SCUOLA NORMALE SUPERIORE

PISA

PERFEZIONAMENTO IN MATEMATICA FINANZIARIA E ATTUARIALE

TESI DI PERFEZIONAMENTO

The impact of contagion on large portfolios. Modeling aspects.

Perfezionando: Marco Tolotti

Relatori: Ch.mo Prof. Paolo Dai Pra
Ch.mo Prof. Wolfgang J. Runggaldier

A Maria
e Federico

Acknowledgments

I must first express my deepest gratitude towards Paolo Dai Pra and Wolfgang Runggaldier for their precious help and their courtesy in reading countless versions of this thesis. Their suggestions and guidance have always encouraged me throughout the study.

A very special thanks goes out to Elena Sartori. I cannot forget the beautiful hours spent discussing about the generators of the Markov processes.

I am very grateful for the valuable comments of Rama Cont and Rüdiger Frey.

I wish to thank Maurizio Pratelli for his patience and kindness, the *Amici della Scuola Normale* and the Scuola Normale Superiore for the financial support.

I was delighted to interact with the Finance group at the Department of Mathematics of the ETH Zurich and the Swiss Banking Institute of the University of Zurich during the *Master of Advanced Studies in Finance*. I would like to thank in particular Philipp Schönbucher for giving me the opportunity to work with him for the Master thesis.

My gratitude goes also to Fulvio Ortu for his trust in offering me a position at the Bocconi University. I must acknowledge the (former) Istituto di Metodi Quantitativi and the Department of Finance at the Bocconi university for the financial support.

Anna Battauz, Elena Catanese, Francesca Beccacece, Giuliana Bordigoni, Marzia De Donno, Claudio Tebaldi, Fabio Maccheroni, Francesco Corielli, Gianluca Fusai and Gino Favero have been supporting me with inspiring discussions and their friendship.

... no words to express how I am grateful to Maria and my parents.

Milano, June 2008

Contents

Introduction

This Ph.D. thesis is part of a research project originated within the Probability group of the Mathematics department of the University of Padova in the year 2005. This project involves the Professors Paolo Dai Pra and Wolfgang J. Runggaldier, a Ph.D. student of the University of Padova, Elena Sartori and Marco Tolotti of the Bocconi University and the Scuola Normale Superiore.

The goal of this project is to integrate techniques and skills coming from different disciplines (Probability, Statistical Mechanics, Finance, Econometrics, Physics) in order to describe, analyze and quantify phenomena related to current issues in the Finance literature.

Financial motivations

The crucial issue that has motivated in particular this thesis is to describe a mathematical framework within which it is possible to explain the financial phenomenon referred to *clustering of defaults* or *credit crisis*.
By clustering of default we mean the situation in which many obligors experience *financial distress* (default or downgrading in a rating system) in a short time period. What we mean by *"many"* defaults in a *"short"* time will be discussed later. It is clear that in order to speak of a credit crisis there must be an unexpected breakdown from the standard economic business cycle. In some sense we have in mind a sudden change in the equilibrium of the credit market.

"Financial distress", "default", "downgrading" are all issues belonging to the field of *credit risk management*.
Managing the risk concerns the identification and the analysis of the randomness intrinsic in the financial world and in particular the capacity to predict and quantify the losses triggered by changes in the variables describing the financial market.
More specifically, when speaking of credit risk, one is dealing with the risks connected with the possible changes in the *credit worthiness* of the obligors.

We shall concentrate, in particular, on the losses related to *large portfolios*, meaning to portfolios of many obligors with similar characteristics. The precise meaning of words such as *"large"* and *"similar"* shall be extensively discussed. When dealing with many obligors, the issue of modeling the *dependence structure* plays a major role.

Our idea is that a credit crisis could be explained as the effect of a *contagion* process. A firm experiencing financial distress may affect the credit quality of business

partners (via direct contagion) as well as of firms in the same sector (due to an information effect). Therefore the mechanism of credit contagion is the crucial mechanism that we are going to develop in order to describe clustering of defaults.
Reduced form models for direct contagion can be found -among others- in Jarrow and Yu (2001) [47] for counterparty risk, Davis and Lo (2001) [22] for infectious default, Kiyotaki and Moore (1987) [48], where a model of credit chain obligations leading to default cascade is considered, Horst (2006) [46] for domino effects, and Giesecke and Weber (2005) [40] for a particle system approach. Concerning the banking sector, a micro-economic liquidity equilibrium is analyzed by Allen and Gale (2000) [1]. Recent papers on *information driven default models* are e.g. Schönbucher (2003) [59], Duffie et al. (2006) [28], Collin-Dufresne et al. (2003) [12].

An important point that we would like to stress is that we are aiming at describing the formation of a credit crisis starting from the influences that the single obligors have among each others, in other words as a *microeconomic* phenomenon.
The standard literature suggests that the aggregate *financial health of the system* is fully described by some *macroeconomic factors* that capture the business cycle and then influence the credit quality of the obligors. These factors are usually *exogenously* specified. One consequence of our *"micro"* point of view is that in our modeling framework a global health indicator is *endogenously* computed and not a priori assigned. Notice that the same philosophy is already standard practice in other disciplines: similar models and techniques as the ones developed in this thesis are applied for instance by Brock and Durlauf in (2001) [8] for modeling social interactions or by Cont and Löwe (1998) [15] in order to describe phenomena as *herding behavior* or *peer pressure*.

The last remark is about a new trend in the literature concerning portfolio credit risk. In the last years *portfolio credit derivatives* such as default basket and Collateralized Debt Obligations (CDO's) have become very popular. In order to treat this kind of structured derivatives a dynamic study of the aggregate losses $L(t)$ caused by the defaults in the underlying pool of assets becomes crucial. It has been documented that an effective study of the dynamics of the aggregate losses $L(t)$ may not necessarily require a full understanding of the single name processes related to the underlying assets. For this reason a new approach has been recently proposed in the literature: the so called *top-down* approach (for more details see Cont and Minca (2008) [16], Giesecke and Goldberg (2007) [39] and Schönbucher (2006) [60]).
When considering Markov chain models similar in spirit to the ones proposed in this work, it is possible -under certain hypothesis- to fully explain the evolution of the system via *aggregate sufficient statistics*. As argued also in Frey and Backhaus (2007) [36], this is exactly the philosophy behind a top-down model. In particular we shall see in Chapter 3 (see Remark 3.5.8) that our approach may be considered as a useful tool also under this new perspective since it naturally exploits the problem of computing approximations of $L(t)$ in a parsimonious way as a function of *aggregate asymptotic variables* that may account for the heterogeneity in the underlying portfolio.

We have thus identified the financial core of this discussion: quantifying the losses connected to the deterioration of credit quality, taking the contagion into account and eventually describing under which market conditions a credit crisis may take place.

Technical aspects

From a more technical point of view there are different tasks that we have to deal with; we briefly illustrate the most important ones:

- Having large portfolios in mind, we are going to describe asymptotic results for a system of *"infinitely many"* firms and then provide finite volume approximations. It basically implies the development of suitable *Laws of Large Numbers* (LLN) and *Central Limit Theorems* (CLT).
  The implementation of LLN and CLT for the study of large portfolios is not new in the risk management literature; for example it is implemented (in a rather basic setup, without considering contagion) in the Basel II accord. More recent generalizations to contagion models have been proposed -among others- by Frey and Backhaus (2006) [36] and by Giesecke and Weber (2005) [40].
  A different approach to the study of large portfolio losses may concern the extreme events (analyzing the tails of the loss distributions). In this case, it is quite common to rely on *Large Deviation* techniques. For a recent survey on these techniques, applied to Finance and risk management, see Pham (2007) [56]. In Dembo et al. (2006) [23] an application to large portfolio losses is proposed.

- The issue of contagion may be described relying on *interacting particle systems* borrowed from Statistical Mechanics. Notice that also in the literature of quantitative risk management the terminology *interacting intensities* is used in order to describe reduced form models where interaction is taken into account. The use of particle systems is rather common in the Social Sciences literature, for instance when modeling social interactions and aggregate behaviors (see the paper by Cont and Bouchaud (2000) [5] for a discussion on herding behavior in Finance). Particle and dynamical systems can be found also in the literature on financial market modeling. It has been shown that some of these models have "thermodynamic limits" that exhibit similar features compared to the limiting distributions (in particular when looking at the tails) of market returns time series. For a discussion on financial market modeling see the survey by Cont (1999) [14] and the papers by Föllmer (1994) [33] and Föllmer et al. (2004) [34] that contain inspiring discussions on interacting agents.

