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Fourth-order geometric flows on manifolds with boundary

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Chapter 1

Introduction

1.1 Context

Geometric flows have become, since their popularization by Hamilton in 1982 ([Ham82]), one of the main topic of interest in geometric analysis. The underlying idea, that analytical features of a geometric object can be "enhanced" in a controlled way by mean of suitable evolution equations, has enabled many important problems to be solved, among which it is of particular fame the proof of the Poincaré conjecture (and indeed of the whole geometrization conjecture) by Perelman in [Per02, Per03a, Per03b].

In order to get a grasp of what advantage a geometric flow can introduce in a Riemannian geometry problem, let us consider the following famous theorem (proved for example in [GHL04, Theorem 3.82]). In this work we will always assume that manifolds are connected.

1.1 Theorem (Killing-Hopf theorem)**.** *Let (M^n, g) be a complete Riemannian manifold of constant sectional curvature. Then it is isometric to the quotient of the sphere S^n, the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$ (all considered with their standard metric up to a constant multiple), respectively when the curvature is positive, zero or negative.*

The applicability of the theorem is unfortunately severely limited by its hypotheses, that appear extremely strong from the point of view of a generic Riemannian manifold. Under most sensible topologies, the set of constant curvature manifolds is nowhere dense, meaning that any metric of constant curvature can be modified with an arbitrarily small perturbation to make it non constant curvature. In order to "fatten" the set on which the Killing-Hopf theorem can be applied, one can imagine to develop a tool that "fixes" a metric by smoothly transforming it to a metric of constant curvature, while at the same time preserving the underlying differential structure. Inheriting the regularization properties of parabolic equations, geometric flows theory often provide a valid candidate for this type of tool.

Many mathematicians have provided examples of this technique, beginning from the seminal paper by Hamilton cited above, in which it is proved that if a three manifold (M^3, g) has positive Ricci curvature, then the normalized Ricci flow starting at g converges to a metric of positive constant sectional curvature. Another famous result, by Brendle and Schoen in 2009 ([BS09]), states that the same happens when (M^n, g) has dimension $n \geqslant 4$ and the maximum sectional curvature is less than 4 times the minimum sectional curvature.

In order to state other theorems that are relevant to this work, let us recall that the Riemann tensor on a manifold admits the following orthogonal decomposition:

$$\mathrm{Riem}_g = S_g + Z_g + W_g := \frac{R_g}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{\mathrm{Ric}}_g \owedge g + W_g,$$

where the Kulkarni-Nomizu product $\owedge$ is defined, for any $a, b \in \mathscr{S}_2(M)$, as

$$(a \owedge b)_{ijk\ell} = a_{ik}b_{j\ell} + a_{j\ell}b_{ik} - a_{i\ell}b_{jk} - a_{jk}b_{i\ell}$$

and the Ricci, scalar and traceless Ricci curuvatures are

$$\begin{aligned} \mathrm{Ric}^g_{ik} &= \mathrm{Riem}^g_{ijk\ell} g^{j\ell}, \\ R_g &= \mathrm{Ric}^g_{ik} g^{ik}, \\ \mathring{\mathrm{Ric}}^g_{ik} &= \mathrm{Ric}^g_{ik} - \frac{R_g}{n} g_{ik}. \end{aligned}$$

We can thus state the following theorem by Margerin ([Mar98]), which is a refinement of a previous theorem by Huisken ([Hui85]).

1.2 Theorem. *Let (M^4, g) a closed Riemannian manifold with positive scalar curvature such that the following pinching conditions holds pointwise on M:*

$$|Z_g|^2 + |W_g|^2 < |S_g|^2.$$

Then the normalized Ricci flow starting at g exists for all times and converges to a metric of positive sectional curvature in the $\mathscr{C}^\infty$ sense as the time goes to infinity.

Together with Theorem 1.1, this implies that M is diffeomorphic to a quotient of the sphere S^4 (and it is known that only two such manifolds exist: S^4 itself and the real projective space $\mathbb{PR}^4$).

Later a generalization of Margerin's theorem was proved by Chang, Gursky and Yang ([CGY03]): the pointwise pinching is replaced by an integral one, and the positive scalar curvature condition is replaced by one based on the Yamabe constant, which, on a closed manifold M, is defined as:

$$Y(M, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_M R_{\tilde{g}} \, dv_{\tilde{g}}}{\mathrm{Vol}(M, \tilde{g})^{1-\frac{2}{n}}}, \tag{1.1}$$

where $[g]$ is the conformal class of the metric g. The Yamabe constant will be discussed in more detail later in this introduction.

1.3 Theorem. *Let (M^4, g) be a closed Riemannian manifold with positive Yamabe constant such that the following pinching condition holds:*

$$\int_M |Z_g|^2 \, dv_g + \int_M |W_g|^2 \, dv_g < \int_M |S_g|^2 \, dv_g. \tag{1.2}$$

Then M is diffeomorphic to either S^4 or $\mathbb{PR}^4$.

The proof by Chang, Gursky and Yang does not immediately rely on a flow argument: instead they first build a metric, conformal to g, that satisfies the pointwise pinching condition, and then invoke Margerin's theorem on it. In his PhD thesis ([Bou12]), Bour proved a (slightly weaker) fact using just one flow, tailored for this specific problem. Bour's proof is based on the observation that one can improve the pinching (1.2) by flowing down the gradient of an energy that penalizes $\int |Z|^2$ and $\int |W|^2$. Since it is known that a Riemannian metric has constant sectional curvature if and only if $Z = W = 0$, it is expected that the Bour flow will converge to such a metric as soon as it can reduce to zero the energy

$$\mathscr{F}^\lambda(g) = (1-\lambda) \int_M |W_g|^2 \, dv_g + \lambda \int_M |Z_g|^2 \, dv_g,$$

defined for $\lambda \in (0,1)$. The Bour flow is thus defined by the equations:

$$\begin{aligned} \partial_t g(t) &= -2 \cdot \nabla \mathscr{F}^{\lambda}(g(t)) \\ g(0) &= g_0. \end{aligned} \tag{1.3}$$

The gradient of an energy is defined (when it exists) to be the operator that satisfies, for every variation h of the metric g, the identity

$$d_g \mathscr{F}^{\lambda}(h) = \int_M \langle \nabla \mathscr{F}^{\lambda}(g) | h \rangle \, dv_g. \tag{1.4}$$

This definition mimics the definition of the gradient on Riemannianm manifolds, but it must be noted that we are not defining an L^2 structure on the space of metrics here. In particular, this "scalar product" depends on the point g itself (via the volume element dv_g). In [Bou12, Chapter 2, Section 2] it is proved that the gradient $\nabla \mathscr{F}^{\lambda}$ actually exists. In the end, Bour proves the following theorem.

1.4 Theorem ([Bou12, Corollary H]). *Let (M^4, g) be a closed Riemannian manifold with positive Yamabe constant such that the following pinching condition holds:*

$$\int_M |Z_g|^2 \, dv_g + \frac{5}{8} \int_M |W_g|^2 \, dv_g < \frac{1}{8} \int_M |S_g|^2 \, dv_g.$$

Then the gradient flow associated to $\mathscr{F}^{\lambda}$ for $\lambda = \frac{4}{13}$ exists for all times and converges to a metric of positive constant sectional curvature. In particular, M is diffeomorphic to either S^4 or $\mathbb{PR}^4$.

Bour's result actually applies to all $\lambda \in (0,1)$, but $\lambda = \frac{4}{13}$ is the value that gives the best pinching.

1.2 Purpose

In this thesis we begin exploring the problem of extending Bour's theorem to manifolds with boundary. This is a first step for later further extension to complete noncompact manifolds: a strategy for studying the behaviour of a flow on a noncompact manifold M is to first decompose it in an exhaustion by compact sets K_j, find a solution of the flow on each of the sets K_j and then glue all the solutions back to a solution of the flow on M. This was done successfully, for example, to prove short time existence of the Ricci flow on noncompact manifolds (with appropriate curvature bounds) in [Shi89]. Short time existence results for the Ricci flow on manifolds with boundary were also given in [Pul13] and [Gia13] (also published in [Gia16]), with different sets of boundary conditions.

From the analytical point of view, the Bour flow (as many others, such as the Ricci flow) can be described by a parabolic system of partial derivatives equations of the form:

$$\partial_t g(t) = P(g(t)),$$

where P is a differential operator depending on the space derivatives of the metric g. We will detail later what structure we assume on the operator P. For the flow equation we want to set an initial value (which corresponds to the initial metric in the discussion above); when the underlying manifold has a boundary, then a certain number of boundary conditions has to be set as well: they can specify an exact value that the unknown g must take at the boundary, but they can also be expressed in terms of a generic operator B that involves the derivatives of the unknowns. The main feature that distinguishes the Bour flow from the Ricci flow is that the operator P has order 4, instead of 2. Unfortunately, to the best of the author's knowledge, very little appears in literature about geometric flows of order higher than 2, let alone about flows on manifolds with boundary. This thesis seeks therefore to give a contribution to the field by presenting a short time existence theorem for parabolic systems of PDEs on manifolds with boundary, that can be used for flows of arbitrary (even) order.

Beside regularity considerations, there are three main requirements that one has to check when trying to find solutions for parabolic equations.

- The *strong parabolicity condition* essentially requires that the operator P behaves, at its highest order, like an elliptic operator. It is known from the theory of parabolic equations that if P has elliptic highest order, then it naturally evolves and has regularization effects towards positive times, while it might lose regularity (or even do not admit a solution altogether) towards negative times. If P is antielliptic, the situation is reversed and the flow naturally evolves towards negative times. If P is not definite, then one usually cannot guarantee existence in neither direction.

- The *complementary condition* can be informally described as the request that boundary conditions specified by the operator B are linearly independent. While the idea is easy to grasp and accept (a dependency relation between different boundary conditions would produce an overdetermined and thus impossible system, in mostly the same way as it does in plain linear algebra), the actual formulation turns out to be rather technical, because it needs to take into consideration the interactions between differential equations, which is more complicated than the interaction between vectors in a vector space.

- The *compatibility condition* regulates the potential overdefinition that could result at the edge $M \times \{0\}$, where both initial and boundary conditions apply and one must be sure they are not in conflict. The compatibility condition can actually be verified at different levels: the higher regularity one seeks to prove at $M \times \{0\}$, the higher level of compatibility they must require to have it.

Taking into account the bigger picture in which this work is situated, selecting the best boundary conditions turns out to be a delicate and critical issue. On one hand, choosing geometrically significant boundary conditions is a key factor to the subsequent analysis of the flow, especially of its long time behaviour. Flows of higher order do not have a maximum principle like the Ricci flow; its absence must be compensated with the heavy use of integral estimates, to control the evolution of geometric quantities during the flow: this translates into a frequent usage of the integration by parts formula, with the incovenient potential proliferation of boundary terms and the need of appropriate boundary conditions to handle them (either by proving that they are zero, or that they have the right sign, or that they can be estimated in some other way). Without claiming to be exhaustive, let us consider some of the more prominent integral geometric quantities and relationships used in [Bou12], to review what is their effect on the boundary ∂M.

- The entire point of using a gradient flow is that the *flow energy* $\mathscr{F}^\lambda$ should be decreasing during the flow itself. On a closed manifold one can use (1.4) to prove

$$\partial_t \mathscr{F}^\lambda(g(t)) = 2\int_M \langle \nabla \mathscr{F}^\lambda(g(t)) | \partial_t g(t)\rangle = -4\int_M |\nabla \mathscr{F}^\lambda(g(t))|^2 \leqslant 0.$$

On a manifold with boundary, equation (1.4) is not true anymore, because in general a boundary term appears. We do not show the whole computation, but by using the Chern-Gauss-Bonnet formula (discussed below) one can prove that the boundary term gives zero contribution if the following conditions are satisfied at the boundary:

$$\begin{aligned} [g^T] &= [g_0^T] \\ \nu &= \nu_{g_0} \\ \Pi &= 0 \\ \nabla_\nu R_g &= 0 \\ \nabla_\nu \mathrm{Ric}^g_{\nu\nu} &= 0. \end{aligned}$$

In the formula, Π is the second fundamental form of the boundary ∂M with respect to the metric g and g^T is the tangential metric at the boundary ∂M. The letter ν indicates the normal vector at the boundary, both when used as an object and as an index. As before, $[\cdot]$ indicates the conformal class of a metric.

- It is shown in [Bou12, Chapter 3, Section 2.3] that the *volume of the manifold* during the flow is constant when the manifold M is closed, because

$$\partial_t \mathrm{Vol}(M, g(t)) = \frac{1}{2}\int_M \mathrm{tr}\, P(g(t))\, dv_g = \frac{\lambda}{8}\int_M \Delta R_g\, dv_g = 0.$$

  Using the divergence theorem, we readily see that the constancy of the volume is valid on a manifold with boundary as soon as

$$\nabla_\nu R_g = 0.$$

- The *Chern-Gauss-Bonnet formula* relates the Euler characteristic of a manifold with the L^2 energy of the curvature tensor S, Z and W introduced above. On a closed manifold the formula reads:

$$8\pi^2\chi(M) = \int_M |S_g|^2\, dv_g - \int_M |Z_g|^2\, dv_g + \int_M |Z_g|^2\, dv_g.$$

  This fromula is useful in that it effectively reduces by 1 the "dimension" of the energy that must be controlled during the flow, because the number $\chi(M)$ is a topological invariant and does not change; for example, an immediate consequence of the Chern-Gauss-Bonnet formula is that if the energy $\mathscr{F}^\lambda$ is bounded during the flow, then the L^2 energy of the Riemann tensor is bounded as well.

  A complete expansion of all the boundary terms that appear when ∂M is not empty can be found in [CQ97, Remark 3.3] or in [CQY00, p. 67] and is the following:

$$\begin{aligned} 8\pi^2\chi(M) = &\int_M |S_g|^2\, dv_g - \int_M |Z_g|^2\, dv_g + \int_M |Z_g|^2\, dv_g \\ &+ 2\int_{\partial M}\Big(-\frac{1}{12}\nabla_\nu R_g + \frac{1}{3}\Delta_{g^T}\mathscr{H} - 2\mathrm{Riem}^g_{a\nu b\nu}\Pi_{ab} + \mathrm{Ric}^g_{ab}\Pi_{ab} + \mathrm{Ric}^g_{\nu\nu}\mathscr{H} \\ &\qquad - \frac{1}{6}R_g\mathscr{H} - \frac{1}{9}\mathscr{H}^3 + \mathscr{H}|\Pi|^2 - \frac{4}{3}\mathrm{tr}\,\Pi^3\Big)\, d\sigma_g. \end{aligned}$$

  Here $\mathscr{H} = \mathrm{tr}_g \Pi$ is the mean curvature.

From the above formulation one sees that an easy boundary conditions that restores the original Chern-Gauss-Bonnet formula is

$$\Pi = 0$$
$$\nabla_\nu R_g = 0.$$

- The *Yamabe constant* is used in [Bou12] to control the evolution of the Sobolev constant; this is important for the long time analysis of the flow, because the Sobolev constant controls the volume of the balls of a manifold, and therefore the injectivity radius. When considering the limit of a sequence of manifolds, such as those appearing in blow-up arguments of flow theory, a lower bound on the injectivity radius guarantees that the limit manifold will not "lose" a dimension, for example collapsing on a hypersurface.

  The Yamabe constant appears in connection with the Yamabe problem, which asks whether, given a metric g on a closed manifold of dimension $n \geqslant 3$, one can find another metric $\tilde{g}$, conformal to g, which has constant scalar curvature. The answer, which is affirmative, was first given by Yamabe in [Yam60]. Yamabe's proof unfortunately had a gap, which was later filled by a series of papers culminating with [Sch84]. See also [Aub98, Chapter 5] for complete proof.

  The Yamabe problem can be reformulated into solving the following variational problem on M:

$$Y(M,[g]) = \inf_{u\in\mathscr{C}^1(M)} \frac{\int_M \left(\frac{4(n-1)}{n-2}|du|^2 + R_g u^2\right) dv_g}{\left(\int_M u^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}. \tag{1.5}$$

  Such problem is equivalent to (1.1), meaning that if the minimum in (1.5) is attained at the function u, then the function u is smooth and the minimum in (1.1) is attained at the metric

$$\tilde{g} = u^{\frac{4}{n-2}} \cdot g, \tag{1.6}$$

  which has constant scalar curvature, equal to $Y(M,[g])$. This special metric is called a *Yamabe metric*, and is the solution of the Yamabe problem. One can also prove that the function u is strictly positive; as a corollary, if the metric g has $R_g > 0$, then also $Y(M,[g]) > 0$, which implies that the positive Yamabe constant hypothesis in Theorem 1.3 is effectively a generalization of the positive scalar curvature hypothesis in Theorem 1.2.

  The existence of Yamabe metrics on manifolds with boundary was proved by Escobar in [Esc92]. The functional $Y(M,[g])$ can be emeneded as it follows:

$$Y(M,[g]) = \inf_{u\in\mathscr{C}^1(M)} \frac{\int_M \left(\frac{4(n-1)}{n-2}|du|^2 + R_g u^2\right) dv_g + 2(n-1)\int_{\partial M} \mathscr{H} u^2\, d\sigma_g}{\left(\int_M u^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}.$$

  As in the closed case, equation (1.6) can be used to recover a Yamabe metric, which has constant scalar curvature and zero mean curvature at the boundary. These considerations show again that the control of the second fundamental form is important for recovering Bour's results on a manifold with boundary.

The choices for boundary conditions we mentioned hitherto is unfortunately limited by the constraints we mentioned above, particularly by the complementary condition. Moreover, even in relatively simple situations, the verification of the complementary condition requires rather long and tedious computations which, beside taking time, tend to conceal what "is actually going on". As a result, informal analysis trying to understand how to fix some boundary conditions that are "nearly but not entirely" working (i.e., satisfying the complementary condition) is in general difficult.

There is another difficulty that arises when imposing boundary conditions: an intermediate step for proving short time existence of geometric flows is the so-called DeTurck trick. The differential systems that describe geometric flows usually do not satisfy the strong parabolicity condition as they are; it can actually be proved that if the operator P is geometric (meaning that it is invariant by diffeomorphisms: $\varphi^* P(g) = P(\varphi^* g)$), then it needs to have a zero eigenvalue, thus contradicting the strong parabolicity condition (see for example [CK04, Chapter 3, Section 2.3]). However, one can produce a modified version of the equation

$$\partial_t \bar{g}(t) = Q(\bar{g}(t)) := P(\bar{g}(t)) + \mathscr{L}_{V(\bar{g}(t))}\bar{g}(t),$$

where $\mathscr{L}$ denotes the Lie derivative and $V(\bar{g})$ is an appropriate operator, chosen so that this modified equation is strongly parabolic. The original flow g can be reconstructed by taking $g(t) := \varphi_t^* \bar{g}(t)$, where φ_t is a family of diffeomorphisms such that

$$\partial_t \varphi_t(x) = V(\bar{g}(t))(\varphi_t(x)).$$

The problem with this approach is that it does not guarantee that boundary conditions are preserved when doing the pullback of $\bar{g}$ along φ_t. The problem can be partially mitigated by adding $V|_{\partial M} = 0$ among the boundary conditions, so that all the diffeomoprhisms φ_t are the identity at the boundary. In this way the control is restored for all boundary conditions that are invarient under this class of diffeomoprhisms. Some conditions though do not, most notably the condition on the normal vector ν. For those conditions we are not yet able to propose an alternative method.

As we mentioned above, the results in this work are preliminary to the proof of a short time existence for the Bour flow on noncompact manifolds, obtained by glueing the solutions on the sequence of sets K_j. For this to be possible, it is essential that such solutions have existence time bounded below by a uniform constant, and that in such uniform time frame the geometry of the flow (as measured, for example, by the Riemann tensor and its derivatives) stay bounded as well. Pushed by these requirements, in this work the dependency on known data of constants arising in estimates is tracked carefully.

In general, an effort has been made to keep the exposition as general and accessible as reasonably possible, hoping that it can be useful both for those willing to study the theory of short term existence for geometric flows and those who are not specifically interested in learning the whole theory, but would just like to apply its results to their problem of study.

1.3 Overview of the work

Let (M^n, g) be a Riemannian manifold. The generic semilinear parabolic PDE system of order $2b$ which we will consider has the form: for $(x,t) \in M \times [0,T]$

$$\begin{aligned}\partial_t u_{I_k}(x,t) = {} & A_{I_k}^{J_{2b}K_k}(x,t,u(x,t),\dots,\nabla^{2b-1}u(x,t)) \cdot \nabla^{2b}_{J_{2b}} u_{K_k}(x,t) \\ & + F_{I_k}(x,t,u(x,t),\dots,\nabla^{2b-1}u(x,t)).\end{aligned}$$

The notation I_k is a shorthand notation for the indices $i_1, \dots, i_k$, and the same happens with all other capital letters when used as tensor indices. Here u is k-covariant a tensor and A

and F are the coefficients of the system. For the moment let us delay the precise definition of all the objects in play. Together with the main equation we impose initial and boundary conditions: for $x \in M$

$$u_{I_k}(x,0) = (u_0)_{I_k}(x)$$

and for $(x,t) \in \partial M \times [0,T]$ and $q = 1,\dots,\upsilon$

$$\begin{aligned}(B^q)^{J_{m_q}K_k}_{I_{d_q}}(x,t,u(x,t),\dots,\nabla^{m_q-1}u(x,t)) \cdot \nabla^{m_q}_{J_{m_q}} u_{K_k}(x,t) \\ = (E^q)_{I_{d_q}}(x,t,u(x,t),\dots,\nabla^{m_q-1}u(x,t)).\end{aligned}$$

Here $0 \leqslant m_q < 2b$ is the order the q-th boundary condition and d_q is its number of indices.

As a further shorthand, we will omit altogether the tensor indices when this does not lead to confusion and we will write $A_w(x,t)$ instead of $A(x,t,w(x,t),\dots,\nabla^{2b-1}w(x,t))$ (and the same for symbols F, B^q and E^q). We can thus rewrite the problem as:

$$\partial_t u(x,t) - A_u(x,t)\cdot\nabla^{2b}u(x,t) = F_u(x,t) \qquad x \in M, t \in [0,T] \tag{1.7}$$

$$u(x,0) = u_0(x) \qquad x \in M \tag{1.8}$$

$$B^q_u \cdot \nabla^{m_q} u(x,t) = E^q_u(x,t). \qquad x \in \partial M, t \in [0,T] \tag{1.9}$$

We begin in Chapter 2 by giving some general definitions, mostly in order to establish the notation that will be used. Compatibility conditions are also introduced.

In Chapter 3 we recall some properties of geometry on manifolds with boundary and define what is a bounded atlas. It is known that many elementary theorems in Riemannian geometry are not necessary true when the manifold has a boundary: this depends on the fact that geometric geodesics (i.e., curves of locally minimal length) are not necessarily analytical geodesics (i.e., curves that solve the geodesics equation). However, if the boundary is convex (meaning that it has nonnegative definite second fundamental form), then identity between geometric and analytical geodesics is restored, and with it most basic results, such as the properties of the exponential map. It is then shown that on a manifold with bounded geometry a *bounded atlas*, having geometric properties particularly well adapted to what will be needed in the following, can be constructed.

In Chapter 4 appropriate function spaces will be defined on the manifolds M, $M \times [0,T]$ and $\partial M \times [0,T]$. Both Hölder-type and Sobolev-type spaces will be defined: as it is customary with parabolic problems, different order of regularity must be imposed in the time and space directions, in order to account for the fact that, roughly speaking, a time derivative is equivalent to $2b$ space derivatives. Also, we need such spaces to support fractional derivatives, which are readily available for Hölder spaces. For Sobolev spaces we turn to the theory of Sobolev-Slobodeckij spaces, which are a special case of Besov spaces. Thanks to appropriate localization arguments, we connect our theory with that presented in [Ama09] for Euclidean spaces, so that we can access the theorems available there. In particular, we present the calculus toolbox that is used in later chapters: it includes embedding theorems, differentiation theorems, product theorems and trace theorems. The important case of spaces of functions with zero initial value is also discussed, as well as the relevant Sobolev embeddings.

In Chapter 5 the case of a linear system of PDEs is studied. Equations (1.7), (1.8) and (1.9) take the following form:

$$\partial_t u(x,t) - A_0(x,t)\cdot\nabla^{2b}u(x,t) = \sum_{p=0}^{2b-1} A_{2b-p}(x,t)\cdot\nabla^p u(x,t) + F_0(x,t) \qquad x\in M, t\in[0,T] \tag{1.10}$$

$$u(x,0) = u_0(x) \qquad x\in M \tag{1.11}$$

$$B_0^q\cdot\nabla^{m_q}u(x,t) = \sum_{p=0}^{m_q-1} B_{m_q-p}^q\cdot\nabla^p u(x,t) + E_0^q(x,t). \qquad x\in\partial M, t\in[0,T] \tag{1.12}$$

The classical theory of linear parabolic equations, exposed in [Sol65], is adapted to the realm of manifolds and presented, both for Hölder and Sobolev-Slobodeckij functions. The coefficients and data of the system must satisfy the *(strong) parabolicity*, *complementary* and *compatibility conditions*, mentioned above. Both parabolicity and complementary conditions are introduced; compatibility conditions, already discussed in Chapter 2, are considered in the special case of linear systems.

In Chapter 6 the generic problem (1.7), (1.8), (1.9) is finally taken into consideration. As it often happens with nonlinear problems, a solution can only be constructed by iterative approximation; the semilinear system is converted into a linear one by "freezing" the coefficients at u_0:

$$\begin{aligned}
\partial_t u(x,t) - A_{u_0}(x,t)\cdot\nabla^{2b}u(x,t) &= F_{u_0}(x,t) && x\in M, t\in[0,T]\\
u(x,0) &= u_0(x) && x\in M\\
B_{u_0}^q\cdot\nabla^{m_q}u(x,t) &= E_{u_0}^q(x,t). && x\in\partial M, t\in[0,T]
\end{aligned}$$

This system is now an instance of (1.10), (1.11) and (1.12), so we can find a solution u, which of course will not in general solve the semilinear system. However, we can replace F_{u_0} and $E_{u_0}^q$ with new data computed from u, trying to correct the deviation of u from an actual solution of the semilinear system. Solving again the frozen system, we obtain another candidate solution, which gives a better approximation of the actual solution. We will show that u is a solution of (1.7), (1.8), (1.9) if and only if it is a fixed point of the discrete dynamical system we have just described, to be considered in a suitable Sobolev-Slobodeckij space. In order to show the existence and uniqueness of such a fixed point, we show that, if T is small enough, the iteration has Lipschitz constant smaller than 1, so that Banch's fixed point theorem can be used. Higher regularity is then proved using the linear theory for Hölder spaces.

At last, in Chapter 7, the theory of existence for solutions of the semilinear problem is applied to the Bour flow. In [Bou12] a class of operators having the following form, of which (1.3) is a special case, is considered:

$$\begin{aligned}
\partial_t g_{ij}(t) = \nabla^k\nabla^\ell \mathrm{Riem}_{ikj\ell}(g(t)) + a_1\cdot\Delta R(g(t))\cdot g_{ij}(t) + a_2\cdot\nabla^2_{ij}R(g(t))\\
+ (\mathrm{Riem}(g(t)) * \mathrm{Riem}(g(t)))_{ij},
\end{aligned} \tag{1.13}$$

where the operator $*$ indicates any contraction between its operands and a_1 and a_2 are real number. In the context of this thesis we restrict for simplicity to the case $a_1 = a_2$ (for the problem (1.3) it holds $a_1 = a_2 = \frac{1-\lambda}{6}$). Equation (1.13) is however not ready to be plugged directly in (1.7), because covariant derivatives are expressed according to the metric g, which is the unknown of the equation itself. Instead we have to distinguish between a background (and time independent) metric $\hat{g}$ and the actual unknown g, playing respectively the roles of g and u in Chapter 6. Moreover, the classical DeTurck trick must be put into action, in order to compensate for the geometric invariance of the operator in (1.13), which makes the linearized equation not strongly parabolic, and therefore not solvable according to the theory in Chapter 5.

Later on, a number of different possible boundary conditions that one might want to impose are introduced and discussed. Here the most relevant difficulty is to check, for any given choice of boundary conditions, whether or not it satisfies the complementary condition described in Chapter 5. The problem is tackled in two stages: first, a reformulation of the complementary condition is proposed, adapting the corresponding reformulation in [Gia13] to the case of a fourth-order operator whose principal symbol is not necessarily the identity. This still leaves the need to do rather time consuming and error prone computations: a computer program was thus written, using the Python language and the SageMath free software and open source framework, in order to take advantage of the higher speed and reliability machines tend to have over human beings for this kind of jobs. The working of the program is described in Appendix B. The culmination of this thesis is Theorem 7.13, that is summarized here.

1.5 Theorem. *Let (M^4, g) be a smooth compact Riemannian manifold with boundary. Let $\gamma(t)$ be a time-depending Riemannian metric on ∂M and let $\gamma_1(t)$ be a time-depending 2-covariant tensor on ∂M. Suppose that $g_0|_{\partial M} = \gamma(0)$, $\Pi_{g_0}|_{\partial M} = \gamma_1(0)$ and $dR_{g_0}|_{\partial M} = 0$. Then for λ sufficiently close to 1 the system (1.3) has a solution for a certain time $T > 0$, subject to the boundary conditions*

$$\begin{aligned} g^T|_{\partial M}(t) &= \gamma(t) \\ \Pi_g|_{\partial M}(t) &= \gamma_1(t). \end{aligned}$$

The solution g is smooth in $M \times (0, T]$ and converges to g_0 for $t \to 0$ in the $\mathscr{C}^\ell$ sense, up to a diffeomorphism that fixes the boundary ∂M, for $\ell < 4\left(1 - \frac{2}{p}\right)$.

Chapter 2

First properties of a parabolic system of PDEs

2.1 Notations

Let M^n be an n-dimensional smooth manifold with boundary ∂M. In this paper we will always use the word "smooth" to indicate $\mathscr{C}^\infty$ objects. The product manifold $M_T = M \times [0,T]$ will be considered, with the axis $[0,T]$ playing the role of the time (and usually denoted by t). We will use the notation $\partial M_T = \partial M \times [0,T]$ (which is different from the actual manifold boundary of M_T, that also includes $M \times \{0,T\}$). The manifold M will be identified with $M \times \{0\} \subset M_T$.

We denote with TM and T^*M the usual tangent and cotangent bundles of M, and with $T_h^k M$ the tensor product $(TM)^{\otimes h} \otimes (T^*M)^{\otimes k}$. In $T_k M$ we call $S_k M$ the subspace of the tensors that are symmetric in all their indices; in $S_2 M$ we further call $S_2^+ M$ the submanifold of symmetric and positive definite matrices. The same goes of $S^k M$ and $S_+^2 M$ in $T^k M$. Let $\pi\colon M_T \to M$ be the projection on M and $\iota\colon \partial M_T \to M_T$ the boundary embedding. Then, by definition,

$$TM_T = \pi^*(TM) \qquad \text{and} \qquad T\partial M_T = \iota^*(TM_T).$$

Geometrically, this means that tensors in M_T are restricted to their space components (they cannot have time components) and tensors in ∂M_T are permitted to have normal components (but, again, no time components). This is sensible for our problem, because tensors of M and ∂M_T will often arise as traces of tensors in M_T (as initial or boundary conditions of a flow). Occasionally we will also need the actual tangent bundle or symmetric tangent bundle of ∂M_T, and we will denote it as $\tilde{T}\partial M_T$ or $\tilde{S}\partial M_T$. From TM_T and $T\partial M_T$ all the other bundles T^*, T_h^k, S and S^+ can be constructed, also in the tilde variants for ∂M_T.

We will denote with $\mathrm{Map}(V;W)$ and $\mathrm{Lin}(V;W)$ respectively the set of bundle maps and the set of linear bundle homomorphism between bundles V and W, assumed to cover the identity map of the base manifold M; in particular, $\mathrm{Lin}(V;W) = V^* \otimes W$. When V is the base manifold M itself (seen as a trivial bundle over itself), then we have the the tensor field spaces, that we will indicate with the symbols $\mathscr{T}$ and $\mathscr{S}$, depending on whether W is a bundle of type T or S. For example, $\mathscr{T}_k^h(M)$ is the usual space of k-covariant and h-contravariant tensor fields on M, $\mathscr{S}_2^+(M)$ is the space of Riemannian metrics on M and $\mathscr{S}_2^+(M_T)$ is the space of curves of Riemannian metrics between times 0 and T. Both for symbols S and T and for symbols $\mathscr{S}$ and $\mathscr{T}$, when more than one symbol appear in a row, we mean that we are working with their product tensor. For example, $\mathscr{T}_i\mathscr{S}_j$ is the space of $(i+j)$-covariant tensor fields, which are symmetric in the last j on them.

The symbols $\mathscr{T}$ and $\mathscr{S}$ only indicate the domain and the shape of the tensors, not their regularity or integrability. We will indicate regularity and integrability with additional symbols, that will be defined later; for instance, the space of smooth Riemannian metrics will be denoted by $\mathscr{C}^\infty\mathscr{S}_2^+(M)$ and the space of L^p integrable vector fields with $L^p\mathscr{T}^1(M)$. Sometimes no indication of regularity will be given, meaning that it is either irrelevant or otherwise explicited.

On M we denote the space coordinate derivatives with ∂_i, and on M_T we denote the time derivative with ∂_t. If $g \in \mathscr{S}_2^+(M)$ is a Riemannian metric, then

$$\Gamma_{ij}^k := \frac{1}{2} g^{k\ell}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

denotes its Christoffel symbols, ∇ the associated Levi-Civita connection, defined so that for $A \in \mathscr{T}_1^1(M)$

$$\nabla_i A_j^k := \partial_i A_j^k + \Gamma_{i\ell}^k A_j^\ell - \Gamma_{ij}^\ell A_\ell^k.$$

The Laplacian Δ is defined by

$$\Delta X := g^{ij}\nabla_i\nabla_j X$$

and the Riemann tensor is

$$\mathrm{Riem}_{ijk\ell}^g := g_{\ell m} \cdot \left(\nabla_j\nabla_i\partial_k^m - \nabla_i\nabla_j\partial_k^m\right).$$

Tracing the Riemann tensor we obtain the Ricci tensor and the scalar curvature:

$$\mathrm{Ric}_{ik}^g = \mathrm{Riem}_{ijk\ell}^g g^{j\ell},$$
$$R_g = \mathrm{Ric}_{ik}^g g^{ik}.$$

On a Riemannian metric we also consider the distance $\mathrm{dist}(x,y)$ between the points x and y of M, the open and closed balls

$$B(x,r) := \left\{ y \in M \,\middle|\, \mathrm{dist}(x,y) < r \right\}$$
$$\bar{B}(x,r) := \left\{ y \in M \,\middle|\, \mathrm{dist}(x,y) \leqslant r \right\},$$

the volume element dv_g and the volume of a set $X \subseteq M$:

$$\mathrm{Vol}(X,g) := \int_X dv_g.$$

Occasionally we will also use the "musical isomorphisms" for raising and lowering incides:

$$(\eta^\sharp)^i = g^{ij}\eta_j$$
$$(X_\flat)_i = g_{ij}X^j.$$

At the boundary ∂M some additional quantities can be defined: ν is the outward pointing normal vector; the second fundamental form and mean curvature are defined:

$$\Pi_{\alpha\beta} = \langle \nabla_\alpha \nu | \partial_\beta \rangle$$
$$\mathscr{H} = \mathrm{tr}_g \Pi.$$

Since we will deal with different metrics on the same manifold, called for example g, $\bar{g}$, $\hat{g}$, $\ldots$, we distinguish between geometric objects computed from different metrics either specifying the metric as subscript or superscript (like in Riem_g, $\mathrm{Riem}_{\bar{g}}$, $\ldots$) or by adding the same diacritic to the object (like in ∇, $\bar{\nabla}$, $\ldots$).

In this work the letter $C(\ldots)$ will always denote a constant which is only determined by its arguments, but can vary in different appearances of this notation, even in the same formula. Also, the letter $\zeta(T)$ will denote a function $(0,\infty) \to (0,\infty)$ which is increasing and infinitesimal at zero. Again, it can represents different functions at different appearances. The set $\mathbb{N}$ of natural numbers is always assumed to contain the zero. The Einstein conventions of summing over repeated indices is in force.

2.2 The evolution differential equation

We are concerned with studying the parabolic system

$$\partial_t u(t) = P(u(t)),$$

where the derivation ∂_t is made in the time axis and P is a differential operator in $u(t)$ and its (space) derivatives. The tensor u is the unknown: it can either belong to $\mathscr{T}_k(M_T)$ or to $\mathscr{S}_2(M_T)$. In the future we will use the letter $\mathscr{U}$ to indicate either $\mathscr{T}$ or $\mathscr{S}$ (and similarly U for T or S) when there is no need to specify one. The symbol $u(t)$ indicates the evaluation of u at time t and belongs to $\mathscr{U}_k(M)$.

We will consider systems that are semilinear and of order $2b$, i.e. described by the expression:

$$\begin{aligned}\partial_t u_{I_k}(x,t) &= P_{I_k}(u(t))(x) \\ &= A_{I_k}^{J_{2b}K_k}(x,t,u(x,t),\ldots,\nabla^{2b-1}u(x,t)) \cdot \nabla^{2b}_{J_{2b}} u_{K_k}(x,t) \\ &\quad + F_{I_k}(x,t,u(x,t),\ldots,\nabla^{2b-1}u(x,t))\end{aligned} \tag{2.1}$$

for

$$\begin{aligned}A &\in \mathrm{Map}(U_k(M_T),\ldots,T_{2b-1}U_k(M_T); T^{2b}U_k^k(M_T)) \\ F &\in \mathrm{Map}(U_k(M_T),\ldots,T_{2b-1}U_k(M_T); U_k(M_T)).\end{aligned}$$

Thus (2.1) is system whose unknown is a k-covariant tensor; contravariant tensors can be treated as well with the same theory, but we refrain from introducing them in this work to avoid further complexity in the notation. Let us call r the dimension of the fibre space of $U_k(M_T)$ (e.g., if $U_k = T_k$ then $r = n^k$; if $U_k = S_2$ then $r = \frac{n(n+1)}{2}$). Then (2.1) is essentially a system of r PDEs in r unknowns.

The shorthand notations presented in Section 1.3 are in force, so system (2.1) can be rewritten as

$$\partial_t u(x,t) = P(u(t))(x) = A_u(x,t) \cdot \nabla^{2b} u(x,t) + F_u(x,t).$$

This equation will be assumed to be true in the whole domain M_T. Together with (2.1) we impose an initial condition on $M \times \{0\}$ of the type

$$u(x,0) = u_0(x), \tag{2.2}$$

with $u_0 \in \mathscr{U}_k(M)$.

We will also impose a number of boundary conditions on ∂M_T. For $q = 1, \dots, v$ let, on ∂M_T,

$$\begin{aligned}(B^q)_{I_{d_q}}^{J_{m_q} K_k}(x,t,u(x,t),\dots,\nabla^{m_q-1}u(x,t)) \cdot \nabla_{J_{m_q}}^{m_q} u_{K_k}(x,t) \\ = (E^q)_{I_{d_q}}(x,t,u(x,t),\dots,\nabla^{m_q-1}u(x,t)),\end{aligned} \tag{2.3}$$

for

$$B^q \in \mathrm{Map}(U_k(\partial M_T), \dots, T_{m_q-1}U_k(\partial M_T); T^{m_q}U_{d_q}^k(\partial M_T))$$
$$E^q \in \mathrm{Map}(U_k(\partial M_T), \dots, T_{m_q-1}U_k(\partial M_T); U_{d_q}(\partial M_T)).$$

Each boundary condition has order m_q, which must satisfy $m_q \in [0, 2b) \cap \mathbb{N}$, and takes values in a space $U_{d_q}\partial M_T$, which can be chosen between $T_{d_q}\partial M_T$, $S_2\partial M_T$, $\tilde{T}_{d_q}\partial M_T$ and $\tilde{S}_2\partial M_T$ (different conditions can use different spaces). We call s_q the dimension of the fibre space of $U_{d_q}(\partial M_T)$ for the q-th boundary condition (e.g., if $U_{d_q} = T_{d_q}$ then $s_q = n^{d_q}$; if $U_{d_q} = \tilde{T}_{d_q}$, then $s_q = (n-1)^{d_q}$; if $U_{d_q} = S_2$ then $s_q = \frac{n(n+1)}{2}$; if $U_{d_q} = \tilde{S}_2$ then $s_q = \frac{n(n-1)}{2}$). The total dimension of all the boundary conditions is thus $s := \sum_q s_q$. We will see that the “natural” condition on s is that it is equal to br.

As before we benefit from the more compact notation writing

$$B_u^q(x,t) \cdot \nabla^{m_q} u(x,t) = E_u^q(x,t).$$

2.3 Compatibility conditions

The two equations (2.2) and (2.3) overspecify the behaviour of the solution u in the region $\partial M \times \{0\}$. For u to exist, it is necessary that such overspecification does not lead to any conflict. In this section we study the relationship that express the absence of such conflict. For the moment all the computations will be just formal, without caring about the actual regularity of the employed maps, which will be discussed in Sections 5.3 and 5.4.

Let us consider

$$G \in \mathrm{Map}(U_k((\partial)M_T), \dots, T_{m-1}U_k((\partial)M_T); T_G((\partial)M_T)),$$

where G is one of A, F, B^q and E^q, the number m is the associated derivation order ($2b$ for A and F; and m_q for B^q and E^q) and T_G is the target tensor bundle. Then G can be differentiated in the time direction:

$$\partial_t G \in \mathrm{Map}(U_k((\partial)M_T), \dots, T_{m-1}U_k((\partial)M_T); T_G((\partial)M_T)).$$

or in the directions of u, ∇u, ..., $\nabla^{m-1}u$:

$$\begin{aligned}
\partial_u G &\in \mathrm{Map}(U_k((\partial)M_T), \dots, T_{m-1}U_k((\partial)M_T); \mathrm{Lin}(U_k((\partial)M_T), T_G((\partial)M_T))) \\
\partial_{\nabla u} G &\in \mathrm{Map}(U_k((\partial)M_T), \dots, T_{m-1}U_k((\partial)M_T); \mathrm{Lin}(T_1U_k((\partial)M_T), T_G((\partial)M_T))) \\
&\vdots \\
\partial_{\nabla^{m-1}u} G &\in \mathrm{Map}(U_k((\partial)M_T), \dots, T_{m-1}U_k((\partial)M_T); \mathrm{Lin}(T_{m-1}U_k((\partial)M_T), T_G((\partial)M_T))).
\end{aligned}$$

Let $u \in \mathscr{U}_k(M_T)$, not necessarily a solution of the system under study. We have already agreed to call

$$G_u(x,t) := G(x,t,u(x,t),\dots,\nabla^{m-1}u(x,t)).$$

We can write a chain rule of the form:

$$\partial_t G_u(x,t) = (\partial_t G)_u(x,t) + \left[(\partial_u G)_u(x,t)\right](\partial_t u(x,t)) + \cdots + \left[(\partial_{\nabla^{m-1}u} G)_u(x,t)\right](\nabla^{m-1}\partial_t u(x,t)).$$

Since g does not depend on the time, the derivatives ∂_t and ∇ commute, and we can repeatedly use the chain rule in order to prove a Faà di Bruno-like formula, which is for example discussed in [CS96, Theorem 2.1] in the case of $\mathbb{R}^n$. We do not really need to work out the exact coefficients; the important point is that $\partial_t^\eta G_u(x,t)$ can be expressed as:

$$\partial_t^\eta G_u(x,t) = H(\nabla^\rho \partial_t^{\eta'} u(x,t), (\partial \ldots \partial G)_u(x,t)), \tag{2.4}$$

where H is a polynomial evaluated in $\nabla^\rho \partial_t^{\eta'} u(x,t)$ (with $\rho \leqslant m$ and $\eta' \leqslant \eta$) and $(\partial \ldots \partial G)_u(x,t)$ (with no more than η partial derivatives, each of which can be in direction t, u, …, $\nabla^{m-1}u$).

