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"Quantum kinetic models of open quantum systems in semiconductors theory"

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Introduction

In this thesis we will present the results achieved in three years of research, performed between Scuola Normale Superiore, under the supervision of Prof. A. M. Anile, and the University of Münster, under the supervision of Prof. A. Arnold. In particular, are collected here three articles; the first of them is in collaboration with Doct. L. Barletti of the University of Florence, while the third with Prof. A. Arnold and E. Dhamo of the University of Münster.

The aim of the developed research is to contribute to the analytical study of quantum kinetic models of certain quantum systems, whose dynamics is time-irreversible due to the interaction with the environment; accordingly, they are called *open*. In particular, the models under examination have a well-grounded application to the simulation of nanoscale semiconductor devices, thus semiconductor physics is the background of reference for our work.
The models are formulated according to the Wigner-function formalism, a well-known tool in both the physical and the mathematical literatures, which provides a quantum-mechanically consistent, phase-space description of the dynamics of the systems of interest.
The leitmotiv of our investigation is the attempt of keeping to a *purely kinetic* analysis. More precisely, our aim is to obtain results that are physically-consistent and in agreement with those achieved via other formalisms, but *independently* of them. The way we pursue that end is by developing new analytical tools, in some cases inspired by formal analogies with other problems. The motivation for that type of study is that, apart from being analytically challenging, it is naturally suitable to numerical reformulation for applications to real devices simulation.
The aspects of our research we have here presented will be widely discussed, in comparison with the existing literature, according to the following sectioning: In Part I, we will present an overview of the possible mathematical descriptions of semiconductor physics. In particular, in Chapter 1, we will introduce the (semi-)classical kinetic approach and start to discuss possible ways of modelling the irreversible dynamics of certain systems. In Chapter 2, we will focus on quantum systems, since we are interested in the applications to semiconductor devices reaching quantum regimes and we will present the quantum-statistical formalism. In Section 2.2 we will briefly introduce the theory of open quantum systems, which constitutes the reference frame of our work, and proceed in the discussion of the literature related to the description of irreversible quantum dynamics. Thus, the background is complete.
The aim of Chapter 3, in Part II, is to present the quantum kinetic description of the quantum systems, and, to compare it with the quantum-statistical one. By that discussion we will derive the motivation for our analytical study: in particular, in Section 3.4, we will describe the starting point of our investigation, namely, the choice of the Hilbert space of L^2-functions defined on the phase-space, as the state space for the successive well-posedness studies. In Chapter 4 are introduced new tools we have employed in the cited articles, which are also promising in view of the resolution of open problems. We will compare them with similar instruments used in *classical kinetic theory*, which in many cases have directly inspired their derivation in the quantum framework. The tools presented constitute a contribution to the discussion in literature about the analogies between the Schrödinger and the Vlasov equations. We anticipate that we will recover a further *a posteriori* motivation for our choice of the functional setting: according to our investigation, the analogy with the classical kinetic formalism can indeed be exploited *just in the L^2-context*.

Part III and Part IV contain the bodies of the above cited articles: in particular, Part III those related to the Wigner-Poisson system on bounded spatial domains, while Part IV, the one about the (all-space) Wigner-Poisson-Fokker-Planck model. At the beginning of each part we will provide both a description of the physical system they are meant to describe and a discussion of the related literature.
We remark that, in the three cases, the well-posedness result will be obtained in the Hilbert space of L^2-functions defined on the phase-space, modified by an appropriate *weight in the velocity-direction*. Accordingly, the result we present in Part IV is a slightly improved version of the one presented in the above cited paper, where also a weight in the space-direction was used.
Relatively to both problems (in the all-space formulation), we also discuss possible perspectives for attaining the same well-posedness theorems in a L^2-setting, *without* using the weights.

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Our acknowledgement is also addressed to Professors, Researchers and PhD-students of the universities of Firenze and Catania, for their constant support, advice and affection.
Then, a special thanks is for the PhD-students of various nationalities we have met during the past three years, at Scuola Normale, other universities, schools and congresses: they have contributed with their pleasant and stimulating company, to make that life period a stirring experience, both from the scientific (more in general, the cultural) and the personal point of views.
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Part I

Preliminaries

Chapter 1

Semiconductor materials

1.1 Quantum description

A semiconductor is a solid-state material; accordingly, its atoms are arranged in a crystal. This structure can be described as an infinite three-dimensional array in which the atoms are located at the points P of a lattice L. L is represented, starting from the basis $\{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ of $\mathbb{R}^3$, as the set of vectors

$$L := \{i\vec{a_1} + j\vec{a_2} + z\vec{a_3},\ i, j, z \in \mathbb{Z}\}$$

(cf. the definition of *Bravais lattice* in Ref. [12]), and $P \equiv (i, j, z) \in \mathbb{R}^3_L$, $i, j, z \in \mathbb{Z}$ (where we indicate with $\mathbb{R}^3_L$, $\mathbb{R}^3$ with the basis of L). Thus, the material consists of a periodic distribution of charges, which generates an electrostatic potential V_{per} with the same period; the motion of a free-electron in the periodic field (usually called *Bloch electron*) is described by the following version of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi(x) + V_{\text{per}}(x)\psi(x) = E\psi(x),\ x \in \mathbb{R}^3, \tag{1.1}$$

where $\psi : x \in \mathbb{R}^3 \to \mathbb{C}$ is the free-electron wave-function, m is the free-electron mass and E the value of its total energy[1]. The physical space $\mathbb{R}^3$, in which the electron move, can be partitioned by translates of an *elementary cell* D with the vectors of L, where

$$D := \{(\alpha, \beta, \gamma)\,,\ \alpha, \beta, \gamma \in [-1/2, 1/2]\}.$$

Correspondingly, it can be defined the dual lattice L^*

$$L^* := \{i\vec{a_1^*} + j\vec{a_2^*} + z\vec{a_3^*},\ i, j, z \in \mathbb{Z}\},$$

with the vectors $\vec{a_1^*}, \vec{a_2^*}, \vec{a_3^*}$ defined by $\vec{a_m} \cdot \vec{a_n^*} = 2\pi\delta_{mn}$, $m, n = 1, 2, 3$ (the Kronecker delta symbol). Accordingly, for all $\vec{l} \in L$, $\vec{l^*} \in L^*$, it holds $e^{i\vec{l^*}\cdot\vec{l}} = 1$. The elementary cell D^* can be defined analogously to D, starting from the lattice L^*. In particular, the *first Brillouin zone* $\mathcal{B}$ is the elementary cell constituted by the points of $\mathbb{R}^3$ closer to the origin, than to

[1] Since this is a many-electrons problem, in the one-electron Schrödinger equation it should appear also a potential taking into account electron-electron interactions. In the *independent* electron approximation, these interactions can be represented by an effective one-electron potential, V_{per}, which has the same periodicity of the crystal (cf. Ref. [12]).

any other point $P^* \equiv (i,j,z) \in \mathbb{R}^3_{L*}$, $i,j,z \in \mathbb{Z}$.
According to such partition of the space, the wave-function ψ can be decomposed into

$$\psi(x) = \frac{1}{\mathrm{vol}(\mathcal{B})} \int_{\mathcal{B}} \psi_k(x)\, dk\,, \tag{1.2}$$

with the function ψ_k k-pseudo-periodic in L, i.e.

$$\psi_k(x+\vec{l}) = \psi_k(x)\, e^{i\vec{k}\cdot\vec{l}}\,,\ \forall\, \vec{l} \in L$$

(cf. the definition of *Bloch decomposition* in Ref. [21]). Observe that, for all $\vec{l^*} \in L^*$ and $\vec{l} \in L$, $e^{i\vec{k}\cdot\vec{l}} = e^{i(\vec{k}+\vec{l^*})\cdot\vec{l}}$, thus, k can be indeed restricted to $\mathcal{B}$. Equivalently,

$$\psi_k(x) = \sum_{\vec{l}\in L} e^{-i\vec{k}\cdot\vec{l}} \psi(x+\vec{l})\,.$$

According to such decomposition[2], it can be proved (cf. Ref. [12]) that the Schrödinger equation (1.1) is equivalent to infinitely many Schrödinger equations indexed by $k \in \mathcal{B}$ posed on the elementary cell D with k-pseudo-periodic boundary conditions on ∂D:

$$-\frac{\hbar^2}{2m}\Delta\psi_k(x) + V_{\mathrm{per}}(x)\psi_k(x) = E_k\psi_k(x),\ x \in D, \tag{1.3}$$

$$\psi_k(x+\vec{l}) = \psi_k(x)\, e^{i\vec{k}\cdot\vec{l}}\,,\ \forall\, x \in \partial D,\ x+\vec{l} \in \partial D\,. \tag{1.4}$$

It can be proved as well that, for all $k \in \mathcal{B}$, there exist a sequence of eigenvalues $\{E_k^n\,,\, n \in \mathbb{N}\}$ and another of eigenfunctions $\{\psi_k^n(x)\,,\, n \in \mathbb{N}\}$. The ψ_k^n are "distorted" waves

$$\psi_k^n = u_k^n(x)\, e^{i\vec{k}\cdot\vec{x}}\,,$$

by u_k^n L-periodic functions (i.e. $u_k^n(x+\vec{l}) = u_k^n(x)$), because of the periodic potential. Accordingly, the vector k is *not* a real wave-vector, but a pseudo wave-vector and the corresponding pseudo momentum[3] $p = \hbar k$ is called *crystal momentum* (cf. Ref. [12]). The function $k \in \mathcal{B} \to E_k^n$ is called dispersion relation and describes the way the energy depends on the pseudo wave-vector k.
An electron that is in the eigenstate ψ_k^n has the energy E_k^n. For each fixed n, E_k^n individuates the n-th energy band: we assume for simplicity that the bands do not cross. It can happen that some energies are not in the range of any of the functions E_k^n: such energies are "forbidden" and fall in intervals that are called energy gaps. In our picture, two successive bands are separated by an energy gap.
Now we specify in what consists a *semiconductor* material. In a crystal at thermodynamical equilibrium, each electron will occupy one[4] of the eigenstate described by ψ_k^n. In particular, at zero temperature, only the lowest energy states will be occupied and the highest value among the energies corresponding to the occupied energy states is called the Fermi energy. The material is an insulator or a semiconductor, in case all the states in the band below the Fermi energy, the *valence* band, are occupied and the energy gap, between the valence band

[2]Precisely, the Hilbert space $L^2(\mathbb{R}^3)$ of the wave-functions is represented as $L^2(\mathcal{B}, L^2(D)_k)$, with $L^2(D)_k$ the space of square integrable k-pseudo-periodic functions.

[3]In fact, due to the presence of the periodic potential, ψ_k^n is *not* simultaneously an eigenfunction of the momentum operator $(\hbar/i)\nabla_x$ with eigenvalue $p = \hbar k$.

[4]The electrons are Fermions, thus, the Pauli's principle excludes that two electrons can occupy the same state (observe that we are ignoring the spin).

and the band above the Fermi energy (the *conduction* band), is large with respect to the energy that can be gained from the electric field. Thus, electron conduction, i.e. electron motion in k-space, is excluded.
However, in the semiconductor case the gap is moderate[5], then typically the absorption of thermal energy is enough to allow the electrons to jump from valence band states to conduction states, due to the electric field. The electrons "moving" to empty states in the conduction band leave behind *vacancies* (empty states in the valence band), which can be filled by valence electrons[6]. Hence, occupation of conduction sites will propagate, namely, conduction will occour, and, simultaneously, vacancies will propagate in the band below. These two simultaneous phenomena can be described as the motion of two different classes of particles: electrons in the conduction band and *holes* in the valence band.
Typical examples of semiconductor materials are Silicon (Si), Germanium (Ge) and Gallium-Arsenide (GaAS).

1.2 Semi-classical description

In the semi-classical picture, an electron belonging to the nth-energy band is described as a point particle individuated by the continuous variables (x, k), moving with velocity v_k^n. In the semi-classical limit, the velocity of the particle will coincide with the group velocity of the wave-packet, made of wave-functions near the (pseudo) wave-vector k ($|k| = \kappa$). Since the group velocity is defined by $v_g = d\omega/d\kappa$, with ω the frequency associated to the wave-function of energy ϵ, $\omega = \epsilon/\hbar$, then

$$v_g = \frac{1}{\hbar}\frac{d\epsilon}{d\kappa} \tag{1.5}$$

(cf. Ref. [36]), accordingly

$$v_k^n = \frac{1}{\hbar}\nabla_k E_k^n \,.$$

Thus, the equation of motion of an electron in the crystal reads

$$\dot{x} = \frac{1}{\hbar}\nabla_k E_k^n \,. \tag{1.6}$$

The dispersion relation for a free-electron in the vacuum reads $\epsilon(\kappa) = (\hbar^2/2m)\kappa^2$, and the curvature of such parabola is $1/m$, the reciprocal mass. On the other hand, an electron in a periodic potential is accelerated relatively to the crystal, say, by an external force F, as if the mass of the electron were equal to an *effective* mass. Indeed,

$$\dot{v_g} = \frac{dv_g}{dt} = \frac{1}{\hbar}\frac{d^2\epsilon}{d\kappa\, dt} = \frac{1}{\hbar}\frac{d^2\epsilon}{d\kappa^2}\frac{d\kappa}{dt}\,,$$

where $\hbar\, d\kappa/dt = F$ for an electron in a crystal (cf. Ref. [36]). Accordingly,

$$\frac{dv_g}{dt} = \left(\frac{1}{\hbar^2}\frac{d^2\epsilon}{d\kappa^2}\right)F\,, \quad \text{or } F = \left(\frac{1}{\hbar^2}\frac{d^2\epsilon}{d\kappa^2}\right)^{-1}\frac{dv_g}{dt}\,.$$

[5] Precisely, comparable to $k_B T$ with T temperature, k_B Boltzmann constant.

[6] A way, alternative to thermal excitation, to improve the conductivicity, consists in inserting impurities in the crystal lattice. They can be either positively or negatively ionized, correspondingly, they will either provide electrons to the conduction band, or trap electrons in the valence band: such process is called *doping*.

which reads like the second Newton's law, if we define the effective mass[7] m^* as

$$\left(\frac{1}{\hbar^2}\frac{d^2\epsilon}{d\kappa^2}\right)^{-1} =: m^* \,.$$

If we call respectively $\epsilon_c(k)$ and $\epsilon_v(k)$ the conduction and valence band energies (i.e. $E_k^n =: \epsilon_v(k)\,,\ E_k^{n+1} =: \epsilon_c(k)$ for some $n \in \mathbb{N}$), and

$$\epsilon_c = \min_{k\in\mathcal{B}} \epsilon_c(k)\,, \quad \epsilon_v = \max_{k\in\mathcal{B}} \epsilon_v(k)\,,$$

then

$$\epsilon_c(k) = \epsilon_c + \frac{\hbar^2}{2m^*}|k|^2\,, \quad \epsilon_v(k) = \epsilon_v - \frac{\hbar^2}{2m^*}|k|^2\,, \quad \forall k \in \mathbb{R}^3$$

are called *parabolic* (or effective mass) band approximations[8]. Accordingly, for what an electron in the conduction band is concerned, (1.6) reads

$$\dot{x} = v_c(k)\,, \quad v_c(k) = \frac{1}{\hbar}\nabla_k \epsilon_c(k) = \frac{\hbar k}{m^*}\,, \tag{1.7}$$

while for a hole in the valence band

$$\dot{x} = v_v(k)\,, \quad v_v(k) = -\frac{1}{\hbar}\nabla_k \epsilon_v(k) = -\frac{\hbar k}{m^*}\,. \tag{1.8}$$

If an external potential $V = V(x)$ is added to the periodic potential V_{per}, the analysis carried in the previous section continues to hold, if the potential is assumed to be *weak*[9]. Then, the motion of an electron in the conduction band is described by

$$\dot{x} = v_c(k)\,, \quad \dot{k} = \frac{q}{\hbar}\nabla_x V(x). \tag{1.9}$$

These are Hamilton's equations for the pair of conjugate variables $(x, p) = (x, \hbar k)$ with Hamiltonian

$$H(x,p) = \epsilon_c\left(\frac{p}{\hbar}\right) - qV(x)\,.$$

Analogously, for the hole in valence band

$$\dot{x} = v_v(k)\,, \quad \dot{k} = -\frac{q}{\hbar}\nabla_x V(x)\,, \quad H_v(x,p) = \epsilon_v\left(\frac{p}{\hbar}\right) + qV(x)\,. \tag{1.10}$$

By the way, we observe that the hole behaves like a particle endowed with effective mass m^* and opposite charge q and opposite velocity $-v_c(k)$ (cf. Ref. [36]).

[7] In case of an anisotropic energy surface, as in Ge and Si, we should introduce components of the reciprocal effective mass *tensor*:

$$\left(\frac{1}{m^*}\right)_{\mu\nu} = \frac{1}{\hbar^2}\frac{d^2\epsilon}{d\kappa_\mu d\kappa_\nu}\,, \quad \frac{dv_\mu}{dt} = \left(\frac{1}{m^*}\right)_{\mu\nu} F_\nu\,.$$

[8] Such approximation is valid indeed in the neighborhood of a conduction (respectively, valence) band minimum (respectively, maximum). The Kane dispersion relation, instead, takes into account the non-parabolicity at high energy, in terms of a parameter α, accordingly

$$\epsilon_c(\kappa)[1 + \alpha\epsilon_c(\kappa)] = \frac{\hbar^2\kappa^2}{2m^*}\,,\ k \in \mathbb{R}^3\,,$$

and analogously for the valence band.

[9] This depends on the fact the Bloch decomposition does not commute with the multiplication by V, thus, the bands are all coupled. However, the effect of the coupling can be disregarded if the additional potential is supposed to be weak enough.

1.3 Semi-classical kinetic equations

Let $z = (z_1, \ldots, z_n)$ individuate an event in terms of n continuous variables, z ranging in a continuous set $Z \subset \mathbb{R}^n$, and, for all $t \in \mathbb{R}$, let $P(.\,,t) : z \in Z \to \mathbb{R}^+$ be the probability distribution function associated to the realization of an event $z \in Z$ at time t. Then, $P(z,t)\,dz$ will be the probability that the event which occours at time t is described by a vector belonging to the infinitesimal volume of Z around z and

$$\int_Z P(z,t)\,dz = 1\,, \quad \forall\, t.$$

Let $f : z \in Z \to \mathbb{R}$ represent some quantity that can be computed starting from the event z, then the average value of f at time t will be

$$< f\,P>(t) = \int_Z f(z)P(z,t)\,dz\,. \tag{1.11}$$

Let $\zeta : z \in Z \to \mathbb{R}^n$ be a vector field such that

$$\dot{z} = \zeta\,, \qquad \operatorname{div}\zeta = 0\,. \tag{1.12}$$

Thus, to all $z_0 \in Z$, it can be associated, for all $t > 0$, $z(t) \in Z$ by integrating Eq. (1.12) with the initial condition $z(0) = z_0$, if ζ is regular enough. Analogously, to the volume dz around z_0 will correspond, for all $t > 0$, the volume $dz(t)$ around $z(t)$ and, accordingly, the probability that the event at time t lies in $dz(t)$ around $z(t)$, $P(z(t),t)\,dz(t) = P(z_0,0)\,dz$, i.e. the probability that it lies at time $t = 0$ in dz around z_0. Moreover $\operatorname{div}\zeta = 0$, thus, by the *Liouville theorem*, $dz(t) = dz$, i.e. the volume is preserved in the correspondence $z_0 \to z(t)$, and then

$$P(z(t),t) = P(z_0,0)\,. \tag{1.13}$$

Differentiating both sides with respect to t and remembering Eq. (1.12), it follows

$$\left(\frac{\partial P}{\partial t} + \zeta\cdot\nabla_z P\right)\bigg|_{(z(t),t)} = 0\,, \quad \forall\, t > 0,\, z \in Z.$$

Due to the time-reversibility of the equations, the mapping $z \to z(t)$ is invertible and

$$\frac{\partial P}{\partial t} + \zeta\cdot\nabla_z P = 0. \tag{1.14}$$

follows. Eq. (1.14) is called the *Liouville* (or *transport*) equation.
Observe that $z = (x,p) \in \mathbb{R}^{2d}$ can be the phase-space description of the state of a system with d degrees of freedom. Let $\zeta = (\partial H/\partial p, -\partial H/\partial x)$ with $H = H(x,p)$ Hamiltonian function (associated to the system) expressed in terms of the canonical variables, then, in case H is regular enough, the Hamilton equations will read as eqs. (1.12) and Eq. (1.14) still holds. Correspondingly, if we call $f_c(x,k,t)dx\,dk$ the probability of finding the electron in the volume $dx\,dk$ of the conduction-band phase-space around (x,k), the Hamilton Eqs. (1.7), (1.9) are analogous to Eqs. (1.12), thus (1.14) will read as

$$\frac{\partial f_c}{\partial t} + v_c(k)\cdot\nabla_x f_c + \frac{q}{\hbar}\nabla_x V(x)\cdot\nabla_k f_c = 0\,, \quad (x,k) \in \mathbb{R}^6\,, t \in \mathbb{R}\,, \tag{1.15}$$

which is called the *one-particle* semi-classical Vlasov equation.
Exactly the same equation describes the evolution in time of the probability distribution

function $f_c^{(N)}$ relative to an ensemble of N non-interacting electrons in a periodic field, subject to an external potential V. Indeed, with $f_c^{(N)}(z) = f_c(\{x_i\}_i, \{k_i\}_i)\,,\ i = 1, \dots 3N$, the Hamilton equations will look exatly the same as Eqs. (1.7), (1.9) in the phase-space $\mathbb{R}^{6N}$, thus

$$\frac{\partial f_c^{(N)}}{\partial t} + v_c(k)\cdot \nabla_x f_c^{(N)} + \frac{q}{\hbar}\nabla_x V(x)\cdot \nabla_k f_c^{(N)} = 0\,, \quad (x,k) \in \mathbb{R}^{6N}\,, t \in \mathbb{R}\,. \tag{1.16}$$

A way to go beyond the independent electrons approximation we have so far applied to derive the Vlasov equation, is to include two-particle interaction forces in the considered dynamics. Then, starting again from a phase-space description of the ensemble of N electrons ($z = (\{x_i\}_i, \{k_i\}_i)\,, i = 1, \dots 3N$), which leads to a transport equation for the electron ensemble, and introducing the “marginal” probability distributions

$$f_c^{(d)}(\{x_i\}_{i=1,\dots 3d}, \{k_i\}_{i=1,\dots 3d}) = \int f_c(z)\, dx_{3d+1} \dots dx_{3N}\, dk_{3d+1} \dots dk_{3N}\,,$$

which are referred to the sub-ensemble of d electrons, it can be derived the BBGKY hierarchy of (transport) equations for the functions $f_c^{(d)}$, $0 < d < N$ (cf. Ref. [41] and the Refs. therein). Then, under the assumption of low correlation, which is reasonable for $d << N$, it can be obtained the following equation

$$\frac{\partial f}{\partial t} + v_c(k)\cdot \nabla_x f + \frac{q}{\hbar}(\nabla_x V(x) + \nabla_x \Phi(x))\cdot \nabla_k f = 0\,, \quad (x,k) \in \mathbb{R}^{6}\,, t \in \mathbb{R}\,, \tag{1.17}$$

for $f := N f_c^{(1)}$, which can be interpreted as the electron number density[10] in the phase space. In such equation, which has the shape of a one-particle equation, the reciprocal effect of the electrons (i.e. the Coulombian repulsive force they exert each on the other due to the charge) is taken into account via a *mean-field* description. Precisely, the field generated by the electrons is represented as if the ensemble constitutes a uniform distribution of negative charge with charge density $-q\,n$ (where n is the electron-ensemble position density), analogously to the *Hartree approximation* in the Schrödinger picture. Thus, to the external field ∇V, it is added the electrical field $\nabla\Phi$ calculated by solving the Poisson equation

$$-\epsilon \Delta_x \Phi(x,t) = q(n^+(x,t) - n(x,t) + (N_A(x) - N_D(x))\,,$$

where ϵ is the permittivity, n, n^+ are, respectively, electron and hole number densities in the position space

$$n(x,t) = \int_{\mathcal{B}} f(x,k,t)\, dk\,,$$

and $N_{A,D}$ are the densities of the (respectively, acceptor and donor) ions that can be implanted in the semiconductor.
Equation (1.17) is known as the *semi-classical Vlasov* equation.
It differs from the *classical* one, since v_c is an assigned function of k (cf. Eq. (1.7)), which takes into account the presence of the crystal.
However, in the present description, the crystal is represented as an *ideal* lattice, thus, we are disregarding both its imperfections (both artificial, due to the introduction of impurities, e.g., and manifactural) and the vibrations due to the thermal energy. These effects can be modelled as collisions of the electrons with *other* “particles”(respectively, with impurities

[10] i.e., the number of electrons per unit volume,

and phonons[11])[12].
Observe that, for the first time, we are taking into account the presence of what is considered, in first approximation, *external* with respect to the electron ensemble, thus, so to say, with the environment. The assumption underlying the previous derivation, precisely the application of the Liouville theorem, is instead, that the electron ensemble can be considered as a *closed* sub-system; as a consequence,

$$f(x(t), k(t), t) = f(x_0, k_0, 0)\,,\ \forall\, t\,, \quad \text{equivalently,} \quad \frac{df}{dt} = 0$$

(cf. Eq. (1.13) equivalent to Eq. (1.14)), hold.
When collisions are taken into account, f can no longer be constant along the paths $(x(t), k(t), t)$, $\forall\, t$, thus it will hold

$$\frac{df}{dt} = \mathcal{C}(f)\,, \tag{1.18}$$

with $\mathcal{C}(f)$ rate of change of the probability distribution function. In particular, the term $\mathcal{C}(f)$ will have the shape of a balance between the probability that an interaction occourring at x causes a transition from the Bloch state k to a Bloch state $k' \in \mathcal{B}$ and the probability that a transition to the state individuated by k from a certain k' takes place. Such evaluation is made in terms of a *scattering probability* $S(k, k')$, which has to be determined according to the type of interaction to be modelled.
We will not provide here the specific shape of the term $\mathcal{C}(f)$, since it would require the introduction of a terminology and of concepts (the detailed balance principle, e.g.) that are not interesting for our porpouse (cf. Refs. [21, 41], e.g.). However, the introduction of the collisional term $\mathcal{C}(f)$ in Eq. (1.18) constitutes an example of a possible way of modeling the *irreversible* interaction of the sub-system under examination with the environment (cf. Section 2.2).

Remark 1.3.1 *Our aim is, indeed, to introduce the "quantum kinetic" description of the irreversible evolution of systems related to semiconductors physics. In Part II, the* quantum transport *equation, namely the Wigner equation, will be derived from the Schrödinger equation, via a series of unitary transforms. Accordingly, the Wigner equation will provide a quantum kinetic description, equivalent to the Schrödinger one, of the* reversible *dynamics of an isolated quantum system. As a consequence, an* a posteriori *modification of such equation via the introduction of a classical collisional term, in the spirit of* (1.18)*, wouldn't be quantum-mechanically consistent. In Section 2.2 we will discuss some alternative strategies to include dissipative effects in the Wigner formalism.*

The modified version of Eq. (1.17) reads

$$\frac{\partial f}{\partial t} + v_c(k) \cdot \nabla_x f + \frac{q}{\hbar}(\nabla_x V(x) + \nabla_x \Phi(x)) \cdot \nabla_k f = \mathcal{C}(f)\,, \quad (x,k) \in \mathbb{R}^6\,,\ t > 0\,, \tag{1.19}$$

and is called the *Boltzmann* equation for semiconductors.
Eq. (1.19) is the analog, in semi-classical kinetic framework, of the Boltzmann equation of gas-dynamics. Accordingly, in the last decade, much interest has been arised by the repetition of the steps of the classical kinetic theory in the semiconductors case. Indeed, a H-Theorem corresponding to its specific collisional term and equilibrium solutions were studied in Refs. [38, 39, 40] and *macroscopic* balance equations were deduced as moments equations from the Boltzmann equation, analogously to the gas-dynamics case (cf. Ref. [18]).

[11] i.e. the independent normal modes of the harmonic oscillations that approximate thermal vibrations.
[12] Observe that, the collisions with other electrons are negligible at the densities tipically encountered in semiconductor devices.

1.4 Macroscopic description

One of the most interesting aspect of the kinetic approach is, indeed, the possibility of passing from a microscopic to a macroscopic description by computing averaged quantities, i.e. *moments*. We mention here some physical quantities that can be computed[13] starting from the electron distribution function f

$$n(x,t) \; = \; <f>(x,t) = \int_{\mathcal{B}} f(x,k,t)\,dk\,, \quad \text{electron position density}\,, \tag{1.20}$$

$$V(x,t) \; = \; <v_c f>(x,t) = \int_{\mathcal{B}} v_c(k)\,f(x,k,t)\,dk\,, \quad \text{electron velocity}\,, \tag{1.21}$$

$$P(x,t) \; = \; <pf>(x,t) = \int_{\mathcal{B}} \hbar k\,f(x,k,t)\,dk\,, \quad \text{crystal momentum}\,, \tag{1.22}$$

$$E_k(x,t) \; = \; <\epsilon_c f>(x,t) = \int_{\mathcal{B}} \epsilon_c(k)\,f(x,k,t)\,dk\,, \quad \text{electron kinetic energy}\,, \tag{1.23}$$

$$S(x,t) \; = \; <v_c\,\epsilon_c f>(x,t) = \int_{\mathcal{B}} v_c(k)\,\epsilon_c(k)\,f(x,k,t)\,dk\,, \text{ kinetic energy flux}\,. \tag{1.24}$$

The evolution equations for such quantities can be deduced from the Boltzmann equation (1.19), by multiplying by the appropriate factors and integrating in the k-space. Thus, in principle it can be created an infinite hierarchy of equations, then, the degree of precision of the description will be decided by the number of equations that are taken into account. The moments, whose time-evolution is disregarded, but appear in the equations that are considered, are to be expressed in terms of the moments that are kept into consideration. Moreover, on the right hand side of the corresponding equations, will appear the moments of the collisional term, which are called the production terms, that are also to be expressed in terms of the choosen moments. This procedure, called *closure*, is usually done by physical assumptions or reductions.

For example, in case only the balance equations for the electron position density is taken into account, the particle flux $J = nV$, is to be expressed via a constitutive equation, under the assumption that the temperature is uniform[14]. The corresponding model is called *drift-diffusion* (cf. Ref. [41], e.g.) and its validity is restricted to quasi-stationary regimes. If, instead, also the energy and energy flux equations are included, *hydrodynamical* models are obtained. One of the earliest is the *energy transport* model: there exist several versions of it, starting from the BBW[15] (cf. Ref. [48] and the Refs. therein), which uses the (questionable) closure assumption that the heat flux vector can be expressed according to the Fourier law. There, the production terms for momentum and energy are assumed to be of relaxation type and the relaxation times are computed via phenomenological arguments. Several extensions[16] of it were studied (cf. Refs. [5, 29], e.g.). However, such description reveals to be quite unsatisfactory, since certain physical relations (namely, the Onsager reciprocity relations) are not fullfilled (cf. Ref. [3]). Accordingly, a great interest has been devoted to the investigation (cf. Refs. [7, 8]) of a different closure strategy, which is based on the application of the *maximum entropy principle*, in the spirit of Extended Thermodynamics (cf. Ref. [42]).

[13]in the parabolic band approximation; otherwise, if the Kane dispersion relation is adopted, in the definition of the crystal momentum, $\hbar k$ is to be replaced with $m^* v_c[1 + 2\alpha\epsilon(k)]$, with α non-parabolicity parameter

[14]i.e., the temperature of the electrons is the same as the lattice temperature,

[15]i.e., the famous Blotekjaer-Baccarani-Wonderman, for the parabolic band approximation case

[16]e.g., to the non-parabolic case,

Constitutive relations are obtained in a systematic way, by using a formalism close to the one adopted in Ref. [37] for the classical kinetic case. Accordingly, *extended hydrodynamical* models can be considered, where high order fluxes and the respective production terms, both in the parabolic and in the Kane cases (cf. Refs. [6, 46]), are closed consistently with the Onsager reciprocity principle of linear irreversible thermodynamics. Such models provide a refined description of the semiconductor physics and constitute a relevant tool in semiconductor devices simulation (cf. Refs. [2, 4, 47]).

1.5 Beyond the semi-classical approach: quantum device simulation

In the recent years, due to the increasing degree of miniaturization and integration the semiconductor technology is pursuing, it has become important to realise transport models suitable for describing quantum phenomena and sufficiently simple to allow for efficient numerical simulations. More precisely, already when the characteristic length of the active region of the device is under 1μm, and when potential variations of $10^6\,\mathrm{V/cm}$ are reached, quantum effects occour. Additionally, are in use semiconductor devices whose performances rely on a quantum-mechanical phenomenon, namely the tunneling effect, e.g., the *resonant tunneling diodes* (RTD).
The behaviour of such devices cannot be adequately described via semi-classical Vlasov or Boltzmann equations. Therefore, it is necessary to use *intrinsically quantum* models.
In the next chapter, we will introduce the quantum-mechanical and quantum-statistical descriptions of (semiconductor) solide-state physics. However, for real device simulations, a phase-space, quantum description would be desirable: that is indeed possible via the Wigner function. Part II of the present work will be devoted to present that tool in view of the successive analytical studies.
Here, we simply cite some articles that collect numerical results obtained via the Wigner-function method, thus proving that this formalism is indeed suitable for numerical implementation: namely, Refs. [14, 31] and the Refs. therein, where different numerical methods are tested on double-barrier semiconductor structures and the physically-expected diagrams are recovered; Ref. [10], where a numerical analysis is performed in a L^2-framework of the coupled Wigner-Poisson problem, and the more recent Ref. [23], where the analogy with the classical Vlasov equation is exploited to build a scheme for a Wigner-model for non-parabolic band profiles.
For a matter of completeness, we also report on the development of a new group of devices, namely *interband* resonant tunneling diodes (IRTD), in which occour transitions of electrons from valence to conduction band, due to the tunneling effect through potential barriers (cf. Ref. [50]). As a response to the increasing simulation interest, multi-band quantum kinetic models are under examination: we just quote, e.g., Ref. [25], where the Wigner formalism is again employed.

Chapter 2

Quantum systems

2.1 Quantum-statistical description

In Section 1.1 we have presented a first example of an *isolated* quantum system, namely, an electron moving in the three-dimensional space under the effect of a periodic potential. We have described its state through a wave-function $\psi \in L^2(\mathbb{R}^3;\mathbb{C})$ and the energy associated to it can be computed via the operator "kinetic plus potential" energy $\hat{E}_{\text{tot}} = -\hbar^2/2m\,\Delta_x + V_{\text{per}}$ acting[1] on $L^2(\mathbb{R}^3;\mathbb{C})$. More precisely, the (real) value E of the energy measured via the operator $\hat{E}_{\text{tot}}$ when the system is in the state ψ is

$$E = <\psi, \hat{E}_{\text{tot}}\psi> = \int \overline{\psi(x)}\,\hat{E}_{\text{tot}}\psi(x)\,dx\,,$$

since with $<.\,,.>$ we represent the scalar product in the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C})$. In general, in Quantum Mechanics, a physical observable quantity is a linear, self-adjoint operator $\hat{A}$ on $L^2(\mathbb{R}^d;\mathbb{C})$, which is the state space of a system with d degrees of freedom, and the result of the measurement[2] of the physical observable, when the system is in the state ψ, is

$$<\psi, \hat{A}\psi>= \int \overline{\psi(x)}\,\hat{A}\psi(x)\,dx\,. \tag{2.1}$$

Let us denote with $\hat{H}$ the operator associated to the observable total-energy of the system ($\hat{H} = \hat{E}_{\text{tot}}$, in the case of the isolated electron in the semiconductor material), then the following differential equation in the space $L^2(\mathbb{R}^d;\mathbb{C})$

$$i\hbar\frac{d}{dt}\psi(t) = \hat{H}\psi(t)\,, \quad t \in \mathbb{R} \tag{2.2}$$

describes the evolution in time of the system and is the *Schrödinger* equation in abstract form (for a derivation of it, consistent with the classical mechanics limit, see Ref. [34] Chapt. I §6 and Chapt. II §8). We have already observed that Eq. (2.2) provides a quantum description of the reversible dynamics of an *isolated* quantum system.

A possible way of describing semiconductor physics, by keeping to the Schrödinger formalism, but going beyond the independent electron approximation, consists in modeling the mutual

[1] The operator potential energy is the multiplication operator by the real-valued function V_{per}. In what follows, we are not specifying the domains of definition of the operators.

[2] a real number, due to the fact the operator $\hat{A}$ is *self-adjoint*,

interaction of the electrons via a mean-field approximation. Accordingly, the dynamics of an ensemble of N electrons in an ideal crystal (represented via the V_{ion} potential) can be described by the following system of equations for the corresponding wavefunctions ψ_i , $i = 1, \dots N$,

$$
\begin{aligned}
&i\hbar \frac{d}{dt}\psi_i(t) = -\hbar^2/2m\,\Delta_x \psi_i(t) + (V_{\text{ion}} - q\,V_{\text{e}})\,\psi_i(t)\,, \quad t \in \mathbb{R}\,,\ x \in \mathbb{R}^3\,,\ i = 1, \dots N,\\
&-\epsilon \Delta_x V_{\text{e}}(x,t) = q\,n(x,t)\,, \qquad n(x,t) = \sum_{i=1}^{N} |\psi_i(x,t)|^2\,,
\end{aligned}
$$

which are known as the *Hartree equations* (cf. Ref. [12]). For a mathematically rigorous derivation of the *Hartree equation* (i.e. of a one-body Schrödinger-Poisson equation) from the N-body Schrödinger equation in the mean-field limit, see Ref. [27].
However, in case we are dealing with a quantum system with *many degrees of freedom* (many-particles system occourring in semiconductor modeling, e.g.), the use of the wave-function formalism is no longer adequate; in fact, more in general, the quantum-mechanical approach reveals to be unfeasible, as, for example, the concept of closed sub-system ceases to be rigorous (cf. Ref. [35] Chapt. I §5 for a satisfactory discussion). Accordingly, it is necessary to adopt the Quantum Statistics' formalism and introduce the concept of *density matrix* (cf. Ref. [34] Chapt. II §12), as the proper instrument for the description of the states of a "macroscopic" quantum system.

Definition 2.1.1 (Density matrix) *The state of a quantum system with d degrees of freedom is described by an operator $\hat{\rho}$ on $L^2(\mathbb{R}^d;\mathbb{C})$ of the form*[3]

$$\hat{\rho} := \sum_{j \in \mathbb{N}} \lambda_j < \psi_j, . > \psi_j$$

where $\{\psi_j |\, j \in \mathbb{N}\}$ is a complete orthonormal set in $L^2(\mathbb{R}^d;\mathbb{C})$, $\lambda_j \geq 0\,, \forall\, j \in \mathbb{N}$ and $\sum_j \lambda_j = 1$. If all but one of the λ_j are zero, the system is in a pure *state, otherwise it is in a* mixed *state.*

(cf. Ref. [28])

Remark 2.1.1 (Physical quantum state) *Due to its definition, a density matrix operator $\hat{\rho}$ is a* positive, self-adjoint, trace-class *operator on $L^2(\mathbb{R}^d;\mathbb{C})$ (cf. Ref. [43] for the definitions). According to* (2.3)*, $\hat{\rho}$ is, indeed, an integral operator with kernel $\rho \in L^2(\mathbb{R}^{2d};\mathbb{C})$*

$$\rho(x,y) = \sum_{j \in \mathbb{N}} \lambda_j \psi_j(x) \overline{\psi_j}(y)\,.$$

Equivalently, $\hat{\rho}$ is a Hilbert-Schmidt *operator; in addition, $\rho(x,y) = \overline{\rho(y,x)}$, equivalently $\hat{\rho}$ is* self-adjoint. *Moreover, $\lambda_j \geq 0\,, \forall\, j \in \mathbb{N}$ and $\{\lambda_j\}_{j \in \mathbb{N}} \in l^1(\mathbb{N})$ imply that $\hat{\rho}$ is* positive *and* trace class.
In what follows, we will consider a physically relevant state of a quantum system *("physical quantum state") to be univocally individuated by a density matrix operator.*

[3]precisely, for all $\phi \in L^2(\mathbb{R}^d;\mathbb{C})$,

$$(\hat{\rho}\phi)(x) = \sum_{j \in \mathbb{N}} \lambda_j \int \psi_j(x) \overline{\psi_j(y)} \phi(y)\, dy\,. \tag{2.3}$$

The expectation of the physical observable $\hat{A}$ relative to the system in the state $\hat{\rho}$ is well-defined[4] as $\mathrm{Tr}(\hat{\rho}\,\hat{A})$ and it coincides with

$$\mathrm{Tr}(\hat{\rho}\,\hat{A}) = \sum_{j\in\mathbb{N}} \lambda_j < \psi_j, \hat{A}\psi_j > \tag{2.4}$$

(cf. Ref. [28]). By comparing (2.4) with (2.1), it follows from Def. 2.1.1 that each λ_j can be interpreted as the probability that the system is in the state ψ_j. In particular, if it exists j such that $\lambda_j = 1$, the system is in the pure state described by the density matrix $\hat{\rho}_j := < \psi_j, . > \psi_j$ and

$$\mathrm{Tr}(\hat{\rho}_j\,\hat{A}) = < \psi_j, \hat{A}\psi_j > = \int \overline{\psi_j(x)}\,\hat{A}\psi_j(x)\,dx\,.$$

According to the previous interpretation of the $\{\lambda_j\,,\, j\in\mathbb{N}\}$, as *occupation probabilities* of the states $\{\psi_j\,,\, j\in\mathbb{N}\}$, the density n relative to the quantum system in the state described by $\hat{\rho}$ can be calculated as

$$n(x) = \sum_{j\in\mathbb{N}} \lambda_j |\psi_j(x)|^2\,, \tag{2.5}$$

and it is a positive function. Moreover, by Def. 2.1.1 and Eq. (2.4),

$$\int n(x)\,dx = \mathrm{Tr}(\hat{\rho}) = 1\,,$$

equivalently, the *total mass* of a physical QS is finite. Analogously, the kinetic energy is defined by

$$E_{\mathrm{kin}}(\hat{\rho}) := \mathrm{Tr}(\hat{H}_0\,\hat{\rho})\,,\quad \hat{H}_0 := -\hbar^2/2\,\Delta_x.$$

If we denote again with $\hat{H}$ the observable total energy, then the evolution in time of a physical quantum system is governed by the *Von Neumann* equation (cf. Ref. [34] Chapt. II §12)

$$i\hbar\frac{d}{dt}\hat{\rho} = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}\,,\ t\in\mathbb{R}\,, \tag{2.6}$$

which is a differential equation in the space of the trace-class operators on $L^2(\mathbb{R}^d;\mathbb{C})$. The corresponding Cauchy problem with initial value the density matrix $\hat{\rho}_0$ is studied, e.g., in Refs. [19, 44][5], and the solution $\hat{\rho}(t)$ is proved to be a density matrix for all times $t\in\mathbb{R}$, equivalently *individuates a physical quantum system for all times* $t\in\mathbb{R}$.
Moreover, the explicit representation that can be given of the evolute of $\hat{\rho}_0$, i.e. of $\hat{\rho}(t)$ (cf. Ref. [9]), shows that the dynamics of $\hat{\rho}$ is fully described by the time evolution of its eigenfunctions $\{\psi_j\,,\, j\in\mathbb{N}\}$, while the eigenvalues are constant in time. Thus, the Von Neumann equation (2.6) is equivalent to countably many Schrödinger equations with the same operator $\hat{H}$. As a by-product, this equivalence shows that Eq. (2.6) provides a quantum statistical description of the *reversible* dynamics of a physical quantum system.
A possible extension of such model consists in including a *Hartree-type nonlinearity* representing a mean-field interaction, namely,

$$i\hbar\frac{d}{dt}\hat{\rho} = \hat{H}(t)\hat{\rho} - \hat{\rho}\hat{H}(t)\,,\ t\in\mathbb{R}\,,\qquad \hat{H}(t) = -\hbar^2/2\,\Delta_x + V(x,t) \tag{2.7}$$

$$-\Delta_x V(x,t) = n(x,t)\,, \tag{2.8}$$

[4] actually if the operator $\hat{\rho}\,\hat{A}$ is also trace class

[5] respectively, with $\hat{H} = \hat{H}_0$ and with $\hat{H}(t) = \hat{H}_0 + V(t)\,,\ V(t)$ a time-dependent, bounded perturbation of $\hat{H}_0$

where n is defined in terms of $\hat{\rho}$ by Eq. (2.5).
Observe that the models (1.17), (2.3) and (2.7) have in common that they belong to the class of *Markovian approximation* of the dynamics of the corresponding systems: in Ref. [49] can be found an extended overview of such derivations for a variety of kinetic equations.
Such picture has to be modified in case we want to include the effect of an *irreversible* interaction of the system with the "environment": in some situations (including semiconductor devices modeling) it could be *not* negligible. Accordingly, the concept of *open* quantum system has to be introduced and we will briefly focus on the available descriptions of it in the literature.

2.2 Open quantum systems

In the final part of Section 1.3, we have introduced the (semi-classical) Boltzmann equation (1.19) that, in the (semi-)classical kinetic framework, constitutes a description of the *irreversible* dynamics of a system (cf. Ref. [18] for a satisfactory discussion). Another possibility is to use the *Fokker-Planck* equation (cf. Ref. [45]): there, instead of a collisional term, are included a *friction* and a *diffusion* terms. In plasma physics, where it is widely employed, these two terms describe respectively the friction between the particles and the grazing collisions, which produce a diffusion term in the velocity direction.
In the quantum picture, a multi-particle system undergoing an *irreversible* interaction with the environment, falls into the class of the open quantum systems (cf. Ref. [20]).
The system under examination is rather small with respect to the "rest of the universe" and the aim is to study in detail the evolution of the former by taking into account the influence of the latter (cf. Ref. [1]). Accordingly, the starting point of the description consists in considering a picture of the universe, in which two parts are distinguishable. The attention is restricted to the evolution of weakly coupled systems, thus to the states of the universe that evolve from an initial state which can be "factorised" by a given reference state of the reservoir (ω_R, a density matrix on the Hilbert space $\mathcal{H}_{\mathcal{R}}$), and an arbitrary state of the system under examination ($\hat{\rho}_0$, a density matrix on $L^2(\mathrm{I\!R}^d)$). The object of the study is the *reduced dynamics*, i.e. the evolution of the state $\hat{\rho}_0$, which can be isolated from the evolution of the initial state of the universe, by computing the partial trace of the universal evolution, i.e. the trace $\mathrm{Tr}_{\mathcal{H}_{\mathcal{R}}}$(cf. Def. (2.4)) of the evolute of the state of the universe.
Accordingly, the evolution of an open quantum system will be governed again by an equation of the following type

$$\frac{d}{dt}\hat{\rho} = \hat{L}(\hat{\rho})\,,\ t > 0\,, \quad \hat{\rho}(t=0) = \hat{\rho}_0\,, \tag{2.9}$$

the so called *Markovian master equation* (cf. Ref. [1] or [49]), with $\hat{\rho}_0$ trace-class operator and $\hat{L}$ the "Liouvillian" operator. A possible class of operators is the so called Lindblad one: since they can be unbounded, they may lead to non unique and non conservative (i.e. non trace-preserving) solutions (cf. Ref. [1, 49] or Ref. [11] for a discussion about the wide related literature). However, among Lindblad operators, there are those that are linear combinations of position and momentum operators: these ones can be used, in particular, to represent the coupling of the quantum system to the reservoir. In Ref. [11] is established the existence and uniqueness of a trace-preserving solution of the system (2.9) with a family of unbounded Lindblad operators of the latter type and with a Hartree interaction term. Correspondingly, the evolute of a quantum physical system, which can describe a many-particle system undergoing an irreversible interaction with the environment, will be a quantum physical system for all positive times.

In literature are also available many *phenomenological* description of the dissipative interaction of a quantum physical system with its environment, via relaxation-type term or quantum-BGK operator. For example, we cite Ref. [9], where a *relaxation-time* term is added to the Von-Neumann equation, coupled with the Poisson one (cf. Eq. (2.7)). There, it is proved a global-in-time well-posedness result for the corresponding Cauchy problem and, moreover, that the evolute of a (initial) physical quantum state is a physical quantum state for all positive times. An example of use of a *quantum-BGK* operator can be found in Ref. [22]. We remark that such modifications of Eq. (2.7) make the cited equivalence with a system of Schrödinger equations no longer valid.