- In order to derive a LLN and a CLT for a particle system we rely on *Large Deviation* techniques. The idea is to consider $\mathcal{M}_1$ the space of probability measures on trajectories endowed with the Skorohod topology. It is quite easy to state a Large Deviation Principle (LDP) for a reference system where there is no interaction between the firms. Then the goal is to find a suitable function $F : \mathcal{M}_1 \to \mathbb{R}^+$ that relates (via Varadhan's lemma) the LDP for the reference system with our interactive model for contagion. This technique has been applied to spin-flip systems by Dai Pra and Den Hollander (1996) [18].
  Concerning the central limit theorem we shall develop two different approaches. The first one (based on large deviations) relies on a theory by Bolthausen (1986) [4]. The second one is based on a weak convergence-type approach based on the uniform convergence of the generators of the associated Markov chains (as developed in the book by Ethier and Kurtz (1996) [32]).

- We are aiming at building a dynamic model. This has to be done in order to describe the time evolution of the variables describing the credit quality of the obligors, and hence (under particular conditions) the formation of a credit crisis.
  We shall propose a model where the credit crisis is connected with the existence of multiple equilibria for the dynamical system. The system may in fact spend some time near an unstable configuration and then suddenly decay to a stable one. We shall see that this effect is related to the *phase transition*, i.e., on the level of interaction in the model.

It is finally worth to spend some words on the *point of view* that we have adopted in this research project (at least in its first step). We have focused our attention on the modeling aspects, trying to build a model as simple as possible where the effects of contagion (as the credit crises) could have been observed. Moreover we have looked for a completely solvable model, where closed form solutions can be provided. Put differently, we have given more relevance to the technical part of the problem and the modeling aspects.
On the other hand we have not developed the validation part of the model, i.e., the calibration and the analysis of real data. Nevertheless many numerical simulations are provided in order to illustrate the shape of the loss distributions and the trajectories of the health indicators under different market conditions.
Although this observation could be seen as a drawback of our work, we would like to argue that what we are aiming at are the *qualitative* aspects more than the *quantitative* ones (at least in the fist step of the research). Indeed, the models we are going to propose are very basic and have to be considered as a starting point for the construction of more realistic models that may better fit real data.

Structure of the thesis and main results

The thesis is divided into five chapters and one appendix.

Chapter 1 and **Chapter 2** are devoted to the introduction of the main concepts and the basic tools for managing credit risk.
In writing this introductive part we had two goals in mind: firstly we aimed at letting this work be -as much as possible- self consistent. On the other hand we have tried to report and briefly discuss the main approaches and models used in the literature. We have focused in particular on those models that can be considered as the starting point for our research.
The guideline to these two chapters may be summarized in the three following fundamental questions:

How can we model the credit quality of one obligor? How can we determine her default probability and the possible losses that the event of default may trigger?

Given a set of $N \in \mathbb{N}$ different obligors, how can dependence be modeled and eventually joint default probabilities computed?

How can we model credit contagion? Is it possible to build a model that explains clustering of defaults (credit crises)?

These questions shall be picked up again in more detail in the two introductive chapters and should find appropriate answers in the further exposition. Notice that they go from the very basic concepts up to the issues that have motivated our research. Hence these questions (and the two chapters themselves) are intended to build a bridge between the existing literature and our point of view.

The succeeding two chapters contain the mathematical implementation and the discussion of our two original models, developed in order to tackle the problem of contagion in a credit risk perspective.

In **Chapter 3** we propose in particular a first attempt to model an interactive system of defaultable counterparties where a *local random environment* enters into play.

We shall see that a crucial object of this work is the so called *empirical measure*. Indeed, suppose that N firms are acting in a market and their default indicators $\underline{\sigma}(t) = (\sigma_i(t);\ i = 1, \ldots, N)$, where $\sigma_i(t) \in \{-1, 1\}$, evolve in time. We denote by $\sigma[0,T] \in \mathcal{D}[0,T]$ a trajectory on the interval $[0,T]$ of such a rating indicator. $\mathcal{D}[0,T]$ denotes the space of càdlàg functions endowed with the Skorohood topology. The empirical measure ρ_N is defined as follows

$$\rho_N(\underline{\sigma}[0,T]) = \frac{1}{N}\sum_{i=1}^{N} \delta_{(\sigma_i[0,T])}.$$

It is a random measure taking values in $\mathcal{M}_1(\mathcal{D}[0,T])$, the space of probabilities on $\mathcal{D}[0,T]$. ρ_N weights the realizations of the N dimensional process $\underline{\sigma}[0,T]$. Put differently the empirical measure can be thought of the physical (historical) measure of the market.
Most of the mathematical results of this thesis are concerned with the sequence of measures $(\rho_N)_N$. In this chapter we state in particular a large deviation principle for this sequence of measures (Theorem 3.3.3) and then we prove a suitable law of large numbers (Theorem 3.3.6), finding a unique Q_* such that $\rho_N \to Q_*$ almost surely.
Significant and very useful for applications is also the characterization of Q_* provided in Proposition 3.3.5.
In Section 3.5 we finally state and prove a functional central limit theorem characterizing the fluctuations of ρ_N around Q_* (Theorem 3.5.6). The proof of this theorem is inspired by the seminal work of E. Bolthausen (see [4]). In [4] a rather general framework is proposed in order to derive central limit theorems in Banach spaces. Since $\mathcal{M}_1$ is not a Banach space we are forced to construct an auxiliary space where a suitable large deviation principle is inherited and a central limit theorem can be proved accordingly. This procedure is summarized in Theorems 3.5.4 and 3.5.5.

Although this first model shows some interesting features and -to some extent-generalizes the present literature on dynamic mean-field models for large portfolio losses, it turns out to be not yet comprehensive enough in order to explain clustering and credit crises.

Chapter 4 is then devoted to the analysis of a new framework that makes possible the explicit identification of the desired clustering effect.
To this aim we introduce a *fundamental indicator of robustness* $\omega_i \in \{-1, 1\}$ that is coupled with σ_i defined before. The $2N$ state variables $(\underline{\sigma}, \underline{\omega})$ evolve in time and their dynamics turn out to be non trivial at all. In particular our model is *non-reversible*. As compared to similar but reversible stochastic interacting systems, more careful arguments have to be used in order to prove the large deviation principle, which represents the basic tool in our approach.
Similarly to Chapter 3, the first main result is a Law of Large Numbers (Theorem 4.3.2) based on a Large Deviation Principle (Proposition 4.3.4).
In Theorem 4.3.11 we shall see that different asymptotic configurations can be found, depending on the values of the parameters. This phenomenon (called *phase transition*) has implications for the description of a credit crisis as we shall explain in more details in Chapter 5.
The last section of this chapter is devoted to the study of the fluctuations of the empirical measure around its limit. Two different approaches are described, the former is based on uniform convergence of generators (Theorem 4.4.1). The latter (Theorem 4.4.5) mimics the functional approach already introduced in the previous chapter.

One remark is needed at this point. Part of these rather technical proofs have been pursued in collaboration with Paolo Dai Pra and Elena Sartori. We shall refer to the Ph.D. thesis of Elena where some explicit computations have not been reported in this dissertation.
To make the exposition less heavy we have postponed to **Appendix A** the most technical proofs of Chapters 3 and 4.

The financial applications are discussed in **Chapter 5**.

The main result of this chapter is Theorem 5.0.8. It is concerned with the computation of risk measures for managing the risk involved in large portfolios. Various examples are provided, some of them have been suggested by the existing literature on the subject.

The second issue that we are going to analyze in this chapter is related to the formation of a credit crisis. To this aim the equilibria of the limiting dynamics and their stability are studied and the phase transition is fully characterized. At the end of the chapter, different graphs and numerical implementations are presented in order to support the financial interpretation of our model.

Finally we conclude this thesis with a brief summary of the main results and mentioning some open problems and possible lines of future research in this area.

Chapter 1

Credit quality and credit risk

The goal of the first two chapters of this thesis is to equip the reader with the basic notations and concepts necessary to enter the world of credit risk. First of all we would like to specify what we intend for *"risk"* in the context of credit and propose a mathematical framework in which a punctual quantitative analysis can be pursued.

