S). The set of minimal left coset representatives of G_T in G is denoted by G^T.

Theorem 16. *[10] Let G be a finite reflection group and S a set of simple reflections for G. Then the cellular chain complex C_*^G of $X^{(\infty)}$ if $\mathbb{Z}[G]$-free with basis $\{e_{\underline{\Gamma}}\}$ indexed by the set of flags of finite length in S*

$$\underline{\Gamma} = (\Gamma_1 \supseteq \cdots \supseteq \varnothing) : \Gamma_1 \subseteq S.$$

Moreover, the boundary homomorphism $\partial_* \colon C_*^G \to C_*^G$ is described by the following formula:

$$\partial_*(e_{\underline{\Gamma}}) = \sum_{\substack{i \geq 1 \\ |\Gamma_i| > |\Gamma_{i+1}|}} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in W_{\Gamma_i}^{\Gamma_i \setminus \{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta \subseteq \Gamma_i \setminus \{\tau\}}} (-1)^{\alpha(\underline{\Gamma}, i, \tau, \beta)} \beta e_{\Gamma'}$$

In the formula above we have fixed a total ordering on S and we have put:

$$\Gamma' = (\Gamma_1 \supseteq \cdots \supseteq \Gamma_{i-1} \supseteq \Gamma_i \setminus \{\tau\} \supseteq \beta^{-1}\Gamma_{i+1}\beta \supseteq \beta^{-1}\Gamma_{i+2}\beta \supseteq \dots)$$
$$\alpha(\underline{\Gamma}, i, \tau, \beta) = i\ell(\beta) + \sum_{j=1}^{i-1} |\Gamma_j| + |\{s \in \Gamma_i : s \leq \tau\}| + \sum_{j=i+1}^{k} \mathrm{sgn}(\Gamma_j \to \beta^{-1}\Gamma_j\beta)$$

The previous result describes an explicit resolution of $\mathbb{Z}$ in $\mathbb{Z}[G]$-modules that is smaller than the so-called standard resolution.

Moreover, note that, if $S = \{s_1, \dots, s_n\}$ has cardinality n, flags $\underline{\Gamma} = (\Gamma_1 \supseteq \cdots \supseteq \Gamma_k \supseteq \varnothing)$ of subsets of S can be indexed with n-tuples $[a_1, \dots, a_n]$ of non-negative integers. In order to do this, we associate to each flag $\underline{\Gamma}$ the n-tuple $[a_1, \dots, a_n]$ whose coordinates are defined by the formula

$$a_i = \max\{k \in \mathbb{N} : s_i \in \Gamma_k\}$$

and observe that this gives a bijection.

There is an alternative description of the same chain complex C_*^G, that we recall below. This description can be found, for example, in Vassiliev [41] or, more recently, in Giusti and Sinha [14], where this is developed only for the reflection groups of Type A_n. See also Bjorner and Ziegler [5] for a different approach.

In order to do this, we observe that, for every $1 \leq d < \infty$, the strata of $\mathcal{L}^{(d)}(\mathcal{A}_G)$ are homeomorphic to open disks of suitable dimension and give rise to a CW structure on the Alexandroff compactification of the component of $\mathcal{A}_G^{(d)}$ $\left(\mathcal{M}(\mathcal{A}_G^{(d)})\right)^+ = \mathcal{M}(\mathcal{A}_G^{(d)}) \cup \{\infty\}$. This CW structure is, once again, G-equivariant. Let $\widetilde{FN}_G^{(d)}$ be its augmented (G-equivariant) cellular chain complex. In this case, the cells are closures $e^F = \overline{F}$ of strata $F \in \mathcal{L}^{(d)}(\mathcal{A}_G)$, together with the basepoint $\{*\}$. Using Atiyah's duality theorem [3], we see that $\left(\mathcal{M}(\mathcal{A}_G^{(d)})\right)^+$ is the Spanier-Whitehead dual of $X^{(d)}$, and thus the homology of $\widetilde{FN}_G^{(d)}$ coincides, up to some degree shifts, with the cohomology of $X^{(d)}$. From this and the fact that $X^{(d)}$ is $(dn-2)$-connected, we easily see that $\widetilde{FN}_G^{(d)}$ is acyclic up to a dimension $dn-2$. Thus, by passing to the limit, we obtain an acyclic $\mathbb{Z}[G]$-free complex $\widetilde{FN}_G^* = \widetilde{FN}_G^{(\infty)}$. In particular, its quotient FN_G^* by the action of G calculates the cohomology of $B(G)$.

We can directly deduce from De Concini and Salvetti's construction in [10] that the given cellular structure on $\left(\mathcal{M}(\mathcal{A}_G^{(d)})\right)^+$ is dual to that of $X^{(d)}$. This means that cells of $\left(\mathcal{M}(\mathcal{A}_G^{(d)})\right)^+$ always intersect transversally cells of $X^{(d)}$, that e_F is the only cell of dimension d in $X^{(d)}$ that intersects the d-codimensional cell in $\left(\mathcal{M}(\mathcal{A}_G^{(d)})\right)^+$ e^F, and that $|e_F \cap e^F| = 1$, as already observed.

This determines an isomorphism between $\widetilde{FN}^{(d)}$ and the dual complex of the cellular chain complex of $X^{(d)}$, and, at the limit, between $\widetilde{FN}_G^*$ and the dual complex of C_*^G. Thus the two approaches are fully equivalent.

1.2 The little cubes operad and iterated loop spaces

In this section, we are going to recall some basics about operads and iterated loop spaces. First, we give the classical definition of loop spaces.

Definition 17. Let $(X, *)$ be a pointed topological space. The *loop* space of $(X, *)$ is the topological space $\Omega(X, *)$, defined as follows. Pointwise, it is the set of continuous functions $\gamma\colon [0,1] \to X$ such that $\gamma(0) = \gamma(1) = *$. Its topology is given as a subspace of the space of continuous functions $C([0,1], X)$ with the compact-open topology.

Note that, for all pointed topological space $(X, *)$, $\Omega(X, *)$ has a natural basepoint, given by the constant function $\gamma\colon [0,1] \to \{*\} \subseteq X$. Thus, this construction defines a functor $\Omega\colon \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$, where $\mathcal{T}\mathrm{op}_*$ is the category of pointed topological spaces.

We can apply the functor Ω recursively to obtain iterated loop spaces. For technical convenience, following [28], we use loop space sequences.

Definition 18. Let $(X, *)$ be a topological space and let $1 \leq n \leq \infty$. An *n-loop sequence* is a sequence of pointed topological spaces $\{(X_i, *)\}_{i=0}^n$ such that, for all i, there exists a homotopy equivalence $\Omega(X_i, *) \simeq (X_{i-1}, *)$. The space X_0 is called a *n-loop space* If $X = \{X_i\}_{i=0}^n$ and $Y = \{Y_i\}_{i=0}^n$ are n-loop sequences, a morphism of n-loop spaces from X to Y is a sequence of based maps $f_i\colon (X_i, *) \to (Y_i, *)$ that makes the following diagram commute for all i, where the vertical arrows are the structural homotopy equivalences:

$$\begin{tikzcd}
(X_i, *) \arrow[r, "f_i"] \arrow[d, "\simeq"] & (Y_i, *) \arrow[d, "\simeq"] \\
\Omega(X_{i+1}, *) \arrow[r, "\Omega(f_{i+1})"] & \Omega(Y_{i+1}, *)
\end{tikzcd}$$

With a little abuse of notation, we describe an n-loop sequence $\{X_i\}_{i=0}^n$ via the n-loop space X_0, omitting X_i for i positive. For example, we refer to the category consisting of n-loop sequences with their morphisms as the category of n-loop spaces.

Iterated loop spaces, and in particular ∞-loop spaces, have a particularly important role in topology. For example, they lay at the foundations of stable homotopy theory, and the homology of many is strongly related to the homology of certain loop spaces. The interested reader can find details, for example, in [4] and [8] for the symmetric group and braid groups, in [40] for mapping class groups, or in [42] for automorphism groups of free groups.

Since we will use it in the following chapters, we proceed by recalling the construction of free iterated loop spaces. Let $\Sigma\colon \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ be the reduced suspension functor. Explicitly,

$$\Sigma(X, *) = \frac{[0,1] \times X}{(\{0,1\} \times X) \cup ([0,1] \times \{*\})}$$

Consider the space $\Omega\Sigma(X)$. There is a natural inclusion $u_1^X \colon X \to \Omega\Sigma(X)$ that maps $x \in X$ into the path γ_x given by $\gamma_x(t) = [t,x]$ for all $t \in [0,1]$. If $Y = \Omega(Z,*)$ is a loop space and $f \colon (X,*) \to (Y,*)$ is a continuous map of pointed spaces, then $\tilde{f} \colon \Sigma(X,*) \to (Z,*)$ given by $\tilde{f}([t,x]) = f(x)(t)$ is the unique continuous map that makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "u_1^X"] \arrow[dr, "f"'] & \Omega\Sigma(X,*) \arrow[d, "\Omega(\tilde{f}_1)"] \\ & Y \end{tikzcd}$$

In this sense, $\Omega\Sigma \colon \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$ can be regarded as the free loop space functor.

Similarly, for all $n \in \mathbb{N}$, $\Omega^n\Sigma^n(X,*)$ is the free loop space over $(X,*)$. In this case, the universal map $u_n^X \colon X \to \Omega^n\Sigma^n(X,*)$ is constructed recursively as the composition

$$(X,*) \overset{u_{n-1}^X}{\to} \Omega^{n-1}\Sigma^{n-1}(X,*) \overset{\Omega^{n-1}u_1^{\Sigma^{n-1}(X,*)}}{\to} \Omega^n\Sigma^n(X,*).$$

Moreover, if $Y = \Omega^n(Z,*)$ is an n-loop space and $f \colon (X,*) \to (Y,*)$ is a based map, the corresponding n-loop space map between $\Omega^n\Sigma^n(X,*)$ and Y is $\Omega^n(\tilde{f}_n)$, defined recursively by putting $\tilde{f}_n = \widetilde{(\tilde{f}_{n-1})_1}$. Note that the following diagram commutes for all $n \in \mathbb{N}$ and for all n-loop space $Y = \Omega^n(Z,*)$:

$$\begin{tikzcd} X \arrow[r, "u_1"] \arrow[d, "f"'] & \Omega\Sigma(X,*) \arrow[r, "\Omega u_1^{\Sigma(X,*)}"] \arrow[dl, "\Omega\tilde{f}_1"'] & \Omega^2\Sigma^2(X,*) \arrow[r, "\Omega^2 u_1^{\Sigma^2(X,*)}"] \arrow[dll, "\Omega^2\tilde{f}_2"'] & \cdots \arrow[r] & \Omega^n\Sigma^n(X,*) \arrow[dllll, "\Omega\tilde{f}_n"] \\ Y \end{tikzcd}$$

Thus, by taking the direct limit $\varinjlim_{n\to\infty} \Omega^n\Sigma^n(X,*)$, we obtain the free ∞-loop space over $(X,*)$, that we denote $Q(X)$.

Iterated loop spaces can be characterized in term of higher homotopy coherence. For example, a loop space $\Omega(X,*)$ is naturally an H-space, with the product given by juxtaposition of loops. This product is associative up to homotopy. For a double loop space $\Omega^2(X,*)$, these homotopies are themselves associative up to higher-order homotopies, and so on.

This idea can be made rigorous with the notion of operad, that can be used as a formal tool to organize these higher-order homotopies. Intuitively, a (topological) operad is a sequence of topological spaces $\{F(n)\}_{n=0}^{\infty}$ such that $F(n)$ parametrizes a certain family of "n-ary operations", and we want these families to be closed under composition. Following Marks–Shnider–Stasheff [27], we give below the precise definitions.

Definition 19. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. A *plain operad* in $\mathcal{C}$ is a sequence $\mathcal{F} = \{F(n)\}_{n=0}^{\infty}$ of objects of $\mathcal{C}$ with *composition morphisms*

$\gamma_{j_1,\dots,j_k} \colon F(k) \otimes F(j_1) \otimes \cdots \otimes F(j_k) \to F(\sum_{i=1}^{k} j_i)$ and a *unital morphism* $e \colon 1_C \to F(1)$ satisfying the following properties:

- for all $n \in \mathbb{N}$, the composition $F(n) \cong 1_C \otimes F(n) \overset{e \otimes \mathrm{id}}{\to} F(1) \otimes F(n) \overset{\gamma_n}{\to} F(n)$ is the identity map
- for all $n \in \mathbb{N}$, $F(n) \cong F(n) \otimes 1_C^{\otimes^n} \overset{\mathrm{id} \otimes e^{\otimes^n}}{\to} F(n) \otimes F(1)^{\otimes^n} \overset{\gamma_{1,\dots,1}}{\to} F(n)$ is the identity map
- for all $k_1, \dots, k_m \in \mathbb{N}$ and for all $j_1, \dots, j_{\sum_{i=1}^{m} k_i} \in \mathbb{N}$, the following equality holds:

$$\gamma_{k_1,\dots,k_m} \circ (\gamma_{j_1,\dots,j_{k_1}} \otimes \cdots \otimes \gamma_{j_{\sum_{i=1}^{m-1} k_i+1},\dots,j_{\sum_{i=1}^{m} k_i}}) = \gamma_{j_1,\dots,j_{\sum_{i=1}^{m} k_i}}$$

Definition 20. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. Let Σ_n be the symmetric group on n objects. Given permutations $\tau_1 \in \Sigma_{j_1}, \dots, \tau_k \in \Sigma_{j_k}$, let $\tau_1 \oplus \cdots \oplus \tau_k$ be the image of $\tau_1 \times \cdots \times \tau_k$ via the obvious morphism of groups $\Sigma_{j_1} \times \dots \Sigma_{j_k} \to \Sigma_{\sum_{i=1}^{k} j_i}$. Given a permutation $\sigma \in \Sigma_k$ and natural numbers $j_1, \dots, j_k \in \mathbb{N}$, let $\sigma(j_1, \dots, j_k)$ be the permutation in $\Sigma_{\sum_{i=1}^{k}}$ that permutes the k blocks of letters determined by the given natural numbers as σ permutes k letters.

A *symmetric operad*, or simply an *operad*, in $\mathcal{C}$, is a plain operad $\mathcal{F} = \{F(n)\}_{n=0}^{\infty}$ in $\mathcal{C}$ with right actions of the symmetric groups $F(n) \curvearrowleft \Sigma_n$ satisfying the following conditions for all $j_1, \dots, j_k \in \mathbb{N}$, $\sigma \in \Sigma_k$, $\tau_i \in \Sigma_{j_i}$:

- $\gamma_{j_1,\dots,j_k} \circ (\_.\sigma \otimes \mathrm{id}_{F(j_1)} \otimes \cdots \otimes \mathrm{id}_{F(j_k)}) = (\_.\sigma(j_1, \dots, j_k)) \circ \gamma_{j_{\sigma^{-1}(1)},\dots,j_{\sigma^{-1}(k)}}$
- $\gamma_{j_1,\dots,j_k} \circ (\mathrm{id}_{F(k)} \otimes \_.(\tau_1 \otimes \cdots \otimes \_.\tau_k))(\_.\tau_1 \oplus \cdots \oplus \tau_k) \circ \gamma_{j_1,\dots,j_k}$

We can think of an operad $\mathcal{F} = \{F(n)\}_{n=0}^{\infty}$ as an abstract description of a particular type of algebraic structures in a monoidal category. A particular specialization of this structure, in which $F(n)$ is interpreted as actual n-ary operations on some object, is called an $\mathcal{F}$-algebra.

Definition 21. Let $\mathcal{C}$ be a monoidal category and $\mathcal{F} = \{F(n)\}_{n=0}^{\infty}$ be a plain operad in $\mathcal{C}$. An *$\mathcal{F}$-algebra* is an object $X \in \mathcal{C}$ together with morphisms $\theta_n \colon F(n) \otimes X^{\otimes^n} \to X$ for all $n \in \mathbb{N}$ satisfying the following properties:

- $\theta_1 \circ e = \mathrm{id}_X$
- $\forall j_1, \dots, j_k \in \mathbb{N} : \theta_{\sum_{i=1}^{k} j_i} \circ \gamma_{j_1,\dots,j_k} \otimes \mathrm{id}_{X^{\otimes \sum_{i=1}^{k} j_i}} = \theta_k \circ (\theta_{j_1} \otimes \cdots \otimes \theta_{j_k}) \cdot \mu$, where $\mu \colon F(k) \otimes F(j_1) \otimes \cdots \otimes F(j_k) \otimes X^{\otimes^{\sum_{i=1}^{k} j_i}} \to F(k) \otimes F(j_1) \otimes X \otimes \cdots \otimes F(j_k) \otimes X$ is the evident shuffle isomorphism

If $\mathcal{F}$ is a symmetric operad, we also require that $\theta_j \circ (\_.\sigma \otimes \mathrm{id}_{X^{\otimes j}}) = \theta_j \circ (\mathrm{id}_{F(j)} \otimes \sigma.\_)$, where we consider the left action of Σ_j on $X^{\otimes^j}$ given by permuting factors.

We only consider topological operads, i.e. symmetric operads in the category of topological spaces, with the symmetric monoidal structure given by the cartesian product. Thus, from now on, when we use the term "operad" we will always implicitly refer to a symmetric operad in this category. We will also refer to an algebra over a topological operad $\mathcal{F}$ as an $\mathcal{F}$-space.

The operads that describe the structure of iterated loop spaces are called little cubes operads. We briefly recall their construction. We refer to the classical book by Peter May [28] for further details.

Definition 22. Let $n \in \mathbb{N} \cup \{\infty\}$. If $n < \infty$, let $J^n = (0,1)^n$ be an open cube in $\mathbb{R}^n$. If $n = \infty$; let $J^n = (0,1)^\infty$ be an open cube in the limit $\mathbb{R}^\infty = \varinjlim_n \mathbb{R}^n$, the vector space of eventually-zero real-valued sequences. A *little n-cube* is an embedding $f\colon J^n \to J^n$ of the form $f = f_1 \times \cdots \times f_n$, where f_i is the restriction of an affine map $\mathbb{R} \to \mathbb{R}$.

Proposition 23. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $j \in \mathbb{N}$. Define $\mathcal{C}_n(j)$ as the space of j-tuples $(c_1, \dots, c_j)$ of little n-cubes with pairwise disjoint images, with the topology induced by the compact-open topology on the space of continuous functions $C\left((J^n)^{\sqcup j}, J^n\right)$. For all $j_1, \dots, j_k \in \mathbb{N}$, consider the composition functions $\gamma\colon \mathcal{C}_n(k) \times \mathcal{C}_n(j_1) \times \mathcal{C}_n(j_k) \to \mathcal{C}_n(\sum_{i=1}^k j_k)$ given by:*

$$\begin{aligned}\gamma\left((c_1, \dots, c_k), (d_{1,1}, \dots, d_{1,j_1}), \dots, (d_{k,1}, \dots, d_{k,j_k})\right) \\ = (c_1 \circ d_{1,1}, \dots, c_1 \circ d_{1,j_1}, c_2 \circ d_{2,1}, \dots, c_k \circ d_{k,j_k})\end{aligned}$$

Moreover, let $e\colon \{*\} \to \mathcal{C}_n(1)$ be the map $* \mapsto \mathrm{id}_{J^n}$. Then $\mathcal{C}_n = \{\mathcal{C}_n(j)\}_{n=0}^\infty$ is an operad.

The little cubes operads are important because they detect iterated loop space structures. Before stating the recognition theorem, note that, for every $n \in \mathbb{N} \cup \{\infty\}$, an n-loop space admits a natural action of $\mathcal{C}_n$. If $n < \infty$, we describe this action by regarding points of X as maps $[0,1]^n \to Y$ and letting

$$\theta_j\left((c_1, \dots, c_j), x_1, \dots, x_j\right) = \begin{cases} x_r(c_r^{-1}(t_1, \dots, t_n)) & \text{if } \exists r : \underline{t} \in \mathrm{im}(c_r) \\ * & \text{if } (t_1, \dots, t_n) \notin \mathrm{im}(c_r) \end{cases}.$$

If $n = \infty$, the action is obtained by a limiting process. With this in mind, we recall the following result.

Theorem 24 (recognition principle, [28])**.** *Let $n \in \mathbb{N} \cup \{\infty\}$. Let X be a connected space. Let $n \in \mathbb{N} \cup \{\infty\}$. X is weakly homotopy equivalent to a n-loop space if and only if it admits a $\mathcal{C}_n$-space structure.*

We also recall a fact about little cubes operads that we will use in the later chapters.

Definition 25. Let X be a topological space. The *(ordered) configuration space on n points in X* is the following subspace of X^n:

$$\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) \in X^n : \forall 1 \leq i < j \leq n : x_i \neq x_j\}$$

The *(unordered) configuration space on n points in X* if the the quotient $\overline{\mathrm{Conf}}_n(X)$ of $\mathrm{Conf}_n(X)$ by the action of the symmetric group Σ_n defined by permutation of the n coordinates.

Proposition 26. *[28] Let $k \in \mathbb{N}$ and $n \in \mathcal{N} \cup \{\infty\}$. Consider the map $\mathcal{C}_n(k) \to \mathrm{Conf}_k((0,1)^n) \cong \mathrm{Conf}_k(\mathbb{R}^n)$ that sends a k-tuple of little n-cubes $(f_1, \ldots, f_k)$ to the configuration of their centers $(f_1(0), \ldots, f_k(0))$ in $\mathrm{Conf}_k((0,1)^n)$. This map is a Σ_k-equivariant homotopy equivalence. Hence, it induces an homotopy equivalence*

$$\frac{\mathcal{C}_n(k)}{\Sigma_k} \simeq \overline{\mathrm{Conf}}_k(\mathbb{R}^n).$$

We conclude this chapter with the classical treatment of the homology of $\mathcal{C}_\infty$-spaces, that we will require in the following chapters. This description has been introduced half a century ago by Dyer and Lashof [11]. We refer to the book of Cohen, May and Lada [8] for a detailed exposition.

In what follows, we will let p be a prime number and we will use the symbol π_p to denote the cyclic group with p elements. Recall that there is a standard free resolution W_* of $\mathbb{Z}$ with $\mathbb{Z}[\pi_p]$-modules:

$$\ldots \mathbb{Z}[\pi_p] \xrightarrow{N} \mathbb{Z}[\pi_p] \xrightarrow{T} \mathbb{Z}[\pi_p] \xrightarrow{N} \mathbb{Z}[\pi_p] \xrightarrow{T} \mathbb{Z}[\pi_p] \to 0$$

where, if g is a generator of π_p, T is the multiplication by $\sum_{i=0}^{p-1} g^i$ and N is the multiplication by $g-1$.

Definition 27. Let p be a prime number and let $\mathbb{F}_p$ be the field with p elements. Let X be a $\mathcal{C}_\infty$-space. Let $i \in \mathbb{N}$. Let W_* be the standard free resolution of $\mathbb{Z}$ with $\mathbb{Z}[\pi_p]$-modules. Let e_i be the generator of the free $\mathbb{Z}[\pi_p]$-module $W_i \cong \mathbb{Z}[\pi_p]$. We define the i^{th} *Kudo–Araki operation* as the linear map

$$Q_i \colon \alpha \in H_d(X; \mathbb{F}_p) \mapsto (\theta_p)_*(e_i \otimes_{\pi_p} \alpha^{\otimes^p}) \in H_{pd+i}(X; \mathbb{F}_p)$$

induced by the structural $\mathcal{C}_\infty$-action map $\theta_p \colon \mathcal{C}_\infty(p) \times X^p \to X$. Here, we embed W_* in the (singular, for example) chain complex $C_*(\mathcal{C}_\infty(p))$ of $\mathcal{C}_\infty(p)$ in a $\mathbb{Z}[\pi_p]$-equivariant way (thus regarding e_i as an element of $C_*(\mathcal{C}_\infty(p))$), and we let π_p act on X^p by cyclic permutation of the factors. By convention, we define $Q_i = 0$ if $i < 0$.

If $p = 2$, the i^{th} *Dyer–Lashof operation* $Q^i \colon H_d(X; \mathbb{F}_p) \to H_{d+i}(X; \mathbb{F}_p)$ is the linear map Q_{i-d}. If $p > 2$, we define the i^{th} *Dyer–Lashof operation* $Q^i \colon H_d(X; \mathbb{F}_p) \to H_{d+2i(p-1)}(X; \mathbb{F}_p)$ as $\mu_d Q_{(p-1)(2i-d)}$, where

$$\mu_d = (-1)^{i + \frac{d(d-1)(p-1)}{4}} \left(\frac{p-1}{2}! \right)^d.$$

Definition 28. Let p be a prime number. The *Dyer-Lashof* algebra $\mathcal{R}_p$ is the unital associative algebra generated by $\{Q^i\}_{i=0}^\infty$ and, if $p > 2$, by $\{\beta Q^i\}_{i=1}^\infty$, with the following relations:

$$Q^r \circ Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} \circ Q^i \text{ if } r > ps$$

and, when $p > 2$,

$$\begin{aligned} Q^r \circ \beta Q^s = & \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta Q^{r+s-i} \circ Q^i \\ & - \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} Q^{r+s-i} \circ \beta Q^i \text{ if } r \geq ps \end{aligned}$$

and those obtained by left-multiplying both members in the previous relations by β and imposing $\beta^2 = 0$.

An *algebra over the Dyer-Lashof algebra* is a graded unital $\mathbb{F}_p$-algebra $(A_*, \cdot)$ with a linear action of $\mathcal{R}_p$ satisfying the following additional conditions:

- $Q^0(1_A) = 1_A$, $Q^i(1_A) = 0$ for all $i > 0$ and, if $p > 2$, $\beta Q^i(1_A) = 0$ for all $i \in \mathbb{N}$
- $Q^i(x \cdot y) = \sum_{j=0}^{i} Q^j(x) \cdot Q^{i-j}(y)$ and, if $p > 2$, $\beta Q^i(x \cdot y) = \sum_{j=0}^{i} \beta Q^j(x) \cdot Q^{i-1}(y) + (-1)^j Q^j(x) \cdot \beta Q^{i-j}(y)$ (Cartan formulas)
- for all $x \in H_d(X; \mathbb{F}_p$, $Q^i(x) = 0$ if $p = 2$ and $i < d$, or $p > 2$ and $2i < d$
- for all $x \in H_d(X; \mathbb{F}_p$, $Q^i(x) = 0$ if $p = 2$ and $i = d$, or $p > 2$ and $2i = d$

The homology of $\mathcal{C}_\infty$-spaces have an algebra structure. In order to see this, fix an element $c \in \mathcal{C}_\infty(2)$ and, given a $\mathcal{C}_\infty$-space X, define the product map $\mu_c = \theta_2(c, \cdot, \cdot) \colon X \times X \to X$. μ_c makes X a H-space. Since $\mathcal{C}_\infty(2)$ is connected, the homotopy class of μ_c and, as a consequence, the induced map in homology $\mu_{c*} \colon H_*(X) \otimes H_*(X) \to H_*(X)$, does not depend on the choice of c. Furthermore, note that there is a free $\mathcal{C}_\infty$-space functor $C \colon \mathcal{T}\mathrm{op}_* \to \mathcal{T}\mathrm{op}_*$, defined by the following construction:

$$CX = \frac{\bigsqcup_{n=0}^{\infty} \mathcal{C}_\infty(n) \times_{\Sigma_n} X^n}{((c_1, \ldots, c_n) \times (x_1, \ldots, x_{n-1}, *)) \sim ((c_1, \ldots, c_{n-1}) \times (x_1, \ldots, x_{n-1}))}$$

Theorem 29. *[8] Let p be a prime number and let X be a $\mathcal{C}_\infty$-space. The homology $H_*(X; \mathbb{F}_p)$, with the Dyer-Lashof operations constructed above and the Bockstein homomorphism β, is an algebra over the Dyer-Lashof algebra. Moreover, for all topological spaces X homotopical equivalent to a CW complex:*

- *$H_*(C(X); \mathbb{F}_p)$ is the free Dyer-Lashof algebra generated by $H_*(X; \mathbb{F}_p)$*
- *$H_*(Q(X); \mathbb{F}_p)$ is the group completion of $H_*(C(X); \mathbb{F}_p)$*

In particular, if X is connected, $C(X)$ and $Q(X)$ are homotopy equivalent.

This result can be used to extract the additive structure and obtain an explicit basis, as a vector space, for $H_*(C(X); \mathbb{F}_p)$. We recall this description below.

Definition 30. If $p = 2$, let $I = (i_1, \ldots, i_r)$ be a sequence of non-negative integers. We let Q^I be the linear operation $Q^{i_1} \circ \cdots \circ Q^{i_r}$. We define the *excess* of I the number $e(I) = i_1 - \sum_{j=2}^{r} i_j$. We say that I is *admissible* if $e(I) \geq 0$ and, for all $1 \leq k < r$, $i_k \leq 2i_{k+1}$. We say that I is *strongly admissible* if, in addition, $e(I) > 0$.

If $p > 2$, let $I = (\varepsilon_1, i_1, \ldots, \varepsilon_r, i_r)$ be a sequence such that $i_1, \ldots, i_r$ are non-negative integers and $\varepsilon_1, \ldots, \varepsilon_r \in \{0, 1\}$. We define the *excess* of I the number $e(I) = 2i_1 - \varepsilon_1 - \sum_{j=2}^{r} 2(p-1)i_j - \varepsilon_j$. We say that I is *admissible* if $e(I) \geq 0$ and, for all $1 \leq k < r$, $i_k \leq i_{k+1}p - \varepsilon_{k+1}$. We say that I is *strongly admissible* if, in addition, $e(I) > 0$.

The Dyer–Lashof operations on the homology of $\mathcal{C}_\infty$-spaces bear some similarity with the Steenrod operation on the cohomology of topological spaces. For example, both can be interpreted as higher order p^{th} powers in their respective context.

Theorem 31. *[8] Let X be a topological space and consider $H_*(C(X);\mathbb{F}_p)$ as an algebra with the product induced by the H-space structure. Let $\mathcal{B}$ be an additive graded basis for $H_*(X;\mathbb{F}_p)$ containing the class of the basepoint $[*]$. Let $\tilde{\mathcal{B}}$ be $\mathcal{B}$ without $[*]$. $H_*(C(X);\mathbb{F}_p)$ is, with the product $*$ induced by its H-space structure, the free graded commutative unital algebra generated by elements $Q^I(x)$, where $x \in \tilde{\mathcal{B}}$ and I is an admissible sequence with excess strictly bigger that the homological dimension of x.*

In particular, an additive basis for the mod p homology of $C(X)$ is given by products $(Q^{I_1}(x_1))^{a_1} \cdots (Q^{I_k}(x_k))^{a_k}$ of such generators, where $a_1, \ldots, a_k > 0$, $Q^{I_1}(x_1), \ldots, Q^{I_k}(x_k)$ are pairwise distinct and, if $p > 2$, $a_j = 1$ whenever $Q^{I_j}(x_j)$ has odd degree. In what follows, we call these products *Nakaoka monomials* and this additive basis *Nakaoka basis.* The name comes from [32], where Nakaoka first described this basis in the particular case of $C(S^0)$.

1.3 Hopf rings

We recall here some basics about Hopf rings. A more complete theoretical background can be found, among others, in Goerss [15] or Hunton–Turner [25]. We describe the theory only the graded setting since we will only deal with graded algebras.

A Hopf ring is a ring object in the category of (graded) coalgebras. This first consists of an abelian group object in this category. Such an object is known as a commutative Hopf algebra. It is a graded R-coalgebra (A, Δ) together with two morphisms of graded coalgebras: a product $\mu\colon A \otimes A \to A$, that should be thought as an "addition" map, and an inversion map $S\colon A \to A$. μ is required to be associative and commutative and to have an identity $\eta\colon R \to A$. In this context, commutativity means that for all graded elements $x, y \in A$ we have that $\mu(x, y) = (-1)^{|x||y|}\mu(y, x)$. S is required to make the following diagram commute, where ε is the counity:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\varepsilon"] & A \otimes A \arrow[dr, "S\otimes\mathrm{id}"] & \\
A \otimes A \arrow[dr, "\mathrm{id}\otimes S"] & R \arrow[dr, "\eta"] & A \otimes A \arrow[d, "\odot"] \\
 & A \otimes A \arrow[r, "\odot"] & A
\end{tikzcd}$$

A graded coalgebra with an associative and commutative product μ, but possibly without an inversion map, is known as a bialgebra. Although there is a notion of a non-commutative bialgebra, all the bialgebras and Hopf algebras that we take into consideration are intended as commutative.

A Hopf ring has an "addition" product that makes it a Hopf algebra, but also an additional "multiplication" product that relates to the previous one via a "distributivity formula". The precise definition is given below.

Definition 32. Let R be a commutative ring. A *(graded) Hopf ring* over $\mathbb{R}$ is the datum of an R-module A, a graded coproduct $\Delta\colon A \to A \otimes A$, two graded products $\odot, \cdot\colon A \otimes A \to A$ and a graded map $S\colon A \to A$ that satisfy the following conditions:

- $(A, \Delta, \cdot)$ is a bialgebra
- $(A, \Delta, \odot, S)$ is a Hopf algebra
- if $\Delta(x) = \sum_i x'_i \otimes x''_i$, then

$$x \cdot (y \odot z) = \sum_i \left[(-1)^{|x''_i||y|} (x'_i \cdot y) \odot (x''_i \cdot z)\right]$$

An alternative, more conceptual, definition of Hopf rings uses the notion of bilinear maps in a category. Following [15], we briefly recall it below.

Definition 33. Let $\mathcal{C}$ be a category and $\mathcal{A} \subseteq \mathcal{C}$ a subcategory of abelian objects that has all the finite products. This means that for all $A \in \mathcal{A}$ the representable functor $\mathrm{Hom}_{\mathcal{C}}(\cdot, A)\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$ factors through a functor $\mathcal{C}^{op} \to \mathcal{A}\mathrm{b}$, where $\mathcal{S}\mathrm{et}$ and $\mathcal{A}\mathrm{b}$ are the categories of sets and groups respectively.

Let $X, Y, Z \in \mathcal{A}$. A morphism $\varphi\colon X \times Y \to Z$ is said to be *bilinear* if, for all $C \in \mathcal{C}$, the following map is a natural bilinear map of abelian groups:

$$\mathrm{Hom}_{\mathcal{C}}(C, X) \times \mathrm{Hom}_{\mathcal{C}}(C, Y) = \mathrm{Hom}_{\mathcal{C}}(C, X \times Y) \xrightarrow{\varphi\circ -} \mathrm{Hom}_{\mathcal{C}}(C, Z)$$

A *tensor product* of $X, Y \in \mathcal{A}$ is a universal bilinear morphism from $X \times Y$, that is an object $X \boxtimes Y$ with a bilinear morphism $u\colon X \times Y \to X \boxtimes Y$ such that every bilinear map $X \times Y \to Z$ factors uniquely through a morphism $X \boxtimes Y \to Z$ in $\mathcal{A}$.

It is not true, in general, that tensor product exists. However, under suitable assumptions, that hold for the categories we are interested in, a tensor product functor exists.

Proposition 34. *[15] Let $\mathcal{C}$ be a category and let $\mathcal{A} \subseteq \mathcal{C}$ a subcategory of abelian objects. Assume that the following two conditions are satisfied:*

- *$\mathcal{C}$ and $\mathcal{A}$ have all limits and colimits*
- *the forgetful functor $\mathcal{A} \to \mathcal{C}$ has a left adjoint*

Then every pair of objects $X, Y \in \mathcal{A}$ has a tensor product $X \boxtimes Y \in \mathcal{A}$.

The categories of bialgebras and Hopf algebras over a ring R, as subcategories of the category of coalgebras, satisfy the hypotesis of the previous proposition, thus they admit all tensors products. In this context, a Hopf ring over R is just a Hopf algebra A over R with a bilinear map $A \boxtimes A \to A$.

Hopf rings arise naturally in the context of algebraic topology and infinite loop spaces. More explicitly, consider an Ω-spectrum $\{G_n\}_{n=0}^{\infty}$ representing a generalized cohomology theory G^*, i.e. a sequence of spaces such that $G^n(X)$ is naturally isomorphic to the homotopy classes $[X, G_n]$. If E^* is a generalized multiplicative cohomology theory, and if for E^* the Künneth isomorphism theorem holds, then $\bigoplus_{n=0}^{\infty} E^*(G_k)$ is a Hopf ring.

Many of these Hopf rings have been calculated. For example, the Hopf ring for complex cobordism has been calculated by Ravenel and Wilson [36], or that for Morava K-theory by Wilson [43].

Chapter 2

Hopf ring for the symmetric groups

2.1 Construction of the structural maps and mod 2 description

We describe here a Hopf ring structure on the cohomology of the finite reflection groups of type A_n, i.e. the symmetric groups Σ_{n+1}. A set of simple reflections for A_n is given by the set $S = \{s_i\}_{i=1}^{n}$, where s_i is the transposition $(i, i+1)$.

Figure 2.1: Coxeter graph for $W_{A_{n-1}} \cong \Sigma_n$

Let $1 \leq k \leq n$. The parabolic subgroup $W_{\{s_i : i \neq k\}}$ is itself a finite Coxeter group, and its Coxeter graph is obtained from A_n by removing the vertex s_k and the edges to which it belongs. Thus, it is isomorphic to the direct product $A_{k-1} \times A_{n-k}$. This isomorphism is explicitly realized as the obvious monomorphism $\mu_{k,n-k+1} \colon \Sigma_k \times \Sigma_{n-k+1} \to \Sigma_{n+1}$ that makes Σ_k act on the first k objects $\{1, \dots, k\}$, and Σ_{n-k+1} acts on the remaining $n-k+1$ objects $\{k+1, \dots, n+1\}$.

Let R be a commutative ring and consider the direct sum of the cohomology groups $A_A = \bigoplus_{n=0}^{\infty} H^*(\Sigma_n; R) = \bigoplus_{n=-1}^{\infty} H^*(W_{A_n}; R)$, where we let, by notational convention, $W_{A_{-1}}$ be the trivial group. If R is a field, via the Künneth isomorphism $H^*(W_{A_n} \times W_{A_m}; R) \cong H^*(W_{A_n}; R) \otimes H^*(W_{A_m}; R)$, thus the previously considered embeddings $\mu_{n,m}$ determine maps in cohomology $H^*(\Sigma_{n+m}; R) \to H^*(\Sigma_n; R) \otimes H^*(\Sigma_m; R)$ that sum up to give a linear coproduct $\Delta \colon A_A \to A_A \otimes A_A$.

Since each $\mu_{n,m}$ is a monomorphism, the corresponding map at the level of classifying spaces $B(\mu_{n,m}) \colon B(\Sigma_{n+m}) \to B(\Sigma_n) \times B(\Sigma_m)$ can be realized as a finite covering. Thus, it also determines a cohomological transfer map $\tau_{n,m} \colon H^*(\Sigma_n \times \Sigma_m; R) \to H^*(\Sigma_{n+m}; R)$. Once again, we obtain a product $\odot \colon A_A \otimes A_A \to A_A$ by summing up the morphisms $\tau_{n,m}$.

Finally, we take into consideration the usual cup product $\cdot\colon A_A \otimes A_A \to A_A$ and the map $S\colon A_A \to A_A$ that acts as the multiplication by $(-1)^n$ on the component $H^*(W_{A_{n-1}}; R) = H^*(\Sigma_n; R)$.