Apart from being physically well-grounded and providing a physically consistent description for all times, the density matrices formalism is not useful for practical applications, namely, for real devices modeling and simulation. To that aim, it would be advisable the introduction of a *quantum kinetic* description and that will be indeed the object of Part II. We expect it to be equivalent or at least to be deducible from the quantum mechanical/statistical approaches, and, at the same time, being a phase-space description, to be suitable for the study of boundary-value problems and for numerical approximations.
As we have anticipated in Remark 1.3.1, it will be introduced a *quantum transport* equation (the Wigner equation) for the reversible dynamics of a quantum system. Possible modifications to describe the irreversible evolution of the system via the Wigner formalism are examined in literature: we briefly discuss here some of them.
In Ref. [17], can be found a tentative derivation of a quantum transport equation with a *scattering term* describing electron-phonon interaction. It starts from the introduction of a Hamiltonian for the coupled system of the electrons and the phonons: due to the scattering events, the number of phonons is not conserved, thus, to deal with a non-constant number of particles, the procedure of second quantization is employed. In Ref. [24], instead, a numerical study is performed of a transport model, based on the Wigner-function approach, which allows a non-parabolic band profile and, moreover, include two different scattering mechanisms (namely, with polar optical and intervalley phonons) via a Boltzmann-like collision operator. We refer the reader again to Ref. [9] and the Refs. therein, for the discussion about a possible use of the *relaxation-time* approximation in the Wigner picture, while, in Ref. [32], this option is investigated from the numerical simulation point of view.
We anticipate that, in the last chapter, we'll study a modified version of the Wigner equation, namely the Wigner-Fokker-Planck equation: the *diffusive term* that is added to the transport equation is a "generalization" of the classical Fokker-Planck term for plasma physics (cf. the introduction of Part IV for a more precise comparison with the classical F-P term and for a detailed description). It is widely used in literature to describe the dissipative interaction of a quantum system with a heath bath, i.e. with an environment in thermodynamic equilibrium. Accordingly, it can model the irreversible interaction of the electrons with the crystal lattice, represented by a phonon bath, in nanoscale semiconductor devices (cf. Refs. [30, 33]), and in quantum Brownian motion, quantum optics and decoherence, as well (cf. Ref. [11] and the Refs. therein). For a formal derivation of various versions of such equation the reader can refer to Refs. [15, 26]), while in Ref. [16] a tentative derivation from many-body Quantum Mechanics is performed. We conclude with the additional remark that the operator that appears in the Master equation (2.9) studied in Ref. [11] is a generalization of the one which corresponds to the quantum Fokker-Planck one. Thus, the description of the evolution of a physical QS provided by the Wigner-Fokker-Planck equation is equivalent to the physically-consistent one which uses the density matrices formalism.

Another physical situation, adherent to real device simulation, which can be modeled via an open QS, is a device coupled to an external resevoir. In that case there is an exchange of particles between the "system" and the "environment" that, in first approximation, can be described as a black-body (see the introduction to Part III for a more complete discussion and for Refs.). Also for such model a quantum kinetic approach seems promising, since it provides the possibility of stating a boundary-value problem in the phase-space (at difference with the density matrices formalism). By the way, we name that, in the recent years, has been widely investigated the possibility of *coupling* different parts of a semiconductor device, distinguishing between those in which quantum regimes are attained and the ones that are well-described by the semi-classical approach. In particular, for what a mixed classical-kinetic and quantum-operatorial description is concerned, we name Ref. [13]: there, an open quantum system is under examination, as well, since are described "particles" in the quantum region interacting with those coming from the classical region of the device.

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Part II

Quantum kinetic theory

Chapter 3

The Wigner-function description

3.1 The concept of quasiprobability

The aim of the present chapter is to introduce a *kinetic* model for the evolution of a quantum system (QS), starting from the dynamics of Quantum Mechanics (QM). Accordingly, the state of the system at a certain time t should be described by a probability distribution function in the phase space[1], $W : (p, q, t) \in \mathbb{R}^{2d} \times \mathbb{R} \mapsto W(p, q, t) \in \mathbb{R}^{+}$.
In the classical kinetic theory, that enables to compute macroscopic physical quantities corresponding to functions $f(p, q)$, as averages, i.e., by

$$< f\, W >= \iint f(p, q) W(p, q)\, dp\, dq \tag{3.1}$$

(cf. Eq. (1.11)). Correspondingly, in the Quantum-Mechanics framework, that would mean to compute the expectation values of a physical observable $\hat{A}$, say at a pure state of our physical system, without resorting to the wave function ψ that describes the state (cf. Def. (2.1)). In particular, a joint (i.e. phase-space) probability density W would make it possible to perform the calculation for observables $\hat{f}$ that are expressed in terms of position and momentum operators, $\hat{f} = f(\hat{p}, \hat{q})$. That would be in contradiction with the Heisenberg commutation relation, which reads

$$\hat{p}\, \hat{q} - \hat{q}\hat{p} = \frac{\hbar}{2\pi} \hat{\imath}$$

(where $\hat{\imath}$ is the identity operator on the appropriate domain). Indeed, even if the operator $f(\hat{p}, \hat{q})$ *differs* from $f_1(\hat{p}, \hat{q})$ by some power of $(\hbar/2\pi)\, \hat{\imath}$, due to the non-commutability of the operators, it can still be $f(p, q) = f_1(p, q)$, for all $(p, q) \in \mathbb{R}^{2d}$. Accordingly, the corresponding observable would have the same average value at an arbitrary (pure) state ψ, when computed with Eq. (3.1), thus

$$< \psi, \hat{f}\psi >= \iint W(p, q)\, f(p, q)\, dp\, dq = \iint W(p, q) f_1(p, q)\, dp\, dq =< \psi, \hat{f}_1\psi > ,$$

which is contradictory, since it can *not* be $< \psi, \hat{f}\psi >=< \psi, \hat{f}_1\psi >$. This consideration suggests that, before introducing a distribution $W(p, q)$ corresponding to a state of the QS, it is necessary to set up rules to assign to each function f of variables p, q, a unique function

[1] Here, as before, d indicates the degrees of freedom of the system, while p are the momentum and q the position variables.

$f(\hat{p}, \hat{q})$ of noncommuting operators: that is the well-known problem of *quantization.* There isn't a unique way to solve this problem; however, to every possible ordering, corresponds the definition of a function W in terms of a state of the QS, such that the use of expressions like (3.1), for observables that are described by *suitably ordered* functions $f(\hat{p}, \hat{q})$, will give the correct quantum-mechanical mean values at that state (cf. Ref. [30]).
The corresponding functions W are called *quasiprobability* distributions, since they are to a large extent analogous to a joint probability densities for coordinate and momentum, although they have also certain peculiarities which do not allow to fully treat them as such. Somehow, during the use of such quasiprobabilities for the kinetic description of the system, their peculiarities will remind us of the quantum origin of the description itself, since their appearance is related to the Heisenberg uncertainty principle, which is one of the distinguishing points of QM.
A possible way to associate to $f(p, q)$ a unique operator is the Weyl-ordering rule (cf. Refs. [16, 31]). Accordingly, we can state the following definition

Definition 3.1.1 (The Wigner function) *Let $f(p, q)$ denote a physical observable and let us indicate with $\{f\} = \{f(\hat{p}, \hat{q})\}$ the Weyl-ordered function of operators $\hat{p}, \hat{q}$, which individuates an operator in QM. The expectation value of the observable f computed at the state ψ is given by*

$$< \psi, \{f\} \psi > = \iint f(p, q)\, w(p, q)\, dp\, dq\,,$$

where the function $w : (p, q) \in \mathbb{R}^{2d} \mapsto w(p, q) \in \mathbb{R}$ is defined in terms of ψ by

$$w(p, q) := \frac{1}{(2\pi\hbar)^d} \int \exp\left(ip \cdot \xi/\hbar\right) \overline{\psi} \left(q + \frac{\xi}{2}\right) \psi \left(q - \frac{\xi}{2}\right) d\xi \tag{3.2}$$

The interested reader can find in Ref. [30] the details of the derivation of the definition in the contest of the choice of the quantization. Instead, we will simply assume Eq. (3.2) to be the definition of a quasiprobability distribution for the QS described by ψ. By Remark 2.1.1, it follows the equivalent definition in terms of density matrices.

Definition 3.1.2 (The Wigner function of a physical QS) *The Wigner function associated to the density matrix*[2] *$\hat{\rho}$ describing the mixed (respectively, pure) state of a physical QS, is defined as*

$$w(p, q) := \frac{1}{(2\pi\hbar)^d} \int \exp\left(-ip \cdot \xi/\hbar\right) \rho \left(q + \frac{\xi}{2}, q - \frac{\xi}{2}\right) d\xi = \mathcal{F}_{\xi \to p}\, \rho \left(q + \frac{\hbar\xi}{2}, q - \frac{\hbar\xi}{2}\right), \tag{3.3}$$

where we indicate with $\mathcal{F}_{\xi \to p}$ the Fourier transform multiplied by a factor $(2\pi)^{-d/2}$.

Remark 3.1.1 (The Wigner transform) *Eq.* (3.3) *defines as well the Wigner transform $\mathcal{W}$, which is an isometry on* $L^2(\mathbb{R}^{2d})$*, by*

$$w =: \mathcal{W}\, \rho\,. \tag{3.4}$$

This function was first introduced by E.Wigner in Ref. [32] to study quantum corrections to thermodynamic equilibrium distributions.
In what follows, we will keep indicating with w the Wigner distribution function and we will consider Definition (3.2), as well as (3.3) as the starting point of the quantum kinetic description of the system under consideration.

[2]In particular, $\hat{\rho}$ is self-adjoint, thus the corresponding kernel ρ satisfies $\rho(x, y) = \overline{\rho(y, x)}$. That is equivalent to the Wigner function (defined by (3.3)) being real-valued.

3.2 Peculiarities of a quasiprobability

In this section we discuss some properties of the Wigner function that can be used as the terms for a comparison with a classical probability distribution function.

3.2.1 Compatibility with the densities

From a joint probability density one naturally expects that its integral over p (respectively, over q) leads to the probability density for the coordinate q (respectively, for the coordinate p). Indeed, by exploiting Def. (3.2),

$$n[w](p) := \int w(p,q)\,dp = |\,\psi(q)|^2\,, \qquad \int w(p,q)\,dq = |\,\mathcal{F}_{q\to p}\,\psi(q)|^2\,, \tag{3.5}$$

and the right hand sides of the equalities above coincide with the position and momentum densities, in case the QS is described by the state ψ.
The discussion for the density matrix case has to be postponed to Section 3.3, since it is by far more delicate.

3.2.2 Admissable distributions

Before comparing the evolution equations for the classical and the quasidistributions, we should discuss the following point. According to the derivation in Section 3.1, *not* every real-valued function defined in the phase space $\mathbb{R}^d_p \times \mathbb{R}^d_q$ can describe a QS.
Therefore, admissable initial data for the Cauchy problems relative to the quantum evolution equation, have to be selected by the *necessary* condition that they describe quantum states. A necessary and sufficient condition for a function $w : (p,q) \in \mathbb{R}^{2d} \mapsto w(p,q) \in \mathbb{R}$ to correspond to a pure state of a QS can be found in Ref. [30]. On Section 3.3, we will discuss possible conditions to select functions which can be associated to density matrices.
Accordingly, such functions will automatically satisfy the uncertanty relation.

3.2.3 Evolution equation

The evolution equation for the quasidistribution function $w(p,q,t)$ can be easily determined starting by the Definition (3.2) and considering the time-dependent function $\psi(q,t)$ as the solution of the Schrödinger equation (2.2). It reads[3]

$$\begin{aligned}\frac{\partial}{\partial t}w(p,q,t) &= -\frac{p}{m}\cdot\nabla_q w(p,q,t) + \\ &- \frac{i}{(2\pi)^d\hbar}\iint\left(V\left(q+\frac{\hbar\xi}{2}\right) - V\left(q-\frac{\hbar\xi}{2}\right)\right) w(p',q,t)\exp\left(-i\,(p-p')\cdot\xi\right)dp'\,d\xi,\end{aligned} \tag{3.6}$$

and it is the *quantum generalization* of the classical Liouville equation (1.14).

Remark 3.2.1 *Observe that, being equivalent to the Schrödinger equation, such equation provides a description of the* reversible *dynamics of an isolated QS (cf. the discussion in*

[3]Here, as before, with m it is indicated the electron mass, while we have substituted the periodic potential V_{per} with the generic one V.

Remark 1.3.1 and in Section 2.2. Equivalently, it could be derived from the density matrices picture, i.e. from Eq. (2.6)*, via Wigner transform* (3.3)*. We can include a Hartree-type non-linearity, analogously to* (2.7)*, in order to model adequately the dynamics of the QS under examination; accordingly, Eq.* (3.6) *can be coupled to the Poisson equation*

$$-\Delta_x V(q,t) = n[w](q,t)\,,$$

with $n[w]$ formally *defined as in Eq.* (3.5)*.*

Indeed, at least for (potential) functions V that are analytic at q, the Wigner equation can be rewritten in the form

$$\frac{\partial}{\partial t} w(p,q,t) = \left(-\frac{p}{m}\cdot\nabla_q - \frac{i}{\hbar}\left(V\left(q + \frac{i\hbar}{2}\nabla_p\right) - V\left(q - \frac{i\hbar}{2}\nabla_p\right)\right)\right) w(p,q,t). \tag{3.7}$$

And, if we expand the potential V in series at the point q, we get the following form

$$\frac{\partial}{\partial t} w(p,q,t) = -\frac{p}{m}\cdot\nabla_q w(p,q,t) + \nabla_q V(q)\cdot\nabla_p w(p,q,t) + \mathcal{O}(\hbar^2)\text{-terms} \tag{3.8}$$

that suggests that, if we can neglect quantities of order $\hbar^2$, the quantum Liouville equation goes over into the classical Vlasov equation. Another remarkable case is when V is a quadratic polynomial, then the Wigner equation will coincide with the classical Liouville equation for the corresponding system.
The previous (formal) limit procedure, apart from fixing in terms of $\hbar$ the order of the error of the classical description of a quantum system, opens the discussion about *the meaning* of the Wigner formulation. Let us rewrite Eq. (3.8) in the form

$$\begin{aligned} \frac{\partial}{\partial t} w(p,q,t) + \frac{p}{m}\cdot\nabla_q w(p,q,t) - \nabla_q V(q)\cdot\nabla_p w(p,q,t) = \Phi(p,q)\, w(p,q,t)\,, \\ \Phi(p,q) := \left[\frac{i}{\hbar}\left(V\left(q + \frac{i\hbar}{2}\nabla_p\right) - V\left(q - \frac{i\hbar}{2}\nabla_p\right)\right) - \nabla_q V(q)\cdot\nabla_p\right]. \end{aligned} \tag{3.9}$$

The solution of the corresponding Cauchy problem with initial condition $w(p,q,0) = w_0(p,q)$, identically satisfies

$$w(p,q,t) = G(p,q,t) * w_0 + \int_0^t G(p,q,s) * (\Phi\, w)(s)\, ds, \tag{3.10}$$

with G the Green's function of the classical Vlasov equation. The function defined by $w^0(p,q,t) = G(p,q,t) * w_0$ is indeed the solution of the Vlasov equation with initial datum w_0. Thus, if we consider the solution of Eq. (3.10) in the form of an iterative series with first term w^0, we are driven to say that its next terms will bring quantum corrections to the "classical" solution w^0. Let us recall instead (cf. Section 3.2.2), that a correct use of the Wigner description requires the use of (admissible) quantum initial datum. That will already carry some "quantum information"; thus, it is not appropriate to call w^0 a classical solution; instead, an initial quantum distribution evolving according to the laws of Classical Mechanics. Thus, the *formal* similarities of classical and quantum kinetic formulation can be misleading.
Moreover, the previous discussion points out the problem that the Wigner equation can be satisfied also by *extraneous* solutions, i.e. which can have nothing to do with quantum systems: again, the quantum-mechanical meaning of the solution of the Wigner equation

has to be guaranteed by the selection of the initial data.
A possible concrete example is the harmonic oscillator: in that case (more generally in the case of a quantum system with quadratic potential), the classical and the quantum Liouville equations coincide: thus, the Wigner equation is satisfied by the solution of the classical equation. However, the solution of the problem of the quantum oscillator is definitely different from the classical one. In Ref. [30], is shown that the quantization condition on the oscillator energy is obtained not from the equation of motion for the corresponding Wigner function, but from the supplementary condition which distinguishes those quasidistributions that correspond to pure quantum states.
In conclusion, the Wigner equation is the quantum equivalent of the Liouville equation and, in certain circumstances, can be *formally* identical to its classical counterpart; however, the conditions coming from its quantum origin have to be kept into consideration, in order to preserve the quantum-mechanical value of its solution.

Remark 3.2.2 (Analogies between the Schrödinger and the Liouville equations) *From a strictly mathematical point of view*[4]*, in case $V \equiv 0$, the transform defined by*

$$\psi \in L^2(\mathbb{R}^d_q;\mathbb{C}) \longmapsto \psi(\,.\,)\,\overline{\psi(\,.\,)} \in L^2(\mathbb{R}^d_\xi \times \mathbb{R}^d_q;\mathbb{C}) \xmapsto{\mathcal{W}} w^\psi \in L^2(\mathbb{R}^d_p \times \mathbb{R}^d_q;\mathbb{R}) \tag{3.11}$$

(cf. Eq. (3.4)*), make the Schrödinger equation* (2.2) *with $V \equiv 0$ correspond to the* free-transport *equation, i.e. the Liouville equation* (1.14) *with only the free-streaming operator. Accordingly, we expect that the two equations present analogies. Indeed, first, in Ref. [21] have been deduced* dispersion *estimates for the Schrödinger equation from a moments lemma for the free-transport equation, via the transform* (3.11)*. Secondly, in Ref. [13], have been recovered estimates for the free-transport equation, which are analogous to the* Strichartz'*estimates for the Schrödinger equation (cf. Section 4.2 for a more detailed discussion and Refs.). Then, in Ref. [28], analogies between the Schrödinger-Poisson and the Vlasov-Poisson systems are discussed: among them a* regularizing *effect on the electric field (proved for the Schrödinger case in Ref. [11]), still due to dispersive properties of the free-streaming operator. Inspired by the strategy there used (which, actually, is to be ascribed to Ref. [20]), we will recover also for the Wigner case (cf. Section 4.3) an analogous effect and that will be crucial for our well-posedness analysis (Ref. [4]).*
However, we observe that, in spite of the correspondences between the different pictures, which motivate to pursue the parallelism, and the similarities of the achieved results, the tecniques used to prove them have to be adequated from case to case in order to take into account the peculiarities of each description. For example, the nonnegativity of the classical distribution probability is lost both at the Schrödinger and at the Wigner level, the Strichartz'inequalities recovered in Ref. [13] are not simply the Wigner-transformed version of those that hold in the Schrödinger picture (cf. Section 4.2), and, finally, roughly speaking, the quantities which are quadratic in the Schrödinger formalism are linear[5] *in the Wigner one (cf. Def.* (3.11)*).*

3.2.4 Positivity and smoothed quasiprobabilities

A fundamental difference between the Wigner function and a probability distribution is the former does *not* satisfy, in general, the condition of being non-negative. In Ref. [30] it is

[4] i.e., neglecting the previous discussion about the admissability of the initial data and considering just the *formal* aspect,

[5] Think, e.g., to the fact that $\psi \in L^2(\mathbb{R}^d_q\,\mathbb{C}) \Rightarrow n[\psi] := |\psi|^2 \in L^+_1(\mathbb{R}^d_q;\mathbb{R})$, while $w \in L^2(\mathbb{R}^d_p \times \mathbb{R}^d_q;\mathbb{R})$ does not yield that $n[w](q) := \int w(p,q)\,dp$ is well-defined.

investigated the reason why: starting from Definition (3.2), the Wigner function can be represented as the difference of two nonnegative quantities, thus its sign depends on their ratio.
The only example of a non-negative Wigner function is the one defined by Eq. (3.2) with ψ equal to a Gaussian function (cf. Ref. [27]). No similar characterization has been found yet for the mixed states case.
However, it is possible to define some *averaged* quasidensities, which do not assume negative values. The use of such substitutes as probability densities will *not* lead to the genuine quantum averages (cf. Ref. [30]); thus, they don't have a remarkable physical meaning.
Nevertheless, they have proved to be the correct instruments to perform the *semiclassical limit*, i.e. the limit for $\hbar \to 0$ (cf. Ref. [30] for the example of the quantum oscillator, and Ref. [19, 25] for a rigorous study). There, a smoothed Wigner function is defined by convolution with an appropriate mollifier; namely, the *Husimi function*

$$\begin{aligned} w^{\hbar}(p,q) &:= w *_q G^{\frac{\hbar}{m}}(q) *_p G^{\frac{\hbar}{m}}(p)\,, \text{ with} \\ G^{\delta}(r) &:= \frac{1}{(2\pi\delta)^{d/2}} \exp\left(-\frac{r^2}{\delta}\right) . \end{aligned} \tag{3.12}$$

As a consequence of the uncertainty principle, it can be proved $w^{\hbar}$ is nonnegative, and exactly on that rely compactness criteria for the convergence. Moreover,

$$\int w(p,q)\,dp\,dq = \int w^{\hbar}(p,q)\,dp\,dq\,;$$

so this distribution is at least compatible with the one-dimensional distributions (see also Section 3.3).

As a conclusion, we can state that the Wigner function is the object, arising from a quantum description, which is *nearest to a classical distribution function*. However, we must be aware of the limits pointed above, before using it for a quantum-mechanically meaningful kinetic description.

3.3 The kinetic characterisation of a physical QS

In Section 2.1, the tools for a quantum-mechanical and for a quantum-statistical description of a QS have been introduced, while in Section 3.1, we have presented the main instrument, namely the Wigner function, to try a quantum kinetic description of a QS; then, in Section 3.2, we have restricted the use of the quasiprobability as a "classical" probability distribution, by listing its peculiarities and the conditions for it to preserve its quantum meaning.
The further questions that naturally arise are whether and to what extent the physical informations that QM conveys, can be transferred to the kinetic framework. Such physical characterization can, indeed, be translated in mathematical features which are consistent with the kinetic formulation, but they can happen *not* to have a meaning in that picture, thus, in the end, to be unexploitable for the analysis of the kinetic model.
Specifically, in the quantum-statistical picture, a physical QS is a *positive, self-adjoint trace-class operator* $\hat{\rho}$ with kernel ρ (cf. Remark 2.1.1) and those informations already yield, e.g., the finite total mass of the system. The corresponding Wigner function can be defined as $w := \mathcal{W}\rho$ (cf. Eq. (3.4)): in the next proposition, we collect the mathematical properties

that can be deduced for the function w, being the Wigner transformed of a physical QS. The next point to be discussed is whether such properties can be used for the kinetic description of the evolution of a physical QS.
The other way round, we investigate the possibility of characterizing a QS *just in terms of* a function w defined on the phase-space. Accordingly, we look for *sufficient* conditions for a function w (and its evolution) to individuate a physical QS (respectively, to describe its evolution).
Thus, the following discussion completes that started in Section 3.2.2, about Wigner functions which are admissable for a quantum kinetic description.

Proposition 3.3.1 *Let $\hat{\rho}$ be an integral operator on $L^2(\mathbb{R}^d_x)$ with kernel ρ*

$$\hat{\rho} f(x) = \int \rho(x,y) f(y)\, dy\,, \quad \rho \in L^2(\mathbb{R}^{2d})\,,$$

and[6] *$w : \mathbb{R}^d_x \times \mathbb{R}^d_y \to \mathbb{C}$ be defined by $w := \mathcal{W}\rho$ (cf. Eq. (3.4)). Then, the following equivalences hold:*

1. *$\hat{\rho}$ is (H-S) self-adjoint $\Leftrightarrow$ $w \in L^2(\mathbb{R}^{2d})$ is real-valued,*
2. *$\hat{\rho}$ is trace class $\Rightarrow$ the particle density n, defined by Eq. (2.5), $n \in L^1(\mathbb{R}^d)$,*
3. *$\hat{\rho}$ is (H-S) self-adjoint, positive*[7] *$\Leftrightarrow < w, w^\psi >_{L^2(\mathbb{R}^{2d};\mathbb{R})} \geq 0$, $\forall \psi \in L^2(\mathbb{R}^d)$, with w^ψ defined by Eq. (3.11) ($\Rightarrow$ $w^\hbar \geq 0$)*[8]*,*
4. *if $\hat{\rho}$ is (H-S) positive, then $\hat{\rho}$ is trace class $\Leftrightarrow$ $n \in L^1(\mathbb{R}^d;\mathbb{R}^+)$ $\Leftrightarrow$ $w^\hbar \in L^1(\mathbb{R}^{2d};\mathbb{R}^+)$ and*
$$\|n\|_{L^1(\mathbb{R}^d)} = \|w^\hbar\|_{L^1(\mathbb{R}^{2d})}\,.$$

Let us carefully discuss the content of the previous proposition, since it motivates our future choices; for the detailed proof of it, the reader can refer to Refs. [1, 7, 19], e.g..
Observe that a trace-class operator is *necessarily* Hilbert-Schmidt. Starting from $\hat{\rho}$ Hilbert-Schmidt, the Wigner function is well-defined by Eq. (3.4) (cf. Remark 2.1.1) and the equivalence in 1. is straightforward from the definition and can be summarized in the following way

$$\|w\|_{L^2(\mathbb{R}^{2d};\,\mathbb{R})} = (4\pi)^{-N/2}\|\rho\|_{L^2(\mathbb{R}^{2d})} < +\infty. \tag{3.13}$$

In the next sections, we will see that the information in Eq. (3.13) is the *only* one to be conveyed during the time-evolution by the kinetic model and it will be crucial for the analysis. Nevertheless, that property doesn't have any direct physical interpretation, apart from being a *necessary* condition for the Wigner function to describe a physical QS. In particular, it does *not* imply that the position density $n[w]$, expressed in the kinetic formulation by

$$n[w](x) := \int w(x,y)\, dy$$

(cf. Eq. (3.5) for the pure-state case), is well-defined. Correspondingly, if we define the particle density n in terms of the eigenvectors $\{\psi_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ and eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}} \subset l^2(\mathbb{N})$ of the H-S operator $\hat{\rho}$, in analogy with the case of a physical QS (cf. Eq. (2.5)), by

$$n(x) := \sum_{j\in\mathbb{N}} \lambda_j |\psi_j(x)|^2\,, \tag{3.14}$$

[6] equivalently, a *Hilbert-Schmidt*(H-S) operator,
[7] equivalently, $< \hat{\rho}\psi, \psi > \geq 0$, $\forall \psi \in L^2(\mathbb{R}^d)$
[8] cf. Eq. (3.12)

the sum in Eq. (3.14) could also *not* converge.
Thus, starting from a function $w \in L^2(\mathbb{R}^{2d};\mathbb{R})$, neither the density n defined in the corresponding operatorial formulation (via $\rho := \mathcal{W}^{-1}w$), nor the "kinetic" one, $n[w]$, are well-defined.
In case $\hat\rho$ *is trace class*, instead, the density n is well-defined by (2.5), since it holds

$$\|n\|_{L^1(\mathbb{R}^d)} \leq \mathrm{Tr}|\hat\rho| \tag{3.15}$$

and it can be proved a *necessary* condition for a function to be the Wigner transformed of a trace-class operator, namely,

$$\hat\rho\,\text{trace class} \;\Rightarrow\; w \in \mathcal{C}_0(\mathbb{R}^d_x;\mathcal{F}L^1(\mathbb{R}^d_y)) \cap \mathcal{C}_0(\mathbb{R}^d_y;\mathcal{F}L^1(\mathbb{R}^d_x)) \tag{3.16}$$

(cf.Ref. [1] and the refs. therein).
However, only if the operator is *trace class and positive* (i.e. corresponds to a physical QS), the $\{\lambda_j\}_j$ in Eq. (3.14) are non-negative and starting from the Fourier expansion of the corresponding kernel ρ

$$\rho(x,y) = \sum_{j\in\mathbb{N}} \lambda_j \psi_j(x)\overline{\psi_j}(y)\,,$$

the equality $n(x) = \rho(x,x)$ has a rigorous meaning by the limit

$$n(x) = \lim_{\epsilon\to 0} \int \rho\left(x+\frac{\eta}{2}, x-\frac{\eta}{2}\right) \frac{e^{-|\eta|^2/(2\epsilon)}}{(2\pi\epsilon)^{d/2}}\,d\eta \;< +\infty\,, \tag{3.17}$$

and $n \in L^1_+(\mathbb{R}^d)$ (see Ref. [1] and the refs. therein). In addition, the definitions of the density in the two formulations are consistent, since

$$n(x) = \rho(x,x) = \left(\mathcal{F}^{-1}_{y\to\eta} w\right)(x,\eta=0) = n[w](x)\,, \tag{3.18}$$

by

$$n[w](x) = \lim_{\epsilon\to 0} \int w(x,y)\, e^{-\epsilon|y|^2/2}\,dy \;< +\infty\,,$$

and $n[w] \in L^1_+(\mathbb{R}^d)$ (cf. (3.17) and Def. (3.3)).
Unfortunately, there is no possible characterization of the positivity of the operator $\hat\rho$ *only in terms of* a function $w \in L^2(\mathbb{R}^{2d};\mathbb{R})$, except for the equivalence stated in 3., which follows directly by the definitions and implies the positivity of many convolutions with the function w, thus, of the Husimi function as well. In particular, in case the operator is trace class and positive, $w^\hbar$ is non-negative, and it holds

$$\int n(x)\,dx = \iint w^\hbar(x,y)\,dx\,dy \geq 0$$

(cf. Eq. (3.18) and Section 3.2.4).
To conclude, let us deal with physically meaningful quantities: in case the operator is trace class *and* positive,

$$\mathrm{Tr}\,\hat\rho = \sum_{j\in\mathbb{N}} \lambda_j = \int n(x)\,dx = \iint w^\hbar(x,y)\,dx\,dy$$

and this quantity, which corresponds to the *total mass* of the system, is positive and finite.
For what the *kinetic energy* is concerned, its expression in terms of the operator $\hat\rho$ is $\mathrm{Tr}\,\hat H_0\hat\rho$ with $\hat H_0 := -\frac{1}{2}\Delta_x$ and it holds

$$\mathrm{Tr}(\hat H_0\hat\rho) < +\infty \;\Leftrightarrow\; |y|^2 w^\hbar \in L^1(\mathbb{R}^{2d})\,. \tag{3.19}$$

Instead, since $\rho \in \mathcal{S}(\mathbb{R}^{2d}) \Leftrightarrow w \in \mathcal{S}(\mathbb{R}^{2d})$, under such assumptions, it holds

$$\mathrm{Tr}(\hat{H}_0\hat{\rho}) = \iint |y|^2 w(x,y)\, dx\, dy\,.$$

Let us derive some conclusions from the previous discussion: First of all, the mathematical properties that characterize a physical QS on the operatorial framework, when translated in the Wigner formalism give the conditions (3.16) and 3. on the Wigner function w, and the desireable property 4. on the (Wigner related) position density $n[w]$, which coincides with the quantum-mechanically correct one. For what the properties of the Wigner function are concerned, instead, neither (3.16) nor 3. are much of interest or use in the kinetic context: we just name that in Ref. Ref. [19], are discussed conditions on the potential, such that the weak solution $w(t)$ of the Wigner equation belongs to $\mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^{2d}_x)) \cap \mathcal{C}_b(\mathbb{R} \times \mathbb{R}^d_x; \mathcal{F}L^1(\mathbb{R}^d_y)) \cap \mathcal{C}_b(\mathbb{R} \times \mathbb{R}^d_y; \mathcal{F}L^1(\mathbb{R}^d_x))$, for all times t. Moreover, condition 3. is very peculiar, if we consider that w can take negative values as well.
The other way round, a characterization of a physical QS *only in terms of* a function w defined on the phase-space is not possible, since both the conditions 1. and (3.16) on the function w, are just *necessary* and, moreover, incomplete, since none of them includes positivity of the corresponding operator, defined via $\rho := \mathcal{W}^{-1}w$. Indeed, the only *sufficient* condition in terms of w, for the positivity of the operator to hold, is 3.. Additionally, the physical quantities relative to a physical QS (e.g., the kinetic energy) are well-defined in terms of w and coincide with the quantum-mechanically correct ones (obtained via the corresponding density matrix), under more restrictive conditions on w with respect to the necessary ones to describe a physical QS.

Remark 3.3.1 (Positivity) *Notice that, in case condition 3. holds for $w \in L^2(\mathbb{R}^{2d}; \mathbb{R})$, the additional information $n[w] \in L^1_x$ will be sufficient to guarantee that the H-S, self-adjoint operator $\hat{\rho}$ corresponding to w, via $\rho := \mathcal{W}^{-1}w$, is also trace class (by 4.). Accordingly, this could be a strategy to obtain* a posteriori *that w provides a description of a physical QS. However, condition 3. looks not so meaningful in the kinetic formalism.*

The characterization obtained for $w^\hbar$, instead, is more practical (cf. the necessary and sufficient condition in 4.), *once positivity is assumed.* Nevertheless, the Husimi function is *not* a proper instrument for a correct quantum description (cf. Section 3.2.4).

3.4 The functional setting

In the literature of semiconductor devices modeling, there are several examples of analysis of the evolution of a QS at the operatorial level (Ref. [1, 6], e.g.): there, the mathematical properties distinguishing a physical QS are preserved during the evolution. Tipically, such result is recovered, by setting the study in some "energy space", namely, in the space of the density matrices such that the corresponding kinetic energy is well-defined (maybe in some weak sense) and bounded. Then, the conservation of the total energy is exploited, to recover a solution for all times, belonging to the energy space.
Similarly, at the Schrödinger level, there is an entire collection of results (cf. Refs. [22, 11] and the references therein) devoted to weakening the regularity assumptions on the wave-functions which guarantee a physically-consistent description of the evolution of the QS. In this framework, the sharpest result is in Ref. [11]; there a L^2-theory is performed, by extending to the mixed state case the use of the Strichartz' inequalities (cf. Ref. [11] and the

Refs. therein).

Our aim would be to prove analogous results *at the quantum kinetic level, by keeping the analysis to it.* Instead, most part of the results achieved till now in Wigner framework, either rely on the reformulation of the kinetic problem in terms of density matrices or of wave-functions (cf. Refs. [10, 22] and the Refs. therein), or, implicity exploit such correspondence, by adding some assumptions that are *not* consistent with the kinetic framework (cf. Refs. [3, 14]).
The discussion in the previous section motivates to choose $L^2(\mathbb{R}^{2d};\mathbb{R})$ as the functional setting for the analysis of a quantum kinetic model, since such Wigner functions are in bijective correspondence with H-S, self-adjoint operators. Thus, they satisfy at least the *necessary* condition to represent physical QSystems. Moreover, we have anticipated that the L^2-norm is preserved during the evolution by the Wigner equation, then the necessary condition is satisfied at any time. Nevertheless, the physical quantities are not even well-defined, nor positive, starting by an L^2-quasidistribution. We recall that the reason is they are not well-defined starting by the corrisponding H-S operator as well. Accordingly, a L^2-analysis of problems in the Wigner formulation cannot rely on any physical conservation law[9], at variance with what happens in the other contexts. In particular, there isn't, as far as we know, any way to plug in the kinetic description, the condition 3. on w in Prop. 3.3.1, which is equivalent to positivity at the operatorial level, and to check whether it is conveyed by the evolution.
In conclusion, even if the L^2-setting consists of admissable quasidistributions, by a L^2-kinetic analysis we will *not* obtain a physical QS, at variance with the results in the other formulations.
Nevertheless, in many cases (cf. Refs. [2, 5, 4, 24, 26]), the L^2-analysis of the Wigner problems proves to be *self-consistent.* Moreover, we will show in the following sections that the study in the L^2-setting permit to recover results that are in agreement with the physically expected ones, *without* exploiting the physical quantities (as well as the conservation laws), which would require to add some non-kinetic assumptions inspired by the alternative formulations. Precisely, we will *not* make any assumption concerning positivity.

Remark 3.4.1 (L^1-analysis) *We mention here that in literature are present some examples of L^1-analysis of problems in Wigner formulation (cf. Refs. [3, 14], as well as in Ref. [29], concerning a L^p-analysis): we remark that in those cases, there is no reason why the models under examination should describe a QS (cf. Section 3.2.2), nor why the Wigner function should be associated to a positive, H-S operator $\hat{\rho}$. Thus, even if $w \in L^1(\mathbb{R}^{2d};\mathbb{R})$ ensures $n[w] \in L^1_x$, there is no reason why $n[w]$ should coincide with the positive density n defined starting from a H-S positive operator. Accordingly, the Husimi function $w^\hbar$, defined starting from $w \in L^1(\mathbb{R}^{2d};\mathbb{R})$, will belong to $L^1(\mathbb{R}^{2d})$, but could take negative values as well. As a consequence, the corresponding kinetic energy (defined via Eq.* (3.19)*) could as well take negative values and cannot be used to state a priori estimates.*
In conclusion, a L^1-analysis doesn't have any intrinsic quantum-mechanical value and moreover it requires additional assumptions (namely, positivity) from the operatorial level to be performed.

[9]In literature there are several examples of boundary value problems for the Wigner function, studied in L^2-context (cf. Refs. [2, 5, 26, 24, 23], e.g.): there arises the additional problem of a quantum-mechanically consistent definition of the physical quantitites, since the bounded domain breaks the equivalence both with the operatorial and with the Schrödinger formalisms.

Chapter 4

Tools in quantum kinetic theory

In this chapter we introduce some analytical tecniques for the study of quantum kinetic problems in the L^2-setting: we will extensively use them in the next chapters, thus we provide a preliminary discussion and a comparison with the related literature. In particular, we collect in Section 4.1 some results presented in Refs. [23, 24], while Section 4.3 contains extracts from Ref. [4]. Here, our model-problem is the three-dimensional version of the Wigner-Poisson (WP) system, already introduced in Remark 3.2.1; namely,

$$\left[\frac{\partial}{\partial t} + v\cdot \nabla_x\right] w(x,v,t) = (\Theta\,[V]\,w)(x,v,t)\,, \quad (x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v\,,\ t \geq 0, \tag{4.1a}$$

$$\Delta_x V(x,t) = \int_{\mathbb{R}^3_v} w(x,v,t)\,\mathrm{d}v\,, \quad x \in \mathbb{R}^3\,,\ t \geq 0\,, \tag{4.1b}$$

with the unknown functions $w : (x,v,t) \in \mathbb{R}^3_x \times \mathbb{R}^3_v \times [0,\infty) \to w(x,v,t) \in \mathbb{R}$ and $V : (x,t) \in \mathbb{R}^3_x \times [0,\infty) \to V(x,t) \in \mathbb{R}$ and the additional initial condition

$$w(x,v,0) = w_0(x,v), \quad (x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v\,.$$

The operator $\Theta\,[V]$ in Eq. (4.1a) is a pseudo-differential operator (cf. Ref. [16]), formally defined, for $V = V(x)$, $x \in \mathbb{R}^3$, by

$$\left(\Theta[V]w\right)(x,v) = \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d_\xi \times \mathbb{R}^3_{v'}} \delta V(x,\xi)\, w(x,v')\, e^{i(v-v')\cdot\xi}\,\mathrm{d}\xi\,\mathrm{d}v', \tag{4.2a}$$

$$\delta V(x,\xi) := V\left(x + \frac{\xi}{2}\right) - V\left(x - \frac{\xi}{2}\right), \quad (x,\xi)\, \in \mathbb{R}^3_x \times \mathbb{R}^3_\xi. \tag{4.2b}$$

Since $\delta V(x,\xi)$ is an odd function with respect to ξ, it is straightforward from the definition that, the operator $\Theta[V]$ with V real-valued, maps a real-valued function w to a real-valued function $\Theta[V]w$. Observe that, in the Fourier space with respect to the v-variable[1], the pseudo-differential operator $\Theta[\Phi]$ has the following "product shape"

$$\left(\mathcal{F}_v\left(\Theta\,[V]\,w\right)\right)(x,\eta) = i\,\delta V(x,\eta)\left(\mathcal{F}_v w\right)(x,\eta). \tag{4.3}$$

[1]We recall that with $\mathcal{F}_{v\to\eta}$ we indicate the Fourier transform multiplied by a factor $(2\pi)^{-3/2}$.

The system[2] of non-lineraly coupled equations (4.1), together with (4.2), is the quantum generalization[3] of the Vlasov-Poisson (VP) system, which we recall here for later reference,

$$\left[\frac{\partial}{\partial t} + v\cdot \nabla_x\right] f(x,v,t) = \nabla_x V(x,t)\cdot \nabla_v f(x,v,t)\,, \quad (x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v\,,\ t \geq 0, \tag{4.4a}$$

$$\Delta_x V(x,t) \;=\; \int_{\mathbb{R}^3_v} f(x,v,t)\,\mathrm{d}v\,, \quad x \in \mathbb{R}^3\,,\ t \geq 0\,, \tag{4.4b}$$

with the unknown functions $f : (x,v,t) \in \mathbb{R}^3_x \times \mathbb{R}^3_v \times [0,\infty) \to f(x,v,t) \in \mathbb{R}^+$, classical probability distribution function, and $V : (x,t) \in \mathbb{R}^3_x \times [0,\infty) \to V(x,t) \in \mathbb{R}$.
For a matter of simplicity, we have introduced the version of the WP system with the physical constants equal to one, in particular $\hbar = 1$. The interested reader can refer to Ref. [19], where VP is recovered as the limit for $\hbar \to 0$ of WP.
The common features with the VP system has inspired to us some strategies to tackle the analytical difficulties the Wigner equation present. In particular, we anticipate here that the Fourier-transformed pseudo-differential operator (4.3) can be written similarly to the non-linear term on the right hand side of the Vlasov Eq. (4.4a), via an appropriate reformulation; namely,

$$\left(\mathcal{F}_v\left(\Theta\,[V]\,w\right)\right)(x,\eta) = i\,W[\nabla_x V](x,\eta)\cdot \nabla_\eta\left(\mathcal{F}_v w\right)(x,\eta), \tag{4.5}$$

where the (vector-valued) function $W[F] : \mathbb{R}^6 \to \mathbb{R}^3$ can be defined starting from an arbitrary field $F : \mathbb{R}^3 \to \mathbb{R}^3$ (cf. Definition (3.29)). The analogy of the shape (4.5) with the non-linear term in the Vlasov equation, $\nabla_x V(x)\cdot \nabla_v f(x,v)$, will play a relevant role in the analysis. Since the reformulation is possible in the Fourier space and the Fourier transform of the unknown w is involved, it is natural to choose a L^2-space as the functional setting for the study and this is a further motivation with respect to the discussion in the previous sections. However, for a correct analysis, it is necessary to keep in mind the peculiarities of the Wigner quasi-distribution function with respect to the classical probability distribution function f.

4.1 The weighted spaces

A natural demand from the state space for the analysis of a system of the type (4.1), is that it consists of functions $u : (x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v \to u(x,v) \in \mathbb{R}$ such that the function $n[u] := \int_{\mathbb{R}^3_v} u(x,v)\,\mathrm{d}v\,,\ x \in \mathbb{R}^3$, on right hand side of Eq.(4.1b), is well-defined. The way to conciliate that expectation with the choice of $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v; \mathbb{R})$ as the functional setting, is to introduce

$$X \;:=\; L^2(\,\mathbb{R}^3_x \times \mathbb{R}^3_v\,, (1+|v|^2)^2 \mathrm{d}x\,\mathrm{d}v\,; \mathbb{R}\,)\,,$$

which is a Hilbert space with scalar product

$$< u, w >_X := \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_x} u(x,v) w(x,v)\,(1+|v|^2)^2\,\mathrm{d}x\,\mathrm{d}v. \tag{4.6}$$

That choice of the functional setting is in the same spirit of Refs. [5, 26] for the one-dimensional (bounded spatial domain) case. A generalization for the d-dimensional (bounded spatial domain) case was first introduced in Ref. [24](Prop. 5.2.1, Chapter 5). The motivation of it is contained in the following lemma:

[2] In the introduction of Part III there is a more detailed description of it.

[3] Remember the discussion in Section 3.2.3, where we discuss in which sense the quantum transport equation can be considered a generalization of the classical one.

Lemma 4.1.1 *Let $u \in X$ and $n[u](x) := \int_{\mathbb{R}^3_v} u(x,v)\,\mathrm{d}v$, for all $x \in \mathbb{R}^3$. Then*

$$\|n[u]\|_{L^2(\mathbb{R}^3_x)} \;\leq\; C\,\|u\|_X, \tag{4.7}$$

with $C := \pi$.

Proof. The estimate follows directly by applying Hölder inequality:

$$\begin{aligned}
\|n\|^2_{L^2(\mathbb{R}^3_x)} &= \int_{\mathbb{R}^3_x}\left|\int_{\mathbb{R}^3_v} u(x,v)\,\mathrm{d}v\right|^2 \mathrm{d}x \leq \int_{\mathbb{R}^3_x}\left(\int_{\mathbb{R}^3_v} |u(x,v)|\frac{1+|v|^2}{1+|v|^2}\,\mathrm{d}v\right)^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} |u(x,v)|^2(1+|v|^2)^2\,\mathrm{d}v\,\mathrm{d}x \int_{\mathbb{R}^3_v}\frac{1}{(1+|v|^2)^2}\,\mathrm{d}v = \pi^2\|u\|^2_X.
\end{aligned}$$

□

The way to get similar estimates for L^p-norms with $p < 2$ is to introduce a weight also in the x-variables. Precisely, we can define

$$X_\alpha \;:=\; L^2(\,\mathbb{R}^3_x \times \mathbb{R}^3_v\,, (1+|x|^{2\alpha}+|v|^2)^2 \mathrm{d}x\,\mathrm{d}v\,; \mathbb{R}\,)\,,$$

and, accordingly, we can prove the following result.

Lemma 4.1.2 *For all $u \in X_\alpha$, the function $n[u]$ belongs to $L^p(\mathbb{R}^3)$, $\frac{6}{3+\alpha} < p \leq 2$, and satisfies*

$$\|n[u]\|_{L^p(\mathbb{R}^3)} \;\;\leq\;\; C(p)\|u\|_{X_\alpha}, \tag{4.8}$$

Proof. By using Hölder inequality first in the v-integral and then in the x-integral, we get

$$\begin{aligned}
\|n[w]\|^p_{L^p(\mathbb{R}^3)} &= \int\left|\int w(x,v)\,dv\right|^p dx \\
&\leq \int\left(\int|w(x,v)|^2(1+|x|^{2\alpha}+|v|^2)^2\,dv\right)^{\frac{p}{2}}\left(\int\frac{dv}{(1+|x|^{2\alpha}+|v|^2)^2}\right)^{\frac{p}{2}} dx \\
&\leq \left(\iint|w(x,v)|^2(1+|x|^{2\alpha}+|v|^2)^2\,dv\,dx\right)^{\frac{p}{2}}\left[\int\left(\int\frac{dv}{(1+|x|^{2\alpha}+|v|^2)^2}\right)^{\frac{p}{2-p}} dx\right]^{\frac{2-p}{2}} \\
&= C\|w\|^p_{X_\alpha}\left[\int\left(\frac{1}{1+|x|^{2\alpha}}\right)^{\frac{p}{2(2-p)}} dx\right]^{\frac{2-p}{2}} \;\leq\; C(p)\|w\|^p_{X_\alpha} \quad \text{for} \quad \frac{6}{3+\alpha} < p \leq 2.
\end{aligned}$$

□

Thus, in principle, by increasing appropriately the index α, it is possible to work in a functional setting such that the corresponding particle density n belongs even to $L^1(\mathbb{R}^3)$ (namely, by choosing $\alpha > 3$).