At the initial boundary $\partial M \times \{0\}$ the behaviour of u is specified by both initial and boundary conditions, therefore appropriate compatibility conditions must be satisfied between the two constraints. The amount of conditions that one has to impose depends on the regularity of the sought solution u; at the very basic level, if u, …, $\nabla^{2b-1}u$ are continuous up to the boundary, we can evaluate (2.3) at time zero and, substituting with (2.2), we have:

$$B^q_{u_0}(x,0) \cdot \nabla^{m_q} u_0(x) = E^q_{u_0}(x,0).$$

If we want some more regularity on u, for instance the continuity of $\partial_t u|_{t=0}$ and its space derivatives, then higher order conditions are necessary: from (2.1) we know that

$$\partial_t u(x,0) = A_u(x,0) \cdot \nabla^{2b} u_0(x) + F_u(x,0),$$

so that, differentiating (2.3) with respect to time and commuting derivatives,

$$\partial_t B^q_u(x,0) \cdot \nabla^{m_q} u_0(x) + B^q_{u_0}(x,0) \cdot \nabla^{m_q} \partial_t u(x,0) = \partial_t E^q_u(x,0).$$

The term $\partial_t B^q_u(x,0)$ can be reworked with (2.4), so that it only depends on the partial derivatives of B at time zero and on $u(x,0) = u_0(x)$ and $\partial_t u(x,0)$, which we have just derived. Thus we have obtained another condition that must be satisfied by the objects u_0, A, F, B^q and E^q to hope to have a solution with sufficient regularity at $\partial M \times \{0\}$.

The procedure above can be repeated as long as the solution u is regular enough at $\partial M \times \{0\}$. More precisely, given u_0, A and F we define a sequence of functions w_k defined on ∂M given by the recursive formulae:

$$\begin{aligned} w_0(x) &= u_0(x) \\ w_{k+1}(x) &= \sum_{i=0}^{k} \binom{k}{i} \cdot \partial_t^i A_w(x) \cdot \nabla^{2b} w_{k-i}(x) + \partial_t^k F_w(x), \end{aligned} \tag{2.5}$$

where $\partial_t^k G_w(x)$ is defined as the polynomial H in (2.4) with $t = 0$ and $\nabla^\rho w_{\eta'}(x)$ instead of $\nabla^\rho \partial_t^{\eta'} u(x,0)$.

2.1 Definition ([Sol65, §14, pag. 98])**.** For $\omega \in \mathbb{R}$, we say that the *compatibility conditions* of order ω are satisfied if, for all $x \in \partial M \times \{0\}$ and for all q and $k \in [0, \frac{\omega - m_q}{2b} + 1] \cap \mathbb{N}$ the following equation is satisfied:

$$\sum_{i=0}^{k} \binom{k}{i} \cdot \partial_t^i B^q_w(x) \cdot \nabla^{m_q} w_{k-i}(x) = \partial_t^k E^q_w(x). \tag{2.6}$$

2.2 Remark. Notice that, according to our definition, the order of compatibility can be negative.

2.3 Remark. At this stage the definitions of the functions w_k and of the compatibility conditions are purely formal. In due time we will detail how many of them actually exist and which is their regularity, depending on the space in which a solution of the equation is looked for.

Chapter 3

Geometry on manifolds with boundary

3.1 Manifolds with convex boundary

Let (M^n, g) be a complete Riemannian manifold with boundary. It is well known that many classical results in Riemannian geometry fail when considered on manifolds with boundary. Many of such failures are ultimately due to the behaviour discrepancy of geodesics close to the boundary. Let us distinguish the two concepts of *analytic geodesic* (i.e., a curve satisfying the geodesic equation) and of *geometric geodesic* (i.e., a curve which is locally a minimizer of the energy functional). Although by Gauss' lemma an analytic geodesic is always a geometric geodesic, the converse is not true in general.

3.1 Example. Let $M = \mathbb{R}^2 \setminus B(0,1)$ and g the restriction of the standard Euclidean metric. Then the curve $\gamma\colon [0,\pi] \to M$ defined by $\gamma(t) = (\cos t, \sin t)$ is clearly not an analytic geodesic. However, it can be seen (for instance integrating in polar coordinates) that it is a curve of minimal length in M between its two endpoints.

Most results can however be retained if the boundary ∂M is convex, i.e., if the second fundamental form Π, taken with respect to the outward normal vector, is positively semidefinite. This, in particular, includes the case of a totally geodesic boundary.

3.2 Proposition. *Let (M^n, g) be a Riemannian manifold with boundary satisfying $\Pi \geqslant 0$ at all points of the boundary. Then for every $x \in \partial M$ and $y \in \mathring{M}$ there is a minimal geodesic joining x and y and it is completely contained in $\mathring{M}$, except for x. Also, its tangent vector at x does not belong to $T_x\partial M$.*

Proof. This is a corollary of [Bis75]; see also [Cap13]. Bishop's theorem implies, in particular, that any geodesic leaving from x tangentially to the boundary (which does not even need to exist!) stays in the boundary: this fact is equivalent to the thesis. □

3.3 Corollary. *Let (M^n, g) be a complete Riemannian manifold with boundary satisfying $\Pi \geqslant 0$ at all points of the boundary. Then all geometric geodesics are analytical geodesics.*

Proof. Let $\gamma\colon [0,1] \to M$ be a geometric geodesic in M. We need to show that it satisfies the geodesics equation:

$$\nabla_{\dot\gamma}\dot\gamma^k = \ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0.$$

We can use the first variation formula for fixed endpoints variations (see for instance [GHL04, Theorem 3.31]). Let $V(s)$ be a variation field along γ with $V(0) = 0$ and $V(1) = 0$ and let $\gamma_t(s)$ be a variation such that $\gamma_0(s) = \gamma(s)$ and $V(s) = \frac{\partial \gamma_t(s)}{\partial t}$. Then the energy E satisfies the formula

$$0 \leqslant \frac{d}{dt} E(\gamma_t) = -\int_0^1 g_{ij}(\gamma(s)) \cdot V^i(s) \cdot \nabla_{\dot\gamma}\dot\gamma^j(s)\, ds.$$

If at least one of the endpoints of γ is in $\mathring{M}$, then the whole geodesic is in $\mathring{M}$ by the previous Proposition. Then the usual argument can be employed: if $\nabla_{\dot\gamma}\dot\gamma \neq 0$ at some point s, an absurd can be obtained by crafting a variation field $V(s)$ pointing in direction $\nabla_{\dot\gamma}\dot\gamma(s)$ near $\gamma(s)$.

The only missing case is when the geodesic γ belongs to ∂M: with the method above we can only infer that $\nabla_{\dot\gamma}\dot\gamma$ points out of the manifold, because for V pointing out of the manifold we cannot build a corresponding variation γ_t. However, since the manifold ∂M has no boundary and γ is also a geometric geodesic of ∂M, we deduce that γ is an analytical geodesic of ∂M. Taking coordinates adapted to ∂M, this implies that γ solves the equation

$$\nabla_{\dot\gamma}\dot\gamma^\delta = \ddot\gamma^\delta + \Gamma^\delta_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta = 0$$

for δ taking values on the tangential directions. Since $\dot\gamma^\nu = 0$ and $\ddot\gamma^\nu = 0$, this is already the geodesic equation in M, except that we need to check that $\nabla_{\dot\gamma}\dot\gamma^\nu = \Gamma^\nu_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta = 0$. The fact that $\Gamma^\nu_{\alpha\beta} \geqslant 0$ has already been shown before using the variational argument in M; the other inequality follows from the hypothesis on the second fundamental form:

$$0 \leqslant \Pi_{\alpha\beta} = g_{\beta i}\Gamma^i_{\alpha\nu} = -\frac{1}{2}\partial_\nu g_{\alpha\beta} = -\Gamma^\nu_{\alpha\beta}.$$ □

A corollary of this result is that most usual properties can be recovered for the exponential map, which is defined as usual as the map $\operatorname{Dom}\exp_x \to M$ sending $v \in \operatorname{Dom}\exp_x \subseteq T_xM$ to the point $\gamma_v(1)$, where γ_v is the geodesic such that $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. The domain $\operatorname{Dom}\exp_x$ does not need to be the whole T_xM, even if M is complete, because the geodesic γ_v can cease to exist because it has hit the boundary ∂M. However on its domain the exponential map is well defined, because of the uniqueness of solutions of the ODE describing analytical geodesics (and we have already ruled out the existence of geometric geodesics which are not analytical). Another consequence is that the exponential map is a surjection

$$\exp_x\colon \operatorname{Dom}\exp_x \cap B(0,r) \twoheadrightarrow B(x,r)$$

The injectivity radius $\operatorname{inj}_g(x)$ can also be defined as the supremum of all r for which the above surjection is actually a diffeomorphism, and it is positive and lower semicontinuous as in the boundaryless case. In particular, it is bounded below on compact manifolds.

With these tools in hand, most of the classical theory of Riemannian geometry can be recovered. We will need, in particular, the Bishop-Gromov theorem for volume comparison.

3.4 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary satisfying* $\operatorname{Ric} \geqslant (n-1)K$ *on M for some $K \in \mathbb{R}$ and $\Pi \geqslant 0$ on ∂M. Let us call* $\operatorname{Vol}(B_r, g_K)$ *the volume of a ball of radius r in the space form of curvature K. Then for any $x \in M$ and $r \in (0,\infty)$ the function*

$$r \longmapsto \frac{\operatorname{Vol}(B(x,r), g)}{\operatorname{Vol}(B_r, g_K)}$$

is nonincreasing. Its limit for $r \to 0$ is 1 if $x \in \mathring{M}$ and $\frac{1}{2}$ if $x \in \partial M$.

Proof. The proof in [Pet06, Chapter 9, Section 1] or [GHL04, Section 3.H.5] are based on integration of Jacobi fields along radial geodesics and can be repeated without modifications thanks to Corollary 3.3 and the subsequent discussion. □

3.5 Corollary. *Let (M^n, g) be a complete Riemannian manifold with boundary satisfying* $\mathrm{Ric} \geqslant (n-1)K$ *on M for some $K \in \mathbb{R}$ and $\Pi \geqslant 0$ on ∂M. Let us call*

$$s(x, r, g) = \int_{S(x,r)} d\sigma_g$$

the area of the sphere of radius r centered at x and $s(r, g_K)$ the area of the sphere of radius r in the space form of curvature K. Then for any $x \in M$ and $r \in (0, \infty)$ it holds

$$s(x, r, g) \leqslant s(r, g_K).$$

Proof. It is easy to show, using the coarea formula (see [Cha93, Exercise III.12, (d)]), that

$$s(x, r, g) = \frac{d}{dr} \mathrm{Vol}(B(x, r), g)$$
$$s(r, g_K) = \frac{d}{dr} \mathrm{Vol}(B_r, g_K).$$

Then Proposition 3.4 implies that

$$0 \geqslant \frac{d}{dr}\left(\frac{\mathrm{Vol}(B(x,r), g)}{\mathrm{Vol}(B_r, g_K)}\right) = \frac{s(x,r,g) \cdot \mathrm{Vol}(B_r, g_K) - s(r, g_K) \cdot \mathrm{Vol}(B(x,r), g)}{(\mathrm{Vol}(B_r, g_K))^2},$$

so

$$s(x, r, g) \leqslant s(r, g_K) \cdot \frac{\mathrm{Vol}(B(x,r), g)}{\mathrm{Vol}(B_r, g_K)} \leqslant s(r, g_K).$$

□

3.2 Bounded geometry

The uniformity of many estimates that we will use in this work depends on the regularity that can be assumed on the base manifold and metric. In this section we define how to measure such regularity and present a way to construct atlases with appropriate geometric and analytical properties. See [Sch01] for a general discussion on different equivalent definitions of bounded geometry, which are however a bit different from the one we use here.

3.6 Definition. We say that a complete Riemannian manifold with boundary (M^n, g) has *geometry bounded* by $C \in (0, \infty)$ up to order $k \in \mathbb{N}$ if:

- it has totally geodesic boundary;
- it admits a collar of width at least $\frac{1}{C}$, i.e., the distance function

  $$d_{\partial M}(x) := \mathrm{dist}(x, \partial M) = \inf_{y \in \partial M} \mathrm{dist}(x, y)$$

  has no critical points when it is smaller than $\frac{1}{C}$; this implies that $d_{\partial M}^{-1}([0, \frac{1}{C}))$ is diffeomorphic to $\partial M \times [0, \frac{1}{C})$;
- the injectivity radius satisfies $\mathrm{inj}_g(x) \geqslant \frac{1}{C}$ for every $x \in M$;
- for all $0 \leqslant i \leqslant k$ and $x \in M$ it holds

  $$|\nabla^i \mathrm{Riem}(x)| \leqslant C.$$

We denote with $\|g\|_k$ the minimum value of C for which the above properties hold.

Let us define:

$$\mathbb{R}^n_+ := \left\{ (x^1, \dots, x^n) \in \mathbb{R}^n \,\middle|\, x^n \geqslant 0 \right\}.$$

We denote with $\mathbb{1}_A$ the indicator function of the set A.

3.7 Proposition. *For any $n \in \mathbb{N} \cap [2, \infty)$ and $L \in (0, \infty)$ there are constants $\tilde{\lambda}(n, L)$ and $N_0(n, L)$ such that every Riemannian manifold with boundary (M^n, g) having geometry bounded by L up to order $k \in \mathbb{N}$ admits, for any $\lambda \in (0, \tilde{\lambda})$, two atlases $(U_i, \Omega_i, \varphi_i)$ and $(\hat{U}_i, \hat{\Omega}_i, \hat{\varphi}_i)$ for $i \in I = I^\partial \sqcup I^\circ$ with the following properties.*

- *The two systems of sets U_i and $\hat{U}_i$ are open coverings of M, such that $U_i \subset \hat{U}_i$ for any $i \in I$ and no more than N_0 sets $\hat{U}_i$ intersect at any given point of M.*

- *Any geodesic ball of radius λ is entirely contained in at least one of the sets U_i and any set $\hat{U}_i$ is contained in a ball of radius 20λ.*

- *There are points $x_i \in M$ such that $U_i = B(x_i, R)$ and $\hat{U}_i = B(x_i, 2R)$, where $R = 3\lambda$ if $i \in I^\circ$ and $R = 9\lambda$ if $i \in I^\partial$. All sets U_i and $\hat{U}_i$ are geodesically convex.*

- *It holds $\Omega_i = B(0, R)$ and $\hat{\Omega}_i = B(0, 2R)$, where the balls are taken in $\mathbb{R}^n$ (if $i \in I^\circ$) or $\mathbb{R}^n_+$ (if $i \in I^\partial$).*

- *The maps $\varphi_i \colon \Omega_i \to U_i$ and $\hat{\varphi}_i \colon \hat{\Omega}_i \to \hat{U}_i$ are the normal charts centered at x_i of the appropriate radius. In particular, $\hat{\varphi}_i|_{\Omega_i} = \varphi_i$.*

- *For each $i \in I$ there is a smooth function $\Psi_i \colon M \to \mathbb{R}$ such that $\mathbb{1}_{U_i} \leqslant \Psi_i \leqslant \mathbb{1}_{\hat{U}_i}$ and such that for each $j \leqslant k$ it holds*
$$\sup_M |\nabla^j \Psi_i| \leqslant \frac{C(n,k,L)}{\lambda^j}.$$

- *For each $i \in I$ and $j \leqslant k$ it holds*
$$\begin{aligned} &\frac{1}{2} g_{\mathbb{R}^n} \leqslant \varphi_i^* g \leqslant 2 g_{\mathbb{R}^n} \\ &|\partial^j g| \leqslant C(n,k,L). \end{aligned} \tag{3.1}$$

  Equation (3.1) also implies that
$$\frac{1}{2^{n/2}}\, dx \leqslant (\varphi_i^{-1})_\# \, dv_g \leqslant 2^{n/2}\, dx. \tag{3.2}$$

- *For each $i \in I^\circ$ the ball $\hat{U}_i$ is compactly contained in $\mathring{M}$.*

- *For each $i \in I^\partial$ it holds $x_i \in \partial M$. Also, let $U_i^\partial := U_i \cap \partial M$ and $\Omega_i^\partial := \Omega_i \cap \{x_n = 0\}$. Then φ_i restricts to a diffeomorphism $\varphi_i^\partial \colon \Omega_i^\partial \to U_i^\partial$, which is again the normal chart with respect to the metric g restricted to ∂M. The same happens for $\hat{\varphi}_i^\partial \colon \hat{\Omega}_i^\partial \to \hat{U}_i^\partial$. The sets U_i^∂ and $\hat{U}_i^\partial$ are the balls of radius R and $2R$ centered at x_i, according to the metric induced on ∂M.*

Proof. For a fixed λ, consider a maximal collection $(x_i)_{i\in I^\partial}$ such that $x_i \in \partial M$ and all the balls $B(x_i,\lambda)$ are disjoint and another maximal collection $(x_i)_{i\in I^\circ}$ such that $\operatorname{dist}(x_i,\partial M) \geqslant 6\lambda$ and, as before, all the balls $B(x_i,\lambda)$ are disjoint. We will now show that they satisfy all the requirements of the theorem, as long as λ is sufficiently small once n and L are given. As a first requirement, we request that 20λ is smaller than the convexity radius of M, so that all the balls mentioned in this proof are embedded and convex. The convexity radius can be estimated in terms of the injectivity radius and the curvature of the manifolds (thus in terms of $\|g\|_0$), as for example it is done in [CE75, Theorem 5.14].

Let $x \in M$: if $\operatorname{dist}(x,\partial M) \geqslant 6\lambda$, then x is distant at most 2λ from an x_i for $i \in I^\circ$ (if not, then the maximality of the collection above is contradicted). Therefore $B(x,\lambda) \subseteq B(x_i,3\lambda) = U_i$. If $\operatorname{dist}(x,\partial M) < 6\lambda$, then x is distant at most 8λ from an x_i for $i \in I^\partial$ (because any point in ∂M is distant at most 2λ from one x_i, as before). Therefore $B(x,\lambda) \subseteq B(x_i,9\lambda) = U_i$.

Once the open charts U_i and $\hat{U}_i$ are established, there is no difficulty in defining the diffeomorphisms φ_i and $\hat{\varphi}_i$; even at the boundary, since we assumed the boundary itself is totally geodesic, the exponential map is nicely defined on a half ball. Then the analysis in [Ham95, Section 4] can be carried out and, particularly by Theorem 4.9, for sufficiently small $\tilde{\lambda}$ the metric g can be bounded above and below by the Euclidean metric in every geodesic chart. Higher derivatives of the metric can be bound using [Eic91, Theorem A and Proposition 2.3].

The uniform local finiteness can be proved along the same lines of [Shu92, Lemma 1.2]: take, for example, x_i for $i \in I^\circ$ and consider all the $j \in I^\circ$ such that $\hat{U}_i \cap \hat{U}_j \neq \emptyset$, i.e., such that $\operatorname{dist}(x_i,x_j) < 12\lambda$. Now, all sets $B(x_j,\lambda)$ are disjoint and wholly contained in $B(x_i,13\lambda)$, so there can be at most $\frac{A}{B}$ of them, where A is the maximum possible volume of a ball of radius 13λ and B is the minimum possible volume of a ball of radius λ. Since we have already proved that g is bounded between two multiples of the identity matrix, such ratio is itself bounded. Analogous reasoning works for i or j (or both) in I^∂.

At last we have to provide the functions Ψ_i. The construction is very classical: let $\hat{\Psi}\colon [0,\infty) \to [0,1]$ be a "master" smooth bump function, such that $\mathbb{1}_{[0,\frac{1}{2}]} \leqslant \hat{\Psi} \leqslant \mathbb{1}_{[0,1]}$. Then define

$$\Psi_i(x) := \hat{\Psi}\left(\frac{d(x_i,x)}{2R}\right).$$

The required estimates are satisfied by virtue of [Eic91, Theorem A]. □

3.8 Definition. An atlas satisfying the thesis of Proposition 3.7 will be called a *bounded atlas*.

3.9 Example. If $M = \mathbb{R}^n$, then the trivial atlas consisting of a single chart of infinite radius around the origin together with its identity function is a bounded atlas. The same can be said for $M = \mathbb{R}^n_+$.

These atlases and bump functions can be extended on M_T by mean of its product structure. With a small abuse of notation, we will not use new symbols for the same objects defined on the whole M_T. Thus we will use the symbol U_i to indicate also $U_i \times [0,T]$, the symbol Ω_i to indicate also $\Omega_i \times [0,T]$, the symbol φ_i to indicate also $\varphi_t \otimes \mathrm{Id}$ and the symbol Ψ_i to indicate also $\Psi_i \otimes 1$. The same convention will be used for $\hat{U}_i$, $\hat{\Omega}_i$ and $\hat{\varphi}_i$ and for the ∂ variants. The properties outlined above for M remain valid for M_T and the extended objects.

Let $u \in \mathcal{T}^h_k$ on any of M, M_T or ∂M_T: we want to define u_{φ_i} to be its "localization" on the chart $U_i^{(\partial)}$. For a function this is easy: if $u \in \mathscr{C}^\infty$, then $u_{\varphi_i} \in \mathscr{C}^\infty(\Omega_i^{(\partial)})$ is defined by

$$u_{\varphi_i}(x) := \varphi_i^* u(x) = u(\varphi_i(x)).$$

By extending the differential $D\varphi_i \colon \Omega_i \times \mathbb{R}^n \to TU_i$ to tensors of all covariance and contravariance orders, we have pullback operator

$$\begin{aligned} \varphi_i^* \colon \mathscr{T}_k^h(M) &\longrightarrow \mathscr{T}_k^h(\mathbb{R}^n) \\ u &\longmapsto u_{\varphi_i}. \end{aligned}$$

The same construction works also on M_T and ∂M_T, being careful again about the fact that tensors have no time components.

3.3 Sobolev and interpolation inequalities

The usual Sobolev and Gagliardo-Nirenberg inequalities can be recovered on a manifold with totally geodesic boundary; their constants can be controlled in terms of the boundedness of the geometry. Let us give the formulation we will use, together with links to the proofs.

3.10 Proposition. *Let (M^n, g) be a compact manifold with boundary and bounded geometry and consider $u \in \mathscr{T}_d(M)$. Take $p, q \in [1,\infty)$ and $m, k \in [0,\infty) \cap \mathbb{N}$ such that $m < k$ and*

$$\frac{1}{p} = \frac{1}{q} - \frac{k-m}{n}.$$

Then

$$\left(\int_M |\nabla^m u|^p \, dv_g\right)^{\frac{1}{p}} \leqslant C(n,d,k,m,p,q,\|g\|_k) \cdot \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^q \, dv_g\right)^{\frac{1}{q}}.$$

Proof. See the proof of [Heb96, Theorem 3.5], where it can be assumed that the atlas in use is a bounded atlas. The argument never depends on the manifold being closed. □

From the Sobolev embedding, the Gagliardo-Nirenberg interpolation inequality can be proven.

3.11 Proposition. *Let (M^n, g) be a compact manifold with boundary and bounded geometry and consider $u \in \mathscr{T}_d(M)$. Take $p, q, r \in [1,\infty)$ and $k, m \in [0,\infty) \cap \mathbb{N}$ such that $m \leqslant k$. Suppose there exists $\alpha \in [\frac{m}{k}, 1]$ such that*

$$\frac{1}{p} = \frac{m}{n} + \alpha\left(\frac{1}{q} - \frac{k}{n}\right) + \frac{1-\alpha}{r}.$$

Then

$$\left(\int_M |\nabla^m u|^p \, dv_g\right)^{\frac{1}{p}} \leqslant C(n,d,k,m,p,q,r,\|g\|_k) \cdot \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^q \, dv_g\right)^{\frac{\alpha}{q}} \cdot \left(\int_M |u|^r \, dv_g\right)^{\frac{1-\alpha}{r}}.$$

Proof. The proof proceeds as in [Man02, Proposition 6.5], substituting inequality (6.4) there with Proposition 3.10. □

Chapter 4

Parabolic function spaces

4.1 Tools for function spaces

Parabolic partial differential equations are characterized by the fact that, in a sense, a time derivative is equivalent to $2b$ space derivatives. This peculiarity requires that we define anisotropic function spaces, for which an order of regularity in the time direction counts "as much" as $2b$ orders of regularity in the space directions. In this chapter we introduce parabolic Hölder and Sobolev-Slobodeckij spaces, borrowing most of the underlying theory from [Ama09]. However, Amann's tratise only covers spaces on Euclidean spaces, so we must undertake the burden of defining equivalent spaces on manifolds with boundary and showing how the Euclidean theory can be brought to manifold by using appropriate localization arguments. In this chapter we will sometimes call u a function, but the same definitions can given for an arbitrary tensor field.

Let $p \in [0,\infty]$. The usual $L^p(M_T)$ space is defined as the set of measurable functions with finite norm

$$\|u\|_{p,M_T} := \left(\iint_{M_T} |u(x,t)|^p \, dv_g \, dt\right)^{\frac{1}{p}} \qquad 1 \leqslant p < \infty$$
$$\|u\|_{\infty,M_T} := \operatorname*{ess\,sup}_{M_T} |u|.$$

The definitions of $\|\cdot\|_p$ and $\|\cdot\|_\infty$ apply with obvious changes also to M and ∂M_T.

When working with local charts it will be useful to have, for each space we will use, many (nearly) equivalent norms: one adapted to the intrinsic geometry of M and not depending on any atlas. The others depend on the chosen atlas and essentially assume that the manifold M locally has the flat metric induced by the charts: they constitute a bridge thanks to which global results for Euclidean spaces $\mathbb{R}^n$ contained in [Ama09] can be applied to functions spaces on manifolds.

Suppose that we have defined a norm $[\cdot]_{p,M_T}$ on any manifold M, having the characteristic exponent $p \in [1,\infty]$. The chart-induced norms are defined in terms of $[\cdot]_{p,\mathbb{R}^n_T}$, in a way which we now detail. In the first chart-induced norm we simply pull back the tensor on the model Euclidean space, which is clearly considered with its standard flat metric:

$$[u]^{\mathrm{Euc},1}_{p,M_T} := \left(\sum_{i\in I} [u_{\varphi_i}]^p_{p,\Omega_{i,T}}\right)^{\frac{1}{p}}. \tag{4.1}$$

When $p = \infty$ we assume that $\left(\sum_i [u_{\varphi_i}]^{\infty}_{\infty,M_T}\right)^{\frac{1}{\infty}} := \max_i [u_{\varphi_i}]_{\infty,M_T}$.

In order to use results from [Ama09] we need to extend functions from $\Omega_{i,T}$ to $\mathbb{R}^n$ or a corner of $\mathbb{R}^n$ (where a *corner* is a subset of $\mathbb{R}^n$ defined by constraints of the type $x^i \geqslant 0$ for some directions i, like in [Ama09, Section 4.3]). This is easier in the space direction, because we can use the bump functions Ψ_i and define:

$$[u]^{\text{Euc},2}_{p,M_T} := \left(\sum_{i\in I^\circ} [(\Psi_i \cdot u)_{\hat\varphi_i}]^p_{p,\mathbb{R}^n_T} + \sum_{i\in I^\partial} [(\Psi_i \cdot u)_{\hat\varphi_i}]^p_{p,\mathbb{R}^n_{+,T}}\right)^{\frac{1}{p}}. \tag{4.2}$$

The function $(\Psi_i \cdot u)_{\hat\varphi_i}$ is naturally defined on $\Omega_{i,T}$, but thanks to the decay introduced by Ψ_i it can be zero extended to $\mathbb{R}^n_T$ or $\mathbb{R}^n_{+,T}$.

This definition is not yet satisfying, because $\mathbb{R}^n_T$ and $\mathbb{R}^n_{+,T}$ are not corners of $\mathbb{R}^n$. Fixing these details is not a mere exercise in style, since in our arguments we will consider spaces with T or λ going to zero, conditions under which Sobolev spaces progressively degenerate (the trivial example of a nonzero constant function is useful: its integral or Sobolev norm go to zero while the domains shrinks, while its supremum or Hölder norm remains constant; thus we would prove the false if we did not take into account the collapsing of the domain).

In order to write an extension from $\mathbb{R}^n_T$ to $\mathbb{R}^n \times [0,\infty)$ we cannot use a partition of the unity trick, but we can define an extension operator $\mathscr{E}$ in this way: let u be defined on $\mathbb{R}^n_T = \mathbb{R}^n \times [0,T]$. For a fixed smooth bump function $\Xi\colon \mathbb{R}\to\mathbb{R}$ such that $\chi_{[\varepsilon,\infty]} \leqslant \Xi \leqslant \chi_{[1-\varepsilon,\infty]}$, the function $u_\Xi(x,t) := \Xi((t+T)/T)\cdot u(x,t+T)$ can be zero extended to $\mathbb{R}^n\times(-\infty,0]$, and then we can use the extension operator e^- defined in [Ama09, Sections 4.1 and 4.2] to obtain a function defined over $\mathbb{R}^n\times\mathbb{R}$. Finally we consider the function $\mathscr{E}u$ with domain $\mathbb{R}^n_\infty$ defined by:

$$(x,t) \longmapsto \begin{cases} u(x,t) & t<T \\ e^- u_\Xi(x,t-T) & t>T-\varepsilon. \end{cases}$$

The definition is well posed, because the two alternatives coincide on the slice $\mathbb{R}^n\times(T-\varepsilon,T)$, where it holds Ξ takes the value 1. It will be evident, once Hölder and Sobolev-Slobodeckij spaces are introduced, that the operator $\mathscr{E}$ preserves the regularity of the function u, except that it might introduce a factor related to $\frac{1}{T}$ into the norm due to the multiplication by Ξ. Considering the example above, this is expected. Also, in the same way we can also define an extension operator, called again $\mathscr{E}$, from $\mathbb{R}^n_+$ to $\mathbb{R}^n_{+,\infty} = \mathbb{R}^n_+ \times [0,\infty)$.

We can finally define:

$$[u]^{\text{Euc},3}_{p,M_T} := \left(\sum_{i\in I^\circ} [\mathscr{E}(\Psi_i \cdot u)_{\hat\varphi_i}]^p_{p,\mathbb{R}^n_\infty} + \sum_{i\in I^\partial} [\mathscr{E}(\Psi_i \cdot u)_{\hat\varphi_i}]^p_{p,\mathbb{R}^n_{+,\infty}}\right)^{\frac{1}{p}}, \tag{4.3}$$

for which the norms appearing in the right hand side are finally computed on a corner of $\mathbb{R}^n$.

Unfortunately $[\cdot]^{\text{Euc},3}_{p,M_T}$ is still not completely satisfying, because of the introduction of the factor $\frac{1}{T}$ mentioned above. We have already shown that this in general cannot be avoided, but there are special cases for which the function $u(x,t+T)$ already extends to $\mathbb{R}^n\times(0,\infty]$ without the need of the mollifier Ξ: this will happen when u is taken from the spaces of functions with zero initial values that will be introduced later. For these cases we define another extension operator $\hat{\mathscr{E}}$, that maps a function u to $\hat{\mathscr{E}}u$ defined by:

$$(x,t) \longmapsto \begin{cases} u(x,t) & t<T \\ e^- u(x,t-T) & t>T-\varepsilon. \end{cases}$$

With that we can define:

$$[u]^{\mathrm{Euc},4}_{p,M_T} := \left(\sum_{i\in I^\circ} [\hat{\mathscr{E}}(\Psi_i \cdot u)_{\hat{\varphi}_i}]^p_{p,\mathbb{R}^n_\infty} + \sum_{i\in I^\partial} [\hat{\mathscr{E}}(\Psi_i \cdot u)_{\hat{\varphi}_i}]^p_{p,\mathbb{R}^n_{+,\infty}} \right)^{\frac{1}{p}}, \tag{4.4}$$

This fourth norm is not well defined in general, because it requires the function u to have zero initial value. When this does not happen, we can assume that the norm takes the value ∞.

When the norm $[\cdot]_{p,M}$ is defined on M, then we can define norms $[\cdot]^{\mathrm{Euc},1}_{p,M}$ and $[\cdot]^{\mathrm{Euc},2}_{p,M}$, simply removing the subscript T everywhere. When the norm $[\cdot]_{p,\partial M_T}$ is defined on ∂M_T, then i only runs over I^∂, $\partial\varphi_i$ is used instead of φ_i, $\partial\Omega_i$ is used instead of Ω_i, $\partial\hat{\Omega}_i$ is used instead of $\hat{\Omega}_i$ and $\mathbb{R}^{n-1}$ is used instead of $\mathbb{R}^n_+$. In this case we can define all the norm $[\cdot]^{\mathrm{Euc},1}_{p,\partial M_T}$ through $[\cdot]^{\mathrm{Euc},4}_{p,\partial M_T}$.

Let us immediately apply these definitions to the norms $\|\cdot\|_{p,M_T}$ and show that they are equivalent. In general we will consider only covariant tensors for ease of notation, but everything works in the same way for contravariant or mixed tensors.

4.1 Lemma. *Let (M^n, g) be a complete manifold with bounded geometry and consider a bounded atlas for g. Let $u \in \mathscr{T}_d(M_T)$. Then*

$$\frac{1}{C(n,d)} \cdot |u_{\varphi_i}|_{g_{\mathbb{R}^n}} \leqslant |u|_g \leqslant C(n,d) \cdot |u_{\varphi_i}|_{g_{\mathbb{R}^n}}.$$

If $x = \varphi_i(\tilde{x})$ and $y = \varphi_i(\tilde{y})$ are points of M belonging to the same chart U_i, then

$$\frac{1}{C} \cdot |\tilde{x} - \tilde{y}| \leqslant \operatorname{dist}(x,y)_g \leqslant C \cdot |\tilde{x} - \tilde{y}|.$$

Proof. Both inequalities follow from (3.1), via inversion of the metric (in the first case) or integration along paths (in the second case). □

4.2 Lemma. *Let (M^n, g) be a complete manifold with bounded geometry and consider a bounded atlas for g. If $f\colon M \to [0,\infty)$, then*

$$\sup_M f = \max_{i\in I} \sup_{\Omega_i} f_{\varphi_i} \tag{4.5}$$

$$\int_M f\, dv_g \leqslant C(n) \cdot \sum_{i\in I} \int_{\Omega_i} f_{\varphi_i}\, dx \tag{4.6}$$

$$\sum_{i\in I} \int_{\Omega_i} f_{\varphi_i}\, dx \leqslant C(n, \|g\|_0) \cdot \int_M f\, dv_g. \tag{4.7}$$

Proof. The first equality is obvious. By definition and using (3.2):

$$\int_M f\, dv_g \leqslant \sum_{i\in I} \int_{U_i} f\, dv_g = \sum_{i\in I} \int_{\Omega_i} f_{\varphi_i} (\varphi_i^{-1})_\# dv_g \leqslant C(n) \cdot \sum_{i\in I} \int_{\Omega_i} f_{\varphi_i}\, dx.$$

On the other hand:

$$\sum_{i\in I} \int_{\Omega_i} f_{\varphi_i}\, dx \leqslant C(n) \cdot \sum_{i\in I} \int_{\Omega_i} f_{\varphi_i} (\varphi_i^{-1})_\# dv_g = C(n) \cdot \int_M \left(\sum_{i\in I} \mathbb{1}_{U_i} \right) f\, dv_g \leqslant C(n) \cdot N_0 \cdot \int_M f\, dv_g$$

and N_0 can be estimated with n and $\|g\|_0$ by Proposition 3.7. □

4.3 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry and consider a bounded atlas for g. Let $u \in \mathcal{T}_d(M_T)$ and $p \in [1, \infty]$. Then*

$$\|u\|^{\mathrm{Euc},1}_{p,M_T} \leqslant \|u\|^{\mathrm{Euc},2}_{p,M_T} \leqslant \|u\|^{\mathrm{Euc},3}_{p,M_T} \leqslant \|u\|^{\mathrm{Euc},4}_{p,M_T}$$
$$\|u\|_{p,M_T} \leqslant C(n,d,p,\|g\|_0) \cdot \|u\|^{\mathrm{Euc},1}_{p,M_T}$$
$$\|u\|^{\mathrm{Euc},4}_{p,M_T} \leqslant C(n,d,p,\|g\|_0) \cdot \|u\|_{p,M_T}.$$

Proof. The claim follows from Lemmata 4.1 and 4.2 and from [Ama09, Lemma 4.1.2]. □

The analogous theorems for spaces over M and ∂M_T are immediate.

***

Let us now introduce the derivatives. Appropriate generalizations of Hölder and Sobolev spaces will be defined, which are suitable for solutions of parabolic problems of order $2b$. For both Hölder-type and Sobolev-type spaces three definitions will be given: one for M_T, one for M and one for ∂M_T.

Let us define the set $\Delta^M_\lambda \subseteq M \times M$ as

$$\Delta^M_\lambda = \left\{ (x,y) \in M \times M \,\middle|\, \mathrm{dist}(x,y) < \lambda \right\}.$$

Most of the spaces we will define will depend on the parameter $\lambda \in (0, \infty]$, which represents the space scale of the fractional derivative. The case $\lambda = \infty$ is perfectly legal: then $\Delta^M_\lambda = M \times M$.

Tensors computed at different points are compared by mean of the parallel transport associated to the background metric: if $T_{x,y}$ is the parallel trasport isomorphism on (M,g) along the unique minimal geodesic connecting x to y, then we define the *transported norm* as

$$\mathrm{tnorm}(u(x), u(y)) := |u(x) - T_{y,x} u(y)|_{g(x)} = |T_{x,y} u(x) - u(y)|_{g(y)}.$$

This definition is always well posed when x and y are sufficiently close to each other (for example, when their distance is smaller than the injectivity radius of the metric g); in particular, it follows from the discussion in Section 3.1 that a unique geodesic always exists between two sufficiently close points. This care will not be necessary when comparing tensors at point which are different only in the time direction, because neither the manifold structure nor the background metric depend on time.

4.4 Lemma. *Let (M^n, g) be a complete manifold with bounded geometry. Let $u, v \in \mathcal{T}_{d_1}(M)$ and $w \in \mathcal{T}_{d_2}(M)$, and $x, y, z \in M$ sufficiently close to each other. The trasported norm has the following properties:*

$$\mathrm{tnorm}(u(x), u(y)) \geqslant 0$$
$$\mathrm{tnorm}(u(x), u(y)) = \mathrm{tnorm}(u(y), u(x))$$
$$\mathrm{tnorm}(u(x), u(z)) \leqslant \mathrm{tnorm}(u(x), u(y)) + \mathrm{tnorm}(u(y), u(z))$$
$$\mathrm{tnorm}(u(x), u(y)) \leqslant |u(x)| + |u(y)|$$
$$\mathrm{tnorm}((u+v)(x), (u+v)(y)) \leqslant \mathrm{tnorm}(u(x), u(y)) + \mathrm{tnorm}(v(x), v(y))$$
$$\mathrm{tnorm}((u \otimes w)(x), (u \otimes w)(y)) \leqslant |u(x)| \cdot \mathrm{tnorm}(w(x), w(y)) + |w(y)| \cdot \mathrm{tnorm}(u(x), u(y)).$$

Also, it holds $\mathrm{tnorm}(u(x), u(y)) = 0$ *if and only if $u(y)$ is the parallel tranport of $u(x)$ along the segmente between x and y.*

Proof. All results follow trivially from the definition. □

In order to use the transported norm in actual proof, we have to see how it can be handled when working with local coordinates. Let $v \in \mathcal{T}_d(M)$ and suppose $x = \varphi_i(\tilde{x})$ and $y = \varphi_i(\tilde{y})$ are two points belonging to the same chart: thanks to Lemma 4.1,

$$\begin{aligned} \operatorname{tnorm}(v(x), v(y)) &= |v(x) - T_{y,x} v(y)|_{g(x)} \\ &\leqslant C(i) \cdot |\varphi_i^*(v(x)) - \varphi_i^*(T_{y,x} v(y))|_{g_{\mathbb{R}^n}} \\ &\leqslant C(i) \cdot \left(|v_{\varphi_i}(\tilde{x}) - v_{\varphi_i}(\tilde{y})|_{g_{\mathbb{R}^n}} + |\varphi_i^*(v(y)) - \varphi_i^*(T_{y,x} v(y))|_{g_{\mathbb{R}^n}}\right). \end{aligned} \tag{4.8}$$

The highest order term here is $|v_{\varphi_i}(\tilde{x}) - v_{\varphi_i}(\tilde{y})|_{g_{\mathbb{R}^n}}$, which coincides with the transported norm defined according to the flat metric of $\mathbb{R}^n$; to control the second addend we will use this lemma.

4.5 Lemma. *Let (M^n, g) be a complete manifold with bounded geometry and consider $x, y \in M$ belonging to the same chart U_i of a bounded atlas and $v \in \mathcal{T}_d(M)$. Then*

$$|\varphi_i^*(v(y)) - \varphi_i^*(T_{y,x} v(y))| \leqslant C(n, d, \|g\|_1) \cdot |v(y)| \cdot \operatorname{dist}(x, y).$$

Proof. For simplicity let us assume that $d = 1$, i.e. v is a covector field and fix $D = \operatorname{dist}(x, y)$. For higher order tensors we just need to add a dependency of the constants on d. Let γ be a segment with unit speed such that $\gamma(0) = y$ and $\gamma(D) = x$. Let $\tilde{v}$ be the field along γ given by the parallel transport equation:

$$\begin{aligned} &\partial_t \tilde{v}^i(t) = \Gamma^i_{jk}(\gamma(t)) \cdot \dot{\gamma}^j(t) \cdot \tilde{v}^k(t) \qquad t \in [0, D] \\ &\tilde{v}(0) = v(y). \end{aligned}$$

Then $T_{y,x} v(y) = \tilde{v}(D)$ and we have to estimate $|\varphi_i^*(\tilde{v}(0)) - \varphi_i^*(\tilde{v}(D))|$.