Strickland and Turner [38] proved that A_A, together with these maps, is a Hopf ring. Without digging into the details of their proof, we want to remark that this fact has a nice geometric interpretation. Indeed, a model for the classifying space $B(W_{A_{n-1}})$ is the configuration space $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$, where we define $\mathbb{R}^\infty = \varinjlim_n \mathbb{R}^n$. Explicitly, take two homeomorphisms $\varphi_+\colon \mathbb{R}^\infty \to (0,\infty)\times\mathbb{R}^\infty$ and $\varphi_-\colon \mathbb{R}^\infty \to (-\infty,0)\times\mathbb{R}^\infty$. The family of group homomorphisms $\mu_{n,m}\colon W_{A_{n-1}} \times W_{A_{m-1}} \to W_{A_{n+m-1}}$ can be modeled, at the geometric level, with the maps $(\varphi_-)^n \times (\varphi_+)^m\colon \overline{\mathrm{Conf}}_n(\mathbb{R}^\infty) \times \overline{\mathrm{Conf}}_m(\mathbb{R}^\infty) \to \overline{\mathrm{Conf}}_{n+m}(\mathbb{R}^\infty)$. Almost all the defining axiomatic properties of Hopf rings can be understood, for A_A, via corresponding (easy to prove) commutative diagrams involving $B(\mu_{n,m})$. For example, the associativity of the coproduct stems from the commutativity up to homotopy of the diagram on the top, while the fact that $\odot$ and Δ form a bialgebra stems from the fact that the diagram on the bottom is homotopy equivalent to a pullback of finite coverings. Here τ is the function that swaps the second and third factor.

$$\begin{tikzcd}
\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)\times\overline{\mathrm{Conf}}_m(\mathbb{R}^\infty)\times\overline{\mathrm{Conf}}_l(\mathbb{R}^\infty) \arrow[r, "B(\mu_{n,m})\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times B(\mu_{m,l})"] & \overline{\mathrm{Conf}}_{n+m}(\mathbb{R}^\infty)\times\overline{\mathrm{Conf}}_l(\mathbb{R}^\infty) \arrow[d, "B(\mu_{n+m,l})"] \\
\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)\times\overline{\mathrm{Conf}}_{m+l}(\mathbb{R}^\infty) \arrow[r, "B(\mu_{n,m+l})"] & \overline{\mathrm{Conf}}_{n+m+l}(\mathbb{R}^\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod_{\substack{p_1+p_3=n\\ p_2+p_4=m}}\prod_{i=1}^4\overline{\mathrm{Conf}}_{p_i}(\mathbb{R}^\infty) \arrow[r, "\coprod(B(\mu_{p_1,p_3})\times B(\mu_{p_2,p_4}))\circ\tau"] \arrow[d, "\coprod B(\mu_{p_1,p_2})\times B(\mu_{p_3,p_4})"] & \overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)\times\overline{\mathrm{Conf}}_m(\mathbb{R}^\infty) \arrow[d, "B(\mu_{n,m})"] \\
\coprod_{r+s=n+m}\overline{\mathrm{Conf}}_r(\mathbb{R}^\infty)\times\overline{\mathrm{Conf}}_s(\mathbb{R}^\infty) \arrow[r, "B(\mu_{r,s})"] & \overline{\mathrm{Conf}}_{n+m}(\mathbb{R}^\infty)
\end{tikzcd}$$

The only formula that does not follow immediately from this space-level construction is the one involving the antipode S. In order to understand it, it is necessary to use a version of these constructions in the category of spectra (see, for example, [6]). S is then induced by the additive inverse map of the sphere spectrum (see [38] for details).

This chapter is devoted to the calculation of the Hopf ring structure on A_A for cohomology with coefficients in the field $\mathbb{F}_p$ with p elements, where p is a prime number. We follow the treatment in [18], which is partially based on the author's master thesis [17]. Additionally, the mod 2 result was achieved by Giusti, Salvatore and Sinha [13]. Hence, we state here their main results.

First, let $n \in \mathbb{N}$ and assume that X is a properly embedded finite-codimensional submanifold $X \subseteq \overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$. Let $d \in \mathbb{N}$ be an arbitrary natural number and consider the standard inclusion $\overline{\mathrm{Conf}}_n(\mathbb{R}^d) \to \overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$ defined by zeroing all the coordinates except the first d. Assume that X is transverse to all $\overline{\mathrm{Conf}}_n(\mathbb{R}^d)$ for d big enough and let X_d be the inverse image of X in $\overline{\mathrm{Conf}}_n(\mathbb{R}^d)$. Then it is defined (for example via triangulations of X_d) a fundamental class $[X_d]$ of X_d in the Borel-Moore homology of $\overline{\mathrm{Conf}}_n(\mathbb{R}^d)$ and thus, by Poincaré duality, X_d determines a cohomology class in $H^*(\overline{\mathrm{Conf}}_n(\mathbb{R}^d))$ of dimension equal to the codimension of X. We call it the *Thom class* of X_d in $\overline{\mathrm{Conf}}_n(\mathbb{R}^d)$. Moreover, our assumptions imply that, for $0 << d < d'$, the restriction of the Thom class of $X_{d'}$ to $\overline{\mathrm{Conf}}_n(\mathbb{R}^d)$ via the standard inclusion $\overline{\mathrm{Conf}}_n(\mathbb{R}^d) \to \overline{\mathrm{Conf}}_n(\mathbb{R}^{d'})$ is the Thom class of X_d. For all $k > 0$, for $d >> 0$, the standard inclusion induces an isomorphism of cohomology groups (with every choice of the ring of coefficients R) $H^k(\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)) \to H^k(\overline{\mathrm{Conf}}_n(\mathbb{R}^d))$. Hence, there is a well-defined Thom class of X in the cohomology of $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$.

Furthermore, note that the free $\mathcal{C}_\infty$-space over S^0, $C(S^0)$, is the disjoint union $\{*\} \cup \coprod_{n=0}^{\infty} B(W_{A_n})$. Thus, its mod 2 homology $H_*(C(S^0);\mathbb{F}_2)$, is the dual graded module $A_A^\vee$ of A_A. Let $*$ be the product of $A_A^\vee$ dual to the coproduct. This algebra has been described in the previous chapter as the free algebra over the Dyer–Lashof algebra. Hence, by Theorem 31, it has an additive basis given by products $(q^{I_1})^{a_1} * \cdots * (q^{I_k})^{a_k}$, where $q^{I_j} = Q^{I_j}(\iota)$, for some strongly admissible sequences I_j, ι being the class of the point in S^0 different from the basepoint.

Definition 35. [13] Let $k \geq 0$ and $m \geq 1$. Let $X_{k,m}$ be the submanifold of $\overline{\mathrm{Conf}}_{m2^k}(\mathbb{R}^\infty)$ consisting of configurations that can be partitioned into m groups of 2^k points sharing the first coordinate. We define $\gamma_{k,m}$ as the Thom class of $X_{k,m}$ in $H^{m(2^k-1)}(W_{A_{m2^k-1}};\mathbb{F}_2) \leq A_A$.

Definition 36. [13] A *gathered block*, or simply a block, in A_A is a product of the form $b = \gamma_{k_1,m_1} \cdot \cdots \cdot \gamma_{k_r,m_r}$, where the elements γ_{k_1,m_1} lie all in the same component.

The *profile* of b is the sequence $(k_1, \ldots, k_r)$, where the k_j are put in non-decreasing order.

A *gathered monomial*, or a *Hopf monomial*, is a transfer product of the form $x = b_1 \odot \cdots \odot b_s$, where $b_1, \ldots, b_s$ are gathered block with pairwise different profiles.

Theorem 37. *[13] Consider the cohomology with coefficients in $\mathbb{F}_2$. For all $k \geq 1$ and $m \geq 0$, $\gamma_{k,m}$ is linear dual to the monomial $q^{(2^{k-1},\ldots,2,1)}$ with respect to the Nakaoka monomial basis in homology. The coproduct of these classes satisfies*

$$\Delta(\gamma_{k,m}) = \sum_{i=0}^{m} \gamma_{k,i} \otimes \gamma_{k,m-i},$$

with the notational convention that $\gamma_{k,0} = 1$, and S acts on them as the identity.

Moreover, A_A is generated, as a Hopf ring, by the set $\{\gamma_{k,m}\}_{k\geq 0, m\geq 1}$ with the following relations:

- $\gamma_{k,m} \odot \gamma_{k,n} = \binom{n+m}{n} \gamma_{k,n+m}$
- *the cup product of generators that lie in different components is zero*

Furthermore, Hopf monomials define an additive basis for A_A as a vector space over $\mathbb{F}_2$.

The authors also calculate the Steenrod algebra action and the restriction of classes to the Dickson algebras.

2.2 Mod p description

We describe here the mod p analog of Giusti, Salvatore and Sinha's result. Note that in Theorem 37 there are no non-trivial relations involving the cup product. This is a reflection of the fact that the dual of the mod 2 Dyer–Lashof algebra is a polynomial algebra. This is not true for $p > 2$. The dual of the mod p Dyer-Lashof algebra has been calculated in Cohen–May–Lada [8], and some additional cup product relations appear. These relations "propagate" by means of Hopf ring distributivity and give rise to all the additional identifications.

First, we define our basic classes and relations. From now on, unless otherwise stated, in this chapter cohomology is taken with coefficients in $\mathbb{F}_p$, the field with p elements, p being an odd prime.

Definition 38. Let the symbol $^\vee$ denote the linear dual with respect to the Nakaoka monomial basis of $A_A^\vee \cong H_*(C(S^0);\mathbb{F}_p)$. We define some classes:

- $\alpha_{j,k} = \left[Q^{p^{k-1}-p^{k-1-j}} \circ \cdots \circ Q^{p^j-1} \circ \beta Q^{p^{j-1}} \circ \cdots \circ Q^p \circ Q^1(\iota)\right]^\vee$
- $\beta_{j,k,m} = \left[\left(\beta Q^{p^{k-1}-p^{k-1-j}} \circ \cdots \circ Q^{p^j-1} \circ \beta Q^{p^{j-1}} \circ \cdots \circ Q^1(\iota)\right)^{*^m}\right]^\vee$
- $\gamma_{k,m} = \left[\left(Q^{p^{k-1}} \circ \cdots \circ Q^p \circ Q^1(\iota)\right)^{*^m}\right]^\vee$

Note that $\alpha_{j,k}$ has odd cohomological degree in A_A, while the degrees of $\beta_{j,k,m}$ and $\gamma_{k,m}$ are even. Recall that passing from the upper-indexed Dyer–Lashof operations to the lower-indexed Kudo–Araki operations and vice-versa is easy. For example, $\gamma_{k,m}$ is the linear dual to

$$(-1)^{km} Q_{2(p-1)}^{\circ^k}(\iota)^{*^m}.$$

Similarly, the linear duals of $\alpha_{j,k}$ and $\beta_{j,k,m}$ can be written as non-zero multiples of the elements in the form

$$Q_{p-1}^{\circ^{k-j}} \circ Q_{2p-3} \circ Q_{2(p-1)}^{\circ^{j-1}}(\iota) \qquad \text{and}$$
$$\left[Q_{p-2} \circ Q_{p-1}^{\circ^{j-i-1}} \circ Q_{2p-3} \circ Q_{2(p-1)}^{\circ^{i-1}}(\iota)\right]^{*^m}.$$

Lemma 39. *The classes defined above satisfy the following cup product relations:*

1. $\alpha_{i,k}\alpha_{j,k} = \gamma_{k,1}\beta_{i,j,p^{k-j}}$ *if* $i < j$.
2. $\beta_{i,j,p^{k-j}}\alpha_{l,k} = (-1)^\rho \beta_{\rho(i),\rho(j),p^{k-\rho(j)}}\alpha_{\rho(l),k}$ *if* i,j,l *are pairwise distinct, where* ρ *is a permutation of the indexes* i,j,l *such that* $\rho(i) < \rho(j)$, *while* $\beta_{i,j,p^{k-j}}\alpha_{l,k} = 0$ *if* i,j,l *are not pairwise distinct.*
3. $\beta_{i,j,m}\beta_{i',j',m'} = [(-1)^\rho]^m \beta_{\rho(i),\rho(j),mp^{j-\rho(j)}}\beta_{\rho(i'),\rho(j'),m'p^{j'-\rho(j')}}$ *if we suppose that* $mp^j = m'p^{j'}$ *and that* i,j,i',j' *are pairwise distinct, where* ρ *is a permutation of the indexes* i,j,i',j' *such that* $\rho(i) < \rho(j)$ *and* $\rho(i') < \rho(j')$, *while* $\beta_{i,j,m}\beta_{i',j',m'} = 0$ *otherwise.*

Example 40. Let us first provide a very simple example to show how the previous relations work. In $H^*\left(\Sigma_{p^2};\mathbb{F}_p\right)$ relation (1) reduces to:

$$\alpha_{2,1}\alpha_{2,2} = \gamma_{2,1}\beta_{1,2,1}$$

Instead, since we do not have three distinct indexes in $\{1,2\}$, the relations in form (2) can be written as $\beta_{1,2,1}\alpha_{1,2} = 0$ and $\beta_{1,2,1}\alpha_{2,2} = 0$. Similarly, (3) only assures that $\beta_{1,2,1}^2 = 0$.

For $H^*\left(\Sigma_{p^3};\mathbb{F}_p\right)$ the relations that can be obtained by 39 are:

- $\alpha_{1,3}\alpha_{2,3} = \gamma_{3,1}\beta_{1,2,p}$, $\alpha_{1,3}\alpha_{3,3} = \gamma_{3,1}\beta_{1,3,1}$ and $\alpha_{2,3}\alpha_{3,3} = \gamma_{3,1}\beta_{2,3,1}$.
- $\beta_{1,2,p}\alpha_{1,3} = \beta_{1,2,p}\alpha_{2,3} = \beta_{1,3,1}\alpha_{1,3} = \beta_{1,3,1}\alpha_{3,3} = \beta_{2,3,1}\alpha_{2,3} = \beta_{2,3,1}\alpha_{3,3} = 0$.
- $\beta_{1,2,p}\alpha_{3,3} = -\beta_{1,3,1}\alpha_{2,3} = \beta_{2,3,1}\alpha_{1,3}$;
- $\beta_{1,2,p}^2 = \beta_{1,2,p}\beta_{1,3,1} = \beta_{1,2,p}\beta_{2,3,1} = \beta_{1,3}^2 = \beta_{1,3,1}\beta_{2,3,1} = \beta_{2,3}^2 = 0$.

In order to prove Lemma 39, we exploit the calculations of Theorem I.3.7 in May–Cohen–Lada [8]. The cited result states that there is a vector space decomposition of the Dyer–Lashof algebra $\mathcal{R}_p = \bigoplus_{k=0}^{\infty}\mathcal{R}_p[k]$, where $\mathcal{R}_p[k]$ is the span os the operations of length k $\beta^{\varepsilon_1}Q^{i_1}\circ\cdots\circ\beta^{\varepsilon_k}Q^{i_k}$, and that the desired relations hold in $\bigoplus_{k=0}^{\infty}\mathcal{R}_p[k]^\vee$. This implies that the two members of each desired equality are indeed equal when paired with elements of the form q^I. Since these homology classes account for all the indecomposables in $A_A^\vee$, the lemma now follows from the bialgebra structure of $A_A^\vee$, with the product $*$ dual to the coproduct in A_A and the coproduct $\Delta.$ dual to the cup product.

In order to describe the structure of A_A with the transfer product alone, we need to recall that there are "pinch" maps $v_n\colon S^n\to S^n\vee S^n$ defined by collapsing a maximal $n-1$-subsphere to a point. The suspension of v_n is v_{n+1}, thus they define what is called a stable map (at the level of spectra). Since the Dyer–Lashof operations commute with the suspension homomorphism, as proved in [8], this stable map induces a morphism in homology $v_*\colon H_*(C(S^0);\mathbb{F}_p)\to H_*(C(S^0\vee S^0);\mathbb{F}_p)\cong H_*(C(S^0);\mathbb{F}_p)\times H_*(C(S^0);\mathbb{F}_p)$. This morphism coincides with the coproduct $\Delta_\odot$ dual to the transfer product on A_A. The proof of this fact is straightforward and can be found, for example, in [6]. It is easily seen that $v_*(\iota) = \iota\otimes 1 + 1\otimes\iota$. All this information together implies the following proposition.

Proposition 41. *Let I be an admissible sequence. Then q^I is primitive in $A_A^\vee$, with respect to $\Delta_\odot$.*

Note that this is the same argument used in [13] for Theorem 4.13.

By a simple dualization process, we can now fully describe the Hopf algebra structure A_A with the transfer product and the coproduct. In order to do this in an efficient way, we use the notion of divided powers Hopf algebra, borrowed from some classical papers, for example from [22] or [2].

Definition 42. Let A be a graded algebra over a field k. Let $A_+ = \bigoplus_{n>0} A_n$ be the positively graded part of A. A *divided powers structure* on A is a sequence of k-linear maps $\{\_^{[n]}\colon A\to A\}_{n\in\mathbb{N}}$ that satisfy the following conditions:

- $\forall n, d \in \mathbb{N} : (A_d)^{[n]} \subseteq A_{nd}$
- $\forall x \in A_+ : x^{[0]} = 1$ and $x^{[1]} = x$
- $\forall n, m \in \mathbb{N} : x^{[n]}x^{[m]} = \left(\begin{array}{c} n+m \\ n \end{array} \right) x^{[n+m]}$
- $\forall n \in \mathbb{N} : \forall x, y \in A_+ : (x+y)^{[n]} = \sum_{r=0}^{n} x^{[r]}x^{[n-r]}$
- $\forall n \in \mathbb{N} : \forall x, y \in A_+ : (xy)^{[n]} = x^n y^{[n]}$
- $\forall n, m \in \mathbb{N} : \forall x \in A_+ : (x^{[n]})^{[m]} = \left(\frac{(nm)!}{(n!)^m m!} \right) x^{[nm]}$
- $\forall n \geq 2 : \forall x \in A_{2d+1} : x^{[n]} = 0$

A *divided powers algebra* over k is the datum $(A, \{\gamma_n\}_n)$ of a graded algebra A over k and a divided powers structure $\{\_^n\}_n$ on A. A morphism of divided powers algebras is a morphism of graded algebras that commutes with $\_^{[n]}$ for all $n \in \mathbb{N}$.

Note that, if $(A, \{\_^n\}_n)$ and $(B, \{\_^n\}_n)$ are divided powers algebras over a field k, there is a unique divided powers structure $\{\_^n\}_n$ on the tensor product $A \otimes B$ such that the natural inclusions $A \hookrightarrow A \otimes B$ and $B \hookrightarrow A \otimes B$ are morphisms of divided powers algebras. Explicitly, we have:

$$\forall x \in A_+ : \forall y \in B : (x \otimes y)^{[n]} = x^{[n]} \otimes y^n$$
$$\forall x \in A : \forall y \in B_+ : (x \otimes y)^{[n]} = x^n \otimes y^{[n]}$$

Definition 43. A *divided powers Hopf algebra* is a graded Hopf algebra A, with a divided powers structure $\{\_^n\}_n$ on A as an algebra, such that the coproduct $\Delta \colon A \to A \otimes A$ is a morphism of divided powers algebras.

Proposition 44. *$(A_A, \odot, \Delta)$ is the free divided powers Hopf algebra generated by the set of $\{(q^I)^\vee\}_I$ indexed by strongly admissible I. The divided powers operations satisfy the formulas:*

$$\left((q^I)^\vee\right)^{[n]} = \left((q^I)^{*^n}\right)^\vee$$

More explicitly, $(A_A, \odot)$ is isomorphic to the tensor product

$$\bigotimes_{\substack{\dim(Q^I)\ even \\ k \in \mathbb{N}}} \frac{\mathbb{F}_p[(Q^I(\iota)^{p^k})^\vee]}{\left([(Q^I(\iota)^{p^k})^\vee]^p\right)} \otimes \bigwedge \left(\{Q^I(\iota)^\vee\}_{\dim(Q^I)\ odd}\right).$$

Note, in particular, that, if $n \in \mathbb{N}$, $\gamma_{k,n} = \gamma_{k,1}^{[n]}$, $\beta_{i,j,n} = \beta_{i,j,1}^{[n]}$ and, for $n \geq 2$, $\alpha_{j,k}^{[n]} = 0$. Thus we obtain, as a direct consequence of this divided powers algebra structure, some formulas involving the transfer product and the coproduct.

Corollary 45. *The following relations hold in A_A:*

4. $\beta_{i,j,m} \odot \beta_{i,j,n} = \left(\begin{array}{c} n+m \\ m \end{array} \right) \beta_{i,j,n+m}$

5. $\gamma_{k,m} \odot \gamma_{k,n} = \begin{pmatrix} n+m \\ m \end{pmatrix} \gamma_{k,n+m}$

Moreover, the coproduct of our generating classes satisfies the identities:

- $\Delta(\alpha_{j,k}) = \alpha_{j,k} \otimes 1 + 1 \otimes \alpha_{j,k}$
- $\Delta(\beta_{i,j,n}) = \sum_{r=0}^{n} \beta_{i,j,r} \otimes \beta_{i,j,n-r}$
- $\Delta(\gamma_{k,n}) = \sum_{r=0}^{n} \gamma_{k,r} \otimes \gamma_{k,n-r}$

These relations, together with the trivial one obtained by imposing that the cup product of elements in different components is 0, suffice to describe A_A as a Hopf ring.

Theorem 46. *As a graded commutative Hopf ring, A_A is generated by the elements $\alpha_{j,k}$, $\beta_{i,j,m}$ and $\gamma_{k,m}$ as defined above under the relations stated in Lemma 39 and in Corollary 45, together with the following:*

6. *The product $\cdot$ between two generators belonging to different components is* 0.

Moreover, the antipode is the multiplication by $(-1)^n$ on the component corresponding to $\Sigma_n = W_{A_{n-1}}$.

First, we need to understand what the Hopf ring generated by that set of generators with the relations stated in the previous theorem looks like. We will call it $\tilde{A}_A$. Consider the natural adaptations of the notions of gathered blocks and Hopf monomials in the mod p case.

Definition 47. A *gathered block*, or simply a *block*, in A_A is a product of the form $b = \gamma_{k_1,m_1} \dots \gamma_{k_r,m_r} \cdot \beta_{i_1,i_2,n_1} \dots \beta_{i_{2s-1},i_{2s},n_s} \cdot \alpha_{j,l}^{\varepsilon}$, where $\varepsilon \in \{0,1\}$, $i_1 < \cdots < i_{2s}$, $i_{2s} < j$ if $\varepsilon = 1$ and all the generators appearing lie in the same component.

The *profile* of b is the $(r+1)$-tuple $(k_0, k_1, \dots, k_r)$, where $k_1, \dots, k_r$ are put in non-decreasing order, and $k_0 = 2s - \sum_{a=1}^{2s} 2p^{-i_a} + \varepsilon(2 - 2p^{-j} - p^{-l})$.

A *gathered monomial*, or *Hopf monomial*, is a transfer product of the form $x = b_1 \odot \cdots \odot b_t$, where $b_1, \dots, b_t$ are gathered blocks with pairwise distinct profiles.

Note that the profile of a block $b = \gamma_{k_1,m_1} \dots \gamma_{k_r,m_r} \cdot \beta_{i_1,i_2,n_1} \dots \beta_{i_{2s-1},i_{2s},n_s} \cdot \alpha_{j,l}^{\varepsilon}$ determines uniquely (up to permutation of $k_1, \dots, k_r$) both the sequences $(k_1, \dots, k_r)$, and $(i_1, \dots, i_{2s})$ if $\varepsilon = 0$ or $(i_1, \dots, i_{2s}, j, k)$ if $\varepsilon = 1$. Thus, a block that has the same profile of b must necessarily be written in the form $b' = \gamma_{k_1,m_1'} \dots \gamma_{k_r,m_r'} \cdot \beta_{i_1,i_2,n_1'} \dots \beta_{i_{2s-1},i_{2s},n_s'} \cdot \alpha_{j,l}^{\varepsilon}$. Moreover, observe that, given a block b, there exists a unique block $\tilde{b}$, among those with the same profile of b, that minimize its component. Note that the component of $\tilde{b}$ is a power of p and that $b = \tilde{b}$ if and only if b is primitive.

Now, observe that relation (3) guarantee that a cup product of generators $\beta_{j_1,j_2,n} \cdot \beta_{j_3,j_4,m}$, not necessarily satisfying $j_1 < j_2 < j_3 < j_4$, is either 0 or can be written, up to a sign, as a gathered block $\beta_{i_1,i_2,n'} \cup \beta_{i_3,i_4,m'}$, with $i_1 < i_2 < i_3 < i_4$. By an evident induction argument on the number s of factors, we immediately see that every arbitrary product $\beta_{j_1,j_2,m_1} \dots \beta_{j_{2s-1},j_{2s},m_s}$, without the condition $j_1 < \cdots < j_{2s}$, is a multiple of a gathered block of the form $\beta_{i_1,i_2,m'_1} \dots \beta_{i_{2s-1},i_{2s},m'_s}$. By a further application of relations (1) and (2), we can write an arbitrary cup product of elements of the form $\beta_{i,j,n}$ and $\alpha_{k,l}$ as a scalar multiple of a gathered block. Thus, gathered blocks span the module generated by arbitrary cup products of our generators.

Furthermore, note that our Hopf ring surjects onto $\bigoplus_{n\geq 0} \mathcal{R}_p[k]^\vee$, the direct sum of the duals of the components of the Dyer–Lashof algebra. Gathered blocks lying in components indexed by powers of p must be linearly independent, since, as we already observed, their images via this surjection are the linear duals of the strongly admissible Dyer–Lashof operations in $\mathcal{R}_p[k]$ and, as a consequence, are linearly independent.

By Hopf distributivity, every element of $\tilde{A}_A$ can be written as a transfer product of blocks $x = b_1 \odot \cdots \odot b_r$, thus $\tilde{A}_A$, as an algebra with the transfer product $\odot$, is generated by gathered blocks. Moreover, note that we can further shrink this set by taking only blocks that lie in a component indexed by powers of p, as we explain below.

Claim. *Let b_1 and b_2 be gathered blocks with the same profile. Let $\tilde{b_1} = \tilde{b_2}$ be the minimal block with the same profile as described above. Assume that $\tilde{b_1}$ belong to the component indexed by p^k, b_1 to that indexed by mp^k, and b_2 by np^k. Let b_3 be the (unique) gathered block with the same profile of b_1 and b_2 belonging to the component $(n+m)p^k$. Then the following divided powers formula holds:*

$$b_1 \odot b_2 = \binom{n+m}{n} b_3$$

Proof of the Claim. First, note that if the class $\alpha_{j,k}$ for some j and k appears in b_1, then $b_2 = b_1 = \tilde{b_1}$, because there are no other blocks with the same profile of b_1. Hence, in this case, the assertion is trivial and thus we can assume, without loss of generality, that b_1 and b_2 are obtained by cup-multiplying classes of the form $\gamma_{*,*}$ and $\beta_{*,*,*}$. We proceed by induction on the number of these cup-product generators appearing in b_1:

- If b_1 (and, as a consequence, b_2) consists of a single generator, then the formula follows from relations 4 and 5.

- Otherwise, we write $b_1 = \gamma_{k,n} b'_1$, $b_2 = \gamma_{k,m} b'_2$, and $b_3 = \gamma_{k,n+m} b'_3$, or $b_1 = \beta_{j,k,n} b'_1$, $b_2 = \beta_{j,k,m} b'_2$ and $b_3 = \beta_{j,k,n+m} b'_3$ for some $1 \leq j < k$, where b'_1, b'_2, and b'_3 are blocks with the same profile and a lower number of cup-product generators. Thus, by induction hypothesis, we have that

$$b'_1 \odot b'_2 = \binom{n+m}{n} b'_3.$$

By applying the Hopf distributivity relation we obtain:

$$\begin{aligned}
b_3 &= \gamma_{k,n+m} \cdot b'_3 \\
&= \binom{n+m}{n} \gamma_{k,n+m} \cdot (b'_1 \odot b'_2) \\
&= \binom{n+m}{n} \sum_{i=0}^{n+m} (\gamma_{k,i} \cdot b'_1) \odot (\gamma_{k,n+m-i} \cdot b'_2) \\
&= \binom{n+m}{n} (\gamma_{k,n} \cdot b'_1) \odot (\gamma_{k,m} \cdot b'_2) \\
&= \binom{n+m}{n} b_1 \odot b_2
\end{aligned}$$

Here, in the second-to-last equality, we used the fact that, by relation 6, many addends are zero because they arise as cup products of elements belonging to different components.

□

Now, let b be a block belonging to the component indexed by np^k, with p^k being the component of $\tilde{b}$. Let $n = \sum_{i=0}^{N} a_i p^i$ be the p-adic expansion of n. For all $i \geq 0$, let b_i be the unique gathered block lying in the component p^{k+i} with the same profile of b and $\tilde{b}$. Due to our previous claim we have:

$$b = \left(\frac{n!}{\prod_{i=0}^{N} (p^i)!^{a_i}} \right) \bigodot_{i=0}^{N} b_i^{\odot^{a_i}}$$

By Kummer's theorem on divisibility of binomial coefficients, the coeffient appearing above is non-zero. Thus, $\tilde{A}_A$ is generated, as an algebra under $\odot$, by gathered blocks belonging to components indexed by powers of p. In particular, the free graded commutative algebra

$$C = \bigotimes_{\substack{d,k \geq 0 \\ b \in H^{2d}\left(\Sigma_{p^k};\mathbb{F}_p\right) \text{ block}}} \left(\frac{\mathbb{F}_p[b]}{b^p} \right) \otimes \bigwedge \left(\left\{ b \text{ block} : b \in H^{2d+1}\left(\Sigma_{p^k};\mathbb{F}_p\right) \right\} \right)$$

generated by these classes surjects onto $\tilde{A}_A$ and, by composition, we have an algebra map $\varphi \colon C \to A_A$. φ is a morphism of divided powers algebras by constructions and, as already noted, the image in A_A of the generating vector space $\bigoplus_k \operatorname{Span}\{b \in H^*(\Sigma_{p^k};\mathbb{F}_p) \text{ block}\}$ pairs perfectly with the module of indecomposables $\bigoplus_k \operatorname{Span}\{q_I : I \text{ str. adm.}\}$.

Theorem 46 now follows from the following, almost immediate, fact.

Claim. *Let k be a field and let V, W be a graded vector spaces of finite type over k. Let $A(V)$ be the free divided powers algebra generated by V and $U(W)$ be the free graded commutative Hopf algebra primitively generated by W. Let $\varphi \colon V \to U(W)^\vee$ be a vector space map such that the corresponding bilinear pairing $V \times W \to k$ is perfect. Then the natural extension of φ defines an isomorphism $A(V) \to U(W)^\vee$.*

Proof of the claim. Fix bases $\mathcal{B}_V$ and $\mathcal{B}_W$ of V and W respectively. Consider the submodule $F^r A(V)$ of $A(V)$ spanned by elements of the form $\prod_{i=1}^s v_i^{[a_i]}$ with $v_i \in \mathcal{B}_V$, $a_i \geq 1$, and $s \leq r$. Let $F_r U(W)$ be the submodule of $U(W)$ spanned by elements of the form $\prod_{i=1}^s w_i^{a_i}$ where $w_i \in \mathcal{B}_W$, $a_i \geq 1$, and $s \leq r$. $F^* A(V)$ and $F_* U(W)$ define increasing filtration whose unions are $A(V)$ and $U(W)$ respectively. An easy induction argument shows that, under our assumptions, $F^r A(V)$ pairs perfectly with $F_r U(W)$, thus the map $A(V) \to U(W)^\vee$ is injective. Since $A(V)$ and $U(W)$ are algebras of finite type and this algebra morphism preserves grades, it is also surjective because the dimensions agree. □

As a byproduct of our proof for Theorem 46 we can describe an additive basis for A_A as an $\mathbb{F}_p$-vector space.

Corollary 48. *The set $\mathcal{M}$ of all Hopf monomials $\bigodot_{i=1}^r b_i$ with the property that the gathered blocks b_i have pairwise distinct profiles is a bigraded basis for A_A as an $\mathbb{F}_p$-vector space.*

To the author's knowledge, the duality pairing between this basis and the Nakaoka monomial basis has not been understood completely. This is caused by the fact that, in order to evaluate a gathered block b on an indecomposable class q^I, we need to apply the coproduct $\Delta.$ on $A_A^\vee$ dual to the cup product on q^I. This coproduct is known (see [8]) to satisfy $\Delta.q^I = \sum_{J+K} q^J \otimes q^K$, where the sum varies on all the tuples J and K whose componentwise sum is I, not necessarily admissible. Thus, the evaluation of b relies on a recursive application of the Adem relations. For example, in the case $p = 3$, $\gamma_{1,3}^4 = \left(Q^8 \circ Q^4 (\iota)\right)^\vee - \left(Q^9 \circ Q^3 (\iota)\right)^\vee$, because the formula for the coproduct $\Delta.$ of $Q^9 \circ Q^3 (\iota)$ yields an addend $Q^3 \circ Q^0 (\iota) \otimes Q^6 \circ Q^3 (\iota)$, that can be written as $-Q^2 \circ Q^1 (\iota) \otimes Q^6 \circ Q^3 (\iota)$.

2.3 Graphical description

The additive basis introduced in the previous section is best understood via a graphical description, similar to what is done in [13]. First, we associate to each Hopf ring generator a rectangle:

- $\gamma_{k,n}$ as a hollow rectangle of width np^k and height $2\left(1 - p^{-k}\right)$.
- $\beta_{j,k,n}$ as a solid rectangle of width np^k and height $2\left(1 - p^{-j} - p^{-k}\right)$.
- $\alpha_{j,k}$ as a solid rectangle of width p^k and height $2\left(1 - p^{-j}\right) - p^{-k}$.

Note that this association has been chosen in order to satisfy the following conditions:

- the area of the rectangle is the cohomological dimension of the corresponding generator
- its width accounts for the component in which the generator lies
- hollow rectangles represent generators whose linear duals in the Nakaoka basis lie in the subalgebra of H generated by sequences of Dyer–Lashof operations $Q^{i_1} \circ \cdots \circ Q^{i_k} (\iota)$ without the Bockstein

- solid rectangles account for generators dual to classes in which the Bockstein homomorphism appears, and their height coincide with the first entry of their profile

The hollow rectangles and their cup products behave very similarly to the generators of the mod 2 case. In terms of the Kudo–Araki lower-indexed operations, they are dual, up to sign, to $Q_{j_1} \circ \cdots \circ Q_{j_k}(\iota)$ where every j_l is even. As will be evident later, the solid rectangles account for the Bockstein part of a gathered block. The cup product of generators of the form $\beta_{j,k,n}$ or $\alpha_{j,k}$ is described by solid boxes whose height is equal to their profile.

We now need to compose these rectangles by using $\cdot$ and $\odot$. Basically, cup product is understood as stacking boxes one on top of the other, while transfer product is understood as placing them next to each other horizontally. Hence, a gathered block is represented as a column made of rectangles all of the same width, and a Hopf monomial is represented as a diagram obtained by juxtaposing columns. In order to conform to the notation used in [13], we will call these objects *skyline diagrams*. Some examples of skyline diagrams are depicted in Figure 2.3 below.

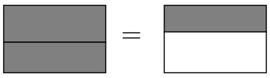
With the aid of skyline diagrams, we can elucidate the formulas in Lemma 39. First, observe that the rectangles of a column associated with a gathered block must satisfy some necessary condition. For example, there must be at most one odd-dimensional (solid) generating rectangle. We will call admissible the columns corresponding to actual gathered blocks. Relation (3) guarantees that the cup product of two even-dimensional solid boxes is, up to sign, equal, if existing, to the unique entirely solid admissible column whose width is equal to the width of the boxes and whose height is the sum of the heights of the two blocks, and is 0 otherwise. An analogous consideration can be made for the cup product of an even-dimensional and an odd-dimensional solid rectangles via relation (2). In particular, a gathered block is completely determined by its hollow part and the overall dimensions of its solid part. This corresponds to the fact that the first entry of the profile of a block completely determines its "β" and "α" part. Relation (1) allows us to replace the cup product of two odd-dimensional solid boxes with two boxes, one hollow and one solid, of suitable dimensions. As an example, we depict the graphical representation of relation 1 in the first example of Figure 2.3. Note that this gives a simple algorithm to write a non-admissible column as a multiple of an admissible one, which is the graphical counterpart of our observations regarding Theorem 46:

- if there are two solid odd-dimensional rectangles, replace them with a hollow rectangle of height $2\left(1 - l^{-1}\right)$ and another solid rectangle to match the column's height

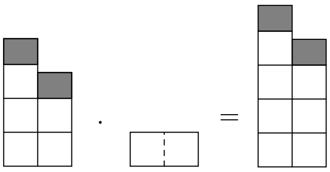
- if there is at most one solid odd-dimensional rectangle and no admissible all-solid column of the same width and height of the solid part exists, then the column is 0

- otherwise, it is equal, up to sign, to the (necessarily unique) admissible column with the same dimensions of the solid and hollow parts

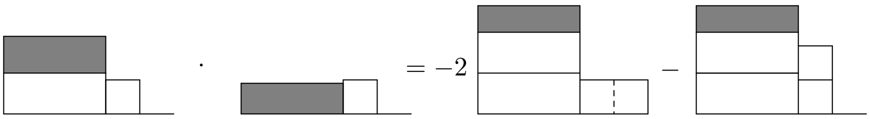
With the skyline diagrams, one can describe the products and the coproduct graphically. Regarding the coproduct, if we have a single hollow rectangle



$\alpha_{1,2}\alpha_{2,2} = \gamma_{2,1}\beta_{1,2,1}$



$\left(\gamma_{1,1}^{i}\alpha_{1,1} \odot \gamma_{1,1}^{j}\alpha_{1,1}\right) \cdot \gamma_{1,2} = \gamma_{1,1}^{i+1}\alpha_{1,1} \odot \gamma_{1,1}^{j+1}\alpha_{1,1}$



$(\gamma_{2,1}\alpha_{2,2} \odot \gamma_{1,1} \odot 1_p) \cdot (\alpha_{1,2} \odot \gamma_{1,1} \odot 1_p) = -2\gamma_{2,1}^{2}\beta_{1,2,1} \odot \gamma_{1,2} - \gamma_{2,1}^{2}\beta_{1,2,1} \odot \gamma_{1,}^{2}$

Figure 2.2: Examples of calculations using the graphical representation. The size of the rectangles is correct only for $p = 3$, but the same calculations with classes understood to be in different degrees are actually true for every p.

of width np^k and height $2(1 - p^{-k})$ or a solid one of width np^k and height $2(1 - p^{-j} - p^{-k})$, we draw dashed lines that divide it into n equal parts. The coproduct is obtained by splitting the rectangle in two along these vertical lines in all the possible ways and summing everything up. Since Δ form a bialgebra with both $\cdot$ and $\odot$, due to relation 6 the coproduct of a general skyline diagram is obtained by drawing dashed lined inside each box as described before, then splitting the diagram into two along full vertical lines, or dashed lines of full height, and summing all the obtained terms.

The transfer product can be described very easily. Given two Hopf monomials $x = b_1 \odot \cdots \odot b_r$ and $y = b'_1 \odot \cdots \odot b'_s$ in $\mathcal{M}$, $x \odot y$ may have gathered blocks with the same profile. However, the transfer product of two even-dimensional gathered blocks with the same profile can be viewed as a scalar multiple of a block via the formula stated in the claim preceding the proof of Theorem 46. Thus, the transfer product corresponds graphically to placing two skyline diagrams next to each other, merging two columns if all their constituent blocks are even-dimensional and have the same height and multiplying by $\binom{n+m}{n}$, where n and m are the widths of the two columns. Moreover, if an odd-dimensional block appears twice, the transfer product is zero.

Finally, we deal with the cup product of two elements of $\mathcal{M}$. We need a preliminary definition.

Definition 49. Let b be a gathered block in A_A. A *partition* of b is a r-tuple of gathered blocks $(b_1, \dots, b_r)$ with the same profile of b, such that the sum of the components of $b_1, \dots, b_r$ is equal to the component of b.