Remark 4.1.1 *In case the power of the x-weight α is $\alpha \neq 1$, for the well-posedness study it is necessary to consider the space $\widetilde{X_\alpha}$ with a symmetric weight in the v-variables, namely $\widetilde{X_\alpha} := L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v\,, (1+|x|^{2\alpha}+|v|^{2\alpha})^2\mathrm{d}x\,\mathrm{d}v; \mathbb{R})$. The free-streaming operator (precisely, the operator $-v\cdot\nabla_x - d(\alpha)I$) is indeed dissipative in $\widetilde{X_\alpha}$, but not in X_α, if $\alpha \neq 1$ (cf. Lemma 7.2.1).*

Remark 4.1.2 *By Lemma 4.1.1, a sufficient assumption for the kinetic energy $E_K[u]$,*

$$E_K[u](x) := \int_{\mathbb{R}^3_v} |v|^2 u(x,v)\,\mathrm{d}v, \quad x \in \mathbb{R}^3\,,$$

to be well-defined (indeed, to belong to $L^2(\mathbb{R}^3)$) is $|v|^2u \in X$. Moreover, $E_K[u] \in L^1(\mathbb{R}^3)$ if $|v|^2u$ belongs to the space X_α, with $\alpha > 3$. However, the kinetic energy is not *non-negative, thus it cannot be exploited in the analysis.*

Let us make some considerations upon the previous Lemmata.
Analogously to Lemma 4.1.1 it can be proved

$$X \hookrightarrow L^2_x(L^1_v)\,, \tag{4.9}$$

where we call $L^p_x(L^r_v) := L^p(\mathbb{R}^3_x; L^r(\mathbb{R}^3_v))$. That means that, if $u \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, for a.e. $x \in \mathbb{R}^3$ we gain the summability of the function $u(x,.) : v \in \mathbb{R}^3 \to u(x,v) \in \mathbb{R}$, by assuming that also $|v|^2u \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$.
In the same spirit, we can gain something also on the p-summability of the function in the x-direction, by adding a symmetric weight in that direction. Namely, by the same proof of Lemma 4.1.2, in the case $\alpha = 1$ it holds

$$X_1 \hookrightarrow L^p_x(L^1_v)\,, \quad 3/2 < p \le 2\,. \tag{4.10}$$

Observe that, in the proof of Lemma 4.1.2, the x and the v variables can be interchanged in order to obtain

$$X_1 \hookrightarrow L^p_v(L^1_x)\,, \quad 3/2 < p \le 2\,, \tag{4.11}$$

In conclusion, the introduction of the symmetric squared weight make us gain, for a.e. fixed x (respectively, v) the summability with respect to v (respectively, x) of the function $u(x,.)$ (respectively, $u(.,v)$) and also the p-summability with $p > 3/2$ ($p < 2$) in the other direction.
We remark that in the embeddings (4.9), (4.10), (4.11) it is always $p > r = 1$. However, by interpolation with $L^2_.(L^2_.)$, we have also the embeddings with $2 = p \ge r \ge 1$.
In addition, it can be easily proved

$$X_1 \hookrightarrow L^a_{x,v}\,, \quad 6/5 < a \le 2\,; \tag{4.12}$$

accordingly, the weight can give in both variables the same gain in a-sommability in both directions, with a varying in the smaller interval $(6/5, 2]$.

Remark 4.1.3 *The use of the x,v-weight to get p-summability of the particle density $n[f]$ is well-known in kinetic theory: as an example, we quote (the three-dimensional version of) the standard estimate for $n[f](t)$, where f is the (non-negative) classical distribution function; namely,*

$$\|n(t)\|_{L^{m+3/3}} \le C\|f(t)\|^{m/(m+3)}_{L^\infty_{x,v}} \||v|^m f(t)\|^{3/m+3}_{L^1_{x,v}} \tag{4.13}$$

(cf. Ref. [20], e.g.): depending on how many moments are employed it is possible to improve the control. In the wide literature concerning the VP problem, are introduced several definitions of solutions corresponding to different assumptions on the initial datum. In particular, we name the distinction between weak *solutions*[4]*, obtained by adding to $f_0 \in L^1_{x,v} \cap L^\infty_{x,v}$ the*

[4] i.e., in the distributional sense,

assumption of initial finite kinetic energy $|v|^2 f_0 \in L^1_{x,v}$, *and* strong *solutions*[5], *corresponding to the additional hypothesis* $|v|^m f_0 \in L^1_{x,v}$, $\forall\, m < m_0$ *with* $m_0 > 3$ *(cf. Ref. [20] for the sharpest result in both sense). In the latter case it holds also* $|v|^m f(t) \in L^1_{x,v}$, $\forall\, t \geq 0$ *(propagation of v-moments).*
Alternatively, by an assumption on the x-moments, precisely $|x|^2 f_0 \in L^1_{x,v}$, *it is possible to obtain solutions*[6] with infinite energy *(cf. Ref. [28]), by exploiting dispersive effects related to the free-streaming operator and a pseudo-conformal law for the VP case (which makes an alternative control*[7] *hold for n, cf. Remark 4.4.1). In the same spirit, the propagation of x-moments is studied in Ref. [12], under the assumption* $|x|^m f_0 \in L^1_{x,v}$.
In the quantum case, instead, an assumption of finite initial kinetic energy (guaranteed by $|v|^2 w_0 \in X$*) is of no help (cf. Remark 4.1.2). Thus, we will search solutions with infinite energy, corresponding to the assumption* $w_0 \in X$ *(or* $w_0 \in X_1$*, as well) and, as well as in Ref. [28], will be exploited* dispersive effects as an alternative to the conservation of energy. *However, we cannot obtain a control of the particle density alternative to* (4.7)*, in analogy with Ref. [28], by the substitution of the assumption* $|v|^2 w_0 \in L^2$ *with some x-weight of the initial datum, since the pseudo-conformal law (which holds in the WP case, as well, cf. Section 4.4) is again of no help, due to the missing non-negativity of the quantities involved.*

Remark 4.1.4 (Conservation of the L^2-norm) *Observe that both the spaces* X *and* X_1*, which we shall use extensively in the next sections, embed in* L^2*. That implies that the chosen state space consists of functions which are* admissible *for the quantum kinetic description, in the sense discussed in the previous section.*
Moreover, it immediately follows, by multiplying Eq. (4.1a) *by* w *and integrating both in* dx *and in* dv*, that* $\frac{d}{dt}\|w(t)\|^2_{L^2_{x,v}} = 0$*, since* $< w, \Theta[V]w >_{L^2_v} = 0$[8] *because* $\delta V(x,\xi)$ *is odd with respect to* ξ.

4.2 Strichartz' estimates for the free-transport equation

Both in the Wigner-Poisson and in the Vlasov-Poisson systems, the linear part consists of the free-streaming operator. In the literature relative to the VP problems, there is a series of results in which a priori estimates for the electric self-consistent field are recovered, by exploiting *dispersive effects* to be referred to the transport operator (cf. Refs. [13, 28],e.g.). In the following section, which contains extracts from Ref. [4], we will show that the same effects can be used to obtain similar estimates in the WP case.
In view of that, here we collect different estimates for the solution of the free-transport equation

$$\partial_t w(x,v,t) + v\cdot \nabla_x w(x,v,t) = 0\,, \quad t \geq 0\,, \quad w(x,v,t=0) = w_0(x,v)\,, \qquad \forall\,(x,v) \in \mathbb{R}^6\,,$$

and, in particular, those presented in the paper [13]. They are the equivalent, via Wigner transform, of the the Strichartz' inequalities for the solution of the free-Schrödinger equation

[5] i.e., $f \in \mathcal{C}(\mathbb{R}^+; L^p_{x,v}) \cap L^\infty(\mathbb{R}^+; L^\infty_{x,v})$, $\forall\, p \in [1,\infty)$,
[6] $f \in L^\infty(\mathbb{R}^+; L^p_{x,v})$, $\forall\, p \in [1,\infty]$,
[7] Precisely,

$$t^{6/5}\|n(t)\|_{L^{5/3}} \leq C(T, \|w_0\|_{L^\infty_{x,v}}, \|(1+|x|^2)w_0\|_{L^1_{x,v}})\,,\ \forall\, t < T,\, \forall\, T > 0$$

cf. Thm. III.3 in Ref. [28].
[8] In case V is regular enough for $\Theta[V]w \in L^2_{x,v}$ to hold ($V \in L^\infty$, e.g.).

(cf. Refs. [15, 18] and the references therein). However, in Ref. [13] they are derived differently, since the norms of w_0 that would appear in the estimate for the Wigner-transformed solution, would be of difficult interpretation[9].

Proposition 4.2.1 (Thm.1-2 in [13]) *The solution of the free-transport equation* $w(x,v,t) = w_0(x-vt,v)$ *satisfies the following estimates, for all* $t \geq 0$,

- *for all* $(p\,,r)$, *such that* $1 \leq r \leq p \leq \infty$,

$$\|w(t)\|_{L^p_x(L^r_v)} \leq t^{-3\left(\frac{1}{r}-\frac{1}{p}\right)} \|w_0\|_{L^r_x(L^p_v)}\,; \tag{4.14}$$

- *for all* (q,p,r,a) *satisfying*

$$\frac{1}{r}-\frac{1}{3} < \frac{1}{p} \leq \frac{1}{r} \leq 1\,, \quad 1 \leq \frac{1}{p}+\frac{1}{r}\,, \quad \frac{2}{q} = 3\left(\frac{1}{r}-\frac{1}{p}\right)\,, \quad a = \frac{2pr}{p+r}\,,$$

$$\|w\|_{L^q_t(L^p_x(L^r_v))} \leq C(3,p,r)\|w_0\|_{L^a(\mathbb{R}^6)}\,. \tag{4.15}$$

Estimate (4.14) shows that, if the initial datum w_0 belongs to $L^r_x(L^p_v)$ with $1 \leq r \leq p$, an effect of the transport is the exchange of summability between the x and the v directions. Indeed, the estimate follows by

$$\|w(t)\|_{L^\infty_x(L^1_v)} \leq t^{-3}\|w_0\|_{L^1_x(L^\infty_v)}\,;$$

(cf. Ref. [8]), which exploits the exact expression of the solution $w(x,v,t) = w_0(x-vt,v)$, and by the conservation of the L^1-norm (cf. Thm. 2 in the paper cited above). The estimate holds for all t with a pole of order $3\,(1/r-1/p)$ at $t=0$.
Estimate (4.15) contains, instead, a mixed information in space, velocity *and* time, and is obtained via a method which generalizes the Strichartz' inequalities (cf. Ref. [18]).
The relation between the indices p, r and a, namely $2/a = (1/p+1/r)$, individuates a hyperbole $\mathcal{H}$ on the plane $\mathbb{R}_p \times \mathbb{R}_r$. Starting from a datum $w_0 \in L^p_x(L^r_v)$ with $(p,r) = (a,a) \in \mathcal{H}$, for a.e. t, the solution $w(t)$ belongs to $L^p_x(L^r_v)$, for all points $(p,r) \in \mathcal{H}$ satisfying $1 \leq r \leq p$ and the additional conditions listed above. Moreover, the function $t \to w(t)$ belongs to L^q_t.

Let us compare the use of the mixed L^p-spaces with the weighted spaces introduced in the previous section. As we have anticipated in Remark 4.1.1, the operator $-v\cdot\nabla_x - \frac{3}{2}$ is dissipative in the weighted space X_1, thus

$$\|w(t)\|_{X_1} = \|w_0(x-vt,v)\|_{X_1} \leq e^{\frac{3}{2}t}\|w_0\|_{X_1}\,, \quad \forall\, t \geq 0\,.$$

Accordingly, if $w_0 \in X_1 (\hookrightarrow L^a\,,\, a \in (6/5,2])$, then the solution $w(t)$ will belong to $L^p_x(L^1_v)$, $3/2 < p \leq 2$, for all $t \geq 0$ and to $L^2_x(L^r_v)$, $1 \leq r \leq 2$, for all $t \geq 0$.
Observe that, for $w_0 \in L^a$, $a \in (6/5,2]$, by Strichartz' inequalities (estimate (4.14)), we get $w(t) \in L^2_x(L^r_v)$ for $1 \leq r \leq 2$, a.e. in time. The choice of $p \in (3/2,2]$ is instead not compatible with $r=1$ and analogously for $a \in (6/5,2]$ with $r=1$ and $a \in (6/5,2]$ with $p > 3/2$.

[9]Remember also the discussion in Section 3.2.2: the initial data that make the equivalence between the free-Schrödinger and the free-transport equation hold, are of very peculiar type.

Moreover, if we want to gain a.e. in time $r = 1$, $p \in [1, 3/2)$, we have to assume $a \in [1, 6/5)$. This is equivalent to increase the weight in x, v. Thus, except by improving the information on the initial datum, we cannot gain anything more in p-summability, even by renouncing to the punctual information in time.
In conclusion, for what the solution of the free-transport equation is concerned, the indices in the estimates in the weighted L^2-spaces are *as sharp as* those in the L^p-mixed ones.
Instead, we are interested in the estimates in Prop. 4.2.1 because of the information about the *time-behaviour* they contain. In particular, they can be used for the particle density relative to the solution of the free-streaming equation, which we indicate by

$$n_0(x,t) := \int_{\mathbb{R}^3_v} w_0(x - vt, v) \, \mathrm{d}v \,, \quad \forall\, x \in \mathbb{R}^3 \,, t \geq 0 \,.$$

Corollary 4.2.1 *The density $n_0(t)$ satisfies*

- *for all $1 \leq p \leq \infty$,*

$$\|n_0(t)\|_{L^p} \leq t^{-3\left(1-\frac{1}{p}\right)} \|w_0\|_{L^1_x(L^p_v)} \,, \; \forall t \geq 0; \tag{4.16}$$

- *for all (q, p, a) satisfying*

$$1 \leq p < \frac{3}{2} \,, \quad \frac{2}{q} = 3\left(1 - \frac{1}{p}\right) \,, \quad \frac{2}{a} - 1 = \frac{1}{p} \,,$$

$$\|n_0\|_{L^q_t(L^p_x)} \leq C(3,p) \|w_0\|_{L^a_{x,v}} \,. \tag{4.17}$$

Again, the previous estimates are to be compared with those obtained by the use of the x-weight: we name here (the three-dimensional version of) the estimate in Lemma I.1 of Ref. [28], namely,

$$\|n_0(t)\|_{L^{5/3}} \leq Ct^{-6/5} \|w_0\|^{2/5}_{L^\infty_{x,v}} \||x|^2 w_0\|^{3/5}_{L^1_{x,v}} \,, \; \forall\, t \geq 0 \tag{4.18}$$

This estimate is in the same spirit of estimate (4.16): the two different assumptions on the inital datum yields the same time-decay of the norm $\|n_0(t)\|_{L^{5/3}}$.

Remark 4.2.1 *In the study of the Vlasov-Poisson problem, such estimate for the density with* $p = 5/3$ *can be used, either in the version* (4.16) *or* (4.18), *in the estimate for the self-consistent electric field. Precisely, the information about the t-decay yields a* reguralizing effect *of the field completely analogous to that in Refs. [11, 22], relative to the Schrödinger-Poisson case (cf. Ref. [28]).*

An estimate concerning the p-summability with $p = 6/5$, can be obtained analogously; it reads:

$$\|n_0(t)\|_{L^{6/5}} \leq Ct^{-1/2} \|w_0\|^{1/6}_{L^\infty} \||x|^{3/5} w_0\|^{5/6}_{L^1} \,, \; \forall\, t \geq 0 \,. \tag{4.19}$$

The alternative version ($p = 6/5$ in (4.16)) reads:

$$\|n_0(t)\|_{L^{6/5}} \leq t^{-1/2} \|w_0\|_{L^1_x(L^{6/5}_v)} \,, \forall\, t \geq 0 \,. \tag{4.20}$$

The index $p = 6/5$ will be crucial, indeed, for the estimate of the self-consistent electric field in the Wigner-Poisson case: we anticipate here that we can recover, by exploiting the

information about the $t^{-1/2}$-decay of $\|n_0(t)\|_{L^{6/5}}$, the same reguralizing effect of the electic field in the Schrödinger-Poisson and Vlasov-Poisson cases (cf. Remark 4.2.1).
We will assume, instead, that it holds, for some $\omega \in [0,1)$,

$$\|n_0(t)\|_{L^{6/5}} \leq C_T t^{-\omega}, \quad \forall\, t \geq 0\,. \tag{$\mathbf{A}$}$$

Observe that if either $w_0 \in L^\infty_{x,v}\,, |x|^2 w_0 \in L^1_{x,v}$ or $w_0 \in L^1_x(L^{6/5}_v)$ holds, then $n_0(t)$ satisfies assumption $(\mathbf{A})$ (by either (4.19) or (4.20)); both the assumption $w_0 \in X_1$ and $w_0 \in X$, instead, do *not* give any information about the time-decay. However, since the previous assumptions on the initial datum have no meaning in the quantum context, we prefer to add assumption $(\mathbf{A})$ to $w_0 \in L^2$ (alternatively, to $w_0 \in X_1$ or to $w_0 \in X$, as well), in order to recover the a priori estimates for the electric field.

4.3 A priori estimates for the electric field

Let us assume that w is a "regular" solution of the WP problem (e.g., let $w(t) \in L^2_x(H^1_v)$, $\nabla_x V[w](t) \in \mathcal{C}_B(\mathbb{R}^3)$, uniformly on bounded t-intervals) for which the Duhamel formula holds:

$$w(x,v,t) \;=\; w_0(x-tv,v) + \int_0^t \Big(\Theta[V[w]]w\Big)(x-sv,v,t-s)\,ds.$$

We formally integrate in v:

$$\begin{aligned} n[w](x,t) \;&=\; \int_{\mathbb{R}^3} w_0(x-tv,v)\,dv + \int_0^t\!\int_{\mathbb{R}^3} \Big(\Theta[V[w]]w\Big)(x-sv,v,t-s)\,dv\,ds \\ &=:\; n_0(x,t) + n_1(x,t), \end{aligned}$$

and split the self-consistent field accordingly:

$$E_0(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x n_0(x,t) \tag{3.22}$$

$$E_1(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x \int_0^t\!\int \Big(\Theta[V[w]]w\Big)(x-sv,v,t-s)\,dv\,ds, \tag{3.23}$$

with $\lambda = \frac{1}{4\pi}$.
Then, we can estimate separately the two terms $E_0(t)$, $E_1(t)$ by exploting the properties of the convolution kernel $1/|x|$, in analogy to the VP case (cf. Ref [20, 28]).
Let us recall the main steps of the derivation of the estimates in the VP case (cf. Ref. [28]) for later comparison.

4.3.1 The Vlasov-Poisson case

Let w^{vp} be the "regular" solution of the VP problem[10]

$$w^{\text{vp}}(x,v,t) \;=\; w_0(x-tv,v) + \int_0^t \Big(\nabla_x V[w^{\text{vp}}]\cdot \nabla_v w^{\text{vp}}\Big)(x-sv,v,t-s)\,ds.$$

[10] Observe that, at this level, there is no reason of distinguishing between a "quantum" initial datum and a classical one (cf. Section 3.2.2, instead), since we are discussing about the formal similarities of the two equations.

and E_0^{vp} , E_1^{vp} be the two terms of the corresponding self-consistent field

$$E_0^{\mathrm{vp}}(x,t) \;:=\; \lambda\frac{x}{|x|^3} *_x n_0(x,t), \tag{3.24}$$

$$E_1^{\mathrm{vp}}(x,t) \;:=\; \lambda\frac{x}{|x|^3} *_x \int_0^t\!\!\int \Big(\nabla_x V[w^{\mathrm{vp}}]\cdot\nabla_v w^{\mathrm{vp}}\Big)(x-sv,v,t-s)\,dv\,ds\,. \tag{3.25}$$

For what the term $E_0^{\mathrm{vp}}(t)$ is concerned, by the properties of $x/|x|^3$ and estimate (4.18) (respectively, estimate (4.16)), it follows

$$\forall\, p\in(3/2,15/4]\,,\quad \|E_0^{\mathrm{vp}}(t)\|_{L^p}\le C_1\,t^{\frac{3}{p}-2}\,,\quad \forall\, t\ge 0 \tag{3.26}$$

with $C_1=C_1(\|w_0\|_{L^\infty},\|(1+|x|^2)w_0\|_{L^1})$ (respectively, $C_1=C_1(\|w_0\|_{L^1},\|w_0\|_{L^{5/3}})$), cf. Lemma III.4 in Ref. [28] (alternatively, Ref. [13]).
The definition latter field $E_1^{\mathrm{vp}}(t)$ can be rewritten as

$$\begin{aligned}(E_1^{\mathrm{vp}})_j(x,t) \;&=\; \lambda\frac{x_j}{|x|^3} *_x \mathrm{div}_x \int_0^t s\int \Big(\nabla_x V[w^{\mathrm{vp}}]w^{\mathrm{vp}}\Big)(x-sv,v,t-s)\,dv\,ds\\ &=\; \lambda\sum_{k=1}^{3}\frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s\int\Big(\nabla_x V[w^{\mathrm{vp}}]w^{\mathrm{vp}}\Big)(x-sv,v,t-s)\,dv\,ds\,,\end{aligned}$$

for all $j=1,2,3$. Since it holds for all $t>s>0$, a.e. $x\in\mathbb{R}^3$,

$$\int\Big(|E^{\mathrm{vp}}|\,w^{\mathrm{vp}}\Big)(x-sv,v,t-s)\,dv\le s^{-3/p'}\|E^{\mathrm{vp}}(t-s)\|_{p'}\Big(\int w^p(x-sv,v,t-s)\,dv\Big)^{1/p}, \tag{3.27}$$

$\forall\, p,q\in(1,\infty)\,,1/p+1/p'=1$, by the properties of the convolution kernel and the conservation of the L^p-norm by the Vlasov equation, it follows

$$\|E_1^{\mathrm{vp}}(t)\|_{L^p}\le\|w_0\|_{L^p_{x,v}}\int_0^t\frac{sds}{s^{3/p'}}\|E^{\mathrm{vp}}(t-s)\|_{L^{p'}}\,.$$

In conclusion, by the use of estimate (3.26) and of an argument of Gronwall type, it can be recovered

$$\forall\, p\in(3/2,3)\,,\quad \|E_1^{\mathrm{vp}}(t)\|_{L^p}\le C_1'\,t^{\frac{3}{p}-2}\,,\quad \forall\, t\ge 0 \tag{3.28}$$

with $C_1'=C_1'(\|w_0\|_{L^\infty_{x,v}},\|(1+|x|^2)w_0\|_{L^1_{x,v}})$ (respectively, $C_1'=C_1'(\|w_0\|_{L^1_{x,v}},\|w_0\|_{L^{5/3}_{x,v}})$), cf. Lemma III.5 in Ref. [28] (respectively, Ref. [13]).
We remark that the time-decay in (3.26), (3.28) is the same obtained for the electric field correponding to the solution of the Schrödinger-Poisson systems (cf. Thm. 5.1 in Ref. [22] and Thm. 2.3 in Ref. [11] for the L^2-study).

4.3.2 The Wigner-Poisson case

We want to proceed analogously to the VP case. To this end, we need an appropriate reformulation of the pseudo-differential operator $\Theta[V]$, in analogy with the operator $\nabla_x V\cdot\nabla_v w$ in the VP equation. The idea is that the latter should be recovered from $\Theta[V]w$ in the semiclassical limit (cf. Remark 4.3.1).

Let us recall that $\Theta[V]w(x,v) = \mathcal{F}^{-1}_{\eta\to v}\Big(i\,\delta V(x,\eta)\mathcal{F}_{v\to\eta}w(x,\eta)\Big)$ (cf. (4.3)). We can rewrite

$$\delta V(x,\eta) \;=\; \int_{x-\eta/2}^{x+\eta/2} \nabla_x V(z)\cdot dz \;=\; \int_{-1/2}^{1/2} \eta\cdot\nabla_x V(x-r\eta)\,dr \;=\; \eta\cdot W(x,\eta)\,, \tag{3.29}$$

with the vector-valued function

$$W(x,\eta) \;:=\; \int_{-1/2}^{1/2} \nabla_x V(x-r\eta)\,dr\,, \quad \forall\,(x,\eta)\in\mathbb{R}^6\,.$$

Then, we define the vector-valued operator

$$\Gamma[\nabla_x V]u(x,v) \;:=\; \mathcal{F}^{-1}_{\eta\to v}\Big(W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big). \tag{3.30}$$

It holds:

Lemma 4.3.1 *Let $\nabla_x V\in\mathcal{C}_B(\mathbb{R}^3)$. Then*

1. $W(x,\eta)\in\mathcal{C}_B(\mathbb{R}^6)$, $\|W\|_{L^\infty_{x,v}} \le \|\nabla_x V\|_{L^\infty}$;
2. $\Gamma[\nabla_x V]: L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)$ *and, for all* $u\in L^2(\mathbb{R}^6)$,
$$\|\Gamma[\nabla_x V]u\|_{L^2_{x,v}} \;\le\; \|\nabla_x V\|_{L^\infty}\|u\|_{L^2_{x,v}};$$
3. $\Gamma[\nabla_x V]: L^2_x(H^1_v)\to L^2_x(H^1_v)$ *and, for all* $u\in L^2_x(H^1_v)$,
$$\|\Gamma[\nabla_x V]u\|_{L^2_x(H^1_v)} \;\le\; \|\nabla_x V\|_{L^\infty}\|u\|_{L^2_x(H^1_v)}. \tag{3.31}$$

Proof. The first and the second assertion are obvious. For (3.31) we use

$$\partial_{v_j}\Gamma[\nabla_x V]u(x,v) \;=\; i\mathcal{F}^{-1}_{\eta\to v}\Big(\eta_j W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big) \;=\; \Gamma[\nabla_x V]\partial_{v_j}u;\;\; j=1,2,3. \tag{3.32}$$

$\square$

Lemma 4.3.2 *Let $\nabla_x V\in\mathcal{C}_B(\mathbb{R}^3)$ and $u\in L^2_x(H^1_v)$. Then*

$$\Theta[V]u(x,v) \;=\; \mathrm{div}_v\left(\Gamma[\nabla_x V]u\right)(x,v) \tag{3.33}$$

Proof. By the definition (3.29) and Lemma 4.3.1,

$$\|\delta V(\,.\,,\eta)\|_{L^\infty_x} \;\le\; |\eta|\,\|W(\,.\,,\eta)\|_{L^\infty_x} \;\le\; |\eta|\,\|\nabla_x V\|_{L^\infty}.$$

Thus, $\|\Theta[V]u\|_{L^2_{x,v}} \le \|\nabla_x V\|_{L^\infty}\|u\|_{L^2_x(H^1_v)}$; the right hand side of equation (3.33) is also well-defined in $L^2(\mathbb{R}^6)$ by estimate (3.31). Equality then follows by equation (3.29) and

$$i\mathcal{F}^{-1}_{\eta\to v}\Big(\eta\cdot W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big) \;=\; \sum_{j=1}^{3}\partial_{v_j}\Gamma_j[\nabla_x V]u(x,v) \;=\; \mathrm{div}_v\left(\Gamma[\nabla_x V]u\right)(x,v).$$

$\square$

Remark 4.3.1 (The semiclassical limit) *The correctly scaled version of the pseudo-differential operator with the reduced Planck constant $\hbar = \frac{h}{2\pi}$ reads*

$$\Theta_\hbar[V]w(x,v) \;=\; \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{V(x+\frac{\hbar}{2}\eta) - V(x-\frac{\hbar}{2}\eta)}{\hbar} \mathcal{F}_{v\to\eta}w(x,\eta)e^{iv\cdot\eta}\,d\eta.$$

Under the assumptions of Lemma 4.3.2, we thus have

$$\begin{aligned}\mathcal{F}_{v\to\eta}(\Theta_\hbar[V]w(x,v)) &= \frac{i}{\hbar}\,\delta V(x,\hbar\eta)\mathcal{F}_{v\to\eta}w(x,\eta)\\ &= i\,W(x,\hbar\eta)\,\cdot\,\eta\,\mathcal{F}_{v\to\eta}w(x,\eta).\end{aligned}$$

The limit $\hbar \to 0$ then yields:

$$i\,W(x,\hbar\eta)\,\cdot\,\eta\,\mathcal{F}_{v\to\eta}w(x,\eta) \;\longrightarrow\; i\nabla_x V(x)\cdot\eta\,\mathcal{F}_{v\to\eta}w(x,\eta) \;=\; \mathcal{F}^{-1}_{\eta\to v}\Big(\nabla_x V(x)\cdot\nabla_v w(x,v)\Big);$$

and hence

$$\Theta_\hbar[V]w(x,v) \;\longrightarrow\; \nabla_x V(x)\cdot\nabla_v w(x,v) \quad \textit{in } L^2(\mathbb{R}^6),$$

which is the non-linear term in the VP equation.

Using the redefinition (3.33) of the pseudo-differential operator, and under the additional assumptions $w \in H^1_x(L^2_v)$, $\Delta V[w] \in \mathcal{C}_B(\mathbb{R}^3)$, we have for $s \in \mathbb{R}$

$$\begin{aligned}\Big(\Theta[V[w]]w\Big)(x-sv,v) &= \mathrm{div}_v\Big(\Gamma[\nabla_x V[w]]w(x-sv,v)\Big)\\ &\quad + s\,\mathrm{div}_x\Big(\Gamma[\nabla_x V[w]]w\Big)(x-sv,v). \end{aligned}\tag{3.34}$$

Thus, also in the WP case, the field E_1 in (3.23) admits the reformulation as ($j = 1,2,3$)

$$\begin{aligned}(E_1)_j(x,t) &:= \lambda\frac{x_j}{|x|^3} *_x \mathrm{div}_x \int_0^t s\int\Big(\Gamma[\nabla_x V[w]]w\Big)(x-sv,v,t-s)\,dv\,ds \qquad (3.35)\\ &= \lambda\sum_{k=1}^{3}\frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s\int\Big(\Gamma_k[\nabla_x V[w]]w\Big)(x-sv,v,t-s)\,dv\,ds.\end{aligned}$$

The following two lemmata are concerned with giving a meaning to the definition (3.35) of the field E_1, independently of the previous derivation.

Lemma 4.3.3 *For all $u \in L^2(\mathbb{R}^6)$ and $E \in L^2(\mathbb{R}^3)$ the following estimate holds*

$$\Big\|\int_{\mathbb{R}^3_v}(\Gamma[E]u)\,(x-sv,v)\,dv\Big\|_{L^2_x} \;\le\; Cs^{-3/2}\|E\|_{L^2}\|u\|_{L^2_{x,v}}, \quad \forall\, s>0. \tag{3.36}$$

Remark 4.3.2 *Observe that the exponent of the variable s recovered in the Lemma is the same as obtained for the VP case in the estimate (3.27) with $p = 2$. The difference between the two estimates is that in the quantum case it has to be derived in the Fourier space, thus the L^2-framework is the only possible. In the classical case, instead, analogous estimates with $p \neq 2$ hold.*

PROOF. Since the operator $\Gamma[.]$ was originally defined for $E \in \mathcal{C}_B(\mathbb{R}^3)$, we shall first derive (3.36) for $E \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ and conclude by a density argument.
By the definition (3.30) and by several changes of variables, the following chain of equalities holds:

$$
\begin{aligned}
(\Gamma[E]u)(x,v) &= (2\pi)^{3/2}\left[\mathcal{F}^{-1}_{\eta\to v}\left(W(x,\eta)\right) *_v u\right](x,v) \\
&= \iiint\int_{-1/2}^{1/2} E(x-r\eta)\, e^{i\eta\cdot z}\, dr\, d\eta\, u(x,v-z)\, dz \;=\; \iiint\int_{-1/2}^{1/2} \frac{1}{|r|^3} E(x-\widetilde{\eta})\, e^{i\widetilde{\eta}\cdot\frac{z}{r}}\, dr\, d\widetilde{\eta}\, u(x,v-z)\, dz \\
&= \iint E(x-\tilde{\eta})\, e^{i\tilde{\eta}\cdot\tilde{z}}\, d\tilde{\eta} \int_{-1/2}^{1/2} u(x,v-r\tilde{z})\, dr\ d\tilde{z} \;=\; \iint E(-\hat{\eta})\, e^{i\hat{\eta}\cdot\tilde{z}}\, d\hat{\eta}\, e^{ix\cdot\tilde{z}} \int_{-1/2}^{1/2} u(x,v-r\tilde{z})\, dr\ d\tilde{z} \\
&= (2\pi)^{3/2}\int \mathcal{F}_{\eta\to\tilde{z}}E(\tilde{z}) \int_{-1/2}^{1/2} u(x,v-r\tilde{z})\, dr\, e^{ix\cdot\tilde{z}}\, d\tilde{z}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int (\Gamma[E]u)\,(x-sv,v)\, dv &= (2\pi)^{\frac{3}{2}}\int \mathcal{F}_{\eta\to\tilde{z}}E(\tilde{z}) \left(\iint_{-1/2}^{1/2} u(x-sv,v-r\tilde{z})\, dr\, e^{-isv\cdot\tilde{z}}\, dv\right) e^{ix\cdot\tilde{z}}\, d\tilde{z} \\
&= \frac{1}{(2\pi s)^3}\int \mathcal{F}_{\eta\to\tilde{z}}E(\tilde{z})\, \mathcal{F}_{v\to\tilde{z}}\left(\int_{-1/2}^{1/2} u(x-v,\frac{v}{s}-r\tilde{z})\, dr\right) e^{ix\cdot\tilde{z}}\, d\tilde{z}\,.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left\|\int (\Gamma[E]u)\,(x-sv,v)\, dv\right\|_{L^2_x} &\le \frac{\|E\|_{L^2}}{(2\pi s)^3}\left\|\int_{-1/2}^{1/2}\mathcal{F}_{v\to\tilde{z}}\Big(u(x-v,\frac{v}{s}-r\tilde{z})\Big)\, dr\right\|_{L^2_{\tilde{z},x}} \\
&\le \frac{\|E\|_{L^2}}{(2\pi s)^3}\left(\int_{-1/2}^{1/2}\Big\|\mathcal{F}_{v\to\tilde{z}}\Big(u(x-v,\frac{v}{s}-r\tilde{z})\Big)\Big\|^2_{L^2_{\tilde{z},x}} dr\right)^{\frac{1}{2}},
\end{aligned}
$$

by applying Hölder's inequality first in the $\tilde{z}$ integral and then in the r integral. Finally, it remains to prove that

$$
\int_{-1/2}^{1/2}\Big\|\mathcal{F}_{v\to z}\Big(u(x-v,\frac{v}{s}-rz)\Big)\Big\|^2_{L^2_{x,z}} dr \;=\; s^3\|u(x,v)\|^2_{L^2_{x,v}}.
$$

This is obtained by using repeatedly Plancherel's equality:

$$
\begin{aligned}
&\int_{-1/2}^{1/2}\Big\|\mathcal{F}_{v\to z}\Big(u(x-v,\frac{v}{s}-rz)\Big)\Big\|^2_{L^2_{x,z}} dr = \int_{-1/2}^{1/2}\Big\|\mathcal{F}_{x\to\xi}\Big[\mathcal{F}_{v\to z}\Big(e^{-iv\xi}u(x,\frac{v}{s}-rz)\Big)\Big]\Big\|^2_{L^2_{\xi,z}} dr \\
&= \int_{-1/2}^{1/2}\Big\|\mathcal{F}_{x\to\xi}\Big(s^3 e^{-is(\xi+z)rz}\mathcal{F}_{v\to s(\xi+z)}u(x,v)\Big)\Big\|^2_{L^2_{\xi,z}} dr = s^6\int_{-1/2}^{1/2}\Big\|\mathcal{F}_{x\to\xi}\Big(\mathcal{F}_{v\to s(\xi+z)}u(x,v)\Big)\Big\|^2_{L^2_{\xi,z}} dr \\
&= s^3\|u(x,v)\|^2_{L^2_{x,v}}.
\end{aligned}
$$

□

Remark 4.3.3 *In Refs. [9, 20], are used more refined versions of the basic estimate* (3.27) *(cf. e.g. estimate (39) in Prop. 1 of Ref. [9]); however, the non-negativity of the classical distribution is a crucial ingredient for their derivation, thus we cannot hope to get analogous estimates in the quantum case.*

The following lemma is an immediate consequence of Lemma 4.3.3. We shall need the notation

$$V_{T,\omega} := \{E \in C((0,T];\ L^2_x(\mathbb{R}^3) \bigm| \|E\|_{V_{T,\omega}} < \infty\}$$

with

$$\|E\|_{V_{T,\omega}} := \sup_{0<t\le T} t^{\omega}\|E(t)\|_{L^2}.$$

Lemma 4.3.4 *For any fixed $T > 0$, let $w \in \mathcal{C}([0,T]; L^2_{x,v})$, and let w_0 be such that for some $\omega \in [0,1)$ it holds*

$$\|n_0(t)\|_{L^{6/5}} \ \le\ C_T t^{-\omega}, \quad \forall\, t \in (0,T]. \tag{$\mathbf{A}$}$$

Then, there exists a unique vector-field $E \in V_{T,\omega-\frac{1}{2}}$ which satisfies the linear equation

$$E_j(x,t) \ = \ \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s \int \Big(\Gamma_k[E_0+E]w\Big)(x-sv,v,t-s)\,dv\,ds;\ j=1,2,3 \tag{4.38}$$

with E_0 defined by ($\lambda = \frac{1}{4\pi}$):

$$E_0(x,t) \ :=\ \lambda \frac{x}{|x|^3} *_x \int w_0(x-tv,v)\,dv.$$

Proof. (4.38) has the structure of a Volterra integral equation of the second kind. Hence, we define the (affine) map $M : V_{T,\omega-\frac{1}{2}} \to V_{T,\omega-\frac{1}{2}}$ by

$$(ME)_j(x,t) \ :=\ \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s \int \Big(\Gamma_k[E_0+E]w\Big)(x-sv,v,t-s)\,dv\,ds.$$

Applying the generalized Young inequality to the definition of E_0 yields

$$\|E_0(t)\|_{L^2} \ \le\ C\|n_0(t)\|_{L^{6/5}}\,, \quad \forall\, t \in (0,T]\,. \tag{3.39}$$

Thus, by Lemma 4.3.3, the second convolution factor in (4.38) is well-defined and

$$\Big\| \int_{\mathbb{R}^3_v} \Big(\Gamma_k[E_0+E]w\Big)(x-sv,v,t-s)\,dv \Big\|_{L^2_x} \le C s^{-3/2} \|(E_0+E)(t-s)\|_{L^2} \|w(t-s)\|_{L^2_{x,v}}\,,\ \forall\, s \in (0,t].$$

By classical properties of the convolution with $\frac{1}{|x|}$ (cf. [St]) and the Young inequality, we get

$$\|(ME)_j(t)\|_{L^2} \ \le\ C \int_0^t \frac{1}{\sqrt{s}} \left(\|E_0(t-s)\|_{L^2} + \|E(t-s)\|_{L^2}\right) \|w(t-s)\|_{L^2_{x,v}}\,ds\,, \quad \forall\, t \in (0,T]. \tag{3.40}$$

Hence, the map M is well-defined from $V_{T,\omega-\frac{1}{2}}$ into itself and satisfies

$$\|ME(t)\|_{L^2} \ \le\ C\Big(C_T + \sup_{s\in(0,T]} s^{\omega-\frac{1}{2}}\|E(s)\|_{L^2}\Big) \sup_{s\in[0,T]} \|w(s)\|_{L^2_{x,v}} \Big(t^{1-\omega} + t^{\frac{1}{2}-\omega}\Big), \quad \forall\, t \in (0,T].$$

Since the map is affine, we have (by induction) for all $t \in (0,T]$

$$
\begin{aligned}
\|M^n E(t) - M^n \widetilde{E}(t)\|_{L^2} &\leq C \sup_{s\in[0,T]} \|w(s)\|_{L^2_{x,v}} \int_0^t \frac{1}{\sqrt{t-s}} \|M^{n-1}E(s) - M^{n-1}\widetilde{E}(s)\|_{L^2}\, ds \\
&\leq \left(C \sup_{s\in[0,T]} \|w(s)\|_{L^2_{x,v}} \right)^n C_{n-1} \int_0^t \frac{s^{\frac{n}{2}-\omega}}{\sqrt{t-s}}\, ds \sup_{s\in(0,T]} \left(s^{\omega-\frac{1}{2}} \|E(s) - \widetilde{E}(s)\|_{L^2} \right),
\end{aligned}
$$

with

$$
\begin{aligned}
\int_0^t \frac{s^{\frac{n}{2}-\omega}}{\sqrt{t-s}}\, ds &= t^{\frac{n+1}{2}-\omega} B\left(\frac{1}{2}, \frac{n+2}{2} - \omega \right), \\
C_{n-1} &= \prod_{j=1}^{n-1} B\left(\frac{1}{2}, \frac{j}{2} + 1 - \omega \right) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{3}{2} - \omega\right)}{\Gamma\left(\frac{n}{2} + 1 - \omega\right)},
\end{aligned}
$$

where B denotes the Beta function and Γ the Gamma function. Thus, the map M^n is contractive for n large enough and admits a unique fixed point $E \in V_{T,\omega-\frac{1}{2}}$. □

With $E = E_1$ the above lemma yields the regularity of the self-consistent field in the WP equation: It satisfies $\nabla_x V[w] = E_1 + E_0 \in V_{T,\omega-\frac{1}{2}}$, under the assumptions that $w \in \mathcal{C}([0,T]; L^2_{x,v})$ and w_0 satisfies $(\mathbf{A})$.

Proposition 4.3.1 *For any fixed $T > 0$, let $w \in \mathcal{C}([0,T]; L^2_{x,v})$ be a mild solution of the WP equation with $\|w(t)\|_2 = \|w_0\|_2$, and with the initial value w_0 satisfying condition $(\mathbf{A})$. Then, the self-consistent field satisfies the following estimates for all $t \in (0,T]$:*

$$
\begin{aligned}
\|E_0(t)\|_{L^2} &\leq C\|n_0(t)\|_{L^{6/5}} \leq C C_T t^{-\omega}, && (3.41)\\
\|E_1(t)\|_{L^2} &\leq C\left(\|w_0\|_{L^2_{x,v}}, \sup_{s\in(0,T]} \{ s^{\omega} \|n_0(s)\|_{L^{6/5}} \}, T \right) t^{\frac{1}{2}-\omega}. && (3.42)
\end{aligned}
$$

Here and in the sequel, the T-dependence of the constants C is continuous (on $T \in \mathbb{R}^+$).

Remark 4.3.4 (No weight) *Observe that the L^2-weighted norm of the Wigner function, correspondingly, the definition of the density n, are* not *needed for the self-consistent field to be well-defined and for the previous estimates to hold. It is sufficient an assumption on the time-decay of n_0, analogously to the classical case (cf. Remark 4.2.1).*
Thus, hypothesis **(A)** *can also be seen as a* kinetic *assumption that allows to overcome the difficulties related to the definition of the density in a L^2-context. However, the possibility of exploiting such an assumption as an alternative to the weight, in order to get well-posedness, is still under investigation. Accordingly, in the previous proposition, it has to be assumed the existence of a mild solution of the WP equation under the assumptions $w_0 \in L^2$, such that* **(A)** *is satisfied (cf. also Remark 4.4.1).*

Proof. The first estimate is (3.39) in Lemma 4.3.4. To derive the second one, we exploit Eq. (3.40), the conservation of the L^2-norm of the solution and (3.41):

$$
\begin{aligned}
\|E_1(t)\|_{L^2} &\leq C \int_0^t s^{-1/2} (\|E_0(t-s)\|_{L^2} + \|E_1(t-s)\|_{L^2}) \|w(t-s)\|_{L^2_{x,v}}\, ds \\
&\leq C\|w_0\|_{L^2_{x,v}} \sup_{s\in(0,T]} \{ s^{\omega} \|n_0(s)\|_{L^{6/5}} \} t^{\frac{1}{2}-\omega} \\
&\quad + C\|w_0\|_{L^2_{x,v}} \int_0^t (t-s)^{-1/2} \|E_1(s)\|_{L^2}\, ds.
\end{aligned}
$$

The thesis follows by Gronwall's Lemma. □

Remark 4.3.5 *Let us compare the a-priori bounds (3.41), (3.42) with their classical counterparts* (3.26), (3.28). *Using either estimate* (4.19) *or* (4.20), *we can obtain the same* $t^{-\frac{1}{2}}$*–singularity of* $\|E(t)\|_2$ *also in the Wigner-Poisson case, by a modification of the proof of Prop. 4.3.1, in the spirit of Lemma III.5 in Ref. [28]. In the Vlasov-Poisson case, similar* L^p*-estimates hold for* p *in a non-trivial interval. One crucial reason for this difference is the conservation of* L^p*-norm of the solution: while the WP equation only conserves the* L^2*-norm, all* L^p*-norms are constant in the VP case.*

Remark 4.3.6 (The WPFP case) *In Chapter 6 we will study an equation which differs from the Wigner equation* (4.1a) *by a uniformly elliptic operator, namely, the Wigner-Fokker-Planck equation* (WFP). *In that case, the diffusive effect adds to the dispersive one and some a priori bounds for the* L^p*-norm of the electrical field hold, with* $p \neq 2$. *These estimates will be a crucial tool to asses a global-in-time well-posedness result, by assuming just* $w_0 \in X$ *and a hypothesis similar to* **(A)**.

4.4 Identities for the WP system

In the Vlasov-Poisson case, it is possible to state some physically-motivated identities, that make evident strong analogies with the Schrödinger-Poisson one. In Ref. [28], e.g., are stated for VP, both a pseudo-conformal law and a dispersive identity, which are inspired respectively by Refs. [15, 17, 22] and Ref. [15, 21]: possible applications of them are to extend the interval of the L^p-a priori estimates for the physical quantities and to deduce decay estimates for large time. In the Wigner-Poisson case, it is possible to state analogous identities; however, due to the fact the Wigner function can assume negative values, the quantities involved are not non-negative, thus such identities cannot be exploited for that aim.

Proposition 4.4.1 (Pseudo-conformal law) *Let* w *be a regular solution of the WP system* (4.1a), (4.1b), *then it satisfies*

$$\frac{d}{dt}\Big[\int |x-vt|^2 w(x,v,t)\,dx\,dv\, + t^2\int |E(x,t)|^2\,dx\Big] = t\int |E(x,t)|^2\,dx\,.$$

Proof. If the solution of the WP problem is smooth enough ($w \in X_1, V \in H^2$, e.g.) the following identities hold by exploiting the equations (4.1a) and (4.1b), and the fact that the function δV is odd in the second group of variables. Precisely,

$$\begin{aligned}
\frac{d}{dt}\int |x|^2 w(x,v,t)\,dx\,dv &= 2\int x\cdot j(x,t)\,dx\,,\\
-2\frac{d}{dt}\int tx\cdot v w(x,v,t)\,dx\,dv &= -2\int x\cdot j(x,t)\,dx - 2t\int |v|^2 w(x,v,t)\,dx\,dv - 2t\int |E(x,t)|^2\,dx\,,
\end{aligned}$$

where is used the current density $j(x,t) := \int v\, w(x,v,t)\,dv$ and

$$\int x\cdot v\,\Theta[V]w(x,v)\,dx\,dv = \int x\cdot \nabla_\eta[\delta V(x,\eta)\hat{w}(x,\eta)]\Big|_{\eta=0} dx = \int x\cdot E(x)n(x)\,dx = \frac{1}{2}\int |E(x)|^2\,dx\,.$$

by applying the Poisson equation (4.1b) for the last equality. Similarly,

$$\int |v|^2 \Theta[V] w(x,v,t)\, dx\, dv = \int \Delta_\eta [\delta V(x,\eta,t) \hat{w}(x,\eta,t)] \Big|_{\eta=0} dx = 2 \int E(x,t) \cdot j(x,t)\, dx$$

and

$$\int E(x,t) \cdot j(x,t)\, dx = -\frac{1}{2} \frac{d}{dt} \int |E(x,t)|^2\, dx\,,$$

since it holds $\partial_t n(x,t) = \nabla_x j(x,t)$, by integrating in dv the Wigner equation. Thus

$$\frac{d}{dt} \int t^2 |v|^2 w(x,v,t)\, dx\, dv = 2t \int |v|^2 w(x,v,t)\, dx\, dv - t^2 \frac{d}{dt} \int |E(x)|^2\, dx\,.$$

The thesis follows by collecting the pieces. □

Remark 4.4.1 *The pseudo-conformal law can be put in the form*

$$\int |x - vt|^2 w(x,v,t)\, dx\, dv \, + t^2 \|E(t)\|^2_{L^2} = \int |x|^2 w_0(x,v)\, dx\, dv \, + \int_0^t s \|E(s)\|^2_{L^2}\, ds\,,$$

and is to be compared with the one that holds in the Schrödinger framework (cf.Refs.[17, 15, 22, 11]), namely

$$\|(x + i\hbar t \nabla_x)\psi(t)\|^2_{L^2} + t^2 \|E(t)\|^2_{L^2} = \|x\psi_0\|^2_{L^2} + \int_0^t s \|E(s)\|^2_{L^2}\, ds$$

and with the one that holds in the Vlasov-Poisson case (for a non-negative distribution function w^{vp})

$$\||x - vt|^2 w^{vp}(t)\|_{L^1_{x,v}} + t^2 \|E^{vp}(t)\|^2_{L^2} = \||x|^2 w_0\|_{L^1_{x,v}} + \int_0^t s \|E^{vp}(s)\|^2_{L^2}\, ds\,,$$

cf. Ref. [28]. Both the identities yield a series of results concerning the asymptotic behaviour in time of the solutions (cf. Refs. [11, 15, 28], e.g.)