A rather concise definition of *risk* in finance could be as follows: we speak about risk when we consider the possibility of having *unexpected* changes in the variables that describe the financial model.
We thus have to define a suitable probability space $\{\Omega, \mathcal{F}, P\}$ that summarizes the states of the world, the interesting events and a possible probability measure on them. We may possibly add a filtration $(\mathcal{F}_t)_{t\geq 0}$ that describes the flow of information when considering dynamic models. Finally we define random variables (eventually processes) $X : \Omega \to \mathbb{R}$, representing the financial variables that we are interested in.
In the context of credit risk, X should basically describe the *credit quality* of a given obligor, where an obligor is somebody who has to pay back a debt to somebody else in the future. For *credit quality* we mean the ability of being able to pay back obligations.

How can we model the credit quality of one obligor? How can we determine her default probability and the possible losses that the event of default may trigger?

Suppose that this obligor is a firm; in this case X could represents the value of the firm at a given time. Thus we could be interested in estimating the probability that X falls below a certain value (also named a threshold level) and this could be a good choice for a credit quality indicator. In an even naiver world, X could simply be an indicator of default (bankruptcy): if $X = 0$ the firm is able to pay back its obligation, if $X = 1$ it is not.
All these issues will be developed and specified in the following chapters. In particular many different models with different specifications for X will be provided as well as different specifications for $\{\Omega, \mathcal{F}, P\}$ and $(\mathcal{F}_t)_{t\geq 0}$.
We would like to stress the fact that the majority of the concepts we are going to state in the first two chapters are not new and can be found in different books dealing with credit risk. Our purpose is to give a glance on the main aspects, focusing the attention on the basic *building blocks* necessary to assess and quantify the riskiness of a portfolio of obligors and to price credit derivatives. We shall describe the existing techniques used to compute these building blocks and the relevance of particular

assumptions often used to provide explicit formulae. In doing this we shall build the bridge between existing literature and our modeling ideas, leading the reader to the new framework that we have introduced in order to solve some open problems that are still debated in the present literature.

1.1 Default probability and loss given default

In this section we briefly recall the very basic tools used to deal with credit risk. Our aim is to focus the attention on the so called *building blocks*, i.e., the basic objects on which the risk measurement and the pricing techniques of large part of more complicated securities rely on. These objects are the so called *default probability* and the *loss given default*.

In a credit risk environment the basic concept is *default*. We do not enter into the discussion of the bankruptcy procedures. We simply say that a default time τ has to be defined. In the next section we shall discuss on how τ is characterized within different models. Our concern now is simply that τ is a random time and in particular a *stopping time* with respect to its own filtration $(\mathcal{H}_t)_{t\geq 0}$[1].

Default probability (PD)

We define the *default probability* between t and T for the obligor as

$$p(t,T) = P(\tau \leq T | \tau > t). \tag{1.1}$$

It is worth to define also the conditional version for the default probabilities meaning the probability at time t of having a default between future dates T_1 and T_2 knowing survival up to T_1:

$$p(t,T_1,T_2) = \frac{p(t,T_2)}{p(t,T_1)}, \quad 0 \leq t \leq T_1 \leq T_2.$$

These quantities are then used in order to determine an important object often referred to the *hazard rate*. We define the hazard rate between t and T as

$$h(t,T) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} p(t,T,T+\Delta t)$$

whenever this limit exists.

Remark 1.1.1 *It should be stressed that in more general models (for example if the interest rate r is stochastic or if other random variables influence the market prices) one has to be careful with the definition of hazard rates. In particular it is very important to study measurability conditions (namely which filtration or information structure are available in the model) and how the total information is related to the filtration generated by the default process (see chapters 5,6 in [3] for a punctual discussion). In particular it can be shown (see for instance Section 6.2 in [3]) that for a random time τ one may consider two different hazard processes (both well defined and mathematically meaningful) and they may not coincide if some regularity hypotheses are not made on the model.*

[1] How $\mathcal{F}$ and $\mathcal{H}$ are related is a crucial point. An extensive discussion is made in [3]. We do not enter into details at this point.

Formally we can also define a *local hazard rate*:

$$\gamma(t) := h(t,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} p(\tau \leq t + \Delta t | \tau > t).$$

Loss given default (LGD)

Suppose that a default happens. Suppose moreover that a bank holds a position issued by the defaulting obligor. At the time of default $\tau < T$ part of the investment of the bank is lost due to the impossibility of the obligor of paying back obligations. The part that the bank (or more general an investor) cannot recover is called *loss given default*. This quantity is often modeled as a random variable $l = \delta e$ where $0 \leq \delta \leq 1$ represents a random proportion of the exposure that is lost and e the actual exposure at default.

A rather general discussion on pricing issues of defaultable claims is also proposed in [3]. There, it is shown that (PD) and (LGD) are the building blocks, necessary to price many types of derivatives such as defaultable fixed-coupon bonds, credit default swaps (and many variants of them), asset swap packages.

1.2 Different approaches to credit risk modeling

Up to now we have considered a given probability space $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P\}$ endowed with a market filtration. On this filtered probability space we have defined a default time τ without taking care of the economic process leading to this definition and without specifying any distributional assumption on it (except for some assumptions on the corresponding filtration).

We now have to implement models that may help in computing the building blocks seen in the previous section. In the literature two typical and rather different point of views are developed.

The first class of models, the so called *structural models* (or firm value models), relies on the precise definition of some economical variables (such as the *"asset value process"* for a firm). The evolution of these variables determine the credit quality of the obligor itself.

The second class of models, the *reduced form models* (or intensity based models), is based on a more *"statistical"* point of view. The idea here is that -having defined a suitable family of processes (without direct economical interpretation)- one tries to calibrate the parameters in order to fit historical time series or other data.

These two approaches lead of course to different modeling frameworks. Both have important advantages: the former gives more direct intuition on what is going on economically; the latter is usually easier to implement and allows for more freedom in modeling parameters and in fitting data[2].

[2] We would like to stress the fact that the "dilemma" on what the best "philosophy" should be, is far from being solved. Moreover, notice that it involves the fundamental debate whether when using random variables one *is* or *is not* allowed to "forget" about the underlying probability space (the underlying "experiment") taking into account only distributional consequences (the numerical results of the experiment).

A. Structural models

The basic idea behind structural models is to consider the evolution in time of an underlying process X that is related to some fundamental indicators of the firm. Then, usually, the default happens when this process hits a predetermined (possibly stochastic) barrier D. So that τ is defined as a first passage stopping time, indeed

$$\tau = \inf_{t \geq 0}\{X_t \leq D\}. \tag{1.2}$$

In the progenitor of these models, due to Merton (1974) [53], the underlying process was the firm asset value process V_t. In particular it is assumed that $V_t = S_t + B_t$ where S_t is the value at time t of the equity and B_t is the value at time t of a zero coupon bond with face value B and maturity T.
In this basic context where the payments to the bond-holders are due at a fixed time T, the event of default is even easier to describe, than in Equation (1.2). We have in fact that default happens if and only if $V_T < D$ so that $\tau = T$. Moreover the recovery at default is simply $B_T = \min(V_T, B)$.
Suppose that V_t evolves according to the differential equation

$$dV_t = \mu_V V_t dt + \sigma_V V_t dW_t$$

where $V_0 = V > 0$, $\mu_V \in \mathbb{R}$ and $\sigma_V > 0$ are constants and W is a standard Brownian motion. Then one can compute the default probability

$$p(0,T) = P(V_T < B) = \mathcal{N}\left(\frac{\ln(B/V) - (\mu_V - \sigma_V^2/2)}{\sigma_V\sqrt{T}}\right)$$

where $\mathcal{N}(\cdot)$ stands for the standard normal distribution function.
We do not want to enter into more details on this topic, we simply mention the fact that this modeling idea is still used in practice. More sophisticated models have been proposed, we shall see some extensions in the next sections (the KMV model, credit migration models, a multivariate extension of Merton model, ...). What remains a milestone in all of them is the precise reference to an underlying explanatory (fundamental) process and the fact that the default time is an hitting time[3].

B. Reduced form models

We have seen what a model for credit risk should be able to provide: probabilities of default and losses given default. In the structural approach, we have seen a basic methodology to compute them.
One different approach is to consider models where the family of the distributions for the probabilities of default is a priori given and the model parameters are then

[3] The fact that τ is an hitting time for a continuous path process makes the time τ be fully predictable. In particular this implies that the local probability of default $h(t)$ is always zero. We shall see that this is not the case in the so called intensity based models (and this reflects better real data). This is probably the main (mathematical) difference between the two approaches and the reason that makes intensity based models so popular. Many authors have tried to relax this hypothesis in order to make τ totally inaccessible. We refer to [58] for a discussion on the accessibility of a stopping time and to [22], [10], [9], [43] and [44] for different "bridge" models between structural and reduced form models.

computed in order to fit market data. In particular it is useful to consider models where a local probability of default is well defined.