First we show, using a Grönwall-styled inequality, that $\tilde{v}$ remains controlled during the parallel transport. Differentiating the logarithm of the norm of $\tilde{v}$ we have that

$$\partial_t \log|\tilde{v}(t)| = \frac{\langle \partial_t \tilde{v}(t) | \tilde{v}(t) \rangle}{|\tilde{v}(t)|^2} \leqslant |\Gamma(\gamma(t))|.$$

So, for every $t \in (0, D)$

$$|\tilde{v}(D)| \leqslant |\tilde{v}(0)| \cdot \exp\left(D \cdot \max_t |\Gamma(\gamma(t))|\right) \leqslant |\tilde{v}(0)| \cdot C(n, \|g\|_1),$$

where $D \leqslant 20\lambda$ and λ is in turn controlled once $\|g\|_1$ and n are known.

We can now do the "linear" estimate:

$$|\tilde{v}(D) - \tilde{v}(0)| \leqslant \int_0^D |\Gamma(\gamma(s))| \cdot |\tilde{v}(s)| \cdot |\dot{\gamma}(s)|\, ds \leqslant C(n, \|g\|_1) \cdot |\tilde{v}(0)| \cdot D. \qquad \square$$

As a corollary, we can quickly recover the definition of the covariant derivative as the limit of the difference quotient, provided that the transported norm is used in the numerator (also, we assume to already know that the covariant derivative is a tensor).

4.6 Corollary. *Let (M^n, g) be a complete manifold with bounded geometry, $u \in \mathcal{T}_d(M)$ be a smooth tensor and $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be a smooth curve. Then*

$$\nabla_{\dot{\gamma}(0)} u(\gamma(0)) = \partial_t \left[T_{\gamma(t),\gamma(0)} u(\gamma(t))\right]_{t=0} = \lim_{t \to 0} \frac{T_{\gamma(t),\gamma(0)} u(\gamma(t)) - u(\gamma(0))}{t}.$$

Proof. Since we assume to already know that both sides of the thesis are tensors, we can use normal coordinates around $\gamma(0)$. So the covariant derivative at $\gamma(0)$ coincides with the coordinate derivative; we thus need to show that

$$\partial_{\dot\gamma(0)} u(\gamma(0)) = \lim_{t\to 0}\left(\frac{u(\gamma(t)) - u(\gamma(0))}{t} + \frac{u(\gamma(t)) - T_{\gamma(t),\gamma(0)} u(\gamma(t))}{t}\right).$$

Since the limit of the first addend is the coordinate derivative, we have reduced ourselves to prove that the second addend is infinitesimal: following the proof of Lemma 4.5 we can see that, for sufficiently small $|t|$,

$$\frac{|u(\gamma(t)) - T_{\gamma(t),\gamma(0)} u(\gamma(t))|}{t} \leqslant C(n,d)\cdot(|\Gamma(\gamma(0))| + \delta) = C(n,d)\cdot\delta,$$

given that we are in normal coordinates. So we are done. □

4.7 Corollary. *Let (M^n, g) be a complete manifold with bounded geometry, $u \in \mathcal{T}_d(M)$ be a smooth tensor and $\gamma\colon [0,D] \to M$ a minimal geodesic with endpoints $x = \gamma(0)$ and $y = \gamma(D)$ (in particular,* $\mathrm{dist}(x,y) = D$*). Then*

$$\mathrm{tnorm}(u(x), u(y)) \leqslant \int_0^D |\nabla u(\gamma(t))|\,dt.$$

Proof. Let us call

$$\tilde u(t) := T_{\gamma(t),x} u(\gamma(t)).$$

Then Corollary 4.6 implies that

$$\partial_t \tilde u(t) = T_{\gamma(t),x} \nabla_{\dot\gamma(t)} u(\gamma(t)).$$

Thus

$$\begin{aligned}
\mathrm{tnorm}(u(x), u(y)) &= |\tilde u(0) - \tilde u(D)|_{g(x)} \\
&\leqslant \int_0^D |T_{\gamma(t),x}\nabla_{\dot\gamma(t)} u(\gamma(t))|_{g(x)}\,dt \\
&= \int_0^D |\nabla_{\dot\gamma(t)} u(\gamma(t))|_{g(\gamma(t))}\,dt \\
&\leqslant \int_0^D |\nabla u(\gamma(t))|_{g(\gamma(t))}\,dt.
\end{aligned}$$ □

4.8 Lemma. *Let (M^n, g) be a complete manifold with bounded geometry. Let $u \in \mathcal{T}_d(M)$, $\ell \in \mathbb{N}$ and $x \in M$ belonging to the chart U_i. Then $\nabla^\ell u(x)$ can be expressed as a linear combination of terms of the form*

$$\partial^j u_{\varphi_i}(x) * \partial^{j_1}\Gamma(x) * \cdots * \partial^{j_k}\Gamma(x), \tag{4.9}$$

where $j + j_1 + \cdots + j_k + k = \ell$. The intervening coefficients are determined by d and ℓ.

Proof. The thesis is obvious when $\ell = 0$, so we continue by induction. Computing $\nabla v = \partial v + \Gamma * v$ for v taking the form (4.9), we see that new terms of the same form are generated:

- if the derivative ∂ is discharged on the u factor, then j is raised by one;
- if the derivative ∂ is discharged on any of the Γ factors, then the corrisponding j_i is raised by one;

- if Γ is multiplied, then k is raised by one and a new j_k is introduced, with value 0.

In each of these cases the sum is kept invariant. □

4.9 Remark. In equation (4.9) Γ is never differentiated more than $\ell-1$ times, because the mere fact that a factor with Γ appears means that $k \geqslant 1$. In particular, the factors containing Γ are bounded by $\|g\|_\ell$.

Lemma 4.8 can also be formulated to represent coordinate derivatives in terms of covariant derivatives.

4.10 Lemma. *Let (M^n, g) be a complete manifold with bounded geometry. Let $u \in \mathscr{T}_d(M)$, $\ell \in \mathbb{N}$ and $x \in M$ belonging to the chart U_i. Then $\partial^\ell u_{\varphi_i}(x)$ can be expressed as a linear combination of terms of the form*

$$\nabla^j u_\varphi(x) * \partial^{j_1}\Gamma(x) * \cdots * \partial^{j_k}\Gamma(x),$$

where $j + j_1 + \cdots + j_k + k = \ell$. The intervening coefficients are determined by d and ℓ.

Proof. The proof is the same as Lemma 4.8, except that the identity $\partial v = \nabla v + \Gamma * v$ is iteratively used. □

The following extension of Lemma 4.2 will be useful.

4.11 Lemma. *Let (M^n, g) be a complete manifold with bounded geometry and consider a bounded atlas for g. If $f\colon M \times M \to [0,\infty)$, then*

$$\sup_{(x,y)\in\Delta_\lambda^M} f(x,y) \leqslant \max_{i\in I} \sup_{(x,y)\in\Omega_i\times\Omega_i} f_{\varphi_i}(x,y)$$

$$\max_{i\in I} \sup_{(x,y)\in\Omega_i\times\Omega_i} f_{\varphi_i}(x,y) \leqslant \sup_{(x,y)\in\Delta_{10\lambda}^M} f(x,y)$$

$$\iint_{\Delta_\lambda^M} f(x,y)\, dv_g(x)\, dv_g(y) \leqslant C(n) \cdot \sum_{i\in I} \iint_{\Omega_i\times\Omega_i} f_{\varphi_i}(x,y)\, dx\, dy$$

$$\sum_{i\in I} \iint_{\Omega_i\times\Omega_i} f_{\varphi_i}(x,y)\, dx\, dy \leqslant C(n, \|g\|_0) \cdot \iint_{\Delta_{20\lambda}^M} f(x,y)\, dv_g(x)\, dv_g(y).$$

Proof. The first two inequalities are obvious. Let us introduce the auxiliary open sets $\tilde{U}_i := U_i \setminus \bar{B}(\partial U_i, \lambda)$. Since, by the definition of bounded atlas, each ball of radius λ is wholly contained in at least one of the balls U_i, the balls $\tilde{U}_i$ are a cover of M. Therefore we can estimate:

$$\begin{aligned}
\iint_{\Delta_\lambda^M} f(x,t)\, dv_g(x)\, dv_g(y) &= \int_M \left(\int_{B(x,\lambda)} f(x,y)\, dv_g(y) \right) dv_g(x) \\
&\leqslant \sum_{i\in I} \int_{\tilde{U}_i} \left(\int_{B(x,\lambda)} f(x,y)\, dv_g(y) \right) dv_g(x) \\
&\leqslant \sum_{i\in I} \int_{\tilde{U}_i} \left(\int_{U_i} f(x,y)\, dv_g(y) \right) dv_g(x) \\
&\leqslant \sum_{i\in I} \int_{U_i} \left(\int_{U_i} f(x,y)\, dv_g(y) \right) dv_g(x) \\
&\leqslant \sum_{i\in I} C(n) \cdot \iint_{\Omega_i\times\Omega_i} f_{\varphi_i}(x,y)\, dx\, dy.
\end{aligned}$$

On the other side:

$$\begin{aligned}\sum_{i\in I}\iint_{\Omega_i\times\Omega_i} f_{\varphi_i}(x,y)\,dx\,dy &\leqslant C(n)\cdot\sum_{i\in I}\int_{U_i}\left(\int_{U_i} f(x,y)\,dv_g(y)\right)dv_g(x)\\ &\leqslant C(n)\cdot\sum_{i\in I}\int_{U_i}\left(\int_{B(x,20\lambda)} f(x,y)\,dv_g(y)\right)dv_g(x)\\ &\leqslant C(n)\cdot N_0\cdot\int_M\left(\int_{B(x,20\lambda)} f(x,y)\,dv_g(y)\right)dv_g(x)\\ &\leqslant C(n)\cdot N_0\cdot\iint_{\Delta^M_{20\lambda}} f(x,y)\,dv_g(x)\,dv_g(y).\end{aligned}$$ □

4.2 Parabolic Hölder spaces

Let us call $[x]\in\mathbb{N}$ the integral part of $x\in[0,\infty)$, i.e., the greatest integer not larger than x. For $\ell\in[0,\infty)$ we will call $\bar\ell=[\ell]$ its integral part and $\hat\ell=\ell-\bar\ell\in[0,1)$ its fractional part, so that $\ell=\bar\ell+\hat\ell$. While the definitions we are about to give are meaningful for $\ell\in(0,\infty)$, most properties we will prove require that ℓ is not an integer. The parabolic Hölder space $\mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ is defined as the set of functions on M_T such that the following norm is finite:

$$\|u\|^{(\lambda)}_{\mathscr{C},\ell,M_T} := \sum_{2b\eta+\rho=\bar\ell}\ \sup_{\substack{(x,y)\in\Delta^M_\lambda\\ t\in[0,T]}} \frac{\operatorname{tnorm}(\partial_t^\eta\nabla^\rho u(x,t),\partial_t^\eta\nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat\ell}} \tag{4.10}$$

$$+\sum_{0<\ell-2b\eta-\rho<2b}\ \sup_{\substack{x\in M\\ t,t'\in[0,T]}} \frac{|\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,t')|}{|t-t'|^{\frac{\ell-2b\eta-\rho}{2b}}} \tag{4.11}$$

$$+\sum_{2b\eta+\rho\leqslant\bar\ell}\ \sup_{M_T}|\partial_t^\eta\nabla^\rho u|. \tag{4.12}$$

The Hölder space $\mathscr{C}^{\ell}(M)$ is defined as the set of functions on M such that the following norm is finite:

$$\|u\|^{(\lambda)}_{\mathscr{C},\ell,M} := \sup_{(x,y)\in\Delta^M_\lambda} \frac{\operatorname{tnorm}(\nabla^{\bar\ell}u(x),\nabla^{\bar\ell}u(y))}{\operatorname{dist}(x,y)^{\hat\ell}} + \sum_{\rho\leqslant\bar\ell}\sup_M|\nabla^\rho u|.$$

The Hölder space $\mathscr{C}^{\ell,\frac{\ell}{2b}}(\partial M_T)$ is defined as the set of functions on ∂M_T such that the following norm is finite:

$$\begin{aligned}\|u\|^{(\lambda)}_{\mathscr{C},\ell,\partial M_T} := &\sum_{2b\eta+\rho=\bar\ell}\ \sup_{\substack{(x,y)\in\Delta^{\partial M}_\lambda\\ t\in[0,T]}} \frac{\operatorname{tnorm}(\partial_t^\eta\nabla^\rho u(x,t),\partial_t^\eta\nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat\ell}}\\ &+\sum_{0<\ell-2b\eta-\rho<2b}\ \sup_{\substack{x\in\partial M\\ t,t'\in[0,T]}} \frac{|\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,t')|}{|t-t'|^{\frac{\ell-2b\eta-\rho}{2b}}}\\ &+\sum_{2b\eta+\rho\leqslant\bar\ell}\ \sup_{\partial M_T}|\partial_t^\eta\nabla^\rho u|.\end{aligned}$$

In all the previous cases we mandated $\ell<\infty$; we can also define the spaces of smooth functions $\mathscr{C}^\infty(M_T)$, $\mathscr{C}^\infty(M)$ and $\mathscr{C}^\infty(\partial M_T)$, in the usual way; it is known this in this case we expect a Fréchet space and not a Banach space, so we cannot put a norm on it. We still define

$$\|u\|_{\mathscr{C},\infty,M_T} := \sum_{k=0}^{\infty} 2^{-k} \frac{\|u\|_{\mathscr{C},k,M_T}}{1+\|u\|_{\mathscr{C},k,M_T}},$$

which is not a norm, but is able to control the norm of all spaces $\mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ up to a constant depending on ℓ.

Norm $\|u\|^{\mathrm{Euc},1}_{\mathscr{C},\ell,(\partial)M_{(T)}}$ to $\|u\|^{\mathrm{Euc},3}_{\mathscr{C},\ell,(\partial)M_{(T)}}$ can be defined (for $\ell < \infty$) with equations (4.1), (4.2) and (4.3), using $p = \infty$, for spaces M, M_T and ∂M_T. Let us see that all Hölder norms defined on M_T up to now are equivalent.

4.12 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$, $\lambda,\mu \in (0,\infty]$ with $\lambda < \mu$ and $u \in \mathscr{T}_d(M_T)$. Then*

$$\|u\|^{(\lambda)}_{\mathscr{C},\ell,M_T} \leqslant \|u\|^{(\mu)}_{\mathscr{C},\ell,M_T} \qquad\qquad \|u\|^{(\mu)}_{\mathscr{C},\ell,M_T} \leqslant (1+2\lambda^{-\hat{\ell}})\cdot\|u\|^{(\lambda)}_{\mathscr{C},\ell,M_T}.$$

Proof. Both claims follow trivially from the definition. □

When the ratio between μ and λ is bounded, we can actually avoid the dependency on λ.

4.13 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$, $\lambda \in (0,\infty]$ and $u \in \mathscr{T}_d(M_T)$. Then*

$$\|u\|^{(2\lambda)}_{\mathscr{C},\ell,M_T} \leqslant C\cdot\|u\|^{(\lambda)}_{\mathscr{C},\ell,M_T}.$$

Proof. Let $x, y \in M$ such that $\lambda < \mathrm{dist}(x,y) \leqslant 2\lambda$. We can find z such that $\mathrm{dist}(x,z) \leqslant \lambda$ and $\mathrm{dist}(z,y) \leqslant \lambda$ and observe that

$$\begin{aligned}\frac{\mathrm{tnorm}(\partial_t^\eta\nabla^\rho u(x),\partial_t^\eta\nabla^\rho u(y))}{\mathrm{dist}(x,y)^{\hat{\ell}}} &\leqslant \frac{\mathrm{tnorm}(\partial_t^\eta\nabla^\rho u(x),\partial_t^\eta\nabla^\rho u(z))}{\mathrm{dist}(x,y)^{\hat{\ell}}} + \frac{\mathrm{tnorm}(\partial_t^\eta\nabla^\rho u(z),\partial_t^\eta\nabla^\rho u(y))}{\mathrm{dist}(x,y)^{\hat{\ell}}} \\ &\leqslant \frac{\mathrm{tnorm}(\partial_t^\eta\nabla^\rho u(x),\partial_t^\eta\nabla^\rho u(z))}{\mathrm{dist}(x,z)^{\hat{\ell}}} + \frac{\mathrm{tnorm}(\partial_t^\eta\nabla^\rho u(z),\partial_t^\eta\nabla^\rho u(y))}{\mathrm{dist}(z,y)^{\hat{\ell}}}.\end{aligned} \tag{4.13}$$

Taking the suprema we have the thesis. □

In light of this result it will be often convenient to avoid making λ explicit. It will be usually clear from the context or irrelevant. Before proving the equivalence of norms $\|u\|^{\mathrm{Euc},\cdot}_{\mathscr{C},\ell,M_T}$ let us introduce a couple of useful results: first, that parabolic Hölder spaces grow when the regularity ℓ decreases; second, that a Leibniz-like formula is valid.

4.14 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell, m \in (0,\infty)$ with $m < \ell$. If $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$, then $u \in \mathscr{C}^{m,\frac{m}{2b}}(M_T)$ and*

$$\|u\|_{\mathscr{C},m,M_T} \leqslant C(m,\ell)\cdot\|u\|_{\mathscr{C},\ell,M_T}.$$

Proof. It is obvious that terms in (4.12) decrease when ℓ decreases. We thus only consider terms with fractional space derivatives (of the form (4.10)), since the reasoning for terms with fractional time derivatives (of the form (4.11)) is analogous.

We also have to consider different cases depending on ℓ and m. The easiest case is when m is integral, because in that case the fractional space derivative is simply disappearing.

If $m \in (\bar{\ell}, \ell)$, then $\bar{m} = \bar{\ell}$ and $\hat{\ell} - \hat{m} = \ell - m$. This implies that the terms appearing in (4.10) are the same, except that they have $\hat{m}$ instead of $\hat{\ell}$ at the denominator. If $\operatorname{dist}(x,y) \leqslant 1$, then simply

$$\frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat{m}}} \leqslant \frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat{\ell}}},$$

while if $\operatorname{dist}(x,y) > 1$, then the denominator can be dropped and the numerator is estimated with the integer derivatives in the term (4.12):

$$\frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat{m}}} \leqslant 2 \sup_{(x,t)\in M_T} |\partial_t^\eta \nabla^\rho u(x,t)|.$$

If $\ell \in \mathbb{N}$ and $m \in (\ell - 1, \ell)$, then, from Corollary 4.7,

$$\begin{aligned}\frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat{m}}} &\leqslant \sup_{(x,t)\in M_T} |\partial_t^\eta \nabla^{\rho+1} u(x,t)| \cdot \operatorname{dist}(x,y)^{1-\hat{m}} \\ &\leqslant \sup_{(x,t)\in M_T} |\partial_t^\eta \nabla^{\rho+1} u(x,t)|.\end{aligned}$$

Again, if $\operatorname{dist}(x,y) > 1$ a direct comparison with (4.12) can be done.

All the other cases for ℓ and m can be obtained by composing the ones above. □

4.15 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_{d_1}(M_T)$ and $v \in \mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_{d_2}(M_T)$ with $d = \max\{d_1, d_2\}$. Then the product $u \otimes v \in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ and*

$$\begin{aligned}\|u\otimes v\|_{\mathscr{C},\ell,M_T} &\leqslant C(n,d,\ell) \cdot \sum_{j=0}^{\bar{\ell}} \Big[\|u\|_{\mathscr{C},j+\hat{\ell},M_T} \cdot \|v\|_{\mathscr{C},\bar{\ell}-j,M_T} + \|u\|_{\mathscr{C},j,M_T} \cdot \|v\|_{\mathscr{C},\bar{\ell}-j+\hat{\ell},M_T}\Big] \\ &\leqslant C(n,d,\ell) \cdot \|u\|_{\mathscr{C},\ell,M_T} \cdot \|v\|_{\mathscr{C},\ell,M_T}.\end{aligned} \tag{4.14}$$

Proof. The second inequality is a corollary of Proposition 4.14, so we turn to the first one, which is an application of the standard idea of splitting the difference quotient of a product in the difference quotients of the two factors. We show how to estimate terms of the form (4.10): by Leibniz' rule and Lemma 4.4 we have that

$$\begin{aligned}&\sum_{2b\eta+\rho=\bar{\ell}} \frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho (u\otimes v)(x,t), \partial_t^\eta \nabla^\rho (u\otimes v)(y,t))}{\operatorname{dist}(x,y)^{\hat{\ell}}} \\ &\leqslant C(n,d,\ell)\cdot \sum \frac{\operatorname{tnorm}(\partial_t^{\eta_1}\nabla^{\rho_1} u(x,t)\otimes \partial_t^{\eta_2}\nabla^{\rho_2} v(x,t), \partial_t^{\eta_1}\nabla^{\rho_1} u(y,t)\otimes \partial_t^{\eta_2}\nabla^{\rho_2} v(y,t))}{\operatorname{dist}(x,y)^{\hat{\ell}}} \\ &\leqslant C(n,d,\ell)\cdot \sum |\partial_t^{\eta_1}\nabla^{\rho_1} u(x,t)| \cdot \frac{\operatorname{tnorm}(\partial_t^{\eta_2}\nabla^{\rho_2} v(x,t), \partial_t^{\eta_2}\nabla^{\rho_2} v(y,t))}{\operatorname{dist}(x,y)^{\hat{\ell}}} \\ &\quad + C(n,d,\ell)\cdot \sum |\partial_t^{\eta_2}\nabla^{\rho_2} v(x,t)| \cdot \frac{\operatorname{tnorm}(\partial_t^{\eta_1}\nabla^{\rho_1} u(x,t), \partial_t^{\eta_1}\nabla^{\rho_1} u(y,t))}{\operatorname{dist}(x,y)^{\hat{\ell}}},\end{aligned}$$

where summations are made on all $(\eta_1, \eta_2, \rho_1, \rho_2)$ such that $2b(\eta_1+\eta_2)+\rho_1+\rho_2 = \bar{\ell}$. Each of the addends in the right hand side is represented in (4.14). We skip the discussion of terms of the form (4.11) or (4.12), which is analogous. □

We can finally prove the most important equivalence theorem.

4.16 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry and $u \in \mathcal{T}_d(M_T)$, and consider a bounded atlas of size λ, for a fixed λ. Then*

$$\|u\|^{\mathrm{Euc},1}_{\mathscr{C},\ell,M_T} \leqslant \|u\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T} \leqslant \|u\|^{\mathrm{Euc},3}_{\mathscr{C},\ell,M_T} \tag{4.15}$$

$$\|u\|_{\mathscr{C},\ell,M_T} \leqslant C(n,d,\ell,\|g\|_{\bar{\ell}+1}) \cdot \|u\|^{\mathrm{Euc},1}_{\mathscr{C},\ell,M_T} \tag{4.16}$$

$$\|u\|^{\mathrm{Euc},1}_{\mathscr{C},\ell,M_T} \leqslant C(n,d,\ell,\|g\|_{\bar{\ell}+1}) \cdot \|u\|_{\mathscr{C},\ell,M_T} \tag{4.17}$$

$$\|u\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T} \leqslant C(n,d,\ell,\min\{1,\lambda\},\|g\|_{\bar{\ell}+1}) \cdot \|u\|_{\mathscr{C},\ell,M_T} \tag{4.18}$$

$$\|u\|^{\mathrm{Euc},3}_{\mathscr{C},\ell,M_T} \leqslant C(n,d,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_{\bar{\ell}+1}) \cdot \|u\|_{\mathscr{C},\ell,M_T}. \tag{4.19}$$

Proof. Identity (4.15) is trivial. In order to prove (4.16) we need to consider independently the three addends (4.10), (4.11) and (4.12) and provide an inequality with the corresponding quantities computed according to the flat metric, using (4.5) from Lemma 4.2 and the corresponding inequalities from Lemma 4.11; let us begin by taking η and ρ such that $2b\eta + \rho \leqslant \bar{\ell}$ and estimating (4.12). Thanks to Lemma 4.8, we have that, for $x \in U_i$,

$$|\partial_t^\eta \nabla^\rho u(x)| \leqslant C(n,d,\bar{\ell},\|g\|_{\bar{\ell}}) \cdot \sum_{\rho' \leqslant \rho} |\partial_t^\eta \partial^{\rho'} u_{\varphi_i}(x)|.$$

The quantity on the right hand side appears as an addend of $\|u\|^{\mathrm{Euc},1}_{\mathscr{C},\ell,M_T}$, so we are done with this one.

Let us now take x and y not more than λ far away and consider the addedend (4.10); by the definition of bounded atlas there is a U_i that contains both. Writing as for (4.8):

$$\frac{\mathrm{tnorm}(\nabla^{\bar{\ell}} u(x), \nabla^{\bar{\ell}} u(y))}{\mathrm{dist}(x,y)^{\hat{\ell}}} \leqslant C(d) \cdot \left(\frac{|\nabla^{\bar{\ell}} u(x) - \nabla^{\bar{\ell}} u(y)|}{\mathrm{dist}(x,y)^{\hat{\ell}}} + \frac{|\nabla^{\bar{\ell}} u(y) - T_{y,x} \nabla^{\bar{\ell}} u(y)|}{\mathrm{dist}(x,y)^{\hat{\ell}}} \right). \tag{4.20}$$

In the first addend of (4.20) we expand both terms in the absolute value with Lemma 4.8. Let us separate the one term with $j = \bar{\ell}$ from all the others:

$$\nabla^{\bar{\ell}} u(x) - \nabla^{\bar{\ell}} u(y) = \partial^{\bar{\ell}} u_{\varphi_i}(x) - \partial^{\bar{\ell}} u_{\varphi_i}(y) + \textsc{lots} \tag{4.21}$$

where LOTs is the sum, for $j < \bar{\ell}$ and $j + j_1 + \cdots + j_k + k = \bar{\ell}$ of terms having the form:

$$\begin{aligned}
&\partial^j u_{\varphi_i}(x) * \partial^{j_1}\Gamma(x) * \cdots * \partial^{j_k}\Gamma(x) - \partial^j u_{\varphi_i}(y) * \partial^{j_1}\Gamma(y) * \cdots * \partial^{j_k}\Gamma(y) \\
&\quad = (\partial^j u_{\varphi_i}(x) - \partial^j u_{\varphi_i}(y)) * \partial^{j_1}\Gamma(x) * \cdots * \partial^{j_k}\Gamma(x) \qquad (4.22) \\
&\qquad + \partial^j u_{\varphi_i}(y) * (\partial^{j_1}\Gamma(x) - \partial^{j_1}\Gamma(y)) * \cdots * \partial^{j_k}\Gamma(x) \qquad (4.23) \\
&\qquad + \dots \\
&\qquad + \partial^j u_{\varphi_i}(y) * \partial^{j_1}\Gamma(y) * \cdots * (\partial^{j_k}\Gamma(x) - \partial^{j_k}\Gamma(y)).
\end{aligned}$$

Again we have to split the discussion in two cases, one for each possible form of the terms in LOTs. Recalling Remark 4.9, terms of the form (4.22) satisfy

$$\left|(\partial^j u_{\varphi_i}(x) - \partial^j u_{\varphi_i}(y)) * \partial^{j_1}\Gamma(x) * \cdots * \partial^{j_k}\Gamma(x)\right| \leqslant C(d,n,\|g\|_{\bar{\ell}}) \cdot \sup_{\Omega_i} |\partial^{j+1} u_{\varphi_i}| \cdot |\varphi_i^{-1}(x) - \varphi_i^{-1}(y)|,$$

while terms of the form (4.23) satisfy

$$\left|\partial^j u_{\varphi_i}(y) * (\partial^{j_1}\Gamma(x) - \partial^{j_1}\Gamma(y)) * \cdots * \partial^{j_k}\Gamma(x)\right| \leqslant C(d,n,\|g\|_{\bar{\ell}+1}) \cdot \sup_{\Omega_i} |\partial^j u_{\varphi_i}| \cdot |\varphi_i^{-1}(x) - \varphi_i^{-1}(y)|.$$

It follows that

$$\frac{|\nabla^{\bar\ell}u(x)-\nabla^{\bar\ell}u(y)|}{\operatorname{dist}(x,y)^{\hat\ell}} \leqslant C(n,d,\|g\|_0)\cdot\frac{|\partial^{\bar\ell}u_{\varphi_i}(x)-\partial^{\bar\ell}u_{\varphi_i}(y)|}{|\varphi_i^{-1}(x)-\varphi_i^{-1}(y)|^{\hat\ell}} + C(n,d,\bar\ell,\|g\|_{\bar\ell+1})\cdot\lambda^{1-\hat\ell}\cdot\sum_{j=0}^{\bar\ell}\sup_{\Omega_i}|\partial^j u_{\varphi_i}|,$$

which accounts for the first addend in (4.20). For the second one we use Lemma 4.5:

$$\frac{|\nabla^{\bar\ell}u(y,t)-T_{y,x}\nabla^{\bar\ell}u(y,t)|}{\operatorname{dist}(x,y)^{\hat\ell}} \leqslant C(n,d,\bar\ell,\|g\|_1)\cdot\lambda^{1-\hat\ell}\cdot|\nabla^{\bar\ell}u(y,t)|,$$

which can then be estimated as above. Up to now we have assumed that $\eta=0$, but, since time derivatives commute with space derivatives, all the proofs can be repeated with the full formulae: we have thus shown that terms of the form (4.10) are bounded by a suitable multiple of $\|u\|^{\mathrm{Euc},1}_{\mathscr{C},\ell,M_T}$. At last we need to estimate terms of the form (4.11): as before the idea is to apply Lemma 4.8 to rewrite covariant derivatives in terms of coordinate derivatives, retain the higher order term and control the others by mean of $\sup|\partial_t^\eta\nabla^\rho u|$. There is nothing new here (actually, it is easier, because the parallel transport term does not appear), so we omit the details again. This concludes the proof of (4.16).

The proof of inequality (4.17) is similar, but Lemma 4.10 is used instead of Lemma 4.8. Since no new ideas are introduced here, we skip the details.

We are finally on inequalities (4.18) and (4.19). Their proof is similar to (4.17), but the presence of the decay and extension operators must be addressed and introduces the dependence on λ and T when they are small. Looking at operands in (4.2) and using Proposition 4.15 we obtain that

$$\|(\Psi_i\cdot u)_{\hat\varphi_i}\|_{\mathscr{C},\ell,\mathbb{R}^n_T} \leqslant C(n,d,\ell)\cdot\|(\Psi_i)_{\hat\varphi_i}\|_{\mathscr{C},\ell,\mathbb{R}^n_T}\cdot\|u_{\hat\varphi_i}\|_{\mathscr{C},\ell,\mathbb{R}^n_T}.$$

By definition of Ψ_i

$$\|(\Psi_i)_{\hat\varphi_i}\|_{\mathscr{C},\ell,\mathbb{R}^n_T} \leqslant \frac{C(n,\|g\|_0)}{\lambda^\ell}, \tag{4.24}$$

so

$$\|u\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T} \leqslant C(n,d,\ell,\min\{1,\lambda\},\|g\|_0)\cdot\sup_{i\in I}\|u_{\hat\varphi_i}\|_{\mathscr{C},\ell,\hat\Omega_{i,T}}.$$

The right hand side has the same form as the norm $\|u\|^{\mathrm{Euc},1}_{\mathscr{C},\ell,M_T}$, except that it is computed on the sets $\hat\Omega_i$ instead of Ω_i. However it can be estimated as for (4.17) and, using Proposition 4.13, inequality (4.18) is established.

We can work in a similar fashion for (4.19): it is a consequence of the discussion in [Ama09, Section 4.4] that the extension operator e^- is bounded by $C(n,d,\ell)$ (the relationship between norms defined in [Ama09] and in this work will be detailed better later on); also, for the function u_Ξ we use again Proposition 4.15 and obtain a factor depending on T from the norm of Ξ. So we have proved that:

$$\|u\|^{\mathrm{Euc},3}_{\mathscr{C},\ell,M_T} \leqslant \|u\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T} + C(n,d,\ell)\cdot\|u_\Xi\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T} \leqslant \|u\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T} + C(n,d,\ell,\min\{1,T\})\cdot\|u\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T},$$

which together with (4.18) concludes the proof of the proposition. □

These theorems are easy to port to the spaces $\mathscr{C}^{\ell,\frac{\ell}{2b}}(\partial M_T)$ and $\mathscr{C}^\ell(M)$. They enable us to access the theory developed in [Ama09], which is dedicated to Euclidean spaces. In general spaces in Amann's book are described by a number s, which corresponds to our ℓ, and two tuples $\boldsymbol{\nu}$ and $\boldsymbol{d}$, which are the *weight system* (their properties are studied in [Ama09, Section 1.3]). Also, by definition, if $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k)$ and $\boldsymbol{d} = (d_1, \dots, d_k)$, then

$$\frac{1}{\boldsymbol{\nu}} = \left(\frac{1}{\nu_1}, \dots, \frac{1}{\nu_k}\right),$$

d is the sum of the values of $\boldsymbol{d}$ (corresponding to the dimension of the underlying domain), $\boldsymbol{\omega}$ is the tuple

$$\boldsymbol{\omega} = (\underbrace{\nu_1, \dots, \nu_1}_{d_1 \text{ times}}, \underbrace{\nu_2, \dots, \nu_2}_{d_2 \text{ times}}, \dots, \underbrace{\nu_k, \dots, \nu_k}_{d_k \text{ times}})$$

and ω is the least common divisor of the values of $\boldsymbol{\omega}$ (or of $\boldsymbol{\nu}$, which is the same). The letter E denotes the codomain space and will not be specified here.

The space $\mathscr{C}^\ell(\mathbb{R}^n)$ corresponds to $C_0^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) = \overset{\triangle}{B}{}_\infty^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E) = \overset{\triangle}{B}{}_{\infty,\infty}^{s/\boldsymbol{\nu}}(\mathbb{R}^d, E)$, defined in [Ama09, Section 3.9 and Section 3.3], with respect to the *isotropic weight system*:

$$\boldsymbol{\nu} = (1) \qquad \boldsymbol{d} = (n) \qquad \boldsymbol{\omega} = (\underbrace{1, \dots, 1}_{n \text{ times}}).$$

The equivalence may be shown by mean of [Ama09, Theorem 3.6.1] and [Tri78, Section 2.7.2, Theorem 2 and Remark 1].

For the other spaces we turn to Amann's theory on corners, contained in [Ama09, Chapter 4]. The space $\mathscr{C}^\ell(\mathbb{R}^n_+)$ corresponds to $C_0^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$, defined in [Ama09, Section 4.4], with the same weight system and $\mathbb{K} = \mathbb{R}^n_+$.

For the time-dependent spaces $\mathscr{C}^{\ell,\frac{\ell}{2b}}(\mathbb{R}^n_\infty)$, $\mathscr{C}^{\ell,\frac{\ell}{2b}}(\mathbb{R}^n_{+,\infty})$ and $\mathscr{C}^{\ell,\frac{\ell}{2b}}(\partial\mathbb{R}^n_{+,\infty})$ (we take $T = \infty$, otherwise the functions are not defined on a corner of $\mathbb{R}^n$) we need to use the *reduced* $2b$-*parabolic weight system* of dimension $n+1$ (for $\mathbb{R}^n_\infty$ and $\mathbb{R}^n_{+,\infty}$):

$$\boldsymbol{\nu} = (1, 2b) \qquad \boldsymbol{\omega} = (\underbrace{1, \dots, 1}_{n \text{ times}}, 2b) \qquad \boldsymbol{d} = (n, 1)$$

or the reduced $2b$-parabolic weight system of dimension n (for $\partial\mathbb{R}^n_{+,\infty}$):

$$\boldsymbol{\nu} = (1, 2b) \qquad \boldsymbol{\omega} = (\underbrace{1, \dots, 1}_{n-1 \text{ times}}, 2b) \qquad \boldsymbol{d} = (n-1, 1).$$

Again, the three spaces correspond to $C_0^{s/\boldsymbol{\nu}}(\mathbb{K}, E)$, with the appropriate choice of the corner $\mathbb{K}$.

In all cases the same results cited for the space $\mathscr{C}^\ell(\mathbb{R}^n)$ can be used to prove the equivalences; Triebel's book does not explicitly concerns itself with anisotropic spaces, but [Ama09, Theorem 3.6.3] can be used to describe an anisotropic space as intersection of many isotropic ones. In [Ama09, Example 3.9.2] the case with $\ell < 2$ is exemplified.

4.17 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell \in (0, \infty) \setminus \mathbb{N}$. Then the spaces $\mathscr{C}^\ell(M)$, $\mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ and $\mathscr{C}^{\ell,\frac{\ell}{2b}}(\partial M_T)$ are Banach spaces. In each of them the set of smooth functions on the corresponding domain is dense.*

Proof. In [Ama09, Theorem 3.3.2] and in the discussion immediately after it is proved that Besov spaces $B^{s/\boldsymbol{\nu}}_{p,q}$ and $\overset{\triangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}$ over Euclidean spaces are Banach spaces; also, by definition, Schwartz functions are dense in $\overset{\triangle}{B}{}^{s/\boldsymbol{\nu}}_{p,q}$. By localization, the thesis follows. □

4.18 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$ and $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$. If $\ell > 1$, then $\partial u, \nabla u \in \mathscr{C}^{\ell-1,\frac{\ell-1}{2b}}(M_T)$; if $\ell > 2b$, then $\partial_t u \in \mathscr{C}^{\ell-2b,\frac{\ell-2b}{2b}}(M_T)$. The following estimates hold:*

$$\|\partial u\|_{\mathscr{C},\ell-1,M_T} \leqslant C(n,d,b,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_{\bar{\ell}+1})\cdot\|u\|_{\mathscr{C},\ell,M_T}$$
$$\|\nabla u\|_{\mathscr{C},\ell-1,M_T} \leqslant C(n,d,b,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_{\bar{\ell}+1})\cdot\|u\|_{\mathscr{C},\ell,M_T}$$
$$\|\partial_t u\|_{\mathscr{C},\ell-2b,M_T} \leqslant C(n,d,b,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_{\bar{\ell}+1})\cdot\|u\|_{\mathscr{C},\ell,M_T}.$$

Proof. By the discussion above, [Ama09, Theorem 4.4.2] can be reformulated as

$$\|\partial u\|^{\mathrm{Euc},3}_{\mathscr{C},\ell-1,M_T} \leqslant C(n,d,b,\ell)\cdot\|u\|^{\mathrm{Euc},3}_{\mathscr{C},\ell,M_T}.$$

By Proposition 4.16, the thesis follows. All the other inequalities can be proved in the same way. □

4.19 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$ and $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$. Then $u|_{t=0} \in \mathscr{C}^{\ell}(M)$ and $u|_{\partial M_T} \in \mathscr{C}^{\ell,\frac{\ell}{2b}}(\partial M_T)$ and*

$$\|u|_{t=0}\|_{\mathscr{C},\ell,M} \leqslant C(n,d,b,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_{\bar{\ell}+1})\cdot\|u\|_{\mathscr{C},\ell,M_T}$$
$$\|u|_{\partial M_T}\|_{\mathscr{C},\ell,\partial M_T} \leqslant C(n,d,b,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_{\bar{\ell}+1})\cdot\|u\|_{\mathscr{C},\ell,M_T}.$$

Proof. Again by the discussion above, [Ama09, Theorem 4.5.4] can be reformulated as

$$\|u|_{t=0}\|^{\mathrm{Euc},3}_{\mathscr{C},\ell,M} \leqslant C(n,d,b,\ell)\cdot\|u\|^{\mathrm{Euc},3}_{\mathscr{C},\ell,M_T}.$$

Again, the thesis follows from Proposition 4.16. The second inequality can be proved in the same way. □

4.20 Remark. Proposition 4.19 can be proved directly from the definitions, even with the benefit of dropping the constant C. However, the same result in the case of Sobolev-Slobodeckij spaces is not trivial, so we retain the reference [Ama09, Theorem 4.5.4] in order to keep the theory homogeneous between Hölder and Sobolev-Slobodeckij spaces.

Propositions 4.18 and 4.19 imply that the map that takes a function $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ and associates to it the collections of time derivatives at time zero $\partial_t^k u|_{t=0}$ for $k = 0,\dots,\left[\frac{\ell}{2b}\right]$ is continuous. The following lemma shows that such map admits a coretraction.

4.21 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$. Suppose that, for each $k = 0,\dots,\left[\frac{\ell}{2b}\right]$, we have $w_k \in \mathscr{C}^{\frac{\ell-2kb}{2b}}\mathscr{T}_d(M)$. Then there is $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$ such that, for each k, it holds $\partial_t^k w|_{t=0} = w_k$ and*

$$\|u\|_{\mathscr{C},\ell,M_T} \leqslant C(n,d,b,\ell,\|g\|_{\bar{\ell}+1},\min\{1,\lambda\})\cdot\sum_{k=0}^{\left[\frac{\ell}{2b}\right]}\|w_k\|_{\mathscr{C},\ell-2kb,M}.$$

Proof. Use [Ama09, Theorem 4.6.3]. □

4.22 Remark. It might be tempting to define

$$u(x,t) = \sum_{k=0}^{\left[\frac{\ell}{2b}\right]}\frac{t^k}{k!}w_k(x).$$

However this naive solution cannot in general be expected to have the required regularity.

The dependency on λ and T of the constants in Proposition 4.16 is rather annoying, because arguments in later chapters will require to shrink λ and T to very small values. Such dependency cannot in general be avoided, as one can simply see from simple examples. However, if we restrict to the subspaces $\mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}(M_T)$ and $\mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}(\partial M_T)$ of functions with zero initial value, the dependency can be removed. The two subspaces are defined as the closures in $\mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ and $\mathscr{C}^{\ell,\frac{\ell}{2b}}(\partial M_T)$ of the set of smooth functions whose support is compactly contained in $M \times (0,T]$ or $\partial M \times (0,T]$. The following proposition justifies the title "zero initial value". As usual, a similar one can be written and proved for ∂M_T.

4.23 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$. Then $u \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$ if and only if $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$ and, for $k = 0,\dots,\left[\frac{\ell}{2b}\right]$, it holds $\partial_t^k u|_{t=0} = 0$.*

Proof. See [Ama09, Theorem 4.7.1]. □

In spaces of functions with zero inital value Proposition 4.14 can be refined so that we gain a power of λ each time we discard some regularity.