Remark 4.4.2 (A priori bound on the density) *The pseudo-conformal law in the VP case can be used to prove the a priori bound on the density*

$$t^{6/5} \|n^{vp}(t)\|_{L^{5/3}} \leq C(T, \|w_0\|_{L^\infty_{x,v}}, \|(1 + |x|^2) w_0\|_{L^1_{x,v}})\,,\ \forall\, t \leq T\,.$$

We remark that such a bound extends the validity of estimate (4.18)*, which gives the rate of dispersion of particles for the solution of the free-transport equation, also to the solution of the VP system; moreover, it allows to avoid the use of a control of the density of type* (4.13)*, i.e. to avoid any assumption on v-weights of the initial datum.*
A further extension to the quantum case is missing and would be a useful tool to asses the global-in-time well-posedness of the three-dimensional WP problem, under the assumption $w_0 \in L^2$ s.t. **(A)** *is satisfied. Precisely, an analogous control of some L^p-norm of the particle density, with $p \leq 3/2$, would give the boundedness of the self-consistent potential $V(x) = 1/|x| * n(x)$, $x \in \mathbb{R}^3$ and, thus, the pseudo-differential operator would be well-defined and bounded in L^2.*

Another identity of interest is the *dispersive identity* stated in Ref. [28] for the VP case: from it can be derived a useful estimate for the VP case, similar to Morawetz' estimate, which holds for the solution of the nonlinear Schrödinger equation (cf. Ref. [15, 18]). In Ref. [21], such an estimate for the solution of a transport-type equation, is employed to deduce the Morawetz' estimate itself.
The identity reads

$$\int_0^t \int w(x,v,s) \left(\frac{|v|^2}{|x|} - \frac{(x\cdot v)^2}{|x|^3} \right) dx\, dv\, ds + \int_0^t \int \frac{(x\cdot E(s,x))^2}{|x|^3} dx\, ds \\ + \int \frac{(x\cdot v)}{|x|} w_0(x,v)\, dx\, dv = \int \frac{(x\cdot v)}{|x|} w(x,v,t)\, dx\, dv$$

for both the WP and the VP cases, and it can be obtained exactly in the same way (cf. Thm. II.2 of Ref. [28]). However, the derivation from it of an estimate of Morawetz' type for the WP equation is impossibile, due to the fact the Wigner function is not non-negative.

Analogously, the *conservation of energy*, which can be formally recovered also for the WP system (cf. Ref. [3]), does not yield any bound for the moments of the solution, as in the classical case, since the kinetic energy (cf. Remark 4.1.2) is not non-negative.

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Part III

Wigner-Poisson systems on bounded domains

The model

In this part we are concerned with a quantum kinetic model for the transport of a charged particles ensemble in a semiconductor device: the active region of the device of interest is a finite region of the physical space (i.e., of the semiconductor material) coupled with the environment through ohmic contacts. Accordingly, the Wigner function, which describes the evolution of the electrons ensemble, will be defined on a subset $\Omega_x \times \mathbb{R}^d_v$ of the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_v$, where Ω_x is a bounded domain in the physical space (relative to the ensemble) $\mathbb{R}^d_x$ and will be the unknow function of a boundary-value problem. In particular, the boundary conditions will model a *time-irreversible* interaction with the environment, which is represented as an ideal particle reservoir (cf. Ref. [15]), accordingly, the system will exchange locally conserved particles with the reservoir.
The evolution of such an *open quantum* system constitutes an example of a problem suitable to be studied with a quantum kinetic approach. Indeed, a reformulation of it, either in terms of Schrödinger wavefunctions or of density matrices, is impossible, because the interaction with the environment we want to model breaks the characterization of the density matrices. For better understanding that point, we refer the reader to Ref. [1], where it is investigated the possibility of constructing a density matrix from a classical incoming distribution function at the boundary between a one-dimensional quantum zone and a classical one.
In the kinetic approach, instead, it is straightforward to model the present situation in which the particles entering the device depend only upon the state of the resevoirs and the particles leaving it depend only upon the state of the device, by assigning *inflow, time-dependent* boundary conditions to the Wigner quasi-distribution function. Precisely,

$$w(s,v,t) = \gamma(s,v,t)\,, \quad (s,v) \in \partial\Omega_x \times \mathbb{R}^d_v\,,\ v\cdot n(s) > 0\,,\ t \geq 0,$$

where $n(s)$ is the inward normal vector to $\partial\Omega_x$ at $s \in \partial\Omega_x$ and $\gamma(t)$ is the prescribed time-dependent inflow. These conditions are the d-dimensional version of those introduced in Ref. [15] in the case of a slab situated between two particle reservoirs, and are those one expects for an analogous classical transport problem formulated in the phase-space.
Inside the active region of the device, the free-transport of the electrons ensemble is modified by the reciprocal repulsive effect due to the charge; the latter can be taken into account through a mean-field self-consistent potential V (cf. the discussion in Part I). Moreover, we can take into account the action of applied potentials and of heterostructures, by including an "external" potential V_e.
Hence, the evolution in time of the function w is described by a version of the Wigner equation (3.6), which is nonlinearly coupled with the Poisson equation for the potential V, because the particle density in the bounded domain Ω_x is expressed by

$$n[w](x) := \int_{\mathbb{R}^d_v} w(x,v)\,\mathrm{d}v, \quad x \in \Omega_x\,. \tag{d}$$

Precisely, by the following d-dimensional version of the WP system (4.1a), (4.1b)

$$\left[\frac{\partial}{\partial t} + v\cdot \nabla_x - \Theta\left[V_e(t) + V(t)\right]\right] w(x,v,t) = 0\,, \quad (x,v) \in \Omega_x \times \mathbb{R}^d_v\,,\ t \geq 0, \tag{WP1}$$

$$\Delta_x V(x,t) = \int_{\mathbb{R}^d_v} w(x,v,t)\,\mathrm{d}v\,, \quad x \in \Omega_x\,,\ t \geq 0\,, \tag{WP2}$$

in addition, the unknown functions w and V satisfy the following boundary conditions (b.c.)

$$w(s,v,t)=\gamma(s,v,t)\,,\quad (s,v)\in\partial\Omega_x\times\mathbb{R}^d_v\,,\ v\cdot n(s)>0\,,\ t\geq 0, \tag{bc1}$$
$$V(x,t)=0,\quad x\in\partial\Omega_x\,,\ t\geq 0, \tag{bc2}$$

and initial condition (i.c.)

$$w(x,v,0)=w_0(x,v),\quad (x,v)\in\Omega_x\times\mathbb{R}^d_v\,. \tag{bc3}$$

According to the definition of the pseudo-differential operator in the present bounded (spatial) domain case,

$$(\Theta[\phi]w)\,(x,v)=\frac{i}{(2\pi)^d}\int_{\mathbb{R}^d_\xi\times\mathbb{R}^d_{v'}}\delta\phi(x,\xi)w(x,v')\,e^{i(v-v')\cdot\xi}\,\mathrm{d}\xi\,\mathrm{d}v',$$
$$\delta\phi(x,\xi):=\phi\left(x+\frac{\xi}{2}\right)-\phi\left(x-\frac{\xi}{2}\right),\quad (x,\xi)\in\Omega_x\times\mathbb{R}^d_\xi,$$

the time-dependent functions $V_e(t)$, which is a datum of our problem, as well as $V(t)$, which is the solution of the Poisson problem (WP2), (bc2), have to be defined in the whole $\mathbb{R}^d_x$. Thus, we add the following condition, which is compatible with (bc2)

$$V(x,t)=0,\quad x\in\mathbb{R}^d_x\setminus\Omega_x\,,\ t\geq 0. \tag{bc4}$$

The time-dependent WP system has been studied in the whole space $\mathbb{R}^3_x\times\mathbb{R}^3_v$ (see Ref. [17] and references therein), in a bounded spatial domain with periodic (Refs. [4, 17]), or absorbing (Ref. [2]), boundary conditions, and on a discrete lattice (Refs. [10, 28]). In most cases, the analysis exploits either the equivalence of Wigner and Schrödinger equations, or some self-adjointness property. In the next two chapters, we will deal respectively with the one-dimensional and the three-dimensional versions of the WP system (WP) with the conditions (bc); as we have already anticipated, the choice of the b.c. will require the use of alternative mathematical techniques.
Moreover, we will have to face the "standard" difficulties related to the WP-type problem we have presented in Chapter 4. Let us recall that in the Fourier space with respect to the v-variable, the pseudo-differential operator $\Theta[\Phi]$ has the "product shape"

$$\left(\mathcal{F}_v\left(\Theta\left[\Phi\right]u\right)\right)(x,\eta)=i\,\delta\Phi(x,\eta)\left(\mathcal{F}_v u\right)(x,\eta), \tag{P-S}$$

which makes its use easier. That consideration and the discussion in the Section 3.4 motivate to choose the space $L^2(\Omega_x\times\mathbb{R}^d_v;\mathbb{C})$ as the functional setting[11]. However, the definition (d) of density $n[w]$ requires the use of some weighted L^2-space $X_k\subset L^2(\Omega_x\times\mathbb{R}^d_v;\mathbb{C})$, namely,

$$X_k:=L^2(\,\Omega_x\times\mathbb{R}^d_v\,,(1+|v|^2)^k\mathrm{d}x\,\mathrm{d}v;\mathbb{C}\,)\,;$$

for an index k s.t. $2k>d$. This is a straightforward generalization for the d-dimensional case of the functional setting used in Refs. [5, 26] (cf. Section 4.1). Observe that, roughly speaking, we are assuming some regularity in the variable η of $\mathcal{F}_{v\to\eta}w$; precisely, that its

[11]Indeed, we shall prove that the Wigner function w (and then V) is real-valued for all times $t\geq 0$ if the initial datum is real-valued, thus we could directly settle the analysis in $L^2(\Omega_x\times\mathbb{R}^d_v;\mathbb{R})$, which is the space of Wigner-transformed H-S operators, i.e. of physically admissable Wigner functions.

derivatives[12] of order k with respect to η still belong to $L^2(\Omega_x \times \mathbb{R}_v^d;\mathbb{C})$.
Accordingly, one can expect, by the product-shape in the Fourier space (cf. Eq. (P-S)), that the pseudo-differential operator $\Theta[\phi]$ will be well-defined from the space X_k in intself, if the derivatives up to order k of the potential function ϕ are (essentially) bounded. Under such hypothesis, the operator $\Theta[\Phi]$ will be bounded from X_k to itself. Thus, a first obstacle will be to prove the sufficient regularity of the solution V of the Poisson equation in the bounded domain Ω_x, for the operator $\Theta[V]$ to be defined on X_k. We anticipate that, while it will be an easy task in the one-dimensional case, in the three-dimensional case we will have to modify the estimate of the pseudo-differential operator, in order to exploit the (previously cited) regularity in the variable η of $\mathcal{F}_{v\to\eta}w$.
For what the linearized[13] problem is concerned, note that this is a non-autonomous "affine" problem (cf. Ref. [5]), because the b.c. (bc1) are non-homogeneous, time-dependent. Accordingly, the use of the semigroup generation property of the streaming operator is more delicate than in Refs. [4, 24] (about the one-dimensional, spatially bounded case with periodic b.c. and with homogeneous b.c., respectively). Thus, we will associate to the "affine" problem one with a linear streaming operator and an additional source term and prove under which assumptions a solution of the former can be recovered from a solution of the latter.
Then, the (local-in-time) solution of the non-linear problem can be obtained by a Banach-type fixed point argument, since the potential term proves to be a locally-Lipschitz perturbation. Accordingly, we will have to recover a priori estimates for the solution in the weighted L^2-norm. Due to the skew-simmetry of the pseudo-differential operator, the L^2-norm is preserved (cf. Remark 4.1.4). For what the other terms in the X_k-norm are concerned, instead, a direct evaluation starting from the equation, may not be enough to get the result, depending on the space-dimension. More precisely, in the one-dimensional case it will be straightforward to prove the boundedness of the norm for all finite time, while in three-dimensional case, it won't be possible.
Also in the present case, hold the same considerations about the impossibility of employing the physically-motivated identities stated in Section 4.4 for recovering a priori bounds for the moments of the solution. Moreover, since the choosen bounded spatial domain breaks the correspondence with the Schrödinger and the density matrices frameworks, there is the additional problem of defining physically-consistent quantities.
We remark that the strategy presented in Section 4.3, to obtain a priori bounds for the self-consitent electric field, cannot be easily modified for the present bounded spatial domain case.
Accordingly, till now, it has not been possible to state the existence for all times of the solution of the three-dimensional version of problem (WP).

[12] Indeed, we are assuming that $\partial_{\eta_i}^k \mathcal{F}_{v\to\eta}w \in L^2(\Omega_x \times \mathbb{R}_v^d;\mathbb{C})$, $\forall\, i = 1,\ldots,d$, $k = m/n$, $m, n \in \mathbb{N}$.
[13] i.e. , Eq. (WP1), with $V = 0$ together with the conditions (bc1),(bc3)

Chapter 5

The one-dimensional Wigner-Poisson system with inflow boundary conditions

5.1 Introduction

In the present chapter we present the content of *Manzini C., Barletti L., An analysis of Wigner-Poisson problem with inflow boundary conditions,* Nonlinear Analysis, **60***(1), 77-100 (2004).* Relatively to the problem (WP) with the conditions (bc), (bc4) introduced in the previous section, we shall prove well-posedness of the linearized d-dimensional version and existence and uniqueness of a global-in-time, regular solution of the one-dimensional nonlinear version.
Here follows an outline of the paper. In Section 5.2 we introduce the spaces $X_k \subset L^2(\Omega_x \times \mathbb{R}^d_v; \mathbb{C})$. Besides, we prove that the pseudo-differential operator $\Theta[\phi]$ is well-defined in such spaces with assumptions on the function ϕ which, in the one-dimensional case, are less restrictive than those in Ref. [24] and yet will ensure well-posedness of the one-dimensional W-P problem.
In Section 5.3, we drop the self-consistent potential V and study the Cauchy problem for the d-dimensional Wigner equation with the time-dependent (external) potential $V_e(t)$ alone, subject to inflow b.c. (bc1).
We prove existence and uniqueness of a classical solution of such problem in the spaces X_k, with $2k > d$, thus improving the result obtained in Ref. [24], relevant to well-posedness of the Cauchy problem with homogeneous b.c. in $L^2(\Omega_x \times \mathbb{R}^d_v; \mathbb{C})$.
In Section 5.4 we discuss the existence of a linear map P which associates to each state $w \in X_k$ the extension with value zero outside Ω_x of the solution V of the Poisson problem (WP2)-(bc2), yielding the self-consistent potential term $\Theta[V]$ in (WP1). However, because of the requirements on the potential V for the operator $\Theta[V]$ to be defined (see Section 2), this attempt is successful only in the one dimensional case. Thus, in the following sections we shall restrict our investigation to the case[1] $d = 1$. In Section 5.5, therefore, we consider

[1] We anticipate that the assumptions on the potential, for the pseudo-differential operator to be well-defined, can be weakened. That modification will make it possible to complete the analysis also in the three-dimensional case (cf. Chapt. 6).

a one-dimensional, nonlinear and non-autonomous Cauchy problem of the following form

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = \mathrm{T}_{\gamma(t)} w(t) + \Theta\left[V_e(t) + \mathrm{P}\, w(t)\right]\, w(t), \quad t \geq 0, \tag{5.1a}$$

$$w(t=0) = w_0, \tag{5.1b}$$

where the affine operator $\mathrm{T}_{\gamma(t)}$ contains the inflow b.c. Note that the problem contains the quadratic nonlinearity $\Theta\,[\mathrm{P}\, w]\, w$. We associate to (5.1) a semilinear, non-autonomous problem where the inflow is transformed into a suitable source term and prove existence and uniqueness of a local-in-time, classical solution of this latter problem.
In Section 5.6, the results of the preceding section, together with a suitable *a priori* estimate, are used to prove the main result of this paper, i.e. existence, uniqueness and regularity of a global-in-time, classical solution of (5.1).
The Appendix is devoted to the construction of an explicit "representation" of the domain $\mathcal{D}(\mathrm{T}_{\gamma(t)})$, according to the concepts introduced in Secs. 5.2 and 5.3.

5.2 The functional setting

Assume that Ω_x is an open, convex and bounded subset of $\mathbb{R}^d_x$ with $\mathcal{C}^1$ boundary $\partial\Omega_x$. Let us introduce, for all $k = 0, 1, 2, \ldots$, the space X_k of the $\mathbb{C}$-valued functions, defined on $\Omega_x \times \mathbb{R}^d_v$, with square summable modulus with respect to the Lebesgue measure in $\Omega_x \times \mathbb{R}^d_v$ with weight $(1 + |v|^2)^k$; in symbols:

$$X_k \;:=\; L^2(\,\Omega_x \times \mathbb{R}^d_v\,, (1 + |v|^2)^k \mathrm{d}x\, \mathrm{d}v\,; \mathbb{C}\,)\,.$$

For all k, X_k is a Hilbert space with scalar product

$$< u, w >_{X_k} := \int_{\mathbb{R}^d_v} \int_{\Omega_x} u(x,v) \overline{w(x,v)}\, (1 + |v|^2)^k \,\mathrm{d}x\, \mathrm{d}v. \tag{5.2}$$

It is straightforward to prove the equivalence of the norm $\|.\|_{X_k}$ with the following

$$\|u\|_{\widetilde{X_k}} \;=\; \|u\|_{X_0} + \sum_{i=1}^{d} \left\|v_i^k u\right\|_{X_0}, \tag{5.3}$$

and X_k is continuously imbedded in $X_0 = L^2(\Omega_x \times \mathbb{R}^d_v\,, \mathrm{d}x\, \mathrm{d}v\,; \mathbb{C})$.
The following proposition is the d-dimensional version of Lemma 4.1.1 for n defined in the domain Ω_x.

Proposition 5.2.1 *Let $u \in X_k$ and $n(x) = \int_{\mathbb{R}^d_v} u(x,v)\,\mathrm{d}v$. If $2k > d$, then $n \in L^2(\Omega_x, \mathrm{d}x)$, and*

$$\|n\|_{L^2(\Omega_x, \mathrm{d}x)} \;\leq\; c(d,k)\; \|u\|_{X_k}, \tag{5.4}$$

where

$$c(d,k) := \max\left\{\, (\omega_d/d)^{1/2},\, (\omega_d/(2k-d))^{1/2} \,\right\} \tag{5.5}$$

and ω_d is the $(d-1)$-dimensional measure of the surface of the d-dimensional unit sphere[2].

[2]Namely,

$$\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

with Γ the Gamma function.

PROOF. From Minkowsky's inequality we have

$$\|n\|_{L^2(\Omega_x,\mathrm{d}x)} \leq \Big(\int_{\Omega_x} \Big| \int_{|v|\leq 1} u(x,v)\,\mathrm{d}v \Big|^2 \mathrm{d}x \Big)^{1/2} + \Big(\int_{\Omega_x} \Big| \int_{|v|>1} u(x,v)\,\mathrm{d}v \Big|^2 \mathrm{d}x \Big)^{1/2}. \tag{5.6}$$

By applying Hölder's inequality on $|v| \leq 1$ in the first summand, we get

$$\int_{\Omega_x} \Big| \int_{|v|\leq 1} u(x,v)\,\mathrm{d}v \Big|^2 \mathrm{d}x \leq \frac{\omega_d}{d}\, \|u\|^2_{X_0}\,. \tag{5.7}$$

Moreover, by using again Hölder's inequality,

$$\begin{aligned} &\int_{\Omega_x} \Big| \int_{|v|\geq 1} u(x,v)\,\mathrm{d}v \Big|^2 \mathrm{d}x \leq \int_{\Omega_x} \Big[\int_{|v|\geq 1} \frac{|u(x,v)|}{(1+|v|^2)^{k/2}} (1+|v|^2)^{k/2}\,\mathrm{d}v \Big]^2 \mathrm{d}x \\ &\leq \int_{\Omega_x} \Big(\int_{|v|\geq 1} |u(x,v)|^2 (1+|v|^2)^k\,\mathrm{d}v \Big) \Big(\int_{|v|\geq 1} \frac{1}{(1+|v|^2)^k}\,\mathrm{d}v \Big) \mathrm{d}x \\ &\leq \Big(\int_{|v|\geq 1} \frac{1}{|v|^{2k}}\,\mathrm{d}v \Big)\, \|u\|^2_{X_k}\,. \end{aligned} \tag{5.8}$$

Since, by changing into polar coordinates,

$$\int_{|v|\geq 1} \frac{1}{|v|^{2k}}\,\mathrm{d}v \;=\; \frac{\omega_d}{2k-d} < \infty \quad \text{if} \quad 2k > d,$$

the bound (5.4) follows from bounds (5.7) and (5.8). □

Remark 5.2.1 If we are dealing with the one-dimensional W-P system (5.1), then the appropriate functional setting is X_1 (see also Ref. [4]) and the suitable estimate is the following

$$\|n(t)\|_{L^2(\,\Omega_x,\mathrm{d}x\,)} \leq c(1,1)\ \|w(t)\|_{X_1} \qquad \forall\, t \geq 0\,, \tag{5.9}$$

where $c(1,1) = \sqrt{2}$. The three-dimensional case has to be dealt with in X_2, and we get the estimate

$$\|n(t)\|_{L^2(\,\Omega_x,\mathrm{d}x\,)} \leq c(3,2)\ \|w(t)\|_{X_2} \qquad \forall\, t \geq 0, \tag{5.10}$$

where $c(3,2) = 2\sqrt{\pi}$. Since our aim is proving that the Wigner equation (WP1) is well-posed in X_k with the appropriate k, we discuss under which assumptions on the function $\phi(t)$ the pseudo-differential operator $\Theta\,[\phi(t)]$ is well-defined from X_k into itself. The following Proposition is a straightforward generalization of (2.4) of Ref. [24].

Proposition 5.2.2 *The map $(u,\phi) \mapsto \Theta\,[\phi]\,u$ is bilinear from $X_k \times W^{k,\infty}(\mathbb{R}^d_x)$ into X_k, and there exists $b > 0$ such that*

$$\|\Theta\,[\phi]\,u\|_{X_k} \leq b\ \|\phi\|_{W^{k,\infty}(\mathbb{R}^d_x)}\, \|u\|_{X_k}\ , \quad \forall\,(u,\phi) \in X_k \times W^{k,\infty}(\mathbb{R}^d_x)\,. \tag{5.11}$$

In particular, if $\phi \in W^{k,\infty}(\mathbb{R}^d_x)$, then $\Theta\,[\phi]$ belongs to the space $\mathcal{B}(X_k)$ of linear, bounded operators on X_k.

PROOF. The proof of bilinearity is immediate; then let us focus on the proof of inequality (5.11). Let $u \in X_k$ and $\phi \in W^{k,\infty}(\mathbb{R}^d_x)$. By using (5.3) we can write

$$\|\Theta[\phi]\, u\|^2_{\widetilde{X}_k} = \|\Theta[\phi]\, u\|^2_{X_0} + \sum_{i=1}^{d} \left\| v_i^k \Theta[\phi]\, u(x,v) \right\|^2_{X_0}. \tag{5.12}$$

Since

$$\left(\mathcal{F}_v\left(\Theta[\phi]\, u\right)\right)(x,\eta) = i\,\delta\phi(x,\eta)\left(\mathcal{F}_v u\right)(x,\eta),$$

the first summand of the right-hand side of (5.12) is, up to a multiplicative constant, $\|\delta\phi\ (\mathcal{F}_v u)\|^2_{X_0}$, while in the second summand there are terms of the type

$$\left\| v_i^k \Theta[\phi]\, u \right\|^2_{X_0} = (2\pi)^d \left\| \partial^k_{\eta_i}\left(\delta\phi\ (\mathcal{F}_v u)\right)\right\|^2_{X_0},$$

Since

$$\begin{aligned}\partial^j_{\eta_i}\delta\phi(x,\eta) = \partial^j_{\eta_i}\left\{\phi\Big(x+\frac{\eta}{2}\Big) - \phi\Big(x-\frac{\eta}{2}\Big)\right\} &= \\ &= \left\{\Big(\frac{1}{2}\Big)^j(\partial^j_{\eta_i}\phi)\Big(x+\frac{\eta}{2}\Big) - \Big(-\frac{1}{2}\Big)^j(\partial^j_{\eta_i}\phi)\Big(x-\frac{\eta}{2}\Big)\right\},\end{aligned}$$

then

$$\left\|\partial^j_{\eta_i}\delta\phi\right\|_{L^\infty(\mathbb{R}^{2d})} \le \Big(\frac{1}{2}\Big)^{j-1}\left\|\partial^j_{\eta_i}\phi\right\|_{L^\infty(\mathbb{R}^d)} < +\infty,$$

for all $j \le k$ as $\phi \in W^{k,\infty}(\mathbb{R}^d_x)$. Moreover, the derivatives $\partial^j_{\eta_i}(\mathcal{F}_v u)$ belong to X_0 since $u \in X_k$, and $\left\|\partial^j_{\eta_i}(\mathcal{F}_v u)\right\|^2_{X_0} = (2\pi)^{-d}\left\|v_i^j u\right\|^2_{X_0}$, for all $j \le k$. Thus, by using the Leibniz's rule

$$\partial^k_{\eta_i}\left(\delta\phi\ (\mathcal{F}_v u)\right) = \sum_{0\le l\le k}\binom{k}{l}\partial^l_{\eta_i}\delta\phi\,\partial^{k-l}_{\eta_i}(\mathcal{F}_v u),$$

we easily obtain the inequality in the thesis. □

Remark 5.2.2 According to Proposition 5.2.2, a sufficient condition for the operator $\Theta[V_e(t) + V(t)]$ appearing in (WP1) to be well-defined on X_k is that $V_e(t)$ and $V(t)$ belong to $W^{k,\infty}(\mathbb{R}^d_x)$. This condition, together with the restriction $2k > d$ coming from Proposition 5.2.1, is a serious obstacle to prove well-posedness of the n-dimensional W-P problem with $d > 1$ (see Sec. 5.4). Actually, we shall prove in the one-dimensional case that, for all $w \in X_1$, there exists a unique solution of the Poisson problem on the bounded domain Ω_x (WP2)-(bc2), whose extension with value zero outside the domain Ω_x belongs to $W^{1,\infty}(\mathbb{R}_x)$. In contrast, an analogous attempt is unsuccessful in the three-dimensional case: in fact the hypothesis $V \in W^{2,\infty}(\mathbb{R}^3_x)$ is in general not satisfied by the extension with value zero on $\mathbb{R}^3 \setminus \Omega_x$ of the self-consistent potential V.
We also remark that, in the one-dimensional case, the requirement $V \in W^{1,\infty}(\mathbb{R}_x)$ is less restrictive than that ensuring well-posedness of W-P in Ref. [24].

A straightforward consequence of Proposition 5.2.2 is the following.

Corollary 5.2.1 *If the function $t \mapsto \phi(t)$ is of class $\mathcal{C}^1([0,\infty)\,;\,W^{k,\infty}(\mathbb{R}^d_x))$, then the function $t \mapsto \Theta[\phi(t)]$ is of class $\mathcal{C}^1([0,\infty)\,;\,\mathcal{B}(X_k))$ and its derivative is $t \mapsto \Theta[\phi'(t)]$.*

Now let us turn our attention to the free-streaming term $-v\cdot\nabla_x$ appearing in Eq. (WP1). Consider the linear operator

$$\mathrm{T}_{\max}u := -v\cdot\nabla_x u,$$

defined on the maximal domain

$$\mathcal{D}(\mathrm{T}_{\max}) := \{\, u\in X_k \,|\, v\cdot\nabla_x u \in X_k \,\}.$$

Since we want to take into account the b.c. (bc1) by defining a suitable subdomain, let us introduce the "inflow trace-space"

$$\mathcal{Y}_k^{\mathrm{in}} := L^2(\Phi^{\mathrm{in}}, v\cdot n(s)\,(1+|v|^2)^k\,\mathrm{d}s\,\mathrm{d}v\,;\mathbb{R}), \tag{5.13}$$

where

$$\Phi^{\mathrm{in}} := \left\{\, (s,v)\in\partial\Omega_x\times\mathbb{R}_v^d \,|\, v\cdot n(s)>0 \,\right\}. \tag{5.14}$$

Under our assumptions on Ω_x, any function belonging to $\mathcal{D}(\mathrm{T}_{\max})$ has a well-defined trace $u_{|\Phi^{\mathrm{in}}}$ on Φ^{in}, belonging to $\mathcal{Y}_k^{\mathrm{in}}$ (see Refs. [16] and [30]). Thus, given a function $t\mapsto\gamma(t)$, with $\gamma(t)\in\mathcal{Y}_k^{\mathrm{in}}$ for all $t\geq 0$, we can define the following time-dependent *affine* operator:

$$\mathrm{T}_{\gamma(t)}u := -v\cdot\nabla_x u, \tag{5.15a}$$

$$\forall\, u\in\mathcal{D}(\mathrm{T}_{\gamma(t)}) := \{u\in\mathcal{D}(\mathrm{T}_{\max}) \mid u_{|\Phi^{\mathrm{in}}}=\gamma(t)\}. \tag{5.15b}$$

In the case $\gamma\equiv 0$, the operator T_0 is linear and represents streaming with null inflow. Note that the sets $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ are $\mathcal{D}(\mathrm{T}_0)$-affine subspaces of X_k, i.e., for all $t\geq 0$ and u_1, $u_2\in\mathcal{D}(\mathrm{T}_{\gamma(t)})$, we have $u_1-u_2\in\mathcal{D}(\mathrm{T}_0)$. Moreover, the operators $\mathrm{T}_{\gamma(t)}$ are T_0-affine operators, i.e., for all $t\geq 0$ and u_1, $u_2\in\mathcal{D}(\mathrm{T}_{\gamma(t)})$, we have $\mathrm{T}_{\gamma(t)}u_1-\mathrm{T}_{\gamma(t)}u_2=\mathrm{T}_0(u_1-u_2)$. According to that, we say that $\{\,\mathrm{T}_{\gamma(t)}\,|\,t\geq 0\,\}$ is a T_0-*affine family* (see Refs. [5, 8]).

Note that if we find a function $p:[0,+\infty)\to X_k$ such that

$$p(t)\in\mathcal{D}(\mathrm{T}_{\gamma(t)}),\quad \forall\, t\geq 0, \tag{5.16}$$

then

$$\mathcal{D}(\mathrm{T}_{\gamma(t)}) = p(t)+\mathcal{D}(\mathrm{T}_0),\quad \forall\, t\geq 0.$$

Such a function $p(t)$ is called a *representation* of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$. In particular, note that, if $p(t)$ is a representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$, then, for any given function $w:[0,+\infty)\to\mathcal{D}(\mathrm{T}_{\gamma(t)})$, a function $u:[0,+\infty)\to\mathcal{D}(\mathrm{T}_0)$ exists such that

$$w(t)=u(t)+p(t),\quad \forall\, t\geq 0.$$

This fact will be exploited in the next section where we associate to the affine evolution equation for the unknown $w(t)$, containing the operators $\mathrm{T}_{\gamma(t)}$, a linear evolution equation for the unknown function $u(t)$, with the operator T_0 and an additional source term depending on $p(t)$.

5.3 The affine evolution problem

In this section we study the d-dimensional, linearized version of Eq. (WP1) with b.c. (bc1) and i.c. (bc3) which, in its abstract form, reads as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) = \mathrm{T}_{\gamma(t)}w(t)+\Theta\,[\phi(t)]\,w(t),\qquad t\geq 0, \tag{5.17a}$$

$$w(0)=w_0, \tag{5.17b}$$

where $\phi(t) = \phi(x,t)$ is an assigned potential. This is an affine (because it contains the T_0-affine operators $\mathrm{T}_{\gamma(t)}$) and non-autonomous evolution problem.
As hinted at the end of the previous section, if we choose an appropriate representation $p(t)$ for the $\mathcal{D}(\mathrm{T}_0)$-affine domains $\mathcal{D}(\mathrm{T}_{\gamma(t)})$, i.e. a function $p : [0,+\infty) \to X_k$ such that (5.16) holds, then we can associate to the affine evolution problem (5.17) a linear problem with an additional source term. More precisely, it can be easily verified that the following holds (see also Theorem 2.1 of Ref. [5]).

Proposition 5.3.1 *Let p be a representation of the family of the $\mathcal{D}(\mathrm{T}_0)$-affine domains $\{\mathcal{D}(\mathrm{T}_{\gamma(t)}) \mid t \geq 0\}$ and assume that p and ϕ are such that $Q_p \in \mathcal{C}(\,[0,+\infty); X_k\,)$, where*

$$Q_p(t) := \mathrm{T}_{\gamma(t)} p(t) + \Theta\,[\phi(t)]\; p(t) - p'(t)\,, \qquad \forall t \geq 0\,. \tag{5.18}$$

If w is a classical solution[3] *of* (5.17)*, then the function $u : [0,+\infty) \to X_k$, defined by $u(t) := w(t) - p(t)$ $\forall t \geq 0$, is a classical solution*[4] *of the following evolution problem*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \mathrm{T}_0 u(t) + \Theta\,[\phi(t)]\; u(t) + Q_p(t), \quad t \geq 0, \tag{5.19a}$$

$$u(t{=}0) = u_0 := w_0 - p(0)\,. \tag{5.19b}$$

Conversely, if u is a classical solution of (5.19)*, then the function $w : [0,+\infty) \to X_k$, defined by $w(t) = u(t) + p(t)$, $\forall\, t \geq 0$, is a classical solution of* (5.17)*.*

In the following, we shall refer to (5.19) as to the "associated problem".

As suggested by the assumptions of Proposition 5.3.1, the representation $p : [0,+\infty) \to X_k$ must grant the required regularity of the function $Q_p(t)$ defined by (5.18) which is the source in Eq. (5.19). Thus, we are led to the following definitions.

Definition 5.3.1 *Let $p : [0,+\infty) \to X_k$ be a representation of the $\mathcal{D}(\mathrm{T}_0)$-affine domains $\mathcal{D}(\mathrm{T}_{\gamma(t)})$. We say that p is* regular *if $p \in \mathcal{C}^1([0,+\infty)\,; X_k)$ and the function $t \mapsto \mathrm{T}_{\gamma(t)} p(t)$ belongs to $\mathcal{C}([0,+\infty)\,; X_k)$. We say that p is* strongly regular *if $p \in \mathcal{C}^2([0,+\infty)\,; X_k)$ and $t \mapsto \mathrm{T}_{\gamma(t)} p(t)$ belongs to $\mathcal{C}^1([0,+\infty)\,; X_k)$.*

Because of Corollary 5.2.1, assuming p regular and ϕ in $\mathcal{C}([0,+\infty)\,; W^{k,\infty}(\mathbb{R}^n_x)$ implies $Q_p \in \mathcal{C}([0,+\infty)\,; X_k)$; similarly, assuming p strongly regular and ϕ in $\mathcal{C}^1([0,+\infty)\,; W^{k,\infty}(\mathbb{R}^n_x))$ implies $Q_p \in \mathcal{C}^1([0,+\infty)\,; X_k)$.

Remark 5.3.1 There is a considerable arbitrariness in the choice of regular and strongly regular representations p of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$. Since, on the other hand, p has been introduced in the associated problem (5.19) in order to obtain the solution of the original affine problem (5.17) as $w = u + p$, then an important point will be proving that w does not depend on the choice of p. Moreover, it may be interesting to look for representations which simplify the source term Q_p. For example, if $v \cdot \nabla_x p = 0$, then $Q_p(t)$ reduces to $\Theta\,[\phi(t)]\; p(t) - p'(t)$. It would be also meaningful to translate the regularity conditions on p into regularity conditions on the inflow datum γ. In the Appendix we give an example of a strongly regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$.

[3] i.e. is such that $w \in \mathcal{C}^1(\,[0,+\infty); X_k\,)\,,\ w(t) \in \mathcal{D}(\mathrm{T}_{\gamma(t)})\ \forall t \geq 0\,,\ w$ satisfies (5.17).
[4] i.e. is such that $u \in \mathcal{C}^1(\,[0,+\infty); X_k\,)\,,\ u(t) \in \mathcal{D}(\mathrm{T}_0)\ \forall t \geq 0\,,\ u$ satisfies (5.19).

From known results on transport operators (see Ref. [16]) it follows that the (linear) null-inflow streaming operator T_0 generates a semigroup of contractions $\{e^{t\mathrm{T}_0} \mid t \geq 0\}$ on X_k. Then, the evolution problem (5.19) contains a time-dependent bounded perturbation $\Theta\,[\phi(t)]$ of the generator T_0, and a source term. The following considerations are standard in the theory of evolution equations, [12].

Definition 5.3.2 (Mild solution of the associated problem (5.19)**)** *Let $p : [\,0,\infty) \to X_k$ be a regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ (in the sense of Definition 5.3.1), let $\phi \in \mathcal{C}(\,[\,0,\infty)\,;\,W^{k,\infty}(\mathbb{R}^d_x)\,)$, and let $u_0 \in X_k$. A continuous solution $u : [0,\infty) \to X_k$ of the integral equation*

$$u(t) = e^{t\mathrm{T}_0}u_0 + \int_0^t e^{(t-s)\mathrm{T}_0}\, Q_p(s)\,\mathrm{d}s + \int_0^t e^{(t-s)\mathrm{T}_0}\,\Theta\,[\phi(s)]\,u(s)\,\mathrm{d}s, \tag{5.20}$$

is called mild solution *of* (5.19)*, [26].*

The assumptions on ϕ and p in Definition 5.3.2 grant that the integral operator appearing in Eq. (5.20) is a Banach-space Volterra operator and that the term

$$e^{t\mathrm{T}_0}u_0 + \int_0^t e^{(t-s)\mathrm{T}_0} Q_p(s)\,\mathrm{d}s$$

is a continuous function of $t \in [0,\infty)$. Moreover, the solution of (5.20) (i.e., the mild solution of (5.19)) is unique and is given by the perturbation series

$$u(t) \;=\; \sum_{n=0}^{+\infty} v_n(t)\,, \quad t \geq 0, \tag{5.21}$$

where the functions $v_n(t)$ are recursively defined by

$$\begin{cases} v_0(t) := e^{t\mathrm{T}_0}u_0 + \displaystyle\int_0^t e^{(t-s)\mathrm{T}_0}\, Q_p(s)\,\mathrm{d}s, \quad t \geq 0, \\ v_{n+1}(t) := \displaystyle\int_0^t e^{(t-s)\mathrm{T}_0}\,\Theta\,[\phi(s)]\,v_n(s)\,\mathrm{d}s, \quad t \geq 0,\; n = 0,1,2,\dots. \end{cases} \tag{5.22}$$

Evidently a classical solution is also a mild solution; under the increased regularity assumptions of the following proposition also the converse is true.

Proposition 5.3.2 (Classical solution of the associated problem (5.19)**)** *Let $p : [\,0,\infty) \to X_k$ be a strongly regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ (in the sense of Definition 6.3.1), let $\phi \in \mathcal{C}^1(\,[\,0,\infty)\,;\,W^{k,\infty}(\mathbb{R}^d_x)\,)$, and let $u_0 \in \mathcal{D}(\mathrm{T}_0)$. If $u : [0,\infty) \to X_k$ is the* continuous *solution of the integral equation* (5.20)*, then u is also a classical solution of the evolution problem* (5.19)*.*

Proof. A proof is obtained by suitably modifying that of Thm. 4.8 in Ref. [7] on account of the presence of the time-dependent bounded perturbation of the free-streaming operator (see Prop. 3.3 in Ref. [21]). Otherwise, we observe that the function $f : [0,+\infty) \times X_k \to X_k$ so defined

$$f(t,u) = \Theta\,[\phi(t)]\,u + Q_p(t),$$

is continuously differentiable in $[0,+\infty) \times X_k$, since

$$f_t(t,u) = \Theta\left[\phi'(t)\right]u + \left(\mathrm{T}_{\gamma(t)}p(t)\right)_t + \Theta\left[\phi'(t)\right]p(t) + \Theta\left[\phi(t)\right]p'(t) - p''(t)$$

and

$$f_u(t,u) = \Theta\left[\phi(t)\right]u,$$

by the definition of Fréchet differentiability. Then the thesis follows by applying Thm. 1.5 in Ref. [26].
□

Finally, we return to our original affine problem (5.17).

Theorem 5.3.1 (Classical solution of the affine problem (5.17)**)** *Let $w_0 \in \mathcal{D}(\mathrm{T}_{\gamma(0)})$ and let p and ϕ be as in Proposition 5.3.2. Then* (5.17) *has a unique classical solution $w : [0,\infty) \to X_k$ given by $w(t) := u(t) + p(t)$, $t \geq 0$ where $u : [0,\infty) \to X_k$ is the unique classical solution of the associated problem* (5.19) *with initial datum $u_0 := w_0 - p(0)$. In particular, w is independent of the choice of the representation p with the required regularity.*

PROOF. The existence is a straightforward consequence of Proposition 5.3.1 and of Proposition 5.3.2. The uniqueness (and, therefore, the independence of the solution of the strongly regular representation p) follows from the uniqueness of the solution of the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = \mathrm{T}_0 z(t) + \Theta\left[\phi(t)\right]\, z(t), \qquad t \geq 0, \tag{5.23a}$$

$$z(0) = 0. \tag{5.23b}$$

□

5.4 The Poisson problem

Let us turn our attention to the Poisson problem for the self-consistent potential V:

$$\Delta_x V(x) = \int_{\mathbb{R}^n_v} w(x,v)\,\mathrm{d}v, \qquad x \in \Omega_x, \tag{5.24a}$$

$$V(x) = 0 \qquad x \in \partial\Omega_x \subset \mathbb{R}^n_x. \tag{5.24b}$$

Throughout this section we do not write the time variable, which plays just the role of a parameter.
Remember that, for all $k \in \mathbb{N}$ such that $2k > d$ and for all $w \in X_k$, the right hand side of Eq. (5.24a) is well-defined and belongs to $L^2(\Omega_x)$ (see (5.4)). In order to proceed in the analysis of the W-P system, we need a solution V of this problem whose extension with value zero out of Ω_x has the properties required by Proposition 5.2.2, i.e., it belongs to $W^{k,\infty}(\mathbb{R}^d_x)$. Such solution grants us that the operator $\Theta\left[V\right]$ in the Wigner equation (WP1) is well-defined from X_k into itself.

In the one-dimensional case, $d = 1$, this program turns out to be successful since we can work with $k = 1$ and, as we shall see in the following, problem (5.24) yields a potential V whose extension to the whole real line belongs to $W^{1,\infty}(\mathbb{R})$.

At higher dimension $d \geq 2$, since in this case we need $k \geq 2$, we should have at least $V \in W^{2,\infty}(\mathbb{R}^3)$ but this in general false.

For these reasons, *from now on we shall assume that the system is one-dimensional* and the analysis will proceed in the case $d = 1$.

Thus, let $\Omega_x = (0, l)$; the solution of

$$V''(x) = n(x) \qquad \text{a.e. } x \in (0,l) \tag{5.25a}$$
$$V(0) \;=\; V(l) \;=\; 0\,, \tag{5.25b}$$

with $n \in L^2((0,l))$, can be easily calculated and is given by

$$V(x) = \int_0^x \int_0^y n(z)\,\mathrm{d}z\,\mathrm{d}y - \frac{x}{l}\int_0^l \int_0^y n(z)\,\mathrm{d}z\,\mathrm{d}y \quad x \in (0,l).$$

V and $\widetilde{V}$, its extension with value zero outside $(0,l)$, are essentially bounded and similarly for V' and for its zero extension $\widetilde{V'}$; actually

$$V'(x) = \int_0^x n(z)\,\mathrm{d}z - \frac{1}{l}\int_0^l \int_0^y n(z)\,\mathrm{d}z\,\mathrm{d}y \quad x \in (0,l).$$

Moreover, for the norm $\left\|\widetilde{V}\right\|_{W^{1,\infty}(\mathbb{R})} := \max\{\left\|\widetilde{V}\right\|_{\infty}, \left\|\widetilde{V'}\right\|_{\infty}\}$, we obtain the following estimate

$$\left\|\widetilde{V}\right\|_{W^{1,\infty}(\mathbb{R})} \;\leq\; \sqrt{l}/3\,\max\{\,3+\sqrt{3}\,,2l\sqrt{3}\,\}\;\|n\|_{L^2(\,(0,l),\mathrm{d}x\,)} \tag{5.26}$$

(see Thm. VIII.5 in Ref. [9]).

Remark 5.4.1 Since the zero extension $\tilde{V}$ of the solution of problem (5.25) belongs to $W^{1,\infty}(\mathbb{R})$, $\Theta\left[\tilde{V}\right]$ belongs to $\mathcal{B}(X_1)$, in virtue of estimate (5.11).

The discussion about the one-dimensional case allows us to state the following lemma.

Lemma 5.4.1 *Let* $\mathrm{P} : X_1 \to W^{1,\infty}(\mathbb{R})$ *be the map* $\mathrm{P}\,w := \tilde{V}$*, where* $\tilde{V}$ *is the extension with value zero outside* Ω_x *of the solution of problem* (5.25)*, with* $n(x) = \int_{\mathbb{R}_v} w(x,v)\,\mathrm{d}v$*. Then* $\mathrm{P} \in \mathcal{B}(X_1\,; W^{1,\infty}(\mathbb{R}))$.

Proof. The map is well-defined because of the estimate (5.26). From (5.26) and bound (5.9), it follows that

$$\|\mathrm{P}\,w\|_{W^{1,\infty}(\mathbb{R})} \leq d\,c(1,1)\,\|w\|_{X_1}\,, \quad \forall\, w \in X_1, \tag{5.27}$$

where

$$d := \sqrt{l}/3\,\max\{\,3+\sqrt{3}\,,2l\sqrt{3}\,\} \tag{5.28}$$

and $c(1,1) = \sqrt{2}$ comes from (5.5). □

As a consequence of Lemma 5.4.1 and Proposition 5.2.2, the nonlinear operator

$$\mathrm{F}\,: X_1 \to X_1 \quad \text{such that} \quad \mathrm{F}(w) := \Theta\,[\mathrm{P}\,w]\,w\,. \tag{5.29}$$

is well-defined for $d = 1$. Moreover, $\mathrm{F}(w)$ depends quadratically on $w \in X_1$.

Remark 5.4.2 In the applications to device modeling a more general, non-homogeneous boundary condition for V may be of interest. In the one-dimensional case such non-homogeneous condition reads as follows:

$$V(0) = V_-, \qquad V(l) = V_+,$$

where V_- and V_+ are constants. The solution of the Poisson problem with such b.c. can be continuously extended, with constant values outside $[0, l]$, in order to have well-defined operators P and F. In this case the operator P is not linear any more but, nevertheless, all the results that we shall obtain for the homogeneous case ($V_- = V_+ = 0$) can be easily extended to the non-homogeneous one. In fact, it can be easily shown that an inequality similar to (5.26) holds and that P is Lipschitz continuous, which implies that the Lipschitz property of F (see Lemma 5.5.1) is still true. However, for higher dimension things would be much more delicate.

5.5 The one-dimensional non-linear evolution problem

This section is devoted to the study of the affine-semilinear evolution problem

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = \mathrm{T}_{\gamma(t)} w(t) + \Theta\left[V_e(t)\right] w(t) + \mathrm{F}(w(t)), \quad t \geq 0, \tag{5.30a}$$

$$w(t{=}0) = w_0. \tag{5.30b}$$

which is the abstract formulation of the (one-dimensional) W-P problem (WP), (bc) in terms of the operators introduced in the previous sections. In particular, we recall that the affine streaming operator $\mathrm{T}_{\gamma(t)}$ is defined by (5.15), the pseudo-differential operator $\Theta\left[\phi\right]$ is defined by (4.2) and the nonlinear operator F is defined by (5.29).

In analogy with Sec. 5.3, by choosing a representation $p(t)$ for the family of $\mathcal{D}(\mathrm{T}_0)$-affine domains $\{\, \mathcal{D}(\mathrm{T}_{\gamma(t)}) \,|\, t \geq 0 \,\}$, we can associate to the affine-semilinear problem (5.30) the following semilinear problem:

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \mathrm{T}_0 u(t) + \Theta\left[V_e(t)\right] u(t) + \mathrm{F}\left((u+p)(t)\right) + Q_p(t), \quad t \geq 0, \tag{5.31a}$$

$$u(t{=}0) = w_0 - p(0), \tag{5.31b}$$

where, as before,

$$Q_p(t) := \mathrm{T}_{\gamma(t)}(p(t)) + \Theta\left[V_e(t)\right] p(t) - p'(t),$$

for all $t \geq 0$. In the following we shall refer to (5.30) as to the "original" problem and to (5.31) as to the "associated" problem. By exploiting a local Lipschitz property of the operator F and applying a Banach's Fixed Point procedure (see Chpt. 6 of Ref. [26]), we shall prove in this section the existence and uniqueness of a local-in-time, mild solution of the associated problem (5.31). Under suitable assumptions, the mild solution is also classical, and, in Section 5.6, we shall derive from it the local-in-time, classical solution of the original problem (5.30). Then, we shall prove *a priori* estimates which yields uniqueness and existence of a global-in-time, classical solution. Lastly, we discuss the independence of the solution on the choice of the representation p, a result which is not evident *a priori*.