Suppose we want to cup-multiply two skyline diagrams x and y, x having r columns and y having s columns. If $r > 1$, we must first compute the iterated coproduct $\Delta^{(r)}\colon A_A \to A_A^{\otimes^r}$ of y and then, by applying the Hopf ring distributivity formula, cup-multiplying each factor with the corresponding column of x. If $s > 1$, then we must re-applying the distributivity relation with the roles of x and y swapped. Explicitly, in this case, the combinatorial formula is the following:

$$(b_1 \odot \cdots \odot b_r) \cdot (b'_1 \odot \cdots \odot b'_s) = \sum_{(\mathcal{P},\mathcal{P}')} (-1)^{\varepsilon_{\mathcal{P},\mathcal{P}'}} \bigodot_{j=1}^{s} \bigodot_{i=1}^{r} \left(b_{i,j} b'_{j,i}\right)$$

Here the sum is over all couples $(\mathcal{P}, \mathcal{P}')$ of sets $\mathcal{P} = \{(b_{i,1}, \dots, b_{i,s})\}_{i=1}^{r}$ and $\mathcal{P}' = \{(b'_{i,1}, \dots, b'_{i,r})\}_{i=1}^{s}$, such that $(b_{i,1}, \dots, b_{i,s})$ is a partition of b_i, while $(b'_{i,1}, \dots, b'_{i,r})$ is a partition of b'_i. The number $\varepsilon_{\mathcal{P},\mathcal{P}'}$ is given by:

$$\varepsilon_{\mathcal{P},\mathcal{P}'} = \sum_{\substack{1\le i<j\le s \\ 1\le k\le r}} \dim\left(b'_{i,k}\right) \dim\left(b_{k,j}\right) + \sum_{\substack{1\le h<k\le r \\ 1\le i\le s}} \dim\left(b'_{i,h}\right) \dim\left(b_{k,i}\right)$$

The coefficient $(-1)^{\varepsilon_{\mathcal{P},\mathcal{P}'}}$ is due to the skew-commutativity of the product. This can be translated into the following graphical procedure.

1. Divide the rectangles with vertical dashed lines as explained before.
2. Divide each diagram into columns using both the boundaries of the original rectangles and the vertical dashed lines of full height.
3. Match each column of the first diagram with a column of the second one in all possible ways up to automorphisms, stack the matched columns one on top of the other and place these newly constructed columns side by side to make new diagrams. In case there are odd-dimensional columns, their position might be permuted by some permutation σ. In this case, we multiply the resulting diagram by the sign of σ.
4. These diagrams may contain a couple of columns with the same profiles. In this case, we must use the transfer product formula to merge them. There may also be non-admissible columns, that we must write as a multiple of admissible ones via the previously described algorithm.

Figure 2.3 contains some examples:

- Let $x = \gamma_{1,1}^{i}\alpha_{1,1} \odot \gamma_{1,1}^{j}\alpha_{1,1}$ and $y = \gamma_{1,2}$. Since x is made of two columns of width p, the only splitting of y which can yield a non-trivial addend in the formula for the cup product is $(\gamma_{1,1}, \gamma_{1,1})$. Hence:

$$x \cdot y = \gamma_{1,1}^{i}\alpha_{1,1}\gamma_{1,1} \odot \gamma_{1,1}^{j}\alpha_{1,1}\gamma_{1,1} = \gamma_{1,1}^{i+1}\alpha_{1,1} \odot \gamma_{1,1}^{j+1}\alpha_{1,1}$$

Working graphically, the rectangle corresponding to y should be divided with a dashed line into two equal parts $(\gamma_{1,1})$. Up to automorphisms, there is only one way to match the columns of x with them. Stacking matched columns is equivalent to adding one hollow rectangle of height $2\left(1-p^{-1}\right)$ to each column of x.

- Let $x = \gamma_{2,1}\alpha_{2,2} \odot \gamma_{1,1} \odot 1_p$ and $y = \alpha_{1,2} \odot \gamma_{1,1} \odot 1_p$. The only two partitions of x that can yield a non-trivial addend in the cup product are $(\gamma_{2,1}\alpha_{2,2}, \gamma_{1,1}, 1_p)$ and $(\gamma_{2,1}\alpha_{2,2}, 1_p, \gamma_{1,1})$. Thus, by our formula:

$$\begin{aligned} x \cdot y &= \gamma_{2,1}\alpha_{2,2}\alpha_{1,2} \odot \gamma_{1,1}^2 \odot 1_p + \gamma_{2,1}\alpha_{2,2}\alpha_{1,2} \odot \gamma_{1,1} \odot \gamma_{1,1} \\ &= -\gamma_{2,1}^2\beta_{1,2,1} - 2\gamma_{2,1}^2\beta_{1,2,1} \odot \gamma_{1,2} \end{aligned}$$

  Graphically, there are two possible matches of the columns of x and y because we only need to ensure that the two largest columns match together. When we stack the two large columns one on top of the other we obtain a non-admissible column that can be transformed as described in the figure. By stacking the remaining columns in the two possible ways, we obtain the two skyline diagrams on the left. In one diagram, two rectangles with the same height have been merged together, and a coefficient of 2 appears.

2.4 Restriction to modular invariants

We reconcile here with a more classical approach to group cohomology involving elementary abelian subgroups. Recall that an elementary abelian p-subgroup of a group G is a subgroup $A \leq G$ such that each element of $A \setminus \{1\}$ has order p. The set $\mathcal{F}_G$ of elementary abelian subgroups of G with morphisms given by inclusions and conjugation maps forms a small category. Cohomology with coefficients in a certain ring R defines a covariant functor on $\mathcal{F}_G$. The inverse limit $\varprojlim_{A \in \mathcal{F}_G} H^*(A; R)$ is often called the *Quillen group* of G and the map $q_G \colon H^*(G; R) \to \varprojlim_{A \in \mathcal{F}_G} H^*(A; R)$ induced by restrictions is called the *Quillen map* because it was introduced by Quillen in [35], where it is also proved that its kernel and cokernel are nilpotent. General sufficient conditions for q_G to be injective or surjective are known. For a general treatment, we refer to [1] and, for the study of the Quillen map of the symmetric groups, to [20] (mod 2 coefficients) and [21] (coefficients modulo odd primes). We devote this section to some calculation regarding the Quillen map that will be useful when dealing with the action of the Steenrod algebra on A_A.

In the case of the symmetric groups, the elementary abelian subgroups can be described as follows. Let V_n be a vector space on $\mathbb{F}_p$ of dimension n. V_n acts on itself via vector space addition. This realizes V_n as a subgroup of Σ_{p^n}, since V_n, as a set, has cardinality p^n. Every maximal elementary abelian p-subgroup of Σ_k is conjugate to a direct product of V_n. Since the restriction to direct products $\Sigma_r \times \Sigma_s$ is codified by the coproduct map described in the previous sections, the Quillen map is completely determined, in our context, by the restrictions $\rho_n \colon H^*(V_n; \mathbb{F}_p) \to H^*(\Sigma_{p^n}; \mathbb{F}_p)$.

First, recall that $H^*(\mathbb{F}_p; \mathbb{F}_p)$ is $\mathbb{F}_p[y] \otimes \Lambda(x)$, the free commutative $\mathbb{F}_p$-algebra generated by non-zero classes x and y of the first and the second cohomology groups respectively. For technical convenience, we chose x and y such that $\beta(x) = y$, where β is the cohomology Bockstein. Hence, by Künneth's theorem:

$$H^*(V_n; \mathbb{F}_p) = H^*(\mathbb{F}_p; \mathbb{F}_p)^{\otimes^n} = \mathbb{F}_p[y_1, \ldots, y_n] \otimes \Lambda(x_1, \ldots, x_n)$$

From a general result in group cohomology (see Adem and Milgram's book [1, Corollary 1.8 page 182]) it is known that the image of ρ_n is contained in $[\mathbb{F}_p[y_1, \ldots, y_n] \otimes \Lambda(x_1, \ldots, x_n)]^{GL_n(\mathbb{F}_p)}$, the invariant subalgebra with respect to the action of the Weil group of V_n in Σ_{p^n}, that coincides with $\mathrm{Gl}_n(\mathbb{F}_p)$.

For coefficients at odd primes, this subalgebra was calculated by Múi in [31]. Explicitly, this can be identified with $\mathbb{F}_p[d_{0,n}, \ldots, d_{n-1,1}] \otimes M$, where M is the $\mathbb{F}_p$-vector space with basis $\{R_{n,\underline{s}} : 0 \leq s_1 < \cdots < s_l < n\}$ indexed by subsets of $\{0, \ldots n-1\}$. The classes $d_{k,n-k}$ and $R_{n,s_1,\ldots,s_l}$ are constructed by Múi by means of some determinants. More precisely, we can define classes $L_{n,k} = \det \left[y_i^{p^{j-\delta_{j\leq k}}} \right]_{1 \leq i,j \leq n}$ and, by letting $\widehat{\ }$ mean 'omit',

$$M_{n,s_1,\ldots,s_l} = \frac{1}{l!} \det \begin{bmatrix} x_1 & \ldots & x_1 & y_1 & \ldots & \widehat{y_1^{p^{s_1}}} & \ldots & \widehat{y_1^{p^{s_l}}} & \ldots & y_1^{p^{n-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n & \ldots & x_n & y_n & \ldots & \widehat{y_n^{p^{s_1}}} & \ldots & \widehat{y_n^{p^{s_l}}} & \ldots & y_n^{p^{n-1}} \end{bmatrix}.$$

Additionally, we have that $d_{k,n-k} = \frac{L_{n,k}}{L_{n,n}}$ and $R_{n,s_1,\ldots,s_l} = M_{n,s_1,\ldots,s_l} L_{n,k}^{p-2}$. Note that the dimensions of $d_{k,n-k}$ and $R_{n,s_1,\ldots,s_l}$ are equal to $2\left(p^n - p^k\right)$ and to $l + 2\left(p^n - 1 - \sum_{j=1}^{l} p^{s_j}\right)$ respectively. These equalities determine the product structure.

We can slightly paraphrase Múi's presentation by saying that the invariant algebra $[\mathbb{F}_p[y_1, \ldots, y_n] \otimes \Lambda(x_1, \ldots, x_n)]^{GL_n(\mathbb{F}_p)}$ is generated by the classical Dickson invariants, plus some elements $R_{n,s_1,\ldots,s_l}$ whose product is determined by the fact that $d_{0,n}$ is a non-zero divisor and the following equalities:

$$\begin{aligned} R^2_{n,s_1,\ldots,s_l} &= 0 \\ R_{n,s_1} \ldots R_{n,s_l} &= (-1)^{\frac{l(l-1)}{2}} R_{n,s_1,\ldots,s_l} d_{0,n}^{l-1} \end{aligned}$$

These classes have been extensively studied. We will require some Steenrod algebra calculations, that has been determined by Hung and Minh. We recall here the relevant theorem:

Theorem 50. *[24] Let $0 \leq r < p^n$. Let $r = \sum_{i=0}^{n-1} a_i p^i$ be the p-adic expansion of r. Let $a_{-1} = 0$ by convention. Then:*

- *$\mathcal{P}^r(d_{s,n-s})$ is 0 unless $a_i \geq a_{i-1}$ for all $0 \leq i < n$ with $i \neq s$, and $a_s + 1 \geq a_{s-1}$. In this case, it is given by the formula*

$$\mathcal{P}^r(d_{s,n-s}) = \lambda_{r,n,s} \prod_{i=0}^{n-1} d_{i,n-i}^{a_i - a_{i-1} + \delta_{i,s}},$$

  where $\delta_{i,s}$ is equal to 1 if $i = s$ and is 0 otherwise, and the following formula for $\lambda_{r,n,s}$ holds:

$$\lambda_{r,n,s} = \frac{(p-1)!}{(p-1-a_{n-1})! \prod_{1 \leq i \leq n-1, i \neq s} (a_i - a_{i-1})! (a_s + 1 - a_{s-1})!} (a_s + 1)$$

- *$\mathcal{P}^r(R_{n,s})$ is 0 unless $a_i \in \{0,1\}$, $a_i \geq a_{i-1}$ for all $i \neq s$ and $a_s = 0$. This condition is equivalent to $r = (p-1)^{-1}(p^n + p^s - p^{t_1} - p^{t_2})$ for some $t_1 \leq s < t_2 \leq n$. In this case:*

$$\mathcal{P}^r(R_{n,s}) = R_{n,t_1} d_{t_2,n-t_2} - R_{n,t_2} d_{t_1,n-t_1}$$

  Here, we use the convention that $R_{n,n} = 0$ and $d_{n,0} = 1$.

- *$\mathcal{P}^r(R_{n,s_1,s_2})$ is 0 unless $a_i \in \{0,1\}$, $a_i \geq a_{i-1}$ for $i \neq s_1, s_2$ and $a_{s_1} = a_{s_2} = 0$. This condition is equivalent to*

$$r = (p-1)^{-1}(p^n + p^{s_1} + p^{s_2} - p^{t_1} - p^{t_2} - p^{t_3})$$

  for some $t_1 \leq s_1 < t_2 \leq s_2 < t_3 \leq n$. In this case, the following formula holds:

$$\mathcal{P}^r(R_{n,s_1,s_2}) = R_{n,t_1,t_2} d_{t_3,n-t_3} - R_{n,t_1,t_2} d_{t_2,n-t_2} + R_{n,t_2,t_3} d_{t_1,n-t_1}$$

  Again, we agree that $R_{n,s,n} = 0$ and $d_{0,n} = 1$.

We now describe how ρ_n maps our generators in the algebra of modular invariants. The formulas are the following.

Proposition 51. *The following equalities hold:*

$$\begin{aligned}
\rho_{j+k}\left(\alpha_{j,j+k}\right) &= (-1)^j R_{j+k,k} \\
\rho_{j+k}\left(\beta_{i,j,p^k}\right) &= (-1)^{k+i} R_{j+k,k,k+j-i} \\
\rho_{j+k}\left(\gamma_{j,p^k}\right) &= (-1)^j d_{k,j}
\end{aligned}$$

In the proof of this result, we will need the aforementioned Steenrod algebra calculations. We will exploit the very construction of the Steenrod operations to relate $\gamma_{j-1,1}$ to $\gamma_{j,1}$, and this will allow us to inductively compute $\rho_j\left(\gamma_{j,1}\right)$. After that, the usual naturality property of the Steenrod operations will let us work out the remaining cases in a straightforward way. This is a simplified version of an idea used by Mann in his thesis [26] to compute $\mathrm{im}\left(\rho_j\right)$. Thus, to a certain extent, we follow his reasoning, but we are also able to reconcile this approach with the Hopf ring structure and to describe in simpler terms the classes in the cohomology of Σ_{p^j} that restrict to $d_{l,j-l}$, $R_{j,l}$, and $R_{j,l,m}$.

The first part of the proof builds on the following technical lemma.

Lemma 52. *Let $k \in \mathbb{N}$. We define J_k as the k-tuple $(2(p-1), \ldots, 2(p-1))$. Let $J = (j_1, \ldots, j_k)$ be a sequence of non-negative integers (not necessarily admissible). Then, if $Q_J = \sum_{J' \textit{admissible}} \lambda_{J,J'} Q_{J'}$ is the expansion of Q_J as a linear combination of admissible sequences of operations, then $\lambda_{J,J_k} = 0$ unless $J = J_k$.*

Proof. We switch to upper indexes because with this notation the Adem relations can be written in a simpler form. Hence $Q_{J_k} = \pm Q^{p^{k-1}} \circ \cdots \circ Q^p \circ Q^1$. Now consider a sequence of operations in $\mathcal{R}$, in general non-admissible. It is expanded in the admissible basis by applying iteratively the Adem relations. Thus, the lemma will follow immediately, if we are able to check that for every $\beta^\varepsilon Q^r \beta^{\varepsilon'} Q^s$ with $r > ps - \varepsilon'$, when we apply the relevant Adem relation, as stated in Section 1.2, the expression we obtain does not contain an addend in the form $\lambda Q^{p^{l+1}}Q^{p^l}$ for some $\lambda \in \mathbb{F}_p \setminus \{0\}$. This is obvious if the Bockstein homomorphism appears. Hence, we can assume that $\varepsilon = \varepsilon' = 0$. In this case $Q^r \circ Q^s = \sum_i c_i Q^{r+s-i} Q^i$ for some coefficient c_i which are different from 0 only if $pi \geq r$. If there exists $\bar{\imath}$, $c_{\bar{\imath}} \neq 0$, then $r + s - \bar{\imath} = p^{l+1}$, and $\bar{\imath} = p^l$. Hence $r + s = p^{l+1} + p^l$ and $r > ps$ implies $r > p^{l+1}$. This is contradictory because $p\bar{\imath} = p^{l+1} < r$. □

We will also need to notice how the transfer product behaves with respect to the restriction map. Recall that the monomorphism $V_n \hookrightarrow \Sigma_{p^n}$ factors through $\mathbb{F}_p \wr (\mathbb{F}_p \wr \cdots \wr (\mathbb{F}_p \wr \mathbb{F}_p))$. A reference to this fact can be found in [1] at page 185. By construction, Dyer–Lashof operations of length n are exactly the classes arising from the homology of V_n, and these elements pair trivially with any $\odot$-decomposable class. Formally:

Remark 53. If $x_1 \in H^*(\Sigma_r; \mathbb{F}_p)$ and $x_2 \in H^*(\Sigma_{p^n - r}; \mathbb{F}_p)$ are Hopf monomials that are different from 1, then $\rho_n^*(x_1 \odot x_2) = 0$.

Proof of Proposition 51. **Step 1.** $\boldsymbol{\rho_j(\gamma_{j,1}) = (-1)^j d_{0,j}}$

Using lower-indexes, this can be written as $\rho_j\left(Q_{J_j}(\iota)^\vee\right) = d_{0,j}$, where J_j is as in Lemma 52.

Identify $H_*(V_j; \mathbb{F}_p)$ with $H_*(\mathbb{F}_p; \mathbb{F}_p) \otimes H_*(V_{j-1}; \mathbb{F}_p)$. The homomorphism $(\rho_n)_* : H_*(V_n; \mathbb{F}_p) \to H_*(\Sigma_{p^n}; \mathbb{F}_p)$ satisfies, for every $x \in H_s(V_{n-1}; \mathbb{F}_p)$ and for every $r \geq 0$, the following equality:

$$\begin{aligned}(\rho_n)_*(e_r \otimes x) = \nu(s) \sum_k (-1)^k Q_{r+2k-s} \circ \mathcal{P}_*^k(x) \\ - \delta(r)\,\nu(s-1) \sum_k (-1)^k Q_{r+p+(2pk-s)(p-1)} \circ \mathcal{P}_*^k \beta(x)\end{aligned}$$

In the expression above, $\mathcal{P}_*^k$ is the linear dual to the k^{th} Steenrod operation $\mathcal{P}^k$, $\nu(2j+\varepsilon) = (-1)^j \left(\frac{p-1}{2}\right)!^\varepsilon$, $\delta(2j+\varepsilon) = \varepsilon$ if $\varepsilon \in \{0,1\}$. This is essentially a rephrasing of the well-known Nishida relations (see [29, Proposition 9.1 at page 205]) and comes from the dualization of the original construction of $\mathcal{P}^k$ made by Steenrod.

Note that, by Lemma 52, all the addends in the previous formula pair trivially with $Q_{J_j}(\iota)^\vee$, except possibly those in the form $Q_{r+2k-s} \circ \mathcal{P}_*^k(x)$ with $r + 2k - s = 2(p-1)$ and $s - 2k(p-1) = 2\left(p^{j-1} - 1\right)$. This means that $r = \left(p^{j-1} - l\right)(p-1)$ and $s = 2\left(p^{j-1} - 1\right) + 2k(p-1)$. Hence, dually, we have:

$$\rho_j\left(Q_{J_j}(\iota)^\vee\right) = \sum_{k=0}^{p^{j-1}-1} (-1)^k \mathcal{P}^k \rho_{j-1}\left(Q_{J_{j-1}}^\vee\right) y_j^{(p-1)\left(p^{j-1}-1\right)}$$

By induction on j, it is easy to prove that the right member is equal to $d_{0,j}$. Explicitly, for $j = 1$ the statement is trivial. For $j > 1$, by induction hypothesis, $\rho_j\left(Q_{J_j}(\iota)^\vee\right)$ is a $GL_j(\mathbb{F}_p)$-invariant polynomial in $H^*(V_{j-1}; \mathbb{F}_p)[y_j]$ whose leading coefficient is $d_{0,j-1}^p$. This must be $d_{0,j}$.

Step 2. Completing the calculation for the other classes.

The action of the Steenrod algebra on the dual of the n^{th} component of Dyer–Lashof algebra $\mathcal{R}_p[n]$ has been calculated in May–Cohen–Lada [8, Theorem 3.9]. When lifted to $H^*(\Sigma_{p^n};\mathbb{F}_p)$, their formulas are true only modulo non-trivial transfer products. Luckily, due to Remark 53, they are nonetheless enough to determine the composition $\rho_n \circ \mathcal{P}^r$ on the Hopf ring generators of A_A.

The formulas involving $\rho_n(x)$ for $x = \gamma_{n-k,p^k}$ with $k > 0$, $x = \alpha_{j,n}$, and $x = \beta_{i,j,p^{n-j}}$ follow directly from those identities and the naturality of the Steenrod powers with respect to the restrictions ρ_n, with a straightforward comparison that utilizes the Steenrod powers of the modular invariants as described in Theorem 50. □

As a direct consequence of Proposition 51, we can prove that, for odd primes, ρ_n is not surjective.

Corollary 54. *[26] The image of ρ_n in $H^*(V_n;\mathbb{F}_p)^{GL_n(\mathbb{F}_p)}$ is the subalgebra generated by $d_{j,n-j}$, $R_{n,j}$, and $R_{n,i,j}$. This can be described as:*

$$\bigoplus_{l=0}^{n} \bigoplus_{0\leq s_1<\cdots<s_l<n} \mathbb{F}_p[d_{0,n},\ldots,d_{n-1,1}]\, d_{0,n}^{[l/2],0} R_{n,\underline{s}}$$

This is in sharp contrast with the mod 2 case, where it is well-known that the surjectivity holds. This allows a simpler proof of the mod 2 analog of Proposition 51, that we recall here for completeness.

Corollary 55. *[13] In the Hopf ring $\bigoplus_{n=0}^{\infty} H^*(W_{A_{n-1}};\mathbb{F}_2)$, the restriction of $\gamma_{l,2^k}$ with $l+k=n$ to the elementary abelian subgroup V_n is the Dickson class $d_{k,l}$.*

2.5 Steenrod algebra action

We conclude this chapter with the calculation of the Steenrod algebra action on A_A. We will achieve this by using the fact, recalled in the previous section, that the Quillen map for the symmetric groups is injective, and comparing the calculations of the Steenrod powers of the elementary abelian subgroups by means of Proposition 51. This is completely analogous to Section 8 of [13], where the authors worked out these computations for mod 2 cohomology.

First, note that, in addition to the usual Cartan formulas for the cup product and the coproduct, since $\odot$ is induced by a stable map, there is a similar identity for it:

$$\mathcal{P}^k(x\odot y) = \sum_{r=0}^{k} \mathcal{P}^r x \odot \mathcal{P}^{k-r} y$$

Thus A_A is a Hopf ring over the mod p Steenrod algebra $\mathcal{A}(p)$. Hence, in order to describe how $\mathcal{A}(p)$ behave with respect to an arbitrary class in A_A, it is sufficient to determine the action of $\beta^\varepsilon\mathcal{P}^l$ on the Hopf ring generators, where $l \geq 0$ and $\varepsilon \in \{0,1\}$.

First, we give a preliminary definition, borrowed from Giusti–Salvatore–Sinha [13].

Definition 56. (partially from [13])

- The height (ht) of a gathered block b is the number of generators that must be cup-multiplied to obtain b. The height of a Hopf monomial is the largest of the heights of its constituent blocks.
- We define the effective scale (effsc) of a gathered block, that we write in the form $b = \gamma_{l_1,n_1} \cdots \gamma_{l_r,n_r} \beta_{i_1,i_2,m_1} \cdots \beta_{i_{2s-1},i_{2s},m_s} \alpha^{\varepsilon}_{j,k}$ ($\varepsilon = 0, 1$), as the largest of the integers $l_1, \ldots, l_r, i_{2s}$ if $\varepsilon = 0$, or as k if $\varepsilon = 1$. In other words, for $b \in H^*(\Sigma_{p^n}; \mathbb{F}_p)$, $\operatorname{effsc}(x)$ is the minimum $k \geq 0$ such that the restriction of x to $\Sigma_{p^k}^{p^{n-k}}$ is not zero. The effective scale of a Hopf monomial is the minimum of the effective scales of its constituent blocks.
- We say that a Hopf monomial is *full-width* if none of its constituent blocks is 1_{Σ_n}.
- We say that a gathered block is of *Type A* if all the Hopf ring generators that must be cup-multiplied to obtain it are in the form $\gamma_{l,n}$, except one that is in the form $\alpha_{j,k}$. For example, $\gamma^3_{1,p^2}\alpha_{1,3}$ is of Type A. A Hopf monomial is of Type A if all its constituent blocks are of Type A.
- We say that a gathered block is of *Type B* if all the Hopf ring generators that appear in it are in the form $\gamma_{l,n}$, except one in the form $\beta_{i,j,m}$. For example, $\gamma^5_{3,1}\gamma^2_{2,p}\beta_{1,2,3}$ is of Type B. A Hopf monomial is of Type B if all its constituent blocks are of Type B.
- We say that a Hopf monomial is of *Type C* if it is obtained by applying $\cdot$ and $\odot$ only to elements in the form $\gamma_{l,n}$.

These definition can be understood graphically. Given a skyline diagram:

- its height is the maximal number of rectangles stacked one on top of the others that appear in the diagram.
- its effective scale is the width of the thinner column among those delineated by the original boundaries and the vertical dashed lines of full height.
- it is full-width if there are not columns of height 0.
- it is of type A if its columns contain exactly one solid rectangle and it is odd dimensional. It is of type B if its columns contain exactly one solid rectangle and it is even dimensional, while it is of type C if it is made only of hollow rectangles.

We briefly recall the mod 2 treatment contained in [13], as the odd primes case is totally similar. For mod 2 coefficients, we do not make distinctions between Types A,B, and C, but we do retain the notions of height, effective scale, and full-width monomial.

Theorem 57. *(partially from [13])* $\mathrm{Sq}^i(\gamma_{l,2^k})$ *is the sum of all full-width monomials of total degree* $2^k(2^l - 1) + i$*, height less than or equal to* 2 *and effective scale at least* l*.*

We also recall that elementary abelian subgroups detect the cohomology of the symmetric groups. In other words, the Quillen map q_{Σ_n} is injective for all n. In the particular case when $n = p^k$ is a power of p, every maximal elementary abelian p-subgroup of Σ_n is conjugate to V_k or to a subgroup of $\Sigma^p_{p^{k-1}}$. Thus, the restriction map $\rho_n \oplus \tau_n \colon H^*(\Sigma_{p^n}; \mathbb{F}_p) \to H^*(V_n; \mathbb{F}_p) \oplus H^*(\Sigma^p_{p^{n-1}}; \mathbb{F}_p)$ is injective. Hence, in order to compute the action of $\beta^\varepsilon \mathcal{P}^l$ on a cohomology class, it is sufficient to restrict to V_n and $\Sigma^p_{p^{n-1}}$, use Theorem 50 and proceed with an induction argument.

Lemma 58. *$\mathcal{P}^r(\gamma_{n-k,p^k})$ can be expressed as a linear combination of full-width Hopf monomials of Type C with a height of at most p and an effective scale of at least $n-k$.*

We call these elements the *outgrowth monomials* of γ_{n-k,p^k}. We denote the set of such monomials as $\mathrm{Outgr}(\gamma_{n-k,p^k})$.

Proof. We follow the reasoning of [13, Theorem 8.3]. We use an induction argument on k. First of all, let $k = 0$. By Theorem 50 and Proposition 51, $\mathcal{P}^r(\gamma_{n,1})$ must map into 0 on $H^*(\Sigma^p_{p^{n-1}}; \mathbb{F}_p)$ and into $(-1)^n \lambda_{r,n,0} \prod_{i=0}^{n-1} d_{i,n-i}^{a_i - a_{i-1} + \delta_{i,0}}$ on $H^*(V_n; \mathbb{F}_p)$. Hence, it must be a multiple of $\prod_{i=0}^{n-1} \gamma_{n-i,p^i}^{a_i - a_{i-1} + \delta_{i,0}}$, as the injectivity of our restriction map implies that this is the only class in $H^*(\Sigma_{p^n}; \mathbb{F}_p)$ that has the desired property. This is the only full-width Hopf monomial of Type C, of degree $2(p^n - 1) + 2r(p-1)$ with a height of at most p and an effective scale of at least n.

Now let $k > 0$. since $\tau_n(\gamma_{n-k,p^k}) = \gamma_{n-k,p^{k-1}}^{\otimes p}$, using the external Cartan formula, we have:

$$\tau_n(\mathcal{P}^r(\gamma_{n-k,p^k})) = \sum_{r_1 + \dots r_p = r} \mathcal{P}^{r_1}(\gamma_{n-k,p^{k-1}}) \otimes \cdots \otimes \mathcal{P}^{r_p}(\gamma_{n-k,p^{k-1}})$$

By induction, this is a linear combination of elements $x_1 \otimes \cdots \otimes x_p$, where each x_i is an outgrowth monomial of $\gamma_{n-k,p^{k-1}}$.

Because of Künneth's theorem, a basis for $H^*(\Sigma_{p^n}; \mathbb{F}_p)$ can be obtained as tensor products of Hopf monomials. We claim that there is exactly one Hopf monomial $x \in H^*(\Sigma_{p^n}; \mathbb{F}_p)$ of height less than n such that $x_1 \otimes \cdots \otimes x_p$ appears in the expansion of $\tau_n(x)$ with respect to that basis. Explicitly, a gathered block b with a given profile appears in x as a constituent block if at least one x_i contains a constituent block with that profile. The component of b is the sum of the component of such blocks appearing in $x_1, \dots, x_p$.

Note that $\mathrm{effsc}(x) \le n-1$ and $x \in \mathrm{Outgr}(\gamma_{n-k,p^k})$, because the coproduct preserves height and being full-width, and the minimum $\{\mathrm{effsc}(x_i)\}_{i=1}^p$ must be equal to $\mathrm{effsc}(x)$. A Hopf monomial $x \notin \mathrm{Outgr}(\gamma_{n-k,p^k})$ with an effective scale less than n cannot appear in the expression of $\mathcal{P}^r(\gamma_{n-k,p^{k-1}})$ since, when applying the iterated coproduct, this would yield addends in $\tau_n(\mathcal{P}^r(\gamma_{n-k,p^k}))$ that are not tensor products of elements in $\mathrm{Outgr}(\gamma_{n-k,p^{k-1}})$. If a Hopf monomial with an effective scale equal to n appears, this must, once again, be an outgrowth monomial of γ_{n-k,p^k}. Otherwise, by first applying the restriction to $H^*(V_n; \mathbb{F}_p)$ and then calculating $\mathcal{P}^r$ via Theorem 50 we would contradict the naturality of the Steenrod powers. $\square$

Thus, $\mathcal{P}^r(\gamma_{n-k,p^k}) = \sum_{x\in \mathrm{Outgr}\,(\gamma_{n-k,p^k}),\deg(x)=2(p^n-p^k)+2r(p-1)} c_{n,k,x}x$. The last things that we need to calculate are the coefficients $c_{n,k,x}$. Applying the restriction to $H^*(V_n;\mathbb{F}_p)$ via Proposition 51 and comparing the result with the formulas in Theorem 50 directly yield the value of $c_{n,k,x}$ in the case in which $x = \prod_{i=0}^{n-1} \gamma_{n-i,p^i}^{a_i-a_{i-1}+\delta_{i,k}}$ is the unique outgrowth monomial consisting of a single block. Explicitly we have

$$c_{n,k,\prod_{i=0}^{n-1}\gamma_{n-i,p^i}^{a_i-a_{i-1}+\delta_{i,k}}} = (-1)^{n-k+\sum_{i=0}^{n-1}(a_i-a_{i-1}+\delta_{i,s})(n-i)}\lambda_{r,n,k},$$

where $r = \sum_i a_i p^i$.

In general, a Hopf monomial $x = b_1 \odot \cdots \odot b_l \in H^*(\Sigma_{p^n};\mathbb{F}_p)$ is written as the transfer product of l blocks with pairwise distinct profiles. Assume that $b_i \in H^*(\Sigma_{p^{n_i}m_i};\mathbb{F}_p)$ with $\mathrm{effsc}(b_i) = n_i$. Recall that, given a gathered block b, we defined $\tilde{b}$ as the block that minimizes the component among those with the same profile of b. The restriction of x to the cohomology of the direct product $\prod_{i=1}^l \Sigma_{p^{n_i}}^{m_i}$ is the symmetrization of $\tilde{b_1}^{\otimes m_1} \otimes \cdots \otimes \tilde{b_l}^{\otimes m_l}$. Observe that $\gamma_{n-k,p^k}|_{\prod_i \Sigma_{p^{n_i}}^{m_i}} = \otimes_i \gamma_{n-k,p^{k-n+n_i}}^{m_i}$. Hence, by applying the naturality of the Steenrod operations and by using the external Cartan formula, we see that $c_{n,k,x} = \prod_{i=1}^l c_{n-n_i,n_i-n+k,\tilde{b_i}}$. This reduces the computation of $c_{n,k,x}$ to the previous case of a single gathered block.

We summarize the previous computations in the following statement.

Proposition 59. *Let $0 \le k < n$. Let $b = \prod_{i=0}^{n-1}\gamma_{n-i,p^i}^{e_i} \in \mathrm{Outgr}(\gamma_{n-k,p^k})$ be the gathered block with an effective scale of n. We define:*

$$\begin{aligned} c_{n,k,b} &= (-1)^{n-k+\sum_i e_i(n-i)}\lambda_{(p-1)^{-1}\left[\sum_i 2(p^n-p^i)-2(p^n-p^k)\right],n,k} \\ &= (-1)^{n-k+\sum_i e_i(n-i)}\frac{(p-1)!}{(p-\mathrm{ht}(b))!\prod_{i=1}^{n-1} e_i!}\sum_{i=1}^k e_i \end{aligned}$$

Let $x \in \mathrm{Outgr}(\gamma_{n-k,p^k})$ be a general outgrowth monomial. Write, as previously described, $x = b_1 \odot \cdots \odot b_s$, with $b_i \in H^*(\Sigma_{l_i};\mathbb{F}_p)$ that are gathered blocks with pairwise distinct profiles. Define:

$$c_{n,k,x} = \prod_{i=1}^l c^{l_i}_{\mathrm{effsc}(b_i),k-n+\mathrm{effsc}(b_i),b_i'}$$

Then $\mathcal{P}^r(\gamma_{n-k,p^k}) = \sum_{x\in\mathrm{Outgr}(\gamma_{n-k,p^k}),\deg(x)=2(p^n-p^k+r(p-1))} c_{n,k,x}x$.

The calculations for $\mathcal{P}^r(\alpha_{j,k})$ and $\mathcal{P}^r(\beta_{i,j,p^k})$ can be done in the exact same way. The definition of the analogous notion of outgrowth monomials for $\alpha_{i,j}$ and β_{i,j,p^k} is required. We define them as the full-width monomials of height one or two with an effective scale of at least j, of Type A and B respectively. As before, we denote the set of such outgrowth monomials with $\mathrm{Outgr}(\alpha_{i,j})$ and $\mathrm{Outgr}(\beta_{i,j,p^k})$, respectively.

Proposition 60. *Let $1 \le j \le n$. For $x = \gamma_{n-u,p^u}\alpha_{n-t,n} \in \mathrm{Outgr}(\alpha_{j,n})$, we define $c'_{n,j,x} = (-1)^{j+t+u}(\delta_{t\le n-j} - \delta_{u \le n-j})$. Here, we allow $u = n$ with the* *convention that* $\gamma_{0,p^n} = 1$*. Then:*

$$\mathcal{P}^r(\alpha_{j,n}) = \sum_{\substack{x \in \mathrm{Outgr}(\alpha_{j,n}) \\ \deg(x) = 2((p-1)r + p^n - p^{n-j}) - 1}} c'_{n,j,x} x$$

Let $1 \leq i < j \leq n$ *and let* $k = n - j$*. Given a gathered block* $b = \gamma_{n-v,v}\beta_{n-u,n-t,p^t}$ *in* $\mathrm{Outgr}(\beta_{i,j,p^k})$*, define*

$$c''_{n,i,j,b} = (-1)^{i+k+t+u+v}(\delta_{v>k+j-i} - \delta_{u>k+j-i})\delta_{t \leq k+j-i}(\delta_{t \leq k} - \delta_{v \leq k})\delta_{u>k}$$

For a general outgrowth monomial $x = b_1 \odot \cdots \odot b_l$ *with* $b_s \in H^*(\Sigma_{m_s}; \mathbb{F}_p)$ *and* $\mathrm{effsc}(b_s) = n_s$ *for all* $1 \leq s \leq l$*, we define* $c''_{n,i,j,x} = \prod_{s=1}^{l} (c''_{n_s,i,j,b_s})^{m_s}$*. Then:*

$$\mathcal{P}^r(\beta_{i,j,p^k}) = \sum_{\substack{x \in \mathrm{Outgr}(\beta_{i,j,p^k}) \\ \deg(x) = 2((p-1)r + p^n - p^{n-j} - p^{n-i})}} c''_{n,i,j,x} x$$

Note that, regarding the previous result, the coefficients $c'_{n,j,x}$ and $c''_{n,i,j,x}$ are always equal to -1, 0 or 1.

We close this section with a proposition that describes the action of the Bockstein homomorphism β on Hopf ring generators. This clearly determines the action of the whole Steenrod algebra on A_A and follows easily from [29, Theorem 3.9 at page 33].

Proposition 61. *The following formulas hold:*

- $\beta(\alpha_{j,k}) = \gamma_{k,1}$ *if* $j = k$ *and is equal to* 0 *otherwise.*
- $\beta(\beta_{i,j,p^k}) = -\alpha_{i,j}$ *if* $k = 0$ *and is equal to* 0 *otherwise.*
- $\beta(\gamma_{j,p^k}) = 0$.

Chapter 3

(Almost)-Hopf rings for the Coxeter groups W_{B_n} and W_{D_n}

The aim of this chapter is to describe Hopf ring-like structures on the graded $\mathbb{F}_p$-vector spaces $A_B = \bigoplus_{n\geq 0} H^*(W_{B_n};\mathbb{F}_p)$ and $A_D = \bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_p)$, where p is a prime number, and W_{B_n}, W_{D_n} are the finite reflection groups described in the first chapter of this thesis.

These sequences of groups have embeddings $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$ and $W_{D_n} \times W_{D_m} \to W_{D_{n+m}}$ that are, in a sense that will be clarified later, well-behaved.

The homomorphisms induced by these maps on mod p cohomology define a coproduct Δ, while the cohomology transfer maps associated to them determine a product $\odot$. In the B_n case and, if $p > 2$, in the D_n case, these, together with the usual cup product $\cdot$ define a Hopf ring. On the contrary, $\bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_2)$ does not constitute a Hopf ring, because the coproduct Δ and the transfer product $\odot$ fail to form a bialgebra. However, the bialgebra property fails in a "controlled" way. This leads us to the definition of a weaker structure that we call almost-Hopf ring.

For this reason, the study of the cohomology of W_{D_n}, with both the cup and transfer product as well as the coproduct, is more complicated. There are analogies with the treatment of the alternating groups given by Giusti and Sinha in [12].

If p is an odd prime, the mod p case will be derived in a very straightforward way via a spectral sequence argument, while for the mod 2 case we will need the following techniques:

- There are maps between W_{B_n}, W_{D_n}, and W_{A_n}; by analyzing their behavior we are able to build the proof of our main theorems on the corresponding result for W_{A_n} described in the previous chapter. When dealing with finite Coxeter groups of Type D, we also exploit the existence of a standard involution on W_{B_n} that preserves W_{D_n}, regarded as a subgroup of W_{B_n}.