4.24 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $\ell, m \in (0,\infty)$ with $m < \ell$ and suppose that $T = \lambda^{2b}\kappa$ for $\lambda \leqslant 1$ and $\kappa \leqslant 1$. If $u \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$, then $u \in \mathring{\mathscr{C}}^{m,\frac{m}{2b}}(M_T)$ and*

$$\|u\|_{\mathscr{C},m,M_T} \leqslant C(m,\ell)\cdot\lambda^{\ell-m}\cdot\|u\|_{\mathscr{C},\ell,M_T}.$$

If, in addition, the condition $m \leqslant \bar{\ell}$ is satisfied, then the thesis can be strengthened to

$$\|u\|_{\mathscr{C},m,M_T} \leqslant C(m,\ell)\cdot T^{\frac{\hat{\ell}}{2b}}\cdot\lambda^{\bar{\ell}-m}\cdot\|u\|_{\mathscr{C},\ell,M_T}.$$

Proof. Let us assume that $m \in (\bar{\ell},\ell)$, which is not difficult to generalize to all the other possible cases, perhaps by repeating the argument more than once. For terms of the form (4.10) it is easy to estimate

$$\frac{\operatorname{tnorm}(\partial_t^\eta\nabla^\rho u(x,t),\partial_t^\eta\nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat{m}}} \leqslant \frac{\operatorname{tnorm}(\partial_t^\eta\nabla^\rho u(x,t),\partial_t^\eta\nabla^\rho u(y,t))}{\operatorname{dist}(x,y)^{\hat{\ell}}}\cdot\lambda^{\ell-m}.$$

For terms of the form (4.11), similarly,

$$\frac{|\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,t')|}{|t-t'|^{\frac{m-2b\eta-\rho}{2b}}} \leqslant \frac{|\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,t')|}{|t-t'|^{\frac{\ell-2b\eta-\rho}{2b}}}\cdot T^{\frac{\ell-m}{2b}}.$$

So far we have not used the hypothesis of being in a space of functions with zero initial value. This becomes important for estimating (4.12), which has the form of an integer derivative and not of a Hölder norm. Thanks to Proposition 4.23 we can write, for $2b\eta+\rho \leqslant \bar{m}$,

$$|\partial_t^\eta\nabla^\rho u(x,t)| = |\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,0)|.$$

If $2b\eta+\rho > \ell-2b$, then

$$\begin{aligned}|\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,0)| &\leqslant \frac{|\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,0)|}{|t|^{\frac{\ell-2b\eta-\rho}{2b}}}\cdot T^{\frac{\ell-2b\eta-\rho}{2b}}\\ &\leqslant \frac{|\partial_t^\eta\nabla^\rho u(x,t)-\partial_t^\eta\nabla^\rho u(x,0)|}{|t|^{\frac{\ell-2b\eta-\rho}{2b}}}\cdot T^{\frac{\ell-m}{2b}}.\end{aligned}$$

If not, then there is at least another integer derivative in the norm, with η incremented of 1. So we can estimate:

$$|\partial_t^\eta \nabla^\rho u(x,t) - \partial_t^\eta \nabla^\rho u(x,0)| \leqslant \sup_{s\in[0,T]} |\partial_t^{\eta+1}\nabla^\rho u(x,s)| \cdot T \leqslant \sup_{s\in[0,T]} |\partial_t^{\eta+1}\nabla^\rho u(x,s)| \cdot T^{\frac{\ell-m}{2b}}.$$

In every case, since by hypothesis $T \leqslant \lambda^{2b}$, the proposition is proved. In particular, when $m \leqslant \bar{\ell}$, then we can do a first step for which $m = \bar{\ell}$: in such case there is no term of the form (4.11), so a factor $T^{\frac{\ell-\bar{\ell}}{2b}} = T^{\frac{\hat{\ell}}{2b}}$ appears, and the second formula in the thesis is proved as well. □

Proposition 4.24 allows a more optimized usage of Proposition 4.15, in a way expressed by the following lemma, whose important point is the fact that C does not depend on λ.

4.25 Lemma ([Sol65, Lemma 4.4])**.** *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Let $T = \lambda^{2b}\kappa$ with $\kappa \leqslant 1$ and $\lambda \leqslant 1$, $\ell \in (0,\infty)\setminus\mathbb{N}$ and $u \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$. Suppose that $\varphi \in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ (in this case φ is an actual function), with $\|\varphi\|_{\mathscr{C},j,M_T} \leqslant \frac{C_1}{\lambda^j}$ for all $j \in [0,\ell]$. Then the following estimates hold:*

$$\begin{aligned}\|\varphi u\|_{\mathscr{C},\ell,M_T} &\leqslant C(n,d,\ell,C_1)\cdot\left[\kappa^{\frac{\hat{\ell}}{2b}} + \|\varphi\|_{\infty,M_T}\right]\cdot\|u\|_{\mathscr{C},\ell,M_T}\\ &\leqslant C(n,d,\ell,C_1)\cdot\|u\|_{\mathscr{C},\ell,M_T}.\end{aligned}$$

Proof. From Propositions 4.15 and 4.24:

$$\begin{aligned}\|\varphi u\|_{\mathscr{C},\ell,M_T} &\leqslant C(n,d,\ell)\cdot\sum_{j=0}^{\bar{\ell}}\left[\|\varphi\|_{\mathscr{C},j+\hat{\ell},M_T}\cdot\|u\|_{\mathscr{C},\bar{\ell}-j,M_T} + \|\varphi\|_{\mathscr{C},j,M_T}\cdot\|u\|_{\mathscr{C},\bar{\ell}-j+\hat{\ell},M_T}\right]\\ &\leqslant C(n,d,\ell,C_1)\cdot\sum_{j=1}^{\bar{\ell}}\left[\frac{1}{\lambda^{j+\hat{\ell}}}\cdot\|u\|_{\mathscr{C},\bar{\ell}-j,M_T} + \frac{1}{\lambda^j}\cdot\|u\|_{\mathscr{C},\bar{\ell}-j+\hat{\ell},M_T}\right]\\ &\quad + C(n,d,\ell,C_1)\cdot\left[\frac{1}{\lambda^{\hat{\ell}}}\cdot\|u\|_{\mathscr{C},\bar{\ell},M_T} + \|\varphi\|_{\infty,M_T}\cdot\|u\|_{\mathscr{C},\ell,M_T}\right]\\ &\leqslant C(n,d,\ell,C_1)\cdot\sum_{j=1}^{\bar{\ell}}\frac{1}{\lambda^{j+\hat{\ell}}}\cdot T^{\frac{\hat{\ell}}{2b}}\cdot\lambda^{\bar{\ell}-(\bar{\ell}-j)}\cdot\|u\|_{\mathscr{C},\ell,M_T}\\ &\quad + C(n,d,\ell,C_1)\cdot\sum_{j=1}^{\bar{\ell}}\frac{1}{\lambda^j}\cdot T^{\frac{\hat{\ell}}{2b}}\cdot\lambda^{\bar{\ell}-(\bar{\ell}-j+\hat{\ell})}\cdot\|u\|_{\mathscr{C},\ell,M_T}\\ &\quad + C(n,d,\ell,C_1)\cdot\left[\frac{1}{\lambda^{\hat{\ell}}}\cdot T^{\frac{\hat{\ell}}{2b}}\cdot\lambda^{\bar{\ell}-\bar{\ell}}\cdot\|u\|_{\mathscr{C},\ell,M_T} + \|\varphi\|_{\infty,M_T}\cdot\|u\|_{\mathscr{C},\ell,M_T}\right]\\ &\leqslant C(n,d,\ell,C_1)\cdot\left[\kappa^{\frac{\hat{\ell}}{2b}} + \|\varphi\|_{\infty,M_T}\right]\|u\|_{\mathscr{C},\ell,M_T}.\end{aligned}$$

The second inequality clearly descends from the first one. □

4.26 Proposition. *Let (M^n, g) be a complete manifold with boundary and bounded geometry. Suppose that $T = \lambda^{2b}\kappa$ with $\kappa \leqslant 1$ and $\lambda \leqslant 1$ and $\ell \in (0,\infty)\setminus\mathbb{N}$. Then, for $u \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}T_d(M_T)$, inequalities (4.18) and (4.19) can be refined to*

$$\begin{aligned}\|u\|^{\mathrm{Euc},2}_{\mathscr{C},\ell,M_T} &\leqslant C(n,d,\ell,\|g\|_{\bar{\ell}+1})\cdot\|u\|_{\mathscr{C},\ell,M_T}\\ \|u\|^{\mathrm{Euc},4}_{\mathscr{C},\ell,M_T} &\leqslant C(n,d,\ell,\|g\|_{\bar{\ell}+1})\cdot\|u\|_{\mathscr{C},\ell,M_T}.\end{aligned}$$

Proof. The proof is the same as the last part of Proposition 4.16; however, using Lemma 4.25 instead of (4.24) allows to avoid the dependency on λ when estimating $\|u\|^{\text{Euc},2}_{\mathscr{C},\ell,M_T}$.

If u has zero initial value, then by definition it can be approximated in $\mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ with functions u_i supported on $M\times[\varepsilon,T]$; each of the u_i can be extended by zero on $\mathbb{R}^n\times(-\infty,0]$, so by continuity u can too. It follows that the operator $\hat{\mathscr{E}}$ is continuous and $\|u\|^{\text{Euc},4}_{\mathscr{C},\ell,M_T}$ is well defined on $\mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}(M_T)$. Once good definition is established, the dependency on T is ruled out by the definition itself. □

Finally, we can prove that we can eliminate the dependency on T and λ of some propositions above in this section if we are working on spaces of functions with zero initial value.

4.27 Proposition. *Let (M^n,g) be a complete manifold with boundary and bounded geometry. Let $\ell\in(0,\infty)\setminus\mathbb{N}$ and suppose that $T=\lambda^{2b}\kappa$ with $\kappa\leqslant 1$ and $\lambda\leqslant 1$. Then Propositions 4.18 and 4.19 remain true on $\mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}(M_T)$, and in addition the dependencies on λ and T of the intervening constants can be removed.*

Proof. All proofs can be repeated using $\|u\|^{\text{Euc},4}_{\mathscr{C},\ell,M_T}$ instead of $\|u\|^{\text{Euc},3}_{\mathscr{C},\ell,M_T}$ and Proposition 4.26 instead of Proposition 4.16. □

4.3 Parabolic Sobolev-Slobodeckij spaces

For $\ell\in 2b\mathbb{N}$ and $p\in[1,\infty)$, the parabolic Sobolev space $W^{(\ell,\frac{\ell}{2b}),p}(M_T)$ is defined as the space of measurable functions on M_T for which the following norm is defined and finite:

$$\|u\|_{W,\ell,p,M_T}:=\sum_{2b\eta+\rho\leqslant\ell}\left(\iint_{M_T}|\partial_t^\eta\nabla^\rho u(x,t)|^p\,dv_g(x)\,dt\right)^{\frac{1}{p}}.$$

Let us now pass to the trace spaces of $W^{(\ell,\frac{\ell}{2b}),p}(M_T)$ on M and ∂M_T. Differently from the Hölder case, the trace operation here introduces a loss of regularity, corresponding to $\frac{1}{p}$ orders of differentiability on the space boundary and $\frac{2b}{p}$ orders of differentiability on the time boundary. A notion of Sobolev spaces with fractional derivatives is therefore necessary, for which we turn to the theory of Sobolev-Slobodeckij spaces, which are a particular case of Besov spaces.

As for the Hölder spaces, we give the definition of Sobolev-Slobodeckij spaces for all $\ell\in(0,\infty)$, although most properties fail when $\ell\notin\mathbb{N}$.

The parabolic Sobolev-Slobodeckij space $W^{\ell,p}(M)$ is defined as the space of measurable functions on M for which the following norm is defined and finite:

$$\|u\|^{(\lambda)}_{W,\ell,p,M}:=\left(\iint_{\Delta^M_\lambda}\frac{\operatorname{tnorm}(\nabla^{\bar\ell}u(x),\nabla^{\bar\ell}u(y))^p}{\operatorname{dist}(x,y)^{n+p\hat\ell}}\,dv_g(x)\,dv_g(y)\right)^{\frac{1}{p}}+\sum_{\rho\leqslant\bar\ell}\left(\int_M|\nabla^\rho u(x)|^p\,dv_g(x)\right)^{\frac{1}{p}}.$$

As mentioned above, when $\ell\in\mathbb{N}$ this is not an actual Sobolev-Slobodeckij spaces, but we use the same notation nevertheless. In that case, the first addend must be dropped.

The Sobolev-Slobodeckij space $W^{(\ell,\frac{\ell}{2b}),p}(\partial M_T)$ is defined as the space of measurable funtions on ∂M_T for which the following norm is defined and finite:

$$\|u\|^{(\lambda)}_{W,\ell,p,\partial M_T} := \sum_{2b\eta+\rho=\bar\ell}\left(\iiint_{\Delta^{\partial M}_\lambda\times[0,T]}\frac{\mathrm{tnorm}(\partial^\eta_t\nabla^\rho u(x,t),\partial^\eta_t\nabla^\rho u(y,t))^p}{\mathrm{dist}(x,y)^{n-1+p\hat\ell}}\,dv_g(x)\,dv_g(y)\,dt\right)^{\frac1p} \tag{4.25}$$

$$+\sum_{\ell-2b\eta-\rho<2b}\left(\iiint_{\partial M\times[0,T]\times[0,T]}\frac{|\partial^\eta_t\nabla^\rho u(x,t)-\partial^\eta_t\nabla^\rho u(x,t')|^p}{|t-t'|^{1+p\frac{\ell-2b\eta-\rho}{2b}}}\,dv_g(x)\,dt\,dt'\right)^{\frac1p} \tag{4.26}$$

$$+\sum_{2b\eta+\rho\leqslant\bar\ell}\left(\iint_{\partial M_T}|\partial^\eta_t\nabla^\rho u(x,t)|^p\,dv_g(x)\,dt\right)^{\frac1p}. \tag{4.27}$$

Again, when $\ell\in\mathbb N$ terms of the form (4.25) are dropped, and when $\ell\in 2b\mathbb N$ terms of the form (4.26) are dropped too.

Norm $\|u\|^{\mathrm{Euc},1}_{W,\ell,p,(\partial)M_{(T)}}$ to $\|u\|^{\mathrm{Euc},4}_{W,\ell,p,(\partial)M_{(T)}}$ can be defined with equations (4.1) through (4.4), for base manifolds M, M_T and ∂M_T.

In the rest of this section we want to develop a theory analogous to that given for Hölder spaces. Since most results and proofs are similar we do not repeat everything, but only outline the differences between last section and this one. Most of the differences are due to the integral nature of the Sobolev-Slobodeckij norms, which require a better control over phenomena of diffusion and concentration than uniform Hölder norms.

Here we will take ∂M_T instead of M_T as the prototypical space, because the latter does not involve fractional derivatives and would thus present the theory in a too simplified form.

4.28 Lemma. *Let (M^n,g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\lambda,\mu\in(0,\infty)$ with $\lambda<\mu$, $\alpha\in\mathbb R$ and $x\in\partial M$. Let us call $A(x):=\partial M\cap(B(x,\mu)\setminus B(x,\lambda))$. Then*

$$\int_{A(x)}\frac{1}{\mathrm{dist}(x,y)^{n-1+\alpha}}\,d\sigma_g(y)\leqslant C(n,\min\{1,\lambda\},\max\{1,\mu\},\alpha,\|g\|_0).$$

If the metric g has nonnegative Ricci curvature and $\alpha>0$, then μ can be taken in $(0,\infty]$ and

$$\int_{A(x)}\frac{1}{\mathrm{dist}(x,y)^{n-1+\alpha}}\,d\sigma_g(y)\leqslant C(\min\{1,\lambda\},\alpha).$$

If instead $\alpha<0$, then λ can be taken in $[0,\infty)$ and

$$\int_{A(x)}\frac{1}{\mathrm{dist}(x,y)^{n-1+\alpha}}\,d\sigma_g(y)\leqslant C(\max\{1,\mu\},\alpha).$$

Proof. Using the coarea formula (see [Cha93, Exercise III.12, (d)]) we can rewrite:

$$\int_{A(x)}\frac{1}{\mathrm{dist}(x,y)^{n-1+\alpha}}\,d\sigma_g(y)=\int_\lambda^\mu\frac{1}{t^{n-1+\alpha}}\left(\int_{S(x,t)}d\tilde\sigma_g(y)\right)dt.$$

Here $S(x,t)$ is the sphere of radius t centered at x inside ∂M, so it has dimension $n-2$; $d\tilde\sigma_g$ is the $(n-2)$-dimensional measure induced over it by g.

Since, by hypothesis, the sectional curvature is bounded in absolute value by $\|g\|_0$, we can use Corollary 3.5 and infer that $\int_{S(x,t)}d\tilde\sigma_g(y)\leqslant C(n,\|g\|_0)$, which implies the thesis. If, in addition, we know that g has nonnegative curvature, then, being the boundary totally geodesic, $g|_{\partial M}$ has nonnegative curvature too and we can use Corollary 3.5 with $K = 0$. In such case, $\int_{S(x,t)} d\tilde{\sigma}_g(y) \leqslant C(n) \cdot t^{n-2}$ and

$$\int_{A(x)} \frac{1}{\operatorname{dist}(x,y)^{n-1+\alpha}}\, d\sigma_g(y) \leqslant C(n) \cdot \int_\lambda^\mu \frac{t^{n-2}}{t^{n-1+\alpha}}\, dt = \int_\lambda^\mu t^{-1-\alpha}\, dt = \frac{1}{\alpha}\left(\lambda^{-\alpha} - \mu^{-\alpha}\right) \leqslant \frac{1}{\alpha \cdot \lambda^\alpha}.$$

The same idea, integrating near 0 instead of near ∞, gives the third inequality. □

4.29 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,\infty) \setminus \mathbb{N}$, $p \in (1,\infty)$, $\lambda, \mu \in (0,\infty)$ with $\lambda < \mu$ and $u \in \mathcal{T}_d(\partial M_T)$. Then*

$$\|u\|^{(\lambda)}_{W,\ell,p,\partial M_T} \leqslant \|u\|^{(\mu)}_{W,\ell,p,\partial M_T}$$
$$\|u\|^{(\mu)}_{W,\ell,p,\partial M_T} \leqslant C(n, d, \ell, p, \min\{1, lambda\}, \max\{1,\mu\}, \|g\|_0) \cdot \|u\|^{(\lambda)}_{W,\ell,p,\partial M_T}.$$

If the metric g has nonnegative Ricci curvature, then μ can be taken in $(0,\infty]$ and

$$\|u\|^{(\mu)}_{W,\ell,p,\partial M_T} \leqslant C(n, \min\{1,\lambda\}, p, \ell) \cdot \|u\|^{(\lambda)}_{W,\ell,p,\partial M_T}.$$

Proof. The first inequality is obvious. For the second we only need to estimate the excess in term (4.25), i.e.,

$$\int_{[0,T]} \int_{\partial M} \int_{A(x)} \frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))^p}{\operatorname{dist}(x,y)^{n-1+p\hat{\ell}}}\, dv_g(y)\, dv_g(x)\, dt \leqslant C \cdot \|u\|^p_{W,\ell,p,\partial M},$$

where $A(x)$ is defined as in Lemma 4.28.

First, let us split the trasported norm:

$$\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))^p \leqslant C(p) \cdot (|\partial_t^\eta \nabla^\rho u(x,t)|^p + |\partial_t^\eta \nabla^\rho u(y,t)|^p).$$

For each of the two terms, commuting integrals and applying Lemma 4.28, we have the thesis. □

4.30 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,\infty) \setminus \mathbb{N}$, $p \in (1,\infty)$, $\lambda \in (0,\infty)$ and $u \in \mathcal{T}_d(\partial M_T)$. Then*

$$\|u\|^{(2\lambda)}_{W,\ell,p,\partial M_T} \leqslant C(n, p, \ell, \|g\|_0) \cdot \|u\|^{(\lambda)}_{W,\ell,p,\partial M_T}.$$

Proof. We would like to use the same idea of Proposition 4.13; however, while dealing with uniform estimates the point z can be selected at will, now we need to do integral estimates, so we have to check that the selection of z does not tend to accumulate too much in a small region: first of all, we can assume that 2λ is smaller than half of the injectivity radius of M, since otherwise the proof of Proposition 4.29 can be repeated without loss of generality. So the function $z(x,y)$ mapping x and y to the midpoint of the segment joining x to y is well defined on $\Delta^{\partial M}_{2\lambda}$. In particular:

$$\operatorname{dist}(x, z(x,y)) = \frac{1}{2}\operatorname{dist}(x,y) < \lambda \qquad \operatorname{dist}(z(x,y), y) = \frac{1}{2}\operatorname{dist}(x,y) < \lambda.$$

The boundedness of the geometry implies that the differentials $D_x z(x,y)$ and $D_y z(x,y)$ and their inverses are all bounded by $C(n, \|g\|_0)$.

Let us now, for simiplicity, call

$$f(x,y) := \frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))^p}{\operatorname{dist}(x,y)^{n-1+p\hat{\ell}}}.$$

Similarly as (4.13), we have that $f(x,t) \leqslant f(x,z(z,y)) + f(z(x,y),y)$. Then

$$\begin{aligned}\iint_{\Delta_{2\lambda}^{\partial M}} f(x,y)\, d\sigma_g(x)\, d\sigma_g(y) \leqslant & \iint_{\Delta_{2\lambda}^{\partial M}} f(x,z(x,y))\, d\sigma_g(x)\, d\sigma_g(y) \\ & + \iint_{\Delta_{2\lambda}^{\partial M}} f(z(x,y),y)\, d\sigma_g(x)\, d\sigma_g(y).\end{aligned}$$

Using the boundedness of the differential of the function z:

$$\iint_{\Delta_{2\lambda}^{\partial M}} f(x,z(x,y))\, d\sigma_g(x)\, d\sigma_g(y) \leqslant C(n, \|g\|_0) \cdot \iint_{\Delta_{\lambda}^{\partial M}} f(x,z)\, d\sigma_g(x)\, d\sigma_g(z),$$

which is controlled by $\|u\|^{(\lambda)}_{W,\ell,p,\partial M_T}$. □

4.31 Proposition. *Let (M^n,g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell, m \in (0,\infty)$ with $m < \ell$ and $p \in (1,\infty)$. If $u \in W^{(\ell,\frac{\ell}{2b}),p}\mathscr{T}_d(\partial M_T)$, then $u \in W^{(m,\frac{m}{2b}),p}(\partial M_T)$ and*

$$\|u\|_{W,m,p,\partial M_T} \leqslant C(m,\ell,p,\lambda) \cdot \|u\|_{W,\ell,p,\partial M_T}.$$

Proof. The proof mimcs that of Proposition 4.14: we consider independently the terms appearing in the norm $\|u\|_{W,m,p,\partial M_T}$ and show that each of them can be estimated with one or more terms in $\|u\|_{W,\ell,p,\partial M_T}$. Terms of the form (4.27) are trivial to estimate and terms of the form (4.26) are similar to those of the form (4.25), so we focus only on the latter.

If m is integer, then there is nothing to prove, because the fractional space derivative is just being dropped. Let us then suppose that $m \in (\bar{\ell}, \ell)$: if $\operatorname{dist}(x,y) \leqslant 1$, then the simple estimate

$$\frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))^p}{\operatorname{dist}(x,y)^{n-1+p\hat{m}}} \leqslant \frac{\operatorname{tnorm}(\partial_t^\eta \nabla^\rho u(x,t), \partial_t^\eta \nabla^\rho u(y,t))^p}{\operatorname{dist}(x,y)^{n-1+p\hat{\ell}}}$$

is enough. If instead $\operatorname{dist}(x,y) > 1$, then we can proceed as in the proof of Proposition 4.29.

If ℓ is integer and $m \in (\ell-1,\ell)$, then Corollary 4.7 can be used analogously to (4.30). All the other combinations of m and ℓ can be obtained by composing these. □

4.32 Proposition. *Let (M^n,g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)$ and $\alpha \in (0, 1-\hat{\ell})$ and suppose that $u \in \mathscr{C}^{\ell+\alpha,\frac{\ell+\alpha}{2b}}\mathscr{T}_{d_1}(\partial M_T)$ and $v \in W^{(\ell,\frac{\ell}{2b}),p}\mathscr{T}_{d_2}(\partial M_T)$ with $d = \max\{d_1,d_2\}$. Then the product $u \otimes v \in W^{(\ell,\frac{\ell}{2b}),p}(\partial M_T)$ and*

$$\begin{aligned}\|u\otimes v\|_{W,\ell,p,\partial M_T} \leqslant & C(n,d,\ell,p,\max\{1,\lambda\},\alpha) \cdot \left(\sum_{j=0}^{\bar{\ell}} \|u\|_{\mathscr{C},j+\hat{\ell}+\alpha,\partial M_T} \cdot \|v\|_{W,\bar{\ell}-j,p,\partial M_T} \right. \\ & \left. + \sum_{j=0}^{\bar{\ell}} \|u\|_{\mathscr{C},j,\partial M_T} \cdot \|v\|_{W,\bar{\ell}-j+\hat{\ell},p,\partial M_T} \right) \\ \leqslant & C(n,d,\ell,p,\max\{1,\lambda\},\alpha) \cdot \|u\|_{\mathscr{C},\ell+\alpha,\partial M_T} \cdot \|v\|_{W,\ell,p,\partial M_T}.\end{aligned}$$

Proof. Following the proof of Proposition 4.15,

$$
\begin{aligned}
&\sum_{2b\eta+\rho=\bar{\ell}} \frac{\operatorname{tnorm}(\partial_t^{\eta}\nabla^{\rho}(u\otimes v)(x,t),\partial_t^{\eta}\nabla^{\rho}(u\otimes v)(y,t))^p}{\operatorname{dist}(x,y)^{n-1+p\hat{\ell}}} \\
&\quad\leqslant C(n,d,p,\ell)\cdot\sum \frac{\operatorname{tnorm}(\partial_t^{\eta_1}\nabla^{\rho_1}u(x,t)\otimes\partial_t^{\eta_2}\nabla^{\rho_2}v(x,t),\partial_t^{\eta_1}\nabla^{\rho_1}u(y,t)\otimes\partial_t^{\eta_2}\nabla^{\rho_2}v(y,t))^p}{\operatorname{dist}(x,y)^{n-1+p\hat{\ell}}} \\
&\quad\leqslant C(n,d,p,\ell)\cdot\sum|\partial_t^{\eta_1}\nabla^{\rho_1}u(x,t)|^p\cdot\frac{\operatorname{tnorm}(\partial_t^{\eta_2}\nabla^{\rho_2}v(x,t),\partial_t^{\eta_2}\nabla^{\rho_2}v(y,t))^p}{\operatorname{dist}(x,y)^{n-1+p\hat{\ell}}} \\
&\qquad+C(n,d,p,\ell)\cdot\sum\frac{|\partial_t^{\eta_2}\nabla^{\rho_2}v(x,t)|^p}{\operatorname{dist}(x,y)^{n-1-p\alpha}}\cdot\frac{\operatorname{tnorm}(\partial_t^{\eta_1}\nabla^{\rho_1}u(x,t),\partial_t^{\eta_1}\nabla^{\rho_1}u(y,t))}{\operatorname{dist}(x,y)^{p(\hat{\ell}+\alpha)}},
\end{aligned}
$$

where summations are made on all $(\eta_1,\eta_2,\rho_1,\rho_2)$ such that $2b(\eta_1+\eta_2)+\rho_1+\rho_2=\bar{\ell}$. Integrating and using Lemma 4.28, one obtains the desired inequality for terms of the form (4.25). The other cases follow in the same way. □

4.33 Proposition. *Let (M^n,g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell\in 2b\mathbb{N}$ and suppose that $u\in\mathscr{C}^{\ell,\frac{\ell}{2b}}\mathscr{T}_{d_1}(M_T)$ and $v\in W^{(\ell,\frac{\ell}{2b}),p}\mathscr{T}_{d_2}(M_T)$ with $d=\max\{d_1,d_2\}$. Then:*

$$
\|u\otimes v\|_{W,\ell,p,M_T}\leqslant C(n,d,\ell,p)\cdot\sum_{j=0}^{\ell}\|u\|_{\mathscr{C},j,M_T}\cdot\|v\|_{W,\ell-j,p,M_T}.
$$

Proof. The proof follows the same schema of Proposition 4.32, except there are no fractional derivatives in this case, so it is actually simpler. □

4.34 Proposition. *Let (M^n,g) be a complete Riemannian manifold with boundary and bounded geometry and consider a bounded atlas of size λ, for a fixed λ. Then*

$$
\begin{aligned}
&\|u\|^{\mathrm{Euc},1}_{W,\ell,p,\partial M_T}\leqslant\|u\|^{\mathrm{Euc},2}_{W,\ell,p,\partial M_T}\leqslant\|u\|^{\mathrm{Euc},3}_{W,\ell,p,\partial M_T}\\
&\|u\|_{W,\ell,p,\partial M_T}\leqslant C(n,d,\ell,p,\max\{1,\lambda\},\|g\|_{\bar{\ell}+1})\cdot\|u\|^{\mathrm{Euc},1}_{W,\ell,p,\partial M_T}\\
&\|u\|^{\mathrm{Euc},1}_{W,\ell,p,\partial M_T}\leqslant C(n,d,\ell,p,\max\{1,\lambda\},\|g\|_{\bar{\ell}+1})\cdot\|u\|_{W,\ell,p,\partial M_T}\\
&\|u\|^{\mathrm{Euc},2}_{W,\ell,p,\partial M_T}\leqslant C(n,d,\ell,p,\lambda,\|g\|_{\bar{\ell}+1})\cdot\|u\|_{W,\ell,p,\partial M_T} && (4.28)\\
&\|u\|^{\mathrm{Euc},3}_{W,\ell,p,\partial M_T}\leqslant C(n,d,\ell,p,\lambda,\min\{1,T\},\|g\|_{\bar{\ell}+1})\cdot\|u\|_{W,\ell,p,\partial M_T} && (4.29)
\end{aligned}
$$

Proof. The proof is similar in the spirit to that of Proposition 4.16, so we do not repeat the whole proof; in lemmata 4.2 and 4.11 the integral estimates are to be used instead of the uniform ones.

Let us see how to handle terms of the form (4.22): by the Hölder inequality

$$
\begin{aligned}
|\partial^j u_{\varphi_i}(x)-\partial^j u_{\varphi_i}(y)|^p&\leqslant\left(\int_0^{|x-y|}\left|\partial^{j+1}u_{\varphi_i}\left(x+t\cdot\frac{y-x}{|x-y|}\right)\right|dt\right)^p\\
&\leqslant|x-y|^{p-1}\cdot\int_0^{|x-y|}\left|\partial^{j+1}u_{\varphi_i}\left(x+t\cdot\frac{y-x}{|x-y|}\right)\right|^p dt.
\end{aligned}
$$

It follows that, commuting the integrals and since the diameter of Ω_i is smaller than 10λ:

$$\begin{aligned}\iint_{\partial\Omega_i\times\partial\Omega_i} \frac{|\partial^j u_{\varphi_i}(x)-\partial^j u_{\varphi_i}(y)|^p}{|x-y|^{n-1+p\hat{\ell}}}\,dx\,dy &\leqslant \int_{\partial\Omega_i}\int_{\mathbb{R}^{n-1}} \mathbb{1}_{\partial\Omega_i - x}(h)\cdot|h|^{-n+p(1-\hat{\ell})} \\ &\quad\cdot\int_0^{|h|}\left|\partial^{j+1}u_{\varphi_i}\left(x+t\frac{h}{|h|}\right)\right|^p dt\,dh\,dx \\ &\leqslant \int_{B_{20\lambda}(0)}|h|^{-n+1+p(1-\hat{\ell})}\,dh\cdot\int_{\partial\Omega_i}|\partial^{j+1}u_{\varphi_i}(x)|^p\,dx \\ &\leqslant C(n,p,\ell)\cdot\lambda^{p(1-\hat{\ell})}\cdot\int_{\partial\Omega_i}|\partial^{j+1}u_{\varphi_i}(x)|^p\,dx.\end{aligned} \tag{4.30}$$

Terms coming from the second addend in (4.20) can be handled again with Lemma 4.5:

$$\begin{aligned}&\iint_{\partial\Omega_i\times\partial\Omega_i}\frac{|\partial^{\bar{\ell}}u_{\varphi_i}(y)-T_{y,x}\partial^{\bar{\ell}}u_{\varphi_i}(y)|^p}{|x-y|^{n-1+p\hat{\ell}}}\,dx\,dy \\ &\qquad\leqslant C(n,d,p,\|g\|_1)\cdot\iint_{\partial\Omega_i\times\partial\Omega_i}|x-y|^{-n+1+p(1-\hat{\ell})}\cdot|\partial^{\bar{\ell}}u(y)|\,dx\,dy,\end{aligned}$$

which is analogous to the above.

Also when proving (4.28) and (4.29) the schema of Proposition 4.16 can be reused, letting Propositions 4.31 and 4.32 play the role of Proposition 4.14 and 4.15. □

Like the Hölder spaces (see there for the notations), the Sobolev-Slobodeckij spaces we have defined have equivalents in [Ama09].

The space $W^{\ell,p}(\mathbb{R}^n)$ corresponds to $W_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d,E) = B_p^{s/\boldsymbol{\nu}}(\mathbb{R}^d,E) = B_{p,p}^{s/\boldsymbol{\nu}}(\mathbb{R}^d,E)$, defined in [Ama09, Section 3.8 and Section 3.3], with respect to the isotropic weight system. The equivalence may be shown by mean of [Ama09, Theorem 3.6.1] and [Tri78, Section 2.5.1, Theorem and Remarks 3 and 4].

The space $W^{\ell,p}(\mathbb{R}^n_+)$ similarly corresponds to $W_p^{s/\boldsymbol{\nu}}(\mathbb{K},E)$, defined in [Ama09, Section 4.4], with the same weight system and $\mathbb{K} = \mathbb{R}^n_+$.

The spaces $W^{(\ell,\frac{\ell}{2b}),p}(\mathbb{R}^n_T)$ and $W^{(\ell,\frac{\ell}{2b}),p}(\mathbb{R}^n_{+,T})$ correspond to $W_p^{s/\boldsymbol{\nu}}(\mathbb{K},E) = H_p^{s/\boldsymbol{\nu}}(\mathbb{K},E)$, defined in [Ama09, Section 3.7 and Section 4.4], with the reduced $2b$-parabolic weight system of dimension $n+1$; compare with [Ama09, Section 3.8] and see the example [Ama09, Example 3.7.4].

At last, the space $W^{(\ell,\frac{\ell}{2b}),p}(\partial\mathbb{R}^n_{+,T})$ corresponds to $W_p^{s/\boldsymbol{\nu}}(\mathbb{K},E)$, with the reduced $2b$-parabolic weight system of dimension n. An explicit norm is described in [Ama09, Proposition 3.8.3].

4.35 Proposition. *Let (M^n,g) be a complete Riemannian manifold with boundary and bounded geometry. Let $p\in[1,\infty)$. Then the spaces $W^{(\ell,\frac{\ell}{2b}),p}(M_T)$ for $\ell\in 2b\mathbb{N}$ and $W^{\ell,p}(M)$ and $W^{(\ell,\frac{\ell}{2b}),p}(\partial M_T)$ for $\ell\in(0,\infty)\setminus\mathbb{N}$ are Banach spaces. In each of them the set of smooth functions on the corresponding domain is dense.*

Proof. As for Proposition 4.17, the result follows from [Ama09, Theorem 3.3.2] and the discussion immediately after, together with [Ama09, Theorem 3.7.1, (iii)] to extend the result to spaces $H_p^{s/\boldsymbol{\nu}}(\mathbb{K},E)$. □

Since we defined Sobolev spaces on M_T only for $\ell\in 2b\mathbb{N}$, we cannot give an equivalent of Proposition 4.18, but we can fuse it together with 4.19, since they are always used together.

4.36 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$, $p \in (1,\infty)$ and $u \in W^{(\ell,\frac{\ell}{2b}),p}\mathcal{T}_d(M_T)$ and take $\eta, \rho \in \mathbb{N}$. If $\ell > 2b\eta + \rho + \frac{1}{p}$, then $\partial_t^\eta \nabla^\rho u|_{\partial M_T} \in W^{(\ell-2b\eta-\rho-\frac{1}{p},\frac{\ell-2b\eta-\rho-\frac{1}{p}}{2b}),p}(\partial M_T)$; if $\ell > 2b\eta + \rho + \frac{2b}{p}$, then $\partial_t^\eta \nabla^\rho u|_{t=0} \in W^{\ell-2b\eta-\rho-\frac{2b}{p}}(M)$. Also, the following equalities respectively hold:*

$$\|\partial_t^\eta \nabla^\rho u|_{\partial M_T}\|_{W,\ell-2b\eta-\rho-\frac{1}{p},p,\partial M_T} \leqslant C(n,d,b,p,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_\ell) \cdot \|u\|_{W,\ell,p,M_T}$$
$$\|\partial_t^\eta \nabla^\rho u|_{t=0}\|_{W,\ell-2b\eta-\rho-\frac{2b}{p},p,M} \leqslant C(n,d,b,p,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_\ell) \cdot \|u\|_{W,\ell,p,M_T}.$$

Proof. See [Ama09, Theorem 4.4.2 and Theorem 4.5.1]. □

4.37 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$. Suppose that, for each $k = 0, \dots, \left[\frac{\ell}{2b} - \frac{1}{p}\right]$, we have $w_k \in W^{(\ell-2bk-\frac{2b}{p},\frac{\ell-2bk-\frac{2b}{p}}{2b}),p}\mathcal{T}_d(M)$. Then there is $u \in W^{(\ell,\frac{\ell}{2b}),p}\mathcal{T}_d(M_T)$ such that, for each k, it holds $\partial_t^k w|_{t=0} = w_k$ and*

$$\|u\|_{W,\ell,p,M_T} \leqslant C(n,d,b,\ell,p,\|g\|_\ell,\min\{1,\lambda\}) \cdot \sum_{k=0}^{\left[\frac{\ell}{2b}-\frac{1}{p}\right]} \|w_k\|_{W,\ell-2kb-\frac{2b}{p},p,M}.$$

Proof. See [Ama09, Theorem 4.6.3]. □

The spaces of functions with zero inital values $\mathring{W}^{(2bk,k),p}(M_T)$ and $\mathring{W}^{(\ell,\frac{\ell}{2b}),p}(\partial M_T)$ are defined, like for Hölder spaces, as the closures of the set of smooth functions whose support is compactly supported in $M \times (0,T]$ or $\partial M \times (0,T]$.

4.38 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$ and $p \in (1,\infty)$. Then $u \in \mathring{W}^{(\ell,\frac{\ell}{2b}),p}(M_T)\mathcal{T}_d$ if and only if $u \in W^{(\ell,\frac{\ell}{2b}),p}(M_T)\mathcal{T}_d$ and, for $k = 0, \dots, \left[\frac{\ell}{2b} + \frac{1}{p}\right]$, it holds $\partial_t^k u|_{t=0} = 0$.*

Proof. See [Ama09, Theorem 4.7.1]. □

4.39 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$ and $p \in (1,\infty)$ and suppose that $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. If $u \in \mathring{W}^{(\ell,\frac{\ell}{2b}),p}(M_T)$, then for all η and ρ such that $2b\eta + \rho < \ell$ it holds*

$$\|\partial_t^\eta \nabla^\rho u\|_{p,M_T} \leqslant C(n,d,\ell,p,\|g\|_\ell) \cdot T^{\frac{\ell-2b\eta-\rho}{2b}} \cdot \|u\|_{W,\ell,p,M_T}.$$

Proof. If $\rho = 0$, the results follows trivially by integration in the time direction. In general we can use Proposition 3.11 on $\partial_t^\eta u$ with $p = q = r$, $m = \rho$, $k = \ell - 2b\eta$ and $\alpha = \frac{m}{k} = \frac{\rho}{\ell-2b\eta}$, followed by Young's inequality:

$$\begin{aligned}\|\partial_t^\eta \nabla^\rho u\|_{p,M_T} &\leqslant C(n,d,\ell,p,\|g\|_\ell) \cdot \|u\|_{W,\ell,p,M_T}^{\frac{\rho}{\ell-2b\eta}} \cdot \|\partial_t^\eta u\|_{p,M_T}^{\frac{\ell-2b\eta-\rho}{\ell-2b\eta}} \\ &= C(n,d,\ell,p,\|g\|_\ell) \cdot \left(T^{\frac{\rho\cdot(\ell-2b\eta-\rho)}{\ell-2b\eta}} \cdot \|u\|_{W,\ell,p,M_T}^{\frac{\rho}{\ell-2b\eta}}\right) \cdot \left(T^{-\frac{\rho\cdot(\ell-2b\eta-\rho)}{\ell-2b\eta}} \cdot \|\partial_t^\eta u\|_{p,M_T}^{\frac{\ell-2b\eta-\rho}{\ell-2b\eta}}\right) \\ &\leqslant C(n,d,\ell,p,\|g\|_\ell) \cdot \left(T^{\ell-2b\eta-\rho} \cdot \|u\|_{W,\ell,p,M_T} + T^{-\rho} \cdot \|\partial_t^\eta u\|_{p,M_T}\right).\end{aligned}$$

And then we conclude using the thesis for $\rho = 0$, which has already been proved. □

4.40 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$ and $p \in (1,\infty)$ and suppose that $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. If $u \in \mathring{W}^{(\ell,\frac{\ell}{2b}),p}\mathscr{T}_d(M_T)$ and $\varphi \in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ (in this case φ is an actual function), with $\|\varphi\|_{\mathscr{C},j,M_T} \leqslant \frac{C_1}{\lambda^j}$ for all $j \in [0,\ell]$, then the following estimates hold:*

$$\begin{aligned}\|\varphi u\|_{W,\ell,p,M_T} &\leqslant C(n,d,\ell,p,C_1)\cdot\left[\kappa + \|\varphi\|_{\infty,M_T}\right]\cdot\|u\|_{W,\ell,p,M_T}\\ &\leqslant C(n,d,\ell,p,C_1)\cdot\|u\|_{W,\ell,p,M_T}.\end{aligned}$$

Proof. See the proof of Lemma 4.25, using Proposition 4.33 to expand the product. □

4.41 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$ and $p \in (1,\infty)$ and suppose that $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. Then, for $u \in \mathring{W}^{(\ell,\frac{\ell}{2b}),p}\mathscr{T}_d(M_T)$, the equivalent of inequalities (4.28) and (4.29) on M_T can be refined to*

$$\begin{aligned}\|u\|^{\mathrm{Euc},2}_{W,\ell,p,M_T} &\leqslant C(n,d,\ell,p,\|g\|_\ell)\cdot\|u\|_{W,\ell,p,M_T}\\ \|u\|^{\mathrm{Euc},4}_{W,\ell,p,M_T} &\leqslant C(n,d,\ell,p,\|g\|_\ell)\cdot\|u\|_{W,\ell,p,M_T}.\end{aligned}$$

Proof. See the proof of Proposition 4.26. □

4.42 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$ and $p \in (1,\infty)$ and suppose that $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. Then Proposition 4.36 remains true on $\mathring{W}^{(\ell,\frac{\ell}{2b}),p}(M_T)$, and in addition the dependencies on λ and T of the intervening constants can be removed.*

Proof. See the proof of Proposition 4.27. □

4.4 Sobolev embeddings in parabolic spaces

At last we connect the Hölder and the Sobolev-Slobodeckij spaces showing the form that Sobolev embedding take for parabolic spaces.