5.5.1 The non-linear operator F

Let us recall that the operator $\mathrm{F} : X_1 \to X_1$ is defined by (5.29). From Proposition 5.2.2 the quadratic structure of F is evident; this is the key point of the proof of the following lemma.

Lemma 5.5.1 *The operator* $\mathrm{F} : X_1 \to X_1$ *has the following properties.*

1. *For all* $R > 0$, F *is Lipschitz-continuous on the ball*

$$B_R(X_1) := \{f \in X_1 \mid \|f\|_{X_1} \leq R\},$$

i.e.

$$\|\mathrm{F}(f) - \mathrm{F}(f_1)\|_{X_1} \leq L(R)\, \|f - f_1\|_{X_1}, \quad \forall f,\ f_1 \in B_R(X_1), \tag{5.32}$$

and the Lipschitz constant $L(R)$ *is bounded by* $2c(1,1)\, b\, d\, R$, *where* b *comes from Proposition 5.2.2 and* d *is given by* (5.28).

2. F *is Fréchet-differentiable on every* $f \in X_1$ *and its Fréchet-derivative* L_f *at* f *depends linearly on* f *and is such that*

$$\|\mathrm{L}_f\, g\|_{X_1} \leq\ 2c(1,1)\, b\, d\, \|f\|_{X_1}\, \|g\|_{X_1}, \tag{5.33}$$

for all f *and* g *in* X_1.

Proof. (1) Let $f,\ f_1 \in B_R(X_1)$; then, because of inequalities (5.11) and (5.27),

$$\begin{aligned} \|\mathrm{F}(f) - \mathrm{F}(f_1)\|_{X_1} &\leq \|\,\Theta\,[\,(\mathrm{P}\, f - \mathrm{P}\, f_1)\,]\, f\,\|_{X_1} + \|\,\Theta\,[\,(\mathrm{P}\, f_1)\,]\,(f - f_1)\,\|_{X_1} \leq \\ &\leq b\,\Big\{\|\,\mathrm{P}\, f - \mathrm{P}\, f_1\,\|_{W^{1,\infty}(\mathbb{R}_x)}\, \|f\|_{X_1} + \|\,\mathrm{P}\, f_1\,\|_{W^{1,\infty}(\mathbb{R}_x)}\, \|f - f_1\|_{X_1}\Big\} \leq \\ &\leq b\, d\, c(1,1)\ \big\{\|f - f_1\|_{X_1}\, \|f\|_{X_1} + \|f_1\|_{X_1}\, \|f - f_1\|_{X_1}\big\}. \end{aligned}$$

Thus, $\|\mathrm{F}(f) - \mathrm{F}(f_1)\|_{X_1} \leq 2c(1,1)\, b\, d\, R\, \|f - f_1\|_{X_1}$ and (5.32) is proved.
(2) For all h, $f \in X_1$, we have

$$\mathrm{F}(f + h) - \mathrm{F}(f) = \Theta\,[\,\mathrm{P}\, h\,]\ f + \Theta\,[\,\mathrm{P}\, f\,]\ h + \Theta\,[\,\mathrm{P}\, h\,]\ h.$$

If we define the linear operator

$$\mathrm{L}_f\, h := \Theta\,[\mathrm{P}\, h]\ f + \Theta\,[\mathrm{P}\, f]\ h, \tag{5.34}$$

then, by using again estimates (5.11) and (5.27), we get

$$\|\mathrm{F}(f + h) - \mathrm{F}(f) - \mathrm{L}_f\, h\|_{X_1} = \|\Theta\,[\mathrm{P}\, h]\ h\|_{X_1} \leq b\, d\, c(1,1)\ \|h\|^2_{X_1} = o\big(\,\|h\|_{X_1}\big)$$

and $\|\mathrm{L}_f\, h\|_{X_1} \leq 2c(1,1)\, b\, d\ \|f\|_{X_1}\ \|h\|_{X_1}$, for $f,\ h \in X_1$, which proves the second part of the Lemma. □

5.5.2 The semi-linear problem

We consider now the semilinear evolution problem (5.31), associated to the original, affine-semilinear evolution problem (5.30). Let us recall the definition of *mild solution* of (5.31), [26].

Definition 5.5.1 *Let $p : [\,0,\infty) \to X_1$ be a regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ (see Definition 6.3.1), let $U \in \mathcal{C}(\,[\,0,\infty)\,;\,W^{1,\infty}(\mathrm{I\!R}_x)\,)$, and let $u_0 \in X_1$. A continuous solution $u : [0,\infty) \to X_1$ of the integral equation*

$$u(t) = e^{t\mathrm{T}_0}u_0 + \int_0^t e^{(t-s)\mathrm{T}_0}\left[\Theta\left[U(s)\right]\,u(s) + \mathrm{F}\left((u+p)(s)\right)\right]\,\mathrm{d}s + \\ + \int_0^t e^{(t-s)\mathrm{T}_0}\,Q_p(s)\,\mathrm{d}s \quad \forall\, t \geq 0, \quad (5.35)$$

is called mild solution *of* (5.31).

Our next goal will be proving existence and uniqueness of a mild solution of (5.31) in some maximal time interval $[0, t_{\max})$.

Lemma 5.5.2 *Under the assumptions of Definition 5.5.1, the map*

$$\begin{aligned} &\mathrm{G} :\; [0,\infty) \times X_1 \to X_1 \\ &\mathrm{G}(t,u) := \Theta\left[V_e(t)\right]\,u + \mathrm{F}\left(u + p(t)\right) + Q_p(t) \end{aligned} \quad (5.36)$$

is well-defined for all $t \geq 0$ and $u \in X_1$, is continuous in t and locally Lipschitz-continuous in u, uniformly for bounded t-intervals.

Proof. The continuity of the map G with respect to the variable t follows from the local Lipschitz property of the operator F (Lemma 5.5.1) and the assumptions on the data of the problem. Moreover, by eqs. (5.11) and (5.32),

$$\left\|\mathrm{G}(t,u) - \mathrm{G}(t,u_1)\right\|_{X_1} \leq b\left\|V_e(t)\right\|_{W^{1,\infty}}\left\|u - u_1\right\|_{X_1} + L(R)\left\|u - u_1\right\|_{X_1}$$

for all u, $u_1 \in B_R(X_1)$, which proves the Lipschitz continuity of G with respect to u. □

Proposition 5.5.1 (Local solution of the associated problem) *Under the assumptions of Lemma 5.5.2 there exists a unique mild solution of the associated problem* (5.31) *on a maximal time interval* $[0, t_{\max})$, *with* $0 < t_{\max} \leq \infty$.
Moreover, if $U \in \mathcal{C}^1([0,\infty); W^{1,\infty}(\mathrm{I\!R}))$, $u_0 \in \mathcal{D}(\mathrm{T}_0)$ and $p : [0,\infty) \to X_1$ is a strongly regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$, then u is the (unique) classical solution of the problem.

Proof. The first statement follows from Lemma 5.5.2 and from Ref. [26], Chp. 6, Thm. 1.4. It can be shown that a lower bound for $t_{\max}$ is given by

$$t_{\max} \geq \sup_{\tau > 0} \frac{r}{\eta_\tau + (2r + \eta_\tau)\left[L(2r + \eta_\tau) + b\delta_\tau\right]}, \quad (5.37)$$

where $r := \|u_0\|_{X_1}$, $\eta_\tau := \|p\|_{\mathcal{C}^1([0,\tau];X_1)}$, $\delta_\tau := \|U\|_{\mathcal{C}([0,\tau];W^{1,\infty}(\mathbb{R}))}$, b comes from (5.11) and $L(R)$ is the Lipschitz constant of F (see Lemma 5.5.1). If $U \in \mathcal{C}^1([0,\infty);W^{1,\infty}(\mathbb{R}))$ and p is a strongly regular representation, then the strong t-derivative

$$\mathrm{G}_t(t,u) = \Theta\,[U'(t)]\,u + \mathrm{L}_{u+p(t)}\,p'(t) + \left(\mathrm{T}_{\gamma(t)}p(t)\right)_t + \\ + \Theta\,[U'(t)]\,p(t) + \Theta\,[V_e(t)]\,p'(t) - p''(t)$$

and the Fréchet u-derivative

$$\mathrm{G}_u(t,u) = \Theta\,[V_e(t)]\,u + \mathrm{L}_{u+p(t)}\,u$$

exist and are continuous functions of both t and u (see Def. 6.3.1, Corollary 5.2.1 and Lemma 5.5.1). Thus, the second part of the Theorem follows from Ref. [26], Chp. 6, Thm. 1.5.[5]

□

In the next section we shall see that the solution $u(t)$ of the associated problem (5.31) (and, therefore, of the original problem (5.30), given by $w(t) = u(t) + p(t)$) is indeed global, i.e. $t_{\max} = \infty$.

We end this section by proving that the (mild or classical) solution $u(t)$ of (5.31) is real-valued at all times $t \in [0, t_{\max})$ if the initial datum u_0 and the the inflow datum $\gamma(t)$ are real-valued.[6] To this aim, let us set $\widetilde{X_1} := L^2(\Omega_x \times \mathbb{R}_v\,,(1+v^2)\mathrm{d}x\mathrm{d}v;\mathbb{R})$.

Corollary 5.5.1 *If $u_0 \in \widetilde{X_1}$ (resp. $u_0 \in \mathcal{D}(\mathrm{T}_0) \cap \widetilde{X_1}$), $p : [0,\infty) \to \widetilde{X_1}$ is a regular (resp. strongly regular) representation, then the mild (respectively, classical) solution $u(t)$ of (5.31) belongs to $\widetilde{X_1}$ for all $t \in [0, t_{\max})$.*

Proof. For all $0 \le t_1 < t_{\max}$, the mild solution of (5.31) is the fixed point of the map $\mathrm{M} : \mathcal{C}(\,[0,t_1]\,;X_1\,) \to \mathcal{C}(\,[0,t_1]\,;X_1)$ so defined

$$\mathrm{M}(u(t)) \;=\; e^{t\mathrm{T}_0}u_0 + \int_0^t e^{(t-s)\mathrm{T}_0}\,\mathrm{G}(s,u(s))\,\mathrm{d}s, \quad \forall t \in [0,t_1],$$

which is a strict contraction from the ball of radius $2\,\|u_0\|_{X_1}$ into itself (see the proof of Thm. 1.4 of Chp. 6 in Ref. [26]). Precisely, u is the unique limit of the sequence $\{v_n \mid n \in \mathbb{N}\} \subset \mathcal{C}(\,[0,t_1]\,;X_1\,)$, recursively defined by $v_0 := u_0\,,\ v_n := \mathrm{M}\,v_{n-1}$.

In the assumptions of the Corollary, since for real potentials ϕ the pseudo-differential operator $\Theta\,[\phi]$ maps real functions into real functions, then $\mathrm{G}(t,u_0) \in \widetilde{X_1}$ for all $t \in [0,t_1]$. Thus, the sequence $\{v_n \mid n \in \mathbb{N}\}$ is contained in the ball of radius $2\,\|u_0\|_{\widetilde{X_1}}$ of $\mathcal{C}(\,[0,t_1]\,;\widetilde{X_1}\,)$ and, consequently, u belongs to $\widetilde{X_1}$ for all $t \in [0,t_1]$. The same holds for the solution extended to $t_{\max}$. □

5.6 Global-in-time solution

In the previous section we have found sufficient conditions on the data of the problem and on the chosen representation $p(t)$ of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ which grant existence and uniqueness of a

[5]In Ref. [21] the interested reader can find a different proof of the differentiability of the mild solution, under the same hypothesis, as well as the proof of (5.37).

[6]Physically meaningful Wigner functions must be real-valued (see Ref. [6]).

local-in-time, classical solution $u(t)$ of the associated semilinear evolution problem (5.31). A classical solution $w(t)$ of the original evolution problem (5.30) can be immediately built from $u(t)$, as stated in the following proposition (the proof is analogous to that of Proposition 5.3.1).

Proposition 5.6.1 *Let $p : [0, +\infty) \to X_1$ be a strongly regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ (see Definition 6.3.1) and let $0 < T \leq +\infty$. Then, $u : [0, T) \to X_1$ is a classical solution of the associated problem* (5.31) *in $[0, T)$ if and only if $w(t) = u(t) + p(t)$ is a classical solution of the original problem* (5.30) *in $[0, T)$.*

Therefore, the following corollary holds.

Corollary 5.6.1 (Local, classical solution of (5.30)**)** *Let the function $t \to \gamma(t)$ be such that $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ has a strongly regular representation $p(t)$ for $t \in [0, +\infty)$. Let $w_0 \in \mathcal{D}(\mathrm{T}_{\gamma(0)})$ and let $U \in \mathcal{C}^1([0, \infty) ; W^{1,\infty}(\mathbb{R}_x))$. Then, the original problem* (5.30) *has a classical solution in a maximal time interval $[0, t_{\max})$, with $0 < t_{\max} \leq +\infty$.*

Proof. Let p be a strongly regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$. By Prop. 5.5.1 there exists a unique classical solution u of (5.31) in a maximal time interval $[0, t_{\max})$. According to Prop. 5.6.1, $w := u + p$ is a classical solution of (5.30) in the same time interval. $\square$

Note that, contrary to the linear case, the uniqueness result cannot be used directly to prove that the solution of (5.30) is independent of the choice of the representation p (see Remark 5.3.1). However, we are going to prove such p-independence result, more in general, for the *mild* solution of (5.30), which implies that the same result holds for the classical solution. Note in fact that the following implications hold:

$$\begin{array}{ccc}
\begin{array}{c} u = w - p \\ \text{mild solution of (5.31)} \end{array} & \Longleftrightarrow & \begin{array}{c} w = u + p \\ \text{mild solution of (5.30)} \end{array} \\
\Downarrow & & \Downarrow \\
\begin{array}{c} u = w - p \\ \text{classical solution of (5.31)} \end{array} & \Longleftrightarrow & \begin{array}{c} w = u + p \\ \text{classical solution of (5.30)} \end{array}
\end{array}$$

Definition 5.6.1 (Mild solution of the original problem) *Let p be a regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$, let $U \in \mathcal{C}([0, \infty) ; W^{1,\infty}(\mathbb{R}_x))$, let $u_0 \in X_1$ and let $0 < T \leq +\infty$. We define* mild solution *of* (5.30) *in $[0, T)$ a continuous function $w : [0, T) \to X_1$ which satisfies*

$$\begin{aligned} w(t) = p(t) + e^{t\mathrm{T}_0}(w_0 - p(0)) - \int_0^t e^{(t-s)\mathrm{T}_0} p'(s)\, \mathrm{d}s \, + \\ + \int_0^t e^{(t-s)\mathrm{T}_0} \left[\mathrm{T}_{\gamma(s)} w(s) + \Theta\left[U(s)\right] w(s) + \mathrm{F}(w(s)) \right] \mathrm{d}s\,, \end{aligned} \tag{5.38}$$

This definition has been given in such a way that $u := w - p$ is the mild solution of the associated problem (5.31), according to Definition 5.5.1. Note, however, that in order to define the mild solution of the original problem (5.30) we have introduced an arbitrary element, i.e. the representation p. Thus, the p-independence of the mild solution is not *a priori* evident. Nevertheless, the following proposition holds.

Proposition 5.6.2 *Let $p : [0,+\infty) \to X_1$ be a regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ and let $0 < T \leq +\infty$. Then, $u : [0,T) \to X_1$ is a mild solution of the associated problem* (5.31) *in* $[0,T)$ *if and only if $w(t) = u(t) + p(t)$ is a mild solution of the original problem* (5.30) *in* $[0,T)$.
Moreover, the mild solution of (5.30) *is unique and independent of the choice of a regular representation p.*

PROOF. For the sake of brevity, we prove only the uniqueness statement, since the equivalence statement can be easily checked. Let p_1 and p_2 be two regular representations of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ and let w_1 and w_2 be two corresponding solutions of (5.38). We want to show that $w_1(t) = w_2(t)$ for all $t \in [0,T)$. From (5.38) we get

$$\begin{aligned}\big(w_1 - w_2\big)(t) = \big(p_1 - p_2\big)(t) - e^{t\mathrm{T}_0}\big(p_1 - p_2\big)(0) + \\ + \int_0^t e^{(t-s)\mathrm{T}_0}\left\{\Theta\left[U(s)\right](w_1 - w_2)(s) + \mathrm{F}(w_1(s)) - \mathrm{F}(w_2(s))\right\}\mathrm{d}s + \\ + \int_0^t e^{(t-s)\mathrm{T}_0}\left\{\mathrm{T}_0(p_1 - p_2)(s) - \big(p_1' - p_2'\big)(s)\right\}\mathrm{d}s.\end{aligned}$$

Since under our regularity assumptions the identity

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{e^{(t-s)\mathrm{T}_0}\big(p_1 - p_2\big)(s)\right\} = e^{(t-s)\mathrm{T}_0}\left\{\big(p_1' - p_2'\big)(s) - \mathrm{T}_0\big(p_1 - p_2\big)(s)\right\}$$

holds for every $s \in [0,t]$ (see Ref. [5]), then

$$\begin{aligned}\big(w_1 - w_2\big)(t) = \int_0^t e^{(t-s)\mathrm{T}_0}\Big\{\Theta\left[U(s)\right](w_1 - w_2)(s) + \\ + \mathrm{F}(w_1(s)) - \mathrm{F}(w_2(s))\Big\}\mathrm{d}s.\end{aligned}$$

Note that, by using (5.11), we have

$$\begin{aligned}\big\|\big(w_1 - w_2\big)(t)\big\|_{X_1} \leq \int_0^t \Big\{b\,\|U(s)\|_{W^{1,\infty}(\mathbb{R})}\,\big\|\big(w_1 - w_2\big)(s)\big\|_{X_1} + \\ + \|\mathrm{F}(w_1(s)) - \mathrm{F}(w_2(s))\|_{X_1}\Big\}\mathrm{d}s,\end{aligned}$$

for all $t \in [0,T)$. Thus, if $R > 0$ is such that both $\|w_1(t)\|_{X_1} \leq R$ and $\|w_2(t)\|_{X_1} \leq R$ hold for all $t \in [0,T)$, since F is Lipschitz-continuous in the ball of radius R (see Lemma 5.5.1), we obtain

$$\big\|\big(w_1 - w_2\big)(t)\big\|_{X_1} \leq \int_0^t \Big\{b\,\|U(s)\|_{W^{1,\infty}(\mathbb{R})} + L(R)\Big\}\,\big\|\big(w_1 - w_2\big)(s)\big\|_{X_1}\,\mathrm{d}s,$$

for all $t \in [0,T)$. Thus, the p-independence statement follows by applying Gronwall's Lemma. If we now assume that w_1 and w_2 are two solutions of (5.38) (with the same p) and repeat the above proof taking $p_1 = p_2 = p$, we also obtain the proof of the uniqueness statement. □

Corollary 5.6.2 *In the assumptions of Corollary 5.6.1, the classical solution w of the original problem* (5.30) *in* $[0, t_{\max})$ *is unique.*

PROOF. It follows from Proposition 5.6.2 and from the fact that a classical solution is also a mild solution. □

Now we can prove the main result of this paper, i.e., existence and uniqueness of a global solution of the affine-semilinear evolution problem (5.30), which is the abstract version of the one-dimensional W-P system (WP) with b.c. and i.c. (bc).

Theorem 5.6.1 (Global solution of (5.30)**)** *Let $u_0 \in \mathcal{D}(\mathrm{T}_0)$ be real-valued. Let $U \in \mathcal{C}^1([0,\infty); W^{1,\infty}(\mathrm{I\!R}_x))$ and let $\gamma \in L^1_{\mathrm{loc}}([0,\infty); \mathcal{Y}_1^{\mathrm{in}})$ be such that $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ has a strongly regular representation $p(t)$ for $t \in [0,+\infty)$. Then there exists a unique global-in-time, classical and real-valued solution $w : [0,+\infty) \to X_1$ of the evolution problem* (5.30).

PROOF. From Corollary 5.6.1, Corollary 5.6.2 and Corollary 5.5.1, there exists a unique classical, real-valued solution u of (5.30) in some maximal time interval $[0, t_{\max})$, with $0 < t_{\max} \le +\infty$. In order to prove that $t_{\max} = +\infty$, we look for an *a priori* estimate for $\|w(t)\|_{X_1}$ for all times t.
If we multiply both left and right hand sides of Eq. (5.30a) by w we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} w^2(t) + w(t)\, v w_x(t) = w(t)\,\Theta\,[V_e(t) + \mathrm{P}\,w(t)]\,w(t).$$

Note that, for ϕ real, the integral over $v \in \mathrm{I\!R}_v$ of the function $w\Theta\,[\phi]\,w$ (as a function of v) vanishes, since an integral over $\mathrm{I\!R}_\xi$ of an odd function of ξ appears (see definition(4.2)). Thus, if we integrate the above equality with respect to $v \in \mathrm{I\!R}_v$ and $x \in (0,l)$, and integrate by parts the term containing w_x, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{X_0} &= -\int_{\mathrm{I\!R}_v} [v w^2(l,v,t) - v w^2(0,v,t)]\mathrm{d}v \\ &\le -\int_{-\infty}^{0} v w^2(l,v,t)\mathrm{d}v + \int_0^{\infty} v w^2(0,v,t)\mathrm{d}v.\end{aligned}$$

(the inequality follows by neglecting something non-positive). The last term of the previous inequality is the $\mathcal{Y}_0^{\mathrm{in}}$-norm of the boundary datum $\gamma(t)$ (see definition (5.13)). Then, from the assumptions on γ we have that

$$\|w(t)\|^2_{X_0} \le \int_0^t \|\gamma(\tau)\|^2_{\mathcal{Y}_0^{\mathrm{in}}}\,\mathrm{d}\tau =: C^2(t) \tag{5.39}$$

for all $t \in [0, t_{\max})$.
Now, since $\|w(t)\|_{X_1} = \|w(t)\|_{X_0} + \|z(t)\|_{X_0}$, with $z(x,v,t) := v\,w(x,v,t)$, we need an estimate for $\|z(t)\|_{X_0}$. The equation satisfied by $z(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) + v z_x(t) - \Theta\,[V_e(t) + \mathrm{P}\,w(t)]\,z = \Omega[U_x(t) + (\mathrm{P}\,w(t))_x] w(t), \tag{5.40}$$

where

$$\begin{aligned}(\Omega[\phi]w)\,(x,v) &:= \frac{i}{4\pi}\int_{\mathrm{I\!R}^2} \delta_+\phi(x,\xi) w(x,v')\, e^{i(v-v')\xi}\,\mathrm{d}\xi\,\mathrm{d}v' \\ \delta_+\phi(x,\xi) &:= \phi\Big(x + \frac{\xi}{2}\Big) + \phi\Big(x - \frac{\xi}{2}\Big)\end{aligned}$$

is a bounded operator from X_0 into itself, whose operatorial norm satisfies

$$\|\Omega[\phi]\| \le \|\phi\|_\infty \tag{5.41}$$

(see Ref. [4]). If we multiply by z the right and the left hand sides of Eq. (5.40) and integrate in both variables x and v, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|z(t)\right\|_{X_0}^2 = -\int_{\mathrm{I\!R}}[v^3 w^2(l,v,t) - v^3 w^2(0,v,t)]\mathrm{d}v + \\ + \int_0^l \int_{\mathrm{I\!R}} z(x,v,t)\,\Omega[V_x(t) + (\mathrm{P}\,w(t))_x]w(x,v,t)\,\mathrm{d}x\mathrm{d}v. \quad (5.42)$$

By using (5.41) and (5.27) we get

$$\int_0^l \int_{\mathrm{I\!R}} z(x,v,t)\,\Omega[V_x(t) + (\mathrm{P}\,w(t))_x]w(x,v,t)\,\mathrm{d}x\mathrm{d}v \le \\ \le \left(\left\|V_e(t)\right\|_{W^{1,\infty}} + \alpha\left\|w(t)\right\|_{X_1}\right)\left\|w(t)\right\|_{X_0}\left\|z(t)\right\|_{X_0}, \quad (5.43)$$

for all $t \in [0, t_{\max})$, where $\alpha := dc(1,1)$. Moreover,

$$-\int_{\mathrm{I\!R}}[v^3 w^2(l,v,t) - v^3 w^2(0,v,t)]\mathrm{d}v \le \\ \le -\int_{-\infty}^0 v^3 w^2(l,v,t)\mathrm{d}v + \int_0^\infty v^3 w^2(0,v,t)\mathrm{d}v \le \left\|\gamma(t)\right\|_{\mathcal{Y}_1^{\mathrm{in}}}, \quad (5.44)$$

for all $t \in [0, t_{\max})$. Therefore, by using (5.43), (5.44) and (5.39), from Eq. (5.42) we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|z(t)\right\|_{X_0} &\le \left\|\gamma(t)\right\|_{\mathcal{Y}_1^{\mathrm{in}}} + C(t)\Big[\left\|V_e(t)\right\|_{W^{1,\infty}} + \alpha\left\|w(t)\right\|_{X_1}\Big]\left\|z(t)\right\|_{X_0} \\ &\le \left\|\gamma(t)\right\|_{\mathcal{Y}_1^{\mathrm{in}}} + C(t)\Big[\left\|V_e(t)\right\|_{W^{1,\infty}} + \alpha\left(C(t) + \left\|z(t)\right\|_{X_0}\right)\Big]\left\|z(t)\right\|_{X_0} \\ &= \left\|\gamma(t)\right\|_{\mathcal{Y}_1^{\mathrm{in}}} + C(t)\left(\left\|V_e(t)\right\|_{W^{1,\infty}} + \alpha C(t)\right)\left\|z(t)\right\|_{X_0} + \alpha\left\|z(t)\right\|_{X_0}^2, \end{aligned} \quad (5.45)$$

which is a differential inequality of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}f^2(t) \le a(t) + b(t)f(t) + c(t)f^2(t), \quad (5.46)$$

with a, b and c non-negative and $f(t) := \left\|z(t)\right\|_{X_0}$. The function $f(t)$ satisfying (5.46) cannot go to $+\infty$ in a finite time; in fact, if $\lim_{t\to t_0} f(t) = +\infty$, with $0 < t_0 < +\infty$, then a $0 < \delta < t_0$ would exist such that $f(t) \ge 1$ for all $t_0 - \delta < t < t_0$. In such interval we could write

$$\frac{\mathrm{d}}{\mathrm{d}t}f^2(t) \le a(t) + d(t)f^2(t),$$

with $d(t) := b(t) + c(t)$; but, therefore, by applying Gronwall's Lemma to the function f^2 we would get $\lim_{t\to t_0} f(t) < +\infty$, which contradicts $\lim_{t\to t_0} f(t) = +\infty$.
From this result and from (5.39) we can conclude that

$$\lim_{t\to t_{\max}} \left\|w(t)\right\|_{X_1} < +\infty, \quad \text{if} \quad t_{\max} < +\infty.$$

Thus, from Theorem 1.4 in Chp. 6 of Ref. [26], we have that $t_{\max} = +\infty$. □

5.7 Appendix

An example of strongly regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$

In this Appendix we explicitly construct a regular representation of the $\mathcal{D}(\mathrm{T}_0)$-affine domains $\{\mathcal{D}(\mathrm{T}_{\gamma(t)}) \mid t \geq 0\}$ (see Def. 6.3.1 and Remark 5.3.1) for a suitable class of inflow data.

Since Ω_x is a convex set, the following functions are well defined for all $(x, v) \in \overline{\Omega}_x \times \mathbb{R}^d_v$ with values in $[0, +\infty)$:

$$t_+(x, v) := \sup\{ t \geq 0 \,|x + tv \in \Omega_x \}, \tag{5.1a}$$
$$t_-(x, v) := \sup\{ t \geq 0 \,|x - tv \in \Omega_x \}. \tag{5.1b}$$

Note that $t_\pm(x, v)$ is the time it takes, starting from $x \in \Omega_x$ and traveling with a constant velocity $\pm v$, to reach the point $y = x \pm t_\pm(x, v)v$ on $\partial\Omega_x$. Recalling (5.14), it is easy to verify that

$$t_-(x, v) = 0 \Longleftrightarrow (x, v) \in \Phi^{\mathrm{in}}\,, \tag{5.2a}$$
$$t_+(x, v) = 0 \Longleftrightarrow (x, v) \in \Phi^{\mathrm{out}} := \{(s, v) \in \partial\Omega_x \times \mathbb{R}^n_v \,|\, v \cdot n(s) < 0\} \tag{5.2b}$$

and, for all $(x, v) \in \Omega_x \times \mathbb{R}^d_v,\ t \in [-t_-(x, v), t_+(x, v)]$,

$$t_-(x + tv, v) = t_-(x, v) + t\,, \quad t_+(x + tv, v) = t_+(x, v) - t\,. \tag{5.3}$$

For $c > 0$ let us introduce the cut-off inflow subspaces $\mathcal{Y}^{\mathrm{in}}_{k,c} \subset \mathcal{Y}^{\mathrm{in}}_k$ defined as follows:

$$\mathcal{Y}^{\mathrm{in}}_{k,c} := \{\gamma \in \mathcal{Y}^{\mathrm{in}}_k \mid \gamma(s, v) = 0 \text{ for a.e. } v \text{ with } |v| < c\}.$$

Lemma 5.7.1 *Setting, for all $\gamma \in \mathcal{Y}^{\mathrm{in}}_{k,c}$,*

$$(\mathrm{B}\,\gamma)\,(x, v) := \gamma(x - t_-(x, v)v, v), \quad \forall\,(x, v) \in \Omega_x \times \mathbb{R}^d_v, \tag{5.4}$$

defines a bounded operator $\mathrm{B} : \mathcal{Y}^{\mathrm{in}}_{k,c} \to X_k$.

Proof. For any given $\gamma \in \mathcal{Y}^{\mathrm{in}}_{k,c}$, let us evaluate $\|\mathrm{B}\,\gamma\|_{X_k}$. To this aim, observe first that, for all (x, v) such that $|v| \geq c$,

$$0 \leq t_+(x, v),\, t_-(x, v) \leq \mathrm{diam}(\Omega_x)/c =: t_c. \tag{5.5}$$

Moreover, for every $(y, v) \in \Phi^{\mathrm{in}}$, from (5.2) and (5.3) we easily obtain

$$(\mathrm{B}\,\gamma)\,(x, v) = \gamma(y, v), \quad \forall\, x \in \Omega_x \quad \text{s.t.} \quad x = y + hv,\ h > 0. \tag{5.6}$$

Now, recalling (5.3), we can write

$$\begin{aligned}\|\mathrm{B}\,\gamma\|^2_{X_k} = \|\mathrm{B}\,\gamma\|^2_{X_0} + \sum_{\substack{|\alpha|=k\\ \alpha\neq(1,0,\ldots,0)}} \binom{k}{\alpha} \left\|v^{\alpha'}\,\mathrm{B}\,\gamma\right\|^2_{X_0} \\ = \int_{\Phi^{\mathrm{in}}} v\cdot n(s)\,\mathrm{d}s\,\mathrm{d}v \int_0^{t_+(s,v)} \left|(\mathrm{B}\,\gamma)\,(s + hv, v)\right|^2 \mathrm{d}h \\ + \sum_{\substack{|\alpha|=k\\ \alpha\neq(1,0,\ldots,0)}} \binom{k}{\alpha} \int_{\Phi^{\mathrm{in}}} v\cdot n(s)\,\mathrm{d}s\,\mathrm{d}v \int_0^{t_+(s,v)} \left|v^{\alpha'}\,(\mathrm{B}\,\gamma)\,(s + hv, v)\right|^2 \mathrm{d}h,\end{aligned} \tag{5.7}$$

where we used the identity

$$\int_{\Omega_x \times \mathbb{R}^d_v} f(x,v)\,\mathrm{d}x\mathrm{d}v = \int_{\Phi^{\mathrm{in}}} v\cdot n(s)\,\mathrm{d}s\,\mathrm{d}v \int_0^{t_+(s,v)} f(s+tv,v)\,\mathrm{d}t,$$

which holds for all $f \in L^1(\Omega_x \times \mathbb{R}^d_v, \mathrm{d}x\mathrm{d}v; \mathbb{C})$ (see Lemma 3.2 and Lemma 3.3 of Ref. [30]). Thus, from (5.5) and (5.6) and (5.7) we have

$$\|\mathrm{B}\,\gamma\|^2_{X_k} \le \int_{\Phi^{\mathrm{in}}} v\cdot n(s)(1+|v|^2)^k \mathrm{d}s\,\mathrm{d}v \int_0^{t_+(s,v)} |\gamma(s,v)|^2\,\mathrm{d}h \le t_c\,\|\gamma\|^2_{\mathcal{Y}^{\mathrm{in}}_k}\,, \tag{5.8}$$

which proves the Lemma. □

Proposition 5.7.1 *Assume that the time-dependent inflow datum* $\gamma : [\,0,\infty) \to \mathcal{Y}^{\mathrm{in}}_k$ *satisfies*

(i) $\gamma \in \mathcal{C}^2([\,0,\infty); \mathcal{Y}^{\mathrm{in}}_k)$*;*

(ii) $\gamma(t) \in \mathcal{Y}^{\mathrm{in}}_{k,c}$*, for all* $t \ge 0$*.*

Then $p(t) := \mathrm{B}\,\gamma(t)$ *(explicitly,* $p(x,v,t) = \gamma(x - t_-(x,v,)v, v, t)$*) is a strongly regular representation of* $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ *such that* $\mathrm{T}_{\gamma(t)}p(t) = 0$ *for all* $t \ge 0$*.*

Proof. From Eq. (5.8) and assumption (ii), we have that $p(t) \in X_k$ for all $t \ge 0$. Moreover, from Eq. (5.6) we have that $p(t)$ is constant along the characteristic lines, i.e.,

$$v\cdot \nabla_x\,(\mathrm{B}\,\gamma)\,(x,v) = 0, \quad \forall\,(x,v) \in \Omega_x \times \mathbb{R}^d_v, \tag{5.9}$$

and thus $p(t) \in \mathcal{D}(\mathrm{T}_{\gamma(t)})$, with $\mathrm{T}_{\gamma(t)}p(t) = 0$. Finally, since B is bounded from $\mathcal{Y}^{\mathrm{in}}_{k,c} \subset \mathcal{Y}^{\mathrm{in}}_k$ to X_k (Lemma 5.7.1), from assumption (i) we can immediately deduce that p belongs to $\mathcal{C}^2([\,0,\infty); X_k)$. Thus, $p(t)$ is a strongly regular representation of $\mathcal{D}(\mathrm{T}_{\gamma(t)})$, according to Def. 6.3.1. □

Chapter 6

The three-dimensional Wigner-Poisson system with inflow boundary conditions

6.1 Introduction

This chapter is devoted the three-dimensional version of problem (WP), namely

$$\left[\frac{\partial}{\partial t} + v\cdot \nabla_x - \Theta\left[V_e(t) + V(t)\right]\right] w(x,v,t) = 0\,, \quad (x,v) \in \Omega_x \times \mathrm{I\!R}^3_v\,,\ t \geq 0, \tag{6.1a}$$

$$\Delta_x V(x,t) \ = n[w](x,t) = \int_{\mathrm{I\!R}^n_v} w(x,v,t)\,\mathrm{d}v\,, \quad x \in \Omega_x\,,\ t \geq 0\,, \tag{6.1b}$$

with the following boundary conditions (b.c.)

$$w(s,v,t) = \gamma(s,v,t)\,, \quad (s,v) \in \partial\Omega_x \times \mathrm{I\!R}^3_v\,,\ v\cdot n(s) > 0\,,\ t \geq 0, \tag{6.2a}$$

$$V(x,t) = 0, \quad x \in \partial\Omega_x\,,\ t \geq 0, \tag{6.2b}$$

and initial condition (i.c.)

$$w(x,v,0) = w_0(x,v), \quad (x,v) \in \Omega_x \times \mathrm{I\!R}^3_v\,. \tag{6.2c}$$

Due to the definition of the pseudo-differential operator, the time-dependent function $V_e(t)$, which is a datum of our problem is defined in the whole $\mathrm{I\!R}^3_x$. Similarly, we will define an appropriate extension of the function V, $\widetilde{V}$ such that $\widetilde{V} \equiv V$ on Ω_x, $\widetilde{V} \equiv 0$ outside the set Σ, where Σ is any open, bounded subset of $\mathbb{R}^3_x$, $\overline{\Omega_x} \subset \Sigma$. Accordingly, equations (6.1a), (6.1b) are nonlinearly coupled in the unknown function w, through the "potential" $\widetilde{V}$.
Relatively to that problem, we have attained in *Manzini C., On the three-dimensional Wigner-Poisson problem with inflow boundary conditions, accepted by Journ. Math. Anal. Appl. (2004)*, a local-in-time, well-posedness result.
Observe that the linearized problem has already been solved in the d-dimensional case in Ref. [22] (cf. Chapter 2). Thus, the only difficulty consists in the definition of the pseudo-differential operator containing the self-consistent potential at dimension three.
Here follows an outline of the paper: in Section 2 we recall the functional setting and the preliminary results relative to the density function n, the pseudo-differential operator $\Theta\,[V_e]$

and the streaming operator $v\cdot\nabla_x$. Section 2.1 is devoted to the study of the (weak formulation of the) three-dimensional Poisson problem with homogeneous Dirichlet b.c. . In order to define the non-linear term in the present three-dimensional case, we will state an *ad hoc* result (Prop.6.2.3), that relies on the combination of the regularity of the state functions (in Fourier transform) with an analogous assumption on the potential. Then, the three-dimensional W-P problem will have the same structure of the one-dimensional one and the local-in-time solution can be obtained again by a fixed point argument.
However, we cannot recover *a priori* estimates as in the one-dimensional case: this fact depends precisely on the modified estimate we use in order to deal with the self-consistent potential in the three-dimensional case.

6.2 The functional setting

Let us introduce the Hilbert space X of the $\mathbb{C}$-valued functions, defined on $\Omega_x \times \mathbb{R}^d_v$, with square summable modulus with respect to the Lebesgue measure in $\Omega_x \times \mathbb{R}^d_v$ with weight $(1+|v|^2)^2$; in symbols:

$$X := L^2(\Omega_x \times \mathbb{R}^d_v, (1+|v|^2)^2 \mathrm{d}x\,\mathrm{d}v\,; \mathbb{C}),$$

with the scalar product

$$< u, w >_X := \left(\int_{\mathbb{R}^3_v} \int_{\Omega_x} u(x,v)\overline{w(x,v)}\,(1+|v|^2)^2\,\mathrm{d}x\,\mathrm{d}v \right)^{1/2}.$$

In our calculations we shall use the following equivalent norm

$$\|u\|^2_{\widetilde{X}} := \|u\|_2^2 + \sum_{i=1}^{3} \|v_i^2 u\|_2^2. \tag{6.3}$$

Let us observe that

$$u \in X \;\Leftrightarrow\; (x,\eta) \to (\mathcal{F}_v u)(x,\eta) \in L^2(\Omega_x, W^{2,2}(\mathbb{R}^3_\eta)), \tag{6.4}$$

where we indicate with $(\mathcal{F}_v u)$ the Fourier transform of the function u with respect to the second group of variables v.
The following proposition motivates our choice of the space X for the analysis: it is the bounded spatial domain version of Lemma 4.1.1

Lemma 6.2.1 *Let $u \in X$ and $n(x) := \int_{\mathbb{R}^3_v} u(x,v)\,\mathrm{d}v$, for all $x \in \Omega_x$. Then*

$$\|n\|_{L^2(\Omega_x,\mathrm{d}x)} \le C\,\|u\|_X, \tag{6.5}$$

with $C := \pi$.

Remark 6.2.1 The choice of the space X as the state space for our analysis is not optimal, in the sense that we could obtain an estimate analogous to eq. (6.5) even under decreased regularity assumption on the function $(\mathcal{F}_v u)$. Precisely, we could assume $u \in X_k := L^2(\Omega_x \times \mathbb{R}^d_v, (1+|v|^2)^k \mathrm{d}x\,\mathrm{d}v\,; \mathbb{C})$, with $3/2 < k < 2$ (cf. Prop.2.1 in Ref. [22]). However, even in the space X_k, we would obtain a *local-in-time* wellposedness result for the W-P problem; on the contrary, the calculations on Prop. 6.2.3 would become more complicated since they would involve derivatives of fractional order.

6.2.1 The Poisson problem

The Poisson problem for the self-consistent potential V is

$$\Delta_x V(x) = n(x) = \int_{\mathbb{R}^3_v} w(x,v)\,\mathrm{d}v \qquad x \in \Omega_x \tag{6.6a}$$

$$V(x) = 0 \qquad x \in \partial\Omega_x; \tag{6.6b}$$

in this subsection we can neglect the time variable, since it plays just the role of a parameter. According to Lemma 6.2.1, for all $w \in X$, the function n in the right hand side of eq. (6.6a) is well-defined and belongs to $L^2(\Omega_x)$. Then we can introduce a weak version of the problem (6.6)

$$\sum_{i=1}^{3} \int_{\Omega_x} \partial_{x_i} V\, \partial_{x_i}\psi\, \mathrm{d}x \;=\; \int_{\Omega_x} n\psi\, \mathrm{d}x \quad \forall \psi \in W_0^{1,2}(\Omega_x), \tag{6.7}$$

where the derivatives are in a weak sense. The most relevant facts about its solution V are collected in the following proposition (cf. Riesz Thm. and Hilbert Regularity Thm., e.g., Thm.6.3.4 in Ref. [13]).

Proposition 6.2.1 *For all $n \in L^2(\Omega_x)$, there exists a unique $V \in W_0^{1,2}(\Omega_x)$, which satisfies eq. (6.7) and $\|V\|_{W^{1,2}(\Omega_x)} \le d\,\|n\|_{L^2(\Omega_x)}$, with d depending only on diam(Ω_x). Moreover, if $\partial\Omega_x \in \mathcal{C}^2$, then $V \in W^{2,2}(\Omega_x)$ and*

$$\|V\|_{W^{2,2}(\Omega_x)} \;\le\; \tilde{d}\,\|n\|_{L^2(\Omega_x)}, \tag{6.8}$$

with $\tilde{d}$ depending only on Ω_x, and $\Delta_x V(x) = n(x)$ a.e. $x \in \Omega_x$.

By the definition of the pseudo-differential operator (cf. eq.(4.2)), the self-consistent potential has to be appropriately extended outside Ω_x. However, we can state the following result:

Corollary 6.2.1 *Let $w \in X$ and V be the solution of problem (6.7). Let Σ be any open and bounded subset of $\mathbb{R}^3_x$, such that $\overline{\Omega_x} \subset \Sigma$. There exists a function $P[w] \in W^{2,2}(\mathbb{R}^3_x)$, such that $P[w](x) = V(x)$ a.e.$x \in \Omega_x$ and $P[w](x) = 0$ for all $x \in \mathbb{R}^3_x \setminus \Sigma$. Moreover,*

$$\|P[w]\|_{W^{2,2}(\mathbb{R}^3_x)} \;\le\; D\,\|V\|_{W^{2,2}(\Omega_x)}.$$

Proof. By Lemma 6.2.1 the function $n \in L^2(\Omega_x)$, then, by Prop. 6.2.1, the solution $V \in W_0^{1,2}(\Omega_x) \cap W^{2,2}(\Omega_x)$, and we get the result by applying an extension theorem (cf., e.g., Thm.5.4.1 in Ref. [13]). □

By the previous discussion and Lemma 6.2.1, one can define a map, which we call again P for simplicity, whose properties are collected in next corollary.

Corollary 6.2.2 (The self-consistent potential) *The map* P *defined by*

$$\begin{aligned} \mathrm{P}: \quad X &\to W^{2,2}(\mathbb{R}^3_x) \\ w &\mapsto \mathrm{P}\,w := P[w], \end{aligned}$$

where the function $P[w]$ is the extension (in the sense of Corollary 6.2.1) of the solution of problem (7.25), is linear and bounded, and the following estimate holds

$$\|\,\mathrm{P}\,w\|_{W^{2,2}(\mathbb{R}^3_x)} \le C\,D\|w\|_X, \quad \forall\, w \in X. \tag{6.9}$$

The constants C, D in estimate(6.9) are the same in Lemma 6.2.1 and Corollary 6.2.1.

6.2.2 The pseudo-differential operator

The discussion of the previous sections enables us to give a less formal definition of the pseudo-differential operator; we will show indeed that this operator is well-defined from the space X to itself under appropriate regularity assumptions on the potentials, and we will also point out the role it plays on the Wigner-Poisson problem.
For what the given external potential V_e is concerned, if we call again V_e the extension whose values are zero outside the bounded spatial domain Ω_x, then we can state the following result.

Proposition 6.2.2 *If $V_e \in W^{2,\infty}(\mathbb{R}^3_x)$, then the map $u \mapsto \Theta\,[V_e]\,u$ is linear and bounded from X to itself and there exists $B > 0$ such that*

$$\|\Theta\,[V_e]\,u\|_X \leq B\,\|V_e\|_{W^{2,\infty}(\mathbb{R}^3_x)}\|u\|_X\,, \quad \forall\, u \in X. \tag{6.10}$$

Proof. cf. proof of Prop.2.3 in Ref. [22].

□

Remark 6.2.2 Let $V_e : [0, \infty) \to W^{2,\infty}(\mathbb{R}^3_x)$ represent a time-dependent potential, then the family of operators $\{\Theta\,[V_e]\,(t) := \Theta\,[V_e(t)]\,,\ t \geq 0\}$ constitutes a time-dependent bounded perturbation, by estimate (6.10).

The previous result is the three-dimensional version of Prop.2.3 in Ref. [22] concerning the d-dimensional Wigner-Poisson problem; in particular, in the one-dimensional case, it is sufficient to assume $V \in W^{1,\infty}(\mathbb{R}_x)$ to get the result. Since in the one-dimensional case the self-consistent potential belongs to $W^{1,\infty}$, the pseudo-differential operator containing it is bounded, by Prop.2.3 in Ref. [22].
On the contrary, on the present three-dimensional case, for all $w \in X$, the self-consistent potential $\mathrm{P}\,w$ belongs to $W^{2,2}(\mathbb{R}^3_x)$ (cf. Corollary 6.2.2), thus it does not satisfy the assumption of Prop. 6.2.2. However, the same conclusion as in Prop. 6.2.2 holds also under weaker hypotheses, as we shall prove in next proposition by exploiting Sobolev Embedding Theorem (cf. ,e.g., Ref. [13]).

Proposition 6.2.3 *If $V \in W^{2,2}(\mathbb{R}^3_x)$, then the map $u \mapsto \Theta\,[V]\,u$ is bounded from X to itself and there exists $B > 0$ such that*

$$\|\Theta\,[V]\,u\|_X \leq B\,\|V\|_{W^{2,2}(\mathbb{R}^3_x)}\|u\|_X\,, \quad \forall\, u \in X. \tag{6.11}$$

Remark 6.2.3 Once we have proved the boundedness of the pseudo-differential operator $\Theta[\mathrm{P}\,w]$, for all $w \in X$ (cf. Corollary 6.2.2), the well-posedness result follows by using exactly the same arguments of the one-dimensional case; thus, Proposition 6.2.3 is the key point of the present work. The new idea in it consists in exploiting the "regularity" of the state functions to which the pseudo-differential operator is applied, instead of the essential boundedness of the potential and its derivatives (as in Refs. [22, 24]), in order to get an estimate of the same type of (6.10).