- We use the geometric description by De Concini and Salvetti recalled in Chapter 1 to describe our morphisms and generators.

- At some point, especially when dealing with the cup product, we analyze the restriction to elementary abelian groups and use the fact that the Quillen map is an isomorphism in our case.

The structure of this chapter roughly mimick that of Chapter 2, with the presentation of the Hopf ring in term of generators and relations first, then the study of the restriction to elementary abelian subgroups, and finally the description of the Steenrod algebra action. However, there is an additional part (Section 3.2) devoted to the geometric description of generators).

The content of this chapter will presumably appear somewhere [16].

3.1 Definition of the algebraic structures

In this section, we describe how those algebraic structures arise for W_{B_n} and W_{D_n}. First, recall from Chapter 1 that W_{B_n} can be realized as the group of isometries of the hypercube $[-1,1]^n \subseteq \mathbb{R}^n$ and, as a consequence, the symmetric group $\Sigma_n = W_{A_{n-1}}$ can be interpreted as a subgroup of W_{B_n} by permutation of coordinates. Moreover, $\mathbb{F}_2^n$ can be embedded in W_{B_n} by letting the i^{th} factor acting by multiplying the i^{th} coordinate by -1. This realizes our group as a semidirect product $W_{B_n} = \mathbb{F}_2^n \rtimes W_{A_{n-1}}$, where the conjugation action of $W_{A_{n-1}}$ on $\mathbb{F}_2^n$ is given by permutation of the n factors. This is generally called the wreath product of $\mathbb{F}_2$ and Σ_n and is denoted by $\mathbb{F}_2 \wr \Sigma_n$.

With this fact in mind, it is easily seen that the structural morphisms $\mu_{k,n-k}\colon W_{A_{k-1}} \times W_{A_{n-k-1}} \to W_{A_{n-1}}$, together with the obvious identifications $\mathbb{F}_2^k \times \mathbb{F}_2^{n-k} \cong \mathbb{F}_2^n$ define embeddings $W_{B_k} \times W_{B_{n-k}} \to W_{B_n}$, that we will still denote, with a little abuse of notation, with the symbol $\mu_{k,n}$. These monomorphisms induce a coproduct $\Delta\colon A_B \to A_B \otimes A_B$ and their cohomological transfer map defines a product $\odot\colon A_B \otimes A_B \to A_B$. Finally, let $\cdot\colon A_B \otimes A_B \to A_B$ the usual cup product.

We claim that these structural morphisms make A_B a Hopf ring. Many of the defining axioms of Hopf rings follows directly from the argument used in 2.1 for A_A and can be checked in essentially the same way, with minor modifications. The only point when an additional argument is needed is the proof that $(A_B, \Delta, \odot)$ form a bialgebra. This fact follows from the following diagram being a pullback of finite coverings, where π indicates the obvious projections.

$$\begin{tikzcd}
\bigsqcup_{p+q=n,r+s=m} \frac{E(B_{n+m})}{B_p \times B_q \times B_r \times B_s} \arrow[r, "\bigsqcup \pi_{p+q,r+s}"] \arrow[d, "\bigsqcup \pi_{p+r,q+s}"] & \frac{E(B_{n+m})}{B_n \times B_m} \arrow[d, "\pi_{n,m}"] \\
\bigsqcup_{k+l=n+m} \frac{E(B_{n+m})}{B_k \times B_l} \arrow[r, "\bigsqcup \pi_{k,l}"] & \frac{E(B_{n+m})}{B_{n+m}}
\end{tikzcd}$$

This can be easily seen by naturally identifying $E(W_{B_n})$ with the space $(\mathbb{S}^\infty)^n \times_{\Sigma_n} \mathrm{Conf}_n(\mathbb{R}^\infty)$, where Σ_n acts diagonally by permuting the factors in $(S^\infty)^n$ and the points of the configurations in $\mathrm{Conf}_n(\mathbb{R}^\infty)$, while $\mathbb{F}_2^n$ acts via the antipodal map on the n factors of $(S^\infty)^n$ and trivially on $\mathrm{Conf}_n(\mathbb{R}^\infty)$.

In a similar way, we can construct some maps on A_D. Indeed, recall from Chapter 1 that W_{D_n} is the index 2 subgroup of W_{B_n} that preserves the "positivity" of the vertices of the hypercube $[-1,1]^n$. Algebraically, consider the morphism $\varepsilon_n\colon \mathbb{F}_2 \wr \Sigma_n = \mathbb{F}_2^n \rtimes \Sigma_n \to \mathbb{F}_2$ defined as the sum $\sum\colon \mathbb{F}_2^n \to \mathbb{F}_2$ on the first factor and as the trivial homomorphism $\Sigma_n \to \mathbb{F}_2$ on the latter. It can be seen immediately that ε_n is well-defined and that $\ker(\varepsilon_n) = W_{D_n}$.

Since the morphisms $\mu_{n,k}\colon W_{B_n} \times W_{B_k} \to W_{B_{n+k}}$ satisfy the condition $\varepsilon_{n+k} \circ \mu_{n,k} = \varepsilon_n + \varepsilon_k$ for all $n, k \geq 0$, $\mu_{n,k}$ induces, by restricting on subgroups, maps $W_{D_n} \times W_{D_k} \to W_{D_{n+k}}$. Let $\Delta\colon A_D \to A_D \otimes A_D$ be the coproduct induced by them, and let $\odot\colon A_D \otimes A_D \to A_D$ the product defined by summing up the transfers of $\mu_{n,k}$ for $n, k \geq 0$. Once again, it is easy to see, via the reasoning explained previously, that $(A_D, \Delta, \cdot)$ form a bialgebra, that $\odot$ is commutative and associative and that the Hopf ring distributivity formula holds. However, in this case, the argument used for A_B to prove that Δ and $\odot$ forms a bialgebra fails. Indeed, as it will soon be evident, $(A_D, \Delta, \odot)$ is not a bialgebra when cohomology is taken with mod 2 coefficients. Thus, in general, A_D will not be a Hopf ring but has the weaker structure of an almost-Hopf ring, defined below.

Definition 62. let R be a commutative ring and A be a graded R-module, with a coproduct $\Delta\colon A \to A \otimes A$ and two products $\cdot, \odot\colon A \otimes A \to A$. We say that A is an *almost-Hopf ring* over R if $(A, \Delta, \cdot)$ is a bicommutative and biassociative bialgebra, $\odot$ is commutative and associative and $\cdot$ satisfy the Hopf ring distributivity formula for $\odot$.

We recall, for completeness, the cohomology of a family of groups is known to possess an almost-Hopf ring structure under certain general conditions. Regarding this, we refer to [12].

3.2 Geometric description of structural morphism and generators

Recall that, since W_{B_n} and W_{D_n} are finite reflection groups generated by sets ($\{s_0, \ldots, s_{n-1}\}$ and $\{t_0, \ldots, t_{n-1}\}$ respectively) of cardinality n, the Fox-Neuwirth complexes $\widetilde{FN}^*_{W_{B_n}}$ and $\widetilde{FN}^*_{W_{D_n}}$ are free (as graded $\mathbb{Z}[W_{B_n}]$-module and $\mathbb{Z}[W_{D_n}]$-module respectively) with basis given by n-tuples of non-negative integers $(a_0, \ldots, a_{n-1})$. We want to describe some morphisms and some cohomology classes geometrically in this context, similarly to what is done by Giusti and Sinha in [14] for the symmetric groups.

We start with W_{B_n}. For all $1 \leq i < j \leq n$ let $H^+_{i,j}$ and $H^-_{i,j}$ be the sets:

$$H^{\pm}_{i,j} = \{\underline{x} \in \mathbb{R}^n : x_i = \pm x_j\}$$

Moreover, for all $1 \leq i \leq n$, define H^0_i as:

$$H^0_i = \{\underline{x} \in \mathbb{R}^n : x_i = 0\}$$

These sets are clearly hyperplanes in $\mathbb{R}^n$ and the reflection arrangement associated to W_{B_n} can be described as $\mathcal{A}_{B_n} = \{H^{\pm}_{i,j}\}_{1 \leq i < j \leq n} \cup \{H^0_i\}$. Moreover, s_0 can be identified with the reflection with respect to H^0_1 and, for every $i > 0$,

s_i with the reflection with respect to $H^+_{i,i+1}$. Thus, simply by unwrapping the construction described in the first section of Chapter 1, we observe that the basis element of the Fox-Neuwirth complex that corresponds to an n-tuple $\underline{a} = (a_0 : \cdots : a_n)$ is the stratum:

$$e(\underline{a}) = \Big\{(\underline{x}_1, \ldots, \underline{x}_n) \in (\mathbb{R}^\infty)^n : \forall 1 \leq i \leq n-1, \forall 1 \leq j \leq a_i : (x_i)_j = (x_{i+1})_j,$$
$$(x_i)_{a_i+1} < (x_{i+1})_{a_i+1}, \forall 1 \leq k \leq a_0 : (x_1)_k = 0, (x_1)_{a_0+1} > 0\Big\}$$

If we consider the quotient by W_{B_n}, we immediately notice that $FN^*_{W_{B_n}}$ has a $\mathbb{Z}$-basis consisting of cochains $[a_0 : \cdots : a_{n-1}] = [e(a_0, \ldots, a_{n-1})]$.

We first describe how the structural morphisms of $A_{W_{B_n}}$ act on this basis. We need to recall from [12] an elementary construction that allows us to associate a cochain to a stratum in $\mathcal{L}^{(\infty)}_{W_{B_n}}$. Consider a manifold X and an embedded submanifold $i\colon W \to X$ of a codimension d in X. A smooth singular chain in X is called transverse to i if, for every singular face $\sigma\colon \Delta^k \to X$ of the chain, σ is transverse to i (in the sense of manifolds with boundary) as well as every subface of codimension 1. Given a d-dimensional singular simplex $\sigma\colon \Delta^d \to X$ transverse to i we can define a number $\tau_W(\sigma)$ given by the cardinality mod 2 of $\sigma^{-1}(W)$. This defines a dual cochain of the chain complex of smooth singular chains transverse to i (with mod 2 coefficients). The inclusion of this subcomplex into the full complex of singular chains is a homotopy equivalence. All the main classical constructions in cohomology can be understood geometrically from this model. For example, if $f\colon Y \to X$ is transverse to i, then the pullback in cohomology can be described as $f^\#(\tau_W) = \tau_{f^{-1}(W)}$ and the Künneth isomorphism is understood as taking the geometric product. When W is a stratum in Φ_m ($1 \leq m \leq \infty$), τ_W represents the cochain associated to W in the cellular cochain complex of $Y^{(m)}$. Thus, we can exploit this construction to compute a morphism inducing the coproduct and the transfer product at the chain level.

Our chain-level description requires a preliminary definition, which is partially borrowed from [14].

Definition 63. Let $\boldsymbol{a} = [a_1 : \cdots : a_n] \in FN^*_{B_n}$ be a basis element of $FN^*_{B_n}$. Let, as a notational convention, $a_{-1} = a_{n+1} = 0$. A *k-block* of $\boldsymbol{a}$ is a tuple of the form $[a_i : \cdots : a_{j-1}]$, where $a_l > k$ for all $i \leq l \leq j-1$ and $\max\{a_{i-1}, a_j\} \leq k$. Here we assume that $i \leq j$ where the condition $i = j$ corresponds to the empty tuple. A k-block $[a_i : \cdots : a_j]$ of $\boldsymbol{a}$ is *principal* if, with the previous notation, $\min_{0 \leq r < i} a_r = k$.

For example, $[3:2:3]$ and $[2]$ are the 1-blocks of $[3:2:3:1:2]$, of which $[2]$ is principal.

Lemma 64. *The coproduct is induced by the morphism of chain complexes $\Delta\colon FN^*_{B_n} \otimes \mathbb{F}_2 \to \bigoplus_{k+l=n} FN^*_{B_k} \otimes FN^*_{B_l} \otimes \mathbb{F}_2$ defined by the formula:*

$$[a_0 : \ldots a_{n-1}] \mapsto \sum_{\substack{-1 \leq k \leq n \\ a_k \leq \min\{a_0, \ldots, a_{k-1}\}}} [a_0 : \cdots : a_{k-1}] \otimes [a_k : \cdots : a_{n-1}]$$

Proof. First, we analyze the classifying space level map corresponding to the group homomorphisms $\mu_{n,m}\colon W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$, with varying n and

m. These are modeled as the functions $f_{n,m}\colon \frac{Y^{(\infty)}_{W_{B_n}}}{W_{B_n}} \times \frac{Y^{(\infty)}_{W_{B_m}}}{W_{B_m}} \to \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_{n+m}}}$ that merges two configurations with n and m points respectively into a single configuration with $n+m$ points. This is not defined in the naive way, because the two configurations can share a point and, as a consequence, this construction may not produce an element of $Y^{(\infty)}_{W_{B_{n+m}}}$. However, by rescaling the norm of vectors in $\mathbb{R}^\infty$ via homeomorphisms $(0,\infty) \cong (0,1)$ and $(0,\infty) \cong (1,\infty)$, we can produce an element of $Y^{(\infty)}_{B_{n+m}}$ from two such configurations by taking the union of n points with norm in $(0,1)$ and m points with norm in $(1,\infty)$. Note that we can do this rescaling continuously, because the configurations of $Y^{(\infty)}_{W_{B_m}}$ never contain the zero vector.

The finite-dimensional approximation $f^{(d)}_{n,m}\colon \frac{Y^{(d)}_{W_{B_n}}}{W_{B_n}} \times \frac{Y^{(d)}_{W_{B_m}}}{W_{B_m}} \to \frac{Y^{(d)}_{W_{B_{n+m}}}}{W_{B_{n+m}}}$ is a local homeomorphism, thus it is transverse to every possible submanifold. Let S be a stratum $e(a_0,\dots,a_{n-1},b_0,\dots,b_{m-1})$ of the De Concini–Salvetti stratification. Then, as noted above, $(f^{(d)})^{\#}(\tau_S) = \tau_{(f^{(d)})^{-1}(S)}$. Thus the lemma simply follows from the following immediate equality:

$$(f^{(d)})^{(-1)}(S) = \begin{cases} e\left(a_0,\dots,a_{n-1}\right) \times e\left(b_0,\dots,b_{m-1}\right) & \text{if } b_0 \leq \min_{k=0}^{n-1} a_k \\ \varnothing & \text{otherwise} \end{cases}$$

$\square$

With a similar argument, we can obtain a geometric description of the transfer product.

Lemma 65. *Let $\odot\colon FN^*_{W_{B_n}} \otimes FN^*_{W_{B_m}} \otimes \mathbb{F}_2 \to FN^*_{W_{B_{n+m}}} \otimes \mathbb{F}_2$ be the morphism of chain complexed defined by assigning to $\boldsymbol{a}\otimes\boldsymbol{b}$ to the sum of all basis elements $\boldsymbol{c}$ whose principal k-blocks are shuffles of the principal k-blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$ for all $k \geq 0$. This is a well-defined map and induces the transfer product in cohomology.*

Proof. The natural inclusion $Y^{(\infty)}_{W_{B_{n+m}}} \to Y^{(\infty)}_{W_{B_n}} \times Y^{(\infty)}_{W_{B_m}}$ is a $W_{B_n} \times W_{B_m}$-equivariant homotopy equivalence. Hence, by taking quotients, this yields a map $\pi\colon \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n}\times W_{B_m}} \to \frac{Y^{(\infty)}_{W_{B_n}}}{W_{B_n}} \times \frac{Y^{(\infty)}_{W_{B_m}}}{W_{B_m}}$ that realizes the natural homotopy equivalence $B(W_{B_n} \times W_{B_m}) \simeq B(W_{B_n}) \times B(W_{B_m})$. Moreover, the quotient map $\pi'\colon \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n}\times W_{B_m}} \to \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_{n+m}}}$ models $B(W_{B_n} \times W_{B_m}) \to B(W_{B_{n+m}})$ as a finite covering.

Let $x = [a_0 : \dots : a_{n-1}] \otimes [b_0 : \dots : b_{m-1}] \in \widetilde{FN}^*_{W_{B_n}} \otimes \widetilde{FN}^*_{W_{B_m}} \otimes \mathbb{F}_2$ be an arbitrary basis element. Given any singular simplex σ transverse to our strata, the evaluation of the transfer product $[a_0 : \dots : a_{n-1}] \odot [b_0 : \dots : b_{m-1}]$ on σ is obtained by evaluating x on the sum of $\pi(\tilde{\sigma})$, where $\tilde{\sigma}$ is a lifting of σ. Thus, in order to prove the lemma is sufficient to check that there exists $\tilde{\sigma}$ such that $\pi(\tilde{\sigma})$ intersects the stratum corresponding to x if and only if σ intersects some stratum $e(\boldsymbol{c})$, where the k-principal blocks of $\boldsymbol{c}$ are shuffles of the k-principal blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Indeed, the liftings of σ are in bijective correspondence with the set $\mathrm{Sh}(n,m)$ of (n,m)-shuffles, that represent the left cosets of $W_{B_n} \times W_{B_m}$ in $W_{B_{n+m}}$, in such a way that, if σ intersects transversally $e(\boldsymbol{c})$ in r points, then the lifting $\tilde{\sigma}_\tau$ corresponding to a shuffle τ intersects transversally the translated stratum $\tau.\boldsymbol{c}$ in r points. For all $i \neq n$, let $m_i = \min_{j=\tau(i)}^{\tau(i+1)-1} c_j$. In this translated stratum, the first m_i coordinates of x_i and x_{i+1} are equal, while the $(m_i+1)^{th}$ coordinate is different. Similarly, the first coordinate that differs in $-x_2$ and x_1 (respectively $-x_{n+2}$ and x_{n+1}) is the m_0^{th} (respectively m_n^{th}), where $m_0 = \min_{j=0}^{\tau(1)-1} c_i$ ($m_n = \min_{j=0}^{\tau(n+1)-1}$). Thus, $\tau.e(\boldsymbol{c})$ is contained in the product $x = e(\boldsymbol{a}) \times e(\boldsymbol{b})$ if and only if $a_i = m_i$ for all $0 \leq i \leq n-1$ and $b_i = m_{i-n}$ for all $0 \leq i \leq m-1$. If we assume that $\boldsymbol{c}$ and x have the same codimension, a simple combinatorial observation prove that these equalities hold only when τ induces a shuffle of the principal blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$. This proves our claim and, as a consequence, the lemma. □

We can give an analog geometric description for $F^*_{W_{D_n}}$. As before, this Coxeter group has n fundamental reflections $t_0, \dots, t_{n-1}$, defined in Chapter 1 and Figure 1.1. Thus, a $\mathbb{Z}[W_{D_n}]$-basis for $\widetilde{FN}^*_{W_{D_n}}$ is given by n-tuples of non-negative integers $\underline{a} = (a_0, \dots, a_{n-1})$.

The inclusion $j_n \colon W_{D_n} \to W_{B_n}$ embeds the hyperplane arrangement $\mathcal{A}_{W_{D_n}}$ into $\mathcal{A}_{W_{B_n}}$ as the subarrangement

$$\{H^{\pm}_{i,j}\}_{1 \leq i < j \leq n}.$$

We can also make a comparison between the fundamental reflections of W_{D_n} and those of W_{B_n}. Explicitly, $t_i = s_i$ for $1 \leq i \leq n_1$ and t_0 is the reflection along $H^{-}_{1,2}$. Hence, an n-tuple $\underline{a} = [a_0 : \cdots : a_{n-1}]$ corresponds to the cell dual to the stratum

$$e(\underline{a}) = \Big\{(\underline{x}_1, \dots, \underline{x}_n) \in (\mathbb{R}^\infty)^n : \forall 1 \leq i \leq n-1, 1 \leq j \leq a_i : (x_i)_j = (x_{i+1})_j, \\ (x_i)_{a_i+1} < (x_i)_{a_i+1}, \forall 1 \leq k \leq a_0 : (x_2)_k = -(x_1)_k, (x_2)_{a_0+1} > (x_1)_{a_0+1}\Big\}.$$

By taking the quotient, a $\mathbb{Z}$-basis for $FN^*_{W_{D_n}}$ is given by elements of the form $[a_0 : \dots a_{n-1}] = [e(a_0, \dots, a_{n-1})]$ as before.

The geometric description of the structural maps of A_D is similar to that we obtained above for A_B. However, there are some additional complications: for example, the argument used to prove Lemma 65 cannot be replicated for $FN^*_{W_{D_n}}$, because a product of strata $\underline{a} \times \underline{b} \in Y^{(\infty)}_{W_{D_n}} \times Y^{(\infty)}_{W_{D_m}}$ cannot necessarily be written as union of strata in $Y^{(\infty)}_{W_{D_{n+m}}}$. For example, the stratum $e([0,0,0,0]) \in \mathcal{L}^{(\infty)}_{W_{D_4}}$ has a non-zero intersection with but is not entirely contained in the product $e([0,0]) \times e([0,0]) \in \mathcal{L}^{(\infty)}_{W_{D_2}} \times \mathcal{L}^{(\infty)}_{W_{D_2}}$.

Fortunately, these ideas can be still used for the cochain complex $\frac{\widetilde{FN}^*_{B_n}}{j_n(D_n)}$, that also calculates the cohomology of D_n. Hence, our strategy is to relate $FN^*_{D_n}$ and $\frac{\widetilde{FN}^*_{B_n}}{j_n(D_n)}$ and to use the previous reasoning to obtain cochain-level formulas for the latter. We refer to this complex with the symbol $FN'^*_{W_{D_n}}$, as we will need to work with this cochain complex for a while.

We can now develop our explicit cochain-level formulas. First of all, recall that the index $[W_{B_n} : j_n(W_{D_n})]$ is equal to 2. Hence, $j_n(W_{D_n})$ is a normal subgroup of W_{B_n}. The left cosets of $j_n(W_{D_n})$ in W_{B_n} are represented by id and s_0, the only fundamental reflection of W_{B_n} that is not contained in $j_n(W_{D_n})$. Hence, if $\mathcal{B}$ is a $\mathbb{Z}[W_{B_n}]$-basis for $\widetilde{FN}^*_{W_{B_n}}$, a $\mathbb{Z}$-basis for $FN'^*_{W_{D_n}}$ can be obtained as the set of classes of the form x and $s_0.x$, where $x \in \mathcal{B}$. If $\mathcal{B}$ is the basis constructed in Section 1.1, indexed by n-tuples of non-negative integers $\boldsymbol{a} = (a_0, \dots, a_{n-1})$, we denote by $[a_0 : \dots : a_{n-1}]$ and $s_0[a_0 : \dots : a_{n-1}]$ the cochains in $FN'^*_{W_{D_n}}$ arising from the basis element corresponding to $\boldsymbol{a}$.

The study of the conjugation c_{s_0} by s_0 on W_{D_n} will also be of particular importance because we require it to describe in which way the transfer product and the coproduct fail to constitute a bialgebra. c_{s_0} acts on W_{D_n} by fixing t_i for $2 \leq i < n$ and switching t_0 and t_1. We will denote by $\iota\colon \bigoplus_{n\geq 0} H^*(D_n;\mathbb{F}_2) \to \bigoplus_{n\geq 0} H^*(D_n;\mathbb{F}_2)$ the homomorphism induced in cohomology by these conjugation maps.

ι is described by the following result.

Lemma 66. *ι is induced by the cochain-level map $\iota^\#\colon FN^*_{W_{D_n}} \to FN^*_{W_{D_n}}$ defined as follows:*

$$\iota^\#[a_0 : a_1 : a_2 : \dots : a_{n-1}] = [a_1 : a_0 : a_2 : \dots : a_{n-1}]$$

Proof. Observe that c_{s_0} maps fundamental reflections in W_{D_n} into fundamental reflections. Hence, for every $\Gamma' \subseteq \Gamma \subseteq \{t_0, \dots, t_{n-1}\}$, minimal-length coset representatives in $W_\Gamma^{\Gamma'}$ are mapped into other minimal-length representatives in $W_{c_{s_0}(\Gamma)}^{c_{s_0}(\Gamma')}$. This implies that the following is a well defined c_{s_0}-equivariant chain map:

$$e(\Gamma_1 \supseteq \dots \supseteq \Gamma_k \supseteq \dots) \in C_*^{W_{D_n}} \mapsto e(c_{s_0}(\Gamma_1) \supseteq \dots \supseteq c_{s_0}(\Gamma_k) \supseteq \dots) \in C_*^{W_{D_n}}$$

This yields, dually, a cochain map $FN^*_{W_{D_n}} \to FN^*_{W_{D_n}}$ that induces ι in cohomology and is easily seen to act as desired on basis elements. $\square$

Alternatively, we can describe ι by means of $FN'^*_{W_{D_n}}$.

Lemma 67. *ι is induced by the cochain-level map $\iota'^\#\colon FN'^*_{D_n} \to FN'^*_{D_n}$ described as follows:*

$$\iota'^\#[a_0 : \dots : a_{n-1}] = s_0[a_0 : \dots : a_{n-1}]$$
$$\iota'^\# s_0[a_0 : \dots : a_{n-1}] = [a_0 : \dots : a_{n-1}]$$

Proof. Since s_0 is a reflection, $\iota'^\#$, as defined in the statement of the lemma, correspond to the multiplication by s_0. It is straightforward to check that this is a well-defined morphism of cochain complexes and that it induces ι. $\square$

We now relate $FN^*_{W_{D_n}}$ and $FN'^*_{W_{D_n}}$.

Lemma 68. *There is a cochain homotopy equivalence $\varphi^*\colon FN^*_{W_{D_n}} \to FN'^*_{W_{D_n}}$ defined by the formula:*

$$\varphi^*[a_0 : \dots : a_{n-1}] = \begin{cases} [a_0 : a_1 : a_2 : \dots : a_{n-1}] & \text{if } a_0 < a_1 \\ [a_0 : a_1 : a_2 : \dots : a_{n-1}] + \\ \quad s_0 \cdot [a_1 : a_0 : a_2 : \dots : a_{n-1}] & \text{if } a_0 = a_1 \\ s_0[a_1 : a_0 : a_2 : \dots : a_{n-1}] & \text{if } a_0 > a_1 \end{cases}$$

induced by the obvious inclusion $Y^{(\infty)}_{B_n} \subseteq Y^{(\infty)}_{D_n}$.

Moreover, the following diagram commutes:

$$\begin{array}{ccc} FN^*_{D_n} & \xrightarrow{\varphi^*} & FN'^*_{D_n} \\ \Big\downarrow \iota^{\#} & & \Big\downarrow \iota'^{\#} \\ FN^*_{D_n} & \xrightarrow{\varphi^*} & FN'^*_{D_n} \end{array}$$

Proof. Note that the embedding $Y^{(\infty)}_{W_{B_n}} \subseteq Y^{(\infty)}_{W_{D_n}}$ is a W_{D_n}-equivariant homotopy equivalence. Moreover, we observe that each stratum of $\mathcal{L}^{(\infty)}_{W_{D_n}}$ can be written as a union of strata in $\mathcal{L}^{(\infty)}_{W_{B_n}}$. Hence, if we take the quotient by the action of W_{D_n}, this yields a map $\varphi\colon \frac{Y^{(\infty)}_{W_{B_n}}}{W_{D_n}} \to \frac{Y^{(\infty)}_{W_{D_n}}}{W_{D_n}}$ that induces a well-defined morphism between the cellular cochain complexes $\varphi^*\colon FN^*_{W_{D_n}} \to FN'^*_{W_{D_n}}$.

φ^* is clearly a cochain homotopy equivalence because it is induced by a topological homotopy equivalence. We now prove that φ^* satisfies the additional required conditions. Observe that every finite-dimensional approximation $\varphi^{(d)}$, being a 0-codimensional immersion, is transverse to strata in $\mathcal{L}^{(d)}_{W_{D_n}}$. For this reason, for any given stratum $S = e(a_0, \dots, a_{n-1})$ in $\mathcal{L}^{(d)}_{W_{D_n}}$, τ_S and $\tau_{(\varphi^{(d)})^{-1}(S)}$ are defined, and $(\varphi^{(d)})^*(\tau_S) = \tau_{(\varphi^{(d)})^{-1}(S)}$. Since φ^* is induced by φ, for any $\varphi^*(S)$ is the sum of all the strata S' of $\mathcal{L}^{(\infty)}_{W_{B_n}}$ that satisfy $S' \subseteq S$ and $\operatorname{codim}(S') = \operatorname{codim}(S)$. Thus, it is sufficient to describe these strata. There are three distinct cases:

- if $a_0 < a_1$, then $(\varphi^{(d)})^{-1}(S) = e(a_0, a_1, a_2, \dots, a_{n-1})$
- if $a_0 > a_1$, then $(\varphi^{(d)})^{-1}(W) = s_0.e\,(a_1, a_0, a_2, \dots, a_{n-1})$
- if $a_0 > a_1$, then $(\varphi^{(d)})^{-1}(W)$ is the union of $e\,(a_0, \dots, a_{n-1})$, of $s_0.e\,(a_0, \dots, a_{n-1})$ and of strata of bigger codimension.

This concludes the proof of the lemma. □

We can now state the cochain-level formulas analogous to those of Lemma 64 and Lemma 65 for W_{D_n}.

Lemma 69. *The coproduct is induced by the cochain complex homomorphism $\Delta\colon FN'^*_{W_{D_n}} \otimes \mathbb{F}_2 \to \bigoplus_{k+l=n} FN'^*_{W_{D_k}} \otimes FN'^*_{W_{D_l}} \otimes \mathbb{F}_2$ defined by the formula:*

$$\begin{aligned} [a_0 : \dots a_{n-1}] \mapsto & \sum_{\substack{-1 \le k \le n \\ a_k \le \min\{a_0, \dots, a_{k-1} << \}}} [a_0 : \cdots : a_{k-1}] \otimes [a_k : \cdots : a_{n-1}] \\ & \qquad + s_0[a_0 : \cdots : a_{k-1}] \otimes s_0[a_k : \cdots : a_{n-1}] \\ s_0[a_0 : \dots a_{n-1}] \mapsto & \sum_{\substack{-1 \le k \le n \\ a_k \le \min\{a_0, \dots, a_{k-1}\}}} [a_0 : \cdots : a_{k-1}] \otimes s_0[a_k : \cdots : a_{n-1}] \\ & \qquad + s_0[a_0 : \cdots : a_{k-1}] \otimes [a_k : \cdots : a_{n-1}] \end{aligned}$$

Proof. The proof is essentially the same as that of Lemma 64, and it, once again, reduces to the calculation of the inverse image of strata via the map $\frac{Y^{(\infty)}_{W_{D_n}}}{W_{D_n}} \times \frac{Y^{(\infty)}_{W_{D_m}}}{W_{D_m}} \to \frac{Y^{(\infty)}_{W_{D_{n+m}}}}{W_{D_{n+m}}}$ defined by merging two configurations. The only difference is that the inverse image of the stratum $e([a_0 : \cdots : a_{n+m-1}])$ is empty if $a_n > \min\{a_0, \dots, a_{n-1}\}$, and the union of $e([a_0 : \cdots : a_{n-1}])$ and $e([a_n : \cdots : a_{n+m-1}])$ otherwise. $\square$

Lemma 70. *Let $s_0^{\varepsilon_1}\boldsymbol{a}$ and $s_0^{\varepsilon_2}\boldsymbol{b}$ be generic basis elements of $FN'^*_{W_{D_n}}$ and $FN'^*_{W_{D_m}}$ respectively, where $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ and $\boldsymbol{a}$ (respectively $\boldsymbol{b}$) is defined by an n-tuple $\underline{a}$ (respectively an m-tuple $\underline{b}$) of non-negative integers. Let $\odot\colon FN'^*_{W_{D_n}} \otimes FN'^*_{W_{D_m}} \otimes \mathbb{F}_2 \to FN'^*_{W_{D_{n+m}}} \otimes \mathbb{F}_2$ be the homomorphism that maps $s_0^{\varepsilon_1}\boldsymbol{a} \otimes s_0^{\varepsilon_2}\boldsymbol{b}$ to the sum of all elements $s_0^{\varepsilon_1+\varepsilon_2}\boldsymbol{c}$, such that the principal k-blocks of $\boldsymbol{c}$ are shuffles of the principal k-blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$ for all $k \geq 0$. This defines a morphism of complexes and induces the transfer product in cohomology.*

Proof. Once again, this formula can be obtained via the same proof as the W_{B_n} case, with some obvious minimal modifications. $\square$

After dealing with the structural morphisms, we now turn to the definition of our generators. First of all, the natural projection $\pi\colon W_{B_n} \to \Sigma_n$ induces a map $\pi^*\colon H^*(\Sigma_n;\mathbb{F}_2) \to H^*(W_{B_n};\mathbb{F}_2)$. Recall from Theorem 37 that, as a Hopf ring, $\bigoplus_{n\geq 0} H^*(\Sigma_n;\mathbb{F}_2)$ is generated by certain cohomology classes $\gamma_{k,n} \in H^{n(2^k-1)}(\Sigma_{n2^k};\mathbb{F}_2)$. With a small abuse of notation, we identify $\gamma_{k,n}$ with its image in $H^*(W_{B_{n2^k}};\mathbb{F}_2)$ under π^*.

We further define some other elements in A_B that cannot be traced back to the symmetric groups.

Definition 71. Consider the vector bundle η defined as the twisted product $E(W_{B_n}) \times_{W_{B_n}} \mathbb{R}^n \to B(W_{B_n})$. Define $\delta_n \in H^n(W_{B_n};\mathbb{F}_2) \subseteq A_{W_{B_n}}$ as the n-dimensional Stiefel–Whitney class of η.

We now give an interpretation of these classes in the context of the complex $C_*^{W_{B_n}}$.

Proposition 72. *Regarded as elements of the complex $FN^*_{B_n}$, cochain representatives of the classes $\tilde{\gamma}_{k,m}$ and δ_n have the following description:*

- $\gamma_{k,m} = [0 : \underbrace{\underbrace{1 : 1 : \cdots : 1}_{2^k-1 \text{ times}} : 0 : \underbrace{1 : 1 : \cdots : 1}_{2^{k-1} \text{ times}} : 0 : \cdots : 0 : \underbrace{1 : 1 : \cdots : 1}_{2^k-1 \text{ times}}}_{m \text{ times}}]$
- $\delta_n = [\underbrace{1 : 1 : \cdots : 1}_{n \text{ times}}]$

Proof. Consider the chain complex $FN^*_{\Sigma_n}$ that computes the cohomology of Σ_n. Since a set of simple reflections for Σ_n is $\{s_1, \dots, s_{n-1}\}$, $FN^*_{\Sigma_n}$ has an additive basis parametrized by the set of $(n-1)$-tuples of non-negative integers. By construction, the homomorphism $\pi^*\colon H^*(\Sigma_n) \to H^*(W_{B_n})$ is induced by the homomorphism $FN^*_{\Sigma_n} \to FN^*_{W_{B_n}}$ that maps the basis element corresponding to $(a_1, \dots, a_{n-1})$ to $[0 : a_1 : \cdots : a_{n-1}]$. The first point follows from this fact and the observations at page 13 in [14].

As regarding the latter point, define:

$$X_n = \{(\underline{x}_1, \dots, \underline{x}_n) \in Y^{(\infty)}_{B_n} : (x_1)_1 = \cdots = (x_n)_1 = 0\}$$

Note that X_n is embedded into $\frac{Y^{(\infty)}_{W_{B_n}}}{W_{B_n}}$ as a proper submanifold and that $e([1 : \cdots : 1])$ is the Thom class T_n of X_n (with the meaning explained in Section 2.1). This can be seen by observing that the only cell of dimension n in the De Concini–Salvetti complex of W_{B_n} that intersects X_n is that corresponding to $[1, \dots, 1]$.

Furthermore, we observe that the normal bundle of X_n in $\frac{Y^{(\infty)}_{W_{B_n}}}{W_{B_n}}$ is isomorphic to $\eta|_{X_n}$. This is easy to see from the fact that the normal bundle to the inverse image $\tilde{X_n} \subseteq \mathrm{Conf}_n(\mathbb{R}^\infty)$ of X_n at a generic point $q = ((0, x_{1,2}, x_{1,3}, \dots), \dots, (0, x_{n,2}, x_{n,3}, \dots)) \in \tilde{X_n}$ is the sub-vector space of $(\mathbb{R}^\infty)^n = T_q(\mathrm{Conf}_n(\mathbb{R}^\infty)$ consisting of vectors $(v_1, \dots, v_n) \in (\mathbb{R}^\infty)^n$ such that all the coordinates of v_i are 0, except possibly the first one.

We are required to prove that T_n is equal to $\delta_n = w_n(\eta)$, the n-dimensional Stiefel-Whitney class of η. Let $\sigma_0 \colon B(\eta) \to E(\eta)$ be the zero section. Let $\Phi \colon H^*(B(\eta); \mathbb{F}_2) \to H^*(E(\eta), E(\eta) \setminus \sigma_0(B(\eta)); \mathbb{F}_2$ be the Thom isomorphism. It is well known to experts that the Thom class $T(\eta)$ and $w_n(\eta)$ are linked by the following formula:

$$w_n(\eta) = \Phi^{-1}(\mathrm{Sq}^n(T(\eta))) = \Phi^{-1}(T(\eta)^2)$$

This result is discussed, for example in Milnor and Stasheff's book [30].

Hence, to prove that $T_n = \delta_n$, it is sufficient to show that $T(\eta) \cup i^*(T(\eta)) = (\sigma_0 \circ j \circ p)^*(T(\eta))$, where i and j are the obvious inclusions of pairs $i \colon (E(\eta), \varnothing) \to (E(\eta), E(\eta) \setminus \sigma_0(B(\eta)))$ and $j \colon (B(\eta), \varnothing) \to (B(\eta), B(\eta) \setminus X_n)$. This can be proved as follows. First, by excision, we can replace $B(\eta) = \frac{Y^{(\infty)}_{W_{B_n}}}{W_{B_n}}$ with a tubular neighborhood N of X_n in $B(\eta)$. Second, we observe that, with this substitution, the maps i and $\sigma_0 \circ j \circ p$ are homotopic via a linear homotopy, hence the desired equality. □

The generators of A_B clearly give rise, by restriction, to classes in A_D. In particular, we need those arising from δ_n.

Definition 73. Let $n \geq 1$ and $m \geq 0$. We define $\delta^0_{n;m} \in H^{n+m}(W_{D_{n+m}}; \mathbb{F}_2)$ as the restriction of $\delta_n \odot 1_m \in H^*(W_{B_{n+m}}; \mathbb{F}_2)$ to the cohomology of $W_{D_{n+m}}$.

We will also require some other classes. Given $k, m \geq 1$, we define two cochains in $FN^*_{W_{D_n}}$:

- $g^+_{k,m} = [0 : \underbrace{\underbrace{1 : \cdots : 1}_{2^k-1 \text{ times}} : 0 : \underbrace{1 : \cdots : 1}_{2^k-1 \text{ times}} : 0 : \cdots : \underbrace{1 : \cdots : 1}_{2^k-1 \text{ times}}}_{m \text{ times}}]$
- $g^-_{k,m} = [1 : 0 : \underbrace{\underbrace{1 : \cdots : 1}_{2^k-2 \text{ times}} : 0 : \underbrace{1 : \cdots : 1}_{2^k-1 \text{ times}} : 0 : \cdots : \underbrace{1 : \cdots : 1}_{2^k-1 \text{ times}}}_{m \text{ times}}]$

A direct calculation of the boundary of these cochains in the dual complex of $C_*^{W_{D_n}}$, as described by De Concini and Salvetti, proves that $g^+_{k,m}$ and $g^-_{k,m}$ are cocycles and thus, as a consequence, they represent cohomology classes.

Definition 74. Let $k, m \geq 1$. We define $\gamma^+_{k,m}$ (respectively $\gamma^-_{k,m}$) as the cohomology class in $H^{m2^k}(D_n; \mathbb{F}_2)$ represented by $g^+_{k,m}$ (respectively $g^-_{k,m}$).