4.43 Proposition (Sobolev embedding)**.** *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,2b)\setminus\mathbb{N}$ and $p > \frac{2b+n}{2b-\ell}$. Then for every $u \in W^{(2b,1),p}\mathscr{T}_d(M_T)$ we have that $u \in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$ and*

$$\|u\|_{\mathscr{C},\ell,M_T} \leqslant C(n,d,b,p,\ell,\min\{1,\lambda\},\min\{1,T\},\|g\|_{2b})\cdot\|u\|_{W,2b,p,M_T}.$$

Proof. See [Ama09, Theorem 3.9.1]. □

When the function u has zero initial value, the dependency of the constant of λ and T can be dropped.

4.44 Proposition. *Let (M^n, g) be a complete Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,2b)\setminus\mathbb{N}$ and $p \in (\frac{2b+n}{2b-\ell},\infty)$ and suppose $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. Then for every $u \in \mathring{W}^{(2b,1),p}(M_T)$ we have that $u \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{T}_d(M_T)$ and*

$$\|u\|_{\mathscr{C},\ell,M_T} \leqslant C(n,d,b,p,\ell,\|g\|_{2b})\cdot\|u\|_{W,2b,p,M_T}.$$

Proof. Same as Proposition 4.43, but using Proposition 4.26 and Proposition 4.41 in order to remove the dependency on λ and T. □

A trivial corollary of the Sobolev embedding is the following, and will become crucial in the solution of semilinear systems.

4.45 Corollary. *Let* (M^n, g) *be a complete Riemannian manifold with boundary and bounded geometry. Let* $p \in (2(2b+n), \infty)$. *If* $u \in W^{(2b,1),p}\mathcal{T}_d(M_T)$, *then* $u \in \mathscr{C}^{2b-\frac{1}{2}, \frac{2b-\frac{1}{2}}{2b}}(M_T)$. *In particular, for* $j \in [0, 2b) \cap \mathbb{N}$ *the function* $\nabla^j u$ *is Hölder continuous with exponent at least* $\frac{1}{2}$ *in the space direction and at least* $\frac{1}{4b}$ *in the time direction.*

Proof. Just use Proposition 4.43 with $\ell = 2b - \frac{1}{2}$. □

4.46 Remark. Thanks to this corollary, we can meaningfully consider the restriction $\nabla^j u|_{t=0}$, which is pointwise defined and again Hölder continuous. Also, $\nabla^j u|_t$ converges uniformly to $\nabla^j u|_{t=0}$ as $t \to 0$.

At the same time, $\nabla^j u$ is not, in general, differentiable in the time direction.

4.47 Corollary. *Let* (M^n, g) *be a complete Riemannian manifold with boundary and bounded geometry. Let* $j \in [0, 2b) \cap \mathbb{N}$ *and* $p \in (2(2b+n), \infty)$. *Let* $u \in W^{(2b,1),p}\mathcal{T}_d(M_T)$ *such that* $u|_{t=0} = 0$. *Then* $u \in \mathring{W}^{(2b,1),p}(M_T)$ *and*

$$\|\nabla^j u\|_{\infty, M_T} \leqslant C(n, d, b, p, \|g\|_{2b}) \cdot T^{\frac{1}{4b}} \cdot \|u\|_{W,2b,p,M_T},$$

Proof. By Proposition 4.38, we have $u \in \mathring{W}^{(2b,1),p}(M_T)$. Then, by Proposition 4.44, $u \in \mathring{\mathscr{C}}^{\ell, \frac{\ell}{2b}}(M_T)$, for $\ell = 2b - \frac{1}{2}$. Writing the definition of the norm, one has the thesis. □

Chapter 5

Linear parabolic systems

5.1 Linear parabolic systems

In this chapter we consider linear parabolic systems, which are special cases of (2.1), (2.2) and (2.3). In particular, A and B^q will depend only of x and t and F^q and E_q will be linear combinations of the lower derivatives of u, plus a forcing term:

$$A_{I_k}^{J_{2b}K_k}(x,t,u(x,t),\dots) = (A_0)_{I_k}^{J_{2b}K_k}(x,t) \tag{5.1}$$

$$F_{I_k}(x,t,u(x,t),\dots) = \sum_{p=0}^{2b-1} (A_{2b-p})_{I_k}^{J_pK_k}(x,t)\cdot\nabla_{J_p}^p u_{K_k}(x,t) + (F_0)_{I_k}(x,t) \tag{5.2}$$

$$(B^q)_{I_{d_q}}^{J_{m_q}K_k}(x,t,u(x,t),\dots) = (B_0^q)_{I_{d_q}}^{J_{m_q}K_k}(x,t) \tag{5.3}$$

$$(E^q)_{I_{d_q}}(x,t,u(x,t),\dots) = -\sum_{p=0}^{m_q-1} (B_{m_q-p}^q)_{I_{d_q}}^{J_pK_k}(x,t)\cdot\nabla_{J_p}^p u_{K_k}(x,t) + (E_0^q)_{I_{d_q}}(x,t). \tag{5.4}$$

Substituting in the general equations we have (as usual, q ranges on $1,\dots,\nu$):

$$\partial_t u_{I_k}(x,t) - \sum_{p=0}^{2b} (A_{2b-p})_{I_k}^{J_pK_k}(x,t)\cdot\nabla_{J_p}^p u_{K_k}(x,t) = (F_0)_{I_k}(x,t) \qquad (x,t)\in M_T \tag{5.5}$$

$$\sum_{p=0}^{m_q} (B_{m_q-p}^q)_{I_{d_q}}^{J_pK_k}(x,t)\cdot\nabla_{J_p}^p u_{K_k}(x,t) = (E_0^q)_{I_{d_q}}(x,t) \qquad (x,t)\in\partial M_T \tag{5.6}$$

$$u(x,0) = u_0(x) \qquad x\in M. \tag{5.7}$$

The coefficients are taken in the following spaces:

$$\begin{aligned}
A_{2b-p} &\in \mathscr{T}^p\mathscr{U}_k^k(M_T) && p=0,\dots,2b\\
F_0 &\in \mathscr{U}_k(M_T) &&\\
B_{m_q-p}^q &\in \mathscr{T}^p\mathscr{U}_{d_q}^k(\partial M_T) && p=0,\dots m_q\\
E_0^q &\in \mathscr{U}_{d_q}(\partial M_T). &&
\end{aligned}$$

The system can be formally rewritten as

$$\mathscr{L}_{I_k}^{K_k}(x,t,\nabla,\partial_t)u_{K_k}(x,t) = F_{I_k}(x,t),$$

where $\mathscr{L}$ is the differential operator given by

$$\mathscr{L}_{I_k}^{K_k}(x,t,\nabla,\partial_t) = \delta_{I_k}^{K_k}\cdot\partial_t - \sum_{p=0}^{2b}(A_{2b-p})_{I_k}^{J_pK_k}(x,t)\cdot\nabla_{J_p}^{p}.$$

We call the *principal part* $\mathscr{L}_0$ of the operator $\mathscr{L}$ the sum of the two addends of maximal order, i.e.

$$(\mathscr{L}_0)_{I_k}^{K_k}(x,t,\nabla,\partial_t) = \delta_{I_k}^{K_k}\cdot\partial_t - (A_0)_{I_k}^{J_{2b}K_k}(x,t)\cdot\nabla_{J_{2b}}^{2b}.$$

Taken $p\in\mathbb{C}$ and $\xi\in\mathscr{T}^1(M_T)$, the *principal symbol* is obtained from the principal part by formally replacing ∂_t with p and ∇ with $i\xi_\flat$:

$$(\mathscr{L}_0)_{I_k}^{K_k}(x,t,i\xi,p) = \delta_{I_k}^{K_k}\cdot p - (-1)^b\cdot(A_0)_{I_k}^{J_{2b}K_k}(x,t)\cdot\xi_{J_{2b}}^{\otimes 2b}. \tag{5.8}$$

Although defined by means of coordinates, having replaced the covariant derivative with a covariant tensor field, the definitions on different charts fit together to defined an actual tensor. In particular, $\mathscr{L}_0(i\xi,p)\in\mathscr{T}_k^k(M_T)$ can be interpreted as a linear endomorphism of $T_k(M_T)$. Its determinant is therefore well defined: we will indicate it with $L(x,t,i\xi,p)$. From the definition we easily see that it satisfies the homogeneity formula:

$$L(x,t,i\lambda\xi,\lambda^{2b}p) = \lambda^{2br}L(x,t,i\xi,p). \tag{5.9}$$

Also, $L(x,t,i\xi,p)$ is a polynomial with real coefficients in p by (5.8).

Similar definitions can be given for the boundary operator, with the small clarification that each of the boundary conditions is considered individually for the sake of determining its principal part (i.e., if there is a boundary condition of order 1 and another boundary condition of order 2, then the principal part of the boundary operator contains the components with 1 derivative of the first condition and the components with 2 derivatives of the second condition). In particular, for each $q=1,\ldots,v$, the q-th condition can be written

$$(\mathscr{B}^q)_{I_{d_q}}^{K_k}(x,t,\nabla)u_{K_k}(x,t) = (E_0^q)_{I_{d_q}}(x,t) \tag{5.10}$$

with

$$(\mathscr{B}^q)_{I_{d_q}}^{K_k}(x,t,\nabla) = \sum_{p=0}^{m_q}(B_{m_q-p}^q)_{I_{d_q}}^{J_pK_k}(x,t)\cdot\nabla_{J_p}^{p}.$$

The principal part is

$$(\mathscr{B}_0^q)_{I_{d_q}}^{K_k}(x,t,\nabla) = (B_0^q)_{I_{d_q}}^{J_{m_q}K_k}(x,t)\cdot\nabla_{J_{m_q}}^{m_q}$$

and the principal symbol

$$(\mathscr{B}_0^q)_{I_{d_q}}^{K_k}(x,t,i\xi) = i^{m_q}(B_0^q)_{I_{d_q}}^{J_{m_q}K_k}(x,t)\cdot\xi_{J_{m_q}}^{\otimes m_q}$$

As above we have that $\mathscr{B}_0^q(i\xi)\in\mathscr{T}_{d_q}^k(\partial M_T)$. In order to collect together all the boundary conditions and express them as a single equality in a vector bundle, we need to define

$$U = \bigoplus_{q=1}^{v}U_{d_q}(\partial M_T),$$

where the symbol $\oplus$ denotes the Whitney sum of vector bundles (see for instance [Lee13, Example 10.7]). Then all the boundary conditions are equivalent to an equality in the space of sections of U:

$$\bigoplus_{q=1}^{\nu} (\mathscr{B}^q)_{I_{d_q}}^{K_k}(x,t,\nabla)u_{K_k}(x,t)\cdot dx^{I_{d_q}} = \bigoplus_{q=1}^{\nu} (E_0^q)_{I_{d_q}}(x,t)\cdot dx^{I_{d_q}}.$$

The equivalent of $\mathscr{L}_0$ above is therefore $\mathscr{B}_0(x,t,i\xi)$, defined as

$$u_{K_k}(x,t)\longmapsto \bigoplus_{q=1}^{\nu} (\mathscr{B}_0^q)_{I_{d_q}}^{K_k}(x,t,i\xi)\cdot u_{K_k}(x,t)\cdot dx^{I_{d_q}},$$

which is a linear bundle homomorphism over ∂M_T from the space $\mathscr{U}_k(\partial M_T)$ to the space of the sections of U, because it is clearly $\mathscr{C}^\infty$-linear (see [Lee13, Lemma 10.29]).

5.2 Parabolicity and complementary conditions

Let us formulate the parabolicity and complementary conditions that a linear system must satisfy so that Solonnikov's theory can work.

5.1 Definition ([Sol65, p. 8, bottom])**.** A parabolic linear system of PDEs is *uniformly strongly parabolic* if there is $\delta > 0$ such that the complex roots in p of $L(x,t,i\xi,p)$ satisfy the equation

$$\operatorname{Re} p \leqslant -\delta|\xi|^{2b}$$

for every $(x,t)\in M_T$ and $\xi \in T_xM$.

Since the strongly parabolicity condition is defined purely in terms of the determinant L, it does not depend on the coordinates chosen on M.

When $x\in\partial M$ we call ν the outward normal vector at x. Then we can decompose $\xi = \tau\nu+\zeta$, where $\tau\in\mathbb{R}$ and $\zeta\in T_x\partial M$.

5.2 Lemma. *For $(x,t)\in\partial M_T$ take $\zeta\in T_x\partial M$ and $p\in\mathbb{C}$ such that they are not both vanishing and* $\operatorname{Re} p \geqslant -\delta|\zeta|^{2b}$. *Then the polynomial $L(x,t,i(\tau\nu+\zeta),p)$ has br roots in τ with positive imaginary part and br roots with negative imaginary part, counted according to multiplicity. In particular, it has no real roots.*

Proof. This follows from the parabolicity condition and the homogeneity formula (5.9) for L. See the details in [Sol65, Theorem 2.2, §7] and its corollary. □

Thanks to the lemma we can define the polynomial in τ

$$M^+(x,t,\zeta,\tau,p) = \prod_{\tilde\tau}(\tau-\tilde\tau), \tag{5.11}$$

where $\tilde\tau$ iterates on the roots of L with positive imaginary part, counted with multiplicity, as defined above. In particular, $M^+(\tau)$ has degree br.

Let us now consider, for $(x,t)\in\partial M_T$, the map

$$\mathscr{B}_0(x,t,i(\tau\nu+\zeta))\circ\hat{\mathscr{L}}_0(x,t,i(\tau\nu+\zeta),p)\colon U_K(\partial M_T)\to U,$$

where $\hat A$ is the adjugate of the linear endomorphism A (which coincides with $\det A\cdot A^{-1}$ when A is invertible). Following Solonnikov's theory, we give this definition.

5.3 Definition ([Sol65, p. 11, top])**.** The *complementary condition* between $\mathscr{B}_0$ and $\mathscr{L}_0$ is satisfied if for each point $(x,t) \in \partial M_T$, $p \in \mathbb{C}$ and $\zeta \in T_x\partial M$ such that p and ζ are not both vanishing and $\operatorname{Re} p \geqslant -\delta_1|\zeta|^{2b}$ we have that the matrix $\mathscr{B}_0 \cdot \hat{\mathscr{L}}_0$ has br rows, and the rows are linearly dependent as polynomials in τ modulo $M^+(x,t,\zeta,\tau,p)$, according to Definition A.8.

A slight extension of the complementary condition can be given: it will be helpful to deal with boundary conditions involving the conformal class of the metric.

5.4 Definition. The *extended complementary condition* between $\mathscr{B}$ and $\mathscr{L}_0$ is satisfied if the ordinary complementary condition is satisfied replacing $\mathscr{B}_0$ with the new matrix $\tilde{\mathscr{B}}_0$, constructed from $\mathscr{B}_0$ by taking a maximal set of rows whose corresponding rows in $\mathscr{B}$ and E_0 as described by (5.10) are linearly independent (over the base field $\mathbb{C}$).

The extended complementary condition implements the usual feature of linear systems, according to which linearly dependent rows of a system can be removed and then recovered from the others. However, in our case, it is important that the linear dependency relation is satisfied on the whole $\mathscr{B}$, not only on $\mathscr{B}_0$, and without the polynomial quotient.

5.5 Definition ([Sol65, p. 97–98])**.** Assume that at some point $(x,t) \in \partial M_T$ normal coordinates according to the metric g are considered. Then the *uniform complementary condition* is satisfied if there is a $\delta_1 \in (0,\delta)$ such that for each $(x,t) \in \partial M_T$, p and ζ as above, under the additional normalization condition that $|\zeta|^{4b} + |p|^2 = 1$, the rows of the matrix $\mathscr{B}_0 \cdot \hat{\mathscr{L}}_0$ are uniformly independent modulo M^+ with constant δ_1, according to Definition A.9.

Because of Remark A.10, the constant δ_1 in the uniform complementary condition does not depend on the coordinates chosen on M, as long as they are normal at the point (x,t).

5.3 Linear systems in Hölder spaces

In this section we describe an existence and uniqueness theory for linear parabolic systems in Hölder spaces. Let us begin by restating compatibility conditions given in Section 2.3, discussing the expected regularity of the functions w_j. Substuting the definitions (5.1), (5.2), (5.3) and (5.4) in (2.5) and (2.6) we have that:

$$\begin{aligned} w_0(x) &= u_0(x) \\ w_{j+1}(x) &= \sum_{i=0}^{j} \binom{j}{i} \cdot \sum_{p=0}^{2b} \partial_t^i A_{2b-p}(x,0) \cdot \nabla^p w_{j-i}(x) + \partial_t^j F_0(x,0) \end{aligned} \tag{5.12}$$

and the compatibility conditions are verified when

$$\sum_{i=0}^{j} \binom{j}{i} \cdot \sum_{p=0}^{m_q} \partial_t^i B^q_{m_q-p}(x,0) \cdot \nabla^p w_{j-i}(x) = \partial_t^j E_0^q(x,0). \tag{5.13}$$

Fulfilling a promise in Remark 2.3, we now need to ascertain the existence and regularity of the function w_i.

5.6 Lemma ([Sol65, Lemma 4.5])**.** *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,\infty) \setminus \mathbb{N}$. Suppose that*

$$\begin{aligned} A, F_0 &\in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T) \\ u_0 &\in \mathscr{C}^{\ell+2b}(M). \end{aligned}$$

Then w_j is well defined for $j = 0,\ldots,\left[\frac{\ell}{2b}\right]+1$, it belongs to $\mathscr{C}^{\ell+2b(1-j)}(M)$ and the following estimate is valid:

$$\|w_j\|_{\mathscr{C},\ell+2b(1-j),M} \leqslant C(n,k,\ell,b,\lambda,T,\|g\|_{\bar{\ell}+2b+1},\|A\|_{\mathscr{C},\ell,M_T})\cdot\big(\|u_0\|_{\mathscr{C},\ell+2b,M}+\|F_0\|_{\mathscr{C},\ell,M_T}\big).$$

Proof. By definition, $w_0 = u_0 \in \mathscr{C}^{\ell+2b}(M)$, so we can proceed by induction; suppose that the thesis is true for $0 \leqslant j \leqslant \left[\frac{\ell}{2b}\right]$: then $w_{j-i} \in \mathscr{C}^{\ell+2b(1-j+i)}(M) \subset \mathscr{C}^{\ell+2b(1-j)}$ by Proposition 4.14 and, since $p \leqslant 2b$, $\nabla^p w_{j-i} \in \mathscr{C}^{\ell+2bj}(M)$ by Proposition 4.18. Similarly, $\partial_t^i A_{2b-p} \in \mathscr{C}^{\ell-2bi,\frac{\ell-2bi}{2b}}(M_T) \subset \mathscr{C}^{\ell-2bj,\frac{\ell-2bj}{2b}}(M_T)$, and so $\partial_t^i A_{2b-p}|_{t=0} \in \mathscr{C}^{\ell-2bj}(M)$ by Proposition 4.19. The same goes for F_0. From (5.12) it follows that $w_{j+1} \in \mathscr{C}^{\ell-2bj}(M)$, so the inductive step is done. □

5.7 Lemma. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$. Suppose that*

$$\begin{aligned} A, F_0 &\in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)\\ B^q, E_0^q &\in \mathscr{C}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}(\partial M_T)\\ u_0 &\in \mathscr{C}^{\ell+2b}(M). \end{aligned}$$

Then the two sides of (5.13) are well defined for $j = 0,\ldots,\left[\frac{\ell-m_q}{2b}\right]+1$ and they belong to the space $\mathscr{C}^{\ell+2b(1-j)-m_q}(\partial M)$. In particular, it is meaningful to require compatibility conditions according to Definition 2.1 for orders up to ℓ.

Proof. The proof runs as for the previous lemma. □

We are ready to state the main theorem of this section.

5.8 Theorem ([Sol65, Theorem 4.9])**.** *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,\infty)\setminus\mathbb{N}$. Suppose that*

$$\begin{aligned} A, F_0 &\in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)\\ B^q, E_0^q &\in \mathscr{C}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}(\partial M_T)\\ u_0 &\in \mathscr{C}^{\ell+2b}(M). \end{aligned}$$

Suppose furthermore that parabolicity and extended complementary conditions are uniformly satisfied with constants δ and δ_1 and compatibility conditions are satisfied up to order ℓ.

Then there is a unique $u \in \mathscr{C}^{\ell+2b,\frac{\ell+2b}{2b}}(M_T)$ such that (5.5), (5.6) and (5.7) are satisfied. Moreover the following estimate is valid:

$$\begin{aligned} \|u\|_{\mathscr{C},\ell+2b,M_T} \leqslant C(n,k,\ell,b,\|A\|_{\mathscr{C},\ell,M_T},\|B^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T},T,\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1})\\ \cdot\left(\|u_0\|_{\mathscr{C},\ell+2b,M}+\|F_0\|_{\mathscr{C},\ell,M_T}+\sum_q\|E_0^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T}\right). \end{aligned}$$

5.9 Remark. In Solonnikov's book compatibility conditions are required up to order $\bar{\ell}$, but it is easy to see that the two requirements are equivalent.

The proof of this theorem can be divided in two main parts: in the first one a simpler problem is considered, where the domain is the Euclidean space or the half space and the coefficients are constant (Proposition 5.10 and Proposition 5.11). We will not go through the proof of this part, which is of totally different nature than the material covered in this work, because we can reuse Solonnikov's result without any modification. We just mention the fact that at the core of the proof there is a Laplace transform (in the time direction) and a Fourier transform (in the space directions), after which the problem is rephrased to an ordinary differential equation.

The second part of the proof has a more geometrical nature: it leverages the first part by locally solving the differential equation on the charts of a sufficiently fine bounded atlas and then seeks to glue all the local solutions. Solonnikov employs an argument of this kind to solve systems on domains of $\mathbb{R}^n$. Here the same ideas are ported to the realm of manifold, taking care of tracking the dependency of the resulting estimates on the underlying geometric object (in particular, on their geometric bounds), a concern that does not make sense for domains of $\mathbb{R}^n$.

Let us begin by stating the two theorems that we directly recover from Solonnikov's theory and give essentially for granted.

5.10 Proposition. *Let $\ell \in (0,\infty)\setminus\mathbb{N}$. Suppose that*

$$A \in (\mathbb{R}^n)^{\otimes(2b+k)} \otimes ((\mathbb{R}^n)^*)^{\otimes k}$$
$$F \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{U}_k(\mathbb{R}^n_T)$$

and consider the following system for $x \in \mathbb{R}^n_T$ and $t \in [0,T]$:

$$\partial_t u_{I_k}(x,t) - A_{I_k}^{J_{2b}K_k} \cdot \partial^{2b}_{J_{2b}} u_{K_k}(x,t) = F_{I_k}(x,t).$$

Suppose also that the parabolicity condition is satisfied with a uniform constant δ.

Then there is a unique $u \in \mathring{\mathscr{C}}^{\ell+2b,\frac{\ell+2b}{2b}}\mathscr{U}_k(M_T)$ that satisfies the system. Moreover the following estimate is valid:

$$\|u\|_{\mathscr{C},\ell+2b,\mathbb{R}^n_T} \leqslant C(n,k,\ell,b,|A|,\delta)\cdot\|F\|_{\mathscr{C},\ell,\mathbb{R}^n_T}.$$

Proof. This is proved in [Sol65, § 15], as part of the proof of Proposition 5.11. See next proof for more details. □

5.11 Proposition. *Let $\ell \in (0,\infty)\setminus\mathbb{N}$. Suppose that*

$$A \in (\mathbb{R}^n)^{\otimes(2b+k)} \otimes ((\mathbb{R}^n)^*)^{\otimes k}$$
$$B^q \in (\mathbb{R}^n)^{\otimes(m_q+k)} \otimes ((\mathbb{R}^n)^*)^{\otimes d_q}$$
$$F \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{U}_k(\mathbb{R}^n_T)$$
$$E^q \in \mathring{\mathscr{C}}^{k+2b-m_q,\frac{k+2b-m_q}{2b}}\mathscr{U}_{d_q}(\partial\mathbb{R}^n_T)$$

and consider the following system:

$$\partial_t u_{I_k}(x,t) - A_{I_k}^{J_{2b}K_k}\cdot\partial^{2b}_{J_{2b}} u_{K_k}(x,t) = F_{I_k}(x,t) \qquad x\in\mathbb{R}^n_T,\ t\in[0,T] \tag{5.14}$$
$$(B^q)_{I_{d_q}}^{J_{m_q}K_k}\cdot\partial^{m_q}_{J_{m_q}} u_{K_k}(x,t) = (E^q)_{I_{d_q}}(x,t) \qquad x\in\partial\mathbb{R}^n_T,\ t\in[0,T]. \tag{5.15}$$

Suppose that parabolicity and extended complementary conditions are satisfied with uniform constants δ and δ_1.

Then there is a unique $u \in \mathring{\mathscr{C}}^{\ell+2b,\frac{\ell+2b}{2b}}\mathscr{U}_k(\mathbb{R}^n_T)$ that satisfies the system. Moreover the following estimate is valid:

$$\|u\|_{\mathscr{C},\ell+2b,\mathbb{R}^n_T} \leqslant C(n,k,\ell,b,|A|,|B|,\delta,\delta_1)\cdot\left(\|F\|_{\mathscr{C},\ell,\mathbb{R}^n_T} + \sum_q \|E^q\|_{\mathscr{C},\ell+2b-m_q,\partial\mathbb{R}^n_T}\right).$$

Proof. This is proved in [Sol65, §15] (see in particular the estimate (4.38)) and is essentialy the culmination of the first half of the book. In order to ease the job for the reader who would want to check the details, let us take the opportunity to explain the mismatch between our and Solonnikov's notation, which is largely given by the fact that PDE systems of much greater generality than ours are treated by Solonnikov. There is no apparent fundamental issue that prevents this higher generality to be ported to manifolds, but this is not done here in order to avoid hindering the main target of this work, which is geometric flows.

Choosing the bases for $\hat{\mathscr{L}}_0$ and $\mathscr{B}_0$ induced by the canonical metric of $\mathbb{R}^n$, one can write the system (5.14) and (5.15) in terms of [Sol65, (4.27)], where $\mathscr{L}_0$ and $\mathscr{B}_0$ are constructed as earlier in this chapter and f and Φ are respectively F and $\bigoplus_q E^q$. If there are rows of $\mathscr{B}$ which are redundant and for which the corresponding values of E^q are identical, then they are removed, and the extended complementary condition is replaced by the ordinary complementary condition. Thanks to the linearity of the system, this operation is fully reversible and we are not introducing loss of generality.

In [Sol65, §1] a number of parameters is defined, for which we now give the value in our case. The number m is the number of equations and unknowns and it is equal to our r. The number r is the homogeneity of the system in ∂_t, and it is again equal to our r (Solonnikov's result allow for more general systems having more than one time derivative, or having more intertwined time and space derivatives). Also the number $2b$, the homogeneity in the space derivatives, coincides with our $2b$. For all k we set $s_k = 0$, thus complying with the requirement $\max_k s_k = 0$. The numbers t_j are then forced to $2b$: it cannot be less because for each column of $\mathscr{L}_0$ there is at least one entry which is an operator of order $2b$, otherwise the parabolicity condition is violated; there is also no need to set it bigger than $2b$, since $2b$ is the maximum order that can appear in the system. With these choices of s_k and t_j, Solonnikov's $\mathscr{L}_0$ corresponds with our $\mathscr{L}_0$ and the determinants L corresponds with our determinant L. Therefore Solonnikov's parabolicity condition is verified, since our is.

Concerning the boundary conditions, for each q the corresponding rows of $\mathscr{B}_0$ have order m_q, so the numbers σ_q are forced to $m_q - 2b$. Again, this implies that the operator $\mathscr{B}_0$ corresponds to our definition and, since the polynomial M^+ is defined in the same way from L, the complementary condition too is satisfied. We do not need to take into account the initial conditions, since for this theorem they are all zero.

Finally, Solonnikov's definitions of function spaces on the Euclidean space are at the beginning of §11, and it is easy to see that they are aligned with ours if the standard atlases described in Example 3.9 are in use. Once all this "notation glue" is set up, one easily sees that the function spaces defined by Solonnikov coincide with ours and all the proving machinery is ready to work.

The dependency of the constant C is unfortunately not tracked explicitly by Solonnikov, so in order to ascertain it one would have to go through all the estimates, which we will not do here. We just mention that the dependency on the most meaningful objects (namely, $|A|$, $|B|$, δ and δ_1) is factored by inequalities (2.4) and (2.88), whose proofs are contained respectively in Appendix I and §8–9 of Solonnikov's work. In particular, the bound on δ is used in (I.4) and (I.7), which are very similar; the bound on δ_1 implies that the estimate (2.50) for D is uniform, so that the symbols $\Delta^{(i)}$ and thus the inverse of $\mathscr{U}$ are controlled, which ultimately leads to (2.64). All the other estimates derive from these, so they benefit of the same uniformity. □

The propositions above have four missing features: they work only for Euclidean spaces (and not for manifolds), they only allow constant coefficients, they allow only highest order terms and they allow only zero initial conditions. We will now show how to overcome these limitations.

The first three missing features are essentially solved by localization, since, when data is regular enough, a small region of a manifold look like a Euclidean space and on such a small region the coefficients of a problem are approximately constant and the lower terms are negligible. We will show that for a sufficiently small atlas the perturbation introduced by discarding geometry information, considering the coefficients constant and the lower orders zero are small. Nonzero initial conditions require instead a global modification of the problem and will be treated later.

Let us then fix a manifold with boundary (M^n, g) and the coefficients

$$A \in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$$
$$B^q \in \mathscr{C}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}(\partial M_T)$$

on it. We can define the map $\mathscr{A}$ in the following way:

$$\mathring{\mathscr{C}}^{\ell+2b,\frac{\ell+2b}{2b}}\mathscr{U}_k(M_T) \longrightarrow \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{U}_k(M_T) \times \bigoplus_q \mathring{\mathscr{C}}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}\mathscr{U}_{d_q}(\partial M_T)$$
$$u \longmapsto \mathscr{A}u = (\mathscr{A}_1 u, \mathscr{A}_2 u) := \left(\partial_t u - \sum_{p=0}^{2b} A_{2b-p}\cdot\nabla^p u, \bigoplus_q \sum_{p=0}^{m_q} B^q_{m_q-p}\cdot\nabla^p u\right).$$

Showing Theorem 5.8 for functions with zero initial values amounts to proving that $\mathscr{A}$ is an isomorphism of Banach spaces. Here we do not need to impose compatibility conditions, because if data is in a space of functions with zero initial value, then compatibility conditions are automatically satisfied.

5.12 Proposition. *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Suppose that $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. Then the map $\mathscr{A}$ is linear and*

$$\|\mathscr{A}\| \leqslant C(n,k,b,\ell,\|g\|_{\bar{\ell}+2b+1}, \|A\|_{\mathscr{C},\ell,M_T}, \|B^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T}).$$

Proof. It follows from Propositions 4.14, 4.15, 4.18 and 4.19 together with Proposition 4.27, similarly to Lemma 5.6. □

We begin by constructing an approximate inverse of $\mathscr{A}$, called the *regularizer* and indicated with $\mathscr{R}$. Reasoning as in [Sol65, §16], we will show how to use it to find an actual inverse for $\mathscr{A}$. The regularizer maps

$$\mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}\mathscr{U}_k(M_T) \times \bigoplus_q \mathring{\mathscr{C}}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}\mathscr{U}_{d_q}(\partial M_T) \longrightarrow \mathring{\mathscr{C}}^{\ell+2b,\frac{\ell+2b}{2b}}\mathscr{U}_k(M_T)$$
$$\left(F_0, \bigoplus_q E_0^q\right) \longmapsto v,$$

where the function v will be constructed shortly. The regularizer samples in space the highest order coefficients of the parabolic problem and locally solves the parabolic problem induced in each chart. Let $F_0 \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}(M_T)$ and $E_0^q \in \mathring{\mathscr{C}}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}(\partial M_T)$. For each $i \in I$ of a bounded atlas, let

$$\psi_i(x) = \frac{\Psi_i(x)}{\sum_{j\in I}\Psi_j^2(x)},$$

so that $\Psi_i(x)\cdot\psi_i(x)$ are a partition of the unity. We define $v_i \in \mathring{\mathscr{C}}^{\ell+2b,\frac{\ell+2b}{2b}}(M_T)$ in the following way (here i is an index in the set of local charts I, and does not indicate the coordinate of a tensor):

- If $i \in I^\circ$, then $\hat{U}_i$ does not intersect the boundary: the following problem can be solved in $\mathbb{R}^n$ thanks to Proposition 5.10:

$$\partial_t\tilde{v}_i - \varphi_i^*(A_0(x_i,0))\cdot\partial^{2b}\tilde{v}_i = (\Psi_i\cdot F_0)_{\hat{\varphi}_i}.$$

- If $i \in I^\partial$, then $\hat{U}_i$ intersects the boundary: the following problem can be solved in $\mathbb{R}^n_+$ thanks to Proposition 5.11:

$$\begin{aligned}\partial_t\tilde{v}_i - \varphi_i^*(A_0(x_i,0))\cdot\partial^{2b}\tilde{v}_i &= (\Psi_i\cdot F_0)_{\hat{\varphi}_i}\\ \varphi_i^*(B_0^q(x_i,0))\cdot\partial^{m_q}\tilde{v}_i &= (\Psi_i\cdot E_0^q)_{\hat{\varphi}_i}.\end{aligned}$$

In either case the parabolicity and complementary conditions are satisfied as a result of A and B satisfying them in the first hand.

We set at last $v_i = (\hat{\varphi}_i^{-1})^*\tilde{v}_i$ and

$$v = \mathscr{R}u\left(F_0, \bigoplus_q E_0^q\right) := \sum_{i\in I}\psi_i v_i.$$

5.13 Proposition. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Suppose that $T=\lambda^{2b}\kappa$, with $\lambda\leqslant 1$ and $\kappa\leqslant 1$. Then the map $\mathscr{R}$ is linear and*

$$\|\mathscr{R}\| \leqslant C(n,k,\ell,b,\|A_0\|_{\infty,M_T},\|B_0\|_{\infty,M_T},\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1}).$$

Proof. The linearity is obvious and the estimates descend from Propositions 5.10 and 5.11 and Lemma 4.25. □

The fact that $\mathscr{A}$ and $\mathscr{R}$ are approximate inverses is expressed by the following two propositions.

5.14 Proposition. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell\in(0,\infty)\setminus\mathbb{N}$. Suppose that $T=\lambda^{2b}\kappa$ with $\lambda\leqslant 1$ and $\kappa\leqslant 1$. Then it holds $\mathscr{A}\mathscr{R}=\mathrm{Id}+T$, where T is a linear map and*

$$\|T\| \leqslant C(n,k,\ell,b,\|A\|_{\mathscr{C},\ell,M_T},\|B^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T},\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1})\cdot\left(\lambda^{\hat{\ell}}+\kappa^{\frac{\hat{\ell}}{2b}}\right). \tag{5.16}$$

Proof. Let us fix F_0 and E_0^q; we use the notations $v = \mathscr{R}u(F_0, \bigoplus_q E_0^q)$, v_i and $\tilde{v}_i$ defined above. Then:

$$\begin{aligned}\mathscr{A}_1\mathscr{R}(F_0, \bigoplus_q E_0^q) &= \partial_t v - \sum_{p=0}^{2b} A_{2b-p} \cdot \nabla^p v \\ &= \sum_{i\in I} \psi_i \partial_t [(\hat{\varphi}_i^{-1})^* \tilde{v}_i] - \sum_{i\in I} \psi_i \sum_{p=0}^{2b} A_{2b-p} \cdot \nabla^p [(\hat{\varphi}_i^{-1})^* \tilde{v}_i] \\ &= \sum_{i\in I} \psi_i \cdot \partial_t \tilde{v}_i - \sum_{i\in I} \psi_i \sum_{p=0}^{2b} A_{2b-p} \cdot \nabla^p \tilde{v}_i,\end{aligned}$$

where for simplicity we avoid expliciting local coordinates $\hat{\varphi}_i$. Substituting the definition of $\tilde{v}_i$ and using Lemma 4.8 we have that:

$$\begin{aligned}\mathscr{A}_1\mathscr{R}(F_0, \bigoplus_q E_0^q) &= \sum_{i\in I} \psi_i \left[A_0(x_i, 0) \cdot \partial^{2b} \tilde{v}_i + \tilde{\Psi}_i \cdot F_0 - \sum_{p=0}^{2b} A_{2b-p} \cdot \nabla^p \tilde{v}_i \right] \\ &= F_0 + \sum_{i\in I} \psi_i \cdot (A_0(x_i,0) - A_0) \cdot \partial^{2b} \tilde{v}_i \\ &\quad - \sum_{i\in I} \psi_i \cdot \left[A_0 \cdot \sum_{p=0}^{2b-1} \nabla^p \tilde{v}_i * \Gamma * \dots * \partial^{2b-1}\Gamma + \sum_{p=0}^{2b-1} A_{2b-p} \cdot \nabla^p \tilde{v}_i \right].\end{aligned}$$

We need to prove that the second addend is small in the $\mathring{\mathscr{C}}^{\ell, \frac{\ell}{2b}}(M_T)$ norm, and to do so we estimate individually its components. We can use Lemma 4.25 and Proposition 5.10 to see that

$$\begin{aligned}\|(A_0(x_i,0) - A_0) \cdot \partial^{2b} \tilde{v}_i\|_{\mathscr{C},\ell,M_T} &\leqslant C(n,k,\ell,b,\|A\|_{\mathscr{C},\ell,M_T}) \\ &\qquad \cdot (\kappa^{\frac{\hat{\ell}}{2b}} + \|A_0(x_i,0) - A_0\|_{\infty,M_T}) \cdot \|\partial^{2b}\tilde{v}_i\|_{\mathscr{C},\ell,M_T} \\ &\leqslant C(n,k,\ell,b,\|A\|_{\mathscr{C},\ell,M_T},\delta) \\ &\qquad \cdot (\kappa^{\frac{\hat{\ell}}{2b}} + \|A_0(x_i,0) - A_0\|_{\infty,M_T}) \cdot \|F_0\|_{\mathscr{C},\ell+2b,M_T}.\end{aligned}$$

The only missing item to obtain (5.16) is showing the cancellation effect for A_0 when x ranges inside the chart U_i: if $\ell < 1$, then

$$\begin{aligned}|A_0(x_i,0) - A_0(x,t)| &\leqslant |A_0(x_i,0) - A_0(x_i,t)| + |A_0(x_i,t) - A_0(x,t)| \\ &\leqslant \frac{|A_0(x_i,0) - A_0(x_i,t)|}{t^{\frac{\hat{\ell}}{2b}}} \cdot T^{\frac{\hat{\ell}}{2b}} + \frac{|A_0(x_i,t) - A_0(x,t)|}{\operatorname{dist}(x_i,x)^{\hat{\ell}}} \cdot \lambda^{\hat{\ell}} \\ &\leqslant C \cdot \lambda^{\hat{\ell}} \cdot \|A_0\|_{\mathscr{C},\ell,M_T}.\end{aligned}$$

We have thus proved that

$$\|(A_0(x_i,0) - A_0) \cdot \partial^{2b} \tilde{v}_i\|_{\mathscr{C},\ell,M_T} \leqslant C(n,k,\ell,b,\|A_0\|_{\mathscr{C},\ell,M_T}) \cdot (\kappa^{\frac{\hat{\ell}}{2b}} + \lambda^{\hat{\ell}}) \cdot \|F_0\|_{\mathscr{C},\ell,M_T}.$$

The other therms and those in $\mathscr{A}_2\mathscr{R}(F_0, \bigoplus_q E_0^q)$ are estimated similarly. □

5.15 Proposition. *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in (0,\infty) \setminus \mathbb{N}$. Suppose that $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. Then it holds $\mathscr{R}\mathscr{A} = \mathrm{Id} + W$, where W is a linear map and*

$$\|W\| \leqslant C(n,i,\ell,b,\|A\|_{\mathscr{C},\ell,M_T}, \|B^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T}, \delta, \delta_1, \|g\|_{\bar{\ell}+2b+1}) \cdot \left(\lambda^{\hat{\ell}} + \kappa^{\frac{\hat{\ell}}{2b}}\right).$$

Proof. The proof is analogous to that of Proposition 5.14. □

We are finally ready to prove that the operator $\mathscr{A}$ is invertible.

5.16 Theorem. *Let $\ell \in (0,\infty) \setminus \mathbb{N}$. Suppose that*

$$\begin{aligned} A &\in \mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T) \\ F_0 &\in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}(M_T) \\ B^q &\in \mathscr{C}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}(\partial M_T) \\ E_0^q &\in \mathring{\mathscr{C}}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}(\partial M_T). \end{aligned}$$

Suppose furthermore that parabolicity and complementary conditions are uniformly satisfied with constants δ and δ_1.

Then there are

$$\begin{aligned} &\lambda(n,k,\ell,b,\|A\|_{\mathscr{C},\ell,M_T},\|B^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T},\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1}) \\ &\kappa(n,k,\ell,b,\|A\|_{\mathscr{C},\ell,M_T},\|B^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T},\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1}) \end{aligned}$$

such that, if $T \leqslant \tilde{T} := \lambda^{2b}\kappa$, then there is a unique $u \in \mathring{\mathscr{C}}^{\ell+2b,\frac{\ell+2b}{2b}}(M_T)$ such that (5.5) and (5.6) are satified. Moreover the following estimate is valid:

$$\begin{aligned} \|u\|_{\mathscr{C},\ell+2b,M_T} \leqslant C(&n,k,\ell,b,\|A_0\|_{\infty,M_T},\|B_0^q\|_{\infty,\partial M_T},\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1}) \\ &\cdot\left(\|F_0\|_{\mathscr{C},\ell,M_T} + \sum_q \|E_0^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T}\right). \end{aligned}$$

Proof. Let us take λ and κ such that $\|T\|$ and $\|W\|$ in Proposition 5.14 and 5.15 are smaller than $\frac{1}{2}$; then suppose that $T \leqslant \tilde{T} := \lambda^{2b}\kappa$. It follows $\mathscr{A}\mathscr{R} = \mathrm{Id}+T$ and $\mathscr{R}\mathscr{A} = \mathrm{Id}+W$ are invertible, with inverses whose norm is not greater than 2. In particular:

$$\begin{aligned} \mathscr{A}\mathscr{R}(\mathrm{Id}+T)^{-1} &= \mathrm{Id} \\ (\mathrm{Id}+W)^{-1}\mathscr{R}\mathscr{A} &= \mathrm{Id}\,. \end{aligned}$$

This means that $\mathscr{A}$ possesses both a left and a right bounded inverse map; then the two inverses actually have to coincide and, thanks to Proposition 5.13, satisfy the estimate

$$\begin{aligned} \|\mathscr{A}^{-1}\| &= \|\mathscr{R}(\mathrm{Id}+T)^{-1}\| = \|(\mathrm{Id}+W)^{-1}\mathscr{R}\| \\ &\leqslant 2\|\mathscr{R}\| = C(n,k,\ell,b,\|A_0\|_\infty,\|B_0\|_\infty,\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1}). \end{aligned}$$ □

Thus we have proved that $\mathscr{A}$ is invertible, meaning that now we know how to solve linear systems with zero initial data, on a time slice whose duration can be estimated knowing the parameters of the system and the norms of the coefficients. We show now how to use these results to finally prove Theorem 5.8.