Proof. Let $u \in X$ and $V \in W^{2,2}(\mathbb{R}^3_x)$.
Since

$$(\mathcal{F}_v\,(\Theta\,[V]\,u)\,)\,(x, \eta) = i\,\delta\phi(x, \eta)\,(\mathcal{F}_v u)\,(x, \eta),$$

by using the equivalent norm (6.3) and then Plancherel Theorem, we can write

$$\begin{aligned}\|\Theta\,[V]\,u\|_{\widetilde{X}}^2 =&\|\Theta\,[V]\,u\|_2^2+\sum_{i=1}^{3}\|v_i^2\Theta\,[V]\,u\|_2^2=\\ &=(2\pi)^6\left(\|\delta V\,(\mathcal{F}_v u)\,\|_2^2+\sum_{i=1}^{3}\|\partial_{\eta_i}^2(\delta V\,(\mathcal{F}_v u))\|_2^2\right).\end{aligned}$$

Since

$$\partial_{\eta_i}^j\delta V(x,\eta)=\left\{\left(\frac{1}{2}\right)^j\partial_i^j V\Big(x+\frac{\eta}{2}\Big)-\left(-\frac{1}{2}\right)^j\partial_i^j V\Big(x-\frac{\eta}{2}\Big)\right\},$$

for all $i=1,2,3,\ j\in\mathbb{N}\cup\{0\}$, then, by the Leibniz rule and the Minkowski inequality

$$\begin{aligned}\|\partial_i^2(\delta V\,(\mathcal{F}_v u))\|_2\leq&\frac{1}{4}\|\delta(\partial_{\eta_i}^2 V)\,(\mathcal{F}_v u)\,\|_2+\|\delta V\partial_{\eta_i}^2((\mathcal{F}_v u))\|_2+\\ &+\left(\int_{\mathbb{R}^6}\Big|\partial_i V\Big(x+\frac{\eta}{2}\Big)+\partial_i V\Big(x-\frac{\eta}{2}\Big)\Big|^2|\partial_{\eta_i}\,(\mathcal{F}_v u)\,|^2dxd\eta\right)^{1/2}.\end{aligned}$$

For all functions $f\in W^{2,2}(\mathbb{R}^3)$, there exist two constants $C_1,\ C_2>0$, independent of f, such that

$$\|f\|_\infty\leq C_1\|f\|_{W^{2,2}} \tag{6.12}$$

$$\|\partial_i f\|_4\leq C_2\|\partial_i f\|_{W^{1,2}}, \tag{6.13}$$

by Sobolev Embedding Theorems. Then, if we apply estimate (6.12) to the function V, we get

$$\|\delta V\,(\mathcal{F}_v u)\,\|_2\leq 2\|V\|_\infty\|u\|_2\leq 2\,C_1\|V\|_{W^{2,2}}\|u\|_2 \tag{6.14a}$$

$$\|\delta V\partial_{\eta_i}^2((\mathcal{F}_v u))\|_2\leq 2\|V\|_\infty\|\partial_{\eta_i}^2((\mathcal{F}_v u))\|_2\leq 2\,C_1\|V\|_{W^{2,2}}\|v_i^2 u\|_2 \tag{6.14b}$$

while, by using estimate (6.12) for the function $(\mathcal{F}_v u)\,(x,.\,)\in W^{2,2}(\mathbb{R}^3_\eta)$, we obtain

$$\begin{aligned}\|\delta(\partial_{\eta_i}^2 V)\,(\mathcal{F}_v u)\,\|_2^2\leq&\int_{\Omega_x}dx\|\,(\mathcal{F}_v u)\,(x,.\,)\|_\infty^2\int_{\mathbb{R}^3}|\delta(\partial_i^2 V)(x,\eta)|^2d\eta\\ &\leq 2^4C_1^2\|\partial_i^2 V\|_2^2\int_{\Omega_x}dx\ \|\,(\mathcal{F}_v u)\,(x,.\,)\|_{W^{2,2}}^2=2^4C_1^2\|V\|_{W^{2,2}}^2\|u\|_X^2.\end{aligned} \tag{6.15}$$

Moreover, we can estimate the remaining addendum as follows

$$\begin{aligned}&\left(\int_{\mathbb{R}^6}\Big|\partial_i V\Big(x+\frac{\eta}{2}\Big)+\partial_i V\Big(x-\frac{\eta}{2}\Big)\Big|^2|\partial_{\eta_i}\,(\mathcal{F}_v u)\,|^2dxd\eta\right)^{1/2}\leq\\ &\leq 2^4\|\partial_i V\|_4\left(\int_{\Omega_x}\|\partial_{\eta_i}\,(\mathcal{F}_v u)\,(x,.\,)\|_4^2dx\right)^{1/2}\leq 2^4C_2\|\partial_i V\|_{W^{1,2}}\left(\int_{\Omega_x}\|\partial_{\eta_i}\,(\mathcal{F}_v u)\,(x,.\,)\|_{W^{1,2}}^2\right)^{1/2}\\ &=2^4C_2\|V\|_{W^{2,2}}\|u\|_X,\end{aligned} \tag{6.16}$$

where the first inequality is obtained by applying Minkowski and Hölder inequalities in the variable η, and the second one by using estimate (6.13) both for the functions V and $(\mathcal{F}_v u)\,(x,.\,)$.
Hence, by collecting pieces, we get the result.

□

Remark 6.2.4 Observe that, on the right hand side of estimates (6.15) and (6.16), unlike in estimate (6.14), there are the *complete* norms of *both* the functions u and V in the spaces X and $W^{2,2}$ respectively. Thus, when they are used for $\|\Theta[\mathrm{P}\,w]w\|_X$, according to estimate (6.9) for the self-consistent potential $P[w]$, on the right hand side there will be the term $\|w\|_X^2$. This technical point is the reason why we cannot recover a priori estimates for the solution w of the three-dimensional W-P system analogously to the one-dimensional case (cf. Thm. 5.6.1, estimate (5.43), in Ref. [22]).

Finally, by Corollary 6.2.2 and Proposition6.2.3, the operator F is well-defined on X, where

$$\begin{aligned} \mathrm{F}: \quad & X \;\rightarrow\; X \\ & w \;\mapsto\; \Theta[\mathrm{P}\,w]w; \end{aligned} \tag{6.17}$$

by estimates (6.11) and (6.9), the operator depends quadratically on the function $w \in X$. Actually, it can be proved analogously to Lemma 5.1 in Ref. [22] the following result:

Corollary 6.2.3 *The operator* F *defined by* (5.29) *satisfies the following properties;*

1. *for all $R > 0$,* F *is Lipschitz-continuous on the ball of radius R of the space X and the Lipschitz constant is $L(R) = 2BCDR$,*
2. F *is Frechét-differentiable on every $w \in X$ and its Frechét-derivative L_w at w depends linearly on w and is such that*

$$\|\,\mathrm{L}_w\,u\|_X \leq 2BCD\|w\|_X\,\|u\|_X,$$

for all w and u in X, where the constants B, C, D are the same as in estimates (6.11) *and* (6.9).

6.3 Local-in-time well-posedness

In this section we shall present the local-in-time existence and uniqueness result relative to the three dimensional Wigner-Poisson problem (6.1). Actually, the preliminary results and the definitions of Section 6.2 allow us to exploit the ideas and the tecniques used in the one dimensional case, then we shall simply outline the procedure, which is explained in detail in Ref. [22](as well as in Ref. [21]), and state the three-dimensional version of the propositions in Sections 5, 6 of that work.

6.3.1 The affine semi-linear problem

Let us reformulate the three dimensional Wigner-Poisson problem (6.1) with conditions (6.2a),(6.2b), in terms of the operators introduced in Section 6.2:

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) = \mathrm{T}_{\gamma(t)}w(t) + \Theta[V_e](t)w(t) + \mathrm{F}(w(t)), \quad t \geq 0 \tag{6.18a}$$

$$w(t=0) = w_0 \in X, \tag{6.18b}$$

with the time-dependent boundary datum $\gamma : [0, \infty) \to \mathcal{Y}^{\mathrm{in}}$, the affine streaming operator $\mathrm{T}_{\gamma(t)}$ defined by eqs. (5.15), the pseudo-differential operators $\Theta[V_e](t) := \Theta\,[V_e(t)]$ characterised for all $t \geq 0$ by Prop. 6.2.2, and the nonlinear operator F (cf. Corollary 6.2.3).

Our aim is to prove existence and uniqueness of a function $w \in \mathcal{C}^1([0,T);X)$ which satisfies eqs. (6.18) and such that $w(t) \in \mathcal{D}(\mathrm{T}_{\gamma(t)})$ for all $t \in [0,T)$, with $T \leq \infty$. Then, if we choose a representation $p(t)$ for the family of affine domains $\{\, \mathcal{D}(\mathrm{T}_{\gamma(t)}) \,|\, t \geq 0 \,\}$, by the decomposition $\mathcal{D}(\mathrm{T}_{\gamma(t)}) = p(t) + \mathcal{D}(\mathrm{T}_0)$, $\forall\, t \geq 0$, we can introduce the following *associated* problem

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \mathrm{T}_0 u(t) + \Theta\,[V_e]\,(t) u(t) + \mathrm{F}\,((u+p)(t)) + Q_p(t), \quad t \geq 0 \tag{6.19a}$$

$$u(t=0) = w_0 - p(0) =: u_0, \tag{6.19b}$$

with the function

$$Q_p(t) := \mathrm{T}_{\gamma(t)} p(t) + \Theta\,[V_e(t)]\, p(t) - p'(t), \quad \forall t \geq 0$$

and with the unknown function $u \in \mathcal{C}^1([0,T);X)$, such that $u(t) \in \mathcal{D}(\mathrm{T}_0)$, for all $t \in [0,T)$. The associated problem contains the linear streaming operator with null inflow T_0, the time-dependent bounded perturbation $\Theta\,[V_e]\,(t)$ (cf. Remark 6.2.2), the non-linear operator F and the source term Q_p. First, we will handle this problem in order to recover the solution of problem (6.18).

6.3.2 Local-in-time solution of the associated problem

After the following definitions, problem (6.19) will prove to be a Lipschitz perturbation of a linear evolution problem.

Definition 6.3.1 *Let* $p : [0,+\infty) \to X$ *be a representation of the* T_0*-affine domains* $\{\mathcal{D}(\mathrm{T}_{\gamma(t)}) \,|\, t \geq 0\}$. *We say that* p *is* regular *if* $p \in \mathcal{C}^1([0,+\infty)\,;X)$ *and the function* $t \mapsto \mathrm{T}_{\gamma(t)} p(t)$ *is* $\mathcal{C}([0,+\infty)\,;X)$. *We say that* p *is* strongly regular *if* $p \in \mathcal{C}^2([0,+\infty)\,;X)$ *and* $t \mapsto \mathrm{T}_{\gamma(t)} p(t)$ *is* $\mathcal{C}^1([0,+\infty)\,;X)$.

Remark 6.3.1 By the previous definition follows that, if p is a regular representation and $V_e \in \mathcal{C}(\,[\,0,\infty)\,;\, W^{2,\infty}(\mathbb{R}^3_x)\,)$, then $Q_p \in \mathcal{C}([0,+\infty)\,;X)$. Moreover, if p is a strongly regular representation and $V_e \in \mathcal{C}^1(\,[\,0,\infty)\,;\, W^{2,\infty}(\mathbb{R}^3_x)\,)$, then $Q_p \in \mathcal{C}^1([0,+\infty)\,;X)$.

Let us make precise what we mean by *mild* solution of the associated problem, before stating the existence result:

Definition 6.3.2 *Let* $p : [\,0,\infty) \to X$ *be a regular representation of* $\mathcal{D}(\mathrm{T}_{\gamma(t)})$, *let* $V_e \in \mathcal{C}(\,[\,0,\infty)\,;\, W^{2,\infty}(\mathbb{R}^3_x)\,)$, *and let* $u_0 \in X$. *A continuous solution* $u : [0,\infty) \to X$ *of the integral equation*

$$u(t) = \mathrm{e}^{t\mathrm{T}_0} u_0 + \int_0^t \mathrm{e}^{(t-s)\mathrm{T}_0} \left[\Theta\,[V_e(s)]\; u(s) + \mathrm{F}\,\big((u+p)(s)\big)\right]\, \mathrm{d}s\, + \\ + \int_0^t \mathrm{e}^{(t-s)\mathrm{T}_0}\, Q_p(s)\, \mathrm{d}s \quad \forall\, t \geq 0,$$

is called mild solution *of problem* (6.19).

Remark 6.3.2 Under the assumptions of Def. 6.3.2, by Remark 6.3.1 and the preliminary results of the previous section, the map

$$\begin{aligned} &\mathrm{G}:\ [0,\infty)\times X \to X \\ &\mathrm{G}(t,u) := \Theta\,[V_e(t)]\ u + \mathrm{F}\left(u + p(t)\right) + Q_p(t) \end{aligned}$$

is well-defined for all $t \geq 0$ and $u \in X$, is continuous in t and locally Lipschitz continuous in u, uniformly for bounded t-intervals (cf. Lemma 5.3 in Ref. [22]).

Hence, the announced result.

Proposition 6.3.1 (Local solution of the associated problem) *Under the assumptions of Def. 6.3.2, there exists a unique mild solution of the associated problem* (6.19)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \mathrm{T}_0 u(t) + G(t, u(t)), \quad t \geq 0 \\ &u(t=0) = u_0, \end{aligned}$$

on a maximal time interval $[0, t_{\max})$*, with* $0 < t_{\max} \leq \infty$*.*
Moreover, if V_e *belongs to* $\mathcal{C}^1([0,\infty); W^{2,\infty}(\mathbb{R}^3))$*,* $p : [0,\infty) \to X$ *is a strongly regular representation and* u_0 *belongs to* $\mathcal{D}(\mathrm{T}_0)$*, then* u *is the unique solution of the problem* (6.19) *in the same time interval.*
In addition, if both u_0 *and* $p(t)$*, for all* $t \geq 0$*, are real-valued, the same will hold for* $u(t)$*, for all* $t \geq 0$*.*

Proof. Under the assumptions of the proposition, we can apply Thm.6.1.4 in Ref. [26] (cf. Remark 6.3.2) and get the first statement in the proposition. The second one can be proved by using Thm.6.1.5, thanks to the increased regularity assumptions (cf. the proof of Prop. 5.4 in Ref. [22] for more details). The last assertion simply follows by the fixed point procedure (cf. Corollary 5.5 in Ref. [22]). □

6.3.3 Local-in-time solution of the original problem

In this section we shall obtain the solution of problem (6.18) from the solution of the associated problem (6.19), by exploiting the definition of representation.

Proposition 6.3.2 *Let* $p : [0,+\infty) \to X$ *be a strongly regular representation of* $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ *and let* $0 < T \leq +\infty$*. Then,* $u : [0,T) \to X$ *is a solution of the associated problem* (6.19) *in* $[0,T)$ *if and only if* $w(t) = u(t) + p(t)$ *is a solution of the original problem* (6.18) *in* $[0,T)$*.*

(cf. Proposition 6.2 in Ref. [21])
Therefore, the following corollary holds.

Corollary 6.3.1 (Local solution of the original problem) *Let* $\gamma(t)$ *be such that* $\mathcal{D}(\mathrm{T}_{\gamma(t)})$ *has a strongly regular and real-valued representation* $p(t)$ *for* $t \in [0,+\infty)$*. Let* w_0 *belong to* $\mathcal{D}(\mathrm{T}_{\gamma(0)})$ *and be real-valued. Let* V_e *belong to* $\mathcal{C}^1(\,[\,0,\infty)\,;\,W^{2,\infty}(\mathbb{R}^3)\,)$*. Then, the problem* (6.18) *has a real-valued solution in a maximal time interval* $[0, t_{\max})$*, with* $0 < t_{\max} \leq +\infty$*.*

By the formulation of the Corollary, it can be inferred a certain arbitrariness in the choice of the representation p, however it can be stated the p-independence of the solution of problem (6.18) (see the discussion in Section 6 in Ref. [22] and Props. 6.4 and 6.5). Moreover, in the Appendix of Ref. [22] we show, under certain assumptions on the datum γ, an example of a representation p which satisfies the assumptions in the Corollary above.

We remark that a lower bound for the maximal time interval $t_{\max}$ can be found in Lemma 5.1 in Ref. [21].
In Ref. [22], when dealing with the one-dimensional case, we proved that $t_{\max} = \infty$ by recovering a priori estimates for $\|w(t)\|_2, \|vw(t)\|_2$ for all times. In the present three-dimensional case, we do not succeed in repeating the same strategy because of *the kind of* estimate we recover for the pseudo-differential operator with the self-consistent potential in Prop. 6.2.3 (cf. Remark 6.2.4).
Once more, we observe that the equivalence with the Schrödinger formulation (as well as the density matrices one) is broken by the boundedness of the spatial domain chosen in our analysis. Thus, we are still working on results assuring the existence of the solution for t on the whole $\mathbb{R}^+$.

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Part IV

The Wigner-Poisson-Fokker-Planck system

The model

In this part it is under examination a second quantum kinetic model of charge transport in a semiconductor device: in addition, it is taken into account the effect of the interaction of the particles with the crystal lattice. That is described as a surrounding phonon bath in thermal equilibrium, the presence of which produces a dissipative phenomenon. Thus, again the evolution of the particle ensemble is time-irreversible because of the coupling with the environment, and we are dealing with an *open* quantum system.
The kinetic model for such a system, namely the Wigner-Fokker-Planck equation, has many possible applications, as we have anticipated in Section 2.2. For a physical derivation of the model from the reversible dynamics of a quantum system, the interested reader can refer to Refs. [CL, CEFM]. There, first of all, it is recovered a Hamiltonian for the coupled system test-particle and reservoir, which consists of an idealized ensemble of harmonic oscillators. Then, a kinetic formulation is derived via Wigner transform, and the equation of interest is obtained by applying certain asymptotic regimes and "tracing out" the reservoir coordinates (cf. Section 2.2 for the terminology).
Mathematically speaking, the WFP equation reads

$$w_t + v\cdot\nabla_x w - \Theta[V]w \;=\; \beta\mathrm{div}_v(vw) + \sigma\Delta_v w + 2\gamma\mathrm{div}_v(\nabla_x w) + \alpha\Delta_x w \qquad \text{(WFP)}$$

with $w = w(x,v,t)$, $(x,v)\in\mathbb{R}^6$, $t>0$; thus, it differs from the Wigner one because of the right hand side, where appear the friction parameter $\beta\geq 0$ and the parameters α, $\gamma\geq 0$, $\sigma>0$, which constitute the phase-space diffusion matrix of the system. The condition

$$\begin{pmatrix} \alpha & \gamma+\frac{i}{4}\beta \\ \gamma-\frac{i}{4}\beta & \sigma \end{pmatrix} \;\geq\; 0, \qquad \text{(C)}$$

guarantees that the system is *quantum mechanically correct*. More precisely, it guarantees that the equation for the density matrix corresponding to the quantum system is in Lindblad form and that the density matrix itself stays a positive operator under temporal evolution (see Ref. [ALMS] for details).
We mention here the relation between such coefficients and the physical constants of interest, namely, the coupling constant with the bath ξ, the temperature of the bath T and the scaled Planck constant $\hbar$:

$$\beta\sim\xi\,,\quad \sigma\sim T\,,\quad \gamma\,,\alpha\sim\frac{\hbar^2}{T}$$

(cf. Ref. [ALMS]) Since $\epsilon := \hbar\xi/T$ is the asymptotic expansion parameter, the FP term can be regarded as a $\mathcal{O}(\epsilon^3)$-accurate description of the dissipative effect (corresponding to a medium temperature situation), while the version of it with $\gamma=\alpha=0$ is a $\mathcal{O}(\epsilon^2)$-accurate description (at high temperature) and it's the classical FP term (cf. Refs. [CSV]). Observe that, in the classical case, the condition (C) is not satisfied. Another case that, instead, is quantum-mechanically correct is the one with $\gamma=\alpha=\beta=0$ and it is a $\mathcal{O}(\epsilon)$-accurate model (at very high temperature).
In the sequel we shall assume

$$\alpha\sigma \;\geq\; \gamma^2+\frac{\beta^2}{16} \quad\text{and}\quad \alpha\sigma \;>\; \gamma^2. \qquad \text{(C1)}$$

Accordingly, the principle part of the Fokker-Planck term will be uniformly elliptic. Hence, we expect the equation under consideration to show some *parabolic* features. In particular, a reguralization of the initial datum for $t>0$, analogously to the classical case (cf. Ref. [Car]);

actually, the presence of a laplacian both in x and v variables should produce "better" diffusive effects in comparison with the classical case: that effect will be indeed recovered.
Precisely, we will consider equation (WFP) self-consistently coupled with the Poisson equation for the (real-valued) potential $V = V(x,t)$:

$$-\Delta V = n[w], \quad x \in \mathbb{R}^3, \quad t > 0, \tag{P}$$

with the particle density

$$n[w](x,t) = \int_{\mathbb{R}^3} w(x,v,t)\, dv.$$

This potential models the repulsive Coulomb interaction within the considered particle system in a mean-field description (cf. the discussion in Part I). Accordingly, the WPFP system is the quantum generalization of the Vlasov-Poisson-Fokker-Planck system, for the diffusive transport of charged particles (in plasmas, e.g.). Our aim is to achieve results comparable to Refs. [Bo1, Bo2, Ca2], concerning the classical case.
In the quantum framework, instead, the following discussion will be complementary to Ref. [ALMS], where the friction-free, hypoelliptic case (with $\alpha = \beta = \gamma = 0$) is analyzed.

Chapter 7

The three-dimensional Wigner-Poisson-Fokker-Planck system

7.1 Introduction

In this chapter we will be concerned with the WPFP system we recall here for later reference

$$\partial_t w(t) = (-v\cdot \nabla_x w + \beta \mathrm{div}_v(vw) + \sigma\Delta_v w + 2\gamma \mathrm{div}_v(\nabla_x w) + \alpha\Delta_x w + \Theta\left[V[w]\right] w)(t) \quad (7.1a)$$

$$-\Delta_x V(t) = n[w](t) \quad (7.1b)$$

$$w(t=0) = w_0, \quad (7.1c)$$

for the unknown functions $w = w(x, v, t)$, $(x, v) \in \mathbb{R}^6$, $t \geq 0$ and $V = V(x, t)$, $x \in \mathbb{R}^3$, $t \geq 0$. The main analytical challenge in tackling such system is again the proper definition of $n[w]$ in appropriate L^p spaces (cf. Section 4.1). By adapting L^1-techniques from the classical Vlasov-Fokker-Planck equation, the 3D WPFP system was analyzed in Ref. [ALMS] (local-in-time solution for the friction-free problem) and Ref. [CLN] (global-in-time solution). The latter paper, however, is not a purely kinetic analysis as it requires to assume the positivity of the underlying density matrix (cf. Remark 3.3.1). In both cases the dissipative structure of the system allows to control $n[w]$.
In Ref. [AS] the 3D WPFP system was reformulated as von-Neumann equations for the density matrix. This implies $n \in L^1(\mathbb{R}^3)$. While this approach is the most natural, both physically and in its mathematical structure, it is restricted to whole space cases. Accordingly, its use for practical applications and numerical analysis seem unfeasible.
Thus, in spite of the various existing well-posedness results for the WPFP problem, there is a need for a purely kinetic analysis that possibly allows for an extension to boundary-value problems. And this is the goal of the joint work we will include here, namely *Arnold A., Dhamo E., Manzini C., "On the three dimensional Wigner-Poisson-Fokker-Planck problem: global-in-time solutions and dispersive effects", preprint in "Angewandte Mathematik und Informatik" 10/04 Univ. Münster* (Ref.[ADMa]). The tools used there are closely related to those used for the classical Vlasov-Poisson-Fokker-Planck equation in the vast body of mathematical literature from the 1990's (cf. Refs. [Bo1, Bo2, CSV, Car, Ca2]).
In particular, the following ones could be important also for other quantum kinetic applications: In all of the existing literature on Wigner-Poisson problems (except Ref. [Ste]) the potential V is bounded, which makes it easy to estimate the operator $\Theta[V]$ in L^2. Here, V is

unbounded and it lies in *no* L^p-space. However, the operator Θ only involves δV, a potential difference, which has better decay properties at infinity. This observation gives rise to new estimates that are crucial for our local-in-time analysis. However, these new estimates rely as well on the regularizing effect of the FP term, thus they cannot be exploited for the well-posedness of the (non-diffusive) WP equation.
In order to establish *global-in-time* solution, we shall extend the use of the dispersive effects related to the free-streaming operator (cf. Refs. [LP, Pe, CP] and Chapter 4) to the WPFP system: we will get, indeed, "better" a priori estimates for the electric field, since the diffusive effects will add to the dispersive ones.
As in the WP case, we shall assume that the initial state lies in a weighted L^2-space, but we shall *not* require that our system has finite mass or finite kinetic energy. Since the energy balance will not be used, this also implies that the sign of the interaction potential does *not* play a role in our analysis.
For what the weight is concerned, the result in Ref. [ADMa] can be improved, in the sense that it is *not* necessary to introduce a x-weight as well, for the well-posedness to hold. Accordingly, the main theorem of the chapter reads as follows

Theorem 7.1.1 *Let* $w_0 \in X := L^2(\mathbb{R}^6; (1+|v|^2)^2 dxdv)$ *satisfy for some* $\omega \in [0,1)$

$$\left\| \int w_0(x - \vartheta(t)v, v)\, dv \right\|_{L^{6/5}(\mathbb{R}^3_x)} \le C_T \vartheta(t)^{-\omega}, \quad \forall\, t \in (0,T], \quad \forall\, T > 0, \qquad (\mathbf{B})$$

with $\vartheta(t) := \frac{1-e^{-\beta t}}{\beta}$ *for* $\beta > 0$, *and* $\vartheta(t) = t$ *for* $\beta = 0$. *Then the WPFP equation* (7.1) *admits a unique* global-in-time *mild solution* $w \in Y_T$, $\forall\, 0 < T < \infty$, *where*

$$Y_T := \left\{ z \in \mathcal{C}([0,T]; X) \,|\, \nabla_x z, \nabla_v z \in \mathcal{C}((0,T]; X), \|\nabla_x z(t)\|_X + \|\nabla_v z(t)\|_X \le C t^{-1/2}, t \in (0,T) \right\}.$$

If, in addition, $|x|^2 w_0 \in L^2(\mathbb{R}^6)$ *(i.e.* $w_0 \in X_1$, *cf. Section 4.1), then the unique solution* $w \in \widetilde{Y_T}$, *with*

$$\widetilde{Y_T} := \left\{ z \in \mathcal{C}([0,T]; X_1) | \nabla_x z, \nabla_v z \in \mathcal{C}((0,T]; X_1), \|\nabla_x z(t)\|_{X_1} + \|\nabla_v z(t)\|_{X_1} \le C t^{-1/2}, t \in (0,T) \right\}$$

The second statement requires more technical effort to be proved and it is the main result in Ref. [ADMa]. It deserves some interest to show it in parallel, in view of understanding the different role played by the "$|x|^2$-moment" with respect to those in the v-variable, and whether the former can be alternative to the latter ones. We anticipate that, analogously to the WP case (cf. Remark 4.1.1), whenever a x-weight is introduced, a symmetric v-weight has to be included as well, for the dissipativity of the linear operator to hold.
The chapter is organized as follows: In Section 2 we introduce a weighted L^2-space for the Wigner function w that allows to define $n[w]$ and the nonlinear term $\Theta[V]w$. In §3 we obtain a local-in-time, mild solution for WPFP using a fixed point argument and the parabolic regularization of the Fokker-Planck term. In §4 we establish a-priori estimates to obtain global-in-time solutions: the key point is to derive first L^p-bounds for the electric field ∇V by exploiting dispersive effects of the free kinetic transport. "Bootstrapping" then yields estimates on the Wigner function in a weighted L^2-space. Finally, we give regularity results on the solution. The technical proofs of several lemmata are defered to the Appendix.

7.2 The functional setting

In this section we shall discuss the functional analytic preliminaries for studying the non-linear problem. First we shall introduce an appropriate "state space" for the Wigner function w, next, we shall discuss the linear Wigner-Fokker-Planck equation and the dissipativity of its (evolution) generator A.

7.2.1 State space and self-consistent potential

Let us recall the definition of the following weighted (real valued) L^2-spaces (cf. Section 4.1)

$$\begin{aligned} X &:= L^2(\mathbb{R}^6; (1+|v|^2)^2\, dx\, dv), \\ X_1 &:= L^2(\mathbb{R}^6; (1+|x|^2+|v|^2)^2\, dx\, dv). \end{aligned}$$

Obviously, $X_1 \hookrightarrow X$ and, by Lemmata 4.1.1, 4.1.2 it follows

$$w \in X \quad \Rightarrow \quad n[w] \in L^2(\mathbb{R}^3) \tag{7.2}$$

$$w \in X_1 \quad \Rightarrow \quad n[w] \in L^p(\mathbb{R}^3),\ \frac{3}{2} < p \le 2\,. \tag{7.3}$$

In this framework the following estimates for the self-consistent potential hold.

Proposition 7.2.1 *Let $w \in X_1$. Then, the (Newton potential) solution $V = V[w]$ of the equation $-\Delta_x V[w] = n[w]$, $x \in \mathbb{R}^3$, satisfies*

$$\|\nabla V[w]\|_{L^r(\mathbb{R}^3)} \le C\|n[w]\|_{L^p(\mathbb{R}^3)}, \quad 3 < r \le 6, \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{3}. \tag{7.4}$$

PROOF. Since $V = -\frac{1}{4\pi|x|} * n$, we have $\nabla V = \frac{x}{4\pi|x|^3} * n$, and the estimate follows from the generalized Young inequality. □

According to Lemma 4.1.1, in particular,

$$w \in X \Rightarrow \|\nabla V[w]\|_{L^6(\mathbb{R}^3)} \le C\|n[w]\|_{L^2(\mathbb{R}^3)}\,. \tag{7.5}$$

Remark 7.2.1 *Note that $n \in L^p(\mathbb{R}^3)$, $3/2 < p \le 2$, does not yield a control of V in any L^r-space (via the generalized Young inequality). However, the operator $\Theta[V]$ involves only the function δV, which is slightly "better behaved".*

Omitting the time-dependence we have

$$\begin{aligned} \delta V(x,\eta) &= V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{n[w](x-\frac{\eta}{2}-\xi) - n[w](x+\frac{\eta}{2}-\xi)}{|\xi|}\, d\xi \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3} f(y;\eta)\, n[w](x-y)\, dy, \end{aligned}$$

with the "dipole-kernel" $f(y;\eta) := \left(\frac{1}{|y-\frac{\eta}{2}|} - \frac{1}{|y+\frac{\eta}{2}|}\right)$.

Proposition 7.2.2 *For all $w \in X_1$ and fixed $\eta \in \mathbb{R}^3$, we have*

$$\|\delta V[w](.,\eta)\|_{L^\infty(\mathbb{R}^3_x)} \;\leq\; C_r|\eta|^{2-\frac{3}{r}}\|n[w]\|_{L^r(\mathbb{R}^3)}, \quad 3/2 < r \leq 2. \tag{7.6}$$

Proof. By using the triangle inequality,

$$|f(y;\eta)| \;\leq\; \frac{|\eta|}{|y-\frac{\eta}{2}||y+\frac{\eta}{2}|},$$

and the transformation $y = |\eta|x$, we estimate for $3/2 < p < 3$

$$\|f(.\,;\eta)\|^p_{L^p(\mathbb{R}^3)} \;=\; |\eta|^{3-p}\int_{\mathbb{R}^3}\frac{dx}{\left(|x-\frac{e}{2}||x+\frac{e}{2}|\right)^p} \;<\; \infty,$$

where $e \in \mathbb{R}^3$ is some unit vector (due to the rotational symmetry of $\|f(.;\eta)\|^p_{L^p(\mathbb{R}^3)}$ with respect to η). Young inequality then gives

$$\begin{aligned}\forall\, w \in X, &\quad \delta V(.,\eta) \in L^q(\mathbb{R}^3),\; 6 < q \leq \infty,\\ \forall\, w \in X_1, &\quad \delta V(.,\eta) \in L^q(\mathbb{R}^3),\; 3 < q \leq \infty\,,\end{aligned}$$

and, in particular,

$$\|\delta V(.\,;\eta)\|_{L^\infty(\mathbb{R}^3)} \;\leq\; C_r|\eta|^{2-\frac{3}{r}}\|n[w]\|_{L^r(\mathbb{R}^3)}, \quad \text{for} \quad 3/2 < r \leq 2.$$

□

Accordingly, for all $w \in X$,

$$\|\delta V(.\,;\eta)\|_{L^\infty(\mathbb{R}^3)} \;\leq\; C_r|\eta|^{\frac{1}{2}}\|n[w]\|_{L^2(\mathbb{R}^3)}\,. \tag{7.7}$$

In most of the literature the pseudo-differential operator Θ is defined on $L^2(\mathbb{R}^d_v)$ for bounded potentials V, cf. [MR, MB, ACD]. For our nonlinear problem (7.1), however, $V \in L^\infty(\mathbb{R}^3)$ does *not* hold. Alternatively, we will use Eq. (7.6) and, as a compensation, it is necessary to assume some additional regularity of the Wigner function. Moreover, we shall exploit the regularity of the elements of the weighted space in Fourier transform, in combination with Prop. 7.2.1(cf. Prop. 6.2.3, [Ma], for a similar strategy).
We remark that we are presenting a modified version of Prop. 2.6 in Ref. [ADMa]; precisely, we shall prove that the nonlinear term $\Theta[V[w]]w$ has values in X, by exploiting only the informations (7.2), (7.5), (7.7) about the potential $V[w]$.

Proposition 7.2.3 *Let $u \in X$ and $\nabla_v u \in X$ Then, the linear operator*

$$z \longmapsto \Theta[V[z]]u,$$

with the function $V[z] = -\frac{1}{4\pi|x|} * n[z]$, is bounded from the space X into itself and satisfies

$$\|\Theta[V[z]]u\|_X \;\leq\; C\{\,\|u\|_X + \|\nabla_v u\|_X\}\,\|z\|_X, \quad \forall z \in X \tag{7.8}$$

Proof. To estimate $\|\Theta[V[z]]u\|_X$ we shall consider separately the two terms of the equivalent norm

$$\|u\|^2_X = \|u\|^2_2 + \sum_{i=1}^{3}\|v_i^2 u\|^2_2.$$

First, by denoting $\hat{u} := \mathcal{F}_{v\to\eta} u$, we get

$$
\begin{aligned}
\|\Theta[V[z]]u\|_2^2 &= \iint |\delta(V[z])(x,\eta)\hat{u}(x,\eta)|^2 dx\, d\eta \;\leq\; \iint \|\delta(V[z])(\,.\,,\eta)\|_\infty^2 |\hat{u}(x,\eta)|^2 d\eta\, dx \\
&\leq C\|z\|_X^2 \iint \left(|\eta|^{1/2}|\hat{u}(x,\eta)|\right)^2 d\eta\, dx \;\leq\; C\|z\|_X^2 \left(\|u\|_2^2 + \|\nabla_v u\|_2^2\right), \qquad (7.9)
\end{aligned}
$$

by applying first the Plancherel Theorem, then Hölder's inequality in the x variable, the estimates (7.2), (7.6) for the function $\delta V[z]$, and finally, Young inequality and the Plancherel Theorem to the last integral.
For the second term of $\|\Theta[V[z]]u\|_X$ we shall use

$$
v_i^2 \Theta[V]w(x,v) = \frac{1}{4}\Theta[\partial_i^2 V]w(x,v) + \Omega[\partial_i V](v_i w)(x,v) + \Theta[V]v_i^2 w(x,v), \qquad (7.10)
$$

with the pseudo-differential operator

$$
\Omega[V] := i(\delta_+ V)\left(x, \frac{\nabla_v}{i}\right), \qquad (\delta_+ V)(x,\eta) := V\left(x+\frac{\eta}{2}\right) + V\left(x-\frac{\eta}{2}\right). \qquad (7.11)
$$

Here and in the sequel we use the abreviation $\partial_i := \partial_{x_i}$. (7.10) is now estimated:

$$
\|v_i^2\Theta[V[z]]u\|_2 \leq \frac{1}{4}\|\delta(\partial_i^2 V[z])\hat{u}\|_2 + \|\delta_+(\partial_i V[z])\partial_{\eta_i}\hat{u}\|_2 + \|\delta V[z]\partial_{\eta_i}^2\hat{u}\|_2 \qquad (7.12)
$$

The first two terms of (7.12) can be estimated as follows:

$$
\begin{aligned}
\|\delta(\partial_i^2 V[z])\hat{u}\|_{L^2(\mathbb{R}^6)} &\leq 2\|\partial_i^2 V[z]\|_{L^2(\mathbb{R}_x^3)}\|\hat{u}\|_{L^2(\mathbb{R}_x^3;L^\infty(\mathbb{R}_\eta^3))} \\
&\leq C\|z\|_X \|(1+|v|^2)u\|_{L^2(\mathbb{R}^6)},
\end{aligned}
$$

by applying Hölder's inequality, (7.2) and the Sobolev imbedding $\hat{u}(x,.) \in H^2(\mathbb{R}_\eta^3) \hookrightarrow L^\infty(\mathbb{R}_\eta^3)$.

$$
\begin{aligned}
\|\delta_+(\partial_i V[z])\partial_{\eta_i}\hat{u}\|_{L^2(\mathbb{R}^6)} &\leq C\|\partial_i V[z]\|_{L^6(\mathbb{R}_x^3)}\|\partial_{\eta_i}\hat{u}\|_{L^2(\mathbb{R}_x^3;L^3(\mathbb{R}_\eta^3))} \\
&\leq C\|z\|_X \|(1+v_i^2)u\|_2, \qquad (7.13)
\end{aligned}
$$

by the Sobolev imbedding and $\nabla_\eta \hat{u}(x,.) \in H^1(\mathbb{R}_\eta^3) \hookrightarrow L^3(\mathbb{R}_\eta^3)$, and by estimate (7.4) for $\nabla V[z]$ and (7.2). For the last term of (7.12) we estimate as in (7.9):

$$
\begin{aligned}
\|\delta V[z]\partial_{\eta_i}^2\hat{u}\|_2^2 &\leq \iint \|\delta V[z](\,.\,,\eta)\|_\infty^2 |\partial_{\eta_i}^2\hat{u}(x,\eta)|^2 d\eta\, dx \\
&\leq C\|z\|_X^2 \iint \left(|\eta|^{1/2}\partial_{\eta_i}^2\hat{u}(x,\eta)\right)^2 d\eta\, dx \\
&\leq C\|z\|_X^2 \left(\|\partial_{\eta_i}^2\hat{u}\|_2^2 + \|\eta\partial_{\eta_i}^2\hat{u}\|_2^2\right) \\
&\leq C\|z\|_X^2 \left(\|\partial_{\eta_i}^2\hat{u}\|_2^2 + \|\partial_{\eta_i}^2(\eta\hat{u})\|_2^2 + \|\partial_{\eta_i}\hat{u}\|_2^2\right) \\
&\leq C\|z\|_X^2 \left(\|(1+v_i^2)u\|_2^2 + \|v_i^2\nabla_v u\|_2^2\right),
\end{aligned}
$$

by interpolation and integration by parts.
This concludes the proof of estimate (5.11). $\square$

Remark 7.2.2 *The previous proposition shows that the bilinear map*

$$(z,u) \longmapsto \Theta[V[z]]u$$

is well-defined for all $z, u \in X$, subject to $\nabla_v u \in X$. The unusual feature of the above proposition is the boundedness of this map with respect to the function z appearing in the self-consistent potential $V[z]$. This is in contrast to most of the existing literature ([ACD, MB, MR]), where the boundedness of the pseudo-differential operator $\Theta[V[z]]$ (with z fixed) is used. However, this can only hold for bounded potentials V.

Remark 7.2.3 *Observe that if, in the proposition the space X is substituted wherever it appears with the space X_1, the result holds as well and with the same proof, except for adding the following estimate*

$$\begin{aligned} \|x_i^2\Theta[V[z]]u\|_2^2 &\leq C\|z\|_X^2 \iint \left(x_i^2|\eta|^{1/2}|\hat{u}(x,\eta)|\right)^2 d\eta\, dx \\ &\leq C\|z\|_X^2 \left(\|x_i^2 u\|_2^2 + \|x_i^2\nabla_v u\|_2^2\right), \qquad i = 1,2,3\,, \end{aligned}$$

since

$$\|u\|_{X_1}^2 \;:=\; \|u\|_2^2 + \sum_{i=1}^{3}\left(\|x_i^2 u\|_2^2 + \|v_i^2 u\|_2^2\right)$$

(cf. Ref. [ADMa]).

7.2.2 Dissipativity of the linear equation

In our subsequent analysis we shall first consider the linear Wigner-Fokker-Planck equation, i.e. Eq. (7.1) with $V \equiv 0$: we shall consider in parallel the cases in which the state space is X_1, as in Ref. [ADMa], and X. Roughly speaking, the proofs of the Lemmata in the case X, are the same as those in Ref. [ADMa], except for omitting the parts related to the x-weight. The generator of this evolution problem is the unbounded linear operator $A_1 : D(A_1) \longrightarrow X_1$, respectively, its extension $A : D(A) \longrightarrow X$,

$$A_1 u \;:=\; -v\cdot\nabla_x u + \beta\,\mathrm{div}_v(vu) + \sigma\Delta_v u + 2\gamma\,\mathrm{div}_v(\nabla_x u) + \alpha\Delta_x u = Au, \tag{7.14}$$

defined on

$$\begin{aligned} D(A_1) &= \{u \in X_1 \mid v\cdot\nabla_x u, v\cdot\nabla_v u, \Delta_v u, \mathrm{div}_v\nabla_x u, \Delta_x u \in X_1\} \\ \subset D(A) &= \{u \in X \mid v\cdot\nabla_x u, v\cdot\nabla_v u, \Delta_v u, \mathrm{div}_v\nabla_x u, \Delta_x u \in X\} \end{aligned}$$

Clearly, $C_0^\infty(\mathbb{R}^6) \subset D(A_1)$. Hence, $D(A_1)$ is dense in X_1 (and in X), while $D(A)$ is dense in X. Next we study whether the operator A is dissipative on the (real) Hilbert space X, i.e. if

$$< Au, u >_X \;\leq\; 0, \quad \forall\, u \in D(A) \tag{7.15}$$

holds; respectively, whether A_1 is dissipative on the (real) Hilbert space X_1.

Lemma 7.2.1 *Let the coefficients of the operator A_1 satisfy $\alpha\sigma \geq \gamma^2$. Then $A - \kappa I$, respectively, $A_1 - \kappa_1 I$, with*

$$\kappa := \frac{3}{2}\beta + 9\sigma\,, \quad \kappa_1 := \frac{3}{2} + \frac{3}{2}\beta + 9\alpha + 9\sigma \tag{7.16}$$

are dissipative in X, respectively in X_1.

The proof is lengthy but straigthforward and deferred to the Appendix. By Theorem 1.4.5b of [Pa] their closure, $\overline{A-\kappa I} = \overline{A} - \kappa I$, $\overline{A_1} - \kappa I$ are also dissipative.
A straightforward calculation using integrations by parts yields

$$< A_1 u, w >_{X_1} \; = \; < u, B_1 w >_{X_1} \; + \; < u, B_2 w >_{X_1}, \quad \forall\, u, w \in D(A_1),$$

with

$$\begin{aligned}
B_1 w \;&=\; v \cdot \nabla_x w - \beta v \cdot \nabla_v w + \sigma \Delta_v w + 2\gamma \mathrm{div}_v(\nabla_x w) + \alpha \Delta_x w, \\
< u, B_2 w >_{X_1} \;&=\; \sum_{i=1}^{3} \left(\frac{4}{3} \iint x_i^3 v_i w u + \frac{8}{3}\gamma \iint x_i^3 w_{v_i} u + \frac{8}{3}\alpha \iint x_i^3 w_{x_i} u \right. \\
&\quad + \frac{12}{3}\alpha \iint x_i^2 w u - \frac{4}{3}\beta \iint v_i^4 w u + \frac{8}{3}\sigma \iint v_i^3 w_{v_i} u \\
&\quad \left. + \frac{12}{3}\sigma \iint v_i^2 w u + \frac{8}{3}\gamma \iint v_i^3 w_{x_i} u \right).
\end{aligned}$$

Analogously, it can be computed

$$< A u, w >_{X} \; = \; < u, B_1^{'} w >_{X} \; + \; < u, B_2^{'} w >_{X}, \quad \forall\, u, w \in D(A),$$

with

$$\begin{aligned}
B_1^{'} w \;&=\; v \cdot \nabla_x w - \beta v \cdot \nabla_v w + \sigma \Delta_v w + 2\gamma \mathrm{div}_v(\nabla_x w) + \alpha \Delta_x w, \\
< u, B_2^{'} w >_{X} \;&=\; \sum_{i=1}^{3} \left(-\frac{4}{3}\beta \iint v_i^4 w u + \frac{8}{3}\sigma \iint v_i^3 w_{v_i} u \right. \\
&\quad \left. + \frac{12}{3}\sigma \iint v_i^2 w u + \frac{8}{3}\gamma \iint v_i^3 w_{x_i} u \right).
\end{aligned}$$

Hence, $A_1^*|_{D(A_1)}$ – the restriction of the adjoint of the operator A_1 to $D(A_1)$ – is given by $A_1^* = B_1 + B_2$. A_1^* is densly defined on $D(A_1^*) \supseteq D(A_1)$, and hence A_1 is a closable operator (cf. Theorem VIII.1.b of [RS1]). Its closure $\overline{A_1}$ satisfies $(\overline{A_1})^* = A_1^*$ (cf. [RS1], Theorem VIII.1.c).
Analogous considerations hold by substituting A_1 with A and A_1^* with $A^* = B_1^{'} + B_2^{'}$.
Since $< A_1^* u, u > = < A_1 u, u >$, $< A^* u, u > = < A u, u >$,the following lemma on the dissipativity of the operator A_1^* restricted to $D(A_1)$, respectively, A^* restricted to $D(A)$, holds.

Lemma 7.2.2 *Let the coefficients of the operator A_1 satisfy $\alpha\sigma \geq \gamma^2$. Then $A^*|_{D(A)} - \kappa I$ and $A_1^*|_{D(A_1)} - \kappa_1 I$ are dissipative.*

Next we consider the dissipativity of this operator on its proper domain $D(A^*)$ (respectively, $D(A_1^*)$), which, however, is not known explicitly. To this end we shall use the following technical lemma whose proof is defered to the appendix. The arguments are inspired by [ACD], [AS], but there are also similar results for FP-type operators in [HelN, HerN], e.g.

Lemma 7.2.3 *Let $P = p(x, v, \nabla_x, \nabla_v)$ where p is a quadratic polynomial and*

$$D(P) \;:=\; C_0^\infty(\mathbb{R}^6) \subset X_1.$$

Then $\bar{P}$ is the maximum extension of P in the sense that

$$D(\bar{P}) \;:=\; \{u \in X_1 |\ \textit{the distribution } Pu \in X_1\}.$$

We now apply Lemma 7.2.3 to $P = A^* - \kappa I, P_1 = A_1^* - \kappa_1 I$, which are dissipative on $D(P) \subset D(A), D(P_1) \subset D(A_1)$. Since A^*, A_1^* are closed, we have $D(A_1^*) = D(\overline{P_1}) = \{u \in X_1 \,|\, A_1^* u \in X_1\}$ and $A_1^* - \kappa_1 I$ is dissipative on all of $D(A_1^*)$ and, analogously, $D(A^*) = D(\overline{P}) = \{u \in X \,|\, A^* u \in X\}$ and $A^* - \kappa I$ is dissipative on all of $D(A^*)$.

Applying Corollary 1.4.4 of [Pa] to $\overline{A} - \kappa I, \overline{A_1} - \kappa_1 I$ (with $(\overline{A})^* = A^*$), then implies that $\overline{A} - \kappa I, \overline{A_1} - \kappa_1 I$ generate a C_0 semigroup of contractions on X, respectively X_1 and the C_0 semigroups generated by $\overline{A}, \overline{A_1}$ satisfies

$$\|e^{t\overline{A_1}} u\|_{X_1} \;\le\; 4e^{\kappa_1 t}\|u\|_{X_1}, \quad u \in X_1, \;\; t \ge 0$$

and

$$\|e^{t\overline{A}} u\|_{X} \;\le\; 4e^{\kappa t}\|u\|_{X}, \quad u \in X, \;\; t \ge 0. \tag{7.17}$$

7.3 Existence of the local-in-time solution

In this section we shall use a contractive fixed point map to establish a local solution of the WPFP system. To this end the parabolic regularization of the linear WFP equation will be crucial to define the self-consistent potential term.
We remark that, in the sequel, we will just consider the case in which the state space is X. However, every statement can be written and proved with X_1, instead than X.