As in the previous sections we assume that a probability space $\{\Omega, \mathcal{F}, P\}$ is given. We consider the nonnegative random variable $\tau : \Omega \to \mathbb{R}^+$, with $P\{\tau = 0\} = 0$ and $P\{\tau > t\} > 0$ for all t, that describes the default time.
For simplicity we suppose that the information available to the market is the natural filtration generated by the default time τ. Indeed, we consider the σ-fields $\mathcal{H}_t := \sigma(\{\tau \leq u\} : u \leq t)$ and the corresponding filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$.
In many reduced form models the default time τ is distributed according to an exponential law, so that the event of default is related to the first jump of a Poisson process $(N(t))_{t\geq 0}$ such that $\tau = \inf\{t : N(t) = 1\}$. After the first jump we stop the Poisson process so that for $t > \tau$ we have $N(t) = N(\tau) = 1$. Summarizing we finally have $N(t) = N(t \wedge \tau) = \mathbb{I}_{\{\tau \leq t\}}$. Notice that $\mathcal{H}_t = \sigma(N(u) : u \leq t)$.
We denote by $F(t) = P\{\tau \leq t\}$ for all $t \in \mathbb{R}^+$ the cumulative distribution function of the default time.

Assumption 1.2.1 *$F(t)$ is absolutely continuous with respect to the Lebesgue measure, that is, it admits a density function $f(t) : \mathbb{R}^+ \to \mathbb{R}^+$ such that $F(t) = \int_0^t f(u)du$, then*

$$F(t) = 1 - e^{-\int_0^t \gamma(u)du},$$

where $\gamma(t) = f(t)(1 - F(t))^{-1}$.

The function γ is called *intensity function*. It can be shown that in this case

$$\gamma(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(\tau \leq t + \Delta t | \tau > t).$$

Put differently, it describes the local probability of default. In the class of models where the intensity function is well defined, it is natural to assume γ as the primitive object to be characterized. These models are usually called *intensity based models*.

1.3 Credit migration

In the previous sections we have described the basic tools when dealing with default risk. In particular we have defined a stopping time τ as the time of default. We want now to consider more general frameworks in which the default event is no longer the only interesting event (and the unique risk to be taken into account).
Instead of looking only at the default event, we shall describe the credit quality of an obligor considering a whole set of *rating classes* where the default is only the last (and worst) class. We are thinking of the so called *credit rating models*. These models are implemented by rating agencies (Moody's, Standard and Poor's, Fitch) but also by the internal rating systems of financial insitutions[4].
The basic idea behind these models is to assign each obligor to one class that is characterized by an estimated probability of default (i.e. of falling down to the default state). This means that all the available information about the probability of default of each obligor is given by its rating class.

[4]In the new Basel II accord the single institutions are encouraged to implement internal systems to asses credit worthiness of obligors, see [2].

For example in Moody's model the rating classes are labelled from the best to the worst as AAA, AA, A, BBB,..., CCC, D, where D indicates the default state. Thus we have to estimate a K-dimensional vector of probabilities of default where K is the number of rating classes (apart form default). Notice that all the *migration probabilities* have to be determined, in the sense that we are interested in the probability that a firm in class AA falls within a year into class B, and this has to be done for each pair of classes. This means that we have to determine a *transition matrix* in which the entries are exactly the probabilities of migration among the $K+1$ classes. To formalize this issue we give the following definitions:

Definition 1.3.1

- *We label the $K+1$ classes as $K, K-1, ..., 1, 0$ where $k = 0$ indicates the default state;*
- *We name $\mathcal{K}_0$ the set of all the rating classes and $\mathcal{K} = \mathcal{K}_0/\{0\}$ the set of all the classes except for default;*
- *Define the rating process $R_{\{t\geq 0\}}$ as the process $R(\omega, t) : \Omega \times [0, \infty] \to \mathcal{K}_0$ where $\{\Omega, \mathcal{F}, \mathcal{F}_t, P\}$ is a probability space where $\mathcal{F}_t$ is the market filtration;*
- *We define $Q(t,T) \in \mathbb{R}^{K+1} \times \mathbb{R}^{K+1}$ as the transition probability matrix for the time interval $[t,T]$, so that*

$$(Q(t,T))_{ij} = P(R(T) = j|\mathcal{F}_t \vee R(t) = i).$$

Notice that $(Q(t,T))_{ij} \geq 0, \ \forall\ i, j \in \mathcal{K}_0;\ \sum_{j=0}^{K} Q(t,T)_{i,j} = 1, \forall\ i \in \mathcal{K}_0;\ Q(t,T)_{0,j} = 0, \forall\ j \in \mathcal{K}$ and $Q(t,t) = I$ where I is the identity matrix.

The problem of credit migration is now to specify a model in order to describe the evolution of the transition matrix $Q(t,T)$ and eventually to calibrate it to the data. In the standard literature as well as in the most famous models used in practice, the rating transition process R is assumed to evolve as a time homogeneous Markov chain. Basically this means that

$$(Q(t,T))_{ij} = P(R(T) = j|R(t) = i);$$

$$Q(t,T) = \tilde{Q}(T-t)\ \forall t \leq T.$$

where $\tilde{Q}$ is a transition probability that depends only on the length of the time interval between t and T.

The Markov hypothesis simplifies the computation of the migration probabilities. It can be proved (see [58] for more details) that in this case

$$Q(t,T) = \exp\{(T-t)\Lambda\}$$

where the exponential is defined formally as the limit of the series expansion

$$\exp\{(T-t)\Lambda\} = \sum_{n=0}^{\infty} \frac{((T-t)\Lambda)^n}{n!}$$

and where $(\Lambda)_{ij} = \lim_{h\to 0} \frac{\tilde{Q}(h)_{ij}}{h}, \forall i \neq j$ are the local transition probabilities and the diagonal elements $(\Lambda)_{ii}$ are defined as $1 - \sum_{j\neq i}(\Lambda)_{ij}$.
A discussion of these models can be found in [58] or in [31]. We want to stress the fact that the Markov assumption is very restrictive and not supported by real data on credit migration time series. In some recent papers different authors have proposed models where this hypothesis is relaxed in various ways. We quote the papers by Lando and Skodeberg (2002) [50], Christensen, Hansen, Lando (2004) [11] and McNeil and Wendin (2006) [54]. In the latter, a credit migration model that allows for dependence between obligors is proposed. We shall describe in Chapter 4 an alternative approach to credit migration that takes into account the *direct* contagion between the obligors.

1.4 Modeling dependence

Suppose that a set of $N \in \mathbb{N}$ obligors are facing credit risk. Clearly each of them has his marginal probability of default at time t (let's say $p_i(t),\ i = 1,..,N$), that can be computed using some techniques shown in the previous sections.

One might be interested in pricing multi-name derivatives (such as CDOs) or to asses the riskiness of a loan portfolio of a bank. In both cases we have to deal with *joint default probabilities* of the whole set (or part of it) of the N obligors. Notice that it would be quite "naive" (and unrealistic) to simply assume that default events of different obligors are independent. There is in fact empirical evidence in the data that reject the independence hypothesis (see for instance Section 3.1 in [29] for a discussion on macroeconomic common factors and [47] for the necessity of direct linkages between firms). Thus a new problem arises when building a credit risk model:

Given a set of $N \in \mathbb{N}$ different obligors, how can dependence be modeled and eventually joint default probabilities computed?

As we have discussed in previous sections, there are basically two approaches to model credit risk, the so called structural models and the reduced form model. This difference applies also in the case of multiple obligors. A new issue is how to define *dependence*. The first possibility is to model directly (in a structural framework) some correlated asset value processes and to try to infer probabilities of default from the dynamics of these processes. The drawback is that even with only two obligors, calculations become quite involved. An example of structural model for two value processes is due to Zhou (see [64]). The second way is to rely on some intensity based approaches.
In the next chapter we shall concentrate on the latter. We shall discuss two different families: the so called *conditionally independent* models and the *contagion* models.

Chapter 2

Portfolio credit risk

In this chapter we explore the basic tools for *managing* the risk, concentrating our attention to the topics related to *credit portfolios*, eventually *large portfolios*.

From the financial viewpoint we are interested in studying models for evaluating riskiness of portfolios and computing related risk measurements.
From the mathematical viewpoint, the core of the discussion remains how to model dependence.
We shall also briefly explore some particular models proposed in the literature for credit risk management, giving examples and addressing the reader to the literature.