Proof of Theorem 5.8. Let us fix $\tilde{T}$, λ and κ given by Theorem 5.16; if $T < \tilde{T}$, we take instead $\tilde{T} = T$, without any loss of generality. We begin by constructing a solution on $M_{\tilde{T}}$; if $\tilde{T} < T$, we then show how to repeat the construction many times, until a solution on the whole M_T has been found.

Let us consider the functions w_j defined by (5.13). By Lemma 5.6 and Proposition 4.21 we can find $w \in \mathscr{C}^{\ell+2b,\frac{\ell+2b}{2b}}(M_{\tilde{T}})$ such that for $j = 0, \dots, \left[\frac{\ell}{2b}\right] + 1$

$$\partial_t^j w|_{t=0} = w_j \tag{5.17}$$

and

$$\|w\|_{\mathscr{C},\ell+2b,M_{\tilde{T}}} \leqslant C(n,k,\ell,b,\|g\|_{\bar{\ell}+2b+1},\|A\|_{\mathscr{C},\ell,M_{\tilde{T}}}) \cdot \left(\|u_0\|_{\mathscr{C},\ell+2b,M} + \|F_0\|_{\mathscr{C},\ell,M_{\tilde{T}}}\right). \tag{5.18}$$

For u to satisfy (5.5), (5.6) and (5.7) it is sufficient that $\sigma = u - w$ satisfies

$$\begin{aligned}
\partial_t \sigma(x,t) - \sum_{p=0}^{2b} A_{2b-p}(x,t) \cdot \nabla^p \sigma(x,t) &= \hat{F}(x,t) \\
\sum_{p=0}^{m_q} B^q_{m_q-p}(x,t) \cdot \nabla^p \sigma(x,t) &= \hat{E}^q(x,t) \\
\sigma(x,0) &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\hat{F}(x,t) &:= F_0(x,t) - \partial_t w(x,t) - \sum_{p=0}^{2b} A_{2b-p}(x,t) \cdot \nabla^p w(x,t) \\
\hat{E}^q(x,t) &:= E_0^q(x,t) - \sum_{p=0}^{m_q} B^q_{m_q-p}(x,t) \cdot \nabla^p w(x,t).
\end{aligned}$$

In the usual way we obtain the estimates

$$\begin{aligned}
\|\hat{F}\|_{\mathscr{C},\ell,M_{\tilde{T}}} &\leqslant C(n,k,\ell,b,\|g\|_{\bar{\ell}+2b+1},\|A\|_{\mathscr{C},\ell,M_{\tilde{T}}}) \cdot \left(\|u_0\|_{\mathscr{C},\ell+2b,M} + \|F_0\|_{\mathscr{C},\ell,M_{\tilde{T}}}\right) \\
\|\hat{E}^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_{\tilde{T}}} &\leqslant C(n,k,\ell,b,\|g\|_{\bar{\ell}+2b+1},\|A\|_{\mathscr{C},\ell,M_{\tilde{T}}},\|B^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_{\tilde{T}}}) \\
&\quad \cdot \left(\|u_0\|_{\mathscr{C},\ell+2b,M} + \|F_0\|_{\mathscr{C},\ell,M_{\tilde{T}}} + \|E_0^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_{\tilde{T}}}\right).
\end{aligned}$$

Also, we can verify that $\hat{F} \in \mathring{\mathscr{C}}^{\ell,\frac{\ell}{2b}}(M_{\tilde{T}})$ and $\hat{E}^q \in \mathring{\mathscr{C}}^{\ell+2b-m_q,\frac{\ell+2b-m_q}{2b}}(\partial M_{\tilde{T}})$ by iteratively differentiating in the time direction, expanding with (5.17) and using (5.12), (5.13) and Proposition 4.23.

By Theorem 5.16 we can then find a unique solution $\sigma \in \mathring{\mathscr{C}}^{\ell+2b,\frac{\ell+2b}{2b}}(M_{\tilde{T}})$, satisfying the estimate

$$\begin{aligned}
\|\sigma\|_{\mathscr{C},\ell+2b,M_{\tilde{T}}} &\leqslant C(n,k,\ell,b,\|A_0\|_{\infty,M_{\tilde{T}}},\|B_0^q\|_{\infty,\partial M_{\tilde{T}}},\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1}) \\
&\quad \cdot \left(\|\hat{F}\|_{\mathscr{C},\ell,M_T} + \sum_q \|\hat{E}^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_T}\right) \\
&\leqslant C(n,k,\ell,b,\|A_0\|_{\mathscr{C},\ell,M_{\tilde{T}}},\|B_0^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_{\tilde{T}}},\delta,\delta_1,\|g\|_{\bar{\ell}+2b+1}) \\
&\quad \cdot \left(\|u_0\|_{\mathscr{C},\ell+2b,M} + \|F_0\|_{\mathscr{C},\ell,M_{\tilde{T}}} + \sum_q \|E_0^q\|_{\mathscr{C},\ell+2b-m_q,\partial M_{\tilde{T}}}\right).
\end{aligned}$$

The function $u = \sigma + w$ is then a solution of (5.5), (5.6) and (5.7) on $M_{\tilde{T}}$.

In order to extend the solution u to the whole interval $[0,T]$, we need to "restart" it at the time $\tilde{T}$; actually, we restart it at $\frac{\tilde{T}}{2}$, to ensure uniformity of estimates. The coefficients and data A, B^q, F_0 and E_0^q can be translated back in time by $\frac{\tilde{T}}{2}$, so they are defined for $t \in [0, T - \frac{\tilde{T}}{2}]$ and clearly retain the same estimates on the norms. Also the solution u already found can be translated back in time, so that it is defined for $t \in [0, \frac{\tilde{T}}{2}]$; its trace at $t = 0$ is then set as the new u_0, using Proposition 4.19.

The first part of the proof can thus be repeated: the new solution u can be translated forward in time by $\frac{\tilde{T}}{2}$, and by uniqueness it fits with the old one for times in $[\frac{\tilde{T}}{2}, \tilde{T}]$. Since the guaranteed existence time $\tilde{T}$ does not depend on the initial data, we have obtained a solution of the original system defined on $[0, \frac{3}{2}\tilde{T}]$. Repeating the same argument for at most $2\frac{T}{\tilde{T}}$ times allows to cover the whole interval $[0, T]$ and prove the theorem. □

5.4 Linear systems in Sobolev-Slobodeckij spaces

Thanks to the theory developed in Chapter 4, all the theory constructed in Section 5.3 is not difficult to adapt to the case of Sobolev-Slobodeckij spaces. First of all, let us check how much regularity we need to have well defined compatibility conditions (5.12) and (5.13).

5.17 Lemma. *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$, $p \in (1, \infty)$ and $\alpha \in (0,1)$ such that for all q the numbers $\frac{\ell - m_q}{2b} - \frac{\frac{1}{2b}+1}{p}$ are not integers. Suppose that*

$$A \in \mathscr{C}^{\ell+\alpha, \frac{\ell+\alpha}{2b}}(M_T)$$
$$F_0 \in W^{(\ell, \frac{\ell}{2b}), p}(M_T)$$
$$u_0 \in W^{\ell+2b-\frac{2b}{p}, p}(M).$$

Then w_j is well defined for $j = 0, \dots, \frac{\ell}{2b}$, it belongs to $W^{\ell+2b(1-j)-\frac{2b}{p}, p}(M)$ and the following estimate is valid:

$$\|w_j\|_{W, \ell+2b(1-j)-\frac{2b}{p}, p, M} \leqslant C(n, k, b, \ell, p, \lambda, T, \|g\|_{\ell+2b}, \|A\|_{\mathscr{C}, \ell+\alpha, M_T}) \cdot (\|u_0\|_{W, \ell+2b-\frac{2b}{p}, p, M} + \|F_0\|_{W, \ell, p, M_T}).$$

Proof. See the proof of Lemma 5.6. □

5.18 Lemma. *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$, $p \in (1, \infty)$ and $\alpha \in (0,1)$ such that for all q the numbers $\frac{\ell - m_q}{2b} - \frac{\frac{1}{2b}+1}{p}$ are not integers. Suppose that*

$$A \in \mathscr{C}^{\ell+\alpha, \frac{\ell+\alpha}{2b}}(M_T)$$
$$F_0 \in W^{(\ell, \frac{\ell}{2b}), p}(M_T)$$
$$B^q \in \mathscr{C}^{\ell+2b-m_q-\frac{1}{p}+\alpha, \frac{\ell+2b-m_q-\frac{1}{p}+\alpha}{2b}}(\partial M_T)$$
$$E_0^q \in W^{(\ell+2b-m_q-\frac{1}{p}, \frac{\ell+2b-m_q-\frac{1}{p}}{2b}), p}(\partial M_T)$$
$$u_0 \in W^{\ell+2b-\frac{2b}{p}, p}(M).$$

Then the two sides of (5.13) are well defined for $j = 0, \dots, \left[\frac{\ell - m_q - \frac{2b+1}{p}}{2b}\right] + 1$ and they belong to the space $W^{\ell+2b(1-j)-m_q-\frac{2b+1}{p}, p}(\partial M)$. In particular, it is meaningful to require compatibility conditions for orders up to $\ell - \frac{2b+1}{p}$.

Proof. See the proof of Lemma 5.7. □

In Sobolev-Slobodeckij spaces the equivalent of Theorem 5.8 is the following.

5.19 Theorem (cfr. [Sol65, Theorem 5.4])**.** *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$, $p \in (1,\infty)$ and $\alpha \in (0,1)$ such that for all q the numbers $\frac{\ell - m_q}{2b} - \frac{\frac{1}{2b}+1}{p}$ are not integers. Suppose that*

$$
\begin{aligned}
A &\in \mathscr{C}^{\ell+\alpha, \frac{\ell+\alpha}{2b}}(M_T)\\
F_0 &\in W^{(\ell, \frac{\ell}{2b}), p}(M_T)\\
B^q &\in \mathscr{C}^{\ell+2b-m_q-\frac{1}{p}+\alpha, \frac{\ell+2b-m_q-\frac{1}{p}+\alpha}{2b}}(\partial M_T)\\
E_0^q &\in W^{(\ell+2b-m_q-\frac{1}{p}, \frac{\ell+2b-m_q-\frac{1}{p}}{2b}), p}(\partial M_T)\\
u_0 &\in W^{\ell+2b-\frac{2b}{p}, p}(M).
\end{aligned}
$$

Suppose furthermore that parabolicity and extended complementary conditions are uniformly satisfied with constants δ and δ_1 and compatibility conditions are satisfied up to order $\ell - \frac{2b+1}{p}$.

Then there is a unique $u \in W^{(\ell+2b, \frac{\ell+2b}{2b}), p}(M_T)$ such that (5.5), (5.6) and (5.7) are satisfied. Moreover the following estimate is valid:

$$
\begin{aligned}
\|u\|_{W,\ell+2b,p,M_T} \leqslant{}& C(n,k,\ell,b,\|A\|_{\mathscr{C},\ell+\alpha,M_T}, \|B^q\|_{\mathscr{C},\ell+2b-m_q-\frac{1}{p}+\alpha,\partial M_T}, T, \delta, \delta_1, \|g\|_{\ell+2b})\\
&\cdot\left(\|u_0\|_{W,\ell+2b-\frac{2b}{p},p,M} + \|F_0\|_{W,\ell,p,M_T} + \sum_q \|E_0^q\|_{W,\ell+2b-m_q-\frac{1}{p},p,\partial M_T}\right).
\end{aligned}
$$

For spaces of functions with zero initial value, the equivalent of Theorem 5.16 is the following.

5.20 Theorem. *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $\ell \in 2b\mathbb{N}$, $p \in (1,\infty)$ and $\alpha \in (0,1)$ such that for all q the numbers $\frac{\ell-m_q}{2b} - \frac{\frac{1}{2b}+1}{p}$ are not integers. Suppose that*

$$
\begin{aligned}
A &\in \mathscr{C}^{\ell+\alpha, \frac{\ell+\alpha}{2b}}(M_T)\\
F_0 &\in \mathring{W}^{(\ell, \frac{\ell}{2b}), p}(M_T)\\
B^q &\in \mathscr{C}^{\ell+2b-m_q-\frac{1}{p}+\alpha, \frac{\ell+2b-m_q-\frac{1}{p}+\alpha}{2b}}(\partial M_T)\\
E_0^q &\in \mathring{W}^{(\ell+2b-m_q-\frac{1}{p}, \frac{\ell+2b-m_q-\frac{1}{p}}{2b}), p}(\partial M_T).
\end{aligned}
$$

Suppose furthermore that parabolicity and extended complementary conditions are uniformly satisfied with constants δ and δ_1.

Then there are

$$
\begin{aligned}
&\lambda(n,k,\ell,p,b,\|A\|_{\mathscr{C},\ell+\alpha,M_T}, \|B^q\|_{\mathscr{C},\ell+2b-m_q-\frac{1}{p}+\alpha,\partial M_T}, \delta, \delta_1, \|g\|_{\ell+2b})\\
&\kappa(n,k,\ell,p,b,\|A\|_{\mathscr{C},\ell+\alpha,M_T}, \|B^q\|_{\mathscr{C},\ell+2b-m_q-\frac{1}{p}+\alpha,\partial M_T}, \delta, \delta_1, \|g\|_{\ell+2b})
\end{aligned}
$$

such that, if $T \leqslant \tilde{T} := \lambda^{2b}\kappa$, then there is a unique $u \in \mathring{W}^{(\ell+2b,\frac{\ell+2b}{2b}),p}(M_T)$ such that (5.5) *and* (5.6) *are satisfied. Moreover the following estimate is valid:*

$$\begin{aligned}\|u\|_{W,\ell+2b,p,M_T} &\leqslant C(n,k,\ell,b,p,\|A_0\|_{\infty,M_T},\|B_0^q\|_{\infty,\partial M_T},\delta,\delta_1,\|g\|_{\ell+2b}) \\ &\cdot\left(\|F_0\|_{W,\ell,p,M_T} + \sum_q \|E_0^q\|_{W,\ell+2b-m_q-\frac{1}{p},p,\partial M_T}\right).\end{aligned}$$

In order to prove the two theorems, we first need to discuss the extension of Propositions 5.10 and 5.11 and the definition of the two operators $\mathscr{A}$ and $\mathscr{R}$.

5.21 Proposition. *Let $\ell \in 2b\mathbb{N}$, $p \in (1,\infty)$ and $\alpha \in (0,1)$ such that for all q the numbers $\frac{\ell-m_q}{2b} - \frac{\frac{1}{2b}+1}{p}$ are not integers. Suppose that in Propositions 5.10 and 5.11 the hypotheses are strenghtened to*

$$\begin{aligned} F &\in \mathring{\mathscr{C}}^{\ell+\alpha,\frac{\ell+\alpha}{2b}}(M_T)\cap \mathring{W}^{(\ell,\frac{\ell}{2b}),p}(M_T)\\ E^q &\in \mathring{\mathscr{C}}^{\ell+2b-m_q+\alpha,\frac{\ell+2b-m_q+\alpha}{2b}}(\partial M_T)\cap \mathring{W}^{(\ell+2b-m_q-\frac{1}{p},\frac{\ell+2b-m_q-\frac{1}{p}}{2b}),p}(\partial M_T).\end{aligned}$$

Then $u \in \mathring{\mathscr{C}}^{\ell+2b+\alpha,\frac{\ell+2b+\alpha}{2b}}(M_T)\cap \mathring{W}^{(\ell+2b,\frac{\ell+2b}{2b}),p}(M_T)$ and

$$\|u\|_{W,\ell+2b,p,M_T} \leqslant C(n,k,\ell,p,b,|A|,|B|,\delta,\delta_1)\cdot\left(\|F\|_{W,\ell,p,M_T} + \sum_q \|E^q\|_{W,\ell+2b-m_q-\frac{1}{p},p,\partial M_T}\right).$$

Proof. This is proved in [Sol65, §21], see in particular inequalities (5.27) and (5.29). □

The operators $\mathscr{A}$ and $\mathscr{R}$, defined as for Hölder spaces, can be constructed between the spaces:

$$\begin{array}{c}\left(\mathring{\mathscr{C}}^{\ell+2b+\alpha,\frac{\ell+2b+\alpha}{2b}}\cap \mathring{W}^{(\ell+2b,\frac{\ell+2b}{2b}),p}\right)(M_T)\\ \Big\downarrow \mathscr{A} \qquad\qquad \Big\uparrow \mathscr{R}\\ \left(\mathring{\mathscr{C}}^{\ell+\alpha,\frac{\ell+\alpha}{2b}}\cap \mathring{W}^{(\ell,\frac{\ell}{2b}),p}\right)(M_T)\times\bigoplus_q\left(\mathring{\mathscr{C}}^{\ell+2b-m_q+\alpha,\frac{\ell+2b-m_q+\alpha}{2b}}\cap \mathring{W}^{(\ell+2b-m_q-\frac{1}{p},\frac{\ell+2b-m_q-\frac{1}{p}}{2b}),p}\right)(\partial M_T).\end{array}$$

From now on, all the proofs are just adaptation of the proofs in Section 5.3, and it would be pointless to repeat all of them. We just mention the inequalities that can be verified on $\mathscr{A}$ and $\mathscr{R}$.

5.22 Proposition. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Suppose that $T = \lambda^{2b}\kappa$ with $\lambda \leqslant 1$ and $\kappa \leqslant 1$. Then $\mathscr{A}\mathscr{R} = \mathrm{Id}+T$ and $\mathscr{R}\mathscr{A} = \mathrm{Id}+W$ with*

$$\begin{aligned}\|\mathscr{A}\| &\leqslant C(n,k,\ell,p,b,\|A\|_{\mathscr{C},\ell+\alpha,M_T},\|B^q\|_{\mathscr{C},\ell+2b-m_q+\alpha,\partial M_T},\|g\|_{\ell+2b+1})\\ \|\mathscr{R}\| &\leqslant C(n,k,\ell,p,b,\|A_0\|_{\infty},\|B_0\|_{\infty},\delta,\delta_1,\|g\|_{\ell+2b+1})\\ \|T\| &\leqslant C(n,k,\ell,p,b,\|A\|_{\mathscr{C},\ell+\alpha,M_T},\|B^q\|_{\mathscr{C},\ell+2b-m_q+\alpha,\partial M_T},\delta,\delta_1,\|g\|_{\ell+2b+1})\cdot\left(\lambda^\alpha+\kappa^{\frac{\alpha}{2b}}\right)\\ \|W\| &\leqslant C(n,k,\ell,p,b,\|A\|_{\mathscr{C},\ell+\alpha,M_T},\|B^q\|_{\mathscr{C},\ell+2b-m_q+\alpha,\partial M_T},\delta,\delta_1,\|g\|_{\ell+2b+1})\cdot\left(\lambda^\alpha+\kappa^{\frac{\alpha}{2b}}\right).\end{aligned}$$

Chapter 6

Semilinear parabolic systems

6.1 Semilinear parabolic systems

We can now move on to consider systems in the general form (2.1), (2.2) and (2.3), which we recall here:

$$\begin{aligned} \partial_t u(x,t) - A_u(x,t)\cdot\nabla^{2b}u(x,t) &= F_u(x,t) && (x,t)\in \mathring{M}_T \\ u(x,0) &= u_0(x) && x\in M \\ B^q_u(x,t)\cdot\nabla^{m_q}u(x,t) &= E^q_u(x,t) && (x,t)\in\partial M_T,\ q=1,\dots,\nu. \end{aligned} \tag{6.1}$$

We also recall that the unknown u is taken in the space $\mathscr{T}_k(M_T)$ and that the condition $0\leqslant m_q<2b$ is required for all boundary conditions.

First we will prove an existence and uniqueness theorem the Sobolev-Slobodeckij space $W^{(2b,1),p}(M_T)$, which is the less regular space we have available that is sufficiently regular to express the problem. Minimal compatibility conditions are required here.

6.1 Theorem. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $p\in(2(2b+n),\infty)$. Suppose that A, F, B^q and F^q are smooth bundle maps of the form defined in Section 2.2 and u_0 is smooth. Suppose that A_{u_0} satisfies uniform parabolicity conditions with constant δ, that A_{u_0} and $B^q_{u_0}$ satisfy uniform complementary conditions with constant δ_1 and that $B^q_{u_0}(x,0)\cdot\nabla^{m_q}u_0=E^q_{u_0}(x,0)$ for all q and $x\in\partial M$.*

Then for every K there is

$$\begin{aligned} T = T(&n,k,b,K,p,\|A\|_{\mathscr{C},\infty,M_T},\|F\|_{\mathscr{C},\infty,M_T},\|B^q\|_{\mathscr{C},\infty,\partial M_T},\|E^q\|_{\mathscr{C},\infty,\partial M_T},\\ &\delta,\delta_1,\|g\|_{2b},\mathrm{Vol}(M,g),\mathrm{Vol}(\partial M,g)) \end{aligned}$$

such that (6.1) has a unique solution $u\in W^{(2b,1),p}(M_T)$ in M_T and

$$\|u-u_0\|_{W,2b,p,M_T}\leqslant K.$$

Then we will try to gain more regularity. To have regularity on the whole manifold M_T we need stronger compatibility conditions.

6.2 Theorem. *Under the same hypotheses as Theorem 6.1, suppose that, for $\ell\in(0,\infty)\setminus\mathbb{N}$, compatibility conditions of order $\bar\ell$ are satisfied.*

Then the solution u belongs to the space $\mathscr{C}^{\ell,\frac{\ell}{2b}}(M_T)$. In particular, if compatibility conditions are satisfied for every order, then $u\in\mathscr{C}^\infty(M_T)$.

If we cannot guarantee stronger compatibility conditions, then we are unable to acheive higher regularity on $\partial M \times \{0\}$, but we still have smoothness everywhere else thanks to the regularizing behaviour of parabolic equations.

6.3 Theorem. *Under the same hypotheses as Theorem 6.1, the solution u belongs to the space $\mathscr{C}^\infty(\mathring{M}_T)$ and to the space $\mathscr{C}^\infty(M \times (0,T])$.*

Following [Gia13], we show that the solution u can be seen as the fixed point of a contracting iteration arising from the linearization of system (6.1). Existence and uniqueness of u then descend from Banach's fixed point theorem. Once the existence is obtained, it is easy to prove regularity by bootstrapping.

Let us extend u_0 on M_T by defining $u_0(x,t) = u_0(x)$. We take a function w and seek the solution u to the problem:

$$\begin{aligned} \partial_t u - A_{u_0} \cdot \nabla^{2b} u &= F_w - (A_{u_0} - A_w) \cdot \nabla^{2b} w \\ u|_{t=0} &= u_0 \\ B^q_{u_0} \cdot \nabla^{m_q} u &= E^q_w + (B^q_{u_0} - B^q_w) \cdot \nabla^{m_q} w \qquad q = 1, \dots, \nu. \end{aligned}$$

Furthermore we substitute $\sigma = u - u_0$ and $\tau = w - u_0$:

$$\begin{aligned} \partial_t \sigma - A_{u_0} \cdot \nabla^{2b} \sigma &= (F_{\tau+u_0} + A_{\tau+u_0} \cdot \nabla^{2b} u_0) - (A_{u_0} - A_{\tau+u_0}) \cdot \nabla^{2b} \tau =: \hat{F}_\tau \\ \sigma|_{t=0} &= 0 \\ B^q_{u_0} \cdot \nabla^{m_q} \sigma &= (E^q_{\tau+u_0} - B^q_{\tau+u_0} \cdot \nabla^{m_q} u_0) + (B^q_{u_0} - B^q_{\tau+u_0}) \cdot \nabla^{m_q} \tau =: \hat{E}^q_\tau \qquad q = 1, \dots, \nu. \end{aligned} \tag{6.2}$$

Solving system (6.2) gives a map $\Phi\colon \tau \mapsto \sigma$ such $\Phi(\tau) = \tau$ if and only if $u = u_0 + \tau$ is a solution of (6.1). So, for $T, K \in (0, \infty)$, we consider the space

$$W^K_T = \left\{ \sigma \in W^{(2b,1),p}\mathscr{U}_k(M_T) \,\middle|\, \sigma|_{t=0} = 0, \|\sigma\|_{W,2b,p,M_T} \leqslant K \right\},$$

which is closed in the complete space $W^{(2b,1),p}(M_T)$, so it is complete. We have to show that if $\tau \in W^K_T$, then $\Phi(\tau) \in W^K_T$ and the mapping has a Lipschitz constant strictly smaller than 1. In order to estimate the Lipschitz constant, we break the mapping in two pieces: $\tau \mapsto (\hat{F}_\tau, \hat{B}^q_\tau) \mapsto \sigma$.

6.4 Remark. Adding and subtracting A_{u_0} is useful to completely separate u and w in the fixed point problem. We want u to appear only on the left (as the solution of the linear problem) and w to appear only on the right (so that it does not impact on the linear problem coefficients). Moreover, it follows from Proposition 4.38 that $W^K_T \subset \mathring{W}^{(2b,1),p}(M_T)$, so we can use 4.42, which is essential for the proofs in this chapter.

Let us now introduce some technical lemmata that will constitute the backbone of the proofs in this chapter. We will use the letters G and m as in Section 2.3. If $u \in W^{(2b,1),p}(M_T)$, then by Proposition 4.43 it has at least $2b-1$ continuous space derivatives. Let us define, for $m \in [1, 2b] \cap \mathbb{N}$,

$$H^{m-1}_u(x,t) = (x, t, u(x,t), \dots, \nabla^{m-1} u(x,t)).$$

If $u = \tau + u_0$, with $\tau \in W^K_T$ and $u_0 \in \mathscr{C}^{2b-1}(M)$, then H^{m-1}_u contains the space derivatives of u_0 and τ up to at most order $2b-1$, which are continuous and bounded respectively by $\|u_0\|_{\mathscr{C}^{2b-1}}$ and $C(n,k,b,p,K,\|g\|_{2b},T)$ thanks to Corollary 4.47. This means that

$$R^{m-1}_{u_0} := \bigcup_{\tau \in W^K_T} \operatorname{Ran}_{(x,t)\in M_T} H^{m-1}_{u_0+\tau}(x,t),$$

is compact (since M_T is compact) and contained in a ball of radius

$$C(n,k,b,p,K,T,\|u_0\|_{\mathscr{C},2b-1,M},\|g\|_{2b}).$$

When G is B^q or E^q, then the definition of $R^{m-1}_{u_0}$ is modified in the following obvious way, retaining the same properties:

$$R^{m-1}_{u_0} := \bigcup_{\tau\in W^K_T} \operatorname{Ran}_{(x,t)\in\partial M_T} H^{m-1}_{u_0+\tau}(x,t).$$

The definition that applies will be clear every time from the context.

Let us recall that we indicate with $\zeta(T)$ any function $\zeta\colon (0,\infty)\to(0,\infty)$ which is increasing and infinitesimal at zero.

6.5 Lemma. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $T,K\in(0,\infty)$, $u_0\in\mathscr{C}^{2b-1}(M)$, $\tau\in W^K_T$ and $j\in\mathbb{N}\cap[0,2b-m]$. Then $\nabla^j G_{u_0}$ and $\nabla^j G_{u_0+\tau}$ are continuous and bounded on M_T and the following inequalities hold:*

$$\|\nabla^j G_{u_0}\|_\infty \leqslant C(n,k,b,\|u_0\|_{\mathscr{C},2b-1,M},\|G\|_{\mathscr{C}^{j+1}(R^{m-1}_{u_0})},\|g\|_{2b})$$
$$\|\nabla^j G_{u_0+\tau}\|_\infty \leqslant C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|G\|_{\mathscr{C}^{j+1}(R^{m-1}_{u_0})},\|g\|_{2b})$$
$$\|\nabla^j (G_{u_0+\tau}-G_{u_0})\|_\infty \leqslant \zeta(T)\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|G\|_{\mathscr{C}^{j+1}(R^{m-1}_{u_0})},\|g\|_{2b}).$$

Proof. It holds $G_u = G\circ H^{m-1}_u(x,t)$. We make use of the generalized Faà di Bruno's formula, for which a statement can be found in [CS96, Theorem 2.1]. Although the general form is rather complicated, for our use it will be enough to note that

$$\begin{aligned}\nabla^j G_u(x,t) &= P(G,H^{m-1}_u)\\ &= P(G(H^{m-1}_u(x,t)),\dots,(\nabla^j G)(H^{m-1}_u(x,t)),H^{m-1}_u(x,t),\dots,\nabla^j H^{m-1}_u(x,t)),\end{aligned} \tag{6.3}$$

where P is polynomial whose structure can be very complicated, but is fixed once we know n, k and the structure of G. In particular, P is uniformly continuous over compact sets.

In equation (6.3), instances of G through $\nabla^j G$ are evaluated over subsets of $R^{m-1}_{u_0}$ when $u=\tau+u_0$. Therefore they are Lipschitz with constant $\|G\|_{\mathscr{C}^{k+1}(R^{m-1}_{u_0})}$. Using again Corollary 4.47, we see that the polynomial P is globally evaluated on a compact set, thus it is Lipschitz continuous. The first two inequalities follow immediately.

The third inequality is proved in a similar way, using the additional fact that

$$H_{u_0+\tau}-H_{u_0} = (0,0,\tau(x,t),\dots,\nabla^m\tau(x,t))$$

and that, again by Corollary 4.47, derivatives of τ are bounded by $\zeta(T)\cdot C(n,k,b,p,K,T,\|g\|_{2b})$. Lipschitz continuity of P and G gives again the thesis. □

6.6 Lemma. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $T,K\in(0,\infty)$, $u_0\in\mathscr{C}^{2b-1}(M)$, $\tau_1,\tau_2\in W^K_T$ and $j\in\mathbb{N}\cap[0,2b-m]$. Then*

$$\begin{aligned}\|\nabla^k(G_{u_0+\tau_1}-G_{u_0+\tau_2})\|_\infty \leqslant{}& C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|G\|_{\mathscr{C}^{j+1}(R^{m-1}_{u_0})},\|g\|_{2b})\\ &\cdot\zeta(T)\cdot\|\tau_1-\tau_2\|_{W,2b,p,M_T}.\end{aligned}$$

Proof. The proof goes as for Lemma 6.5. With the same notations, we have to estimate $G\circ(H_{u_0+\tau_1}-H_{u_0+\tau_2})$. Since:

$$H^{m-1}_{\tau_1+u_0}-H^{m-1}_{\tau_2+u_0}=(0,0,(\tau_1-\tau_2)(x,t),\dots,\nabla^m(\tau_1-\tau_2)(x,t)),$$

the magnitudes of its derivatives is bounded by

$$\zeta(T)\cdot\|\tau_1-\tau_2\|_{W,2b,p,M_T}\cdot C(n,k,b,p,K,\|g\|_{2b}).$$

As usual the Lipschitz continuity of P and G implies the conclusion. □

6.7 Lemma. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $T,K\in(0,\infty)$ and $\tau\in W^K_T$. Then*

$$\|\hat{F}_\tau\|_{p,M_T}\leqslant\zeta(T)\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b,M},\|A\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|F\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b},\mathrm{Vol}(M,g)).$$

Proof. By Lemma 6.5:

$$\begin{aligned}
\|A_{\tau+u_0}\|_{\infty,M_T}&\leqslant C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|A\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b})\\
\|A_{u_0}-A_{\tau+u_0}\|_{\infty,M_T}&\leqslant\zeta(T)\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|A\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b})\\
\|F_{\tau+u_0}\|_{\infty,M_T}&\leqslant C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|F\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b}).
\end{aligned}$$

Also, by definition $\|\nabla^{2b}\tau\|_{p,M_T}\leqslant K$. So we have that

$$\begin{aligned}
\|F_{\tau+u_0}+A_{\tau+u_0}\cdot\nabla^{2b}u_0\|_{p,M_T}&\leqslant(T\cdot\mathrm{Vol}(M,g))^{\frac{1}{p}}\cdot\|F_{\tau+u_0}+A_{\tau+u_0}\cdot\nabla^{2b}u_0\|_{\infty,M_T}\\
\|(A_{u_0}-A_{\tau+u_0})\cdot\nabla^{2b}\tau\|_{p,M_T}&\leqslant\|A_{u_0}-A_{\tau+u_0}\|_{\infty,M_T}\cdot\|\nabla^{2b}\tau\|_{p,M_T}
\end{aligned}$$

and the thesis follows. □

6.8 Lemma. *Let (M^n,g) be a compact Riemannian manifold with boundary and bounded geometry. Let $T,K\in(0,\infty)$ and $\tau_1,\tau_2\in W^K_T$. Then*

$$\begin{aligned}
\|\hat{F}_{\tau_1}-\hat{F}_{\tau_2}\|_{p,M_T}\leqslant{}& C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b,M},\|A\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|F\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b},\mathrm{Vol}(M,g))\\
&\cdot\zeta(T)\cdot\|\tau_1-\tau_2\|_{W,2b,p,M_T}.
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
\hat{F}_{\tau_1}-\hat{F}_{\tau_2}&=(F_{\tau_1+u_0}-F_{\tau_2+u_0})+(A_{\tau_1+u_0}-A_{\tau_2+u_0})\cdot\nabla^{2b}u_0\\
&\quad-(A_{u_0}-A_{\tau_1-u_0})\cdot\nabla^{2b}\tau_1+(A_{u_0}-A_{\tau_2-u_0})\cdot\nabla^{2b}\tau_2\\
&=(F_{\tau_1+u_0}-F_{\tau_2+u_0})+(A_{\tau_1+u_0}-A_{\tau_2+u_0})\cdot\nabla^{2b}u_0\\
&\quad+(A_{\tau_1+u_0}-A_{u_0})\cdot\nabla^{2b}(\tau_1-\tau_2)+(A_{\tau_1+u_0}-A_{\tau_2+u_0})\cdot\nabla^{2b}\tau_2.
\end{aligned}$$

So we just need to estimate the various terms using Lemma 6.5 and Lemma 6.6 as above and then convert uniform estimates to L^p estimates gaining the infinitesimal factor $\zeta(T)$. In particular:

$$\begin{aligned}
\|A_{\tau_1+u_0}-A_{\tau_2+u_0}\|_{\infty,M_T}&\leqslant\|\tau_1-\tau_2\|_{W,2b,p,M_T}\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|A\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b})\\
\|A_{u_0}-A_{\tau_1+u_0}\|_{\infty,M_T}&\leqslant\zeta(T)\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|A\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b})\\
\|F_{\tau_1+u_0}-F_{\tau_2+u_0}\|_{\infty,M_T}&\leqslant\|\tau_1-\tau_2\|_{W,2b,p,M_T}\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M},\|F\|_{\mathscr{C}^1(R^{2b-1}_{u_0})},\|g\|_{2b}).
\end{aligned}$$ □

6.9 Lemma. *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $T, K \in (0,\infty)$ and $\tau \in W_T^K$. Then*

$$\|\hat{E}^q_\tau\|_{W,2b-m_q-\frac{1}{p},\partial M_T} \leqslant C \cdot \zeta(T)$$

where

$$C = C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M}, \|B^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|E^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|g\|_{2b}, \mathrm{Vol}(\partial M, g)).$$

Proof. By Lemma 6.5, and for every $j \leqslant 2b - m_q$:

$$\|\nabla^j B^q_{\tau+u_0}\|_{\infty,\partial M_T} \leqslant C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M}, \|B^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|g\|_{2b})$$
$$\|\nabla^j B^q_{u_0} - \nabla^j B^q_{\tau+u_0}\|_{\infty,\partial M_T} \leqslant \zeta(T)\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M}, \|B^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|g\|_{2b})$$
$$\|\nabla^j E^q_{\tau+u_0}\|_{\infty,\partial M_T} \leqslant C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M}, \|E^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|g\|_{2b}).$$

By Propositions 4.36 and 4.42, it holds $\|\nabla^{m_q}\tau|_{\partial M_T}\|_{W,2b-m_q-\frac{1}{p},\partial M_T} \leqslant C(n,k,p,b,\|g\|_{2b})\cdot K$. Also:

$$\begin{aligned}\|E^q_{\tau+u_0} - B^q_{\tau+u_0}\cdot\nabla^{m_q}u_0\|_{W,2b-m_q-\frac{1}{p},p,\partial M_T} &\leqslant (T\cdot \mathrm{Vol}(\partial M,g))^{\frac{1}{p}}\\ &\quad\cdot \sum_{j=0}^{2b-m_q}\|\nabla^j(E^q_{\tau+u_0} - B^q_{\tau+u_0}\cdot\nabla^{m_q}u_0)\|_{\infty,\partial M_T}\\ \|(B^q_{u_0} - B^q_{\tau+u_0})\cdot\nabla^{m_q}\tau\|_{W,2b-m_q-\frac{1}{p},p,\partial M_T} &\leqslant \|\nabla^j B^q_{u_0} - \nabla^j B^q_{\tau+u_0}\|_{\infty,\partial M_T}\\ &\quad\cdot\|\nabla^{m_q}\tau|_{\partial M_T}\|_{W,2b-m_q-\frac{1}{p},\partial M_T},\end{aligned}$$

from which the thesis follows. □

6.10 Lemma. *Let (M^n, g) be a compact Riemannian manifold with boundary and bounded geometry. Let $T, K \in (0,\infty)$ and let $\tau_1, \tau_2 \in W_T^k$. Then*

$$\begin{aligned}\|\hat{E}^q_{\tau_1} - \hat{E}^q_{\tau_2}\|_{W,2b-m_q-\frac{1}{p},p,\partial M_T} &\leqslant \zeta(T)\cdot\|\tau_1-\tau_2\|_{W,2b,p,M_T}\\ &\cdot C(n,k,b,K,\|B^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|E^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|u_0\|_{\mathscr{C},2b-1,M}, \|g\|_{2b}, \mathrm{Vol}(\partial M,g)).\end{aligned}$$

Proof. We have that

$$\begin{aligned}\hat{E}^q_{\tau_1} - \hat{E}^q_{\tau_2} &= (E^q_{\tau_1+u_0} - E^q_{\tau_2+u_0}) - (B^q_{\tau_1+u_0} - B^q_{\tau_2+u_0})\cdot\nabla^{m_q}u_0\\ &\quad + (B^q_{u_0} - B^q_{\tau_1+u_0})\cdot\nabla^{m_q}\tau_1 - (B^q_{u_0} - B^q_{\tau_2+u_0})\cdot\nabla^{m_q}\tau_2\\ &= (E^q_{\tau_1+u_0} - E^q_{\tau_2+u_0}) - (B^q_{\tau_1+u_0} - B^q_{\tau_2+u_0})\cdot\nabla^{m_q}u_0\\ &\quad - (B^q_{\tau_1+u_0} - B^q_{u_0})\cdot\nabla^{m_q}(\tau_1-\tau_2) - (B^q_{\tau_1+u_0} - B^q_{\tau_2+u_0})\cdot\nabla^{m_q}\tau_2.\end{aligned}$$

So we just need to estimate the various terms using Lemma 6.5 and Lemma 6.6 as above and then convert uniform estimates to L^p estimates gaining the infinitesimal factor $\zeta(T)$. In particular:

$$\begin{aligned}\|\nabla^j B^q_{\tau_1+u_0} - \nabla^j B^q_{\tau_2+u_0}\|_{\infty,\partial M_T} &\leqslant C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M}, \|B^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|g\|_{2b})\\ &\quad\cdot\|\tau_1-\tau_2\|_{W,2b,p,M_T}\\ \|\nabla^j B^q_{u_0} - \nabla^j B^q_{\tau_1+u_0}\|_{\infty,\partial M_T} &\leqslant \zeta(T)\cdot C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M}, \|B^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|g\|_{2b})\\ \|\nabla^j E^q_{\tau_1+u_0} - \nabla^j E^q_{\tau_2+u_0}\|_{\infty,\partial M_T} &\leqslant C(n,k,b,K,p,\|u_0\|_{\mathscr{C},2b-1,M}, \|E^q\|_{\mathscr{C}^{2b-m_q+1}(R^{m_q-1}_{u_0})}, \|g\|_{2b})\\ &\quad\cdot\|\tau_1-\tau_2\|_{W,2b,p,M_T}.\end{aligned}$$

□

We can finally proof the theorems stated at the beginning of the chapter.

Proof of Theorem 6.1. Let us consider the space W_T^K defined above, for T to be determined, and $\tau \in W_T^K$. Let us define $\sigma(\tau)$ to be the solution the the system (6.2). Then, using Lemma 6.7, Lemma 6.9 and Theorem 5.20 we have that

$$\|\sigma(\tau)\|_{W,2b,M_T} \leqslant C \cdot \zeta(T),$$

where

$$\begin{aligned} C = C(&n, k, b, K, p, \|A\|_{\mathscr{C},\infty,M_T}, \|F\|_{\mathscr{C},\infty,M_T}, \|B^q\|_{\mathscr{C},\infty,\partial M_T}, \|E^q\|_{\mathscr{C},\infty,\partial M_T}, \\ &\delta, \delta_1, \|g\|_{2b}, \mathrm{Vol}(M,g), \mathrm{Vol}(\partial M, g)). \end{aligned}$$

For T sufficiently small, $\sigma(\tau)$ is contained in W_T^K, so that σ is a map $W_T^K \to W_T^K$. Using also Lemma 6.8 and Lemma 6.10 we have that

$$\|\sigma(\tau_1) - \sigma(\tau_2)\|_{W,2b,M_T} \leqslant C \cdot \zeta(T) \cdot \|\tau_1 - \tau_2\|_{W,2b,M_T},$$

for a constant C having the same dependencies as above. Perhaps after reducing T, the map σ is therefore Lipschitz continuous with a constant that can be taken strictly smaller than 1. Since W_T^K is a complete metric space, it follows from Banach's fixed point theorem that σ has a unique fixed point, which we call $\tilde{\tau}$. Therefore $u = u_0 + \tilde{\tau}$ is a solution of problem (6.1) and the theorem is proved. □

Proof of Theorem 6.2. From Theorem 6.1 we know that u is in $W^{2b,1}(M_T)$. By Proposition 4.43 it also belongs to $\mathscr{C}^{2b-\frac{1}{2},\frac{2b-\frac{1}{2}}{2b}}(M_T)$, which in turn implies that A_u and F_u are in $\mathscr{C}^{\frac{1}{2},\frac{\frac{1}{2}}{2b}}(M_T)$ and B_u^q and E_u^q are in $\mathscr{C}^{\frac{1}{2}+2b-m_q,\frac{\frac{1}{2}+2b-m_q}{2b}}(M_T)$. By Theorem 5.8, and since sufficiently high compatibility conditions are satisfied, u is then proved to be in $\mathscr{C}^{2b+\frac{1}{2},\frac{2b+\frac{1}{2}}{2b}}(M_T)$, and we have gained half an order of regularity from the initial space.