The last point required to complete our geometric setup is the relation between $\gamma^+_{k,m}$ and $\gamma^-_{k,m}$, and between them and the natural morphisms involving W_{D_n}, W_{B_n}, and Σ_n. We begin by fixing some notations. Let, with a tiny abuse of notation, $\pi\colon W_{D_n} \to \Sigma_n$ be the composition of the epimorphism $W_{B_n} \xrightarrow{\pi} \Sigma_n$ with the inclusion $W_{D_n} \hookrightarrow W_{B_n}$. Moreover, observe that there are at least two natural ways to embed Σ_n in W_{D_n}. In term of the Coxeter generators $t_0, \dots, t_n$ of Figure 1.1, we can map the elementary transposition $(i, i+1)$ into t_i if $i \geq 2$ for all $1 \leq i \leq n-1$. This defines a morphisms $i_+\colon \Sigma_n \to W_{D_n}$. Similarly, we obtain a map $i_-\colon \Sigma_n \to W_{D_n}$ by mapping $(i, i+1)$ into t_i if $2 \leq i \leq n-1$, and $(1, 2)$ into t_0. There are two evident properties of this construction:

- $\pi \circ i_+ = \pi \circ i_- = \mathrm{id}_{\Sigma_n}$
- $c_{s_0} \circ i_+ = i_-$, where $c_{s_0}\colon W_{D_n} \to W_{D_n}$ is the conjugation by s_0, the generating reflection of W_{B_n} that does not belong to W_{D_n}

In our geometric framework, i_+ can be described as described below. Let $\alpha\colon \mathbb{R} \to (0, +\infty)$ be a homeomorphism. This, in addition, determines, by taking direct products, some homeomorphisms $\alpha^{(m)}\colon \mathbb{R}^m \to \mathbb{R}^m$ for each $1 \leq m \leq \infty$. The classifying map $B(i_+)\colon \overline{\mathrm{Conf}}_n(\mathbb{R}^\infty) \to \frac{Y^{(\infty)}_{W_{D_n}}}{W_{D_n}}$ of i_+ can be modeled as

$$B(i_+)[x_1, \dots, x_n] = [\alpha^{(\infty)}(x_1), \alpha^{(\infty)}(x_2), \dots, \alpha^{(\infty)}(x_n)].$$

In a similar way, a model for $B(i_-)$ is given by

$$B(i_-)[x_1, \dots, x_n] = [-\alpha^{(\infty)}(x_1), \alpha^{(\infty)}(x_2), \dots, \alpha^{(\infty)}(x_n)].$$

The relation between $\gamma^{\pm}_{k,n}$ and π^* is explained with the following proposition.

Proposition 75.

$$\pi^*(\gamma_{k,n}) = \gamma^+_{k,n} + \gamma^-_{k,n}.$$

Furthermore, the following formulas hold:

$$\begin{aligned} i_+^*(\gamma^+_{k,n}) &= \gamma_{k,n} \\ i_+^*(\gamma^-_{k,n}) &= 0 \\ i_-^*(\gamma^+_{k,n}) &= 0 \\ i_-^*(\gamma^-_{k,n}) &= \gamma_{k,n} \end{aligned}$$

Proof. π^* is the composition of the morphism $A_A \to A_B$ induced by the obvious projection, and the map $A_B \to A_D$ induced by the inclusions j_n. At the chain level, the map $A_A \to A_B$ is given by $[a_1 : \dots : a_{n-1}] \mapsto [0 : a_1 : \dots : a_{n-1}]$.

The morphism $A_B \to A_D$ is modeled, at the cochain level, via the function $FN^*_{W_{B_n}} \to FN'^*_{W_{D_n}}$ that maps $e(\boldsymbol{a})$ to $e(\boldsymbol{a}) + s_0 e(\boldsymbol{a})$. Then, we can convert classes in $FN'^*_{W_{D_n}}$ to classes in $FN^*_{W_{D_n}}$ via the morphism φ^* of Lemma 68. Thus, in order to prove the result is sufficient to observe that the composition of these three morphisms of cochain complexes maps the representative of $\gamma_{k,n}$ described in the statement of Proposition 72 into the sum $g^+_{k,n} + g^-_{k,n}$. □

3.3 Presentation of A_B and A_D as (almost-)Hopf rings

In this section, we deduce our basic relations for A_B and A_D. This will be obtained by combining two techniques: the geometric results of Section 3.2 and the exploitation of the restriction to the modular invariants.

We begin with A_B. First of all, the transfer product relations in A_A involving the classes $\gamma_{k,n}$ naturally move to A_B. This happens because the natural maps linking A_A and A_B preserve the Hopf ring structures.

Proposition 76. *Consider the inclusions $i \colon \Sigma_n \to W_{B_n}$ and the projections $\pi \colon W_{B_n} \to \Sigma_n$. The induced maps $i^* \colon A_B \to A_A$ and $\pi^* \colon A_A \to A_B$ are Hopf-ring homomorphisms.*

Proof. The following diagrams are pullbacks of finite coverings, where p indicate covering maps:

$$\begin{array}{ccc} \frac{\mathrm{Conf}_{n+m}((0,+\infty)^\infty)}{\Sigma_n \times \Sigma_m} & \xrightarrow{i} & \frac{Y^{(\infty)}_{B_{n+m}}}{B_n \times B_m} \\ \downarrow p & & \downarrow p \\ \frac{\mathrm{Conf}_{n+m}((0,+\infty)^\infty)}{\Sigma_{n+m}} & \xrightarrow{i} & \frac{Y^{(\infty)}_{B_{n+m}}}{B_{n+m}} \end{array} \qquad \begin{array}{ccc} \frac{E(\mathbb{F}_2)^{n+m} \times E(\Sigma_{n+m})}{(\mathbb{F}_2 \wr \Sigma_n) \times (\mathbb{F}_2 \wr \Sigma_m)} & \xrightarrow{\pi} & \frac{E(\Sigma_{n+m})}{\Sigma_n \times \Sigma_m} \\ \downarrow p & & \downarrow p \\ \frac{E(\mathbb{F}_2)^{n+m} \times E(\Sigma_{n+m})}{\mathbb{F}_2 \wr \Sigma_{n+m}} & \xrightarrow{\pi} & \frac{E(\Sigma_{n+m})}{\Sigma_{n+m}} \end{array}$$

We now show that this implies our result. The coverings that appear in the previous diagrams induce the coproducts of A_A and A_B when passing to cohomology. Thus, the commutativity of these diagrams implies that i^* and π^* preserve the coproduct. Being pullbacks, the naturality of the transfer maps imply that i^* and π^* preserve the transfer product. The naturality of i^* and π^* with respect to the cup product is automatic, since they are induced by maps of spaces. □

Corollary 77. *The following formulas hold in A_B:*

$$\Delta(\gamma_{k,n}) = \sum_{i+j=n} \gamma_{k,i} \otimes \gamma_{k,j}$$

$$\gamma_{k,n} \odot \gamma_{k,m} = \begin{pmatrix} n+m \\ n \end{pmatrix} \gamma_{k,n+m}$$

Proof. The desired relations hold in A_A. As a consequence of Proposition 76, we still obtain equalities if we apply π^* to their left and right members. This gives the desired formulas in A_B.

An alternative proof can be obtained by direct application of the cochain-level formulas for the coproduct and transfer product on the cochain representative of $\gamma_{k,n}$ introduced in the previous section. □

This accounts for the classes coming from A_A. However, the generators of the form δ_n do not arise as restrictions of classes in the cohomology of the symmetric groups. Fortunately, we can nonetheless obtain some coproduct and transfer product formulas via our geometric description.

Proposition 78.

$$\Delta(\delta_n) = \sum_{k+l=n} \delta_k \otimes \delta_l$$

$$\delta_n \odot \delta_m = \binom{n+m}{n} \delta_{n+m}$$

Proof. By Proposition 72, δ_n is represented by the cochain $e([1 : \cdots : 1])$. The coproduct can be calculated via the formula described in Lemma 64 that, when applied to $[1 : \cdots : 1]$, gives

$$\sum_{k=0}^{n} \underbrace{[1 : \cdots : 1]}_{k \text{ times}} \otimes \underbrace{[1 : \cdots : 1]}_{(n-k) \text{ times}} .$$

This yields the desired coproduct relations, and those regarding the transfer product are obtained via the same reasoning, by means of Lemma 65. □

As we will prove in Section 3.5, these suffice to completely describe A_B. We state the result below, postponing the proof.

Theorem 79. *The Hopf ring A_B is generated by classes $\gamma_{k,n}$ and δ_n with the relations described in Corollary 77 and Proposition 78, together with the following additional relation:*

the product $\cdot$ of generators from different components is 0

We now turn our attention to A_D, that is sensibly more difficult to treat. In order to write our almost-Hopf ring presentation in the most concise way, we adopt a nice trick borrowed from Giusti and Sinha [12]. Recall that there is an involution $\iota \colon A_D \to A_D$. Define A'_D as the bigraded $\mathbb{F}_2$-module given by $(A'_D)_{n,d} = H^d(D_n; \mathbb{F}_2)$ if $(n,d) \neq (0,0)$ and $(A'_D)_{0,0} = \mathbb{F}_2\{1^+, 1^-\}$. Clearly, A_D is a subspace of A'_D in a natural way, by identifying the non-zero class in $H^0(D_0; \mathbb{F}_2)$ with 1^+. This "extended" module of the cohomology of W_{D_n} is useful because it still retains the structure of an almost-Hopf ring.

Proposition 80. *By letting $1^- \cdot 1^+ = 0$, $1^- \cdot 1^- = 1^-$ and $1^- \odot 1^- = 1^+$, the almost-Hopf ring structure on A_D extends to an almost-Hopf ring structure on A'_D such that $1^- \odot x = \iota(x)$ for every $x \in A_D$ and $1^- \cdot x = 0$ for every $x \in A_D$ lying in a component different from that containing 1. The coproduct of x in A'_D is given by the coproduct of x in A_D plus two terms $1^- \otimes \iota(x) + \iota(x) \otimes 1^-$.*

Proof. ι behave nicely with respect to the transfer product and the coproduct. Explicitly, the following diagrams induce pullbacks of finite coverings at the level of classifying spaces.

$$\begin{tikzcd} W_{D_n} \times W_{D_m} \arrow[r] \arrow[d, "s_0 \times \mathrm{id}"] & W_{D_{n+m}} \arrow[d, "s_0"] \\ W_{D_n} \times W_{D_m} \arrow[r] & W_{D_{n+m}} \end{tikzcd} \qquad \begin{tikzcd} W_{D_n} \times W_{D_m} \arrow[r] \arrow[d, "s_0 \times s_0"] & W_{D_{n+m}} \arrow[d, "\mathrm{id}"] \\ W_{D_n} \times W_{D_m} \arrow[r] & W_{D_{n+m}} \end{tikzcd}$$

This, together with the naturality of transfer maps, imply that the commutativity and associativity of $\odot$ hold also in A'_D.

As regarding Hopf ring distributivity, since we already know that it holds in A_D, it is sufficient to prove that for all $x, y \in A_D$ we have that $(1^- \odot x) \cdot y = 1^- \odot (x \cdot (1^- \odot y))$ or, in other words, $\iota(x) \cdot y = \iota(x \cdot \iota(y))$. This follows immediately from the fact that, being the conjugation c_{s_0} realizeable by a map between the relevant classifying spaces, $\iota \circ \cdot = \cdot \circ (\iota \otimes \iota)$.

This same remark about ι and the cup product immediately proves that Δ and $\cdot$ still form a bialgebra in A'_D. □

We can, in principle, obtain directly a presentation for A_D, but it is easier to calculate a presentation for A'_D because it can be written more concisely. For example, $\gamma^-_{k,m} = 1^- \odot \gamma^+_{k,m}$ in A'_D. Hence we can omit $\gamma^-_{k,m}$ from the set of generators, and the relations involving for these classes will follow directly from those regarding $\gamma^+_{k,m}$, as a direct consequence of the existence of an extended almost-Hopf ring structure on A'_D. Thus, our generators for $A'_{W_{D_n}}$ will be 1^-, $\gamma^+_{k,m}$, and $\delta_{i:j}$. We denote $1^- \odot \gamma^+_{k,m}$ with $\gamma^-_{k,m}$ because this is actually coherent with our previous construction of $\gamma^-_{k,m}$.

We will prove some of our relations by comparison with already known formulas in A_B, via the morphism induced by the natural inclusions $j_n \colon W_{D_n} \hookrightarrow W_{B_n}$. We must be careful, though, because this is not an almost-Hopf ring morphism.

Proposition 81. *The restriction map $\rho\colon H^*(W_{B_n}; \mathbb{F}_2) \to H^*(W_{D_n}; \mathbb{F}_2)$ preserves coproducts, while the transfer map $tr\colon H^*(W_{D_n}; \mathbb{F}_2) \to H^*(W_{B_n}; \mathbb{F}_2)$ preserves transfer products.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} W_{D_n} \times W_{D_m} \arrow[r, "\mu_{n,m}"] \arrow[d, "j_n \times j_m"] & W_{D_{n+m}} \arrow[d, "j_{n+m}"] \\ W_{B_n} \times W_{B_m} \arrow[r, "\mu_{n,m}"] & W_{B_{n+m}} \end{tikzcd}$$

Taking cohomology we still obtain a commutative diagram that proves that ρ preserves coproducts. Similarly, by taking cohomology transfer maps we check that tr preserves transfer products. □

Lemma 82. *Let $k, m \geq 1$. The following coproduct formulas hold:*

$$\Delta(\gamma^+_{k,m}) = \sum_{l=0}^{m} \gamma^+_{k,l} \otimes \gamma^+_{k,m-l} + \gamma^-_{k,l} \otimes \gamma^-_{k,m-l}$$
$$\Delta(\delta^0_{n:m}) = \sum_{i=0}^{n} \sum_{j=0}^{m} \delta^0_{i:j} \otimes \delta^0_{k-i:m-j}$$

Moreover, the transfer product in A_D satisfies the following formulas for every choice of indexes:

$$\gamma^+_{k,a} \odot \gamma^+_{k,b} = \binom{a+b}{a} \gamma^+_{k,a+b}$$
$$b \odot b' = 0 \text{ whenever } b \text{ and } b' \text{ are cup products of “0” generators}$$
$$\delta^0_{n:m} \odot 1^- = \delta^0_{n:m}$$

Proof. The easiest way to check the coproduct formulas for $\gamma^+_{k,m}$ is to evaluate the composition of the cochain homotopy equivalence of Lemma 68 and the cochain-level coproduct formula of Lemma 69 on the cochain $g^+_{k,m}$ representing $\gamma^+_{k,m}$. This clearly gives $\sum_{l=0}^{m} g^+_{k,l} \otimes g^+_{k,m-l} + g^-_{k,l} \otimes g^-_{k,m-l}$.

The coproduct formula for $\delta^0_{k:m}$ can also be deduced via a similar cochain-level calculation. However, it is also an immediate consequence of our calculations in A_B and the fact that the restriction map $\rho \colon A_B \to A_D$ preserves coproducts.

The first transfer product identity can be deduced geometrically. Indeed, the representative $g^+_{k,m}$ for $\gamma^+_{k,m}$ in $FN'^*_{W_{D_{m2^k}}}$ has no principal k-blocks for $k > 0$ and has m identical principal 0-block of length 2^k made only by 1s. Since all the 0-blocks are equal, for every shuffle $\sigma \in \mathrm{Sh}(a, b)$, the cochain obtained by permuting the principal blocks of $g^+_{k,a}$ and $g^+_{k,b}$ according to σ is always $g^+_{k,a+b}$. Thus, the formula clearly holds, since the cardinality of $\mathrm{Sh}(a, b)$ is exactly the desired binomial coefficient.

Finally, the relation $\delta^0_{n:m} \odot 1^- = \delta^0_{n;m}$ is equivalent to the fact that $\delta^0_{n:m}$, being the restriction of a class in A_B, in invariant with respect to the standard involution ι. □

There are some additional cup product relations not arising from A_B that are also difficult to prove geometrically because the Fox-Neuwirth–type cell complex does not behave well with respect to cup product. For this reason, we found that it is simpler to obtain these formulas via restriction to elementary abelian subgroups. This approach is fruitful because of a detection theorem for these subgroups. We postpone the proof of the following proposition to Section 3.5, where we will explain this in details.

Lemma 83. *Then the following formulas hold in A_D:*

- $\forall n, m, k \geq 1, h \geq 2 : \gamma^+_{k,n} \cdot \gamma^-_{h,m} = 0$

- $\forall m \geq 1 : \gamma^+_{1,m}\gamma^-_{1,m} = (\gamma^+_{1,m-1})^2 \odot \delta^0_{2:0}$
- *the $\cdot$ product of generators belonging to different components is* 0
- $\forall m \geq 0 : \delta^0_{1:m} = 0$
- $\forall n, k, l > 0, m \geq 0 : \delta^0_{n:m} \cdot \gamma^+_{k,l} = \delta^0_{n:0} \cdot \gamma^+_{k,\frac{n}{k}} \odot \gamma^+_{k,\frac{m}{k}}$*, where we understand that* $\gamma^+_{k,r} = 0$ *if* r *is not an integer*

The last relation we require involves the relative behavior of the coproduct and the transfer product and how they fail to be a bialgebra. First, we need to fix some notation. We consider elements $b \in A_D$ that are obtained by cup-multiplying some of our generators $\delta^0_{n;m}$ and $\gamma^+_{k,m}$, with the condition that at least one factor of the form $\gamma^+_{k,m}$ appears in b. Equivalently, we ask that b does not belong to the cup-product algebra generated by the "0" generators. Lemma 82 and the fact that the cup product and the coproduct form a bialgebra give a formula for the coproduct of b of the form $\Delta(b) = \sum_i b'_i \otimes b''_i$, where b'_i and b''_i are obtained by cup-multiplying classes of the form $\delta^0_{n:m}$ and $\gamma^+_{k,m}$, or classes of the form $\delta^0_{n:m}$ and $\gamma^-_{k,m}$. We denote with the symbol $\Delta'(b)$ the sum of all addends $b'_i \otimes b''_i$ such that b'_i does not contain any "$-$" class. Informally, $\Delta'(b)$ is obtained from $\Delta(b)$ by choosing only the "positively charged" addends.

Lemma 84 (cf. [12], Theorem 3.21)**.** *Let $\tau \colon \alpha \otimes \beta \in A_D \otimes A_D \mapsto \beta \otimes \alpha \in A_D \otimes A_D$ be the morphism that exchanges the two factors. Let $x \in A_D$ an arbitrary class and $b \in A_D$ a class obtainable by cup-multiplying the generators $\delta^0_{n:m}$ and $\gamma^+_{k,m}$, with at least one "+" generator appearing. Then, with the previous notation:*

$$\Delta(b \odot x) = (\odot \otimes \odot) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta'(b) \otimes \Delta(x))$$

Moreover, the following formula holds:

$$\Delta(1^- \odot x) = (\odot \otimes \odot) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(1^- \otimes 1^+ \otimes x)$$

This result can be proved by analyzing some spectral sequence, but we found it easier to do that via restriction to elementary abelian subgroups. For this reason, we postpone the proof until Section 3.5.

We can now state our presentation theorem for A'_D.

Theorem 85. *A'_D is generated, as an almost-Hopf ring, by the classes $\delta^0_{n:m}$ ($n \geq 2$, $m \geq 0$), $\gamma^+_{k,m}$ ($k, m \geq 1$), and 1^- defined above, under the relations described in Lemmas 82, 83 and 84 and the relation $1^- \odot 1^- = 1^+$ coming from Proposition 80.*

Remark 86. We have described A_B and A_D only for cohomology with coefficients in $\mathbb{F}_2$ because, for $p > 2$, the description of the cohomology of the Coxeter groups of Type B as a Hopf ring and the cohomology of the Coxeter groups of Type D with coefficients in $\mathbb{F}_p$ as an almost-Hopf ring follows easily from the results of the previous chapter, as stated in the following proposition.

Proposition 87. *Let $p > 2$ be a prime number. Let $A_{A,p}$, $A_{B,p}$ and $A_{D,p}$ be the (almost)-Hopf rings corresponding to the mod p cohomology of the finite reflection groups of Type A, B and D respectively. Then $A_{D,p}$ is a full Hopf ring and the maps $i^* \colon A_{B,p} \to A_{A,p}$, $\pi^* \colon A_{A,p} \to A_{B,p}$, $\rho \colon A_{B,p} \to A_{D,p}$ and $\mathrm{tr} \colon A_{D,p} \to A_{B,p}$ are Hopf ring isomorphisms.*

Proof. Recall that $W_{B_n} \cong \mathbb{F}_2^n \rtimes W_{A_{n-1}}$. Hence, there is a Hochschild-Serre spectral sequence converging to $H^*(W_{B_n};\mathbb{F}_p)$ whose E^2 page is isomorphic to $H^*(\mathbb{F}_2^n; H^*(W_{A_{n-1}};\mathbb{F}_p))$. Since $H^*(\mathbb{F}_2^n;\mathbb{F}_p)$ is trivial, the non-zero elements in this page are concentrated in the 0^{th} line. Thus, the spectral sequence degenerates at E^2 and yields a pair of isomorphisms $A_{A,p} \to A_{B,p}$ and $A_{B,p} \to A_{A,p}$ that are one the inverse of the other and that are identified with i^* and π^*.

A similar argument shows that ρ and tr are isomorphisms. Since they preserve $\cdot$ and $\odot$ respectively, as well as Δ, and since $A_{B,p}$ is a full Hopf ring, it follows that also $A_{D,p}$ is a Hopf ring and ρ, tr are Hopf ring homomorphisms. □

3.4 Additive basis and graphical description

Before completing the proof of Theorem 79 and Theorem 85, we explain how they imply the existence of an additive basis with a combinatorial and graphical description, similar to what happens for the symmetric groups. Therefore, in this section, we assume that the statements of these two theorems hold true, and we describe how an (almost-)Hopf ring expressed by the given presentation looks like.

We begin with A_B. The presentation of Theorem 79 is formally very similar to that of Theorem 37. The only difference is the introduction of some new generators δ_n. Thus, we can adapt the definitions of gathered block and gathered monomial given for the symmetric group by allowing the presence of there new classes as factors.

Definition 88. A *gathered block*, or simply a *block*, in A_B is an element of the form

$$b = \delta_m^{t_0} \prod_{k=1}^{n} \gamma_{k,\frac{m}{2^k}}^{t_k}$$

where m is a positive integer, 2^n divides m and n is the maximal index such that $\gamma_{n,\frac{m}{2^n}}$ appears in b with a non-zero exponent. The *profile* of b is the $(n+1)$-tuple $(t_0,\dots,t_n)$.

A *Hopf monomial*, or *gathered monomial*, in A_B is a transfer product of gathered blocks $x = b_1 \odot \cdots \odot b_r$ with pairwise different profiles. We denote with $\mathcal{M}$ the set of Hopf monomials.

The products and the coproduct of gathered monomials behave similarly to the case of the symmetric groups. In particular, given a block b, there is a unique block $\tilde{b}$ that minimize the component, among those with the same profile of b. Moreover, $b = \tilde{b}$ if and only if b is primitive.

Furthermore, the fact that $\cdot$ and Δ form a bialgebra implies, as already noted for A_A, that the coproduct of a block b is the sum of all tensor products $b' \otimes b''$ of blocks with the same profile of b that lies in the "correct" component.

With the exact same reasoning used at the end of Section 2.2, the following claim can be proved.

Claim. *Let b_1 and b_2 be blocks with the same profile. Assume that the components of b_1 and b_2 are $n2^k$ and $m2^k$ respectively, where 2^k is the component of* *the minimal block $\tilde{b}$ with that profile. Let b_3 be the block with the same profile and component $(n+m)2^k$. Then the following formula holds:*

$$b_1 \odot b_2 = \binom{n+m}{n} b_3$$

This, together with Kummer's theorem, implies that the Hopf ring described with the presentation of Theorem 79 is generated, under the transfer product alone, by blocks that lie in component indexed by powers of 2 and that, with $\odot$ and Δ, it has the structure of a divided powers Hopf algebra.

We make another remark regarding the similarity between the cup product in A_A and A_B. In our context, Definition 49 still makes sense. With this definition, the same formula for the cup product of Hopf monomials holds both in A_A and A_B:

$$(b_1 \odot \dots b_r) \cdot (b'_1 \odot \dots \odot b'_s) = \sum_{(\mathcal{P},\mathcal{P}')} \bigodot_{j=1}^{s} \bigodot_{i=1}^{r} (b_{i,j} b'_{j,i}) \qquad (*)$$

Here the sum is over all couples $(\mathcal{P}, \mathcal{P}')$ of sets $\mathcal{P} = \{(b_{i,1}, \dots, b_{i,s})\}_{i=1}^r$ and $\mathcal{P}' = \{(b'_{i,1}, \dots, b'_{i,r})\}_{j=1}^s$, such that $(b_{i,1}, \dots, b_{i,s})$ is a partition of b_i and $(b'_{j,1}, \dots, b'_{j,r})$ is a partition of b'_j.

Proposition 89. *$\mathcal{M}_B$ is an additive basis for A_B.*

Proof. This is completely analogous to what we observed in the proof of Theorem 46. Consider the free divided powers Hopf algebra $(C, \odot, \Delta)$ generated by the primitive blocks:

$$C = \bigotimes_{b \in H^*(W_{B_{2^k}};\mathbb{F}_2) \text{ block}} \frac{\mathbb{F}_2[b]}{(b^2)}$$

There is an obvious surjective morphism of divided powers Hopf algebras $\varphi\colon C \to A_B$.

In order to prove that this is an isomorphism, it is sufficient to note that there is a Hopf ring structure on C extending its Hopf algebra structure, such that φ is a morphism of Hopf rings. Let b_1 and b_2 be blocks with profile $(t_0, \dots, t_n)$ and $(r_0, \dots, r_m)$ respectively. The second product $\cdot$ on C is defined on gathered blocks by imposing that $b_1 \cdot b_2$ is the block that lies in the same component with profile $(t_0{+}r_0, \dots, t_n{+}r_n)$. Since the coproduct of a block b is a sum of elements of the form $b' \otimes b''$, where b' and b'' are gathered blocks, there is a unique way to extend $\cdot$ to all C in such a way that the Hopf ring distributivity formula holds. Explicitly, we extend this product on Hopf monomials in C by means of $(*)$. To prove that $\cdot$ is associative and commutative and that it forms a bialgebra with Δ, we reduce to check these properties when the relevant maps are applied to gathered blocks. $\square$

We can describe this additive basis graphically. We represent $\gamma_{k,n}$ as a rectangle of width $n2^k$ and height $1-2^{-k}$ and δ_n as a rectangle of width n and height 1. The width of a box is the number of the component to which the class belongs, while its area is its cohomological dimension. The cup product of two generators is to be understood as stacking the corresponding boxes one on top of the other, while their transfer product corresponds graphically to placing them next to each other horizontally. The profile of a gathered block is described by the height of the rectangles of the corresponding column. Thus, every gathered block is described as a column made of boxes that have the same width. Hence, an element of $\mathcal{M}_B$ is a diagram consisting of columns placed next to each other, such that there are not two columns that consist of rectangles of the same height. Following the notation we used in Chapter 2, we call these objects *B-skyline diagrams*, or simply *skyline diagrams* where there it is understood that we are taking into consideration the Hopf ring A_B.

The formulas for the calculation of the coproduct and the two products have graphical counterparts that are, once again, formally identical to those for A_A.

- We divide rectangles corresponding to δ_n or $\gamma_{k,n}$ in n equal parts via vertical dashed lines. The coproduct is then given by dividing along all vertical lines (dashed or not) of full height and then partitioning the new columns into two to make two new skyline diagrams.
- The transfer product of two skyline diagrams is given by placing them next to each other, and merging every two column with constituent boxes of the same heights, with a coefficient of 0 if the widths of these columns share a 1 in their binary expansion.
- To compute the cup product of two diagrams, we consider all possible ways to split each into columns, along vertical lines (dashed or not) of full height. We then match columns of each in all possible ways up to automorphism and stack the resulting matched columns to build a new diagram.

Some examples of calculations with skyline diagrams are depicted in Figure 3.4.

We now turn our attention to A_D. In this case, the additive basis $\mathcal{M}_D$ we obtain is sensibly different. We will construct $\mathcal{M}_D$ as the union of three subsets: some elements will be restrictions of certain classes in $\mathcal{M}_B$, and will be invariant under the action of the involution ι. The restriction of the remaining basis elements in $\mathcal{M}_B$ will be written as a sum of a "positively charged" class and a "negatively charged" class, that are swapped by ι. We need to add these two additional families of classes to obtain $\mathcal{M}_D$.

First, we give the definition of gathered block and gathered monomial in A_D.

Definition 90. A *gathered block*, or simply a *block*, in A_D is an element $b \in A_D$ that can be written as cup product of classes of the form $\delta^0_{n:m}$, with $n \geq 2$ and $m \geq 0$, or of the form $\delta^0_{n:0}$ and $\gamma^+_{k,m}$, or of the form $\delta^0_{n:0}$ and $\gamma^-_{k,m}$. We call these three types of classes *neutrally charged*, *positively charged* and *negatively charged* blocks respectively. The *profile* of a positively or negatively charged block $b = \delta^{t_0}_{2^n:0} \prod_{i=1}^n \gamma^{t_i}_{i,2^{n-i}}$ is $(t_0, \ldots, t_n)$. A *Hopf monomial* is a transfer product of gathered blocks $x = b_1 \odot \cdots \odot b_r$ with the following conditions:

- at most one of the b_i is non-positively charged
- the non-neutrally charged blocks have pairwise different profiles

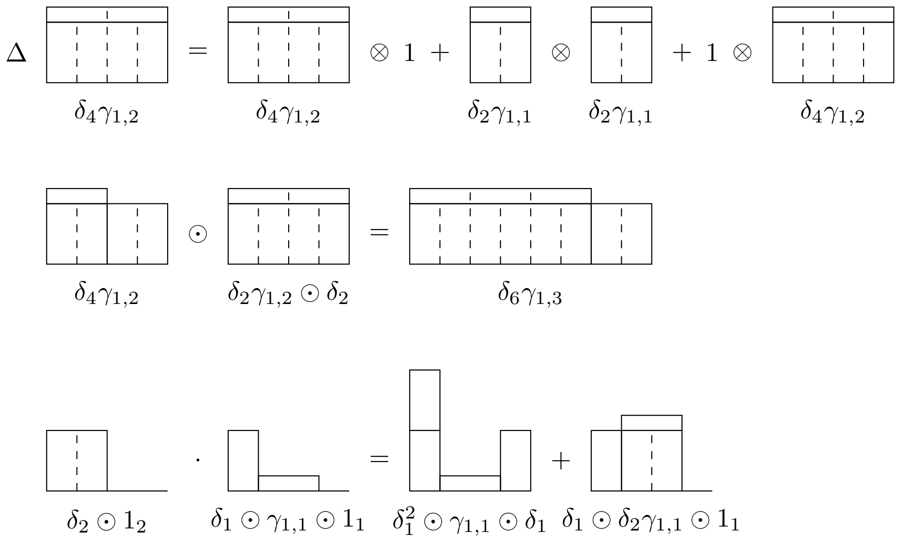


Figure 3.1: Computations via skyline diagrams

- if there is a negatively charged b_i, it is the block with the biggest profile (in a predetermined ordering, e.g. lexicographic)

We let $\mathcal{M}_D$ be the set of Hopf monomials in A_D.

First, note that gathered blocks, as defined above, accounts for all the classes that can be obtained by cup-multiplying the elements $\delta^0_{n:m}$, $\gamma^+_{k,m}$, and $\gamma^-_{k,m}$. More precisely, every such product is a gathered block or a transfer product of gathered blocks. In order to see this, observe that the cup product of a "+" generator and a "−" generator is either 0 (by the first relation of Lemma 83) or is a transfer product of two gathered blocks (by the second relation in the statement of that same Lemma). Thus, we only need to check that the cup product of "0" and "+" generators, or of "0" and "−" generators has that form. Since the property of being a transfer product of gathered blocks is not changed by the application of ι and ι swaps positively and negatively charged generators, it is sufficient to prove this for "0" and "+" generators. In this particular case, the claim follows from the last point of Lemma 83 if a positively charged generator and a neutrally charged one different from $\delta_{n:0}$ appear as factors. If only "+" generators and $\delta_{n:0}$ appear, the resulting class is a positively charged block. Otherwise, only "0" generators appear. In this last case, we need to prove that the resulting class is a linear combination of neutrally charged blocks. With this goal in mind, let $\tilde{\mathcal{B}}^0$ be the set of Hopf monomials $x \in A_B$ of the form $x = \delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}$, with $a_1 > \cdots > a_r$ and $k_1 \geq 2$. Then, let $\mathcal{B}^0$ be the image of $\tilde{\mathcal{B}}^0$ in A_D. Our claim in this last case is an immediate consequence of the following lemma.

Lemma 91. *Let $n \geq 0$. Every element of $\tilde{\mathcal{B}}^0 \cap H^*(W_{B_n}; \mathbb{F}_2)$ lies in the cup product subalgebra generated by $\delta_n, \delta_{n-1} \odot 1_1, \ldots, \delta_1 \odot 1_{n-1}$. Moreover, $\mathcal{B}^0$ is a vector space basis for the sub-Hopf ring generated by elements of the form $\delta^0_{n:m}$ for $n, m \geq 0$.*

Proof. First, we define $\overline{\mathcal{B}}^0$ as the set of Hopf monomials $x \in A_B$ of the form $x = \delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}$ with $a_1 > \cdots > a_r$, without the additional condition $k_1 \geq 2$. We can construct a function $\varepsilon_n \colon \overline{\mathcal{B}}^0 \cap H^*(W_{B_n}; \mathbb{F}_2) \to \mathbb{N}^n$ given by

$$\varepsilon_n(\delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}) = (\underbrace{a_1, \ldots, a_1}_{k_1 \text{ times}}, a_2, \ldots, a_{r-1}, \underbrace{a_r, \ldots, a_r}_{k_r \text{ times}})$$

ε_n is obviously injective. By identifying $\overline{\mathcal{B}}^0 \cap H^*(W_{B_n}; \mathbb{F}_2)$ with a subset of $\mathbb{N}^n$ this way, the lexicographic ordering on $\mathbb{N}^n$ induces a total ordering on $\overline{\mathcal{B}}^0$. Note that, by iteratively applying the Hopf distributivity formula, $\prod_{i=1}^n (\delta_k \odot 1_{n-k})^{a_k}$ can be written as a linear combination in $\overline{\mathcal{B}}^0$. The biggest non-zero Hopf monomial that appears in this linear combination with respect to the previously defined ordering corresponds to $(\sum_{i=1}^n a_i, \sum_{i=2}^n a_i \ldots, a_{n-1} + a_n, a_n)$. To prove this claim, note that, if $x = \bigodot_{i=1}^r \delta_{k_i}^{a_i}$ belong to $\overline{\mathcal{B}}^0$, then Hopf ring distributivity gives

$$x \cdot (\delta_h \odot 1_{n-h}) = \sum_{b_i + c_i = k_i} \bigodot_{i=1}^r (\delta_{b_i}^{a_i+1} \odot \delta_{c_i}^{a_i}).$$

Let j be the index such that $\sum_{i=1}^j k_j \leq h < \sum_{i=1}^{j+1} k_j$. The biggest of the addends appearing in the expansion above corresponds, via ε_n, to

$$\Big(\underbrace{a_1 + 1, \ldots, a_1 + 1}_{k_1 \text{ times}}, a_2 + 1, \ldots, a_j + 1, \underbrace{a_{j+1} + 1, \ldots, a_{j+1} + 1}_{h - \sum_{i=1}^j k_i \text{ times}}, \underbrace{a_{j+1}, \ldots, a_{j+1}}_{\sum_{i=1}^{j+1} k_i - h \text{ times}}, \ldots, \underbrace{a_r, \ldots, a_r}_{k_r \text{ times}}\Big)$$

and it appears with a coefficient of 1. By proceeding inductively on the number of factors and exploiting this calculation the claim is easily proved.

Moreover, note that this biggest addend of $\prod_{i=1}^n (\delta_k \odot 1_{n-k})^{a_k}$ is an element of $\tilde{\mathcal{B}}^0$ if and only if $a_1 = 0$, i.e. if and only if $\delta_1 \odot 1_{n-1}$ does not appear as a factor. Thus, the "leading term" functions $\{(a_1, \ldots, a_n) \in \mathbb{N}^n\} \to \overline{\mathcal{B}}^0$ and $\{(a_2, \ldots, a_n) \in \mathbb{N}^{n-1}\} \to \tilde{\mathcal{B}}^0$ are well-defined and injective. Hence, the classes $\delta_n, \delta_{n-1} \odot 1_1, \ldots, \delta_1 \odot 1_{n-1}$ generate, under the cup product, a polynomial sub-algebra with basis $\overline{\mathcal{B}}^0$, and $\mathcal{B}^0$ is a basis for the cup product sub-algebra generated by the elements $\delta_{n:m}^0$. Since the transfer products of these elements are trivial, the lemma follows. $\square$

Hence, gathered blocks generate the almost-Hopf ring described by the presentation of Theorem 85 as an algebra under $\odot$. We now check that Hopf monomials account for all elements obtained by applying transfer products to gathered blocks and, as a consequence, generate this almost-Hopf ring additively. We need to prove that an arbitrary transfer product of blocks $x = b_1 \odot \cdots \odot b_r$, without the constraints required by the definition of Hopf monomial, can indeed be written as a linear combination of Hopf monomials. First, by Lemma 82, transfer products of neutrally charged generators are zero. Thus, if more than one of the b_is is negatively charged, then $x = 0$. Hence, we can assume that all the blocks except possibly one are positively or negatively charged. Furthermore, if we have a negatively charged block, changing the charge of any other b_i yields the same Hopf monomial x, because of the last relation of Lemma 82. We can thus suppose that all the other b_is are positively charged. Recall that the identity $1^- \odot 1^- = 1^+$ holds and that each negatively charged block can be written as $1^- \odot b$, where b is another positively charged block. This implies that, if at least two of the b_is are negatively charged, for example, b_1 and b_2, we can actually rewrite $b_1 \odot b_2$ as a transfer product of positively charged blocks. Thus, we can assume that x only have at most one neutrally or negatively charged block. Finally, if two of the b_is have the same profile, we can use the exact same algorithm that we explained for A_B to reduce their transfer product to a multiple of a single gathered block. Thus, we conclude that $\mathcal{M}_D$ additively generates A_D.

We already remarked that a certain family of gathered blocks in A_B, that we denoted with $\tilde{\mathcal{B}}^0$, restrict to the set of neutrally charged blocks in A_D. The following lemma described the behavior of the remaining blocks in A_B.

Lemma 92. *Let $b = \delta_n^{a_0} \prod_{k\geq 1} \gamma_{k,\frac{n}{2^k}}^{a_k} \in A_B$ be a gathered block with $a_k > 0$ for at least one $k \geq 1$, i.e. containing some $\gamma_{k,m}$ for some k and m. Then the restriction of b to A'_D is the sum of the two elements*

$$b^+ = (\delta^0_{n:0})^{a_0} \prod_{k\geq 1} (\gamma^+_{k,\frac{n}{2^k}})^{a_k}$$
$$b^- = (\delta^0_{n_0})^{a_0} \prod_{k\geq 1} (\gamma^-_{k,\frac{n}{2^k}})^{a_k}$$

that are switched by the action of ι. Moreover, the profile of b^+ and b^- is the profile of b.