Credit risk management is related to the ability of hedging against changes in the credit worthiness of the obligors. In particular one has to quantity the *risk capital* that a bank is requested to hold in order to cover *unexpected losses*. In this framework, we shall also see how credit risk is treated in the famous *Basel II Accord* (see [2]), and how the formulas suggested to compute Capital Risk adequacies are based on some extensions of one of the models we are going to illustrate in this chapter (the asymptotic behavior of the Bernoulli mixture model).

2.1 Basic definitions and concepts

Relying on the book by Embrechts, Frey and McNeil [31], we explore in this section some more technical details related to the existing models for portfolio credit risk management.
We state in the next definition some general concepts for modeling dependent defaults and portfolio losses. Consider a certain number N of obligors for which we are interested in assessing credit quality (e.g. their probability of default and their exposure at a given time).

Definition 2.1.1 (Default variables)
For any fixed time horizon T (eventually $T \to \infty$)

- $S_i(t) \in \{0, 1, ..., \mathcal{K}\}$ *for* $i = 1, ..., N$ *and* $t \in [0, T]$ *is the* state indicator *of obligor* i *at time* t*. Zero is the default state.*

- $Y(t) := (Y_1(t), ..., Y_N(t))$*.* $Y_i(t)$ *is the default event:* $Y_i(t) = 1 \Leftrightarrow S_i(t) = 0$*.* $M(t) := \sum_{i=1}^{N} Y_i(t)$ *is the default counting variable.*

Notice that at this level we did not assign to the variable S_i a precise financial meaning. In could be any kind of *credit quality* indicator. For example, we shall specialize this general setting to fit a multivariate credit migration model (see Section 2.2.3). It could also represent any fundamental indicator of financial health for a given firm (a liquidity indicator, the sign of the cash balances...).

Remark 2.1.2 *In standard practice and in most of the literature on credit risk management, a* static *approach to default risk is often preferred. The reason being that the time horizon for risk management is usually fixed to be one year. Therefore it is implicitly assumed that $t = 1$ and therefore the time dependence is suppressed in all the definitions in 2.1.1. We have preferred to propose them in a dynamic context since our final aim is to provide a dynamic view to the risk management issues.*
We believe that the dynamic point of view may help in explaining phenomena such as credit crises*, credit migration, and even help in pricing credit derivatives (e.g. CDOs). For these reasons we shall provide in chapters 3 and 4 models where the time evolution of the loss distribution is taken into account.*

A second remark is related to the reduction of complexity of these models. When dealing with fully heterogeneous models one looses analytical tractability. Moreover, some homogeneity assumptions are commonly used in practice and in most of the literature on the topic. Think for instance at the segmentation in different rating classes or in sector groups. These categories are built in order to cut the high dimensionality of the model.

Assumption 2.1.3

(A.1) ***(Exchangeability)*** *The obligors $i \in \{1, ..., m\}$ for $m \leq N$ are said to belong to a* homogeneous group *if the state indicators S_i, $i = 1, .., m$ (and the default indicators Y) are* exchangeable*.*

S_i, $i = 1, .., m$ *are said* exchangeable *if* $(S_1, ..., S_m) \overset{d}{=} (S_{\Pi(1)}, ..., S_{\Pi(m)})$*, for any permutation* $(\Pi(1), ..., \Pi(m))$ *of* $(1, ..., m)$*.*

(A.2) ***(Mean field assumption)*** *The default probability of obligor i depends on the vector $Y(t)$ only via $M(t) = \sum_i F(Y_i(t))$, where $F(\cdot)$ is some smooth function.*

$(A.1)$ is related to the homogeneity within a group of obligors; $(A.2)$ states that the probability of default of one single obligor depends on the others only via one aggregated variable that summarizes the health of the financial system. We shall see that these homogeneity assumptions are often crucial to explicitly solve different classes of models.
We are now going to state a definition of *portfolio losses.* When speaking of portfolio losses, we mean the losses that a financial institution may suffer in a credit portfolio due to deterioration of the credit quality of the obligors. Many specifications may be chosen to this aim. Some general rules are now stated.
Basically one has to describe *marginal* losses (the losses due to single obligors). Then one looks at the aggregate credit portfolio. Thus we need random variables (or stochastic processes when in a dynamic set up) describing marginal and aggregate losses at a given time for the different obligors.

Definition 2.1.4 (Portfolio losses)

- *We define* $L_i(t)$ *for* $i = 1, ..., N$ *and* $t \in [0, T]$ *(eventually* $T \to \infty$*) the* marginal loss *due to obligor* i*.* $L_i(t)$ *can be any non negative random variable defined on the probability space* $\{\Omega, \mathcal{F}, P\}$ *and measurable with respect to an appropriate market filtration* $(\mathcal{F}_t)_{t \geq 0}$*.*

- *We define the* portfolio loss *at time* $t \in [0, T]$ *(eventually* $T \to \infty$*) as the random variable*

$$L^{(N)}(t) = \sum_{i=1}^{N} L_i(t). \tag{2.1}$$

A particular choice of $L_i(t)$*:*

- *We name* $\delta_i(t)e_i(t)$ *the* loss given default *of obligor* i *at time* $t \in [0, T]$*, where* $e_i \in \mathbb{R}^+$ *represents the* exposure *(the value of the position* i *in the portfolio) and where* $\delta_i \in (0; 1]$ *is the (random) proportion of the exposure which is lost in case of default. In this case we have*

$$L^{(N)}(t) = \sum_{i=1}^{N} e_i(t)\delta_i(t)Y_i(t). \tag{2.2}$$

Before starting an extended illustration on different frameworks for modeling dependence, we would like to quote some technical aspects of the Basel II accord (see [2]), related to the computation of the so called *risk capital*, that a bank is supposed to hold in order to hedge against unexpected losses in credit portfolios.

2.1.1 Basel II insights

The standard way in which a financial institution may prevent itself against bankruptcy is to allocate (to keep aside) part of the capital to cover unexpected losses (of a predetermined magnitude) that may occur during a fixed time-period (see Remark 2.1.2). How is this fact formalized?
The simplest way to define it is to hold 8% of the so called *risk weighted* assets (RWA) of its credit portfolio as risk capital (RC). For each investment a risk weight a_i is defined, so that $RWA^{tot} = \sum_i RWA_i$, where $RWA_i = a_i e_i$, being e_i the exposure of obligor i. And finally $RC = 0,08 \times RWA$.
The weights a_i can be determined either simply by the type and the credit of the counterpart (being 50% for a corporation in the range $A+$ to $A-$ of Moody's), or by more sophisticated mechanisms of calculation, the so called *internal rating* based approach. In this case the weights take the form

$$a_i = (0,08)^{-1} c \delta_i \mathcal{N}\left(\frac{\mathcal{N}^{-1}(\bar{p}_i) + \sqrt{\rho}\mathcal{N}^{-1}(0,999)}{\sqrt{1-\rho}}\right), \tag{2.3}$$

where c is a technical factor, $\bar{p}_i$ represents the marginal probability of default, δ_i the percentage loss given default as seen in Definition 2.1.4 and ρ is a "correlation" parameter. The latter parameter is fixed by the Basel II Accord (as well as c), whereas

$\bar{p}_i$, e_i, δ_i can be estimated by the bank with internal models. Finally we write the formula for the RC related to obligor i:

$$RC_i = (0,08)RWA_i = c\delta_i e_i \mathcal{N}\left(\frac{\mathcal{N}^{-1}(\bar{p}_i) + \sqrt{\rho}\mathcal{N}^{-1}(0,999)}{\sqrt{1-\rho}}\right), \tag{2.4}$$

In the next section we shall explore different approaches which lead an institution to come up with estimates for the above mentioned parameters.
Two important classes of models are the *conditional independent defaults models* and the *contagion models*. The first class focuses on the dependence of the entire set of obligors on some exogenous variables (macroeconomic factors), whereas the second is dealing with direct economic links among firms, and how financial distress is transmitted from one stressed firm to a linked one directly. In short one could say that the former is a *macroscopic* point of view and the latter a *microscopic* point of view. It can be argued that, given the enormous size of a typical loan portfolio, direct business relations play a less prominent role in explaining default dependence. This issue has been analyzed in details for instance by Giesecke and Weber (see [40] and [41] for details). They propose a comprehensive model where both effects are taken into account. In our perspective the discussion about the importance of including direct linkages when looking at portfolio losses is still far from being closed. We shall propose our approach in the next chapters.

2.2 Conditionally independent models

The main assumption in this class of models is that after conditioning on some variables (or sigma algebra), the default events are independent. Following [31] we distinguish two important families of conditionally independent models.