The argument can be repeated *ad libitum*, as long as suitable compatibility conditions are satisfied. □

Proof of Theorem 6.3. Let us choose $\varepsilon > 0$ and prove $u \in \mathscr{C}^\infty(M \times [\varepsilon, T])$. We can select a smooth function $\varphi\colon [0,T] \to [0,1]$ such that $\chi_{[\varepsilon,T]} \leqslant \varphi \leqslant \chi_{[\frac{\varepsilon}{2},T]}$ and write the following equation for φu:

$$\begin{aligned} \partial_t(\varphi u) - A_u \cdot \nabla^{2b}(\varphi u) &= \varphi(\partial_t u - A_u \cdot \nabla^{2b} u) + \partial_t \varphi \cdot u = \varphi \cdot F_u + \partial_t \varphi \cdot u \\ (\varphi u)|_{t=0} &= 0 \\ B_u^q \cdot \nabla^{m_q}(\varphi u) &= \varphi(B_u^q \cdot \nabla^{m_q} u) = \varphi \cdot E_u^q. \end{aligned}$$

Proceeding as in the proof of Theorem 6.2, we can show that φu is smooth: in particular, compatibility conditions are satisfied at any order, because both initial data and boundary data are zero at time zero. This implies that u is smooth in $[\varepsilon, T]$, and, since ε is arbitrary, in $(0, T]$.

A similar argument can be produced for proving smoothness in $\mathring{M} \times [0, T]$, by using a function φ that depends on space coordinates, is smooth and has support that does not touch ∂M. □

Chapter 7

The Bour flow

7.1 A class of geometric flows on manifolds with boundary

We now show how to apply theorems proved in the previous chapters to an actual geometric flow. In [Bou12, Section 2.1] the following class of flows on a manifold without boundary is discussed:

$$\begin{aligned} \partial_t g(t) &= P(g(t)) \qquad\qquad t \in [0,T] \\ g(0) &= g_0, \end{aligned} \tag{7.1}$$

where

$$P_{ij}(g) = \nabla^p \nabla^q \mathrm{Riem}^g_{piqj} + a_1 \Delta R_g \cdot g_{ij} + a_2 \nabla^2_{ij} R_g + (\mathrm{Riem}_g * \mathrm{Riem}_g)_{ij} \tag{7.2}$$

and a_1 and a_2 are two real parameters such that $a_1 < \frac{1}{2(n-1)}$.

Two adjustements must be performed on equation (7.1) before it can be plugged into Theorem 6.1.

- The operator P, because of its geometrical nature, is invariant by diffeomorphisms (i.e., $P(\varphi^* g) = \varphi^* P(g)$ when φ is a diffeomorphism of M). This is known to cause zero eigenvalues appear in the principal symbol, thus violating the strong parabolicity condition.

  This behaviour can be compensated by using the classical DeTurck trick: a new equation, whose unknown will be indicated by $\bar{g}$, can be written by suitably modifying (7.1), in a way that lets one recover g once the modified equation has been solved. The new equation depends on the choice of a time dependent family of metrics $\tilde{g}$, which acts as a gauge selection for breaking the diffeomorphism invariance.

- Theorem 6.1 requires us to write the equation for $\bar{g}$ so that covariant derivatives are not computed according to the unknown $\bar{g}$ itself, but according to a time independent background metric that we call $\hat{g}$.

Since the proliferation of different metrics can be tricky to understand at the beginning, let us recall again what is the role of each of them:

- $\hat{g}$ is the time independent background metric, which is used to define function spaces and express the parabolic equation; it plays the role of g in the previous chapters;

- $\tilde{g}$ is a family of metrics depending on the time, which is used to break the geometric invariance of the equation by mean of the DeTurck trick; it can be chosen at will, although its time derivatives at $M \times \{0\}$ must meet some requirements to satisfy compatibility conditions and avoid losing regularity of the solution;

- $\bar{g}$ is a family of metrics depending on the time obtained by solving the parabolic system modified with the DeTurck trick; it depends on the choice of metrics $\tilde{g}$; it plays the role of u in the previous chapters;

- g is a family of metrics depending on the time and solving equation (7.1); it is recovered from $\bar{g}$ by pulling it back along an appropriate family of diffeomoprhisms, reversing the effects of the DeTruck trick.

Let us begin by seeing how to rewrite (7.1) in terms of the background metric $\hat{g}$.

7.1 Lemma. *Let $a_1, a_2 \in \mathbb{R}$ and let P be a smooth operator of the form (7.2). Then there are*

$$A \in \mathscr{C}^\infty \mathrm{Map}(S_2(M_T); T^4 S_2^2(M_T))$$
$$F \in \mathscr{C}^\infty \mathrm{Map}(S_2(M_T), \dots, T_{2b-1} S_2(M_T); S_2(M_T))$$

such that

$$P(g(t))_{I_2}(x) = A_{I_2}^{J_4 K_2}(x,t,g(x,t)) \cdot \hat{\nabla}^4_{J_4} g_{K_2}(x,t) + F_{I_2}(x,t,g(x,t),\dots,\hat{\nabla}^3 g(x,t)).$$

Moreover

$$A_{I_2}^{J_4 K_2}(x,t,g(x,t)) = -\frac{1}{2}(\delta^{j_3}_{\cdot}\delta^{j_4}_{\cdot} \owedge \delta^{k_1}_{\cdot}\delta^{k_2}_{\cdot})_{H_4} \cdot (g^{j_1 h_1} g^{j_2 h_3} \delta^{h_2}_{i_1} \delta^{h_4}_{i_2} + a_1 \cdot g^{h_1 h_3} g^{h_2 h_4} g^{j_1 j_2} g_{i_1 i_2} + a_2 \cdot g^{h_1 h_3} g^{h_2 h_4} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}). \tag{7.3}$$

Proof. Let us define

$$D^k_{ij} := \Gamma^k_{ij} - \hat{\Gamma}^k_{ij}.$$

While Christoffel coefficients are not in general tensors, their difference, and therefore D, is. It measures the error introduced when switching between computing the covariant derivative in terms of $\hat{g}$ and g. For example:

$$\nabla_i X^k - \hat{\nabla}_i X^k = D^k_{ij} X^j$$
$$\nabla_i \eta_j - \hat{\nabla}_i \eta_j = -D^k_{ij} \eta_k.$$

Similar formula can be written for tensors of any order, in the usual way. Then D^k_{ij} can be written purely in terms of derivatives $\hat{\nabla}$ with the following formula, which is [CLN06, (2.48)]:

$$D^k_{ij} = \frac{1}{2} g^{k\ell} \cdot \left(\hat{\nabla}_i g_{j\ell} + \hat{\nabla}_j g_{i\ell} - \hat{\nabla}_\ell g_{ij}\right). \tag{7.4}$$

There are also formulas for switching the metric underlying a Riemann tensor:

$$(\mathrm{Riem}_g)^\ell_{ijk} - (\mathrm{Riem}_{\hat{g}})^\ell_{ijk} = \hat{\nabla}_j D^\ell_{ik} - \hat{\nabla}_i D^\ell_{jk} + D^p_{ik} D^\ell_{jp} - D^p_{jk} D^\ell_{ip}, \tag{7.5}$$

which is derived in [CLN06, (2.49)] (notice that we use the opposite sign convention for the Riemann tensor). By applying repeatedly the formulae above, it is clear that $P(g)$ can be rewritten in term of derivatives $\hat{\nabla}$ of the metric g.

Let us focus now on the highest order terms: as usual, this enables us to commute derivatives and distribute them by recursively selecting the highest orders in the differentiated expression, because both operations require corrections only at lower orders. So it follows from (7.4) and (7.5) that

$$\begin{aligned} \mathrm{Riem}^g_{ijk\ell} &= g_{\ell p}(\hat\nabla_j D^p_{ik} - \hat\nabla_i D^p_{jk}) + \text{LOTs} \\ &= \frac{1}{2}\left[\hat\nabla_j(\cancel{\hat\nabla_i g_{k\ell}} + \hat\nabla_k g_{i\ell} - \hat\nabla_\ell g_{ik}) - \hat\nabla_i(\cancel{\hat\nabla_j g_{k\ell}} + \hat\nabla_k g_{j\ell} - \hat\nabla_\ell g_{jk})\right] + \text{LOTs} \\ &= -\frac{1}{2}(\hat\nabla \owedge g)_{ijk\ell} + \text{LOTs} \\ &= -\frac{1}{2}\hat\nabla^2_{pq} g_{rs} \cdot (\delta^p_\cdot \delta^q_\cdot \owedge \delta^r_\cdot \delta^s_\cdot)_{ijk\ell} + \text{LOTs}. \end{aligned} \tag{7.6}$$

Looking again at (7.4), we see that when we differentate a tensor T of order higher than 1, the rest given by D is of lower order, so that

$$\nabla T = \hat\nabla T + \text{LOTs}.$$

If follows that:

$$\nabla^2_{j_1 j_2}\mathrm{Riem}^g_{H_4} = -\frac{1}{2}\hat\nabla^4_{J_4} g_{K_2} \cdot (\delta^{j_3}_\cdot \delta^{j_4}_\cdot \owedge \delta^{k_1}_\cdot \delta^{k_2}_\cdot)_{H_4} + \text{LOTs}.$$

The thesis then follows by expanding the definition of $P(g)$:

$$P(g) = \nabla^2_{j_1 j_2}\mathrm{Riem}^g_{H_4} \cdot (g^{j_1 h_1} g^{j_2 h_3} \delta^{h_2}_{i_1} \delta^{h_4}_{i_2} + a_1 \cdot g^{h_1 h_3} g^{h_2 h_4} g^{j_1 j_2} g_{i_1 i_2} + a_2 \cdot g^{h_1 h_3} g^{h_2 h_4} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2}) + \text{LOTs}.$$

□

7.2 The DeTurck trick

Let us now discuss the application of the DeTurck trick to (7.1). The material covered here, at least before the boundary is introduced, is essentially that in [Bou12, Section 2.1 and Appendix A]. For similar discussions in the case of the Ricci flow, see [CLN06, § 2.6] (with explicit computations) and [CK04, Chapter 3, Section 2.3] (more focused on the relationship between the diffeomorphism invariance and the zero eigenvalues of the operator). The DeTurck trick was originally described in [DeT83].

The symbol $\mathscr{L}_V T$ will denote the Lie derivative of the tensor T with respect to the vector field V.

7.2 Lemma. *Let M^n be a smooth compact manifold with boundary, $\bar g(t)$ be a smooth family of metrics and φ_t be a smooth family of diffeomorphisms, for $t \in [0,T]$. Then*

$$\partial_t(\varphi_t^* \bar g(t)) = \varphi_t^*(\partial_t \bar g(t) + \mathscr{L}_{V_t} \bar g(t)),$$

where $V_t = \partial_t \varphi_t \circ \varphi_t^{-1}$.

Proof. See [Bou12, Lemma A.1]. □

Lemma 7.2 implies that $g(t)$ solves the equation

$$\partial_t g(t) = P(g(t))$$

at point $(x,t) \in M_T$ if and only if $\bar g(t) = (\varphi_t^{-1})^* g(t)$ solves the equation

$$\partial_t \bar g(t) = P(\bar g(t)) - \mathscr{L}_{V_t} \bar g(t)$$

at point $(\varphi_t(x),t)\in M_T$, where V_t is as in the lemma. The DeTruck trick consists in choosing V_t to be a differential operator depending on $\bar g(t)$ so that $P(\bar g(t))-\mathscr{L}_{V(\bar g(t))}\bar g(t)$ is a uniformly elliptic operator and Theorem 6.1 can then be used to obtain the existence of the flow $\bar g(t)$. Then $g(t)$ can be recovered as $g(t)=\varphi_t^*(\bar g(t))$, where φ_t is defined by

$$\partial_t\varphi_t = V(\bar g(t)) \tag{7.7}$$
$$\varphi_\varepsilon = \mathrm{Id}_M, \tag{7.8}$$

for $\varepsilon\in[0,\infty)$.

7.3 Remark. When the manifold M has no boundary it is natural to choose $\varepsilon=0$, which ensures that $g(0)=\bar g(0)$ and it avoids the introduction on another parameter. However when M has a boundary the solution $\bar g$ is not always guaranteed to be regular in $\partial M\times\{0\}$, so it is better to avoid the time zero. See also [Gia13, Remark 4.4.2].

So we are now left with choosing the operator V. For the Ricci flow the following field is used:

$$(V_R)_i(\bar g)=\frac{1}{2}\bar g_{i\ell}\bar g^{pq}(\bar\Gamma^{\ell}_{pq}-\tilde\Gamma^{\ell}_{pq}). \tag{7.9}$$

According to [Bou12, Proposition 2.1], a valid choice for the Bour flow is

$$(V_B)_i(\bar g)=\bar\Delta(V_R)_i(\bar g)+\frac{2(a_2-a_1)-1}{4}d_iR_{\bar g}. \tag{7.10}$$

Technically the tensors V_R and V_B are defined as covectors, but we will sometimes call them vector fields because of how they are interpreted in Lemma 7.2.

As said above, $\tilde g$ is a smooth family of metric depending on time, which at the present stage can be completely arbitrary. Different choices for $\tilde g$ will, in general, induce different solutions $\bar g$ for the modified equation, corresponding however to the same flow g once the DeTurck trick has been reversed. The Bour-DeTurck operator is defined as

$$Q(\bar g):=P(\bar g)+\mathscr{L}_{V_B^\sharp(\bar g)}\bar g$$

and Lemma 7.1 can be reformulated for it.

7.4 Lemma. *Let $a_1,a_2\in\mathbb{R}$ and let P be a smooth operator of the form (7.2). Then there are*

$$A\in\mathscr{C}^\infty\mathrm{Map}(S_2(M_T);T^4S_2^2(M_T))$$
$$F\in\mathscr{C}^\infty\mathrm{Map}(S_2(M_T),\dots,T_{2b-1}S_2(M_T);S_2(M_T))$$

such that

$$(Q(\bar g(t)))_{I_2}(x)=A_{I_2}^{J_4K_2}(x,t,\bar g(x,t))\cdot\hat\nabla^4_{J_4}\bar g_{K_2}(x,t)+B_{I_2}(x,t,\bar g(x,t),\dots,\hat\nabla^3\bar g(x,t)).$$

Moreover

$$A_{I_2}^{J_4K_2}(x,t,\bar g(x,t))=-\frac{1}{2}\delta^{k_1}_{i_1}\delta^{k_2}_{i_2}\bar g^{j_1j_2}\bar g^{j_3j_4}+a_1\cdot\left(\bar g^{j_1k_1}\bar g^{j_2k_2}-\bar g^{j_1j_2}\bar g^{k_1k_2}\right)\cdot\left(\delta^{j_3}_{i_1}\delta^{j_4}_{i_2}-\bar g^{j_3j_4}\bar g_{i_1i_2}\right). \tag{7.11}$$

Proof. We can rewrite

$$\begin{aligned}(V_R)_i&=\frac{1}{2}\bar g_{i\ell}\bar g^{pq}D^{\ell}_{pq}+\frac{1}{2}\bar g_{i\ell}\bar g^{pq}(\hat\Gamma^{\ell}_{pq}-\tilde\Gamma^{\ell}_{pq})\\&=\frac{1}{4}\bar g_{i\ell}\bar g^{pq}\bar g^{\ell k}\left(\hat\nabla_p\bar g_{kq}+\hat\nabla_q\bar g_{kp}-\hat\nabla_k\bar g_{pq}\right)+\text{LOTs}\\&=\frac{1}{4}\bar g^{pq}\left(2\hat\nabla_p\bar g_{qi}-\hat\nabla_i\bar g_{pq}\right)+\text{LOTs}.\end{aligned}$$

Using (7.6) we also have that

$$\begin{aligned} R_{\bar g} &= \mathrm{Riem}^{\bar g}_{ijk\ell}\,\bar g^{ik}\bar g^{j\ell} \\ &= \bar g^{ik}\bar g^{j\ell}\cdot(\hat\nabla_i\hat\nabla_\ell \hat g_{jk} - \hat\nabla_i\hat\nabla_k g_{j\ell}) + \text{LOTs}. \end{aligned}$$

We can use the fact that $\bar g$ is a Riemannian metric, so that for $V \in \mathscr{T}_1(M)$ it holds:

$$(\mathscr{L}_{V^\sharp}\bar g)_{ij} = \bar\nabla_i V_j + \bar\nabla_j V_i.$$

Combining the two previous formulae:

$$\begin{aligned} (V_B)_i = \frac{1}{4}\bar g^{jk}\bar g^{pq}\,[2\hat\nabla_j\hat\nabla_k\hat\nabla_p\bar g_{qi} - \hat\nabla_j\hat\nabla_k\hat\nabla_i\bar g_{pq} + (2(a_2-a_1)-1)\cdot(\hat\nabla_i\hat\nabla_j\hat\nabla_p\bar g_{kq} - \hat\nabla_i\hat\nabla_j\hat\nabla_k\bar g_{pq})] \\ + \text{LOTs}, \end{aligned} \tag{7.12}$$

from which, using the symmetries of the principal symbol,

$$\begin{aligned} \left(\mathscr{L}_{V_B^\sharp}\bar g\right)_{I_2} &= \frac{1}{2}\hat\nabla^4_{J_4}\bar g_{K_2}\cdot\Bigg[\bar g^{j_1j_2}\bar g^{j_3k_1}(\delta^{k_2}_{i_1}\delta^{j_4}_{i_2} + \delta^{k_2}_{i_2}\delta^{j_4}_{i_1}) \\ &\quad + \left((a_2-a_1)-\frac{1}{2}\right)\cdot\bar g^{j_1k_1}\bar g^{j_2k_2}(\delta^{j_3}_{i_1}\delta^{j_4}_{i_2} + \delta^{j_4}_{i_1}\delta^{j_3}_{i_2}) \\ &\quad + (a_2-a_1)\cdot\bar g^{j_1j_2}\bar g^{k_1k_2}(\delta^{j_3}_{i_1}\delta^{j_4}_{i_2} + \delta^{j_4}_{i_1}\delta^{j_3}_{i_2})\Bigg] + \text{LOTs} \\ &= \hat\nabla^4_{J_4}\bar g_{K_2}\cdot\Bigg[\bar g^{j_1j_2}\bar g^{j_3k_1}\delta^{k_2}_{i_1}\delta^{j_4}_{i_2} + \left((a_2-a_1)-\frac{1}{2}\right)\cdot\bar g^{j_1k_1}\bar g^{j_2k_2}\delta^{j_3}_{i_1}\delta^{j_4}_{i_2} \\ &\quad + (a_2-a_1)\cdot\bar g^{j_1j_2}\bar g^{k_1k_2}\delta^{j_3}_{i_1}\delta^{j_4}_{i_2}\Bigg] + \text{LOTs}. \end{aligned}$$

It is now a simple computation to show that (7.11) can be found as difference of (7.3) and the above. □

Substituting (7.11) in (5.8), the principal symbol of the operator Q for $\xi \in \mathscr{T}_1(M)$ is found to be

$$\begin{aligned} \mathscr{L}_0(x,t,i\xi,p) &= \left(p - \frac{1}{2}|\xi|^4\right)\mathrm{Id}_{S_2(M)} + a_1\langle R_\xi|\cdot\rangle R_\xi, \\ &= \left(p - \frac{1}{2}|\xi|^4\right)\mathrm{Id}_{S_2(M)} + a_1 R_\xi\otimes R_\xi^t, \end{aligned} \tag{7.13}$$

where

$$R_\xi(g) = \xi\otimes\xi - |\xi|^2 g$$

and A^t denotes the transpose of the matrix or vector A.

7.5 Proposition. *Let $a_1 \in (-\infty, \frac{1}{2(n-1)})$ and $a_2 \in \mathbb{R}$. Then the principal symbol $\mathscr{L}_0$ satisfies the strong parabolicity condition with a constant $\delta(n,a_1) > 0$.*

Proof. We have to estimate the eigenvalues of $-\frac{1}{2}|\xi|^4\,\mathrm{Id} + a_1\langle R_\xi|\cdot\rangle R_\xi$. Clearly, on the orthogonal space to R_ξ, the morphism $\mathscr{L}_0$ behaves as $-\frac{1}{2}|\xi|^4\,\mathrm{Id}$, so the eigenvalue $-\frac{1}{2}|\xi|^4$ has multiplicity $r-1$. Also, the vector R_ξ is an eigenvector, with eigenvalue

$$-\frac{1}{2}|\xi|^4 + a_1|R_\xi|^2 = -\frac{1}{2}|\xi|^4 + a_1(n-1)|\xi|^4.$$

The thesis follows immediately. □

Lemmata 7.2 and 7.4 and Proposition 7.5 have purely local content, so they apply to manifolds with boundary without problems. The only element in this section that requires some care when there is a boundary is the existence of the diffeomorphisms φ_t, since the integral flow of a vector field can stop existing because it is "required" to cross the boundary. We then add $V|_{\partial M_T} = 0$ among the boundary conditions. In line of principle we could just require that $V|_{\partial M_T}$ is tangent to the boundary, but making it zero will ensure that the diffeomorphisms φ_t are the identity when restricted to ∂M, which in turns guarantees that boundary conditions imposed on the Bour-DeTurck flow $\bar{g}$ can be brought back to the actual geometric Bour flow g. Once V is required to be at least tangential to the boundary, the existence of φ_t can be proved using [Lee13, Theorem 9.34].

7.3 The complementary condition

The complementary condition presented in Section 5.2 can be reformulated in a simplified manner taking into account the specific form of $\mathscr{L}_0$ given by (7.13). Let us recall that $r = \frac{n(n+1)}{2}$, $b = 2$ and a_1 and a_2 are two real numbers with $a_1 < \frac{1}{2(n-1)}$. Since we will use a_1 much more often than a_2, we stipulate that $a = a_1$. We also introduce the constant $\gamma = \frac{1}{1-2(n-1)a}$, which will be useful to shorten formulae. The condition on a is equivalent to $\gamma > 0$ and $a = 0$ corresponds to $\gamma = 1$.

From (7.13) and using Lemma A.1 we have that

$$\begin{aligned} L(x,t,i\xi,p) &= \det \mathscr{L}_0(x,t,i\xi,p) \\ &= \left(p + \frac{1}{2}|\xi|^4\right)^{r-1} \cdot \left(p + \frac{1-2(n-1)a}{2}|\xi|^4\right) \\ &= \left(p + \frac{1}{2}(|\zeta|^2+\tau^2)^2\right)^{r-1} \cdot \left(p + \frac{1}{2\gamma}(|\zeta|^2+\tau^2)^2\right). \end{aligned}$$

In the second expression we have substituted $\xi = \zeta + \tau\nu$, which implies $|\xi|^4 = (|\zeta|^2+\tau^2)^2$.

7.6 Remark. The expression for $|\xi|^4$ must be considered a polynomial with real coefficients in τ which has been subsequently complexified (and for which complex roots in τ will be searched). In particular, by writing τ^2 we mean the actual squaring operation, and not the squared complex norm $|\tau|^2 = \tau\bar{\tau}$. In other words, when expanding the norm of $|\xi|$ the expression is not "aware" of being a complex expression, so does not have any conjugation operation.

It will be convenient to introduce a few notations: given $p \in \mathbb{C}$, let us fix a certain square root of p, which we will call $\sqrt{p}$. This selection is not continuous in p, but it does not need to be. For fixed p and ζ (such that $|\zeta|^8 + |p|^2 = 1$, according to Definition 5.5), let us search the roots in τ of $p + \frac{1}{2\gamma}(|\zeta|^2+\tau^2)^2$. Clearly they have to satisfy at least one of

$$\tau^2 = T_\gamma^+ := -|\zeta|^2 + i\cdot\sqrt{2\gamma}\cdot\sqrt{p} \qquad \text{or} \qquad \tau^2 = T_\gamma^- := -|\zeta|^2 - i\cdot\sqrt{2\gamma}\cdot\sqrt{p}.$$

Neither T_γ^+ nor T_γ^- can have real roots (this is also mandated by Lemma 5.2). Let us call τ_γ^+ the root of T_γ^+ with positive imaginary part and τ_γ^- the root of T_γ^- with positive imaginary part. Furthermore call $P_\gamma^+(\tau) = (\tau-\tau_\gamma^+)(\tau-\tau_\gamma^-)$ and $P_\gamma^-(\tau) = (\tau+\tau_\gamma^+)(\tau+\tau_\gamma^-)$ (notice that + and

– are used with different meanings between τ and P), so that

$$p+\frac{1}{2}|\xi|^4=\frac{1}{2}P_1^+(\tau)\cdot P_1^-(\tau)$$
$$p+\frac{1}{2\gamma}|\xi|^4=\frac{1}{2\gamma}P_\gamma^+(\tau)\cdot P_\gamma^-(\tau).$$

Applying (5.11) we obtain:

$$L(x,t,i(\zeta+\tau\nu),p)=\frac{1}{2^r\gamma}\cdot\left(P_1^+(\tau)P_1^-(\tau)\right)^{r-1}\cdot P_\gamma^+(\tau)P_\gamma^-(\tau)$$
$$M^+(x,t,\zeta,\tau,p)=\left(P_1^+(\tau)\right)^{r-1}\cdot P_\gamma^+(\tau).$$

7.7 Remark. If $p=0$, then for every $\gamma>0$ it holds

$$\tau_\gamma^+=\tau_1^+=\tau_\gamma^-=\tau_1^-=i|\zeta|.$$

If $p\neq 0$ and $\gamma=1$, then it holds

$$\tau_\gamma^+=\tau_1^+\neq\tau_\gamma^-=\tau_1^-,$$

because $T_1^+\neq T_1^-$.

If $p\neq 0$ and $\gamma\neq 1$, then the four numbers τ_γ^+, τ_1^+, τ_γ^- and τ_1^- are all different.

7.8 Lemma. *Let $\gamma\in(0,\infty)$. Then for $\delta<\tilde{\delta}(\gamma)$*

$$\operatorname{Im}\tau_\gamma^+\geqslant C(\gamma)\qquad\qquad \operatorname{Im}\tau_\gamma^-\geqslant C(\gamma).$$

Proof. Let $a+ib$ be either τ_γ^+ or τ_γ^-, with $a\in\mathbb{R}$ and $b\in(0,\infty)$; we know that it solves $p+\frac{1}{2\gamma}\left((a+ib)^2+|\zeta|^2\right)^2=0$. Then

$$p=-\frac{1}{2\gamma}\left(a^2-b^2+|\zeta|^2+2iab\right)^2$$
$$\operatorname{Re}p=-\frac{1}{2\gamma}\left((a^2-b^2+|\zeta|^2)^2-4a^2b^2\right)\tag{7.14}$$
$$=-\frac{1}{2\gamma}\left(a^4+b^4+|\zeta|^4+2a^2(|\zeta|^2-3b^2)-2b^2|\zeta|^2\right)\tag{7.15}$$
$$|p|^2=\frac{1}{4\gamma^2}\left((a^2-b^2+|\zeta|^2)^2+4a^2b^2\right)^2.\tag{7.16}$$

Since by hypothesis $|\zeta|^8+|p|^2=1$ we divide the proof in two cases, depending on which of $|\zeta|^8$ and $|p|^2$ is greater than $\frac{1}{2}$. Let us begin by assuming that $|\zeta|^8\geqslant\frac{1}{2}$. By hypothesis, $\operatorname{Re}p>-\delta|\zeta|^4$, so, from (7.15),

$$2b^2\geqslant(-2\delta\gamma+1)|\zeta|^2+\frac{a^4+b^4}{|\zeta|^2}+2\frac{a^2}{|\zeta|^2}(|\zeta|^2-3b^2)$$
$$\geqslant\frac{1}{2}|\zeta|^2+2\frac{a^2}{|\zeta|^2}(|\zeta|^2-3b^2)$$

as soon as $\tilde{\delta}<\frac{1}{4\gamma}$. If $|\zeta|^2-3b^2$ is positive, then the inequality implies that $b\geqslant\frac{1}{2}|\zeta|\geqslant\frac{1}{2\sqrt[8]{2}}$. If $|\zeta|^2-3b^2$ is nonpositive, then automatically $b\geqslant\frac{\sqrt{3}}{3}|\zeta|\geqslant\frac{\sqrt{3}}{3\sqrt[8]{2}}$.

This closes the case $|\zeta|^8 \geqslant \frac{1}{2}$, so we pass to assuming $1 \geqslant |p|^2 \geqslant \frac{1}{2}$. From (7.14) and (7.16), using again $\operatorname{Re} p \geqslant -\delta|\zeta|^4$ we have that

$$\begin{aligned} \sqrt{2}\gamma \leqslant (a^2 - b^2 + |\zeta|^2)^2 + 4a^2b^2 \leqslant 2\gamma \\ (a^2 - b^2 + |\zeta|^2)^2 - 4a^2b^2 \leqslant 2\delta\gamma|\zeta|^4. \end{aligned} \tag{7.17}$$

By subtracting the two inequalities we get

$$\sqrt{2}\gamma \leqslant 8a^2b^2 + 2\delta\gamma|\zeta|^4,$$

from which, if $\tilde{\delta} < \frac{1}{2}$,

$$4a^2b^2 \geqslant \frac{\sqrt{2}}{4}\gamma. \tag{7.18}$$

From (7.17) we also get

$$b^2 \geqslant a^2 - \sqrt{2\gamma}. \tag{7.19}$$

If $a^2 \geqslant 2\sqrt{2\gamma}$, then (7.19) implies $b \geqslant \sqrt[4]{2\gamma}$. If $a^2 \leqslant 2\sqrt{2\gamma}$, then we can substitute in (7.18) and obtain

$$b^2 \geqslant \frac{\sqrt{2}}{16}\gamma \cdot \frac{1}{a^2} \geqslant \frac{\sqrt{\gamma}}{32}.$$

In either case the theorem is finally settled. □

Using Lemma A.1 we can also compute the adjugate matrix $\hat{\mathscr{L}}_0$:

$$\begin{aligned} \hat{\mathscr{L}}_0(x,t,i\xi,p) &= \left(p + \frac{1}{2}|\xi|^4\right)^{r-1} \cdot \left[\mathrm{Id} - \frac{a}{p + \frac{1}{2}|\xi|^4} R_\xi \otimes R_\xi\right]^\wedge \\ &= \left(p + \frac{1}{2}|\xi|^4\right)^{r-1} \cdot \left[\left(1 - \frac{a}{p + \frac{1}{2}|\xi|^4}|R_\xi|^2\right) \cdot \mathrm{Id} + \frac{a}{p + \frac{1}{2}|\xi|^4} R_\xi \otimes R_\xi^t\right] \\ &= \left(p + \frac{1}{2}|\xi|^4\right)^{r-2} \cdot \left[\left(p + \frac{1 - 2(n-1)a}{2}|\xi|^4\right) \cdot \mathrm{Id} + aR_\xi \otimes R_\xi^t\right] \\ &= \frac{1}{2^{r-2}}(P_1^+(\tau)P_1^-(\tau))^{r-2} \cdot \left[\frac{1}{2\gamma}P_\gamma^+(\tau)P_\gamma^-(\tau) \cdot \mathrm{Id} + aR_\xi \otimes R_\xi^t\right]. \end{aligned}$$

Leveraging this structure, we can give a first simplification of the complementary condition: in general we are required to check that

$$\mathscr{B}_0 \circ \hat{\mathscr{L}}_0 = \frac{1}{2^{r-2}}(P_1^+(\tau)P_1^-(\tau))^{r-2} \cdot \left[\frac{1}{2\gamma}P_\gamma^+(\tau)P_\gamma^-(\tau) \cdot \mathscr{B}_0(\tau) + a\,[\mathscr{B}_0(\tau)]\,(R_\xi) \otimes R_\xi^t\right] \tag{7.20}$$

has independent rows modulo

$$M^+ = (P_1^+(\tau))^{r-1} \cdot P_\gamma^+(\tau). \tag{7.21}$$

However the next proposition shows that when $\gamma = 1$ the polynomials can be greatly simplified.

7.9 Proposition. *If $\gamma = 1$ and $\delta < \tilde{\delta}$ for a universal $\tilde{\delta} > 0$, then it holds*

$$\Sigma\left((\mathscr{B}_0 \circ \hat{\mathscr{L}}_0) \bmod M^+\right) \geqslant C(n,r) \cdot \left|\det\left(\mathscr{B}_0(\tau) \bmod P_1^+(\tau)\right)\right|^2.$$

Proof. If $\gamma = 1$, then $a = 0$ and

$$(\mathscr{B}_0 \circ \hat{\mathscr{L}}_0) \bmod M^+ = \frac{1}{2^{r-1}} \left((P_1^+(\tau) P_1^-(\tau))^{r-1} \cdot \mathscr{B}_0(\tau) \right) \bmod (P_1^+(\tau))^r.$$

We can repeatedly use Propositions A.11 and A.12 to simplify all the factors $P_1^+(\tau)$ and $P_1^-(\tau)$ close to $\mathscr{B}_0(\tau)$. In the end we have that

$$\left((P_1^+(\tau) P_1^-(\tau))^{r-1} \cdot \mathscr{B}_0(\tau) \right) \bmod (P_1^+(\tau))^r = \left[(\mathscr{B}_0(\tau)) \bmod P_1^+(\tau) \right] \cdot Q_1' \cdot \dots \cdot Q_{2(r-1)}' \cdot Q_1 \cdot \dots \cdot Q_{2(r-1)},$$

where Q_i and Q_i' are copies of the matrices Q and Q' appearing in Propositions A.11 and A.12.

When computing Σ on such matrix, the factor $\mathscr{B}_0(\tau) \bmod P_1^+(\tau)$ is square, because $\deg P_1^+(\tau) = 2$, so we can use Lemma A.6. Each of the matrices Q_i' is square too, and has determinant equal to a power of $P_1^+(-\tau_1^+)$ or $P_1^+(-\tau_1^-)$: in the first case we have that

$$P_1^+(-\tau_1^+) = (-\tau_1^- - \tau_1^+)(-\tau_1^- - \tau_1^-) = 2\tau_1^+(\tau_1^+ + \tau_1^-).$$

By Lemma 7.8, the imaginary part of τ_1^+ and τ_1^- is bounded below, so

$$|\tau_1^+(\tau_1^+ + \tau_1^-)| = |\tau_1^+| \cdot |\tau_1^+ + \tau_1^-| \geqslant \operatorname{Im} \tau_1^+ \cdot (\operatorname{Im} \tau_1^+ + \operatorname{Im} \tau_1^-)$$

is bounded below. The same goes for $P_1^+(-\tau_1^-)$. So we can drop all the matrices Q_i' by modifying appropriately the constant C.

In the end we only need to evaluate $\Sigma(Q_1 \cdot \dots \cdot Q_{2(r-1)})$. However, looking at their form described in Proposition A.11, one can readily see that the tail minor of maximum order is a triangular matrix with all eigenvalues 1. So $\Sigma(Q_1 \cdot \dots \cdot Q_{2(r-1)}) \geqslant 1$ and we are done. □

Proposition 7.9 implies that, in order to check the complementary condition, it is enough to check that the determinant of the matrix $\mathscr{B}_0(\tau) \bmod P_1^+(\tau)$ stays bounded away from zero. Equivalently, according to Proposition A.13, instead of considering the remainders of $\mathscr{B}_0(\tau)$ when divided by $P_1^+(\tau)$, we can evaluate it on the roots of $P_1^+(\tau)$.

7.10 Proposition. *Suppose that $\gamma = 1$. Then there is a universal $\tilde{\delta} > 0$ such that if $\delta < \tilde{\delta}$ then the complementary condition is satisfied as soon as the $2r \times 2r$ matrix*

$$\left(\begin{array}{c|c} \mathscr{B}_0(\tau_1^+) & \mathscr{B}_0(\tau_1^-) \end{array} \right) \tag{7.22}$$

is nonsingular for all $p \in \mathbb{C} \setminus \{0\}$ and $\zeta \in T\partial M$ such that $\operatorname{Re} p \geqslant -\delta_1 |\zeta|^4$ and $|\zeta|^8 + |p|^2 = 1$ and its determinant decays to 0 as $p \to 0$ not faster than $|p|^{\frac{r}{2}}$.

Proof. The result follows from Propositions 7.9 and A.13. The squared determinant of P'' is 1, because P'' is a permutation matrix. Also, the matrix Q'' contains r blocks, each of which is the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ \tau_1^+ & \tau_1^- \end{pmatrix},$$

having a total determinant $(\tau_1^+ - \tau_1^-)^r$. By Lemma 7.8, $\tau_1^+ + \tau_1^-$ is bounded away from zero, so

$$\sqrt{|p|} = -\frac{i}{\sqrt{2}} (\tau_1^+)^2 - (\tau_1^-)^2 = -\frac{i}{\sqrt{2}} (\tau_1^+ + \tau_1^-)(\tau_1^+ - \tau_1^-)$$

has the same order of magnitude of $\tau_1^+ - \tau_1^-$ and the proposition is proved. □

7.11 Remark. The formulation of Proposition 7.10, while still rather abstract, already suggests a few informal "guiding principle" about how to effectively choose a boundary operator. In particular, suppose that, for a certain operator $\tilde{\mathscr{B}}$, the boundary operator $\mathscr{B}$ features normal derivatives of k different (and possibly zero) orders of $\tilde{\mathscr{B}}$:

$$\nabla_\nu^{\alpha_1}\tilde{\mathscr{B}} = E_1$$

$$\vdots$$

$$\nabla_\nu^{\alpha_k}\tilde{\mathscr{B}} = E_k.$$

Then, if $\tilde{\mathscr{B}}_0(\tau)$ is the principal simbol of $\tilde{\mathscr{B}}$, the rows in $\mathscr{B}_0(\tau)$ corresponding to the conditions above are

$$\left(\begin{array}{c|c} (\tau_0^+)^{\alpha_1}\cdot\tilde{\mathscr{B}}_0(\tau_0^+) & (\tau_0^-)^{\alpha_1}\cdot\tilde{\mathscr{B}}_0(\tau_0^-) \\ \vdots & \vdots \\ (\tau_0^+)^{\alpha_k}\cdot\tilde{\mathscr{B}}_0(\tau_0^+) & (\tau_0^-)^{\alpha_k}\cdot\tilde{\mathscr{B}}_0(\tau_0^-) \end{array}\right) = \begin{pmatrix} (\tau_0^+)^{\alpha_1}\cdot\mathrm{Id} & (\tau_0^-)^{\alpha_1}\cdot\mathrm{Id} \\ \vdots & \vdots \\ (\tau_0^+)^{\alpha_k}\cdot\mathrm{Id} & (\tau_0^-)^{\alpha_k}\cdot\mathrm{Id} \end{pmatrix}\cdot\begin{pmatrix} \tilde{\mathscr{B}}_0(\tau_0^+) \\ \tilde{\mathscr{B}}_0(\tau_0^-) \end{pmatrix}.$$

If $k > 2$ (or, in general, if $k > b$), then the rows are necessarily linearly dependent, thus in general one cannot put in a boundary symbol more than b different order of normal derivation of the same equation. Instead, if $k = b$ a condition of this type allows to "split" $\tilde{\mathscr{B}}_0(\tau_0^+)$ and $\tilde{\mathscr{B}}_0(\tau_0^-)$, and therefore it appears particularly natural. The "determinant price" paid, which is the determinant of the matrix $((\tau_0^\pm)^{\alpha_i})$, is comparable with $(\tau_0^+ - \tau_0^-)^d$, where d is the dimension of $\tilde{\mathscr{B}}$; this is exactly the expected order to satisfy the decay speed mandated by Proposition 7.10.

The propositions above give a relatively simple criterion to check the complementary condition when $\gamma = 1$. Unfortunately the general case is more difficult, and we do not have yet a complete result for it. However, because of continuity reasons, we can still obtain the following proposition.

7.12 Proposition. *Suppose that the complementary condition is satisfied for $\gamma = 1$. Then there is a neighbourhood U of 1 such that the complementary condition is satisfied for each $\gamma \in U$.*

Proof. Equations (7.20) and (7.21) show that the coefficients of $\mathscr{B}_0 \circ \hat{\mathscr{L}}_0$ and M^+ are continuous in γ, therefore the coefficients of $\mathscr{B}_0 \circ \hat{\mathscr{L}}_0 \bmod M^+$ are continuous too. Finally, it is clear from the definition that $\Sigma(\mathscr{B}_0 \circ \hat{\mathscr{L}}_0 \bmod M^+)$ is itself continuous in γ. It follows that if $\Sigma(\mathscr{B}_0 \circ \hat{\mathscr{L}}_0 \bmod M^+)$ is positive for $\gamma = 1$, it is positive for an entire neighbourhood of 1. □

7.4 Possible boundary conditions

In the previous section we have shown that complementary conditions can be established by computing the determinant of a suitably constructed matrix. We shall now pass to enumerate a few geometrically significant boundary conditions the we might desire to impose on the boundary of our flow. From now on we assume that M has dimension $n = 4$; also, for simplicity we assume that $a_1 = a_2$ in (7.2).

For each of the conditions below we will indicate the principal symbol with respect to the tangential covector $\zeta \in \mathscr{T}_1(M)$. The variable p does not appear, because all the boundary operators we consider do not depend on the time derivative of the flow. The point (x, t) is never mentioned explicitly, because all the conditions are indepdendent of it. The components of the variation of the metric will be indicated with h_{ij}, and will be, of course, symmetric in i

and j. The Latin indices will be assumed to range on all the directions (for 0 to 3), while the Greek indices will be assumed to range on the tangential directions (for 1 to 3).

By comparison, let us recall that Gianniotis, in [Gia13], proves that the Ricci-DeTurck flows satisfies the complementary condition with the following boundary conditions (using the notation below):

$$[g^T(x,t)] = [\gamma(x,t)]$$
$$\mathscr{H}(x,t) = \sigma(x,t)$$
$$V_R(x,t) = 0.$$

In the case of the Ricci flow he studies, only 10 conditions have to be set in dimension 4, because $b = 1$ (the Ricci flow has order 2).

Tangential metric tensor Possibly the most natural condition to require on the boundary is that the tangential metric is assigned during the evolution; for a given $\gamma \in \tilde{\mathscr{S}}_2^+(\partial M_T)$ we require

$$g^T(x,t) = \gamma(x,t)$$

(we recall that $\tilde{\mathscr{S}}_2^+(\partial M_T)$ is the space of metrics on the boundary ∂M_T, while $\mathscr{S}_2^+(\partial M_T)$ would have allowed to have also a normal component).

While in line of principle the assigned datum γ can depend on the time (like any of the conditions), it can be particularly convenient to set it constantly equal to the initial tangential metric $g^T(0)$.

The associated symbol is clearly the identity on the tangential directions as soon as a local chart that is normal for γ is chosen. Its dimension is therefore 6 and the symbol is:

$$h_{\alpha\beta} \qquad\qquad \alpha,\beta = 1,2,3.$$

Conformal class of the tangential metric tensor Assigning the tangential metric tensor can often violate boundary conditions. We will thus considered a more relaxed condition, namely that the conformal class of the tangential metric tensor is assigned; for $\gamma \in \tilde{\mathscr{S}}_2^+(M_T)$ we require

$$[g^T(x,t)] = [\gamma(x,t)],$$

where $[\cdot]$ denotes the conformal class of a tensor.