Proof. If b has a single factor, the claim has been already proved in Proposition 75, that is a geometric interpretation of this fact. The general case follows by induction on the number of factors, since ι and the restriction preserve the cup product. □

It is also noteworthy to observe explicitly how the charge behave when we compute the transfer product of blocks with the same profile. It basically follows the rule of signs used for the multiplication of integer numbers. The transfer product of two blocks with the same charge and the same profile is positively charged, while the transfer product of two blocks with opposite charge and the same profile is negatively charged.

Before proving that $\mathcal{M}_D$ is actually a basis for our almost-Hopf ring, we summarize how Hopf monomials in $\mathcal{M}_B$ give rise to elements of $\mathcal{M}_D$. Given $x \in \mathcal{M}_B$, we can write $x = x' \odot x''$, where x' lies in the sub-Hopf ring generated by $\{\delta_n\}_{n\in\mathbb{N}}$ and no constituent block of x'' lies in this sub-Hopf ring. We have three cases:

- If $x' \neq 1$ and its column of maximal height (as skyline diagram) has width 1, then its image in A'_D is 0.
- If $x' \neq 1$ and we are not in the previous case, then the restriction x^0 to A_D is a neutrally charged gathered block. By Lemma 91 all the restrictions of elements of $\tilde{\mathcal{B}}^0$ arise this way.

- If $x' = 1$, $x = b_1 \odot \cdots \odot b_{r-1} \odot b_r$, then the image of x in A'_D is the sum of two elements of $\mathcal{M}_D$, one positively charged and one negatively charged, that are switched by the action of ι. Explicitly, these elements are $x^+ = b_1^+ \odot \cdots \odot b_r^+$ and $x^- = b_1^+ \odot \cdots \odot b_{r-1}^+ \odot b_r^-$.

We denote with the symbols $\mathcal{M}^0$, $\mathcal{M}^+$ and $\mathcal{M}^-$ the set of elements x^0, x^+ and x^- respectively arising from Hopf monomials in A_B as described above. Moreover, $\mathcal{M}_D$ is the disjoint union of $\mathcal{M}^0$, $\mathcal{M}^+$, and $\mathcal{M}^-$.

Proposition 93. *$\mathcal{M}_D \cup \{1^-\}$ is a basis for the Hopf ring described by the presentation of Theorem 85.*

Proof. We already noted that $\mathcal{M}_D$ is a set of vector-space generators of A_D. Thus, it is sufficient to prove that this set is linearly independent. The strategy is the same as A_B: we construct a Hopf ring structure on the $\mathbb{F}_2$-vector space generated by $\mathcal{M}_D$ and check that all the desired relations are satisfied. Let $\mathcal{M}' \subseteq \mathcal{M}_B$ the set of Hopf monomials $x \in A_B$ such that at least one constituent block of x belongs to the sub-Hopf ring generated by $\{\delta_n\}_{n \geq 2}$. Recall that the restriction map determines a surjection $x \in \mathcal{M}' \mapsto x^0 \in \mathcal{M}^0$. Let $\mathcal{M}'' \subseteq \mathcal{M}_B$ be the set of Hopf monomials $x \in A_B$ that no constituent block of x belongs to the sub-Hopf ring generated by $\{\delta_n\}_{n \geq 1}$. Recall that there are bijections $x \in \mathcal{M}'' \mapsto x^+ \in \mathcal{M}^+$ and $x \in \mathcal{M}'' \mapsto x^- \in \mathcal{M}^-$. As a notational convention, if $x \in \mathcal{M}_B \setminus (\mathcal{M}' \cup \mathcal{M}'')$, put $x^0 = 0$.

Let V be the $\mathbb{F}_2$-vector space with basis $\mathcal{M}_D$. Define a coproduct map $\Delta\colon V \to V \otimes V$ by letting, for all $x \in \mathcal{M}'$, $\Delta(x^0) = (\_^0 \otimes \_^0) \circ \Delta|_{\mathcal{M}'}$. This is well-defined because $\Delta(\mathcal{M}') \subseteq \mathrm{Span}(\mathcal{M}')^{\otimes^2}$. For all $x \in \mathcal{M}'$, we can write $\Delta(x) = \sum_i x'_i \otimes x''_i$, for some $x'_i, x''_i \in \mathcal{M}_B$. For any $x \in \mathcal{M}_B$, let $\alpha^+(x) = x^+$ and $\alpha^-(x) = x^-$ if $x \in \mathcal{M}''$, $\alpha^+(x) = x^0$ and $\alpha^-(x) = 0$ otherwise. Define $\Delta(x^+)$ via the formula:

$$\Delta(x^+) = \sum_i \alpha^+(x'_i) \otimes \alpha^+(x''_i) + \alpha^-(x'_i) \otimes \alpha^-(x''_i)$$

Let $\Delta(x^-) = (\iota \otimes \mathrm{id}) \circ \Delta(x^+)$.

Moreover, note that, for all $x, y \in \mathcal{M}_B$, $x \odot y$, when not zero, is a multiple of a Hopf monomial in $\mathcal{M}''$ if and only if $x, y \in \mathcal{M}''$. Hence, we can construct a well-defined product $\odot\colon V \otimes V \to V$ by letting:

- $x^+ \odot y^+ = x^- \odot y^- = (x \odot y)^+$, $x^+ \odot y^- = x^- \odot y^+ = (x \odot y)^-$ if $x, y \in \mathcal{M}''$
- $x^+ \odot y^0 = y^0 \odot x^+ = (x \odot y)^0$ if $x \in \mathcal{M}''$, $y \notin \mathcal{M}''$
- $x^0 \odot y^0 = 0$ if $x, y \notin \mathcal{M}''$

With these definitions, it is tedious, but completely straightforward, to check that $(V, \odot, \Delta)$ satisfies the relations of Lemma 82 and 84.

Finally, define a second product $\cdot\colon V \otimes V \to V$ as follows. On gathered blocks let $x^+ \cdot y^+ = (xy)^+$ and $x^- \cdot y^- = (xy)^-$ for all $x, y \in \mathcal{M}''$ that are gathered blocks, and $x^0 \cdot y^0 = (xy)^0$ if $x, y \in \mathcal{M}'$ gathered blocks. Construct all other cup products of gathered blocks in $\mathcal{A}_D$ via the formulas of Lemma 83. Then, extend $\cdot$ on all $\mathcal{M}_D$ by imposing the Hopf ring distributivity formula.

The resulting $(A, \odot, \cdot, \Delta)$ is automatically an almost-Hopf ring and all our relations are satisfied. Moreover, the map $V \to A_D$ is, by construction, a morphism of almost-Hopf rings. Thus, the map $V \to A_D$ must be an isomorphism. $\square$

3.5 Restriction to elementary abelian subgroups and proof of the main theorems

In this section, we recall a description the elementary abelian 2-subgroups of W_{B_n} and W_{D_n}, and we calculate the restriction of our generators to these subgroups, effectively computing the Quillen map. In this regard, much has been done in Swenson's thesis [39]. We then exploit this framework to complete the proof of Theorem 79 and Theorem 85.

First of all, recall that the Quillen map of a finite group has nilpotent kernel and cokernel. In the case of finite reflection groups, an even stronger result holds, that Swenson attributes to Feshbach and Lannes.

Theorem 94. *[39, Theorem 11, page 2] If G is a finite reflection group, then the mod 2 Quillen map q_G is an isomorphism.*

For this reason, Swenson has calculated the elementary abelian 2-subgroups of W_{B_n} and W_{D_n} with the goal of obtaining some (highly recursive) description of $H^*(W_{B_n};\mathbb{F}_2)$ and $H^*(W_{D_n};\mathbb{F}_2)$ as rings. In this regard, recall from Section 2.4 that to any partition π of n whose parts are powers of 2 is associated an elementary abelian 2-subgroup $V_\pi \leq \Sigma_n$. When $n = 2^k$ and $\pi = (n)$, V_π is given by the regular representation of $\mathbb{F}_2$ and is, up to conjugation, the unique transitive elementary abelian 2-subgroup of Σ_{2^k}. In general, if $\pi = (2^{k_1}, \dots, 2^{k_r})$, then V_π is the direct product $V_{k_1} \times \cdots \times V_{k_r}$, embedded in $\Sigma_{\sum_{i=1}^r 2^{k_i}}$ by factoring in the obvious way through $\Sigma_{2^{k_1}} \times \cdots \times \Sigma_{2^{k_r}}$. We recalled in Section 2.4 that, in the case of odd prime coefficients, the invariant algebra of $H^*(V_k;\mathbb{F}_2)$ with respect to the action of Σ_{p^k} is described by Múi invariants. In the mod 2 case, this invariant subalgebra is simpler and can be described as the free commutative algebra over $\mathbb{F}_2$ generated by the classical Dickson invariants $d_{i,k-i} \in H^{2^k-2^i}(V_k;\mathbb{F}_2)$, for $0 \leq i < k$. Moreover, recall that every maximal abelian subgroup of Σ_n is conjugated to V_π for some π partition of n.

We can build upon this result to carry on our calculation of the elementary abelian 2-subgroups of W_{B_n}. We begin with a definition:

Definition 95. Let π be a partition of n. Define $A_\pi \leq W_{B_n}$ as follows. If $n = 2^k$ and $\pi = (2^k)$, let $A_\pi = V_k \times \langle \tau_k \rangle$, where V_k is embedded in $W_{B_{2^k}}$ via the natural inclusion $\Sigma_{2^k} \to W_{B_{2^k}}$ and τ is the image of the non-zero element $1 \in \mathbb{F}_2$ via the iterated diagonal morphism $\Delta^{2^k} : \mathbb{F}_2 \to (\mathbb{F}_2)^{2^k}$. In the general case, where $\pi = (2^{k_1}, \dots, 2^{k_r})$, define A_π as the product $A_{(2^{k_1})} \times \cdots \times A_{(2^{k_r})}$, embedded in Σ_n by factoring in the obvious way through $\Sigma_{2^{k_1}} \times \cdots \times \Sigma_{2^{k_r}}$.

Note that, we the previous notation, the cardinality of A_π is $2^{\sum_{i=1}^r (k_i+1)}$, and that A_π is the semidirect product $(\mathbb{F}_2)^n \rtimes V_\pi$.

Now, consider the action of W_{B_n} on $\{1, \dots, n\}$ determined by the natural projection $\pi \colon W_{B_n} \to \Sigma_n$. Let A be a maximal elementary abelian 2-subgroup of W_{B_n} that is transitive on $\{1,\dots,n\}$. Then, clearly, π must, once again, be transitive. Hence, $n = 2^k$ and $\pi(A)$ is conjugate to V_k. Furthermore, note that $\langle\tau_k\rangle$ is the center of $W_{B_{2^k}}$, and thus it must be contained in A, since A is maximal. In other words, there is a conjugate A' of A in $W_{B_{2^k}}$ such that $A_{(2^k)} \subseteq A'$. Consider an element $x = (\underline{t},\sigma) \in (\mathbb{F}_2)^{2^k} \rtimes \Sigma_{2^k}$. By construction, x commutes with every element of $A_{(2^k)}$ if and only if t and σ commute with every element of V_k. Since V_k is transitive and maximal as an elementary abelian 2-subgroup, this implies that $t \in \langle\tau\rangle$ and $\sigma \in V_k$. Due to this fact, $A_{(2^k)}$ is itself maximal, hence $A' = A_{(2^k)}$.

In general, let A a maximal elementary abelian 2-subgroup of W_{B_n}, not necessarily transitive. Up to conjugation, we can assume that the orbits of the action of A on $\{1,\dots,n\}$ are

$$\{1,\dots,m_1\},\{m_1+1,\dots,m_1+m_2\},\dots,\{\sum_{i=1}^{r-1} m_i+1,\dots,n=\sum_{i=1}^{r} m_i\}.$$

For all $1 \leq i \leq r$, let $A_i = A \cap (\mathbb{F}_2)^{m_i} \rtimes \mathrm{Sym}\{\sum_{j=1}^{i-1} m_j+1,\dots,\sum_{j=1}^{i} m_i\}$. By maximality, we must have $A = \prod_{i=1}^{r} A_i$. Since A_i is transitive on the i^{th} orbit, we must have that $m_i = 2^{k_i}$ for some k_i and that A_i is conjugate to V_{k_i} or, equivalent, A is conjugate to A_π, where $\pi = (2^{k_1},\dots,2^{k_r})$. The A_πs are pairwise non-conjugate because their orbits have pairwise different cardinalities, thus they constitute a set of representatives for the conjugacy classes of maximal abelian 2-subgroups. This fact also appears in [39], where the invariant subalgebra of $H^*(A_\pi;\mathbb{F}_2)$ is also calculated.

Proposition 96. *[39] Let n be an integer. $\{A_\pi\}_{\pi \text{partition of } n}$ is a set of representatives for the conjugacy classes of maximal elementaries abelian 2-subgroups in W_{B_n}. Let $d_{2^i-1},\dots,d_{2^i-2^{i-1}}$ the Dickson invariants in $H^*(V_i;\mathbb{F}_2) \hookrightarrow H^*(A_{(2^i)};\mathbb{F}_2)$ and define*

$$f_{2^i} = \prod_{y\in H^1(V_i;\mathbb{F}_2)} (x+y)$$

where $x \in H^1(A_{2^i};\mathbb{F}_2)$ is the linear dual to the non-trivial element in the C_i-factor of A_{2^i}. There is a natural isomorphism:

$$[H^*(A_\pi;\mathbb{F}_2t)]^{N_{W_{B_n}}(A_\pi;\mathbb{F}_2)} \cong \bigotimes_i (\mathbb{F}_2[f_{2^i},d_{2^i-1},\dots,d_{2^i-2^{i-1}}]^{\otimes m_i})^{\Sigma_{m_i}}$$

We can calculate the restriction of our generating classes $\gamma_{k,n}$ and δ_n to these abelian subgroups.

Proposition 97. *Let $l,n \geq 1$. Let π be a partition of $n2^l$ consisting of powers of 2, $\pi = (2^{k_1},\dots,2^{k_r})$. Then:*

$$\gamma_{l,n}|_{A_\pi} = \begin{cases} \otimes_{i=1}^{r} d_{2^{k_i}-2^{k_i-l}} & \text{if } k_i \geq l \ \forall 1 \leq i \leq r \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since the subgroup $(\mathbb{F}_2)^n \leq W_{B_n} = (\mathbb{F}_2)^n\rangle\Sigma_n$ is mapped to 0 via $\pi\colon W_{B_n} \to \Sigma_n$, the restriction of $\gamma_{l,n}$ pairs trivially with classes arising from the homology of $(\mathbb{F}_2)^n$. This implies that this restriction must be contained in the subspace $[H^*(A_\pi \cap \Sigma_n; \mathbb{F}_2)]^{\Sigma_n} \leq [H^*(A_\pi; \mathbb{F}_2)]^{W_{B_n}}$. The restriction to $A_\pi \cap \Sigma_n$ is calculated by Giusti, Salvatore, and Sinha in [13, Corollary 7.6, page 189]. $\square$

Proposition 98. *Let $n \geq 0$. Let $\pi = (2^{k_1}, \dots, 2^{k_r})$ be a partition of n consisting of powers of 2. The restriction of δ_n to the cohomology of the maximal elementary abelian 2-subgroup $\tilde{A}_{(\pi)}$ is equal to $\otimes_{i=1}^r f_{2^{k_i}}$. Moreover, δ_n is the unique class in $H^n(B_n; \mathbb{F}_2)$ that has this property for every π.*

Proof. We observe that the restrictions of a cohomology class to A_π for all the partitions π of n determine its restriction to every elementary abelian 2-subgroup (not necessarily maximal). Hence, by Theorem 94, a class that satisfies the condition in the statement for every π is necessarily unique.

Let $U_n = \mathbb{R}^n$ be the reflection representation of W_{B_n}. Recall that, if $n = 2^k$ and $\pi = (2^k)$, then $A_\pi = V_k \times C_k$, where $C_k = \langle t \rangle$ is a cyclic group of order 2, the center of W_{B_n}. Given $a \in A_\pi$, let ε_a, sgn_a, and $\mathbb{R}\langle a \rangle$ be the 1-dimensional trivial representation, the signum representation and the regular representation of $\langle a \rangle \cong \mathbb{F}_2$ respectively. We first observe that, since t acts on U_n as the multiplication by -1, $U_n|_{A_\pi} \cong \mathrm{sgn}_t \otimes U_n|_{V_k}$. Moreover, the inclusion of V_k in Σ_{2^k} is given by the regular representation, hence:

$$U_n|_{V_k} \cong \bigotimes_{i=1}^n \mathbb{R}\langle v_i \rangle \cong \bigoplus_{S \subseteq \{1,\dots,n\}} \bigotimes_{i=1}^n U_{S,i}$$

where $U_{S,i}$ is equal to sgn_{v_i} if $i \in S$, to ε_{v_i} if $i \notin S$. Thus, with the notation used before in this document, the Stiefel-Whitney class of $U_n|_{A_\pi}$ is:

$$\prod_{S \subseteq \{1,\dots,n\}} (1 + x + \sum_{i \in S} y_i)$$

Its n-dimensional part is exactly f_{2^k}. Hence, the thesis for $\pi = (2^k)$ follows from the naturality of the characteristic classes.

In the case of a general partition $\pi = (2^{k_1}, \dots, 2^{k_r})$, the proposition follows from the fact that $A_\pi \cong \prod_{i=1}^r A_{(2^{k_i})}$ and $U_n|_{A_\pi} \cong \oplus_{i=1}^r U_{2^{k_i}}|_{A_{(2^{k_i})}}$. $\square$

With a similar reasoning, we can calculate how the Quillen map acts on the generators of A_D. We recall, as a preliminary result, Swenson's description of the elementary abelian 2-subgroups of W_{D_n} and their cohomology.

Theorem 99. *[39] Let π be an admissible partition of n. Let m_1 and m_2 be the multiplicities of 1 and 2 in π. We write $\pi = (2)^{m_1} \cup (4)^{m_2} \cup \pi'$. Let $A_\pi \leq W_{B_n}$ the maximal elementary abelian 2-subgroup corresponding to π and let $\widehat{A}_\pi = A_\pi \cap W_{D_n}$. Then $\widehat{A}_\pi$ is maximal as an elementary abelian subgroup of W_{D_n} if and only if $m_1 \neq 2$. Moreover:*

- *If $m_1 > 0$, then $\widehat{A}_\pi = \ker(\sum : \mathbb{F}_2^{m_1} \to \mathbb{F}_2) \times A_{(2)^{m_2} \cup \pi'}$. If $e_1, \dots, e_{m_1}$ are the elementary symmetric functions in $H^*(\mathbb{F}_2^{m_1}; \mathbb{F}_2) = H^*(A_{(1)^{m_1}}; \mathbb{F}_2)$, we define $\bar{e}_i = e_i + e_1 e_{i-1}$ if $2 \leq i < m$ and $\bar{e}_m = e_1 e_{m-1}$. There is an isomorphism:*

$$[H^*(\widehat{A}_\pi; \mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)} \cong \mathbb{F}_2[\bar{e}_2, \dots, \bar{e}_m] \otimes [H^*(A_{(2)^{m_2} \cup \pi'}; \mathbb{F}_2)]^{N_{W_{B_{n-m_1}}}}$$

Moreover, the restriction from the cohomology of $A_{(1)^m}$ to that of $\widehat{A}_{(1)^m}$ is given by $e_1 \mapsto 0$ and $e_i \mapsto \overline{e}_i$ if $2 \leq i \leq m$.

- *If $m_1 = 0$ and $m_2 > 0$, then $\widehat{A}_\pi = A_\pi$. Identifying $H^*(A_{(2)}; \mathbb{F}_2)^{\otimes m_2}$ as $\otimes_{i=1}^{m_2} \mathbb{F}_2[x_i, y_i]$, we can define:*

$$h_{m_2} = \sum_{\substack{S \subseteq \{1,\dots,m_2\} \\ |S|=2l}} \prod_{i \notin S} (x_i + y_i) \prod_{j \in S} x_j$$

Then $[H^*(\widehat{A}_\pi; \mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)}$ is the free $[H^*(A_\pi; \mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)}$-module with basis $\{1, h_{m_2}\}$.

- *If $m_1 = m_2 = 0$, then $\widehat{A}_\pi = A_\pi$ and $N_{W_{D_n}}(A_\pi) = N_{W_{B_n}}(A_\pi)$, hence:*

$$[H^*(\widehat{A}_\pi; \mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)} = [H^*(A_\pi; \mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)}$$

Moreover, if $m_1 \neq 0$ or $m_2 \neq 0$, then A_π is W_{B_n}-conjugate to A' if and only if $\widehat{A}_\pi$ is W_{D_n}-conjugate to $A' \cap W_{D_n}$. Conversely, if $m_1 = m_2 = 0$, then the W_{B_n}-conjugacy class of A_π contains exactly two W_{D_n}-conjugacy classes of elementary abelian 2-subgroups.

With the help of the previous theorem, we calculate the Quillen map for A_D.

Proposition 100. *Let $n = 2^k m$, for some $k, m \geq 1$. Let π be a partition of n with parts integral powers of 2. Let m_1 and m_2 be the multiplicities of 1 and 2 in π. Then:*

- *for every $k \geq 1$, if $m_1 = m_2 = 0$, then $\gamma^+_{k,m}|_{A_\pi} = \gamma_{k,m}|_{A_\pi}$, $\gamma^-_{k,m}|_{A_\pi^{s_0}} = 0$, $\gamma^-_{k,m}|_{A_\pi} = 0$, $\gamma^-_{k,m}|_{A_\pi^{s_0}} = \gamma_{k,m}|_{A_\pi^{s_0}}$*
- *for every $k \geq 2$, if $m_1 \neq 0$ or $m_2 \neq 0$, or for $k = 1$ if $m_1 \neq 0$, then $\gamma^\pm_{k,m}|_{\widehat{A}_\pi} = 0$.*
- *if $m_1 = 0$ but $m_2 \neq 0$, that is $\pi = (2)^{m_2} \sqcup \pi'$, then the restriction of $\gamma^+_{1,m}$ (respectively $\gamma^-_{1,m}$) to $\widehat{A}_\pi = A_{(2)^{m_2}} \times A_{\pi'}$ is $h_{m_2} \otimes \gamma_{1,m-m_2}|_{A'_\pi}$ (respectively $(d_1^{\otimes m_2} + h_{m_2}) \otimes \gamma_{1,m-m_2}|_{A'_\pi}$)*
- *if $\pi = (1)^{m_1} \sqcup \pi'$, then the restriction of $\delta^0_{k:m}$ to $\widehat{A}_\pi = \widehat{A}_{(1)^{m_1}} \times A_{\pi'}$ is $\sum_{i=2}^k \overline{e}_i \otimes (\delta_{k-i} \odot 1_{D_{m-m_1+i}})|_{A_{\pi'}}$.*

Proof. Assume that the partition π has at least 2 parts. In this case, by construction, the restriction to $\widehat{A}_\pi$ or $\widehat{A}_\pi^{s_0}$ factors through the coproduct. This allows us to use an induction argument on the number of parts of π, together with Lemma 82, to reduce to the base case consisting of partitions with only one part.

Thus, in this proof, we assume, from now on, that $\pi = (2^n)$. For all $2 \leq k \leq n$, let $l = n - k$. $\gamma^\pm_{k,2^l}$ restricts onto A_π $(n = k + l)$ to an $N_{D_{2^n}}(A_\pi)$-invariant class. The only classes with the right degree are 0 and $d_{2^n - 2^l}$. Note that $i_+^*(\gamma^+_{k,2^l}) = \gamma_{k,2^l}$ and $i_+^*(\gamma^-_{k,2^l}) = 0$. For this reason, Proposition 75 guarantees that the restriction of $\gamma^+_{k,2^l}$ (respectively $\gamma^-_{k,2^l}$) to $A_\pi \cap \Sigma_{2^n}$ is $d_{2^n - 2^l}$

(respectively 0). Thus, $\gamma^+_{k,2^l}|_{A_\pi} = d_{2^n-2^l} = \gamma_{k,2^l}|_{A_\pi}$ and $\gamma^-_{k,2^l}|_{A_\pi} = 0$. By using the same argument, with i_+ replaced by i_+, we carry on the calculation for the restriction to $A^{s_0}_\pi$ and we complete the proof of the first point.

We now turn our attention to the second point. The coproduct $\gamma^\pm_{k,2^l}$ does not have addends in the component $H^*(W_{D_1};\mathbb{F}_2) \otimes H^*(W_{D_{2^{k+l}-1}};\mathbb{F}_2)$ and, if $k \geq 2$, in the component $H^*(W_{D_2};\mathbb{F}_2) \otimes H^*(W_{D_{2^{k+l}-2}};\mathbb{F}_2)$.

As regarding the third point, observe that $A_{(2)} = W_{D_2}$ and that, with this identification, $\gamma^+_{1,1}$ is equal to h_1, while $\gamma^-_{1,1,}$ to $d_1 + h_1$.

Finally, Proposition 98 and Theorem 99 directly imply the last point. □

The remainder of this section is devoted to the proof of Theorems 79 and 85, describing the structure of A_B and A_D. We begin with A_B.

Proof of Theorem 79. Let $\tilde{A}_B$ be the Hopf ring generated by $\gamma_{k,m}$ and δ_m with the desired relations. Since the aforementioned relations hold in A_B, there exists an obvious morphism $\varphi\colon \tilde{A}_B \to A_B$. Moreover, an additive basis $\mathcal{M}_B$ for $\tilde{A}_B$ is described in Section 3.4. By Theorem 94 it is sufficient to prove that the Quillen map q maps $\mathcal{M}_B$ in a basis for the Quillen group.

We begin with gathered blocks $b \in A_B$. There exists n, m such that $b = \prod_{i=1}^n \gamma^{a_i}_{i,2^{n-i}m} \delta^{a_0}_{2^n m}$ with $a_n \neq 0$. Note that n and m are necessarily unique. Let $\pi_b = (2^n, \ldots, 2^n)$ be the partition of $2^n m$ consisting only of parts equal to 2^n. Note that the primitive gathered block $\tilde{b}$ with the same profile of b lies in the component indexed by 2^n, thus the coproduct of b has an addend in the component $H^*(W_{B_k};\mathbb{F}_2) \otimes H^*(W_{B_{2^n m-k}};\mathbb{F}_2)$ if and only if k is a multiple of 2^n. As a consequence of this fact, Proposition 97 and Proposition 98, since the restriction to the cohomology of $W_{B_k} \times W_{B_{2^n m-k}}$ is given by the coproduct, π_b is the maximal partition (with respect to refinement) among those having parts that are powers of 2 and such that $b|_{A_\pi} \neq 0$. Indeed, we have:

$$b|_{A_\pi} = \left(\tilde{b}|_{A_{(2^n)}}\right)^{\otimes m} = \left(f^{a_0}_{2^n} \prod_{i=1}^n d^{a_i}_{2^n-2^{n-i}}\right)^{\otimes m}$$

More generally, if $x = b_1 \odot \cdots \odot b_r \in \mathcal{M}_B$ is a Hopf monomial, define $\pi_x = \sqcup_{i=1}^r \pi_{b_i}$. Once again, π_x is the maximal partition with parts of the form 2^k such that $x|_{A_\pi} \neq 0$. In this general case, $x|_{A_{\pi_n}}$ is the symmetrization of $\bigotimes_{i=1}^r b_i|_{A_{\pi_{b_i}}}$.

This construction determines a decomposition of the set $\mathcal{M}_B$ as a disjoint union of the sets $\mathcal{M}_\pi = \{x \in \mathcal{M}_B : \pi_x = \pi\}$, with π varying among the considered partitions.

We can now prove the injectivity of φ. We give a proof by contradiction. Assume that there exists a non-trivial sum $\sum_i x_i$ of elements of $\mathcal{M}$ such that $\varphi(\sum_i x_i) = 0$. Then this sum is 0 when restricted to an arbitrary maximal elementary abelian 2-subgroup. Let π be a maximal element, with the ordering given by refinement, of the set $\{\pi_{x_i}\}_i$. The restriction of $\sum_i x_i$ on A_π is equal to the restriction of $\sum_{i:x_i \in \mathcal{M}_\pi} x_i$, because the contribution of all the other terms is zero by the maximality of π. Hence, under our assumptions, the latter must be 0. It is sufficient to check that the following claim is true.

Claim. *For all π, the restrictions of the elements of $\mathcal{M}_\pi$ to A_π are linearly independent.*

This clearly gives a contradiction and proves injectivity.

In order to prove the claim, recall from Proposition 96 that the invariant subalgebra $[H^*(A_{(2^k)};\mathbb{F}_{\not\asymp})]^{N_{W_{B_{2^k}}}(A_{(2^k)})}$ is a polynomial algebra generated by $f_{2^k}, d_{2^k-1}, \ldots, d_{2^{k-1}}$. Let $\mathcal{B}_k$ be the monomial basis of this polynomial ring. Assume that $\pi = (2^{k_1}, \ldots, 2^{k_r})$ and let $\mathcal{B}_\pi$ be the tensor product of these bases $\bigotimes_{i=1}^r \mathcal{B}_{k_i}$, that define a linearly independent set in $H^*(A_\pi;\mathbb{F}_2)$. An element of $\mathcal{B}_\pi$ appears as an addend of at most one Hopf monomial $x \in \mathcal{M}_\pi$. Indeed, if $\pi = (2^k)$, then $\mathcal{M}_\pi$ is the set of primitive gathered blocks in $H^*(W_{B_{2^k}};\mathbb{F}_2)$ and every monomial of $\mathcal{B}_k$ is the restriction of exactly one block $b_z \in \mathcal{M}$. In the general case, an element of $\mathcal{B}_\pi$ is a tensor product $\bigotimes_{j=1}^s z_j^{\otimes m_j}$, where $z_j \in \mathcal{B}_{k_j}$. The only possible Hopf monomial whose restriction to the cohomology of A_π contains $\bigotimes_{j=1}^s z_j^{\otimes m_j}$ is $\bigodot_{j=1}^s b_{z_j,m_i}$, where $b_{z_j,l}$ is the unique gathered block that has the same profile of b_z and belongs to the component $2^{k_j}l$.

We now turn our attention to the proof of the surjectivity of φ. By Theorem 94, we only need to prove that an element α of the Quillen group $\mathcal{F}^*_{W_{B_n}}$ can be written as the image via $q_{W_{B_n}}$ of a linear combination of elements of $\mathcal{M}$. Recall that such α is a family $\{\alpha_\pi\}_{\pi\in\mathcal{P}}$, parametrized by the set $\mathcal{P}$ of partitions of n with parts that are powers of 2, of classes $\alpha_\pi \in [H^*(A_\pi;\mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)}$ that are compatible with restrictions.

Order $\mathcal{P}$ by refinement, as before, and extend this poset with a total ordering. Let $\overline{\pi}_\alpha = \max\{\pi \in \mathcal{P} : \alpha_\pi \neq 0\}$, where the maximum is taken with respect to this total ordering. Write $\overline{\pi}_\alpha = (2^{r_1}, \ldots, 2^{r_k})$ with $r_1 \leq \cdots \leq r_k$. We prove that α belongs to the image of $\operatorname{Span}\mathcal{M}_B$ by induction on $\overline{\pi}_\alpha$. $\alpha_{\overline{\pi}_\alpha}$ is a sum of tensor products of classes $c_1 \otimes \cdots \otimes c_k$, with $c_i \in H^*(A_{(2^{r_i})};\mathbb{F}_2)$ invariant by the action of the normalizer. Hence, we can assume that each c_i has the form $c_i = \prod_{l=1}^{r_i-1} d_{2^{r_i}-2^{r_i-l}}^{a_{i,l}} f_{2^{r_i}}^{a_0}$. We must have $a_{i,r_i} \neq 0$. Otherwise, by splitting the part 2^{r_i} in two equal part $(2^{r_i-1}, 2^{r_i-1})$, we obtain a partition π' bigger than $\overline{\pi}$ such that $\alpha_{\pi'} \neq 0$. This is enforced by the compatibility of the system $\{\alpha_\pi\}_\pi$, since $\alpha_{\overline{\pi}}$ restricts to a non-zero class on $H^*(A_{\overline{\pi}} \cap A_{\pi'};\mathbb{F}_2)$. This would contradict the maximality of $\overline{\pi}_\alpha$. Hence $a_{i,r_i} \neq 0$ for all i.

By our calculations above, there exists a Hopf monomial $x \in \mathcal{M}_{\overline{\pi}}$ such that $x|_{A_{\pi_x}} = c_1 \otimes \cdots \otimes c_k$. We have thus constructed some elements $x_j \in \mathcal{M}$ such that $\overline{\pi}_{\alpha - \sum_j q_{B_n}(x_j)} < \overline{\pi}_\alpha$. By applying induction, this proves that φ is surjective. □

We now deal with A_D. Before starting the proof of Theorem 85, it is necessary to complete those of Lemmas 83 and 84.

The proof of Lemma 83 is easy: Proposition 100 states that the desired relations hold true when we restrict all classes to elementary abelian subgroups. The injectivity of the Quillen map implies they are actually satisfied in A_D.

We now introduce a preliminary notion, that we will use to prove Lemma 84. First, if we operate the change of variables $z = x + y, w = x$, we observe that $H^*(A_{(2)};\mathbb{F}_2) = \mathbb{F}_2[z, w]$.

Definition 101. A monomial $z^a w^b$ in the classes z and w described above is called *positive* if $a > b$, *negative* if $a < b$. Let π be a partition of an integer n whose parts are powers of 2. Let m_1 be the multiplicity of 1 and m_2 be the multiplicity of 2. Assume that $m_1 = 0$ and $m_2 > 0$. We can write $\pi = (2)^{m_2} \sqcup \pi'$. A class $\alpha \in H^*(A_\pi;\mathbb{F}_2)$ is called *positive* (respectively *negative*) if it admits an expression of the form

$$\alpha = \sum_i g_{i,1} \otimes \cdots \otimes g_{i,m_2} \otimes \alpha_i$$

where $\alpha_i \in H^*(A_{\pi'};\mathbb{F}_2)$, $g_{i,j} \in H^*(A_{(2)};\mathbb{F}_2)$ are monomials in the variables z and w and, among the monomials $g_{i,1},\dots,g_{i,m_2}$, an even (respectively odd) number are negative, while the others are positive.

Lemma 102. *Let $x \in A_B$ be a Hopf monomial such that none of its constituent gathered blocks lies in the cup-product subalgebra generated by δ_n $(n \geq 1)$. Assume that x belongs to the l^{th} component. Let $x^+ \in A_D$ be the element obtained by substituting every instance of $\gamma_{k,m}$ in x with $\gamma^+_{k,m}$ and δ_m with $\delta_{m:0}$. Let π be a partition of l with parts that are powers of 2. If $1 \in \pi$, then the restriction of x^+ to $\widehat{A}_\pi$ is 0. If $1 \notin \pi$ and $2 \in \pi$, the restriction of x^+ to A_π is positive. If $1, 2 \notin \pi$, then the restriction of x^+ to $A_\pi^{s_0}$ is 0.*

Proof. First of all, consider the case of a partition π of l with $1, 2 \notin \pi$. Let $A = A_\pi^{s_0}$. We prove that the restriction of x^+ to A is 0 by induction on the number of constituent blocks in x:

- if x is a single gathered block b then, for some k and n, $\gamma^+_{k,n}$ appears in b as a cup-product factor. By Proposition 100 $\gamma^+_{k,n}$ restricts to 0 on A. Since restriction maps preserve cup products, also the restriction of x must be 0.
- if x has more than one constituent block, we can write $x = y \odot z$, where y and z are Hopf monomials with a lesser number of blocks, thus satisfying the lemma. Then $x^+ = y^+ \odot z^+$, as we have seen in the previous section. Let n and m be the component of y and z respectively. Consider a set of representatives $\mathcal{R}$ for the double cosets $A \backslash W_{D_r}/W_{D_n} \times W_{D_m}$. The classical Cartan-Eilenberg double coset formula yields:

$$\rho_A^{W_{D_l}}(y^+ \odot z^+) = \sum_{r \in \mathcal{R}} \mathrm{tr}^A_{A \cap r(W_{D_n} \times W_{D_m})r^{-1}} c_r \rho^{(W_{D_n} \times W_{D_m})}_{r^{-1}Ar \cap (W_{D_n} \times W_{D_m})}(y^+ \otimes z^+)$$

  Recall from Adem and Milgram's book [1] that the transfer map $\mathrm{tr}^A_{A \cap r(W_{D_n} \times W_{D_m})r^{-1}}$ between elementary abelian subgroups is 0 unless $A \cap r(W_{D_n} \times W_{D_m})r^{-1}$ and A are equal or, equivalently, $r^{-1}Ar$ is contained in $W_{D_n} \times W_{D_m}$. Note that if a subgroup is W_{D_l}-conjugated to A and is contained in $W_{D_n} \times W_{D_m}$, it can be written as $A^{s_0}_{\pi_1} \times A_{\pi_2}$ or $A_{\pi_1} \times A^{s_0}_{\pi_2}$ such that $\pi_1 \sqcup \pi_2 = \pi$. The restrictions of $y^+ \otimes z^+$ to these subgroups are all trivial by the induction hypothesis. The Cartan-Eilenberg formula completes the proof of the claim.

The same reasoning proves the assertion for $A = A_\pi$ with $1 \in \pi$ or $1 \notin \pi$ and $2 \in \pi$. More explicitly, it is true for single generators, because we already computed their restriction to elementary abelian subgroups, and it holds for gathered blocks because positivity is preserved by the cup product. Finally, in the case of a general Hopf monomial, we can inductively apply the same Cartan-Eilenberg formula as before, by observing that the conjugation $c^*_{s_0}$ maps positive monomials into negative monomials and vice versa. $\square$

Proof of Lemma 84. Consider the following diagram, for every $n, m > 0$.

$$\begin{tikzcd} W_{D_n} \times W_{D_m} \arrow[r] \arrow[d] & W_{D_n} \times W_{B_m} \arrow[d] \\ W_{D_{n+m}} \arrow[r] & W_{B_{n+m}} \end{tikzcd}$$

When we apply the classifying space functor to it, we obtain another a pullback of finite coverings (up to homotopy equivalence). Thus, if we let tr be the cohomology transfer map $H^*(W_{D_n}; \mathbb{F}_2) \to H^*(W_{B_n}; \mathbb{F}_2)$, the following formula holds:

$$b \odot \rho(y) = \rho(\mathrm{tr}(b) \odot y)$$

Assume now that b and x satisfy the hypothesis of the lemma. In A_D we have $\iota \circ \odot = \odot \circ (\iota \otimes \mathrm{id})$. Thus, up to an application of ι, we can assume, without loss of generality, that b is positively charged. This is equivalent to say that, with the notation of Lemma 102, $b = \tilde{b}^+$ for some block $\tilde{b} \in A_B$. In this case $\mathrm{tr}(b) = \tilde{b}$. We can write $\Delta(\tilde{b}) = \sum_i \tilde{b}'_i \otimes \tilde{b}''_i$. Then, by definition $\Delta'(b) = \sum_i (\tilde{b}'_i)^+ \otimes (\tilde{b}''_i)^+$.

We consider two cases separately:

- if x is fixed by ι, then $x = \rho(y)$ is the restriction of a class in A_B. In this case $b \odot x = \rho(\tilde{b} \odot y)$ by our formula, hence the previous remark suffices to prove our claim because both ρ and the transfer product in A_B commute with coproducts.