7.3.1 The linear equation

First let us consider the linear equation

$$w_t \;=\; \overline{A} w(t), \quad t > 0, \qquad w(t=0) \;=\; w_0 \in X. \tag{7.18}$$

By the discussion in Subsection 7.2.2, its unique solution $w(t) = e^{t\overline{A}} w_0$ satisfies

$$\|w(t)\|_X \;\le\; 4e^{\kappa t}\|w_0\|_X, \quad \forall\, t \ge 0. \tag{7.19}$$

Actually, the solution of the equation can be expressed as

$$w(x,v,t) \;=\; \iint w_0(x_0,v_0) G(t,x,v,x_0,v_0)\, dx_0\, dv_0, \quad \forall\, (x,v) \in \mathbb{R}^6, \tag{7.20}$$

where the Green's function G satisfies (in a weak sense) the equation (7.18) and the initial condition

$$\lim_{t \to 0} G(t,x,v,x_0,v_0) \;=\; \delta(x - x_0, v - v_0),$$

for any fixed $(x_0, v_0) \in \mathbb{R}^6$ (cf. Def. 2.1 and Prop. 3.1 in [SCDM]).
The Green's function reads

$$G(t,x,v,x_0,v_0) \;=\; e^{3\beta t} F(t, X_{-t}(x,v) - x_0, \dot{X}_{-t}(x,v) - v_0), \tag{7.21}$$

with

$$F(t,x,v) \;=\; \frac{1}{(2\pi)^3 \left(4\lambda(t)\nu(t) - \mu^2(t)\right)^{3/2}} \cdot \exp\left\{ -\frac{\nu(t)|x|^2 + \lambda(t)|v|^2 + \mu(t)(x \cdot v)}{4\lambda(t)\nu(t) - \mu^2(t)} \right\}.$$

The characteristic flow $\Phi_t(x,v) = [X_t(x,v), \dot{X}_t(x,v)]$ of the first order part of (7.14), is given for $\beta > 0$ by

$$
\begin{aligned}
X_t(x,v) &= x + v\left(\frac{1-e^{-\beta t}}{\beta}\right), \\
\dot{X}_t(x,v) &= v e^{-\beta t},
\end{aligned}
$$

and $\Phi_t(x,v) = [x+vt, v]$, for $\beta = 0$. The asymptotic behaviour of the functions $\lambda(t)$, $\nu(t)$, $\mu(t)$ for small t is described (also for $\beta = 0$) by

$$
\begin{aligned}
\lambda(t) &= \alpha t + \sigma\left[\tfrac{e^{2\beta t}-4e^{\beta t}+3}{2\beta^3} + \tfrac{1}{\beta^2}\,t\right] + \gamma\left[\tfrac{2}{\beta}t - \tfrac{2}{\beta^2}(e^{\beta t}-1)\right] &\sim&\quad \alpha t, \quad t \to 0, \\
\nu(t) &= \sigma\,\tfrac{e^{2\beta t}-1}{2\beta} &\sim&\quad \sigma t, \quad t \to 0, \\
\mu(t) &= \sigma\left(\tfrac{1-e^{\beta t}}{\beta}\right)^2 + \gamma\,\tfrac{2(1-e^{\beta t})}{\beta} &\sim&\quad -2\gamma t, \quad t \to 0.
\end{aligned}
$$

And hence:

$$
f(t) \;:=\; 4\lambda(t)\nu(t) - \mu^2(t) \quad\sim\quad 4(\alpha\sigma - \gamma^2)t^2 > 0.
$$

With these preliminaries, the following parabolic reguralization result can be deduced.

Proposition 7.3.1 *For each parameter set $\{\alpha,\beta,\gamma,\sigma\}$, there exist two constants $B = B(\alpha,\beta,\gamma,\sigma)$ and $T_0 = T_0(\alpha,\beta,\gamma,\sigma)$, such that the solution of the linear equation (7.18) satisfies*

$$
\|\nabla_v w(t)\|_X \;\le\; B t^{-1/2} e^{\kappa t}\|w_0\|_X, \quad \forall\, 0 < t \le T_0, \tag{7.22}
$$

$$
\|\nabla_x w(t)\|_X \;\le\; B t^{-1/2} e^{\kappa t}\|w_0\|_X, \quad \forall\, 0 < t \le T_0, \tag{7.23}
$$

for all $w_0 \in X$.

The proof is similar to [Car] and it will be defered to the Appendix.

Remark 7.3.1 *(a) Observe that the functions $\nabla_x w, \nabla_v w \in \mathcal{C}((0,\infty); X)$. The local boundedness of $\nabla_x w, \nabla_v w$ on any interval $(\tau, \tau + T_0]$ follows from (7.19) and Prop. 7.3.1.*
(b) Note that the strategy of the next section will not *work in the degenerated parabolic case $\alpha\sigma - \gamma^2 = 0$, since the decay rates of Prop. 7.3.1 would then be $t^{-3/2}$, which is not integrable at $t = 0$. Alternative strategies for this degenerate case were studied in [ALMS].*

7.3.2 The non-linear equation: local solution

Our aim is to solve the following non-linear initial value problem

$$
w_t(t) \;=\; \overline{A}w(t) + \Theta[V[w(t)]]w(t), \quad \forall\, t > 0, \qquad w(t=0) \;=\; w_0 \in X, \tag{7.24}
$$

where the pseudo-differential operator Θ is formally defined by (WFP) and the potential $V[w(t)]$ is the (Newton potential) solution of the Poisson equation

$$
-\Delta_x V(t,x) \;=\; n[w(t)](x) \;=\; \int_{\mathbb{R}^3} w(t,x,v)\,dv, \qquad x \in \mathbb{R}^3, \tag{7.25}
$$

for all $t > 0$. Actually, if we assume $w(t) \in X$ for all $t \geq 0$, then the function $n[w(t)]$ is well-defined for all $t \geq 0$ (cf. Eq. (7.2)), and the solution $V[w(t)]$ satisfies the estimates of Propositions 7.2.1, 7.2.2 for all $t \geq 0$.

The Propositions 5.2.2 and 7.3.1 motivate the definition of the Banach space

$$\begin{aligned} Y_T \;&:=\; \Big\{ z \in \mathcal{C}([0,T];X) \mid \\ &\nabla_x z,\, \nabla_v z \in \mathcal{C}((0,T];X) \text{ with } \|\nabla_x z(t)\|_X + \|\nabla_v z(t)\|_X \leq C t^{-1/2} \text{ for } t \in (0,T) \Big\}, \end{aligned}$$

endowed with the norm

$$\|z\|_{Y_T} \;:=\; \sup_{t\in[0,T]} \|z(t)\|_X + \sup_{t\in[0,T]} \|t^{1/2}\nabla_x z(t)\|_X + \sup_{t\in[0,T]} \|t^{1/2}\nabla_v z(t)\|_X,$$

for every fixed $0 < T < \infty$. We shall obtain the (local-in-time) well-posedness result for the problem (7.24) by introducing a non-linear iteration in the space Y_T, with an appropriate (small enough) T.

Remark 7.3.2 *In case we want to asses an analogous result by working with the state space X_1, we have to introduce the space*

$$\widetilde{Y_T} := \left\{ z \in \mathcal{C}([0,T];X_1) \mid \nabla_x z, \nabla_v z \in \mathcal{C}((0,T];X_1), \|\nabla_x z(t)\|_{X_1} + \|\nabla_v z(t)\|_{X_1} \leq C t^{-1/2}, t \in (0,T) \right\},$$

as in Ref.[ADMa]

For a given $w \in Y_T$ we shall now consider the linear Cauchy problem for the function z,

$$z_t \;=\; \overline{A} z(t) + \Theta[V[z(t)]]w(t), \quad \forall\, t \in (0,T], \qquad z(t=0) \;=\; w_0 \in X, \tag{7.26}$$

with $0 < T \leq T_0$ and T_0 is defined in Prop. 7.3.1. According to Prop. 5.2.2 the (time-dependent) operator $\Theta[V[.]]w(t)$ is, for each fixed $t \in (0, T_0]$, a well-defined, linear and bounded map on X, which we shall consider as a perturbation of the operator $\overline{A}$.

Lemma 7.3.1 *For all $w_0 \in X$ and $w \in Y_T$, with $T \leq T_0$, the initial value problem*

$$z_t \;=\; \overline{A} z(t) + \Theta[V[z(t)]]w(t), \quad \forall\, t \in (0,T], \qquad z(t=0) \;=\; w_0,$$

has a unique mild solution $z \in \mathcal{C}([0,T];X)$, which satisfies

$$z(t) \;=\; \mathrm{e}^{t\overline{A}} w_0 + \int_0^t \mathrm{e}^{(t-s)\overline{A}} \Theta[V[z(s)]]w(s)\, ds, \quad \forall\, t \in [0,T]. \tag{7.27}$$

Moreover, the solution z belongs to the space Y_T.

Proof. The first assertion follows directly by applying (a trivial extension of) Thm. 6.1.2 in [Pa]:
For any fixed $w \in Y_T$, the function $g(t,.) := \Theta[V[.]]w(t)$ is a bounded linear operator on X

for all $t \in (0,T)$, and it satisfies $g \in L^1((0,T);\mathcal{B}(X)) \cap C((0,T];\mathcal{B}(X))$ (by Prop. 5.2.2). Moreover, by estimates (7.17), (5.11), the following inequalities hold

$$
\begin{aligned}
\|z(t)\|_X &\leq 4\mathrm{e}^{\kappa t}\|w_0\|_X + 4\int_0^t \mathrm{e}^{\kappa(t-s)} C\{\,\|w(s)\|_X + \|\nabla_v w(s)\|_X\}\|z(s)\|_X\, ds && (7.28)\\
&\leq 4\mathrm{e}^{\kappa t}\|w_0\|_X + 4C\mathrm{e}^{\kappa T}\|w\|_{Y_T}\int_0^t (1+s^{-1/2})\|z(s)\|_X\, ds, && (7.29)
\end{aligned}
$$

for all $t \in [0,T]$. Then, by Gronwall's Lemma,

$$
\|z(t)\|_X \;\leq\; 4e^{\kappa T}\|w_0\|_X\left[1 + 4C\|w\|_{Y_T} e^{\left(\kappa T + 4Ce^{\kappa T}\|w\|_{Y_T}(T+2T^{1/2})\right)}(t+2t^{1/2})\right], \qquad (7.30)
$$

for all $t \in [0,T]$. By differentiating equation (7.27) in the v-direction, we obtain

$$
\nabla_v z(t) \;=\; \nabla_v e^{t\overline{A}} w_0 + \int_0^t \nabla_v e^{(t-s)\overline{A}} g(s,z(s))\, ds, \quad \forall\, t \in [0,T]. \qquad (7.31)
$$

Using the estimates (7.22), (5.11), and (7.30) then yields

$$
\begin{aligned}
\|\nabla_v z(t)\|_X &\leq Bt^{-1/2}\mathrm{e}^{\kappa t}\|w_0\|_X \\
&\quad + B\|w\|_{Y_T}\int_0^t (t-s)^{-1/2}\mathrm{e}^{\kappa(t-s)} C\{\,1+s^{-1/2}\}\,\|z(s)\|_X\, ds \\
&\leq Bt^{-1/2}\mathrm{e}^{\kappa t}\|w_0\|_X + 4BCe^{2\kappa T}\|w_0\|_X\|w\|_{Y_T}\Big[\,\pi + 2t^{1/2} \\
&\quad + 4C\|w\|_{Y_T} e^{\left(\kappa T + 4Ce^{\kappa T}\|w\|_{Y_T}(T+2T^{1/2})\right)}\left(4t^{1/2} + \frac{3}{2}\pi t + \frac{4}{3}t^{3/2}\right)\Big], \qquad (7.32)
\end{aligned}
$$

for all $t \in [0,T]$. The continuity in time of $\nabla_v z$ can be derived from (7.31) by using Remark 7.3.1 and the fact that $g(t,z(t)) \in \mathcal{C}((0,T];X)$.

By differentiating Eq. (7.27) in the x-direction, we get the same estimate for $\|\nabla_x z(t)\|_X$, by exploiting (7.23) and (7.30). Hence, the function z belongs to the space Y_T. $\square$

We now define the (affine) linear map M on Y_T (for any fixed $0 < T \leq T_0$):

$$
w \longmapsto Mw := z,
$$

where z is the unique mild solution of the initial value problem (7.26). According to Lemma 7.3.1, $z \in Y_T$. Next we shall show that M is a strict contraction on a closed subset of Y_T, for T sufficiently small. This will yield the local-in-time solution of the non-linear equation (7.24).

Proposition 7.3.2 *For any fixed $w_0 \in X$, let $R > \max\{4,B\}\mathrm{e}^{\kappa}\|w_0\|_X$, with the constant B defined in Prop. 7.3.1. Then there exists a $\tau := \tau(\|w_0\|_X, B) > 0$ such that the map M,*

$$
(Mw)(t) \;=\; \mathrm{e}^{t\overline{A}} w_0 + \int_0^t \mathrm{e}^{(t-s)\overline{A}}\Theta[V[Mw(s)]]w(s)\, ds, \quad \forall\, t \in [0,\tau], \qquad (7.33)
$$

is a strict contraction from the ball of radius R of Y_τ into itself.

Proof. By (the proof of) Lemma 7.3.1, the function $z = Mw \in Y_\tau$ satisfies (7.30). Under the assumption $\|w\|_{Y_\tau} \le R$, this estimate reads

$$\|Mw(t)\|_X \;\le\; 4e^{\kappa\tau}\|w_0\|_X \left[1 + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau + 2\tau^{1/2})\right)}(t + 2t^{1/2})\right], \quad \forall t \in [0,\tau].$$

If we assume

$$4e^{\kappa\tau}\|w_0\|_X \left[1 + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau + 2\tau^{1/2})\right)}(\tau + 2\tau^{1/2})\right] \;\le\; \frac{R}{3}, \tag{7.34}$$

then $\|Mw(t)\|_X \le \frac{R}{3}$. Similar to (7.32) we have

$$\|\nabla_v Mw(t)\|_X \;\le\; Bt^{-1/2}\mathrm{e}^{\kappa t}\|w_0\|_X + 4BCRe^{2\kappa\tau}\|w_0\|_X \Big[\pi + 2t^{1/2} + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau + 2\tau^{1/2})\right)}\left(4t^{1/2} + \frac{3}{2}\pi t + \frac{4}{3}t^{3/2}\right)\Big].$$

If we assume

$$Be^{\kappa\tau}\|w_0\|_X + 4BCRe^{2\kappa\tau}\|w_0\|_X \Big[\pi\tau^{1/2} + 2\tau + + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau + 2\tau^{1/2})\right)}\left(4\tau + \frac{3}{2}\pi\tau^{3/2} + \frac{4}{3}\tau^2\right)\Big] \;\le\; \frac{R}{3}, \tag{7.35}$$

then

$$t^{1/2}\|\nabla_v Mw(t)\|_X \;\le\; \frac{R}{3}, \quad \forall\, t \in [0,\tau].$$

Under the condition (7.35) the same decay also holds for $\|\nabla_x Mw(t)\|_X$.

Let us now choose

$$\tau \;:=\; \min\left\{1, \left(\frac{R/3 - 4e^{\kappa}\|w_0\|_X}{48CR\|w_0\|_X e^{2\kappa + 12CRe^{\kappa}}}\right)^2, \left(\frac{R/3 - Be^{\kappa}\|w_0\|_X}{4BCRe^{2\kappa}\|w_0\|_X\left[\pi + 2 + 4CRe^{(\kappa + 12CRe^{\kappa})}(\frac{3}{2}\pi + \frac{16}{3})\right]}\right)^2\right\}, \tag{7.36}$$

which is positive since $\max\{4, B\}e^{\kappa}\|w_0\|_X < R$. Then, the estimates (7.34) and (7.35) hold, and hence the operator M maps the ball of radius R of Y_τ into itself.

To prove contractivity we shall estimate $\|Mu - Mw\|_{Y_\tau}$ for all $u, w \in Y_\tau$ with $\|u\|_{Y_\tau}, \|w\|_{Y_\tau} \le R$. Since

$$Mu(t) - Mw(t) \;=\; \int_0^t \mathrm{e}^{(t-s)\overline{A}}\Theta[V[(Mu - Mw)(s)]]u(s)\,ds + \int_0^t \mathrm{e}^{(t-s)\overline{A}}\Theta[V[Mw(s)]](u - w)(s)\,ds, \quad \forall\, t \in [0,\tau],$$

by analogous estimates,

$$\|Mu(t) - Mw(t)\|_X \;\le\; 4CR\,\mathrm{e}^{\kappa\tau}\left\{\int_0^t (1 + s^{-1/2})\,\|(Mu - Mw)(s)\|_X\,ds + \|u - w\|_{Y_\tau}\int_0^t (1 + s^{-1/2})\,ds\right\},$$

and, by applying Gronwall's Lemma:

$$\begin{aligned}\|Mu(t)-Mw(t)\|_X &\leq 4CR\mathrm{e}^{\kappa\tau}\left[t+2t^{1/2}+\right.\\ &\left.+\,4CR\mathrm{e}^{\left(\kappa\tau+4CR\mathrm{e}^{\kappa\tau}(\tau+2\tau^{1/2})\right)}\left(2t+2t^{3/2}+\frac{1}{2}t^2\right)\right]\|u-w\|_{Y_\tau},\quad \forall\, t\in[0,\tau].\end{aligned}$$

By using $0\leq t\leq\tau\leq 1$, we obtain

$$\|Mu(t)-Mw(t)\|_X \;\leq\; 4CR\,\mathrm{e}^{\kappa}\left[3+18CR\mathrm{e}^{(\kappa+12CR\mathrm{e}^{\kappa})}\right]\tau^{1/2}\|u-w\|_{Y_\tau}. \tag{7.37}$$

Similarly,

$$\begin{aligned}\|\nabla_v Mu(t)-\nabla_v Mw(t)\|_X &\leq CBR\mathrm{e}^{k\tau}\left\{\int_0^t (t-s)^{-1/2}(1+s^{-1/2})\,\|(Mu-Mw)(s)\|_X\,ds\right.\\ &\left.+\int_0^t (t-s)^{-1/2}(1+s^{-1/2})\,ds\|u-w\|_{Y_\tau}\right\},\end{aligned}$$

and, by using estimate (7.37),

$$\begin{aligned}\|\nabla_v Mu(t)-\nabla_v Mw(t)\|_X &\leq CBR\mathrm{e}^{\kappa\tau}\left[1+4CR\,\mathrm{e}^{\kappa}\left(3+18CR\mathrm{e}^{(\kappa+12CR\mathrm{e}^{\kappa})}\right)\tau^{1/2}\right]\\ &\cdot\left(\pi+2t^{1/2}\right)\|u-w\|_{Y_\tau},\quad \forall\, t\in[0,\tau].\end{aligned}$$

Then, by exploiting $0<\tau\leq 1$,

$$\begin{aligned}t^{1/2}\|\nabla_v Mu(t)-\nabla_v Mw(t)\|_X &\leq CBR\mathrm{e}^{\kappa}(\pi+2)\left[1+4CR\,\mathrm{e}^{\kappa}\right.\\ &\left.\cdot\left(3+18CR\mathrm{e}^{(\kappa+12CR\mathrm{e}^{\kappa})}\right)\right]\tau^{1/2}\|u-w\|_{Y_\tau}.\end{aligned} \tag{7.38}$$

The same holds for $t^{1/2}\|\nabla_x Mu(t)-\nabla_x Mw(t)\|_X$.
When choosing $\tau>0$ small enough, estimates (7.37), (7.38) imply

$$\|Mu-Mw\|_{\mathcal{C}([0,\tau];X)} \;\leq\; C\|u-w\|_{\mathcal{C}([0,\tau];X)},$$

for some $C<1$, and the assertion is proved. □

Corollary 7.3.1 *There exists a $t_{\max}\leq\infty$ such that the initial value problem (7.24) has a unique* mild *solution w in Y_T, $\forall\, T<t_{\max}$, which satisfies*

$$w(t) \;=\; \mathrm{e}^{t\overline{A}}w_0+\int_0^t \mathrm{e}^{(t-s)\overline{A}}\Theta[V[w(s)]]w(s)\,ds,\quad \forall\, t\in[0,T]. \tag{7.39}$$

Moreover, if $t_{\max}<\infty$, then

$$\lim_{t\nearrow t_{\max}}\|w(t)\|_X \;=\; \infty.$$

Proof. The solution of the problem is the fixed point of the map M previously introduced. By Prop. 7.3.2 this solution exists for a time interval of length τ (depending only on $\|w_0\|_X$) and it belongs to the space Y_τ. Since, in particular, $w(\tau)\in X$, the solution can be repeatedly continued up to the maximal time $t_{\max}$. It will then belong to Y_T, $\forall T<t_{\max}$.
If the second assertion of the corollary would not hold, there would be a sequence of times $t_n\uparrow t_{\max}$ such that $\|w(t_n)\|_X\leq C$ for all n. Then, by solving a problem with the initial value $w(t_n)$, with t_n sufficiently close to $t_{\max}$, we would extend the solution up to a certain time $t_n+\tau(\|w(t_n)\|_X)>t_{\max}$. This construction would contradict our definition of $t_{\max}$.
The uniqueness of the mild solution follows by arguments analogous to those in the proof of Thm. 6.1.4 in [Pa]. □

Remark 7.3.3 *Note that the last statement in the thesis of the Corollary 7.3.1 differs from the standard setting (cf. Thm. 6.1.4 in [Pa]). For $t_{\max} < \infty$ we conclude the 'explosion' of $w(t)$, $t \to t_{max}$ in X and* not only *in Y_t. This is due to the parabolic regularization of the problem (7.24).*

Observe that an analogous result holds by substituting the space X with X_1 and Y_T with $\widetilde{Y_T}$.

7.4 Global-in-time solution, a-priori estimates

In this section we shall exploit dispersive effects of the free transport equation to derive an a-priori estimate on the electric field. This is the key ingredient for proving the main result of the paper, the global solution for the WPFP system:

Theorem 7.4.1 *Let $w_0 \in X$ satisfy for some $\omega \in [0,1)$*

$$\left\| \int w_0(x - \vartheta(t)v, v)\, dv \right\|_{L^{6/5}(\mathbb{R}^3_x)} \;\leq\; C_T \vartheta(t)^{-\omega}, \quad \forall\, t \in (0,T], \quad \forall\, T > 0, \tag{B}$$

with $\vartheta(t) := \frac{1-e^{-\beta t}}{\beta}$ for $\beta > 0$, and $\vartheta(t) = t$ for $\beta = 0$. Then the WPFP equation (7.24) admits a unique global-in-time *mild solution $w \in Y_T$, $\forall\, 0 < T < \infty$.*

In order to prove that $t_{\max} = \infty$, we have to show that $\|w(t)\|_X$ is finite for all $t \geq 0$ (cf. Corollary 7.3.1). To this end, we shall derive a-priori estimates for $\|w(t)\|_2$, $\||v|^2 w(t)\|_2$. Thus, the proof of Thm. 7.4.1 will be a consequence of a series of Lemmata, in particular of Lemma 7.4.1 and Lemma 7.4.4.

In case we want to prove the analogous result in $\widetilde{Y_T}$, under the additional assumption $|x|^2 w_0 \in L^2$, we have just to obtain an a priori bound for $\||x|^2 w(t)\|_2$ as well. To that aim we will prove Lemma 7.4.5.

In the sequel, $w(t)$ denotes the unique mild solution for $0 \leq t \leq T$, for an arbitrary $0 < T < t_{\max}$.

Lemma 7.4.1 *For all $w_0 \in X$, the mild solution of the WPFP equation (7.24) satisfies*

$$\|w(t)\|_2^2 \;\leq\; e^{3\beta t} \|w_0\|_2^2, \quad \forall\, t \in [0,T]. \tag{7.40}$$

Proof. Roughly speaking, this follows from the dissipativity of the operator $\overline{A} - \frac{3\beta}{2}$ in $L^2(\mathbb{R}^6)$ (cf. (7.2)) and the skew-symmetry of the pseudo-differential operator. However, since we are dealing only with the *mild* solution of the equation, the proof requires an approximation of w by classical solutions.

Since the solution satisfies $w \in Y_T$, $\forall\, T < t_{\max}$, Prop. 5.2.2 shows that the function $f(t) := \Theta[V[w(t)]]w(t)$, $t \in (0, t_{\max})$ is well defined and it is in $\mathcal{C}((0,t_{\max}); X) \cap L^1((0,T); X)$, $\forall\, 0 < T < t_{\max}$.

For $0 < T < t_{\max}$ fixed, let us consider the following linear inhomogeneous problem:

$$\frac{d}{dt} y(t) \;=\; \overline{A} y(t) + f(t), \quad t \in [0,T], \qquad y(t=0) \;=\; w_0 \in X. \tag{7.41}$$

Its mild solution in $[0,T]$ is the function w, due to the uniqueness of the mild solution of problem (7.24). For this linear problem, we can apply Thm. 4.2.7 of [Pa]: The mild solution w is the uniform limit (on $[0,T]$) of *classical* solutions of problem (7.41). More precisely, there is a sequence $\{w_0^n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\overline{A})$, $w_0^n \to w_0$ in X, and a sequence $\{f_n(t)\} \subset \mathcal{C}^1([0,T];X)$, $f_n(t) \to f(t)$ in $L^1((0,T);X)$. And the classical solutions $y_n \in \mathcal{C}^1([0,T];X)$ of the corresponding problems

$$\frac{d}{dt}y_n(t) \;=\; \overline{A}y_n(t) + f_n(t), \quad t\in[0,T], \qquad y_n(t=0) \;=\; w_0^n, \tag{7.42}$$

converge in $C([0,T];X)$ to the solution w of problem (7.41).

We shall need these approximating classical solutions y_n in order to justify the derivation of the a-priori estimate: Multiplying both sides of (7.42) by $y_n(t)$ and integrating yields

$$\frac{1}{2}\frac{d}{dt}\|y_n(t)\|_2^2 \;\le\; \frac{3\beta}{2}\,\|y_n(t)\|_2^2 + \iint y_n(t)f_n(t)\,dx\,dv,$$

since the operator $\overline{A} - \frac{3\beta}{2}$ is dissipative in $L^2(\mathbb{R}^6)$ (cf. (7.2)). By integrating in t and letting $n\to\infty$, we have

$$\|w(t)\|_2^2 \;\le\; \|w_0\|_2^2 + 3\beta\int_0^t \|w(s)\|_2^2\,ds + 2\int_0^t\iint w(s)f(s)\,dx\,dv\,ds, \quad \forall\, t\in[0,T].$$

The second integral is equal to zero since the pseudo-differential operator Θ is skew-symmetric. Hence, applying Gronwall's Lemma yields

$$\|w(t)\|_2^2 \;\le\; e^{3\beta t}\|w_0\|_2^2, \quad \forall\, t\in[0,T]. \tag{7.43}$$

□

In order to recover similar estimates for $\||v|^2w(t)\|_2$, and in case for $\||x|^2w(t)\|_2$, we first need a-priori bounds for the self-consistent field $E=\nabla V$. To this end, we are going to exploit dispersive effects of the free streaming operator. We shall adapt to the Wigner-Poisson-Fokker-Planck problem the strategies introduced for the Wigner-Poisson case in Section 4.3, inspired by the (classical) Vlasov-Poisson problem ([LP, Pe]) and the Vlasov-Poisson-Fokker-Planck problem ([Bo1, Bo2, Ca2]).

7.4.1 A-priori estimates for the electric field: the WPFP case

According to Corollary 7.3.1, the mild solution of the WPFP problem satisfies for all $t\in[0,T]$ $(0<T<t_{\max})$

$$\begin{aligned} w(x,v,t) \;&=\; \iint G(t,x,v,x_0,v_0)w_0(x_0,v_0)\,dx_0\,dv_0 \\ &\quad + \int_0^t\iint G(s,x,v,x_0,v_0)(\Theta[V]w)(x_0,v_0,t-s)\,dx_0\,dv_0\,ds \end{aligned}$$

with the Green's function G from (7.21). According to [SCDM] we have

$$\int_{\mathbb{R}^3} G(t,x,v,x_0,v_0)\,dv \;=\; R(t)^{-3/2}\mathcal{N}\left(\frac{x-x_0-\vartheta(t)v_0}{\sqrt{R(t)}}\right),$$

with

$$
\begin{aligned}
\mathcal{N}(x) &:= (2\pi)^{-3/2}\exp\left(-\frac{|x|^2}{2}\right), && (7.44)\\
\vartheta(t) &= \frac{1-e^{-\beta t}}{\beta} = \mathcal{O}(t), \quad \text{for } t\to 0, && (7.45)\\
R(t) &:= 2\alpha t + \sigma\left[\frac{4e^{-\beta t}-e^{-2\beta t}+2\beta t-3}{\beta^3}\right] + 4\gamma\left[\frac{e^{-\beta t}+\beta t-1}{\beta^2}\right] = \mathcal{O}(t), \quad \text{for } t\to 0. && (7.46)
\end{aligned}
$$

By exploiting the redefinition (3.33) of the pseudo-differential operator, we obtain the following expression for the density $n[w]$

$$
\begin{aligned}
n[w](x,t) &= \int_{\mathbb{R}^3} w(x,v,t)\,dv\\
&= \frac{1}{R(t)^{3/2}}\iint \mathcal{N}\left(\frac{x-x_0-\vartheta(t)v_0}{\sqrt{R(t)}}\right) w_0(x_0,v_0)\,dx_0\,dv_0\\
&\quad + \int_0^t \frac{1}{R(s)^{3/2}}\iint \mathcal{N}\left(\frac{x-x_0-\vartheta(s)v_0}{\sqrt{R(s)}}\right) \mathrm{div}_{v_0}\left(\Gamma[\nabla_{x_0}V]w\right)(x_0,v_0,t-s)\,dx_0\,dv_0\,ds\\
&= \frac{1}{R(t)^{3/2}}\iint \mathcal{N}\left(\frac{x-x_0}{\sqrt{R(t)}}\right) w_0(x_0-\vartheta(t)v_0,v_0)\,dx_0\,dv_0\\
&\quad + \int_0^t \frac{\vartheta(s)}{R(s)^2}\iint (\nabla_x\mathcal{N})\left(\frac{x-x_0-\vartheta(s)v_0}{\sqrt{R(s)}}\right)\cdot\left(\Gamma[\nabla_{x_0}V]w\right)(x_0,v_0,t-s)\,dx_0\,dv_0\,ds\\
&= n_0(x,t)+n_1(x,t),
\end{aligned}
$$

where

$$
\begin{aligned}
n_0(x,t) &:= \frac{1}{R(t)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x n_0^{\vartheta}(x,t), \quad \text{with} \quad n_0^{\vartheta}(x,t) := \int w_0(x-\vartheta(t)v,v)\,dv,\\
n_1(x,t) &:= \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x \mathrm{div}_x\int\Big(\Gamma[\nabla_x V]w\Big)(x-\vartheta(s)v,v,t-s)\,dv\,ds. \quad (7.47)
\end{aligned}
$$

Correspondingly, we can split the field (with $\lambda = \frac{1}{4\pi}$):

$$
\begin{aligned}
E_0(x,t) &:= \lambda\frac{x}{|x|^3} *_x n_0(x,t) = \frac{1}{R(t)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x E_0^{\vartheta}(x,t), && (7.48)\\
&\quad\text{with}\quad E_0^{\vartheta}(x,t) := \lambda\frac{x}{|x|^3} *_x n_0^{\vartheta}(x,t),\\
E_1(x,t) &:= \lambda\frac{x}{|x|^3} *_x n_1(x,t). && (7.49)
\end{aligned}
$$

Remark 7.4.1 (The density) *Note that the splitting of the density (and of the electric field) is the same as in [Bo1, Bo2, Ca2]: analogously, in the WPFP case the two components of the decomposition (n_0, n_1, as well as E_0, E_1) are smoothed versions (in fact, convolutions with a Gaussian) of those appearing in the WP case (cf. Section 4.3). Actually, the density*

$n_0^\vartheta(x,t)$ (which is convoluted with the Gaussian to give n_0) already differs from the corresponding term in the WP case because the shift contains the function ϑ, which is due to friction (and analogously for $E_0^\vartheta(x,t)$.

From Lemma 4.3.3 we directly get

$$\left\| \int_{\mathbb{R}^3_v} (\Gamma[E]u)\,(x-\vartheta(s)v, v, t-s)\, dv \right\|_{L^2(\mathbb{R}^3_x)} \;\le\; C\vartheta(s)^{-3/2} \|E(t-s)\|_2 \|u(t-s)\|_2\,, \quad \forall\, t \ge s > 0. \tag{7.50}$$

Remark 7.4.2 (The density (cont.)) *In spite of the convolution with the Gaussian in its definition, we cannot derive any estimates for the density; precisely, the problematic term is n_1. Indeed, in order to estimate $\|n_1(\,.\,,t)\|_2$ from its definition (7.47), the use of (7.50), together with the L^1-norm of $(\nabla_x \mathcal{N})(x/\sqrt{R(s)})$ does not give an integrable function of s.*

To derive an L^2-estimate on the field we shall proceed as in the WP case (Lemma 4.3.4, Proposition 4.3.1).

Lemma 7.4.2 *Let w be the mild solution of the WPFP equation (7.24) and let $w_0 \in X$ satisfy* **(B)** *for some $\omega \in [0,1)$. For any fixed $T > 0$ the electric field then satisfies $\nabla_x V \in V_{T,\omega-\frac{1}{2}}$ and the following estimates hold:*

1. *for $2 \le p \le 6$, $\theta = \frac{3(p-2)}{2p}$*

$$\|E_0(t)\|_p \;\le\; C(T) \|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{L^{6/5}}^{1-\theta} = O(t^{-\omega(1-\theta)}), \quad \forall\, t \in (0,T]; \tag{7.51}$$

2.

$$\|E_1(t)\|_2 \;\le\; C\left(T, \|w_0\|_2\,, \sup_{s\in(0,T]} \left\{ \vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}} \right\} \right) t^{\frac{1}{2}-\omega}, \quad \forall\, t \in (0,T]. \tag{7.52}$$

PROOF. The estimate for $\|E_0(t)\|_p$, $p \in [2,6]$ is obtained by applying first the generalized Young inequality and then the Young inequality to the expression (7.48)

$$\begin{aligned}
\|E_0(t)\|_p &\le C \left\| \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}} \right) *_x n_0^\vartheta(x,t) \right\|_q \\
&\le C \left\| \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}} \right) \right\|_1 \|n_0^\vartheta(x,t)\|_q \\
&= C\|n_0^\vartheta(t)\|_q, \quad \text{with} \quad q = \frac{3p}{p+3} \in [6/5, 2].
\end{aligned}$$

Next we interpolate n_0^ϑ between L^2 and $L^{6/5}$, use (7.2) and the dissipativity of the operator $-v\cdot\nabla_x - \frac{3}{2}$ in X (cf. Lemma 7.2.1):

$$\begin{aligned}
\|n_0^\vartheta(t)\|_q &\le C\|w_0(x-\vartheta(t)v, v)\|_X^\theta \|n_0^\vartheta(t)\|_{6/5}^{1-\theta} \\
&\le C\, e^{\frac{3}{2}\theta\vartheta(t)} \|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{6/5}^{1-\theta},
\end{aligned}$$

with $\theta = \frac{5}{2} - \frac{3}{q}$. Hence

$$\|E_0(t)\|_p \;\leq\; C(T)\|w_0\|_X^{\theta}\|n_0^{\vartheta}(t)\|_{L^{6/5}}^{1-\theta}.$$

We rewrite the function $E_1(x,t)$ as

$$(E_1)_j(x,t) \;=\; \lambda \sum_{k=1}^{3} \frac{-3x_jx_k + \delta_{jk}|x|^2}{|x|^5} *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F_k(x,t,s)\, ds, \quad (7.53)$$

$$\text{with} \quad F_k(x,t,s) \;:=\; \int (\Gamma_k[E_0+E_1]w)\,(x-\vartheta(s)v, v, t-s)\, dv,$$

For estimating it we exploit classical properties of the convolution with the kernel $\frac{1}{|x|}$ and apply the Young inequality:

$$\begin{aligned}
\|E_1(t)\|_2 \;&\leq\; C\int_0^t \vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F(x,t,s)\right\|_2 ds \\
&\leq\; C\int_0^t \vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_1 \|F(x,t,s)\|_2\, ds \\
&\leq\; C(T)\|w_0\|_2\int_0^t \frac{\|E_0(t-s)\|_2 + \|E_1(t-s)\|_2}{\sqrt{\vartheta(s)}}\, ds,
\end{aligned}$$

where the last inequality follows from (7.50) and the L^2–a-priori estimate on the solution w (cf. Lemma 7.4.1). By applying the estimate (7.51) to $\|E_0(t)\|_2$, we get

$$\begin{aligned}
\|E_1(t)\|_2 \;\leq\; C(T)\|w_0\|_2 \Bigg(&\sup_{t\in(0,T]}\left\{\vartheta(t)^{\omega}\|n_0^{\vartheta}(t)\|_{L^{6/5}}\right\}\int_0^t \vartheta(s)^{-\frac{1}{2}}\vartheta(t-s)^{-\omega}\, ds \\
&+ \int_0^t \frac{\|E_1(t-s)\|_2}{\sqrt{\vartheta(s)}}\, ds\Bigg),
\end{aligned} \quad (7.54)$$

where the function $\vartheta(s) = \mathcal{O}(s)$ as $s \to 0$. Thus the integrals are finite.
To establish a solution of (7.53) we introduce the fixed point map

$$\begin{aligned}
(ME)_j(x,t) \;:=\; &\lambda \sum_{k=1}^{3} \frac{-3x_jx_k + \delta_{jk}|x|^2}{|x|^5} *_x \\
&*_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x \int (\Gamma_k[E_0+E]w)\,(x-\vartheta(s)v, v, t-s)\, dv\, ds.
\end{aligned}$$

By using $0 < \frac{\vartheta(T)}{T}t \leq \vartheta(t)$, $\forall\, t \in (0,T]$ and (7.54), a simple fixed point argument as in the proof of Lemmma 4.3.4 with the contractivity estimate:

$$\|M^nE(t) - M^n\widetilde{E}(t)\|_2 \leq \left(C\sqrt{\frac{T}{\vartheta(T)}}\|w_0\|_2\right)^n t^{\frac{n+1}{2}-\omega}\frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{3}{2}-\omega\right)}{\Gamma\left(\frac{n+3}{2}-\omega\right)} \sup_{s\in(0,T]}\left(s^{\omega-\frac{1}{2}}\|E(s)-\widetilde{E}(s)\|_2\right)$$

shows that the linear equation (7.53) has a unique solution $E_1 \in V_{T,\omega-\frac{1}{2}}$. Hence $\nabla_x V = E_0 + E_1 \in V_{T,\omega-\frac{1}{2}}$ and Gronwall's Lemma then yields estimate (7.52). $\square$

Remark 7.4.3 *For the derivation of the a-priori bound on $\|E\|_2$, we proceed analogously to the WP case: accordingly, we did not use any moments of w (neither in x nor v), nor pseudo-conformal laws (cf. [Bo1, Bo2, Pe, Ca2] for the classical analogue, i.e. VPFP). Moreover, the convolution with the Gaussian did not play a role there; the estimate (7.52) relies just on the dispersive effect of the free-streaming operator. The parabolic regularization will be exploited in the "post-processing" Proposition 7.4.1.*

The above lemma was the first crucial step towards proving global existence of the WPFP solution. Next we shall extend this estimates on the field E to a range of L^p-norms:

Proposition 7.4.1 *Let w be the mild solution of the WPFP equation (7.24) and let $w_0 \in X$ satisfy* **(B)** *for some $\omega \in [0,1)$. Then, we have for any fixed $T>0$:*

1. *for all $r \in [0,1)$:*

$$\|E_1(t)\|_{H^r} \;\le\; C_r\left(T, \|w_0\|_2, \sup_{s\in(0,T]}\left\{\vartheta(s)^{\omega}\|n_0^{\vartheta}(s)\|_{L^{6/5}}\right\}\right) t^{-\omega+\frac{1-r}{2}}, \quad \forall\, t\in(0,T]; \tag{7.55}$$

2. *for all $p \in [2,6)$:*

$$\|E(t)\|_{L^p} \;\le\; C_p\left(T, \|w_0\|_2, \|w_0\|_X^{\theta}, \sup_{s\in(0,T]}\left\{\vartheta(s)^{\omega}\|n_0^{\vartheta}(s)\|_{L^{6/5}}\right\}\right)\left(t^{-\omega(1-\theta)} + t^{-\omega+\frac{1-\theta}{2}}\right), \tag{7.56}$$

   $\forall t \in (0,T]$; where $\theta = \frac{3(p-2)}{2p}$.

Proof. For the first assertion we shall estimate $E_1(t)$ (cf. (7.53)) by using classical properties of the convolution by the kernel $\frac{1}{|x|}$ and Plancharel's identity:

$$\begin{aligned}
\|E_1(t)\|_{H^r} &= \|(1+|\xi|^2)^{r/2}\mathcal{F}_{x\to\xi}E_1(\xi,t)\|_2 \;=\; \lambda\Bigg\|\sum_{k=1}^{3}\mathcal{F}_{x\to\xi}\left(\frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5}\right)\cdot \\
&\quad\cdot\int_0^t (1+|\xi|^2)^{r/2}\frac{\vartheta(s)}{R(s)^{3/2}}\,\mathcal{F}_{x\to\xi}\left(\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right)\mathcal{F}_{x\to\xi}\Big(F_k(x,t,s)\Big)\,ds\Bigg\|_2 \\
&= \lambda\Bigg\|\sum_{k=1}^{3}\frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5} *_x \\
&\quad *_x\mathcal{F}^{-1}_{\xi\to x}\left(\int_0^t (1+|\xi|^2)^{r/2}\frac{\vartheta(s)}{R(s)^{3/2}}\,\mathcal{F}_{x\to\xi}\left(\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right)\mathcal{F}_{x\to\xi}\Big(F_k(x,t,s)\Big)ds\right)\Bigg\|_2 \\
&\le C\left\|\int_0^t (1+|\xi|^2)^{r/2}\frac{\vartheta(s)}{R(s)^{3/2}}\,\mathcal{F}_{x\to\xi}\left(\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right)\mathcal{F}_{x\to\xi}\Big(F(x,t,s)\Big)\,ds\right\|_2 \\
&\le C\int_0^t \frac{\vartheta(s)}{R(s)^{3/2}}\left\|(1+|\xi|^r)\mathcal{F}_{x\to\xi}\left(\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right)\right\|_\infty \|F(x,t,s)\|_2\,ds.
\end{aligned}$$

Then, applying the Hausdorff-Young inequality and the Lemmata 4.3.3, 7.4.2 and 7.4.1 yields

$$
\begin{aligned}
\|E_1(t)\|_{H^r} &= \\
&\le C\int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \left(\left\| \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_1 + \left\| |\xi|^r \mathcal{F}_{x\to\xi}\left(\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right)\right\|_\infty \right) \|F(x,t,s)\|_2\, ds \\
&\le C\int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \left(R(s)^{3/2} + R(s)^{3/2-r/2}\right) \|F(x,t,s)\|_2\, ds \\
&\le C\int_0^t \frac{1}{\sqrt{\vartheta(s)}} \left(1 + R(s)^{-r/2}\right) \Big(\|E_0(t-s)\|_2 + \|E_1(t-s)\|_2\Big) \|w(t-s)\|_2\, ds \\
&\le C\left(T, \|w_0\|_2, \sup_{t\in(0,T]} \left\{ \|\vartheta(t)^\omega n_0^\vartheta(t)\|_{L^{6/5}} \right\} \right) \\
&\quad \cdot \int_0^t \frac{1}{\sqrt{\vartheta(s)}} \left(1 + R(s)^{-r/2}\right) \left((t-s)^{-\omega} + (t-s)^{\frac{1}{2}-\omega}\right) ds.
\end{aligned}
$$

Since $\vartheta(t) = O(t)\,, R(t) = O(t)$ for $t\to 0$ (cf. (7.45), (7.46)), the last integral is finite for all $t>0$ and for $r\in[0,1)$. In fact the integral is $O(t^{-\omega+\frac{1-r}{2}})$.
The second estimate (7.56) of the thesis follows from (7.51) and (7.55) by the Sobolev embedding $H^\theta(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$. $\square$

Remark 7.4.4 *Proposition 7.4.1 provides a non-trivial interval of L^p-estimates for the electric field in the WPFP case. This is due to the regularizing effect of the FP term. We remark that the corresponding Gaussian is "better behaved" than the classical one, since the quantum FP operator is uniformly elliptic in both x and v variables. On the other hand, exactly as in the WP case, the range of L^p-estimates for the WPFP equation is smaller in comparison to the counterpart VPFP and that depends again on the non-negativity of the classical distribution function.*

It is possible also to exploit the convolution with the Gaussian differently and this yields the following additional a priori bound. We remark that, unlike in Prop. 7.4.2, we will *not* make use of the weighted L^2-norm of the Wigner function. Accordingly, the following estimate is to be considered a useful tool in view of a well-posedness result for the WPFP problem in a L^2-context.

Proposition 7.4.2 *Let w be the mild solution of the WPFP equation (7.24) with $w_0\in L^2$ satisfying* **(B)** *for some $\omega\in[0,1)$ and $\|w(t)\|_2 = \|w_0\|_2$. Then, we have for any fixed $T>0$,*

- *for all $r\in\mathbb{R}$, $p\in[6,+\infty)$, $1/q = 1/2 - 1/p$:*

$$\|E_0(t)\|_{L^{r,p}} \le C_p(T)\, t^{-\frac{3+r}{2q}} \vartheta(t)^{-\omega}, \quad \forall\, t\in(0,T]\,. \tag{7.57}$$

- *If, in addition $\omega\in[0,1/2)\,,$ then $\forall\, r\in\mathbb{R}$, $p\in(2,+\infty]$, $1/q = 1/2-1/p$, such that $\frac{1}{2} - \frac{3+r}{2q} - \omega > 0$:*

$$\|E_1(t)\|_{L^{r,p}} \le C_p\left(T, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}, \|w_0\|_2\right) t^{\frac{1}{2}-\frac{3+r}{2q}-\omega}, \quad \forall\, t\in(0,T]\,. \tag{7.58}$$

Accordingly, for $0 \le \mu \le r - 3/p < 1$, $E_0(t), E_1(t) \in C^{0,\mu}$. *Thus,* $E_0(t), E_1(t) \in \mathcal{C}_B$ *and*

$$\begin{aligned}
\|E_0(t)\|_{L^\infty} &\le C_p(T)\; t^{-\frac{3}{4}}\vartheta(t)^{-\omega}, \quad \forall\, t \in (0,T] \\
\|E_1(t)\|_{L^\infty} &\le C_p\left(T,\, \sup_{s\in(0,T]} \left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}, \|w_0\|_2\right)\; t^{\frac{1}{2}-\epsilon-\omega}, \quad \forall\, t \in (0,T]\,,
\end{aligned}$$

with $\epsilon > 0$.

Proof. By the definition of $L^{r,p}$-norm, with $r \in \mathbb{R}$, $1 \le p \le \infty$, and by Eq.(7.48), it holds

$$\|E_0(t)\|_{L^{r,p}} := \|\mathcal{F}^{-1}_{\xi\to x}\left((1+|\xi|^2)^{r/2}\mathcal{F}_{x\to\xi}E_0(\xi,t)\right)\|_p = \left\|\lambda\frac{x}{|x|^3} *_x \mathcal{F}^{-1}_{\xi\to x}\left((1+|\xi|^2)^{r/2}\mathcal{F}_{x\to\xi}n_0(x,t)\right)\right\|_p. \tag{7.59}$$

Moreover,

$$\mathcal{F}_{x\to\xi}n_0(x,t) = \frac{1}{R(t)^{3/2}}\,\mathcal{F}_{x\to\xi}\mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right)\mathcal{F}_{x\to\xi}n_0^\vartheta(x,t)$$

and, for all $1 \le q < \infty$,

$$\left\|(1+|\xi|^2)^{r/2}\mathcal{N}(\xi\sqrt{R(t)})\right\|_q \le C_q\left(R(t)^{-\frac{3}{2q}} + R(t)^{-\frac{3+r}{2q}}\right). \tag{7.60}$$

Accordingly, by applying successively to (7.59) generalized Young, Haussdorf-Young and Hölder inequalities, we get

$$\|E_0(t)\|_{L^{r,p}} \le C_p\left(R(t)^{-\frac{3}{2q}} + R(t)^{-\frac{3+r}{2q}}\right)\|n_0^\vartheta(x,t)\|_{6/5}\,, \text{ with } \frac{1}{q} = \frac{1}{2} - \frac{1}{p}\,,\ p \in [6,\infty)\,.$$

Thus, the first statement easily follows by using assumption **(B)** and $\vartheta(t) = O(t)\,, R(t) = O(t)$ for $t \to 0$ (cf. (7.45), (7.46)).
For what the second estimate is concerned, we start by (7.53)), use the definition of $L^{r,p}$-norm and the previuos computations and apply Young and Haussdorf-Young inequalities to get

$$\|E_1(t)\|_{L^{r,p}} \le \int_0^t \left\|(1+|\xi|^2)^{r/2}\vartheta(s)\,\mathcal{N}\left(\sqrt{R(s)}\xi\right)\mathcal{F}_{x\to\xi}\!\left(F_k(x,t,s)\right)\right\|_{p'} ds\,,$$

with $1 = 1/p + 1/p'$, $p' \in [1,2]$. Then, Hölder inequality with $1/p' = 1/2 + 1/q$ and estimates (7.60) (with $q \in [1,\infty) \Rightarrow p' \in [1,2)$) and (7.50) yield

$$\begin{aligned}
\|E_1(t)\|_{L^{r,p}} &\le C\int_0^t \frac{1}{\sqrt{\vartheta(s)}}\left(R(s)^{-\frac{3}{2q}} + R(s)^{-\frac{3+r}{2q}}\right)\Big(\|E_0(t-s)\|_2 + \|E_1(t-s)\|_2\Big)\|w(t-s)\|_2\, ds \\
&\le C\left(T,\, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\right\}, \|w_0\|_2\right) \\
&\quad \cdot \int_0^t \frac{1}{\sqrt{\vartheta(s)}}\left(R(s)^{-\frac{3}{2q}} + R(s)^{-\frac{3+r}{2q}}\right)\left(\vartheta(t-s)^{-\omega} + (t-s)^{\frac{1}{2}-\omega}\right) ds,
\end{aligned}$$

with $p \in (2,\infty]$, where the latter inequality follows instead by Lemmata 7.4.2 and the conservation of the L^2-norm. Since $\vartheta(t) = O(t)\,, R(t) = O(t)$ for $t \to 0$ (cf. (7.45), (7.46)), the last integral is finite for all $t > 0\,,\ r \in \mathbb{R}\,,\ q \in [1,\infty)$ such that $\frac{1}{2} - \frac{3+r}{2q} - \omega > 0$.
The last statements in the thesis follow by "Sobolev" embedding (cf. Ref.[Ad]): in particular,

for what the term E_1 is concerned, for $q \to \infty \Leftrightarrow p \to 2^+$ there exists $3/2 < r < 5/2$ such that

$$\|E_1(t)\|_{L^\infty} \le \|E_1(t)\|_{L^{r,p}} \le C_p \left(T, \sup_{s\in(0,T]} \left\{ \vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}} \right\}, \|w_0\|_2 \right) t^{\frac{1}{2}-\epsilon-\omega}, \quad \forall\, t \in (0,T]\,,$$

with $\epsilon > 0$. $\square$

Remark 7.4.5 *We observe that the assumption* "Let w be the mild solution of the WPFP equation (7.24) with $w_0 \in L^2$ and $\|w(t)\|_2 = \|w_0\|_2$" *could be substituted by* "Let $w_0 \in X$ and $w(t)$ be the mild solution of the WPFP equation (7.24)", *which exists by Corollary 7.3.1, for all $t \in [0,T]$ and satisfies* (7.40). *However, we want to stress that, analogously to Prop. 4.3.1 for the WP case (cf. Remark 4.3.4, as well), no weighted L^2-norm, equivalently, no information about the density n, is necessary for the estimate for the electric field to be proved, once the existence of a mild solution in L^2, which satisfies* (7.40) *is assumed.*
The assessment of the global-in-time well-posedness result for the WPFP problem in L^2, under the assumption $w_0 \in L^2$ satisfying **(B)** *is currently in progress.*
We observe also that, here, by comparison with the WP case, the regularizing effect of the convolution with the Gaussian is by far evident.