This condition does not fit the general form for boundary conditions as it is, because it does not take values in a tensor space. We therefore rewrite it as

$$g - \frac{1}{3}\operatorname{tr}_\gamma g \cdot \gamma = 0.$$

From the PDE viewpoint, this means that γ is not considered a forcing term anymore, but it is embedded in the boundary operator itself. The consequences of this are not significant from our perspective.

Using a local chart that is normal for γ the symbol turns out to be:

$$h_{\alpha\alpha} - \frac{1}{3}\sum_\beta h_{\beta\beta} \qquad\qquad \alpha = 1,2,3$$
$$h_{\alpha\beta} \qquad\qquad \alpha,\beta = 1,2,3, \alpha \neq \beta.$$

Its dimension is 5, and not 6 as it might look, because one equation can be removed by linearity. When using this boundary condition, the resulting boundary symbol is ensured to satisfy the extended complementary condition, according to which one of the first three equations must be removed.

Second fundamental form The second fundamental form Π appear in virtually every boundary term as a byproduct of a integration by parts, thus it is very important to be able to control it. For instance, it appears as a boundary term of the Chern-Gauss-Bonnet formula and it also implied when defining the Yamabe constant on a manifold with boundary (see Chapter 1). For $\gamma_1 \in \tilde{\mathscr{S}}_2(\partial M_T)$ we require that

$$\Pi(x,t) = \gamma_1(x,t),$$

though in general particular emphasis will be given to the case $\gamma_1 = 0$. In order to compute its symbol we have to apply the argument in the proof of Lemma 7.1 to the definition of the second fundamental form. We obtain that

$$\begin{aligned}\Pi_{\alpha\beta} &= \langle \nabla_\alpha \nu | \partial_\beta \rangle \\ &= g_{\beta i} \cdot \nabla_\alpha \nu^i \\ &= g_{\beta i} \cdot \hat{\nabla}_\alpha \nu^i + g_{\beta i} D^i_{\alpha k} \nu^k.\end{aligned}$$

It is simple to show that $\nu^i = \frac{g^{0i}}{\sqrt{g^{00}}}$. So

$$\begin{aligned}2\Pi_{\alpha\beta} &= -2 g_{\beta i} \frac{1}{\sqrt{g^{00}}} g^{\ell i} g^{k0} \cdot \hat{\nabla}_\alpha g_{\ell k} + g_{\beta i} \frac{1}{(\sqrt{g^{00}})^3} g^{i0} g^{\ell 0} g^{k0} \cdot \hat{\nabla}_\alpha g_{\ell k} \\ &\quad + g_{\beta i} g^{i\ell} v^k \left(\hat{\nabla}_\alpha g_{k\ell} + \hat{\nabla}_k g_{\alpha\ell} - \hat{\nabla}_\ell g_{\alpha k} \right) \\ &= v^k \cdot \hat{\nabla}_k g_{\alpha\beta} - \nu^k \cdot \hat{\nabla}_\alpha g_{k\beta} - \nu^k \cdot \hat{\nabla}_\beta g_{k\alpha} + \cancel{g_{\beta i} \nu^i} \nu^k \nu^\ell \hat{\nabla}_\alpha g_{k\ell}.\end{aligned}$$

Once coordinates adapted to the boundary are chosen, the symbol is

$$\tau h_{\alpha\beta} - \zeta_\alpha h_{\beta 0} - \zeta_\beta h_{\alpha 0} \qquad \alpha, \beta = 1,2,3 \tag{7.23}$$

and the dimension is clearly 6.

Controlling the second fundamental form also appears to be natural in view of Remark 7.11, since the second fundamental form is related, although not exactly the same thing, to the normal derivative of the tangential metric tensor.

Normal derivative of the mean curvature Let us recall again the criterion in Remark 7.11: when we assign the conformal class of the metric and the second fundamental form at the boundary, it appears that one condition is missing: every tangential component of g appears in the symbol with two different order of derivation (0 and 1), except its trace which only appears with only 1 derivative. It would be pointless to add it with 0 derivatives, since this would just resurrect the condition on the whole tangential metric tensor, which we might want to exclude when it does not allow the complementary condition to be satisfied. The next sensible thing to do is then to add it with 2 derivatives, which, in the geometric interpretation, means to assign the normal derivative of the mean curvature at the boundary. Such condition is expressed by

$$\nabla_\nu \mathscr{H}(x,t) = \sigma(x,t),$$

where σ is a function $\partial M_T \to \mathbb{R}$. As before, it will usually be sensible to take $\sigma = 0$. The condition has dimension 1 and the symbol

$$\tau^2 \sum_\alpha h_{\alpha\alpha} - 2\tau \sum_\alpha \zeta_\alpha h_{\alpha 0}$$

can be obtained by tracing (7.23) and multiplying by τ (representing a normal derivative).

Scalar curvature The scalar curvature of M at the boundary is another geometrically relevant boundary condition we will take into consideration:

$$R_g(x,t) = r(x,t).$$

It has dimension 1 and the symbol can again be recovered reasoning as in the proof of Lemma 7.1:

$$\sum_{i,j} \xi_i \xi_j h_{ij} - |\xi|^2 \sum_i h_{ii}$$

Normal derivative of the scalar curvature The normal derivative of the scalar curvature appears in the Chern-Gauss-Bonnet formula, in the control of the flow energy and in the control of the volume of the manifold during the flow (see again Chapter 1):

$$\nabla_\nu R_g(x,t) = r_1(x,t).$$

It has dimension 1 and symbol:

$$\tau \sum_{i,j} \xi_i \xi_j h_{ij} - \tau |\xi|^2 \sum_i h_{ii}$$

Normal derivative of the normal Ricci curvature The normal derivative of the normal Ricci curvature appears in the control of the flow energy (see again Chapter 1):

$$\nabla_\nu \mathrm{Ric}^g_{\nu\nu} = r_2(x,t).$$

It has dimension 1 and symbol:

$$-\tau |\xi|^2 h_{00} - \tau^3 \sum_i h_{ii} + 2\tau^2 \sum_i \xi_i h_{i0}$$

Bour-DeTurck field The Bour-DeTurck field V_B is discussed in Section 7.2, at the end of which it is said that it needs to be zero at the boundary, so that the integral flow of V_B exists for a positive time:

$$V_B(x,t) = 0.$$

The field has dimension 4 and its symbol was computed in (7.12) as part of the proof of Lemma 7.4. Here we use the assumption $a_1 = a_2$:

$$2|\xi|^2 \sum_j \xi_j h_{ij} - \xi_i \sum_{j,k} \xi_j \xi_k h_{jk} \qquad i = 0,1,2,3.$$

Ricci-DeTurck field The Ricci-DeTurck field, defined in (7.9), has no direct geometric meaning when considering the Bour flow. However, in practice it seems to be useful to build boundary operators that satisfy the complementary condition:

$$V_R(x,t) = 0.$$

It has dimension 4; the symbol can again be extracted from the proof of Lemma 7.4, and is:

$$2\sum_j \xi_j h_{ij} - \xi_i \sum_j h_{jj} \qquad\qquad i = 0,1,2,3.$$

As a variant, we can reduce it to dimension 3 by taking only its tangential component, dropping the row with $i = 0$ in the symbol.

7.5 Existence theorems

We have at last all the tools in order to give an existence theorem for the Bour flow.

7.13 Theorem. *Let $(M^4, \hat{g})$ be a smooth compact Riemannian manifold with boundary, where $\hat{g}$ is a smooth metric with geometry bounded at order 4, used as background metric for spaces of functions. Take moreover $g_0 \in \mathscr{C}^\infty\mathscr{S}_2^+(M)$, $\gamma \in \mathscr{C}^\infty\tilde{\mathscr{S}}_2^+(\partial M_T)$ and $\gamma_1 \in \mathscr{C}^\infty\tilde{\mathscr{S}}_2(\partial M_T)$, requiring that $g_0^T|_{\partial M} = \gamma(0)$, $\Pi_{g_0}|_{\partial M} = \gamma_1(0)$ and $dR_{g_0}|_{\partial M} = 0$. Take at last $K \in (0,\infty)$ and $p \in (16,\infty)$.*

Then there is

$$T(K, p, \|g_0\|_{\mathscr{C},\infty,M}, \|\gamma\|_{\mathscr{C},\infty,\partial M_T}, \|\gamma_1\|_{\mathscr{C},\infty,\partial M_T}, \|g\|_4, \mathrm{Vol}(M,\hat{g}), \mathrm{Vol}(\partial M,\hat{g}))$$

such that, for $a \in \mathbb{R}$ sufficiently close to zero and an operator $P(g)$ of the form (7.2) with $a_1 = a_2 = a$, there is $g \in W^{(2b,1),p}\mathscr{S}_2^+(M_T)$ such that

$$\partial_t g(t) = P(g(t)) \tag{7.24}$$
$$g^T|_{\partial M}(t) = \gamma(t) \tag{7.25}$$
$$\Pi_g|_{\partial M}(t) = \gamma_1(t). \tag{7.26}$$

Also, $\|g - g_0\|_{W,2b,p,M_T} \leqslant K$, $g(t)$ converges to g_0 for $t \to 0$ in the $\mathscr{C}^\ell(M)$ metric for all $\ell < 4\left(1 - \frac{2}{p}\right)$ up to a diffeomorphism of M that fixes ∂M, and g is smooth in $M_T \setminus \partial(M \times \{0\})$.

Proof. First we need to set up a system for $\bar{g}$, modified with the DeTurck trick as we have discussed in Section 7.2:

$$\partial_t \bar{g}(t) = Q(\bar{g}) \tag{7.27}$$
$$\bar{g}(0) = g_0$$
$$\bar{g}^T|_{\partial M}(t) = \gamma(t) \tag{7.28}$$
$$\Pi_{\bar{g}}|_{\partial M}(t) = \gamma_1(t) \tag{7.29}$$
$$V_R(\bar{g}|_{\partial M})(t) = 0 \tag{7.30}$$
$$V_B(\bar{g}|_{\partial M})(t) = 0, \tag{7.31}$$

where V_R and V_B are defined by (7.9) and (7.10). By Proposition 7.5, the system (7.27) satisfies the parabolicity condition with a uniform constant δ.

Let us check that the compatibility conditions expressed by Theorem 6.1 are satisfied: by hypothesis, equations (7.28) and (7.29) are satisfied at time zero, so we are done with them. Looking again at (7.9), it is evident that in order to satisfy compatibility conditions for (7.30) it is enough to take $\tilde{g}|_{t=0} = g_0$ (at this stage we can even take $\tilde{g} = g_0$ independent from the time). Then, since by hypothesis we have $dR_{g_0}|_{\partial M} = 0$, compatibility conditions are also satisfied for (7.31).

At last, let us concern ourselves with the complementary condition. Thanks to Proposition 7.10, the complementary condition can be reduced to checking that the determinant of a 20×20 matrix is not zero. This is a long and tedious task, but fortunately it is purely mechanical and can be delegated to a computer, which is able to solve it in a handful of seconds. So here we limit ourselves to stating that one can check that the complementary condition is satisfied; further details on how to actually do the computation are given in Appendix B.

All the hypotheses being satisfied, we can use Theorem 6.1 to show the existence of a solution $\bar{g}$ on M_T, for an appropriate time T. Also, by Theorem 6.3, $\bar{g}$ is smooth on $M_T \setminus (\partial M \times \{0\})$. To conclude, we need to reconstruct the actual geometric flow g from $\bar{g}$. Let us consider the vector field

$$V(x,t) := V_B(\bar{g}(t))(x),$$

which is defined on M_T and smooth at least on $M \times (0,T]$. Since $\bar{g} \in W^{(4,1),p}(M_T) \subset \mathscr{C}^{\frac{7}{2},\frac{7}{8}}(M_T)$ (by Proposition 4.43), it follows that V is uniformly continuous near $M \times \{0\}$. For a small $\varepsilon > 0$, the system of ordinary differential equations given by (7.7) and (7.8) then has a solution, which converges to a diffeomoprhism φ_0 for $t \to 0$. See for instance [Lee13, Theorem 9.34].

At last, we can define the flow as $g(t) := \varphi_t^*(\bar{g}(t))$. By Lemma 7.2, the equation (7.24) is satisfied on M_T. The boundary conditions (7.25) and (7.26) are also satisfied, on ∂M_T, because the diffeomorphisms φ_t fix the boundary ∂M. All the other claims in the theorem follow easily. □

7.14 Remark. If we want to have more regularity at $\partial M \times \{0\}$, then all we need is to ensure that higher order compatibility conditions are satisfied by the DeTurck modified system (7.27)–(7.31). We have in particular a degree of freedom in selecting the metrics $\tilde{g}(t)$: if they are taken so that $\partial_t^k \tilde{g}|_{t=0}$ coincides with $\partial_t^k \bar{g}|_{t=0}$ as given by equation (2.5), then the compatibility conditions for V_R are automatically satisfied.

7.15 Remark. The introduction of the condition V_R in the proof does not appear to have any deep geometric sense; however, the author could not find any set of boundary conditions satisfying the complementary condition that did not include V_R or at least its tangential component. This fact will be subject of further investigation.

Similar theorems can be written for different sets of boundary conditions. All the proofs are identical, so will not be repeated: the only steps that changes each time is the verification of the complementary condition; Appendix B contains details on how the related computations can be verified. In some of the cases below instead of adding the condition on V_R in the proof, we only add its tangential component V_R^T, which makes room for an additional boundary condition.

7.16 Theorem. *Theorem 7.13 remains true if we substitute* (7.25) *and* (7.26) *with the following:*

$$[g^T|_{\partial M}(t)] = [\gamma(t)]$$
$$\Pi_g|_{\partial M}(t) = \gamma_1(t)$$
$$\nabla_\nu \mathscr{H}(t) = \sigma(t).$$

The compatibility conditions at time zero and the dependencies of T must also be updated accordingly.

7.17 Theorem. *Theorem 7.13 remains true if we substitute (7.25) and (7.26) with the following:*

$$[g^T|_{\partial M}(t)] = [\gamma(t)]$$
$$\Pi_g|_{\partial M}(t) = \gamma_1(t)$$
$$R_g(t) = r(t).$$

The compatibility conditions at time zero and the dependencies of T must also be updated accordingly.

7.18 Theorem. *Theorem 7.13 remains true if we substitute (7.25) and (7.26) with the following:*

$$[g^T|_{\partial M}(t)] = [\gamma(t)]$$
$$\Pi_g|_{\partial M}(t) = \gamma_1(t)$$
$$\nabla_\nu R_g(t) = r_1(t).$$

The compatibility conditions at time zero and the dependencies of T must also be updated accordingly.

7.19 Theorem. *Theorem 7.13 remains true if we substitute (7.25) and (7.26) with the following:*

$$[g^T|_{\partial M}(t)] = [\gamma(t)]$$
$$\Pi_g|_{\partial M}(t) = \gamma_1(t)$$
$$\nabla_\nu \mathrm{Ric}^g_{\nu\nu}(t) = r_2(t).$$

The compatibility conditions at time zero and the dependencies of T must also be updated accordingly.

7.20 Theorem. *Theorem 7.13 remains true if we substitute (7.25) and (7.26) with the following:*

$$[g^T|_{\partial M}(t)] = [\gamma(t)]$$
$$\Pi_g|_{\partial M}(t) = \gamma_1(t)$$
$$\nabla_\nu \mathscr{H}(t) = \sigma(t)$$
$$\nabla_\nu R_g(t) = r_1(t).$$

The compatibility conditions at time zero and the dependencies of T must also be updated accordingly.

Appendix A

Linear algebra lemmata

A.1 Rank one perturbations of the identity

A.1 Lemma. *Let A be a matrix with is a rank 1 perturbation of the identity, i.e, $A = \mathrm{Id} + X \otimes Y$. Let us indicate with $\hat{A}$ the adjugate matrix, i.e., the matrix such that $\hat{A}A = A\hat{A} = \det A \cdot \mathrm{Id}$. Then*

$$\det A = 1 + \langle X|Y\rangle \tag{A.1}$$

$$A^{-1} = \mathrm{Id} - \frac{1}{\det A} X \otimes Y \qquad \textit{when } \det A \neq 0 \tag{A.2}$$

$$\hat{A} = \det A \cdot \mathrm{Id} - X \otimes Y \tag{A.3}$$

$$\operatorname{Ker} A = \operatorname{span} X \qquad \textit{when } \det A = 0 \tag{A.4}$$

$$\operatorname{Ran} A = Y^{\perp} \qquad \textit{when } \det A = 0. \tag{A.5}$$

Proof. Identity (A.1) can be proved by induction using the Laplace expansion for the determinant. It is clearly true if the matrix has size 1, so let us suppose it is true for all matrices of size strictly less than n: we want to prove it for matrices of size n. By doing Gauss moves we have that (we can assume $y_1 \neq 0$ up to permuting X and Y, otherwise $Y = 0$ and the result is again trivial):

$$\det \begin{pmatrix} 1 + x_1 y_1 & x_1 y_2 & \dots & x_1 y_n \\ x_2 y_1 & 1 + x_2 y_2 & \dots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \dots & 1 + x_n y_n \end{pmatrix} = \det \begin{pmatrix} 1 + x_1 y_1 & -\frac{y_2}{y_1} & \dots & -\frac{y_n}{y_1} \\ x_2 y_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & 0 & \dots & 1 \end{pmatrix}. \tag{A.6}$$

Let us take X' and Y' such that $X = (X', x_n)$ and $Y = (Y', y_n)$. Taking the Laplace expansion with respect to the last column, we need to consider two cases: the determinant of the minor associated to the 1 in the corner is $1 + \langle X'|Y'\rangle$, by the inductive hypothesis (using (A.6) right to left); the minor associated to the entry $-\frac{y_n}{y_1}$ is

$$\begin{pmatrix} x_2 y_1 & 1 & 0 & \dots & 0 \\ x_3 y_1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} y_1 & 0 & 0 & \dots & 1 \\ x_n y_1 & 0 & 0 & \dots & 0 \end{pmatrix},$$

whose determinant is $(-1)^n x_n y_1$ (notice that this last matrix has size $n-1$). In the end we have that

$$\det A = (-1)^{2n} \cdot 1 \cdot (1 + \langle X' | Y' \rangle) + (-1)^{n+1} \cdot \left(-\frac{y_n}{y_1} \right) \cdot (-1)^n x_n y_1 = 1 + \langle X | Y \rangle .$$

All the other formulae are trivial to derive. □

A.2 Matrices of polynonials with maximum rank

In Section 5.2 we need a tool to establish quantitatively how much a matrix is far from not having maximum rank. For square matrices a good measure is the square of the determinant: it is always nonnegative, it depends continuously on the coefficients and it is zero if and only if the matrix is singular (i.e., it does not have maximum rank). These properties of the squared determinant can be generalized to rectangular matrices be mean of the following lemma.

A.2 Lemma. *Let $A \in \mathscr{M}^{m \times n}(\mathbb{C})$ with $n \geqslant m$ and let $\sigma_1, \dots, \sigma_m$ be its singular values. Then:*

$$\prod_{i=1}^{m} \sigma_i^2 = \det(AA^*) = \sum_{\substack{B \in \mathscr{M}^{m \times m} \\ B \prec A}} |\det B|^2, \tag{A.7}$$

where B ranges on all the $m \times m$ submatrices of A.

In particular, A has maximum rank if and only if any of those numbers is non zero.

Proof. Let $A = USV^*$ be a singular value decomposition. Then $AA^* = USS^*U^*$ and, since S is diagonal and U is unitary, the first equality follows.

The second equality descends from the Cauchy-Binet formula, proved for example in [MM88, Chapter 2, Theorem 6.1] (it is said that the base field must be $\mathbb{R}$, but the proof works for $\mathbb{C}$ as well, appropriately replacing transposition with conjugate transposition). □

A.3 Definition. For a matrix A, let us call $\Sigma(A)$ the common value in (A.7).

When $A \colon V \to W$ is a linear morphism (without a selection of default bases on V and W to see it as a matrix), then Σ cannot be defined in general, because changes of basis will not necessarily preserve values in (A.7). However, if reference Hermitian products are considered on V and W, then they induce the isomorphisms $\varphi_V \colon V \to V^*$ and $\varphi_W \colon W \to W^*$.

A.4 Definition. Let $A \colon V \to W$ be a linear morphism and consider, for some chosen Hermitian products, the two isomoprhisms φ_V and φ_W defined above. Then we can define:

$$\Sigma(A) := \det(A \circ \varphi_V^{-1} \circ A^* \circ \varphi_W).$$

A.5 Remark. If coordinates on V and W are taken so that the two Hermitian products are the standard Hermitian products (i.e., they are described by the identity matrices), then φ_V and φ_W are trivial and the two definitions of Σ coincide.

In general, differently from the determinant, the function Σ is far from being multiplicative. An easy example is

$$A = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \qquad A \cdot B = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

for a small ε. Clearly $\Sigma(A) = \Sigma(A \cdot B) = 1$, but $\Sigma(B) = \varepsilon^2$. However, Σ is multiplicative if the left factor is square (but not if the right one is square, as the example above shows).

A.6 Lemma. *Suppose that* $A \in \mathcal{M}^{m\times m}(\mathbb{C})$ *and* $B \in \mathcal{M}^{m\times n}(\mathbb{C})$. *Then*

$$\Sigma(A \cdot B) = \Sigma(A) \cdot \Sigma(B) = |\det A|^2 \cdot \Sigma(B).$$

Proof. Just a computation:

$$\Sigma(A \cdot B) = \det(ABB^* A^*) = \det A \cdot \det A^* \cdot \det(BB^*) = |\det A|^2 \cdot \Sigma(B). \qquad \square$$

Another tool needed in Section 5.2 is the concept of linear independence modulo a polynomial, which now we define and for which we give the essential properties.

A.7 Definition. Let $A \in \mathcal{M}^{m\times n}(\mathbb{C}[t])$ be a matrix of polynomials in t with complex coefficients. Let also $p \in \mathbb{C}[t]$ be a polynomial of degree d. The *matrix of remainders*, also indicated with $A \bmod p$, is the matrix $B \in \mathcal{M}^{m\times dn}(\mathbb{C})$ whose entries are the coefficients of the remainders of the entries of A divided by p. More formally, B is the (clearly unique and existing) matrix such that, for all $i = 1,\dots,m$, $j = 1,\dots,n$,

$$p \mid A_{ij} - \sum_{k=0}^{d-1} B_{i,dj+(k-d+1)} \cdot t^k,$$

where $a \mid b$ means that the polynomial a divides the polynomial b.

A.8 Definition. Let $A \in \mathcal{M}^{m\times n}(\mathbb{C}[t])$ and $p \in \mathbb{C}[t]$. We say that the rows of A are *linearly independent* modulo p is the rows of $A \bmod p$ are linearly independent on the base field, according to the usual definition.

A.9 Definition. Let $A \in \mathcal{M}^{m\times n}(\mathbb{C}[t])$ and $p \in \mathbb{C}[t]$. We say that the rows of A are *uniformly linearly independent* modulo p with constant δ_1 if it holds $\Sigma(A \bmod p) \geqslant \delta_1$.

A.10 Remark. In a fashion similar to Definition A.4, we see that if we change the coordinates with a unitary complex matrix, then $\Sigma(A \bmod p)$ is preserved.

In general we expect to be able to simplify polynomials in the usual way: for example, we expect that $A(t)$ has linearly independent rows modulo $p(t)$ if and only if $(t-t_1)A(t)$ has linearly independent rows modulo $(t-t_1)p(t)$; and we expect $A(t)$ to have linearly independent rows modulo $p(t)$ is and only if $(t-t_1)A(t)$ has linearly independent rows modulo $p(t)$ it t_1 is not a root of $p(t)$. We therefore proceed to study to what extent these simplifications hold when uniformity of $\Sigma(A \bmod p)$ is taken into consideration.

A.11 Proposition. *Let* $A \in \mathcal{M}^{m\times n}(\mathbb{C}[t])$ *and* $p \in \mathbb{C}[t]$, *with* $\deg p = d$. *For* $t_1 \in \mathbb{C}$ *it holds*

$$\big[(t-t_1)\cdot A(t)\big] \bmod \big[(t-t_1)\cdot p(t)\big] = \big[A(t) \bmod p(t)\big] \cdot Q,$$

where Q *is a matrix of size* $nd \times n(d+1)$ *made of* n *identical blocks of size* $d \times (d+1)$ *in this way:*

$$Q = \begin{pmatrix} \boxed{T} & & & \\ & \boxed{T} & & \\ & & \ddots & \\ & & & \boxed{T} \end{pmatrix}$$

$$T = \begin{pmatrix} -t_1 & 1 & & & \\ & -t_1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -t_1 & 1 \end{pmatrix}.$$

Also, $\Sigma(T) \geqslant 1$ *and* $\Sigma(Q) \geqslant 1$.

Proof. Let us call $\mathbb{C}_d[t]$ the space of polynomials in $\mathbb{C}$ of degree less than d. It is a vector space of dimension d, which we consider with the canonical basis $(1, t, t^2, \ldots, t^{d-1})$. Then the multiplication by $t - t_1$ is a linear morphism from $\mathbb{C}_d[t]$ to $\mathbb{C}_{d+1}[t]$, whose matrix according to the canonical bases is T (when vectors are represented as row matrices).

Let $a(t)$ be any entry of $A(t)$. Clearly, as polynomials,

$$\big[(t-t_1)\cdot a(t)\big] \bmod \big[(t-t_1)\cdot p(t)\big] = (t-t_1)\cdot\big[a(t) \bmod p(t)\big].$$

Passing to the matrices of remainders (which is equivalent to expressing the remainder polynomials in the basis of $\mathbb{C}_d[t]$) and repeating the reasoning above for all the entries, the identity is proved.

Using the rightmost expression in (A.7), we have $\Sigma(T) \geqslant 1$, because the last d columns of T have determinant equal to 1. The inequality $\Sigma(Q) \geqslant 1$ follows in the same way. □

A.12 Proposition. *Let $A \in \mathcal{M}^{m\times n}(\mathbb{C}[t])$ and $p \in \mathbb{C}[t]$, with $p(t) = t^d + c_{d-1}t^{d-1} + \cdots + c_0$. For $t_1 \in \mathbb{C}$ it holds*

$$\big[(t-t_1)\cdot A(t)\big] \bmod p(t) = \big[A(t) \bmod p(t)\big]\cdot Q',$$

where Q' is a matrix of size $nd \times nd$ made of n identical blocks of size $d \times d$ in this way:

$$Q' = \begin{pmatrix} \boxed{T'} & & & \\ & \boxed{T'} & & \\ & & \ddots & \\ & & & \boxed{T'} \end{pmatrix}$$

$$T' = \begin{pmatrix} -t_1 & 1 & & & \\ & -t_1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -t_1 & 1 \\ -c_0 & -c_1 & \cdots & -c_{d-2} & -t_1 - c_{d-1} \end{pmatrix}.$$

Also, $\Sigma(T') = |\det T'|^2 = |p(t_1)|^2$ and $\Sigma(Q') = |p(t_1)|^{2n}$.

Proof. The proof is similar to Proposition A.11: now, for each entry $a(t)$ in $A(t)$, we have to consider the polynomial

$$b(t) = \big[(t-t_1)\cdot a(t)\big] \bmod p(t).$$

If $a(t)$ is already reduced modulo $p(t)$, then $b(t)$ is obtained multiplying it by $t - t_1$ and then subtracting a scalar multiple of $p(t)$ to reduce it again. This operation is described by the matrix T', from which the matrix Q' is obtained in the same way as above.

Since T' is square, $\Sigma(T') = |\det T'|^2$. Also, if T' is considered as a polynomial in t_1, then its determinant is the characteristic polynomials of the companion matrix

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{d-2} & -c_{d-1} \end{pmatrix},$$

which is the polynomial $p(t_1)$ itself (this can be proved by induction, see for example [HJ13, Theorem 3.3.14]). □

In general a monic polynomial p (say of degree d) can be equivalently described by its coefficients or by its roots (counting the multiplicity). When considering congruence classes modulo that polynomial, two preferred bases can be considered, depending on whether one wants to place emphasis on coefficients or roots:

- the first basis consists of the congruence classes of the monomials up to degree $d-1$, i.e. $([1],[t],[t^2],\dots,[t^{d-1}])$; the associated coordinates of a polynomial q are the coefficients of q after reduction modulo p;
- the second basis consists of the polynomials $[r_i]$ that have value 1 on the i-th root of p and have value 0 on all the other roots; the associated coordinates of a polynomial q are the evaluations of q on the roots of p.

When p has multiple roots the second basis is ill-defined: one can fix it by considering the derivatives of q at the multiple roots, but we leave aside this discussion; we will assume that p has distinct roots. In this lemma we discuss the determinant of the change of basis between this two representations.

A.13 Proposition. *Let $p \in \mathbb{C}[t]$ be a monic polynomial of degree d, with d distinct roots t_1, $\dots$, t_d. Let also $A \in \mathcal{M}^{m\times n}(\mathbb{C}[t])$. We call $A(t_1,\dots,t_d)$ its evaluation on the roots of p, i.e. the $m \times dn$ matrix*

$$A(t_1,\dots,t_d) = \left(\begin{array}{c|c|c|c} A(t_1) & A(t_2) & \dots & A(t_d) \end{array}\right).$$

Then

$$A(t_1,\dots,t_d) = \big[A(t) \bmod p(t)\big] \cdot Q'' \cdot P'',$$

where Q'' is a matrix of size $nd \times nd$ made of n identical blocks of size $d \times d$ in this way:

$$Q'' = \begin{pmatrix} \boxed{T''} & & & \\ & \boxed{T''} & & \\ & & \ddots & \\ & & & \boxed{T''} \end{pmatrix}$$

$$T'' = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_d \\ t_1^2 & t_2^2 & \dots & t_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{d-1} & t_2^{d-1} & \dots & t_d^{d-1} \end{pmatrix}$$

and P'' is a permutation matrix of size $nd \times nd$ made of $n \times d$ blocks of size $d \times n$ and with the following structure:

$$P'' = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1d} \\ P_{21} & P_{22} & \dots & P_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \dots & P_{nd} \end{pmatrix},$$

P_{ij} being a matrix with just one entry equal to one in position j,i and all the other entries equal to zero.

Also, $\Sigma(T'') = |\det T''|^2 = \prod_{i<j}(t_i - t_j)$ and $\Sigma(Q'') = \left(\prod_{i<j}(t_i - t_j)\right)^{2n}$. For P'', evidently, $\Sigma(P'') = |\det P''|^2 = 1$.

Proof. As usual, first we need to compute the matrix that expresses the coordinate transformation on each single polynomial, then replicate it multiple times in the fashion of Q and Q'.

Let $q(t)$ be a polynomial; then the map

$$q(t) \mapsto (q(t_1), \ldots, q(t_d))$$

is a linear morphism, whose kernel contains the polynomial $p(t)$. This implies that the map

$$[q(t)] \mapsto (q(t_1), \ldots, q(t_d))$$

is well defined, and of course linear again. By evaluating it on the basis $([1], [t], \ldots, [t^{d-1}])$, the Vandermonde matrix T'' appears. The value of its determinant is a known fact, see for example [HJ13, Section 0.9.11]. The complicated matrix P'' appears as a result of entries in $A(t_1, \ldots, t_d)$ being "packed" by their original column in A instead of by the root they are computed onto. □

Appendix B

Practically verifying the complementary condition

B.1 The difficulty with the complementary condition

In the proof of the theorems in Section 7.5 the details of the computation used to verify the complementary condition are not reported. They are, unfortunately, very long and involved, and it turns out that it is impractical not only write them down in this work, but even to perform them by hand.

Let us recall that, thanks to Proposition 7.10, in order to verify the complementary condition one first has to compose the 20×10 matrix $\mathscr{B}_0$ by stacking a subset of the rows described in Section 7.4. Then, for any selection of p, a complex number, and ζ, a vector tangent to ∂M, subject to the conditions mentioned in Section 7.3, the entries in the matrix $\mathscr{B}_0$ can be seen as polynomial with complex coefficients in τ. By substituting τ_1^+ and τ_1^- and stacking the two outcomes horizontally, one obtains the matrix (7.22), which has entries in the complex field and dimension 20×20. What we need to check is that the determinant of this matrix is not zero, and does not go to zero quicker than $|p|^5$ when $p \to 0$.

Expanding the determinant of a 20×20 matrix depending on the parameters p and ζ gives expressions that are very difficult to handle by hand, even when one takes into account the symmetries and structure of the matrix $\mathscr{B}_0$. On the other hand, there is essentially nothing creative or smart in this computation: one just needs to expand polynomials with a lot of terms in them and then check where their roots are located. Activities of this type are usually much better performed by a programmable computer instead of by a human being, not only in terms of time wasted, but also in terms of probability of errors. The author thus wrote a computer program to do the computations and automatically check whether some set of boundary conditions verifies or not the complementary condition.

The program is written in the popular Python programming language and is based on the SageMath software [Sag16]. SageMath is a free and open source mathematics software, which self-describes its mission as "creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab", four proprietary software packages often used for doing scientific and technical computations. Version 7.2, the latest available when these computations were began, was used, but the same program is expected to work, perhaps with minor modifications, on more recent ones. Also, SageMath has the advantage of working consistently on all the major operating systems available today.

It is worthwhile to mention that, although numerical computations with the computer are often associated to approximation errors, internal algorithms in SageMath use either interval arithmetic or exact representations of rational and algebraic numbers, so they are guaranteed to give a correct answer, unless a programming bug is present in the implementation. Even in that case, being the source code of SageMath (and of the program described below) available to everyone for audit and modification, chances are that potential bugs are eventually found and corrected. The SageMath developers community has furthermore adopted a peer review scheme for all the modifications that are accepted in SageMath, similarly to what happens for academic papers.

The SageMath package is already able to handle a lot of common mathematical objects, like polynomials, equations, different rings and fields, functions, and many other. All of them can be seen as objects of the Python programmin language, and, by calling appropriate methods or operators on them, a programmer can automate a procedure for performing certain operations. For example, an object of type "polynomial" has a method named "factor", that returns a factorization of that polynomial (provided that the base field is one for which SageMath knows a factorization algorithm). Similarly, an object of type "matrix" will have methods for computing the determinant, adding, multiplying, selecting some rows or columns; or an object of type "complex" will support arithmetic operations, comparison with other object of compatible types and, for instance, operators that check if a certain complex number is in fact real or rational. There also exists a SageMath package called "SageManifolds" ([GBM15]), at the moment of writing still in development, for working with differential manifolds and other differential geometry objects, but it was not used for this work.

Thanks to this formidable ground, the program that checks the complementary condition turns out to be relatively straightforward. It runs in less than a second on a reasonably modern computer, so the user can easily perform many different experiments changing the boundary conditions as they please, in search of the one with the best geometric properties for the problem in study. At the same time, it must be mentioned that the computations performed by the computer tend to be rather opaque: one knows whether the answer is "yes" or "no", but it is difficult to understand the "deep" reasons for which a set on boundary conditions is acceptable and another, maybe seemingly similar, is not.

B.2 How does the program work

The author's program is distributed in a repository on the GitHub platform, at the address `https://github.com/giomasce/complementary-cond`. It consists of a single file named `checker.sage`; in order to run it, a working installation of SageMath is required: see installation instructions on the website `http://www.sagemath.org/`. As mentioned above, the version 7.2 of SageMath was used by the author; the same code should work in more recent versions, unless some noncompatible changes were introduced. In any case, it should not be difficult to adapt the program to more recent versions of SageMath. In this section we briefly go through the program's code to explain what it does; the description here is rather high level and does not cover the details of what each line does, which is better described in the documentation of the SageMath package. Also, the program contains some comments that document what each individual method is supposed to compute.

By default, when executed with SageMath, the program `checker.sage` does the computation required to prove Theorem 7.13. It outputs some details on what it is actually doing, and the last line of output says whether the complementary condition was actually verified or not (in particular, it says it is in the default configuration). If one wants to perform the check for different boundary conditions, one just has to tweak the content of the matrix `B0_eqs`, generated at the beginning of the method `main`, for example commenting and decommenting the example lines, or even adding new ones. Each element of `B0_eqs` is an homogeneous polynomial of first order in the variables h_{ij}, whose coefficients depend on ξ_i, exactly in the same way they are represented in Section 7.4. The metric is assumed to be adapted to the boundary, so $\tau = \xi_0$ and $\zeta_\alpha = \xi_\alpha$. In the code, h_{ij} is represented with `h`ij (aliased also as `h`ji or `hvars_mat[`i`][`j`]`); τ is represented with `t` and ζ_i is represented with `z`i (both are also aliased to `x`i or `xvars_real[`i`]`).

All the variables above are defined in the first lines of the program `checker.sage`. Also, for each variable there is a variant with an additional `t` at the beginning, which represent the second instance of the same variable when the matric $\mathscr{B}_0$ is doubled as per Proposition 7.10. So, variable `t` will be evaluated to τ_1^+ and variable `tt` will be evaluated to τ_1^-. No doubling is done for variables ζ_i, since in that case they are always evaluated on the same vector ζ_i. Another variable introduced at the beginning of the program is `q`, that stands for $\sqrt{-2p}$. Immediately after, some relevant polynomials rings are defined, because SageMath often needs to be explicitly told what is the exact algebraic structure on which it needs to operate.

Let us now consider what happens in `main` after the list `B0_eqs` has been created. First, the matrix `D0`, representing the matrix (7.22), is created, by doubling the entries on each row of `B0_eqs` and replacing variables in the second copy with their `t` variants (which corresponds, as said above, to evaluating on τ_1^- instead of τ_1^+). Then the determinant is computed and stored in `det_poly`; from the proof of Proposition 7.10, we expect the determinant to be divisible by $(\tau_1^+ - \tau_1^-)^{10}$, so we check this fact and divide the determinant by that known factor. After having removed the degenerate term, we expect the quotient to have no roots on $\operatorname{Re} p > -\delta_1 |\zeta|^4$, and the rest of the program is devoted to verify this hypothesis.

The polynomial `det_poly` is factored and processing continues independently for each of the factors (this is not essential, the whole polynomial could be processed at once, but working on individual factors helps speeding up computations and possibly catching a glimpse of how much "bad" the complementary condition fails when this is the case). Each factor is passed to the method `check_factor`, which will in the end return `True` or `False` depending on whether the factor satisfies or not the complementary condition. The whole program will terminate with success if all the factors were checked with success.

Let us now consider what happens inside the method `check_factor`, where the factor to check is passed in the argument variable `factor`. Let us recall that `factor` is a polynomial in $\tau := \tau_1^+$, $\tilde{\tau} := \tau_1^-$, and the components ζ_i. The variable p is not explicitly present, but will appear as a result of substituting the definition of τ_1^+ and τ_1^-. But first we would like to get rid of the variables ζ_i: first, since the boundary condition we have is geometric, we expect that the dependence of `factor` on ζ_i can actually be factored by $|\zeta|^2$; second, we can spend the homogeneity of the principal symbol and impose $|\zeta|^2 = 1$. The dependency on $|\zeta|^2$ is recognized thanks to SageMath's support for symmetric polynomials (calles "symmetric functions"); SageMath supports many different bases for expressing a symmetric function: we use the "powersum basis", which is generated by the polynomials

$$\begin{aligned} p_0(\zeta) &= 1 \\ p_1(\zeta) &= \zeta_1 + \cdots + \zeta_n \\ p_2(\zeta) &= \zeta_1^2 + \cdots + \zeta_n^2 \\ &\cdots \end{aligned}$$

When expressed according to this basis, `factor` depends on $|\zeta|^2$ if and only if it only has components along $p_0(\zeta)$, $p_2(\zeta)$ and their powers. While SageMath can automatically represent a symmetric polynomial as a polynomial of the symmetric functions in the chosen base, it is not able to compute a minimal expression using the fact that ζ is a 3-dimensional vector. This simplification stage was then implemented in the method `simplify_SF`. Immediately after, the method `evaluate_sym` is called, that evaluates the symmetric polynomial on $|\zeta|^2 = 1$ (and, in particular checks that it does not depend on $p_i(\zeta)$ for i different from 0 and 2).

At this point, the variable `atz1` containes the original polynomial `factor` after the evaluation at $|\zeta|^2 = 1$. Now we need to substitute the definitions

$$\tau^2 = (\tau_1^+)^2 := -|\zeta|^2 + i\sqrt{2p} = -1 + q \tag{B.1}$$

$$\tilde{\tau}^2 = (\tau_1^-)^2 := -|\zeta|^2 - i\sqrt{2p} = -1 - q. \tag{B.2}$$

This requires some care, because `atz1` will not in general be a function of τ^2 and $\tilde{\tau}^2$, meaning that in general it will also contain odd powers of τ and $\tilde{\tau}$. In theory one can ask SageMath to exactly solve a system of arbitrary polynomial equations, which is implemented using Gröbner bases and the Buchberger's algorithm. However, it turns out that the computation is really slow, and actually the author could not see a finished computation with this technique. So, instead of using a generic one, a specialized algorithm for the specific polynomials in use was devised. Suppose that we have a polynomial $p(\tau, \tilde{\tau})$, in which we want to substitute (B.1) and (B.2). Without loss of generality, we can assume to already have substituted all the even powers, so all monomials of p depend on either τ, $\tilde{\tau}$ or $\tau\tilde{\tau}$. We rewrite

$$p(\tau, \tilde{\tau}) = p_1(\tau, \tilde{\tau}) + p_2(\tilde{\tau}),$$

basically collecting all terms containing τ in p_1 and all the others in p_2. Then $\tau \mid p_1(\tau, \tilde{\tau})$, so $\tau^2 \mid p_1^2(\tau, \tilde{\tau})$ and we can substitute all instances of τ^2 in it with (B.1). So, if we consider the polynomial

$$p(\tau, \tilde{\tau}) \cdot (p_1(\tau, \tilde{\tau}) - p_2(\tilde{\tau})) = p_1^2(\tau, \tilde{\tau}) - p_2^2(\tilde{\tau}), \tag{B.3}$$

we have that all its instances if τ can be substituted. After another iteration of the same method, this time with $\tilde{\tau}$, we obtain a polynomial that depend only on $q = \sqrt{-2p}$, whose roots can be enumerated thanks to SageMath's factorization algorithms for univariate polynomials. The algorithm described here is performed by the methods `separate_vars` (which compute the $p_1 + p_2$ decomposition), `mangle` (which substitutes (B.1) and (B.2)) and `super_mangle` (which implements the iterative scheme and is directly called by `check_factor`).

In the end, the variable `final` contains a polynomial in q, whose roots are the values $\sqrt{-2p}$ for the numbers p for which the matrix describing the complementary condition becomes singular. However, in the procedure above for substituting τ and $\tilde{\tau}$ some new roots are likely to have been added in the step (B.3). So for each root in q we need to compute p and the associated values $\tau = \tau_1^+$ and $\tilde{\tau} = \tau_1^-$, recalling that they are specified by the fact of having positive imaginary part. The program computes two boolean flags: `valid` is true when the original equation is satisfied by the chosen τ_1^+ and τ_1^-; `parabolic` is true when p has negative real part. In the end, a factor is accepted if there are not roots q that are `valid` but not `parabolic`. Also, as already mentioned, the whole boundary condition is accepted if all its factors are accepted.

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