2.2.1 Mixture models

Here the default probabilities depend on a set of common economic factors labeled as Ψ, for instance explanatory macroeconomic variables. Given a realization of Ψ, defaults of different obligors are independent.

Definition 2.2.1 (Bernoulli mixture models)
Given $p < N$ and a p-dimensional random vector Ψ, Y follows a Bernoulli mixture model with factor vector Ψ, if there are functions Q_i such that conditional on Ψ, the default indicator Y is a vector of independent Bernoulli random variables with $P(Y_i = 1|\Psi = \psi) = Q_i(\psi)$. Then for $y \in \{0,1\}^N$ we have

$$P(Y = y|\Psi = \psi) = \Pi_{i=1}^N Q_i(\psi)^{y_i}(1 - Q_i(\psi))^{1-y_i}$$

Example 2.2.2 (One factor Bernoulli mixture models) *To give an intuition, we explicitly describe the simple case in which we only have one random variable Ψ and suppose that all the functions Q_i coincide. In this case the Bernoulli-mixture model is* exchangeable*, since the random vector Y is exchangeable (recall Assumption 2.1.3).*
In this case the common background variable has the effect of randomizing the probability of the binomial and induces dependence.

If $Y_i, i = 1, ..., N$ are i.i.d. Bernoulli random variables with parameter $\tilde{p}$, where $\tilde{p}$ is again random taking values on $[0,1]$ with a well defined density f and $E[\tilde{p}] = \bar{p}$. We have

$$E[Y_i] = \bar{p} \ ; \qquad Var(Y_i) = \bar{p}(1-\bar{p})$$

The distribution of the random variable counting the number of defaults $M^{(N)} = \sum_{i=1}^{N} Y_i$ has mean $EM^{(N)} = N\bar{p}$ and variance

$$Var(M^{(N)}) = N\bar{p}(1-\bar{p}) + N(N-1)(E\tilde{p}^2 - \bar{p}^2)$$

The conditional probability of $M^{(N)}$ is given by

$$P(M^{(N)} = k | \tilde{p} = q) = \binom{N}{k} q^k (1-q)^{N-k}$$

and then

$$P(M^{(N)} = k) = \binom{N}{k} \int_0^1 q^k (1-q)^{N-k} df(q)$$

Observe also that for a large portfolio the variance is determined only by the randomness of $\tilde{p}$, in fact we have

$$Var\left(\frac{M^{(N)}}{N}\right) \to E\tilde{p}^2 - \tilde{p}^2 \ ; \quad N \to \infty.$$

One can also see that

$$P\left(\frac{M^{(N)}}{N} < \Theta\right) \to \int_0^{\Theta} f(p)dp = F(\Theta).$$

This results suggest that in this kind of model, the basic goal is to find a suitable distribution for p. A possible (and commonly used) specification is $p \sim \beta(\alpha, \beta)$ the beta distribution.
It is also possible to relax the homogeneity assumption to homogeneity within groups. In this case the obligors are exchangeable only inside their group. See for instance [36] for such an extension.

2.2.2 Threshold models

Another very popular class of models are the so called *threshold models* which refer again to the larger class of conditionally independent models. They are called also *latent variable models* because defaults are triggered by some exogenous latent random variables.
In this case default occurs when a random variable X falls below some threshold. An example is the Merton firm value model (described in the previous chapter) where X is the firm asset value.

Definition 2.2.3 *Let $X = (X_1, ..., X_N)$ be an $N-$dimensional random vector and $d_1^i < ... < d_n^i$ a sequence of thresholds, with $d_0^i = -\infty$, $d_{n+1}^i = +\infty$. Then we set*

$$S_i = j \Longleftrightarrow d_j^i < X_i \leq d_{j+1}^i; \quad j \in \{0, ..., n\}, \ i \in \{1, ..., N\}. \tag{2.5}$$

(X, D) is said to be a threshold model for the state vector S where $(X_i, (d_j^i)_{1\leq j\leq n})_{1\leq i\leq N}$ is a latent vector and the i-th row of D contains the critical thresholds for obligor i.

In this kind of models it becomes crucial to know the marginal distribution functions of X: $F_i(x) = P(X_i \le x)$ and how these are correlated. Notice that $\bar{p}_i = F_i(d_1^i)$.
A natural way to define a correlation structure between the latent variables is to use a *copula* function. We refer to [31] for further explanations and examples of latent variable models and copulae. We only notice that two of the most popular credit models used in practice (Credit Metrics and KMV model) are refined versions of the above definition.

Example 2.2.4 (A simple one factor model) *We describe in this example a simple one-factor reduction of the multivariate factor model, which illustrates the modeling intuition behind the KMV/CreditMetrcis model.*
Consider a model with N Gaussian variables $X_1, ..., X_N$. These variables depend on a factor $F \sim \mathcal{N}(0, \sigma)$ so that we can write $\mathbf{X} = \mathbf{b}F + \epsilon$, where $\mathbf{b}$ is a vector of loadings and ϵ the idiosyncratic errors such that $\epsilon_i \sim \mathcal{N}(0, v_i)$.
We would like to normalize the variances of $\mathbf{X}$ so that $Var(X_i) = 1, \forall i$. Hence, if $\beta_i = (b_i)^2\sigma^2$ identifies the systematic risk of X_i then $v_i = 1 - \beta_i$ gives the idiosyncratic risk of X_i.
Hence it can be shown that

$$\bar{p}_i(F) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(\bar{p}_i) - b_i F}{\sqrt{1-\beta_i}}\right) \tag{2.6}$$

In the particular case where all loadings coincide and are equal to $\sqrt{\rho}$ and $F = -\Psi$ (an exogenous variable) so that $X_i = -\sqrt{\rho}\ \Psi + \sqrt{1-\rho}\ \epsilon_i$, equation (2.6) becomes

$$\bar{p}_i(\psi) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(\bar{p}_i) + \sqrt{\rho}\ \psi}{\sqrt{1-\rho}}\right) \tag{2.7}$$

2.2.3 A view on dependent credit migration models

In the framework of threshold models McNeil and Wendin (see [54]) have specified a credit migration model where the evolution of the transition probability is not Markovian and takes into account the dependence of the obligors in a vector of macroeconomic (common) factors and a set of latent variables (which of course break the independence). We remind to Definition 1.3.1 and we add the following:

- We have N_t obligors whose rating class has to be assigned at each time $t \ge 0$;
- We have threshold values $(\mu_{\kappa,l})_{\kappa \in \mathcal{K}, l \in \mathcal{K}_0}$ such that for all $\kappa \in \mathcal{K}$

  $$-\infty = \mu_{\kappa,-1} \le \mu_{\kappa,0} \le \mu_{\kappa,1} \le ... \le \mu_{\kappa,K-1} \le \mu_{\kappa,K} = \infty$$

  and $\mu_{0,l} = \infty$ for all $l \in \mathcal{K}_0$;
- We have a vector of latent factors $\mathbf{b}_t$. A design vector $\mathbf{z}_{ti}$ and an idiosyncratic risk variable ϵ_{ti} for each obligor $i = 1, ..., N_t$. ϵ_{ti} are i.i.d and independent of $\mathbf{b}_t$. Finally we set $X_{ti} := \epsilon_{ti} + \mathbf{z}_{ti}'\mathbf{b}_t$ for $i = 1, ..., N_t$.

In view of these definitions, we reformulate Definition 2.2.3 as follows:

$$R_{ti} = l \Longleftrightarrow X_{ti} \in (\mu_{\kappa(t,i),l-1}, \mu_{\kappa(t,i),l}] \tag{2.8}$$

where R_{ti} represents the rating class at the end of period t and $\kappa(t,i)$ the rating at the beginning thereof. For details on the implications of this model and possible generalizations see [54].
What we would like to stress is that the introduction of latent (and possibly also known) risk factors introduces dependence in the model. As we have already seen in this section, this kind of models belong to the family of the so called conditionally independent models. The events of downgrades for the firms are not independent but they become independent once we condition on the realization of $\mathbf{b}_t$.
We shall propose in Chapter 4 a very basic credit migration model where the dependence structure is built in such a way that it does not allow for conditional independence. This latter approach can be linked to the family of the so called *contagion models*.