- if x is not fixed by ι, then, once again, up to an application of ι, we may assume that $x = y^+$. Thus $b \odot x = (\tilde{b} \odot y)^+$. Now let π_1, π_2 be partitions whose parts are powers of 2 that do not contain 1 or 2. The previous lemma states that the restriction of $\Delta(b \odot x)$ to $A_{\pi_1} \times A^{s_0}_{\pi_2}$ and $a^{s_0}_{\pi_1} \times A_{\pi_2}$ is 0. Similarly, if $1 \in \pi_1$ or $1 \in \pi_2$, its restriction to $\widehat{A}_{\pi_1} \times \widehat{A}_{\pi_2}$ is 0 and, if π_1 and π_2 do not contain 1 but at least one of them contains 2, the restriction to $A_{\pi_1} \times A_{\pi_2}$ is positive. Since we will exploit this property often during this proof, we give it a name: we say that a class satisfying this property has positive restriction to elementary abelian 2-subgroups. We observe that also the following class has positive restriction to elementary abelian 2-subgroups:

$$(\odot \otimes \odot) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta'(b) \otimes \Delta(x))$$

  In order to see this, observe that, being x positively charged, the coproduct of x contains only addends of the form $x' \otimes x''$, where x' and x'' are Hopf monomials with the same charge, while $\Delta'(b)$ has, by definition, only addends that are tensor products of positively charged gathered blocks. Thus, the cohomology class considered above is a sum of tensor products of Hopf monomials with the same charge. This, together with Lemma 102, suffices to prove the positive restriction to elementary abelian subgroups.

Now observe that, by applying ι, if $1,2 \notin \pi_1,\pi_2$ then $\Delta(b \odot \iota(x))$ restricts to 0 to $A_{\pi_1} \times A_{\pi_2}$ and $A_{\pi_1}^{s_0} \times A_{\pi_2}^{s_0}$, if $1 \in \pi_1$ or $1 \in \pi_2$ then it restricts to 0 to $\widehat{A}_{\pi_1} \times \widehat{A}_{\pi_2}$ and if $1 \notin \pi_1,\pi_2$, $2 \in \pi_1 \cup \pi_2$ then its restriction to $A_{\pi_1} \times A_{\pi_2}$ is negative. Once again, we use a special name for this property and we say that a class satisfying it has negative restriction to elementary abelian 2-subgroups. We observe that also the following class has negative restriction to elementary abelian 2-subgroups:

$$(\odot \otimes \odot) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta'(b) \otimes \Delta(\iota(x)))$$

This can be proved by using the same argument as before.

If a class $c \in A_D$ can be decomposed as the sum of a class with positive restriction and one with negative restriction to elementary abelian 2-subgroups, then this decomposition is clearly unique by Theorem 94.

Since $x + \iota(x)$ is the restriction of a class in A_B we have:

$$\Delta(b \odot x) + \Delta(b \odot \iota(x)) = (\odot \otimes \odot) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta' \otimes \Delta) \circ (\mathrm{id} + \mathrm{id} \otimes c_{s_0}^*)(b \otimes x)$$

Our claim now follows from the uniqueness of the considered decomposition.

□

Proof of Theorem 85. Let $\tilde{A}_D$ be the almost Hopf ring generated by elements of the form $\delta_{n:m}$, $\gamma_{k,m}^{\pm}$ and 1^- with the desired relations. Let $\varphi' \colon \tilde{A}_D \to A'_D$ be the obvious morphism. We want to prove that φ' is an isomorphism, or equivalently, that the composition $\varphi \colon A''_D \xrightarrow{\varphi} A'_D \twoheadrightarrow A_D$ is surjective and has kernel $\{0, 1^-\}$.

We first prove that the kernel contains only 0 and 1^-. Let us consider a sum of elements of the basis $\mathcal{M}_D$:

$$L = \sum_i x_i^0 + \sum_j y_j^+ + \sum_k z_k^-$$

with x_i, y_j and z_k Hopf monomials in A_B Assume that $\varphi(L) = 0$. Then

$$\varphi(L + \iota(L)) = \varphi(L) + c_{s_0}^* \varphi(L) = 0.$$

Note that $\varphi(L + \iota(L)) = \sum_j \rho(y_j) + \sum_k \rho(z_k)$ and, since y_j and z_k give rise to non-neuter monomials, their restriction to A_π for every partition π containing 1 is 0. Since $1 \notin \pi$ implies $\widehat{A}_\pi = A_\pi$, if $\varphi(L) = 0$ Theorem 94 guarantees that $\sum_i x_i + \sum_j y_j = 0$. Thus L must assume the following form:

$$L = \sum_i x_i^0 + \sum_j (y_j^+ + y_j^-)$$

We recall, from the proof of the presentation Theorem for A_B above, that for every $x \in \mathcal{M}$ there is a unique admissible partition π_x, maximal with respect to refinement, such that $x|_{A_{\pi_x}} \neq 0$ and that the set $\mathcal{M}_\pi$ of the basis elements x such that $\pi_x = \pi$ maps to a linearly independent set to the cohomology of A_π. Since for every π not containing 1 we have that $\widehat{A}_\pi = A_\pi$, this implies that, if $\varphi(L) = 0$, the elements x_i, y_j must satisfy $1 \in \pi_{x_i}$, $1 \in \pi_{y_j}$. Since elements

x such that $1 \in \pi_x$ always yield a "0" generator for A_D, we conclude that $L = \sum_i x_i^0$. Let π be a partition in which 1 has multiplicity $m > 0$. We write $\pi = (1)^m \sqcup \pi'$, thus $\widehat{A}_\pi = \widehat{A}_{(1)^m} \times A_{\pi'}$. In every x_i $\delta^0_{1:m}$ cannot appear, thus the restriction of x_i to A_π belongs to the subalgebra of $H^*(A_\pi;\mathbb{F}_2)$ generated by the elementary symmetric functions $e_2,\dots,e_m \in H^*(A_{(1)^m};\mathbb{F}_2)$ and the cohomology of $A_{\pi'}$. By Theorem 99 this subalgebra restricts injectively to $H^*(\widehat{A}_\pi;\mathbb{F}_2)$. Hence, $\varphi(L) = 0$ implies $L = 0$ and this proves that the kernel of φ contains only 0 and 1^-, or equivalently that φ' is injective.

We now turn our attention to the proof of the surjectivity. Let $\alpha \in \mathcal{F}^*_{W_{D_n}}$ be an element of the Quillen group for W_{D_n}. α can be described as a family of classes $\{\alpha_\pi\}_{\pi\in\mathcal{P}} \cup \{\alpha_{\pi,s_0}\}_{\pi\in\mathcal{P}:1,2\notin\mathcal{P}}$, where $\mathcal{P}$ of admissible partitions of n, such that α_π is a class in the cohomology of $\widehat{A}_\pi$ invariant by the action of its normalizer, and α_{π,s_0} is such a class in the cohomology of $\widehat{A}^{s_0}_\pi$. The proof of Theorem 79 gives us a class $x \in H^*(W_{B_n};\mathbb{F}_2)$ such that, for every partition $\pi \in \mathcal{P}$ for which $1 \in \pi$, $x|_{\widehat{A}_\pi} = \alpha_\pi$. Hence we will assume, from now on, that $\alpha_\pi = 0$ if $1 \in \pi$.

Let P_m and N_m be the subspaces of $H^*(A_{(2)^m};\mathbb{F}_2)$ generated by positive and negative monomials respectively, as defined in Section 3.5. For every m, let $g_m \colon H^*(A_{(2)^m};\mathbb{F}_2) \to H^*(A_{(2)^m};\mathbb{F}_2)$ be the vector space homomorphism that sends a monomial that does not belong to P_m to 0 and a positive monomial $z^a w^b \in P_m$ to $z^a w^b + z^b w^a$. It can be checked immediately that the image of g_m is contained in the subalgebra fixed by the action of the normalizer in WB_{2m}.

We now define, for every $\pi \in \mathcal{P}$, a class $\beta_\pi \in H^*(A_\pi;\mathbb{F}_2)$ as follows:

$$\beta_\pi = \begin{cases} \alpha_\pi & \text{if } 1,2 \notin \pi \\ 0 & \text{if } 1 \in \pi \\ g_m \otimes \mathrm{id}(\alpha_\pi) & \text{if } \pi = (2)^m \sqcup \pi', 1,2 \notin \pi' \end{cases}$$

We have $\beta_\pi = \sum_j (a_{j,\pi} + b_{j,\pi} d_1^{\otimes^m}) \otimes c_{j,\pi}$. We claim that $\beta = \{\beta_\pi\}_{\pi\in\mathcal{P}}$ defines an element in $\mathcal{F}^*_{W_{B_n}}$. To prove this claim, we first observe that the restrictions of h_2 and $d_1 \otimes d_1$ to the cohomology of $A_{(4)} \cap A_{(2,2)}$ coincide, which can be easily proved by a direct computation. Then, we note that if $\pi, \pi' \in \mathcal{P}$, $1 \notin \pi, \pi'$ and the multiplicity of 2 in π (respectively π') is $m \geq 0$ (respectively $m' \geq 0$), then $m' - m$ is even and, assuming $m' \geq m$,

$$A_\pi \cap A_{\pi'} \subseteq A_{(2)^m \sqcup (4)^{\frac{m'-m}{2}} \sqcup \pi''} \cap A_{(2)^{m'} \sqcup \pi''} = A_{(2)^m} \times \left(A_{(2,2)} \cap A_{(4)}\right)^{\frac{m'-m}{2}} \times A_{\pi''}$$

where $\pi' = (2)^m \sqcup \pi''$. Using the previous calculation for h_2 and $d_1 \otimes d_1$, we have by a simple induction argument that

$$\beta_\pi|_{A_{(2)^m} \times (A_{(2,2)} \cap A_{(4)})^{\frac{m'-m}{2}} \times A_{\pi''}} = \beta_{\pi'}|_{A_{(2)^m} \times (A_{(2,2)} \cap A_{(4)})^{\frac{m'-m}{2}} \times A_{\pi''}}$$

Thus $\beta \in \mathcal{F}^*_{W_{B_n}}$. Due to Theorem 79, there exists a unique sum $\sum_i x_i$ of Hopf monomials $x_i \in \mathcal{M}$ such that $\beta = q_{W_{B_n}}(\sum_i x_i)$. Since every x_i restricts to 0 on every A_π such that $1 \in \pi$, the element $\sum_i x_i^+$ is well defined and the following conditions are satisfied:

- if $1,2 \notin \pi$, then $(\sum_i x_i^+)|_{A_\pi} = \alpha_\pi$ and $(\sum_i x_i^+)|_{A_\pi^{s_0}} = 0$

- if $\pi = (2)^m \sqcup \pi'$ with $1, 2 \notin \pi'$, then $\alpha_\pi + (\sum_i x_i^+)|_{A_\pi} = (p \otimes \mathrm{id})(\alpha_\pi)$, where $p\colon H^*(A_{(2)^m}; \mathbb{F}_2) \to H^*(A_{(2)^m}; \mathbb{F}_2)$ is the projection onto N_m.
- if $1 \in \pi$, then $(\sum_i x_i^+)|_{A_\pi} = 0$

We now apply the conjugation by s_0 to $\alpha + q_{W_{D_n}}(\sum_i x_i^+)$ and repeat the same construction to obtain a sum $\sum_j y_j^+$ such that

$$q_{W_{D_n}}(\sum_j y_j^+) = c_{s_0}^*(\alpha + q_{W_{D_n}}(\sum_i x_i^+))$$

or, equivalently, $\alpha = q_{W_{D_n}}(\sum_i x_i^+ + \sum_j y_j^-)$. Thus φ and, as a consequence, φ' are surjective. □

3.6 Steenrod algebra action

In this final section of the chapter we are going to determine the Steenrod algebra action on A_B and A_D. As a preliminary step, we need to observe that, since the structural morphisms of A_B and A_D are induced by (stable) maps, they satisfy a Cartan formula with respect to Steenrod squares. This means that these structures are almost-Hopf rings over the Steenrod algebra. Hence, we are only required to compute the Steenrod squares on the generators.

The results are similar to those of Section 2.5. We basically follow the argument used there. The first step is to define the A_B and A_D analogs of the notions of height, effective scale and full-width monomials.

Definition 103 (from [13])**.** Define:

- the *height* (ht) of a gathered block in A_B or A_D as the number of generators that are cup-multiplied to obtain it, and the height of a Hopf monomial $x = b_1 \odot \cdots \odot b_r$ as $\max_{i=1}^r \mathrm{ht}(b_i)$
- the *effective scale* (effsc) of a gathered block in the cohomology of W_{B_n} (respectively W_{D_n}) as the least k such that $n/2^l$ is an integer and its restriction to $W_{B_{2^l}}^{n/2^l}$ (respectively $W_{D_{2^l}}^{n/2^l}$) is non-zero, and the effective scale of a Hopf monomial $x = b_1 \odot \cdots \odot b_r$ as $\min_{i=1}^r \mathrm{effsc}\,(b_i)$
- a *full-width* monomial as a Hopf monomial in A_B (respectively A_D) of which no constituent block is of the form $1_{W_{B_n}}$ (respectively $1_{W_{D_n}}$)

Theorem 104. *[13, Theorem 8.3, page 191] Let $k, n \geq 1$ and $i \geq 0$. Then, in A_B, the following formulas hold:*

- $\mathrm{Sq}^i(\gamma_{k,2^n})$ *is the sum of all the full-width monomials $x \in \mathcal{M}$ of degree $2^{n+k} - 2^n + i$ with* $\mathrm{ht}(x) \leq 2$ *and* $\mathrm{effsc}(x) \geq l$ *in which generators of the form δ_k do not appear*
- $\mathrm{Sq}^i(\delta_{2^n})$ *is the sum of all the full-width monomials $x \in \mathcal{M}$ of degree $2^n + i$ with* $\mathrm{ht}(x) \leq 2$ *and* $\mathrm{effsc}(x) \geq 1$ *such that in every constituent gathered block of x a generator of the form δ_k appear*

Proof. The formulas for $\mathrm{Sq}^i(\gamma_{k,2^n})$ are an obvious consequence of [13, Theorem 8.3, page 191], which is the mod 2 counterpart of Theorem 57. $\mathrm{Sq}^i(\delta_{2^n})$ requires some more reasoning. By construction, δ_{2^n} is the top-dimensional Stiefel-Whitney class of the reflection representation U_{2^n}, by Wu's formula $\mathrm{Sq}^i(\delta_{2^n}) = w_i(U_{2^n})\delta_{2^n}$. As a notational convention, assume $\gamma_{k,0} = 1$. Let

$$u_i = \sum_{\substack{j_0,\ldots,j_n\geq 0, \sum_{r=1}^{n-1} 2^r j_r + j_n + j_0 = 2^n \\ \sum_{r=1}^{n-1}(2^r-1)j_r + j_n = i}} \bigodot_{r=1}^{n-1} \gamma_{r,j_r} \odot \delta_{r_n} \odot 1_{B_{j_0}}.$$

Note that the proof of Proposition 98 gives the computation of the restriction of $2_i(U_{2^n})$ on the subgroup A_π as a byproduct. We have already carried out a calculation based on Proposition 97 that proves that this restriction coincides with the restriction of u_i. Hence,

$$\mathrm{Sq}^i(\delta_{2^n}) = w_i(U_{2^n})\delta_{2^n} = u_i\delta_{2^n}$$

and this is exactly the sum of all the desired Hopf monomials x. □

As regarding the calculation of the Steenrod squares on the generators of A_D, We observe that the calculation for $\mathrm{Sq}^i\,(\delta_{n:m})$ is implicit in Theorem 104 since $\delta_{n:m} = \rho\,(\delta_n \odot 1_m)$ and ρ commute with Steenrod operations. Thus we only need to consider generators of the form $\gamma^{\pm}_{k,n}$.

Theorem 105. *Let $k, n \geq 1$ and $i \geq 0$. Then, in A_D, $\mathrm{Sq}^i(\gamma^+_{k,n})$ (respectively $\mathrm{Sq}^i(\gamma^-_{k,n})$) is the sum of all the full-width monomials $x \in \mathcal{M}^+$ (respectively $x \in \mathcal{M}^-$) of degree $2^{n+k} - 2^n + i$ with $\mathrm{ht}(x) \leq 2$ and $\mathrm{effsc}(x) \geq l$ in which generators of the form $\delta_{n:m}$ do not appear.*

Proof. First of all, we claim that $\mathrm{Sq}^i(h_n) \in H^*(A_{(2)^n};\mathbb{F}_2)$ is a positive monomial for all possible i and n. If $n = 1$ this is obvious. If $n > 1$, observe that

$$h_n = h_{n-1} \otimes h_1 + (d_1^{\otimes^{n-1}} + h_{n-1}) \otimes (d_1 + h_1).$$

Thus, by an induction argument on n, an application of the Cartan formula proves the claim.

Restrictions maps clearly commute with Sq^i.Thus, $\mathrm{Sq}^i(\gamma^+_{k,n})$ (respectively $\mathrm{Sq}^i(\gamma^-_{k,n})$) has positive (respectively negative) restriction to elementary abelian 2-subgroups. Furthermore

$$\mathrm{Sq}^i(\gamma^+_{k,n}) + \mathrm{Sq}^i(\gamma^-_{k,n}) = \rho(\mathrm{Sq}^i(\gamma_{k,n})).$$

This implies that $\mathrm{Sq}^i(\gamma^+_{k,n}) = (\mathrm{Sq}^i(\gamma_{k,n}))^+$ and $\mathrm{Sq}^i(\gamma^-_{k,n}) = (\mathrm{Sq}^i(\gamma_{k,n}))^-$ and the statement is reduced to Theorem 104. □

Chapter 4

Some remarks on the general structure of $H^*(DX;\mathbb{F}_2)$ and $H^*(QX;\mathbb{F}_2)$

In this last chapter, we are going to explain how Hopf ring structures arise in a more general context. Explicitly, we prove here that the cohomology of $DX = \bigvee_{n\geq 0} E(\Sigma_n) \wedge_{\Sigma_n} X^{\wedge^n}$, the extended powers of X, possesses both a structure of Hopf ring and a divided powers structures, that relate well to each other. This leads to the definition of an algebraic object called divided powers Hopf ring.

For mod 2 coefficients, we can also give a presentation of $H^*(DX;\mathbb{F}_2)$ in these terms. For odd primary coefficients, some complications arise, and we do not have a description in terms of generators and relations yet.

The space DX is strongly related to QX, the free ∞-loop space over X. For this reason, the knowledge of the cohomology of DX gives some information about the cohomology of QX.

The results of this last Chapter have been obtained in collaboration with Paolo Salvatore and Dev Sinha [19].

4.1 Divided powers Hopf rings

We define in this section our main algebraic object, that we will use to describe $H^*(CX;\mathbb{F}_2)$. First, we need to recall the notion of a component bialgebra.

Definition 106. A *bigraded component bialgebra* over a ring R is a bialgebra $(H,\Delta,\cdot)$ over R satisfying the following additional properties:

- $H = \bigoplus_{n,d\in\mathbb{N}} H_{n,d}$ is a bigraded R-module
- the coproduct and the product preserve the second degree d
- the coproduct preserves also the first degree n
- the product of two bigraded elements having different first degree is always 0

We also define $\overline{H} = \bigoplus_{n>0,d\geq 0} H_{n,d}$, we call the first degree (n) of an element its *component* and the second degree (d) its *dimension*.

We can also define a *bigraded component algebra* as a bigraded algebra whose product satisfies the last three conditions above. As a notational convention, we let R-*bcalg* and R-*bcbialg* be the categories of bigraded component algebras and bigraded component bialgebras respectively that satisfy the following connectedness condition:

$$H = \mathrm{im}(\eta) \bigoplus \overline{H}$$

where $\eta\colon R \to H_{0,0}$ is the unity. The usefulness of this requirement will be apparent later.

We now define our main algebraic notion.

Definition 107. A *bigraded component Hopf ring* over a ring R is a Hopf ring $(H, \odot, \cdot, \Delta)$ such that:

- $H = \bigoplus_{n,d} H_{n,d}$ is a bigraded R-module
- $\odot$ and Δ preserve the two degrees
- H, with the product $\cdot$ alone, is a bigraded component algebra

We say that a bigraded component Hopf ring H over R is *connected* if it satisfies the connectedness condition as a bigraded component algebra. We denote by R-*bchrng* the category of connected bigraded component Hopf rings over R.

Note that the Hopf rings A_A and A_B that we have encountered in the previous chapters are bigraded component Hopf rings. These are indeed particular cases of a more general structure theorem, as we will clarify in the following section.

Recall that during the proof of Theorems 46 and 79 we also observed that A_A and A_B have, in addition to this Hopf ring structure, divided powers operations $\{\_^{[n]}\}_{n\in\mathbb{N}}$ that make them, together with the coproduct Δ and the transfer product $\odot$, divided powers Hopf algebras. Moreover, in both those cases, the relevant divided powers Hopf algebra is freely generated by the modules of primitive elements $P(A_A)$ and $P(A_B)$, consisting of gathered blocks that are minimal among those with the same profile. Actually, every gathered block b can be obtained by applying a certain divided powers operation $\_^{[n]}$ to the primitive block $\tilde{b}$ with the same profile. This gives rise to a formula that links the operations $\_^{[n]}$ with the cup product. Explicitly, given two blocks $b = (\tilde{b})^{[n]}$ and $b' = (\tilde{b}')^{[m]}$, since the profile of the gathered block $b \cdot b'$ is the sum of the profiles of the two factors, we have that:

$$b \cdot b' = \left((\tilde{b})^{[\frac{n}{\mathrm{GCD}(n,m)}]} \cdot (\tilde{b}')^{[\frac{m}{\mathrm{GCD}(n,m)}]} \right)^{[\mathrm{GCD}(n,m)]}$$

This leads us to the following definition.

Definition 108. A *divided powers component bigraded Hopf ring* is a bigraded component Hopf ring $(A, \odot, \cdot, \Delta)$ with a divided powers structure $\{\_^{[n]}\}_{n\in\mathbb{N}}$ such that:

- the given divided powers operations make $(A, \odot, \Delta)$ a divided powers Hopf algebra

- for every $n, m \in \mathbb{N}$ and for every $x \in \overline{A}, y \in \overline{A}$, with x primitive, the following formula holds:

$$\left(x^{[n]} \cdot y^{[m]}\right) = \left(x^{\left[\frac{n}{\mathrm{GCD}(n,m)}\right]} y^{\left[\frac{m}{\mathrm{GCD}(n,m)}\right]}\right)^{[\mathrm{GCD}(n,m)]}$$

The rest of this section is technical and is devoted to the construction a free bigraded component divided powers Hopf ring functor $DPHR$: *R-bcbialg* $\rightarrow$ *R-bchrng.* We assume in the following that the base ring R is a field. In our context, this is not restrictive, since we are going to apply our algebraic construction to the cohomology of DX with coefficients in $\mathbb{F}_p$.

First, we start with a given component bialgebra $B \in$ *R-bcbialg.* Let D be the free divided powers Hopf algebra generated by B, regarded as a coalgebra with its coproduct. D is naturally bigraded. Let $(E, \cdot, \Delta)$ be the free graded commutative bialgebra generated by the coalgebra (D, Δ), quotiented by the relations:

- for all $x, y \in B$, the product $x \cdot y$ in E coincides with that in B
- for all $x, y \in D$, $x \cdot y = 0$ if x and y belong to different components

These relations preserve the gradings. Thus, E is naturally a bigraded component bialgebra. Finally, let F ve the free divided powers Hopf algebra over the coalgebra (E, Δ) and define $DPHR(B)$ as the quotient of the bigraded Hopf algebra $(F, \odot, \Delta)$ by the following relations:

- for all $x, y \in D$, the product $x \odot y$ in F coincides with that in D
- for all $x, y \in E$ primitives, and for all $n, m \in \mathbb{N}$:

$$x^{[n]} \cdot y^{[m]} - \left(x^{\left[\frac{n}{\mathrm{GCD}(n,m)}\right]} \cdot y^{\left[\frac{m}{\mathrm{GCD}(n,m)}\right]}\right)$$

- the Hopf ring distributivity relation

As a notational convention, we denote with the letter I the ideal generated by these relations. Our relations clearly preserve the coproduct. As a direct consequence of this fact, we obtain the following lemma.

Lemma 109. *I is also a coideal in F, thus the algebra $DPHR(B)$ constructed as before is naturally a Hopf algebra.*

Hence, we have a conjectural description of the free component divided powers Hopf ring over B as a Hopf algebra. We must now prove that this object can indeed be given the structure of an Hopf ring. More explicitly, we need to extend the map $\cdot\colon E \otimes E \rightarrow E$ to a product defined on the bigger space $DPHR(B) \otimes DPHR(B) \rightarrow DPHR(B)$. This requires some preliminary remarks.

Lemma 110. *With the notation used above, D is generated, as a divided powers algebra, by its module of primitive elements $P(D)$.*

Proof. First, we fix some notation. We denote with $\pi\colon D \to \overline{D}$ be the quotient map. If $k \in \mathbb{N}$ is a natural number, we let $\Delta^{(k)}\colon D \to D^{\otimes^k}$ denote the coproduct iterated $k-1$ times. Note that the these iterated coproducts determine an increasing sequence of subspaces $\{F_k(D)\}$, where $F_k(D)$ is the kernel of the linear map α_k defined as the composition:

$$\alpha_k\colon D \xrightarrow{\Delta^{(k+1)}} D^{\otimes^{k+1}} \xrightarrow{\pi^{\otimes^{k+1}}} \overline{D}^{\otimes^{k+1}}$$

Since the coproduct preserves the grading, for all $x \in D_k$ that lie in the k^{th} component, the iterated coproduct $\Delta^{(n)}(x)$ must belong to the subspace $\bigoplus_{k_1+\cdots+k_n=k} D_{k_1} \otimes \cdots \otimes D_{k_n}$. Observe that D is connected, thus $\overline{D}$ is trivial in the 0^{th} component. As a consequence, $D_k \subseteq F_{k+1}(D)$. In particular, $\bigcup_k F_k(D) = D$.

Let D' be the subalgebra with divided powers generated by $P(D)$. In order to prove that $D' = D$, it is sufficient to assure that $F_n(D) \subseteq D'$ for all $n \in \mathbb{N}$. This is done by induction on n:

- If $n = 1$, then $F_1(D) = P(D) \subseteq D'$.
- Since the coproduct is cocommutative, if $n > 1$ and $x \in F_n(D)$, we can write
$$\Delta(x) = \sum_{\pi} \sum_{i=1}^{l_\pi} \mathrm{Sym}(\bigotimes_{j=1}^{r} a_{\pi,i,j}^{\otimes^{m_j}})$$
for some $a_{\pi,i,j} \in P(D)$, where the sum is over tuples $\pi == (m_1, \dots, m_r)$ of non-negative integers such that $m_1 + \cdots + m_r = n$. Let $x' = x - \sum_\pi \sum_i \bigodot_j a_{\pi,i,j}^{[m_j]}$ and observe that $x' \in F_{n-1}(D)$. Thus, by induction hypothesis, x' and, as a consequence, x, belong to D'.

□

Lemma 111. *Let R be a field of characteristic $p \geq 0$. Let A be a divided powers bigraded component Hopf ring over R. Let $x, y \in P(A)$ be bigraded primitive elements belonging to strictly positive components. Let $n, m \in \mathbb{N}$. If the ratio q of the component of x and the component of y is not a power of p, then $x^{[n]} \cdot y^{[m]} = 0$ in A.*

Proof. Since the roles of x and y are exchangeable, we can assume that the component of x is bigger than the component of y, and thus $n \leq m$. Because of our relation between $\cdot$ and the divided powers operations, we can also assume $\mathrm{GCD}(n, m) = 1$.

We distinguish $p = 0$ and $p > 0$, because the proof is sensibly different in the two cases.

$p = 0$.

We must prove that, if x and y lie in different components, then $x^{[n]} \cdot y^{[m]} = 0$. Recall from the proof of Lemma 110 that there exists an increasing filtration $\{F_k(A)\}_{k\in\mathbb{N}}$ of D. Since the coproduct is a morphism of divided powers algebras, $\Delta(x^{[n]}) = \sum_{k=0}^{n} x^{[k]} \otimes x^{[n-k]}$. As a consequence, $x^{[n]} \in F_n(A)$. Hence, since with our assumptions $m > n$, an application of Hopf ring distributivity yields $x^{[n]} \cdot y^{\odot^m} = 0$. $m!$ is invertible in fields of characteristic 0, thus $x^{[n]} \cdot y^{[m]} = 0$.

$p > 0$.

There exists an invertible field element u and a sequence of non-negative integers $(l_1, \ldots, l_r)$ such that $x^{[n]} = u \bigodot_{i=1}^r x^{[p^{l_i}]}$. This follows from an argument that we already used for A_A and A_B: if $n = \sum_{i=0}^N a_i p^i$ is the p-adic expansion of n, each i appears in the r-tuple $(l_1, \ldots, l_r)$ with multiplicity a_i, and u is the binomial coefficient

$$u = \left(\begin{array}{c} n! \\ \prod_{i=0}^N (p^i!)^{a_i} \end{array} \right)$$

If necessary, we rearrange the sequence in such a way that $l_1 \leq \cdots \leq l_r$. Now apply the Hopf distributivity formula and the fact that $\cdot$ between two elements in different components is 0 and obtain:

$$x^{[n]} \cdot y^{[m]} = \begin{cases} u \bigodot_{i=1}^r y^{[p^{l_i} q]} \cdot x^{[p^{l_i}]} & \text{if } qp^{l_1} \in \mathbb{N} \\ 0 & \text{if } qp^{l_1} \notin \mathbb{N} \end{cases}$$

Hence we can assume that $qp^{l_1} \in \mathbb{N}$. Since $\mathrm{GCD}(n, m) = 1$, we must have $p \nmid qp^{l_1}$. If q is not a power of p, then there exist an invertible $u' \in R$ such that $y^{[qp^{l_1}]} = u' y^{[qp^{l_1}-1]} \odot y$. This implies that $y^{[qp^{l_1}]} \cdot x^{[p^{l_1}]} = 0$ because the component of x is bigger than the component of y. □

Lemmas 110 and 110 directly imply the following technical result. This is the analog of the fact that A_A and A_B are generated by Hopf monomials as vector spaces.

Lemma 112. *With the notation above, $DPHR(B)$ is generated, as an R-module, by elements of the form $\bigodot_{i=1}^r x_i^{[n_i]}$, with $r \geq 0$, $x_i \in P(E)$ bihomogeneous and $n_i \geq 1$.*

This allows to extend naturally extend the product $\cdot$ to $DPHR(B)$ with the desired properties. Explicitly, for all $x \in P(E)_{n,d}$ and $y = \bigoplus_{i=1}^r y_i^{[m_i]} \in G_{n',d'}$ bihomogeneous, with each $y_i \in P(E)_{n_i,d_i}$, we put:

$$x^{[l]} \cdot y = \begin{cases} \bigodot_{i=1}^r \left(x^{\left[m_i \frac{n_i}{n}\right]} \cdot y^{[m_i]} \right) & \text{if } m_i \frac{n_i}{n} \in \mathbb{N} \text{ and } \sum_i n_i m_i = ln \\ 0 & \text{otherwise} \end{cases}$$

This defines a map $\sum_{n \geq 0} P(E)^{[n]} \otimes DPHR(B) \to DPHR(B)$. We can then extend it to a full product on $DPHR(B)$ by means of the Hopf distributivity relation and linearity.

Proposition 113. *For any $B \in R$-bcbialg, $DPHR(B)$, with its natural Hopf algebra structure and the second product defined above, is the free divided powers bigraded component Hopf ring generated by B.*

The particular case in which B is a connected component bialgebra in which all elements are primitive, the structure of $DPHR(B)$ as an R-module is particularly simple. Moreover, this is a case of interest, as the cohomology of CX arise this way. For these reasons, it is worth defining the following concept.

Definition 114. Let $A \in$ *R-bcalg*. The *divided powers bigraded component Hopf ring primitively generated by A* is $DPHR(A^{pr})$, where A^{pr} is the connected bigraded component bialgebra that coincides with A as an algebra and is endowed with the primitive coproduct.

We conclude this section with the description of an additive basis for $FPHR(A^{pr})$, when R is a field. Our basis elements are a generalization of the Hopf monomials introduced in Chapters 2 and 3.

Definition 115. Let R be a field. Let A be a connected bigraded component algebra and let $n,p \in \mathbb{N}$. Let $\mathcal{B} = \bigsqcup_{n,d} \mathcal{B}_{n,d}$ be a bigraded additive basis for A as a R-vector space. We define $A^p_n = R \oplus \bigoplus_{l,d\in\mathbb{N}} A_{np^l,d}$.

We say that $b \in DPHR(A^{pr})$ is a *gathered block* if it is an element of the form $b = \prod_{i=1}^r b_i^{[m_i]}$ for some $b_i \in \mathcal{B}_{n_i,d_i}$ pairwise distinct, and $m_i \geq 1$ satisfy $m_i n_j = m_j n_i$ and $m_i < m_j$ for all $1 \leq i < j \leq r$. We define the *profile* of x as the sequence $(b_1,\dots,b_r)$.

We say that $x \in DPHR(A^{pr})$ is a *gathered monomial* if it is of the form $x = \bigodot_{i=1}^r x_i$, where the x_is are gathered blocks with pairwise different profiles.

Proposition 116. *Let R be a field and let $A \in R$-bcalg. For any $n \in \mathbb{N}$ and p prime number, let $A^p_n = R \oplus \bigoplus_{l,d\in\mathbb{N}} A_{np^l,d}$. The following holds:*

- *if* $\mathrm{char}(R) = 0$*, then $DPHR(A^{pr})$ coincides, as an algebra with the $\odot$ product, with $DP(A)$*
- *if* $\mathrm{char}(R) = p > 0$*, then $DPHR(A^{pr}) = \bigotimes_{n\in\mathbb{N},p\nmid n} DPHR\left((A^p_n)^{pr}\right)$ as a Hopf algebra with $\odot$ and Δ, while $\cdot$ extends the corresponding product on each factor via the rule*

$$\forall n \neq m : DPHR\left((A^p_n)^{pr}\right) \cdot DPHR\left((A^p_m)^{pr}\right) = 0.$$

- *if* $\mathrm{char}(R) = p > 0$*, given a bigraded additive basis for A^p_n, the set of gathered monomials is an additive basis for $DPHR\left((A^p_n)^{pr}\right)$*

Proof. First, assume $\mathrm{char}(R) = 0$. In this case, for any bialgebra $B \in R$-*bcbialg*, $DPHA(B)$ is the polynomial algebra over the R-module B, because $n!$ is invertible for all $n \in \mathbb{N}$ and the divided powers operations do not actually define a richer structure. By applying the Hopf ring distributivity formula, we can easily extend the product of B to a product $\cdot\colon DPHA(B)\otimes DPHA(B) \to DPHA(B)$ that makes $DPHA(B)$ a Hopf ring. Since B generates $DPHA(B)$ as a Hopf algebra, we can do that in a unique way. We claim that a bigraded component Hopf ring over a field of characteristic 0, with the obvious divided powers operations $\{x \mapsto \frac{x^{\odot n}}{n!}\}_{n\in\mathbb{N}}$, is automatically a divided powers bigraded component Hopf ring. We only need to check the axiom:

$$x^{[n]} \cdot y^{[m]} = \left(x^{\left[\frac{n}{\mathrm{GCD}(n,m)}\right]} \cdot y^{\left[\frac{m}{\mathrm{GCD}(n,m)}\right]}\right)^{[\mathrm{GCD}(n,m)]}$$

for $x,y \in P(DPHA(B))$ and $n,m \in \mathbb{N}$. This is automatically satisfied because $n!$ and $m!$ are invertible in R. Thus, there is a morphism of Hopf rings $DPHR(B) \to DPHA(B)$, which is the inverse of the natural map $DPHA(B) \to DPHR(B)$. In particular, $DPHR(B)$ and $DPHA(B)$ are isomorphic.

We now turn to the more difficult case, in which $\mathrm{char}(R) = p > 0$. Let $n \in \mathbb{N}$ such that $p \nmid n$. Define $G_n = DPHR\left((A^p_n)^{pr}\right)$. Let $G = \bigotimes_{n:p\nmid n} G_n$ the usual tensor product Hopf algebra (with $\odot$ and Δ). Extend the $\cdot$ product of the G_ns to G by imposing that $G_n \cdot G_m = 0$ if $n \neq m$. Due to Lemma 111, in this way G satisfies the axioms of a divided powers bigraded component Hopf ring. We can identify G with the categorial coproduct of the G_ns in the category of divided powers Hopf rings. $DPHR$ is a free functor, hence it is cocontinuous and $A = \bigoplus_n A_n^p$. This implies that $DPHR(A) = G$.

In order to prove the last point, we remain in the case $\mathrm{char}(R) = p > 0$, but we also assume that $A = A_n^p$ for some n. Up to dividing all the components by n (that clearly do not modify the structure of A or the statement we want to prove in any sensible way), we can assume $n = 1$, or equivalently that A is concentrated in components that are powers of p. Let $\mathcal{B}$ be a bigraded basis of A as a vector space, let $G = DPHR(A)$, let $(G', \odot, \Delta)$ be the free divided powers Hopf algebra generated by the submodule of primitive gathered blocks and let $(D, \odot) = DP(A)$. There is an obvious map $\varphi\colon G' \to G$. It is sufficient to prove that φ is an isomorphism.

The first step is observing that the set of gathered monomials is an additive basis for G'. We now use this fact to show that we can define a product $\cdot\colon G' \otimes G' \to G'$, extending the product of A, that makes G' a bigraded component divided powers Hopf ring. It is sufficient to define it on the basis of gathered monomials in G'. Given $x = \bigodot_{i=1}^N a_i^{[n_i]}$ and $y = \bigodot_{j=1}^M b_j^{[m_j]}$ gathered monomials, we put

$$x \cdot y = \sum_{k_{i,j}, h_{j,i}} \bigodot_{i=1}^N \bigodot_{j=1}^M \left(a_i^{\left[\frac{k_{i,j}}{\mathrm{GCD}(k_{i,j},h_{j,i})}\right]} b_j^{\left[\frac{h_{j,i}}{\mathrm{GCD}(k_{i,j},h_{j,i})}\right]} \right)^{[\mathrm{GCD}(k_{i,j},h_{j,i})]}$$

where the sum is over NM-tuples $\{k_{i,j}\}_{1\leq i\leq N, 1\leq j\leq M}$ and $\{h_{j,i}\}_{1\leq i\leq N, 1\leq j\leq M}$ such that for all i $\sum_{j=1}^M k_{i,j} = n_i$ and for all j $\sum_{i=1}^N h_{j,i} = m_j$. It is easy to check that this defines a divided powers bigraded component Hopf ring structure on G'. Note that, in this way, φ is a Hopf ring morphism.

Due to the relation involving $x^{[n]} \cdot y^{[m]}$, every non-primitive gathered block b can be written in G as a suitable divided power of the unique primitive gathered block that has the same profile of b. This implies that G is generated, as a Hopf algebra, by gathered blocks and, as a consequence, φ is surjective. Moreover, being G' generated by A as a divided powers Hopf ring, there is a morphism $\psi\colon G \to G'$ such that $\psi \circ \varphi = \mathrm{id}_{G'}$. Hence, φ is an isomorphism with inverse ψ. □

4.2 The mod 2 cohomology of DX and some remarks on QX

In this section we study a divided powers Hopf ring structure on the mod 2 cohomology of DX. Recall that, when $X = S^0$, $DX = \bigsqcup_{n\geq 0} B(\Sigma_n)$ and, when $X = \{*\} \sqcup \mathbb{P}^\infty(\mathbb{R})$, $DX = \bigsqcup_{n\geq 0} B(W_{B_n})$. Indeed, in these two particular cases, the Hopf ring structures we obtain in this section coincide with A_A and A_B. However, A_D cannot be recovered as the cohomology of DX for some space X. In this section and in the following, for simplicity, we assume all topological spaces to be homotopically equivalent to CW complexes.

Let $A(X) = H^*(DX;\mathbb{F}_2)$. We need to define a bigraded $\mathbb{F}_2$-vector space structure on $A(X)$. In order to do so, we need to recall a classical result about DX, proved by Cohen, May and Taylor in [7].