7.4.2 A-priori estimates for the weighted L^2-norms

A first consequence of the a-priori estimates for the electric field is the following

Lemma 7.4.3 *For all $w_0 \in X$ such that* **(B)** *holds for some $\omega \in [0,1)$, the mild solution of the WPFP equation (7.24) satisfies*

$$\|vw(t)\|_2^2 \;\le\; C\left(T, \|w_0\|_X, \sup_{s\in(0,T]} \left\{ \vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}} \right\} \right), \quad \forall\, t \in [0,T]. \tag{7.61}$$

Proof. In order to justify the derivation of this a-priori estimate we need again the approximating classical solutions y_n introduced in the proof of Lemma 7.4.1. Mutiplying both sides of (7.42) by $v_i^2 y_n(t)$ and integrating yields

$$\frac{1}{2}\frac{d}{dt}\|v_i y_n(t)\|_2^2 \;=\; \iint v_i^2 y_n(t) \overline{A} y_n(t)\, dx\, dv + \iint v_i^2 y_n(t) f_n(t)\, dx\, dv.$$

By analogous calculations as in the proof of Lemma 7.2.1 (cf. also (7.4)) we get,

$$\iint |v|^2 y_n(t) \overline{A} y_n(t)\, dx\, dv \;\le\; 3\sigma \|y_n(t)\|_2^2 + \frac{\beta}{2}\|v y_n(t)\|_2^2,$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|v y_n(t)\|_2^2 \;\le\; 3\sigma \|y_n(t)\|_2^2 + \frac{\beta}{2}\|v y_n(t)\|_2^2 + \iint |v|^2 y_n(t) f_n(t)\, dx\, dv, \quad \forall\, t \in [0,T].$$

By integrating in t, letting $n \to \infty$, and using (7.40), we have

$$\begin{aligned} \|vw(t)\|_2^2 \;&\le\; \|vw_0\|_2^2 + \frac{2\sigma}{\beta}(e^{3\beta t} - 1)\|w_0\|_2^2 + \beta \int_0^t \|vw(s)\|_2^2\, ds \\ &\quad + 2\int_0^t \iint |v|^2 w(s) f(s)\, dx\, dv\, ds, \quad \forall\, t \in [0,T]. \end{aligned}$$

Using again the skew-symmetry of the pseudo-differential operator and the Hölder inequality yields

$$\begin{aligned}\int_0^t\iint v_i w(s) v_i f(s)\,dx\,dv\,ds &= \frac{1}{2}\int_0^t\iint v_i w(s)\Omega[\partial_i V[w(s)]]w(s)\,dx\,dv\,ds\\ &\leq \frac{1}{2}\int_0^t \|v_i w(s)\|_2\,\|\Omega[\partial_i V[w(s)]]w(s)\|_2\,ds,\end{aligned}$$

with the operator Ω defined in (7.11). Estimating as in (7.13) and using the Sobolev inequality we obtain for $t\in[0,T]$:

$$\begin{aligned}\|\Omega[\partial_i V[w(t)]]w(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)} &\leq C\|\partial_i V[w(t)]]\hat{w}(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\eta)}\\ &\leq C\|\partial_i V[w(t)]]\|_3\|\hat{w}(t)\|_{L^2(\mathbb{R}^3_x;L^6(\mathbb{R}^3_\eta))}\\ &\leq C\|\partial_i V[w(t)]]\|_3\|\nabla_\eta\hat{w}(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\eta)}\\ &\leq C\|\partial_i V[w(t)]]\|_3\|vw(t)\|_2, \qquad (7.62)\end{aligned}$$

where $\hat{w}(x,\eta,t):=\mathcal{F}_{v\to\eta}(w(x,v,t))$. Finally, using Prop. 7.4.1 (estimate (7.56) with $p=3$) yields

$$\begin{aligned}\|vw(t)\|_2^2 \;\leq\; & C(T)\left(\|vw_0\|_2^2+\|w_0\|_2^2\right)+C\left(T,\|w_0\|_X,\sup_{s\in(0,T]}\left\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right)\\ &\cdot\int_0^t\left(s^{-\frac{\omega}{2}}+s^{-\omega+\frac{1}{4}}+\beta\right)\|vw(s)\|_2^2\,ds,\quad t\in[0,T], \qquad (7.63)\end{aligned}$$

and the Gronwall Lemma gives the result. □

With this result we can proceed to derive the a-priori estimate for $\||v|^2w(t)\|_2$.

Lemma 7.4.4 *For all $w_0\in X$ such that* **(B)** *holds for some $\omega\in[0,1)$, the mild solution of the WPFP equation (7.24) satisfies*

$$\||v|^2w(t)\|_2^2 \;\leq\; C\left(T,\|w_0\|_X,\sup_{s\in(0,T]}\left\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right),\quad \forall\, t\in[0,T]. \qquad (7.64)$$

PROOF. In order to control the term $\||v|^2w(t)\|_2$, we shall use the same strategy as in the Lemmata 7.4.1 and 7.4.3. Multiplying both sides of (7.42) by $v_i^4 y_n(t)$ and integrating we get by using (7.4) and repeating the same limit procedure as in the previous lemma:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3\iint v_i^4 w(t)^2\,dx\,dv \;\leq\; & 9\sigma\|w(t)\|_2^2+\left(3\sigma-\frac{1}{2}\beta\right)\sum_{i=1}^3\iint v_i^4 w(t)^2\,dx\,dv\\ &+\sum_{i=1}^3\iint v_i^4 w(t)f(t)\,dx\,dv,\quad\forall\, t\in[0,T].\end{aligned}$$

By integrating in t, using $C_1|v|^4\leq\sum v_i^4\leq C_2|v|^4$ and (7.40), we have

$$\begin{aligned}\||v|^2w(t)\|_2^2 \;\leq\; & C\Bigg(\||v|^2w_0\|_2^2+\frac{6\sigma}{\beta}(e^{3\beta t}-1)\|w_0\|_2^2+(6\sigma-\beta)\int_0^t\||v|^2w(s)\|_2^2\,ds\\ &+2\sum_{i=1}^3\int_0^t\iint v_i^4 w(s)f(s)\,dx\,dv\,ds\Bigg),\quad\forall\, t\in[0,T]. \qquad (7.65)\end{aligned}$$

Using again the skew-symmetry of the pseudo-differential operator Θ, the equation (7.10) and the Hölder inequality, we have

$$\begin{aligned}\int_0^t\iint v_i^2 w(s) v_i^2 f(s) dx\, dv\, ds \;&\leq\; \frac{1}{4}\int_0^t \|v_i^2 w(s)\|_2 \,\|\Theta[\partial_i^2 V[w(s)]]w(s)\|_2\, ds \\ &\quad + \int_0^t \|v_i^2 w(s)\|_2 \,\|\Omega[\partial_i V[w(s)]]v_i w(s)\|_2\, ds. \qquad (7.66)\end{aligned}$$

Since $\hat{w}(x,.,t) \in H^2(\mathrm{I\!R}^3_\eta)$, the Gagliardo-Nirenberg inequality yields for $t \in [0,T]$

$$\begin{aligned}\|\hat{w}(x,.,t)\|_{L^\infty(\mathrm{I\!R}^3_\eta)} \;&\leq\; C\|\hat{w}(x,.,t)\|^{1/2}_{L^6(\mathrm{I\!R}^3_\eta)} \|\widehat{|v|^2 w}(x,.,t)\|^{1/2}_{L^2(\mathrm{I\!R}^3_\eta)} \\ &\leq\; C\|\widehat{vw}(x,.,t)\|^{1/2}_{L^2(\mathrm{I\!R}^3_\eta)} \|\widehat{|v|^2 w}(x,.,t)\|^{1/2}_{L^2(\mathrm{I\!R}^3_\eta)}. \qquad (7.67)\end{aligned}$$

Using

$$\|\Delta V[w(t)]\|_2 \;=\; \|n[w(t)]\|_2 \;=\; C\|\hat{w}(.,\eta=0,t)\|_{L^2(\mathrm{I\!R}^3_x)} \;\leq\; C\left(\int \|\hat{w}(x,.,t)\|^2_{L^\infty(\mathrm{I\!R}^3_\eta)}\, dx\right)^{1/2},$$

(7.67), the Hölder inequality, and (7.61) we can estimate:

$$\begin{aligned}\|\Theta[\partial_i^2 V[w(t)]]w(t)\|_2 \;&\leq\; C\|\Delta V[w(t)]\|_2 \left(\int \|\hat{w}(x,.,t)\|^2_{L^\infty(\mathrm{I\!R}^3_\eta)}\, dx\right)^{1/2} \\ &\leq\; C\int \|\hat{w}(x,.,t)\|^2_\infty\, dx \\ &\leq\; C\int \|\widehat{vw}(x,.,t)\|_{L^2(\mathrm{I\!R}^3_\eta)}\, \|\widehat{|v|^2 w}(x,.,t)\|_{L^2(\mathrm{I\!R}^3_\eta)}\, dx \\ &\leq\; C\left(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right) \||v|^2 w(t)\|_2. \qquad (7.68)\end{aligned}$$

For the second term of the r.h.s. of (7.66) we proceed as in (7.62) and use the estimate (7.56):

$$\begin{aligned}\|\Omega[\partial_i V[w(t)]]v_i w(t)\|_2 \;&\leq\; C\|\partial_i V[w(t)]\|_3 \|\widehat{v_i w}(t)\|_{L^2(\mathrm{I\!R}^3_x; L^6(\mathrm{I\!R}^3_\eta))} \\ &\leq\; C\left(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right)\left(t^{-\frac{\omega}{2}} + t^{-\omega+\frac{1}{4}}\right) \||v|^2 w(t)\|_2, \quad \forall\, t\in[0,T]. \qquad (7.69)\end{aligned}$$

Analogously to (7.63), combining the estimates (7.66), (7.68) and (7.69) the Gronwall Lemma gives the assertion. □

Finally, we can obtain an a-priori estimate for the term $\||x|^2 w(t)\|_2$.

Lemma 7.4.5 *For all $w_0 \in X_1$ such that* **(B)** *holds for some $\omega \in [0,1)$, the mild solution of the WPFP equation (7.24) satisfies*

$$\||x|^2 w(t)\|_2^2 \;\leq\; C\left(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right), \quad \forall\, t\in[0,T]. \qquad (7.70)$$

PROOF. To control $\||x|^2 w(t)\|_2$, we shall use again the strategy of the Lemmata 7.4.1, 7.4.3 and 7.4.4. We multiply both sides of equation (7.42) by $x_i^4 y_n(t)$, integrate and get by using (7.3) and repeating the same limit procedure as in the lemmata above:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\iint x_i^4 w(t)^2\,dx\,dv \;\le\; & 9\alpha\|w(t)\|_2^2 + \left(\frac{3}{2}+\frac{3}{2}\beta+3\alpha\right)\sum_{i=1}^{3}\iint x_i^4 w(t)^2\,dx\,dv \\ & + \frac{1}{2}\sum_{i=1}^{3}\iint v_i^4 w(t)^2\,dx\,dv + \sum_{i=1}^{3}\iint x_i^4 w(t) f(t)\,dx\,dv,\end{aligned}$$

$\forall\, t\in[0,T]$. Due to the skew-symmetry of Θ the last integral disappears. Then, by integrating in t, using $C_1|x|^4 \le \sum x_i^4 \le C_2|x|^4$ and (7.40), we have

$$\||x|^2 w(t)\|_2^2 \;\le\; C(T)\left(\||x|^2 w_0\|_2^2 + \|w_0\|_2^2 + \int_0^t \||x|^2 w(s)\|_2^2\,ds + \int_0^t \||v|^2 w(s)\|_2^2\,ds\right). \tag{7.71}$$

Using the a-priori estimate (7.64) and the Gronwall Lemma we obtain (7.70). □

PROOF OF THEOREM 7.4.1.
The Lemmata 7.4.1, 7.4.4 and 7.4.5 show that

$$\|w(t)\|_X \;\le\; C\left(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\right), \quad \forall\, t\in[0,T], \quad \forall\, 0<T<t_{\max},$$

with C being continuous in $T\in[0,t_{\max}]$. Then, Corollary 7.3.1 shows that the mild solution w exists on $[0,\infty)$. □

In conclusion, it holds as well

Theorem 7.4.2 *Let $w_0\in X_1$ satisfy for some $\omega\in[0,1)$*

$$\left\|\int w_0(x-\vartheta(t)v, v)\,dv\right\|_{L^{6/5}(\mathbb{R}_x^3)} \;\le\; C_T\vartheta(t)^{-\omega}, \quad \forall\, t\in(0,T], \quad \forall\, T>0, \tag{\textbf{B}}$$

with $\vartheta(t) := \frac{1-e^{-\beta t}}{\beta}$ for $\beta>0$, and $\vartheta(t)=t$ for $\beta=0$. Then the WPFP equation (7.24) admits a unique global-in-time *mild solution $w\in\widetilde{Y_T}$, $\forall\, 0<T<\infty$.*

7.4.3 Regularity

The following result concerns the smoothness of the solution of WPFP, the macroscopic density and the force field, for positive times.

Corollary 7.4.1 *Under the assumptions of Theorem 7.4.1, the mild solution of the WPFP equation (7.24) satisfies*

$$w\in\mathcal{C}((0,\infty);\mathcal{C}_{\mathcal{B}}^\infty(\mathbb{R}^6)),$$

$$n(t), E(t)\in\mathcal{C}((0,\infty);\mathcal{C}_{\mathcal{B}}^\infty(\mathbb{R}^3)).$$

PROOF. Obviously, $w(t) \in \mathcal{C}(\mathbb{R}^6)$ $\forall t > 0$, because of the Green's function representation in (7.39), (7.20). If we differentiate equation (7.24) with respect to x_i and, resp., v_i, we obtain the following linear, inhomogeneous problems for any fixed $t_1 > 0$.

$$\begin{aligned} z_t(t) &= \overline{A}z(t) + \Theta[V[z(t)]]w(t) + \Theta[V[w(t)]]z(t), \quad \forall\, t > t_1, \quad z(t_1) = \partial_{x_i} w(t_1) \in X, \\ y_t(t) &= \overline{A}y(t) + \beta y(t) - \partial_{x_i} w(t) + \Theta[V[w(t)]]y(t), \quad \forall\, t > t_1, \quad y(t_1) = \partial_{v_i} w(t_1) \in X. \end{aligned}$$

By arguments analogous to Lemma 7.3.1, there exists a unique mild solution

$$z = \partial_{x_i} w \in \mathcal{C}([t_1, \infty); H^1(\mathbb{R}^6; (1 + |v|^2)^2\, dx\, dv)). \tag{7.72}$$

By an induction procedure, the derivatives $\nabla_x^\alpha \nabla_v^\beta w$, for $\alpha, \beta \in \mathbb{N}^3$, $|\alpha| + |\beta| = m > 1$ are also mild solutions of similar problems with additional well-defined inhomogeneities and with initial times $0 < t_1 < t_2 < \dots < t_m$. This yields $\nabla_x^\alpha \nabla_v^\beta w \in \mathcal{C}([t_m, \infty); H^1(\mathbb{R}^6; (1 + |v|^2)^2\, dx\, dv))$, and thus $\nabla_x^\alpha \nabla_v^\beta w \in \mathcal{C}((0, \infty); X)$. Hence, the statement about smoothness of the density and the electric field is straightforward from Eq.(7.2) and 7.2.1 and Sobolev embeddings.

7.5 Appendix

The Proofs are the same as in Ref. [ADMa]. They can be easily adatted to the "simpler" case in which the state space is X, by omitting the parts relative to the x-weight.

Proof of Lemma 7.2.1

For $u \in D(A)$ we have

$$< Au, u >_X \;=\; < Au, u >_{L^2(\mathbb{R}^6)} + \sum_{i=1}^{3} \iint x_i^4 u A u + \sum_{i=1}^{3} \iint v_i^4 u A u, \tag{7.1}$$

where $\iint f$ denotes the integral $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x, v)\, dv\, dx$. Using integrations by parts we shall calculate the three terms on the right hand side separately.

$$\begin{aligned} < Au, u >_{L^2(\mathbb{R}^6)} &= \sum_{i=1}^{3} \Big(- \iint v_i u_{x_i} u + \beta \iint (v_i u)_{v_i} u \\ &\qquad + \sigma \iint u_{v_i v_i} u + 2\gamma \iint u_{x_i v_i} u + \alpha \iint u_{x_i x_i} u \Big) \\ &\le \sum_{i=1}^{3} \Big[3\beta \iint u^2 + \beta \iint v_i u_{v_i} u - \sigma \iint u_{v_i}^2 \\ &\qquad + \gamma \left(\epsilon \iint u_{x_i}^2 + \frac{1}{\epsilon} \iint u_{v_i}^2 \right) - \alpha \iint u_{x_i}^2 \Big] \\ &= \frac{3}{2} \beta \|u\|_2^2 + \left(\frac{\gamma}{\epsilon} - \sigma \right) \|\nabla_v u\|_2^2 + (\epsilon\gamma - \alpha) \|\nabla_x u\|_2^2. \end{aligned}$$

With $\epsilon = \frac{\gamma}{\sigma}$ we obtain

$$< Au, u >_{L^2(\mathbb{R}^6)} \;\le\; \frac{3}{2} \beta \|u\|_2^2. \tag{7.2}$$

Next we estimate the second term of (7.1):

$$\begin{aligned}
\sum_{i=1}^{3}\iint x_i^4 uAu &= \sum_{i,j=1}^{3}\left(-\iint x_i^4 v_j u_{x_j} u + \beta\iint x_i^4 (v_j u)_{v_j} u\right. \\
&\quad \left. +\sigma\iint x_i^4 u_{v_j v_j} u + 2\gamma\iint x_i^4 u_{x_j v_j} u + \alpha\iint x_i^4 u_{x_j x_j} u\right) \\
&\le \sum_{i,j=1}^{3}\left[\frac{2}{3}\iint x_i^3 v_i u^2 + \beta\iint x_i^4 u^2 + \beta\iint x_i^4 v_j u_{v_j} u - \sigma\iint x_i^4 u_{v_j}^2\right. \\
&\quad \left. +\gamma\left(\epsilon\iint x_i^4 u_{x_j}^2 + \frac{1}{\epsilon}\iint x_i^4 u_{v_j}^2\right) - \alpha\iint x_i^4 u_{x_j}^2 - \frac{4}{3}\alpha\iint x_i^3 u_{x_i} u\right] \\
&\le \sum_{i=1}^{3}\left(2\iint x_i^3 v_i u^2 + \frac{3}{2}\beta\iint x_i^4 u^2 + 6\alpha\iint x_i^2 u^2\right),
\end{aligned}$$

where we chose $\epsilon = \frac{\gamma}{\sigma}$ in the last step. With an interpolation inequality this yields

$$\sum_{i=1}^{3}\iint x_i^4 uAu \;\le\; 9\alpha\|u\|_2^2 + \left(\frac{3}{2} + \frac{3}{2}\beta + 3\alpha\right)\sum_{i=1}^{3}\iint x_i^4 u^2 + \frac{1}{2}\sum_{i=1}^{3}\iint v_i^4 u^2. \tag{7.3}$$

We proceed similarly for the third term of (7.1):

$$\begin{aligned}
\sum_{i=1}^{3}\iint v_i^4 uAu &= \sum_{i,j=1}^{3}\left(-\iint v_i^4 v_j u_{x_j} u + \beta\iint v_i^4 (v_j u)_{v_j} u\right. \\
&\quad \left. +\sigma\iint v_i^4 u_{v_j v_j} u + 2\gamma\iint v_i^4 u_{x_j v_j} u + \alpha\iint v_i^4 u_{x_j x_j} u\right) \\
&\le \sum_{i,j=1}^{3}\left[\beta\iint v_i^4 u^2 + \beta\iint v_i^4 v_j u_{v_j} u - \sigma\iint v_i^4 u_{v_j}^2\right. \\
&\quad \left. -\frac{4}{3}\sigma\iint v_i^3 u_{v_i} u + \gamma\left(\epsilon\iint v_i^4 u_{x_j}^2 + \frac{1}{\epsilon}\iint v_i^4 u_{v_j}^2\right) - \alpha\iint v_i^4 u_{x_j}^2\right] \\
&\le \sum_{i=1}^{3}\left(-\frac{1}{2}\beta\iint v_i^4 u^2 + 6\sigma\iint v_i^2 u^2\right) \\
&\le 9\sigma\|u\|_2^2 + \left(-\frac{1}{2}\beta + 3\sigma\right)\sum_{i=1}^{3}\iint v_i^4 u^2, \qquad (7.4)
\end{aligned}$$

by choosing $\epsilon = \frac{\gamma}{\sigma}$ and by an interpolation.
Collecting the three estimates yields

$$\begin{aligned}
< Au, u >_{\tilde{X}} &\le \left(\frac{3}{2}\beta + 9\alpha + 9\sigma\right)\|u\|_2^2 \\
&\quad + \left(\frac{3}{2} + \frac{3}{2}\beta + 3\alpha\right)\sum_{i=1}^{3}\iint x_i^4 u^2 \\
&\quad + \left(\frac{1}{2} + 3\sigma\right)\sum_{i=1}^{3}\iint v_i^4 u^2 \\
&\le \left(\frac{3}{2} + \frac{3}{2}\beta + 9\alpha + 9\sigma\right)\|u\|_{\tilde{X}}^2.
\end{aligned}$$

Thus, the operator $A - \kappa I$ is dissipative. $\square$

Proof of Lemma 7.2.3

To prove the assertion we shall construct for each $f \in D(\overline{P}) \subset L^2(\mathbb{R}^6)$ a sequence $\{f_n\} \subset D(P)$ such that $f_n \to f$ in the graph norm
$\|f\|_P = \|f\|_{L^2} + \||x|^2 f\|_{L^2} + \||v|^2 f\|_{L^2} + \|Pf\|_{L^2} + \||x|^2 Pf\|_{L^2} + \||v|^2 Pf\|_{L^2}$.
To shorten the proof we shall consider here only the case

$$P = \theta + \nu v \cdot \nabla_x + \mu x \cdot \nabla_v + \beta v \cdot \nabla_v + \alpha \Delta_x + \sigma \Delta_v + \gamma \mathrm{div}_v \nabla_x$$

(cf. the definition of the operator A in (7.14)), but exactly the same strategy extends to the case, where P is a general quadratic polynomial.

First we define the mollifying delta sequence

$$\phi_n(x,v) := n^6 \phi(nx, nv), \qquad n \in \mathbb{N},\ x, v \in \mathbb{R}^3,$$

where

$$\begin{aligned} &\phi \in C_0^\infty(\mathbb{R}^6), \quad \phi(x,v) \geq 0, \\ &\iint \phi(x,v)\, dx dv = 1, \quad \text{and} \quad \operatorname{supp} \phi \subset \{|x|^2 + |v|^2 \leq 1\}. \end{aligned}$$

By definition we have the following properties:

(I) $\phi_n \to \delta$ in $\mathcal{D}'(\mathbb{R}^6)$,

(II) $\frac{1}{n}\partial_{x_i}\phi_n$, $\frac{1}{n}\partial_{v_i}\phi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$, $i = 1, 2, 3$,

(III) $(x,v)^\alpha \partial^\beta_{(x,v)} \left[(x,v)^\gamma \phi_n(x,v)\right] \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$, with $\alpha, \beta, \gamma \in \mathbb{N}_0^6$ multi-indexes and $|\gamma| > 0$, since $(x,v)^\gamma \phi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$.

The cutoff sequence is

$$\psi_n(x,v) := \psi\left(\frac{|(x,v)|}{n}\right), \qquad n \in \mathbb{N},\ x, v \in \mathbb{R}^3,$$

where ψ satisfies

$$\psi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \psi(z) \leq 1, \quad \mathrm{supp}\psi \subset [-1, 1], \quad \psi|_{[-\frac{1}{2}, \frac{1}{2}]} \equiv 1,$$

and

$$|\psi^{(j)}(z)| \leq C_j, \quad \forall z \in \mathbb{R},\ j = 1, 2.$$

The sequence ψ_n has the following properties:

(IV) $\psi_n \to 1$ pointwise,

(V) $(x,v)^\alpha \partial^\beta_{(x,v)} \psi_n(x,v) = \frac{1}{n} \frac{(x,v)^\alpha (x,v)^\beta}{|(x,v)|} \psi'\left(\frac{|(x,v)|}{n}\right)$, with $\alpha, \beta \in \mathbb{N}_0^6$, $|\alpha| = |\beta| = 1$, are supported in the annulus

$$\operatorname{supp}\left(\psi'\left(\frac{|(x,v)|}{n}\right)\right) \;=\; \{(x,v) \mid n/2 \le |(x,v)| \le n\} \;=:\; V_n,$$

and they are in $L^\infty(\mathbb{R}^6)$, uniformly in $n \in \mathbb{N}$.

(VI) $n\partial^\alpha_{(x,v)}\psi_n(x,v) = \frac{(x,v)^\alpha}{|(x,v)|}\psi'\left(\frac{|(x,v)|}{n}\right)$, with $\alpha \in \mathbb{N}_0^6$, $|\alpha| = 1$,

are uniformly bounded in $L^\infty(\mathbb{R}^6)$.

(VII) $\partial^\alpha_{(x,v)}\psi_n(x,v) = \frac{(x,v)^\alpha}{n^2|(x,v)|^2}\psi''\left(\frac{|(x,v)|}{n}\right) + \left(\frac{1}{n^2|(x,v)|} - \frac{(x,v)^\alpha}{n^3|(x,v)|^3}\right)\psi'\left(\frac{|(x,v)|}{n}\right)$,

with $|\alpha| = 2$ have support on V_n and converge uniformly to 0 in $L^\infty(\mathbb{R}^6)$.

We now define the approximating sequence

$$f_n(x,v) := (f * \phi_n)(x,v) \cdot \psi_n(x,v), \quad n \in \mathbb{N},$$

where '$*$' denotes the convolution in x and v.
By construction we have $f_n \in C_0^\infty(\mathbb{R}^6) = D(P)$.

Since we can split our operator as

$$\begin{aligned} P \;&=\; \sum_{i=1}^{3}\left[\frac{\theta}{3} + \nu v_i\partial_{x_i} + \mu x_i\partial_{v_i} + \beta v_i\partial_{v_i} + \alpha\partial^2_{x_i} + \sigma\partial^2_{v_i} + \gamma\partial_{v_i}\partial_{x_i}\right] \\ &=\; \sum_{i=1}^{3}\tilde{p}(x_i, v_i, \partial_{x_i}, \partial_{v_i}), \end{aligned}$$

we shall in the sequel only consider

$$\tilde{P} \;=\; \tilde{p}(y, z, \partial_y, \partial_z), \quad y, z \in \mathbb{R}$$

acting in one spatial direction $y = x_j$ and one velocity direction $z = v_j$.
We have to prove that $f_n(x,v) \to f(x,v)$ in the graph norm

$$\|f\|_{\tilde{P}} = \|f\|_{L^2} + \||x|^2 f\|_{L^2} + \||v|^2 f\|_{L^2} + \|\tilde{P}f\|_{L^2} + \||x|^2\tilde{P}f\|_{L^2} + \||v|^2\tilde{P}f\|_{L^2}.$$

According to the 6 terms of the graph norm we split the proof into 6 steps:

Step 1: By applying (P1) and (P4), we have

$$f_n \to f \quad \text{in} \quad L^2(\mathbb{R}^6).$$

Step 2: For the second term of the graph norm we write

$$x_i^2 f_n = (x_i^2 f * \phi_n)\psi_n + 2(x_i f * x_i\phi_n)\psi_n + (f * x_i^2\phi_n)\psi_n.$$

The first summand converges to $x_i^2 f$ in $L^2(\mathbb{R}^6)$ and both the second and the third terms converge to 0 by (III), since also $x_i f$ belongs to $L^2(\mathbb{R}^6)$ by interpolation.

Step 3: For the third term of the graph norm the same argument as in previous step can be used. Hence we have

$$f_n \to f \qquad \text{in } X.$$

Step 4: To prove that $\tilde{P}f_n \to \tilde{P}f$ in $L^2(\mathbb{R}^6)$ we write:

$$\begin{aligned}
\tilde{P}f_n &= \frac{\theta}{3}(f*\phi_n)\psi_n + \nu(zf_y*\phi_n)\psi_n + \mu(yf_z*\phi_n)\psi_n + \beta(zf_z*\phi_n)\psi_n \\
&\quad +\alpha(f_{yy}*\phi_n)\psi_n + \sigma(f_{zz}*\phi_n)\psi_n + \gamma(f_{yz}*\phi_n)\psi_n + r_n^1(y,z) \\
&= (\tilde{P}f*\phi_n)\psi_n + r_n^1(y,z).
\end{aligned}$$

As we shall show, all thirteen terms of the remainder

$$\begin{aligned}
r_n^1 &= \nu(f*\partial_y(z\phi_n))\psi_n + \nu(f*\phi_n)z\partial_y\psi_n + \mu(f*y\partial_z\phi_n)\psi_n \\
&\quad +\mu(f*\phi_n)y\partial_z\psi_n + \beta(f*\partial_z(z\phi_n))\psi_n + \beta(f*\phi_n))z\partial_z\psi_n \\
&\quad +2\alpha(f*(\frac{1}{n}\partial_y\phi_n))(n\partial_y\psi_n) + \alpha(f*\phi_n))(\partial_y^2\psi_n) + 2\sigma(f*\frac{1}{n}\partial_z\phi_n)n\partial_z\psi_n \\
&\quad +\sigma(f*\phi_n)\partial_z^2\psi_n + \gamma(f*(\frac{1}{n}\partial_z\phi_n))(n\partial_y\psi_n) + \gamma(f*(\frac{1}{n}\partial_y\phi_n))(n\partial_z\psi_n) \\
&\quad +\gamma(f*\phi_n)\partial_y\partial_z\psi_n
\end{aligned}$$

converge to 0 in $L^2(\mathbb{R}^6)$.
The first, the third and the fifth terms converge to 0 in $L^2(\mathbb{R}^6)$ by (III).
In the second, fourth and the sixth terms, exploiting (V) we have

$$\|(f*\phi_n)(z\partial_y\psi_n)\|_{L^2(R^6)} \le C\|f*\phi_n - f\|_{L^2(V_n)} + \|f\|_{L^2(V_n)} \to 0, \tag{7.5}$$

because $\|f\|_{L^2(R^6)} = \|f\|_{L^2(B_{1/2}(0))} + \sum_{k=0}^{\infty}\|f\|_{L^2(V_{2^k})}$.
For what the seventh, ninth, eleventh and twelfth terms are concerned, we can exploit (VI) and then (II).
The remaining terms can be handled thanks to (VII).

Step 5: To prove that $|x|^2\tilde{P}f_n \to |x|^2Pf$ in $L^2(\mathbb{R}^6)$ we write:

$$\begin{aligned}
x_i^2\tilde{P}f_n &= \frac{\theta}{3}(x_i^2f*\phi_n)\psi_n + \nu(x_i^2zf_y*\phi_n)\psi_n + \mu(x_i^2yf_z*\phi_n)\psi_n + \beta(x_i^2zf_z*\phi_n)\psi_n \\
&\quad +\alpha(x_i^2f_{yy}*\phi_n)\psi_n + \sigma(x_i^2f_{zz}*\phi_n)\psi_n + \gamma(x_i^2f_{yz}*\phi_n)\psi_n + r_n^2(y,z) \\
&= (x_i^2\tilde{P}f*\phi_n)\psi_n + r_n^2(y,z).
\end{aligned}$$

The remainder r_n^2 can be split in the following way ($y = x_j$, $z = v_j$):

$$\begin{aligned}
r^2_{n,\theta} &= \frac{2}{3}\theta(x_i f * y\phi_n)\psi_n + \frac{\theta}{3}(f * x_i^2\phi_n)\psi_n \\
r^2_{n,\nu} &= 2\nu(zx_i f * \partial_y(x_i\phi_n))\psi_n - 2\nu\delta_{ij}(zf * x_i\phi_n)\psi_n + \nu(zf * \partial_y(x_i^2\phi_n))\psi_n \\
&\quad +\nu(x_i^2 f * z\partial_y\phi_n)\psi_n + 2\nu(x_i f * x_i z\partial_y\phi_n)\psi_n + \nu(f * x_i^2 z\partial_y\phi_n)\psi_n \\
&\quad +\nu(x_i^2 f * \phi_n)z\partial_y\psi_n + 2\nu(x_i f * x_i\phi_n)z\partial_y\psi_n + \nu(f * x_i^2\phi_n)z\partial_y\psi_n \\
r^2_{n,\mu} &= 2\mu(x_i y f * \partial_z(x_i\phi_n))\psi_n + \mu(yf * \partial_z(x_i^2\phi_n))\psi_n + \mu(x_i^2 f * y\partial_z\phi_n)\psi_n \\
&\quad +2\mu(x_i f * x_i y\partial_z\phi_n)\psi_n + \mu(f * x_i^2 y\partial_z\phi_n)\psi_n + \mu(x_i^2 f * \phi_n)y\partial_z\psi_n \\
&\quad +2\mu(x_i f * x_i\phi_n)y\partial_z\psi_n + \mu(f * x_i^2\phi_n)y\partial_z\psi_n \\
r^2_{n,\beta} &= 2\beta(x_i z f * x_i\partial_z\phi_n)\psi_n - 2\beta(x_i f * x_i\phi_n)\psi_n + \beta(zf * x_i^2\partial_z\phi_n)\psi_n - \beta(f * x_i^2\phi_n)\psi_n \\
&\quad +\beta(x_i^2 f * \partial_z(z\phi_n))\psi_n + 2\beta(x_i f * x_i\partial_z(z\phi_n))\psi_n + \beta(f * x_i^2\partial_z(z\phi_n))\psi_n \\
&\quad +\beta(x_i^2 f * \phi_n)z\partial_z\psi_n + 2\beta(x_i f * x_i\phi_n)z\partial_z\psi_n + \beta(f * x_i^2\phi_n)z\partial_z\psi_n \\
r^2_{n,\alpha} &= 2\alpha(x_i f * \partial_y^2(x_i\phi_n))\psi_n - 4\alpha\delta_{ij}(f * \partial_y(x_i\phi_n))\psi_n + \alpha(f * \partial_y^2(x_i^2\phi_n))\psi_n \\
&\quad +2\alpha(x_i^2 f * \frac{1}{n}\partial_y\phi_n)n\partial_y\psi_n + 4\alpha(x_i f * \frac{x_i}{n}\partial_y\phi_n)n\partial_y\psi_n + 2\alpha(f * \frac{x_i^2}{n}\partial_y\phi_n)n\partial_y\psi_n \\
&\quad +\alpha(x_i^2 f * \phi_n)\partial_y^2\psi_n + 2\alpha(x_i f * x_i\phi_n)\partial_y^2\psi_n + \alpha(f * x_i^2\phi_n)\partial_y^2\psi_n \\
r^2_{n,\sigma} &= 2\sigma(x_i f * x_i\partial_z^2\phi_n)\psi_n + \sigma(f * x_i^2\partial_z^2\phi_n)\psi_n + 2\sigma(x_i^2 f * \frac{1}{n}\partial_z\phi_n)n\partial_z\psi_n \\
&\quad +4\sigma(x_i f * \frac{x_i}{n}\partial_z\phi_n)n\partial_z\psi_n + 2\sigma(f * \frac{x_i^2}{n}\partial_z\phi_n)n\partial_z\psi_n + \sigma(x_i^2 f * \phi_n)\partial_z^2\psi_n \\
&\quad +2\sigma(x_i f * x_i\phi_n)\partial_z^2\psi_n + \sigma(f * x_i^2\phi_n)\partial_z^2\psi_n \\
r^2_{n,\gamma} &= 2\gamma(x_i f * \partial_y(x_i\partial_z\phi_n))\psi_n - 2\gamma\delta_{ij}(f * x_i\partial_z\phi_n)\psi_n + \gamma(f * \partial_y\partial_z(x_i^2\phi_n))\psi_n \\
&\quad +\gamma(x_i^2 f * \frac{1}{n}\partial_z\phi_n)n\partial_y\psi_n + 2\gamma(x_i f * \frac{x_i}{n}\partial_z\phi_n)n\partial_y\psi_n + \gamma(f * \frac{x_i^2}{n}\partial_z\phi_n)n\partial_y\psi_n \\
&\quad +\gamma(x_i^2 f * \frac{1}{n}\partial_y\phi_n)n\partial_z\psi_n + 2\gamma(x_i f * \frac{x_i}{n}\partial_y\phi_n)n\partial_z\psi_n + \gamma(f * \frac{x_i^2}{n}\partial_y\phi_n)n\partial_z\psi_n \\
&\quad +\gamma(x_i^2 f * \phi_n)\partial_y\partial_z\psi_n + 2\gamma(x_i f * x_i\phi_n)\partial_y\partial_z\psi_n + \gamma(f * x_i^2\phi_n)\partial_y\partial_z\psi_n.
\end{aligned}$$

By the properties (I)-(VII) and estimate like (7.5), it can be easily seen that each term converges to 0 in $L^2(\mathbb{R}^6)$.

Step 6: In analogy to $|x|^2\tilde{P}f_n$, the sequence $|v|^2\tilde{P}f$ can be split as

$$v_i^2\tilde{P}f_n = (v_i^2\tilde{P}f * \phi_n)\psi_n + r_n^3(y,z).$$

Due to the symmetry of the operator $\tilde{P}$ in x and v, the terms of the remainder r_n^3 can be obtained from r_n^2 by interchanging y and z (and changing the coefficients), except for the following term

$$v_i^2[\beta z\partial_z\left((f * \phi_n)\psi_n\right)] \;=\; \beta(v_i^2 z f_z * \phi_n)\psi_n + r^3_{n,\beta},$$

where

$$\begin{aligned}
r^3_{n,\beta} &= 2\beta(v_i z f * \partial_z(v_i\phi_n))\psi_n - 2\beta(1+\delta_{ij})(v_i f * v_i\phi_n)\psi_n + \beta(zf * \partial_z(v_i^2\phi_n))\psi_n \\
&\quad -\beta(f * v_i^2\phi_n)\psi_n + \beta(v_i^2 f * \partial_z(z\phi_n))\psi_n + 2\beta(v_i f * v_i\partial_z(z\phi_n))\psi_n \\
&\quad +\beta(f * v_i^2\partial_z(z\phi_n))\psi_n + \beta(v_i^2 f * \phi_n)z\partial_z\psi_n + 2\beta(v_i f * v_i\phi_n)z\partial_z\psi_n \\
&\quad +\beta(f * v_i^2\phi_n)z\partial_z\psi_n
\end{aligned}$$

converges to 0 in $L^2(\mathbb{R}^6)$, since (I)-(VII) and (7.5) can be used.

Proof of Proposition 7.3.1

First, we shall prove the following estimates on the derivatives of the Green's function (7.21):

$$
\begin{aligned}
|\nabla_v G(t,x,v,x_0,v_0)| &\leq b\frac{G(t,\frac{x}{2},\frac{v}{2},\frac{x_0}{2},\frac{v_0}{2})}{\sqrt{t}}, \quad \forall\, t \leq t_0, && (7.6)\\
|\nabla_x G(t,x,v,x_0,v_0)| &\leq b'\frac{G(t,\frac{x}{2},\frac{v}{2},\frac{x_0}{2},\frac{v_0}{2})}{\sqrt{t}}, \quad \forall\, t \leq t_1. && (7.7)
\end{aligned}
$$

with $b = b(\alpha,\gamma,\sigma), t_0 = t_0(\alpha,\beta,\sigma,\gamma), b' = b'(\alpha,\gamma,\sigma)$ and $t_1 = t_1(\alpha,\beta,\sigma,\gamma)$. The v-derivative of G is given by

$$
\begin{aligned}
\nabla_v G(t,x,v,x_0,v_0) &= G(t,x,v,x_0,v_0)\Bigg[-\frac{\left(\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right)(x - \frac{e^{\beta t}-1}{\beta}v - x_0)}{f(t)}\\
&\quad - \frac{\left(2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right)(e^{\beta t}v - v_0)}{f(t)}\Bigg]. \qquad (7.8)
\end{aligned}
$$

For all real $a,b,c > 0$ such that $c/\sqrt{a} \leq b\sqrt{2e}$, one easily verifies that

$$
c|x| \leq b e^{a|x|^2}, \quad \forall\, x \in \mathbb{R}^3. \qquad (7.9)
$$

Since $\alpha,\sigma > 0$, we have for $t > 0$ small enough

$$
\nu(t) - \frac{1}{2}\mu(t) > 0, \qquad \lambda(t) - \frac{1}{2}\mu(t) > 0.
$$

In order to apply the estimate (7.9) to the two terms inside the squared bracket in (7.8) we shall use for t small:

$$
\frac{c_1}{\sqrt{a_1}} := \frac{\frac{\sqrt{t}}{f(t)}\left|\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right|}{\sqrt{\frac{3}{4}\frac{\nu(t)-\frac{1}{2}\mu(t)}{f(t)}}} \sim \frac{2\gamma}{\sqrt{3(\alpha\sigma-\gamma^2)(\sigma+\gamma)}} \leq b_1\sqrt{2e},
$$

with $b_1 = \gamma/\sqrt{3(\alpha\sigma-\gamma^2)(\sigma+\gamma)}$. Similarly,

$$
\frac{c_2}{\sqrt{a_2}} := \frac{\frac{\sqrt{t}}{f(t)}\left|2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right|}{\sqrt{\frac{3}{4}\frac{\lambda(t)-\frac{1}{2}\mu(t)}{f(t)}}} \sim \frac{2\alpha}{\sqrt{3(\alpha\sigma-\gamma^2)(\alpha+\gamma)}} \leq b_2\sqrt{2e},
$$

with $b_2 = \alpha/\sqrt{3(\alpha\sigma-\gamma^2)(\alpha+\gamma)}$. Then, there exists some $t_0 > 0$ such that, for all $t \leq t_0$, the two inequalities can be combined with $b = \max\{b_1,b_2\}$ to give

$$
\left|\frac{\left(\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right)\left(x - \frac{e^{\beta t}-1}{\beta}v - x_0\right) + \left(2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right)(e^{\beta t}v - v_0)}{f(t)}\right|\sqrt{t} \leq
$$

$$\le \frac{\sqrt{t}}{f(t)}\left\{\left|\mu(t)e^{\beta t}-2\nu(t)\frac{e^{\beta t}-1}{\beta}\right|\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|+\left|2\lambda(t)e^{\beta t}-\mu(t)\frac{e^{\beta t}-1}{\beta}\right||e^{\beta t}v-v_0|\right\}$$

$$\le b\exp\left\{\frac{\left(\nu(t)-\frac{1}{2}\mu(t)\right)\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|^2+\left(\lambda(t)-\frac{1}{2}\mu(t)\right)\left|e^{\beta t}v-v_0\right|^2}{\frac{4}{3}f(t)}\right\}$$

$$\le b\exp\left\{\frac{\nu(t)\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|^2+\lambda(t)\left|e^{\beta t}v-v_0\right|^2+\mu(t)\left(x-\frac{e^{\beta t}-1}{\beta}v-x_0\right)\cdot\left(e^{\beta t}v-v_0\right)}{\frac{4}{3}f(t)}\right\}.$$

Hence,

$$|\nabla_v G(t,x,v,x_0,v_0)| \;\le\; b\frac{G(t,x,v,x_0,v_0)}{\sqrt{t}}$$

$$\times\exp\left\{\frac{3}{4}\frac{\nu(t)\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|^2+\lambda(t)\left|e^{\beta t}v-v_0\right|^2+\mu(t)\left(x-\frac{e^{\beta t}-1}{\beta}v-x_0\right)\cdot\left(e^{\beta t}v-v_0\right)}{f(t)}\right\}$$

and the decay (7.6) follows by comparison with (7.21).

Next we consider the x-derivative of the Green's function,

$$\nabla_x G(t,x,v,x_0,v_0) \;=\; G(t,x,v,x_0,v_0)\left[-\frac{2\nu(t)(x-\frac{(e^{\beta t}-1)}{\beta}v-x_0)+\mu(t)(e^{\beta t}v-v_0)}{f(t)}\right].$$

Analogously, the decay (7.7) follows by exploiting that for t small enough

$$\frac{\frac{\sqrt{t}}{f(t)}2\nu(t)}{\sqrt{\frac{3}{4}\frac{\nu(t)-\frac{1}{2}\mu(t)}{f(t)}}} \;\sim\; \frac{2\sigma}{\sqrt{3(\alpha\sigma-\gamma^2)(\sigma+\gamma)}} \;\le\; b_1'\sqrt{2e},$$

$$\frac{\frac{\sqrt{t}}{f(t)}|\mu(t)|}{\sqrt{\frac{3}{4}\frac{\lambda(t)-\frac{1}{2}\mu(t)}{f(t)}}} \;\sim\; \frac{2\gamma}{\sqrt{3(\alpha\sigma-\gamma^2)(\alpha+\gamma)}} \;\le\; b_2'\sqrt{2e},$$

with appropriate $b_1'(\alpha,\gamma,\sigma), b_2'(\alpha,\gamma,\sigma)$.

Since

$$e^{t\overline{A}}w_0(x,v) \;=\; \iint G(t,x,v,x_0,v_0)w_0(x_0,v_0)\,dx_0\,dv_0,$$

we have

$$\begin{aligned}|\nabla_v e^{t\overline{A}}w_0(x,v)| &\le \iint |\nabla_v G(t,x,v,x_0,v_0)|\,|w_0(x_0,v_0)|\,dx_0\,dv_0\\ &\le bt^{-1/2}\iint G\left(t,\frac{x}{2},\frac{v}{2},\frac{x_0}{2},\frac{v_0}{2}\right)|w_0(x_0,v_0)|\,dx_0\,dv_0\\ &= 64b\,t^{-1/2}\iint G(t,\tilde{x},\tilde{v},\tilde{x_0},\tilde{v_0})|w_0(2\tilde{x_0},2\tilde{v_0})|\,d\tilde{x_0}\,d\tilde{v_0}\\ &= 64b\,t^{-1/2}e^{t\overline{A}}\tilde{w}_0(\tilde{x},\tilde{v}),\quad \forall\, t\le t_0. \end{aligned}\tag{7.10}$$

Here we used the decay (7.6), and we put $\tilde{x} = \frac{x}{2}$, $\tilde{v} = \frac{v}{2}$ and $\tilde{w}_0(\tilde{x}, \tilde{v}) = |w_0(2\tilde{x}, 2\tilde{v})|$. The assertion (7.22) follows directly by applying the estimate (7.19) to (7.10) and choosing $T_0 = \min\{t_0, t_1\}$.
The estimate (7.23) can be obtained analogously.

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