2.3 Contagion models

Following the standard literature on quantitative credit risk management we have provided the basic concepts and the standard models usually considered when dealing with credit risk. In particular we have focused attention on how to model the dependence structure between different obligors. The conditional independence framework that we have explored has many advantages, in particular it is very tractable and easy to fit. The problems arise when one is interested in studying phenomena where the *connection* between the obligors is not only related to macroeconomic factors.
We are thinking of direct business relationships, lending-borrowing networks, primary-secondary ties and other relationships that have an important consequence: the effect (observed in the data) that default intensities of non-defaulted firms change (usually increase) at the time of default of other "near" firms, where near means that the firms are related with some lending-borrowing linkages or simply that they are in the same sector or group.
Think at the effect of the bankruptcy of the energy company Enron in 2001: investors had the feeling that illegal accounting practices could have been in place in other debt-issuers on similar markets, causing a rising effect on credit spreads of other firms. This effect is called *information effect* in the sense that having new information on the safety of one sector, may change the feeling about a particular firm.[1]
When the connections are more than informational (as in a borrowing-lending network) we speak about *counterparty risk*. The consequences of these business ties are (maybe) rare events but highly significant from the perspective of a credit manager. These effects are credit crises, default clustering, credit cascades, domino effects, bank runs [2]. In one word *credit contagion.*

How can we model credit contagion? Is it possible to build a model that explains clustering of defaults (credit crises)?

In the remaining part of this thesis we shall try to answer this question.

[1]Information driven default is discussed for instance in [59], [27] and [28].

[2]We shall extensively discuss about these effects in the next chapters especially in Chapter 4 where the concept of credit crisis becomes crucial. When speaking of a credit crisis we intend a period of time (short compared to a standard business cycle) during which many firms default (or are downgraded by rating agencies).

The literature on this topic is rather recent. To our knowledge the first paper that appeared in this area is due to Jarrow and Yu [47] (2001) and contains an interesting (and negative) answer to the question above:

"A default intensity that depends linearly on a set of smoothly varying macroeconomic variables is unlikely to account for the clustering of defaults around an economic recession".

This is the starting point for the contagion modeling: the dependence can not be related simply to macroeconomic exogenous factors.

When speaking of intensity based models we have to assign marginal local probabilities of default $\lambda_i(t)$ in some reasonable way (see section 1.2 for details) and we try to infer joint probabilities of default and distribution of losses in credit portfolios.
We are now going to introduce the basic concept for modeling interacting intensities, providing also two examples treated in the literature, which are the starting point for the discussion of the next chapters.
The intuition behind an *interacting intensity model* is that the N obligors are now seen as a "network of interacting agents", meaning that they have a kind of "physical" influence on each other. The probability of having a default somewhere in the network depends explicitly on the state of all the obligors.
The way in which this "philosophy" is then applied, sensibly depends on the specific features of the particular model. It is quite intuitive that the idea of a network where agents interact leads naturally to the literature of *particle systems* used in Statistical Physics.
This point of view is quite new in the world of financial mathematics especially when dealing with credit risk management. Some very recent papers have appeared in the last years. Among them we would like to mention the works by Giesecke and Weber [40], and [41] for an interacting particle approach; the papers by Frey and Backhaus [36] and [35] on credit derivatives pricing and Horst [46] on cascade processes[3].
For a structural model developing the idea of a net of obligors where contagion is taken into account we refer to Egloff, Leippold and Vanini [30].

As already pointed out the intensity λ_i depends on the state of all the obligors and possibly on some other variables (e.g. a vector of macroeconomic factors). We write $\lambda_i(\Psi_t; Y_t)$ for some observable (possibly multidimensional) background process Ψ. This is all what we can define at a very general level. We are now going to explore some particular models proposed in the literature.

Example 2.3.1 (A Markovian approach) *This approach, due to Frey and Backhaus [36], is based on the (conditional)* Markovianity *of the default indicator process Y. The authors show that under this assumption it is possible to provide, via the Kolmogorov backward equation, a useful tool for pricing basket credit derivatives. We briefly state the basic ideas.*
Suppose for simplicity that λ_i (the default intensity for the $i-th$ obligor) depends on Y_t only via an aggregate quantity $M_t^{(N)}$ that counts the number of defaulted firms in the portfolio of size N. This mean-field assumption is in line with Assumption (A.2)

[3] [36] and [40] are summarized as examples below.

in 2.1.3. Generalizations to more sophisticated models are provided in [36]. We then simply have $\lambda_i(t) = \lambda(\Psi_t, M_t^{(N)})$.
The process Y_t *behaves as a time-inhomogeneous continuous time Markov chain with state space* $S = \{0,1\}^N$ *and transition rates* $\lambda(y,x) = \mathbb{I}_{\{y_i=0\}}\lambda_i(\psi_t, y)$ *if* $x = y^i$ *(where* y^i *is the vector* y *where the* $i-th$ *component has changed value), for some* i *and zero otherwise.*
It can be shown that under these hypotheses the process $M_t^{(N)}$ *follows a continuous time Markov process with generator*

$$G_{[\psi]}^{(N)} f(l) = (N-l)\lambda(\psi, l)(f(l+1/N) - f(l)) \tag{2.9}$$

If we name $p(t,s,k,l) = P(M^{(N)}(s) = l | M^{(N)}(t) = k)$ *for any* k, l *in*$[0,1]$ *and* $0 \le t \le s < \infty$, *we know that the function* $(t,k) \to p(t,s,k,l)$ *satisfies the following Kolmogorov backward equation*

$$\frac{\partial}{\partial t} p(t,s,k,l) + G_{[\psi]}^{(N)} p(t,s,k,l) = 0$$

with terminal condition $p(s,s,k,l) = \mathbf{I}_{\{l\}}(k)$.
This equation can be used to compute numerically joint (and marginal) probabilities of default, and also to compute conditional expected values, i.e., to price derivatives. We refer to [36] for details and further generalizations of the model.

We show now an example where an interacting particle model is used to compute credit portfolio losses. It is a comprehensive model in the sense that it takes into account both macroeconomic factors and contagion. It is worth to notice that in this kind of models, it can be useful to look at a *financial distress* indicator (indicated in the following by σ) instead of at a pure *default indicator* Y. In particular it is assumed that a firm may come back to an *healthy* state after having experienced a period of distress. In other words there is no "coffin state" (corresponding to the default state $y_i = 1$ of the previous example).

Remark 2.3.2 *The market value of a portfolio position is subject to the credit quality of the issuers or counterparties, in other words to their ability to generate the required cash flows in the future. Due to adverse changes in a counterparty's credit quality the market valuation of the corresponding position can be severely reduced. Then risk measurements aim to evaluate the potential loss induced by credit quality deterioration (instead of looking at the number of default events).*

Example 2.3.3 (Credit contagion and cyclical correlation) *This approach proposed by Giesecke and Weber (based on [40] and [41]) shows the first attempt to link a "pure" interacting model with the more diffused concept of mixture model (see section 2.2.1). The idea is to take into account both cyclical default correlation (due to a exogenous macroeconomic variable* Ψ*) and direct contagion (due to a interacting particle set up).*
Even if the results show that the cyclical correlation explains the biggest part of the losses, local interaction causes additional fluctuations of losses around their average. Nevertheless, we shall discuss at the end of Chapter 4 how this secondary effect may determine (under particular conditions of the parameters) credit crises and default clustering in the financial systems.

In this model a geometrical structure is proposed: a $d-$*dim lattice* $S = (\mathbb{Z}^d)$ *where* $d \geq 2$. *Each firm represents a node in this lattice and its financial health is represented by a binary variable* $\sigma \in X := \{-1, 1\}^S$.
On the other hand we have perfect homogeneity (all the firms have the same marginal characteristics). One firm i *is directly influenced by nearby firms:* $j \in N(i)$ *where near means that we look at* j *s.t.* $|i - j| = 1$ *(being* $|\cdot|$ *the length of the shortest path between two firms on the lattice).*
Notice that (given the definition of $N(i)$*) the parameter* d *characterizes the "degree of complexity" of the partner network.*
Transition rates are defined as follows

$$\lambda_i(\sigma) = \frac{1}{4d} \sum_{j:|i-j|=1} |\sigma_i - \sigma_j|. \tag{2.10}$$

This model is basically the voter model *used in the literature of particle systems* [4].