Theorem 117 (Cohen–May–Taylor)**.** *Let X be a topological space. For all $n \in \mathbb{N}$, let $D_nX = (E(\Sigma_n) \sqcup \{*\}) \wedge_{\Sigma_n} X^{\wedge^n}$. Let $DX = \bigvee_{n\geq 0} D_nX$. Let CX be the free $\mathcal{C}_\infty$-space over a topological space X, where $\mathcal{C}_\infty$ is the operad of infinite-dimensional little cubes. There is a stable homotopy equivalence $CX \simeq DX$ and, as a consequence, isomorphisms on all homology and cohomology groups.*

Since DX is the wedge of D_nX for $n \in \mathbb{N}$, its (reduced) cohomology splits as a direct sum

$$\tilde{H}^*(DX;\mathbb{F}_2) \cong \bigoplus_{n\geq 0} \tilde{H}^*(D_nX;\mathbb{F}_2).$$

Hence, we see that the cohomology of DX is naturally bigraded, where the degree of a class is its cohomological dimension, and its component locates the addend $\tilde{H}^*(D_nX;\mathbb{F}_2)$ to which it belongs. Because of Theorem 117 the same holds for $H^*(DX;\mathbb{F}_2)$. Theorem 117 also allows us to use CX or DX interchangeably when we deal with the vector space structure on cohomology.

We now construct the structural morphisms that for our divided powers Hopf ring structure. Let $f_{n,m}\colon E(\Sigma_n) \times E(\Sigma_m) \to E(\Sigma_{n+m})$ be the map induced by the obvious inclusion $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. Consider the function $\mu_{n,m}\colon D_nX \times D_mX \to D_{n+m}X$ defined by:

$$\mu_{n,m}(c \times_{\Sigma_n} (x_1,\dots,x_n), c' \times_{\Sigma_m} (x'_1,\dots,x'_m)) = f_{n,m}(c,c') \times_{\Sigma_{n+m}} (x_1,\dots,x'_m)$$

Note that these maps determine a morphism $DX \times DX \to DX$. When passing to cohomology, these determine a coproduct $\Delta\colon A(X) \to A(X) \otimes A(X)$. Similar, the cohomology transfer associated to these functions define a product $\odot\colon A(X) \otimes A(X) \to A(X)$. Finally, $\cdot\colon A(X) \otimes A(X) \to A(X)$ is the usual cup product.

It still remains to define the divided powers operations. Observe that the map $\nu_{n,k}\colon (D_nX)^k \to D_{nk}X$ obtained by iterating the product maps $\mu_{*,*}$ factors through the homotopy quotient $E_{\Sigma_k} \times_{\Sigma_k} (D_nX)^k$ and thus defines a map $\overline{\nu}_{n,k} E_{\Sigma_k} \times_{\Sigma_k} (D_nX)^k \to D_{nk}X$. Given a class $x \in H^*(D_nX;\mathbb{F}_2)$, we can define $x^{[n]}$ as follows. Since $x^{\otimes^k} \in H^*((D_nX)^k;\mathbb{F}_2)$ is Σ_k-invariant, it can be identified with a class of $\frac{(D_nX)^k}{\Sigma_k}$ via the pullback of the projection map. We define $x^{(k)}$ to be the pullback of this class in $E_{\Sigma_k} \times_{\Sigma_k} (D_nX)^k$. Define $x^{[k]}$ as the image of $x^{(k)}$ via the cohomology transfer homomorphism determined by $\overline{\nu}_{n,k}$.

Proposition 118. *Let X be a topological space. With the morphisms defined above, $H^*(DX;\mathbb{F}_2)$ is a divided powers bigraded component Hopf ring.*

Proposition 118 will be proved in an almost completely topological space-level fashion. The only exception is the cup product relation

$$x^{[n]} \cdot y^{[m]} = \left(x^{\left[\frac{n}{\mathrm{GCD}(n,m)}\right]} \cdot y^{\left[\frac{m}{\mathrm{GCD}(n,m)}\right]}\right)^{[\mathrm{GCD}(n,m)]}.$$

For this, we require an auxiliary Lemma.

Lemma 119. *With the notation used in Proposition 118, the primitives in $A(X)$ are concentrated in components corresponding to powers of 2.*

Proof. $H^*(D_nX;\mathbb{F}_2)$ is naturally isomorphic to $H^*(\Sigma_n;\tilde{H}^*(X;\mathbb{F}_2)^{\otimes^n})$, where $\tilde{H}^*(X;\mathbb{F}_2)^{\otimes^n}$ is regarded as a representation of Σ_n acting by permutation of factors. If n is not a power of 2, there exist $1 \leq k < n$ such that the index $[\Sigma_n : \Sigma_k \times \Sigma_{n-k}]$ is coprime to 2. As a consequence of this fact and Lemma I.5.1 in [1], the restriction from the cohomology of Σ_n to that of $\Sigma_k \times \Sigma_{n-k}$, with coefficients in any representation, is injective. This imply that the coproduct of each non-zero cohomology class $x \in H^*(D_nX;\mathbb{F}_2) = A(X)_{n,*}$ has a non-trivial summand in the component $A(X)_{k,*} \otimes A(X)_{n-k,*}$. Thus $P(A(X))_{n,*} = 0$. □

Proof of Proposition 118. The proof that $(A(X), \odot, \cdot, \Delta)$ is a Hopf ring is essentially the same proof we adopted for A_B. The only difference is that, in general, we replace $(\mathbb{P}^\infty(\mathbb{R}))^n$ with $X^{\wedge^n}$. Thus, we are left to check that the maps $\_^{[k]}$ satisfy the axioms of a divided powers Hopf algebra and our relation with respect to cup product that characterizes divided powers bigraded component Hopf ring structures. First of all, note that we can define a function $\nu_k\colon DX^k \to DX$ by gluing the maps $\nu_{n,k}$.

- For all $x \in A_{n,d}(X)$, if $k = 0$, then $x^{\otimes^k} = 1$, the unity in the cohomology of the 0^{th} component $H^*(D_0X;\mathbb{F}_2) = H^*(\{*\};\mathbb{F}_2)$, and $\nu_0 = \mathrm{id}_{D_0X}$. Hence, $x^{[0]} = 1$.

- For all $x \in A_{n,d}(X)$, if $k = 1$, then $x^{\otimes^k} = x \in \tilde{H}^*(D_nX;\mathbb{F}_2)$. Since $\nu_1 = \mathrm{id}_{DX}$, $x^{[1]} = x$.

- Let $x \in A(X)$. Let $k, h \in \mathbb{N}$. By construction, the class $x^{(k)} \otimes x^{(h)}$ in $H^*\left(E(\Sigma_k) \times_{\Sigma_k} (DX)^k \times E(\Sigma_h) \times_{\Sigma_h} (DX)^h;\mathbb{F}_2\right)$ transfers to $x^{[k]} \otimes x^{[h]}$ in $H^*(DX \times DX)$ via $\overline{\nu}_k \times \overline{\nu}_h$. Similarly, $x^{(k+h)}$ transfers to $x^{[h+k]}$ in $H^*(DX;\mathbb{F}_2)$ via $\overline{\nu}_{k+h}$. We observe that the following diagram commutes:

$$\begin{tikzcd} \frac{E(\Sigma_k)\times(DX)^k}{\Sigma_k} \times \frac{E(\Sigma_h)\times(DX)^h}{\Sigma_h} \arrow[r, "\pi_1=\overline{\nu}_k\times\overline{\nu}_h"] \arrow[d, "\pi_2"] & DX \times DX \arrow[d, "\pi_3=\bigvee_{n,m}\mu_{n,m}"] \\ \frac{E(\Sigma_{h+k})\times(DX)^{k+h}}{\Sigma_{h+k}} \arrow[r, "\pi_4=\overline{\nu}_{k+h}"] & DX \end{tikzcd}$$

For $1 \leq i \leq 4$, let τ_i be the transfer map in cohomology associated to π_i. By using the standard properties of the transfer, we obtain:

$$\begin{aligned} x^{[k]} \odot x^{[h]} &= \tau_3 \circ \tau_1(x^{(k)} \otimes x^{(h)}) \\ &= \tau_4 \circ \tau_2(x^{[h]} \otimes x^{[k]}) \\ &= \tau_4 \circ \tau_2 \circ \pi_2^*(x^{(k+h)}) \\ &= \left(\begin{array}{c} k+h \\ k \end{array} \right) \tau_4(x^{(k+h)}) \\ &= \left(\begin{array}{c} k+h \\ k \end{array} \right) x^{[k+h]} \end{aligned}$$

- Let $x, y \in A(X)$ and $k \in \mathbb{N}$. For all $0 \leq i \leq k$, consider the map:

$$\pi_{i,k} \colon \frac{E(\Sigma_i) \times DX^i}{\Sigma_i} \times \frac{E(\Sigma_{k-i}) \times DX^{k-i}}{\Sigma_{k-i}} \to \frac{E(\Sigma_k) \times DX^k}{\Sigma_k}$$

  By identifying the domain of $\pi_{i,k}$ with $\frac{E(\Sigma_k) \times DX^k}{\Sigma_i \times \Sigma_{k-i}}$ up to homotopy, we see that $\pi_{i,k}$ is homotopy equivalent to a finite covering. Let $\tau_{i,k}$ be the cohomology transfer map associated to $\pi_{i,k}$. We note that the formula $(x+y)^{(k)} = \sum_{i=0}^{k} \tau_{i,k}(x^{(i)} \otimes x^{(k-i)})$ holds in $A(X)$. The fact implies the equality $(x+y)^{[k]} = \sum_{i=0}^{k} x^{[i]} \odot y^{[k-i]}$, after applying the naturality of the transfer maps to the following commutative diagram:

$$\begin{tikzcd}
\frac{E(\Sigma_i)\times(DX)^i}{\Sigma_i} \times \frac{E(\Sigma_{k-i})\times(DX)^{k-i}}{\Sigma_{k-i}} \arrow[r, "\pi_1=\overline{\nu}_i\times\overline{\nu}_{k-i}"] \arrow[d, "\pi_{i,k}"] & DX \times DX \arrow[d, "\bigvee_{n,m}\mu_{n,m}"] \\
\frac{E(\Sigma_k)\times(DX)^k}{\Sigma_k} \arrow[r, "\overline{\nu}_k"] & DX
\end{tikzcd}$$

- We consider the following commutative diagram:

$$\begin{tikzcd}
(DX)^h \arrow[rr, "\pi_1"] & & \frac{E(\Sigma_h)\times(DX)^h}{\Sigma_h} \arrow[rr, "\pi_2=\overline{\nu}_h"] & & DX \\
\left(E(\Sigma_k) \times \frac{(DX)^k}{\Sigma_k}\right)^h \arrow[u, "\pi_3=(\overline{\nu}_k)^h"] \arrow[r, "\pi_6"] & E(\Sigma_h) \times_{\Sigma_h} \left(\frac{E(\Sigma_k)\times(DX)^k}{\Sigma_k}\right)^h \arrow[ur, "\pi_4=(\overline{\nu}_k)^h/\Sigma_h"] \arrow[r, phantom, "="] & \frac{E(\Sigma_{hk})\times(DX)^{hk}}{\Sigma_k\wr\Sigma_h} \arrow[rr, "\pi_7"] & & \frac{E(\Sigma_{hk})\times(DX)^{hk}}{\Sigma_{hk}} \arrow[u, "\pi_5=\overline{\nu}_{hk}"]
\end{tikzcd}$$

  For $1 \leq i \leq 7$, let τ_i be the transfer map induced in cohomology by π_i. The leftmost square is homotopy equivalent to a pullback, thus $(x^{[k]})^{(h)} = \tau_4 \left(x^{(k)}\right)^{(h)}$. Hence, by naturality of the transfer maps, we have:

$$\begin{aligned}
\left(x^{[k]}\right)^{[h]} &= \tau_2 \left(x^{[k]}\right)^{(h)} \\
&= \tau_2 \circ \tau_4 \left(x^{(k)}\right)^{(h)} \\
&= \tau_2 \circ \tau_4 \circ \pi_6^*(x^{(hk)}) \\
&= \tau_7 \circ \tau_6 \circ \pi_6^*(x^{(hk)}) \\
&= \frac{(hk)!}{h!(k!)^h} \tau_7(x^{(hk)}) \\
&= \frac{(hk)!}{h!(k!)^h} x^{[hk]}
\end{aligned}$$

- In order to prove that for all $a \in A(X)$, for all $x \in \overline{A(X)}$ and for all $k \in \mathbb{N}$ $(ax)^{[k]} = a^k \odot x^{[k]}$, it is sufficient to prove that for all $a, x \in \overline{A(X)}$

and for all $k \in \mathbb{N}$ $(a \odot x)^{[k]} = k!a^{[k]} \odot x^{[k]}$, since the equality is trivial if $a \in \mathbb{F}_2$. It follows again from the commutativity of a diagram:

$$\begin{tikzcd}
\frac{E(\Sigma_k)\times(DX)^k}{\Sigma_k} \times \frac{E(\Sigma_k)\times(DX)^k}{\Sigma_k} \arrow[dr, "\pi_3=(\overline{\nu}_k)^2"] & \frac{(DX\times DX)^k}{\Sigma_k} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2=\mathrm{id}\times\mu/\Sigma_k"] \arrow[d, "\pi_4=(\nu_k)^2/\Sigma_k"] & \frac{E(\Sigma_k)\times(DX)^k}{\Sigma_k} \arrow[d, "\pi_5=\nu_k/\Sigma_k"] \\
 & DX \times DX \arrow[r, "\pi_6=\mu"] & DX
\end{tikzcd}$$

Explicitly:

$$\begin{aligned}
(a \odot x)^{[k]} &= \tau_2 \circ \tau_5 \circ \pi_1^* \left(a^{(k)}, x^{(k)}\right) \\
&= \tau_6 \circ \tau_3 \circ \tau_1 \circ \pi_1^* \left(a^{(k)}, x^{(k)}\right) \\
&= k!\tau_6 \circ \tau_3 \left(a^{(k)}, x^{(k)}\right) \\
&= k!a^{[k]} \odot x^{[k]}
\end{aligned}$$

- The coproduct Δ being a morphism of divided powers structures is a direct consequence of the following diagram being a pullback:

$$\begin{tikzcd}
\frac{E(\Sigma_k)\times(DX\times DX)^k}{\Sigma_k} \arrow[r, "\mu/\Sigma_k"] \arrow[d, "\mathrm{id}\times(\nu_k)^2/\Sigma_k"] & \frac{E(\Sigma_k)\times DX}{\Sigma_k} \arrow[d, "\overline{\nu}_k"] \\
DX \times DX \arrow[r, "\mu"] & DX
\end{tikzcd}$$

- Finally, we must prove that for all $x, y \in P(A(X))$ bigraded and for all $n, m \in \mathbb{N}$, the following formula holds:

$$x^{[n]} \cdot y^{[m]} = \left(x^{\left[\frac{n}{\mathrm{GCD}(n,m)}\right]} \cdot y^{\left[\frac{m}{\mathrm{GCD}(n,m)}\right]}\right)^{[\mathrm{GCD}(n,m)]}$$

Because of Lemma 119, it is sufficient to prove that for all $x \in P(A(X))$, for all $y \in A(X)$ divided power of a primitive and for all $n \in \mathbb{N}$ we have that $x^{[n]} \cdot y^{[n]} = (x \cdot y)^{[n]}$. Furthermore, we can restrict to the case where $n = 2^b$ is a power of 2. Indeed, we already observed that, for a general n, we can write $\_^{[n]}$ as a linear multiple of transfer products of operations $\_^{[2^r]}$, depending on the diadic expansion of n. We also assume that the component of x and y is 2^a.

Recall that $H^*(D_{2^a}X;\mathbb{F}_2) \cong H^*(\Sigma_{2^a}; \tilde{H}^*(X;\mathbb{F}_2)^{\otimes^{2^a}})$. Fix an additive graded basis $\mathcal{B}$ for $\tilde{H}^*(X;\mathbb{F}_2)$ and let M be the $\mathbb{F}_2[\Sigma_{2^a}]$-submodule of $\tilde{H}^*(X;\mathbb{F}_2)^{\otimes^{2^a}}$ generated by elements of the form $\alpha^{\otimes^{2^a}}$ for some $\alpha \in \mathcal{B}$ and we let N be the $\mathbb{F}_2[\Sigma_{2^a}]$-submodule generated by $\alpha_1 \otimes \cdots \otimes \alpha_{2^a}$ where for $1 \leq i \leq 2^a$ $\alpha_i \in \mathcal{B}$ not all equal.

Clearly $\tilde{H}^*(X;\mathbb{F}_2)^{\otimes 2^a}$ is naturally isomorphic to the direct sum of M and N as Σ_{2^a}-representation. As a consequence

$$H^*(D_{2^a}X;\mathbb{F}_2) \cong H^*(\Sigma_{2^a};M) \oplus H^*(\Sigma_{2^a};N).$$

Note that Σ_{2^a} acts trivially on M. Moreover, for any partition $\pi = (m_1,\dots,m_r) \vdash 2^a$, let N_π be the induced Σ_{2^a}-representation of the trivial representation of the subgroup $\prod_{i=1}^r \Sigma_{m_i}$. Observe that N is isomorphic to a direct sum of addends of the form N_π, by identifying the subrepresentation generated by $\bigotimes_{i=1}^r \alpha_i^{\otimes m_i}$, with $\alpha_1,\dots,\alpha_r \in \mathcal{B}$ pairwise distinct, with $N_{(m_1,\dots,m_r)}$. Recall that, in these cases, the isomorphism in Saphiro's lemma can be identified with the restriction maps. A proof of this fact can be found in Neukirch–Schmidt–Wingberg [33] at pages 60-61. Thus, by and application of Saphiro's lemma, we see that, if x is primitive, it must belong to the direct summand corresponding to $H^*(\Sigma_{2^a};M)$. Hence, we can assume that x is of the form $x = \gamma \otimes \alpha^{\otimes 2^a}$ for some $\gamma \in H^*(\Sigma_{2^a};\mathbb{F}_2)$ and $\alpha \in \tilde{H}^*(X;\mathbb{F}_2)$. By construction, the divided powers of every such class has the same form, thus we can also assume that $y = \gamma' \otimes \alpha'^{\otimes 2^a}$ for some $\gamma' \in H^*(\Sigma_{2^a};\mathbb{F}_2)$ and $\alpha' \in \tilde{H}^*(X;\mathbb{F}_2)$.

We observe that $x^{(2^b)} \otimes y^{(2^b)}$ can be seen as an element of the cohomology ring $H^*((\Sigma_{2^a} \wr \Sigma_{2^b})^2; \langle \alpha^{\otimes 2^{a+b}} \rangle \otimes \langle \alpha'^{\otimes 2^{a+b}} \rangle$ and thus, by a slight abuse of notation, as an element of $H^*((\Sigma_{2^a} \wr \Sigma_{2^b})^2;\mathbb{F}_2)$. With this identification, it is sufficient to prove that $d_1^* \circ \tau_1(x^{(n)} \otimes y^{(n)}) = \tau_2 \circ d_2^*(x^{(n)} \otimes y^{(n)})$, where τ_i and d_i are the transfer and diagonal maps of the following diagram:

$$\begin{tikzcd}
H^*(\Sigma_{2^a} \wr \Sigma_{2^b};\mathbb{F}_2) \arrow[r, "\tau_1"] \arrow[d, "d_2^*"] & H^*(\Sigma_{2^{a+b}};\mathbb{F}_2) \arrow[d, "d_1^*"] \\
H^*((\Sigma_{2^a} \wr \Sigma_{2^b})^2;\mathbb{F}_2) \arrow[r, "\tau_2"] & H^*\Sigma_{2^{a+b}}^2;\mathbb{F}_2)
\end{tikzcd}$$

Note that, in general, this diagram is *not* commutative. However, as we are going to prove below, it does commute when the maps are applied to primitive elements. Let $L = \mathbb{F}_2^a$. The regular representation allows us to identify L as a subgroup of Σ_{2^a}. Let $\rho\colon H^*(\Sigma_{2^{a+b}};\mathbb{F}_2) \to H^*(L^{2^b};\mathbb{F}_2)$ and $\rho'\colon H^*(\Sigma_{2^{a+b}}^2;\mathbb{F}_2) \to H^*(d(L^{2^b});\mathbb{F}_2)$ be the usual restriction maps. Since the coproduct of $(x \cdot y)^{[2^b]}$ and $x^{[2^b]} \cdot y^{[2^b]}$ is concentrated in the components $A(X)_{k2^a,*} \otimes A(X)_{(2^b-k)2^a,*}$, they are identified by their images under ρ. Hence it is sufficient to prove that $\rho \circ d_1^* \circ \tau_1 = \rho \circ \tau_2 \circ d_2^*$.

In order to do this, we let $K = \Sigma_{2^a} \wr \Sigma_{2^b}$ and $G = \Sigma_{2^{a+b}}$ and we recall that the triples $(L^{2^b}, \Sigma_{2^a} \wr \Sigma_{2^b}, \Sigma_{2^{a+b}})$ and, as a direct consequence, $(L^{2^b} \cong d(L^{2^b}), (\Sigma_{2^a} \wr \Sigma_{2^b})^2, \Sigma_{2^{a+b}})$ are closed systems. The normalizers of L in G and $d(L)$ in G^2 are $N_G(L) = L \rtimes GL_a(\mathbb{F}_2) = GA(\mathbb{F}_2^a)$ and $N_{G^2}(d(L)) = L^2 \rtimes d(GL_a(\mathbb{F}_2))$ respectively, where L acts trivially on itself and $GL_a(\mathbb{F}_2)$ acts as usual on $\mathbb{F}_2^a = L$. The normalizer subgroups of L in K and $d(L)$ in K^2 are $N_K(L) = L \rtimes (GL_a(\mathbb{F}_2) \cap K)$

and $L^2 \rtimes d(GL_a(\mathbb{F}_2) \cap K)$ respectively. Thus, by identifying L and $d(L)$, $H^*(L;\mathbb{F}_2)^{N_G(L)} = H^*(L;\mathbb{F}_2)^{N_{G^2}(L)}$, $H^*(L;\mathbb{F}_2)^{N_K(L)} = H^*(L;\mathbb{F}_2)^{N_{K^2}(L)}$ and the two transfer maps $H^*(L;\mathbb{F}_2)^{N_K(L)} \to H^*(L;\mathbb{F}_2)^{N_G(L)}$ are equal.

Now, an application of Kuhn's lemma (see [1], section III.5) suffices to prove our desired identity $\rho \circ d_1^* \circ \tau_1 = \rho \circ \tau_2 \circ d_2^*$.

□

We are now ready to state the main theorem of this chapter.

Theorem 120. *Let X be a topological space. Consider the divided powers bigraded component Hopf ring primitively generated by A, where:*

$$A = \tilde{H}^*(X;\mathbb{F}_2) \otimes \mathbb{F}_2\{\gamma_{n,1}\}_{n \geq 1}$$

Here, an element $\alpha \in \tilde{H}^d(X;\mathbb{F}_2)$ is intended to have bidegree $(1,d)$, while $\gamma_{n,1}$ have bidegree $(2^n, 2^n - 1)$. $H^*(DX;\mathbb{F}_2)$ is naturally isomorphic to the sub-Hopf ring with divided powers consisting of Hopf monomials whose constituent gathered blocks contain a divided power of a class $\alpha \in \tilde{H}^*(X)$ as a factor (i.e., that are not only a product of generators of the form $\gamma_{n,m}$).

Proof. We are going to use a classical description of the homology of CX, that we recall below. Recall that $H_*(CX;\mathbb{F}_2)$, with the product dual to the coproduct Δ in homology, is the algebra over the Dyer-Lashof algebra freely generated by $H_*(X;\mathbb{F}_2)$. In other words, choosing a graded basis $\mathcal{B}$ for $\tilde{H}_*(X;\mathbb{F}_2)$, $H_*(CX;\mathbb{F}_2)$, with the product $*$ dual to the coproduct Δ in homology, is the free commutative $\mathbb{F}_2$-algebra over strongly admissible sequences of Dyer-Lashof operations $Q_{i_1} \circ \cdots \circ Q_{i_r}(x)$, with $1 \leq i_1 \leq \cdots \leq i_r$ and $x \in \mathcal{B}$. Via the natural isomorphism of vector space $H_*(CX;\mathbb{F}_2) \cong H_*(DX;\mathbb{F}_2)$, we can view the classes $Q_{i_1} \circ \cdots \circ Q_{i_r}(x)$ as elements of $H_*(DX;\mathbb{F}_2)$.

In order to prove the Theorem, we need an auxiliary space $\tilde{D}X$, that we define here. For any $n \in \mathbb{N}$, let $\tilde{D}_n X = E(\Sigma_n) \times_{\Sigma_n} X^n$ (and, by convention, $\tilde{D}_0 X$ is a single point). Let $\tilde{D}(X) = \bigsqcup_{n \geq 0} \tilde{D}_n X$. We observe that $H^*(\tilde{D}X;\mathbb{F}_2)$ is a divided powers component Hopf rings, with structural maps defined in the same way we did for DX. This can be seen in many ways. For example, we can observe that $\tilde{D}X = D(X_+)$, where X_+ is the topological space obtained from X by adding a disjoint basepoint. Alternatively, this can be seen by means of the same argument used in the proof of Proposition 118, with some obvious modifications (mainly replacing smash products with cartesian products). It is immediate that the natural projection $\tilde{D}X \to DX$ induces in cohomology a map of Hopf rings that preserves the divided powers structure.

We first prove that $H^*(\tilde{D}X;\mathbb{F}_2)$ is the free divided powers component Hopf ring primitively generated by the given algebra A. First, map a class $\alpha \in \tilde{H}^*(X)$ into the corresponding class in $H^*(\tilde{D}_1 X;\mathbb{F}_2) \cong H^*(X;\mathbb{F}_2)$, and $\gamma_{n,1}$ into the cross product $\gamma_{n,1} \times 1_{H^*(X^n)} \in H^*(\tilde{D}_n X;\mathbb{F}_2)$. In this way, we have a map $DPHR(A^{pr}) \to H^*(\tilde{D}X;\mathbb{F}_2)$. We must prove that this is an isomorphism.

Recall, once again, that we can interpret $H^*(\tilde{D}_n X;\mathbb{F}_2)$ as the group cohomology $H^*(\Sigma_n; H^*(X;\mathbb{F}_2))$. Let $\mathcal{B}^\vee$ be the additive basis of $H^*(X;\mathbb{F}_2)$ linear dual to $\mathcal{B}$. In this way, by construction, the additive basis of Hopf monomials for $DPHR(A^{pr})$ is made by transfer products of gathered blocks in the mod 2 cohomology of some symmetric group cup-multiplied with a suitable divided power of a class in $\mathcal{B}$. Fix a total ordering of $\mathcal{B}^\vee$. Let $\mathcal{J}_r$ be the set of sequences of r strictly positive natural numbers $(j_1,\dots,j_r)$. Let $\mathcal{K}_r$ be the set of chains in $\mathcal{B}^\vee$ of length r, i.e. sequences $(\alpha_1,\dots,\alpha_r)\in(\mathcal{B}^\vee)^r$ such that $\alpha_i<\alpha_{i+1}$ for all $1\le i<r$. For any $\underline{j}\in\mathcal{J}_r$ and $\underline{\alpha}\in\mathcal{K}_r$, let $n=\sum_{i=1}^r j_i$. Let $M_{\underline{\alpha},\underline{j}}$ be the Σ_n-subrepseresentation generated by $\bigotimes_{i=1}^r \alpha_i^{\otimes^{j_i}}$. Then

$$\tilde{H}^*(X;\mathbb{F}_2)^{\otimes^n}=\bigoplus_{\underline{\alpha},\underline{j}:\sum j_i=n} M_{\underline{\alpha},\underline{j}}$$

as representations, and $M_{\underline{\alpha},\underline{j}}$ is isomorphism to the induced Σ_n-representation of the trivial representation of the subgroup $\prod_{i=1}^r\Sigma_{j_i}$. Hence, $H^*(D_nX;\mathbb{F}_2)=\bigoplus_{\underline{\alpha},\underline{j}:\sum j_i=n}H^*(\Sigma_n;M_{\underline{\alpha},\underline{j}})$.

In order to prove that our map $DPHR(A^{pr})\to H^*(\tilde{D}X;\mathbb{F}_2)$ is an isomorphism, it is sufficient to prove that this induces a perfect pairing between the additive basis $\mathcal{M}$ of $DPHR(A^{pr})$ described in Proposition 116 and the Nakaoka monomial basis $\mathcal{N}$ of $H_*(\tilde{D}X;\mathbb{F}_2)$.

Let $\underline{\alpha}=(\alpha_1,\dots,\alpha_r)\in\mathcal{K}_r$ and $\underline{j}=(j_1,\dots,j_r)\in\mathcal{J}_r$ such that $\sum_{i=1}^r j_i=n$. By construction, the addend $H^*(\Sigma_n;M_{\underline{\alpha},\underline{j}})$ is the image of the linear span of gathered monomials of the form $x_1\odot\cdots\odot x_r\in DPHR(A)_{n,*}$, where each x_i is a gathered monomial in $DPHR(A)_{j_i,*}$ that can be written as a transfer product of gathered blocks having profiles of the form $\gamma_{l_1,1}^{[m_1]}\cdot\dots\cdot\gamma_{l_s,1}^{[m_s]}\cdot\alpha_i^{[m_{s+1}]}$. This gives a decomposition of $\mathcal{M}$ as a disjoint union $\bigsqcup_{\underline{\alpha},\underline{j}}\mathcal{M}_{\underline{\alpha},\underline{j}}$ indexed by couples $(\underline{\alpha},\underline{j})\in\mathcal{K}_r\times\mathcal{J}_r$.

Fix $\underline{\alpha}\in\mathcal{K}_r$ and $\underline{j}\in\mathcal{J}_r$ such that $\sum_{i=1}^r j_i=n$. Consider a Nakaoka monomial:

$$Q_{I_{1,1}}(x_1)*\cdots*Q_{I_{1,s_1}}(x_1)*Q_{I_{2,1}}(x_2)*\cdots*\cdots*Q_{I_{r,s_r}}(x_r),$$

with $x_1,\dots,x_r\in\mathcal{B}$. Let $\alpha_1,\dots,\alpha_r\in\mathcal{B}^\vee$ be the linear duals of $x_1,\dots,x_r$ respectively. Since the product is associative, we can assume, without loss of generality, that $\alpha_1<\alpha_2<\cdots<\alpha_r$. By the way Dyer–Lashof operations are constructed, such a Nakakoka monomial is the image of a certain homology class in

$$H_*\left(\Sigma_{j_1}\times\cdots\times\Sigma_{j_r};M_{(\alpha_1),(j_1)}\otimes\cdots\otimes M_{(\alpha_r),(j_r)}\right).$$

In order to simplify notation, we use the symbols $\Sigma_{\underline{j}}=\Sigma_{j_1}\times\cdots\times\Sigma_{j_r}$ and $M'_{\underline{\alpha},\underline{j}}=M_{(\alpha_1),(j_1)}\otimes\cdots\otimes M_{(\alpha_r),(j_r)}$.

The discussion above implies the existence of a disjoint union decomposition $\mathcal{N}=\bigsqcup_{\underline{\alpha},\underline{j}}\mathcal{N}_{\underline{\alpha},\underline{j}}$ indexed by couples $(\underline{\alpha},\underline{j})\in\mathcal{K}_r\times\mathcal{J}_r$ such that $\mathcal{M}_{\underline{\alpha},\underline{j}}$ and $\mathcal{N}_{\underline{\alpha}',\underline{j}'}$ pair non-trivially only if $\underline{\alpha}=\underline{\alpha}'$ and $\underline{j}=\underline{j}'$.

Thus, it is sufficient to prove that, for all $\underline{\alpha}$ and $\underline{j}$, the pairing between $\mathcal{M}_{\underline{\alpha},\underline{j}}$ and $\mathcal{N}_{\underline{\alpha},\underline{j}}$ is perfect. Since $\mathcal{N}_{\underline{\alpha},\underline{j}}$ is the bijective image in $H_*(DX;\mathbb{F}_2)$ of an additive basis of $H_*(\Sigma_{\underline{j}};M_{\underline{\alpha},\underline{j}})$, it is sufficient to check that the pairing is perfect when $\mathcal{M}_{\underline{\alpha},\underline{j}}$ is restricted to $H_*(\Sigma_{\underline{j}},M'_{\underline{\alpha},\underline{j}})$. Since $M'_{\underline{\alpha},\underline{j}}$ is the trivial representation, the latter homology group is isomorphic to the standard mod 2 homology of $\Sigma_{\underline{j}}$. With this identification, $\mathcal{M}_{\underline{\alpha},\underline{j}}$ and $\mathcal{N}_{\underline{\alpha},\underline{j}}$ become the product of the Hopf monomials bases and the Nakaoka monomial bases respectively for the mod 2 cohomology of $\Sigma_{j_1},\dots,\Sigma_{j_r}$. Since the pairing becomes the usual pairing between the mod 2 homology and cohomology of the symmetric groups, we know that this is perfect from previous results.

The Theorem now follows by observing that the cohomology of the extended powers $H^*(DX;\mathbb{F}_2) \cong \bigoplus_{n\geq 0} H^*(\Sigma_n;\tilde{H}^*(X;\mathbb{F}_2)^{\otimes^n})$ is realized, via the natural projection $\tilde{D}X \to DX$, as the desired subspace of $DPHR(A^{pr}) \cong H^*(\tilde{D}X;\mathbb{F}_2)$. □

Remark 121. There is a skew-commutative version of all the algebraic structures defined above, when the relations $x \cdot y = (-1)^{|x||y|} y \cdot x$ and $x \odot y = (-1)^{|x||y|} y \odot x$ are required instead of plain commutativity. In this case, we also require $x^{[n]} = 0$ whenever x has odd degree and $n \geq 1$. All the constructions made above are essentially the same in this new setting, up to taking free skew-commutative algebras instead of free commutative ones and making other very minor adaptations. With the same proof, the obvious analogue Proposition 116 is still true, with the exception of the last point. To obtain an additive basis for a graded-commutative bigraded divided powers component Hopf ring A over a field R of characteristic p, if A is concentrated in components indexed by powers of p, we still use the set $\mathcal{M}$ of gathered monomials as defined before. However, we must discard from $\mathcal{M}$ all the elements that have a constituent block of odd dimension different from its profile. The remaining elements form an additive basis $\mathcal{M}'$ for A.

With essentially the same proof used in the mod 2 case, one can prove that $H^*(CX;\mathbb{F}_p)$ is a divided powers bigraded component Hopf ring over $\mathbb{F}_p$ for all primes p. The only difference is that, obviously, if p is an odd prime, this divided powers Hopf ring is skew-commutative. The obvious analog of Theorem 120 would be the existence of an isomorphism between $H^*(DX;\mathbb{F}_p)$ and the free skew-commutative divided powers component Hopf ring primitively generated by $\tilde{H}^*(X;\mathbb{F}_p)$ and classes $\gamma_{*,*}$, $\beta_{*,*,*}$ and $\alpha_{*,*}$ arising from the mod p cohomology of the symmetric groups, with cup product relations coming from $H^*(X;\mathbb{F}_p)$ and Lemma 39. However, this is not true in general (a counterexample is $X = S^1$, for which the Poincaré polynomial of this free divided powers Hopf ring differs from that of $H^*(CS^1;\mathbb{F}_p)$, for $p = 3$). This seems to be related to some non-trivial relations in the dual of the mod p Kudo–Araki algebra that, to the author's knowledge, are not fully understood yet.

Theorem 120 gives a complete description of $H^*(DX;\mathbb{F}_2)$ as a Hopf ring. We can use this to also obtain some information about the cohomology of QX, the free ∞-loop space over a topological space X. Recall that Proposition 116 gives a bigraded additive basis for the mod 2 cohomology of DX, via the identification with a free bigraded component divided powers Hopf ring generated by a suitable component algebra.

The cohomologies of DX and CX may differ as rings. In general, $H^*(DX)$ is the associated graded to the filtration of $H^*(CX)$ given by number of points. There are, however, notable cases in which the cohomology rings of CX and DX coincide (for instance, if X has a disjoint basepoint). In such cases, we can fully describe the cohomology of QX starting from Theorem 120. In the remaining part of this section, we always implicitly assume that X has a disjoint basepoint, so that we can fully identify CX and DX.

Recall that $H_*(QX;\mathbb{F}_2)$ is strongly related to $H_*(CX;\mathbb{F}_2)$. In particular, in [8], the authors prove that $H_*(QX;\mathbb{F}_2)$, as a Hopf algebra, is the group completion of $H_*(CX;\mathbb{F}_2)$. In the particular case in which X is connected, the two homology groups are actually isomorphic.

We can relate in a similar way the mod 2 cohomology of QX and the mod 2 cohomology of DX. Instead of group completion, in this case we need a suitable "stabilization" process.

Definition 122. We say that a gathered monomial $x \in \mathcal{M}$ is *full width* if none of its constituent gathered blocks have cohomological dimension 0. Let $\mathcal{M}_{FW} \subseteq \mathcal{M}$ be the set of full width gathered monomials. We define the *stabilization* of $H^*(DX;\mathbb{F}_p)$ as the free $\mathbb{F}_p$-vector space over $\mathcal{M}_{FW}$. We denote it by $A^s(X)$.

Remark 123. We observe that $A^s(X)$ has the structure of a Hopf ring, induced by that of $H^*(CX;\mathbb{F}_2)$.

In order to see this, note that if X is connected, then $A^s(X) = H^*(CX;\mathbb{F}_2)$, since there are no gathered blocks of dimension 0. If X is not connected, let $S = \pi_0(X) \setminus \{[*]\}$ the set of the connected components of X that do not contain the basepoint $*$, and chose $w_C \in C$ for all $C \in S$. Let $M(S)$ be the free commutative monoid generated by S. Since CX is an H-space, the set of connected components $\pi_0(CX)$ is a monoid, naturally isomorphic to $M(S)$. For any $m \in M(S)$, let C_m be the component of CX corresponding to m via this isomorphism. We define a partial ordering on $M(S)$ by letting $m \leq m'$ if and only if there exists $m'' \in M(S)$ such that $m' = mm''$.

Consider the inverse system $(\{V_m\}, \omega_{m,m'})$ of $\mathbb{F}_2$-vector spaces on the poset $M(S)$ defined as follows. We put $V_m = H^*(C_m;\mathbb{F}_2)$. If $m > m'$, then $m = m'm''$, with $m'' = \prod_{s\in S} s^{a_s}$. For any gathered monomial $\alpha \in H^*(C_m;\mathbb{F}_2)$, we put

$$\omega_{m,m'}(\alpha) = \begin{cases} \alpha' & \text{if } \alpha = \alpha' \odot \bigodot_{s\in S}[w_s]^{\odot^{a_s}} \\ 0 & \text{otherwise} \end{cases}$$

Note that $\omega_{m,m'}$ is always surjective, and is also injective in a fixed dimension d if the length of m is big enough. $A^s(X)$ can be identified with the limit of this inverse system, and this induces a Hopf ring structure on $A^s(X)$.

In the following result, X_+ denotes the pointed space obtained from X by adding a disjoint basepoint.

Corollary 124. *Let G be the group completion of the monoid $M(S)$ defined above. Let $p \geq 2$ be any prime and let X be a topological space and let QX_+ be the free ∞-loop space over X_+. There is an isomorphism:*

$$H^*(QX;\mathbb{F}_2) \cong G \times A^s(X)$$

Proof. It is known that $H_*(QX_+;\mathbb{F}_2)$, as a Hopf algebra, is isomorphic to the group completion $H_*(CX_+;\mathbb{F}_p) \otimes_{\mathbb{Z}_p[M(S)]} \mathbb{F}_p[G]$. This implies that the homology of each connected component of QX_+ is the limit of the direct system $(\{W_m\}_m, \{f_{m,m'}\})$ on $M(S)$ given by $W_m = H_*(C_m;\mathbb{F}_2)$ and $f_{m,ms} = w_s * \_$.

The linear dual of this direct system is exactly the inverse system described above, thus the mod 2 cohomology of each component of QX_+ is isomorphic to $A^s(X)$. □

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