A significant difference with respect to example 2.3.1 is that the indicator σ *may come back from state* -1*; in fact* $\sigma \in \{-1; 1\}$ *does not represent a default indicator, but a liquidity state indicator, i.e. an indicator of financial health or financial distress of a firm (it characterizes its ability to repay obligations). When firm* i *is in state* -1 *it means that the portfolio containing a position of firm* i *suffers a possible loss.*
The liquidity state indicator $(\sigma_t)_t$ *evolves as a continuous time Markov process. The stationary distributions are described and hence losses are studied at the equilibrium.*

The aggregated losses are given by $L_N = \sum_{i \in \Lambda_N} L_i$ *where* Λ_N *indicates the* $[-N, N]^d$ *cube in the lattice. The marginal loss* L_i*, i.e., the loss caused by firm* i *depends on the liquidity state of firm* i *and on macroeconomic factors* Ψ. Ψ *is intended to capture the economic business environment. The key assumption is that conditional on the variable* Ψ *and on the liquidity profile* σ*, losses are independent and identically distributed, so that we can write* $L_i = L(\sigma(i), \Psi)$ *(the dependence on the single firm is only via its liquidity indicator).*
A possible representation for L_i *is of a Bernoulli type (generalized to the contagion case):*

$$L(\sigma(i), \psi) = \begin{cases} 1 & \textit{with probability } P(\sigma(i), \psi), \\ 0 & \textit{with probability } 1 - P(\sigma(i), \psi) \end{cases}$$

so that conditionally on (σ, ψ)*, the position loss is a Bernoulli random variable.*

2.4 Large portfolio losses

The last concept we want to introduce concerns *large portfolios*. After defining a large portfolio we briefly analyze how the problem of computing large portfolio losses is treated in the literature on risk management, in particular we shall recall the standard static conditional independence case (Subsection 2.4.1) and two extensions based on interacting intensities: the first one is still static and introduces local interactions (Subsection 2.4.2), the latter is a dynamic conditionally Markov model (Subsection 2.4.3). In Subsection 2.4.4 we finally explain why these models are unsatisfactory (in

[4]In the standard voter model (and in [40] as well) the state variable σ takes values in $\{0; 1\}$. To our purposes it is more comfortable to work in the state space $\{-1; 1\}$. See the book by Ligget [52] for more details on the voter model.

our perspective) and how we are going to develop these issues in the rest of the thesis.

To define a large portfolio we rely on Definition 2.1.4. There, we defined marginal losses as $L_i(t)$, $i = 1, ..., N$ and aggregate losses as $L^{(N)}(t) = \sum_i L_i(t)$ so that

Definition 2.4.1 (Asymptotic portfolio) *Let N be the size of a credit portfolio and $L^{(N)}(t) = \sum_i L_i(t)$ the associated losses. The asymptotic portfolio is defined as P^∞ (i.e. when $N \to \infty$). The corresponding losses at any time $t \in [0, T]$ are computed as*

$$L^\infty(t) = \lim_{N\to\infty} L^{(N)}(t)$$

Notice that, in this context, "large" means that we look at approximations (for a large but finite number N of obligors involved) of the asymptotic portfolio. We shall see that some version of the *law of large numbers* and *central limit theorem* should enter into play.
Depending on the nature of the model, the features of the limiting results and the methodologies may change. Referring to [31] and [37], we firstly show how to get asymptotic results in a conditionally independent and static model, then we state some aspects regarding contagion models of different kind.

2.4.1 Conditionally independent static models

Following [31], we now provide an asymptotic result for large portfolios in the Bernoulli mixture model (see Definition 2.2.1). This approximation is very natural when referring to a mixture model, or in general to conditionally independent default models.
Generalizing Definition 2.1.4 we now have $N \in \mathbb{N}$ obligors and sequences of positive exposures $(e_i)_{i\leq N}$, $(\delta_i)_{i\leq N}$ and default indicators $(Y_i)_{i\leq N}$, hence a sequence of losses $L_i = \delta_i e_i Y_i$. Following [31] we make the following assumptions:

Assumption 2.4.2

(B.1) There is a $p-$dimensional random vector Ψ and functions $l_i : \mathbb{R}^p \to [0, 1]$ such that, conditional on Ψ, the $(L_i)_{i\leq N}$ form a sequence of independent random variables with mean $l_i(\psi) = E(L_i|\Psi = \psi)$.

(B.2) There is a function $\bar{l} : \mathbb{R}^p \to \mathbb{R}^+$ such that

$$\lim_{N\to\infty} \frac{1}{N} E(L^{(N)}|\Psi = \psi) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} l_i(\psi) = \bar{l}(\psi)$$

for each $\psi \in \mathbb{R}^p$. We call $\bar{l}(\psi)$ the asymptotic conditional loss function.

(B.3) There is some $C < \infty$ such that $\sum_{i=1}^N \left(\frac{e_i}{i}\right)^2 < C$ for all N.

We state now two important propositions that show how the limiting loss is characterized.

Proposition 2.4.3 *Consider a sequence $L^{(N)}$ satisfying Assumptions $(B.1) - (B.3)$ above. Denote by $P(\cdot|\Psi = \psi)$ the conditional distribution of the sequence $(L_i)_{i\leq N}$ given $\Psi = \psi$. Then*

$$\lim_{N\to\infty} \frac{1}{N} L^{(N)} = \bar{l}(\Psi) \ , \quad P(\cdot|\Psi = \psi) \ a.s.$$

Notice that when $\delta_i = e_i = 1$ for all i, the proposition applies for $M^{(N)} = \sum_i Y_i$ and $(B.2)$ becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} p_i(\psi) = \bar{p}(\psi),$$

for some $\bar{p}$.
This result can be used to obtain an estimate for the quantiles of $L^{(N)}$. For any random variable X we define $q_\alpha(X) = inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$ as the α-quantile of the distribution function F_X.

Proposition 2.4.4 *Consider the sequence $L^{(N)} = \sum_i L_i$ satisfying $(B.1)$–$(B.3)$ with a one-dimensional mixing variable Ψ with distribution function G. Assume moreover that the conditional asymptotic loss function $\bar{l}(\psi)$ is strictly increasing and right continuous and that G is strictly increasing at $q_\alpha(\Psi)$; then*

$$\lim_{N\to\infty} \frac{1}{N} q_\alpha(L^{(N)}) = \bar{l}(q_\alpha(\Psi)).$$

For a proof of these propositions see [37].

The latter proposition shows in particular how quantiles of the loss distribution (in mixture models) are driven by the tail of the mixing variable.
One can verify that, using a one-factor threshold model (see Example 2.2.4) and supposing moreover that the single losses given default $\delta_i e_i$ are deterministic, according to Proposition 2.4.4 and using Equation (2.7), one has:

$$q_\alpha(L^{(N)}) \approx \sum_{i=1}^{N} \delta_i e_i p_i(q_\alpha(\psi)) = \sum_{i=1}^{N} \delta_i e_i \mathcal{N}\left(\frac{\mathcal{N}^{-1}(\bar{p}_i) + \sqrt{\rho}\mathcal{N}^{-1}(\alpha)}{\sqrt{1-\rho}}\right).$$

This formula corresponds to Equation (2.4), given in Chapter 2.1.1, concerning the internal-ratings-based approach in Basel II Accord, useful to compute the Risk weighted assets (see Equation 2.3).

2.4.2 A static model with local interaction

An example of large portfolio losses computed considering direct contagion and local (non mean-field) interaction, is proposed by Giesecke and Weber in [40]. The model and the notations have been introduced in Example 2.3.3.
Depending on d, the complexity parameter, the authors describe two different behaviors of the limiting distribution. The interesting case is when $d > 2$ (dense partner network): define the empirical proportion of low liquidity firms (i.e., in the state $\sigma_i = -1$) as

$$m^{(N)} = \frac{1}{|\Lambda_N|} \sum_i \frac{1-\sigma_i}{2}$$

where Λ_N indicates the $[-N, N]^d$ cube in the lattice. Let $\bar{m} = \lim_N m^{(N)}$. The following result holds true

Proposition 2.4.5 *Given a network of firms structured as a N dimensional lattice $S = \mathbb{Z}^d$, $d > 2$ and a vector of liquidity indicators $\sigma_i \in \{-1, 1\}$, $i = 1, ..., |\Lambda_N|$ (being $|\Lambda_N|$ the number of firms). Suppose that the transition intensities $\lambda_i(\sigma)$ are as* *defined in Equation (2.10), then the sequence of aggregate losses* $(L^N)_{N\geq 0}$ *satisfies the following law of large numbers*

$$\lim_{N\to\infty}\frac{L^{(N)}}{|\Lambda_N|}=\bar{m}l_{-1}(\psi)+(1-\bar{m})l_1(\psi)$$

where $l_\sigma(\psi)=E[L(\sigma_i,\Psi)|\sigma(i)=\sigma,\Psi=\psi]$.

Moreover the following central limit theorem (CLT) also holds true

Proposition 2.4.6 *Let* $d>2$, *then* $(L^{(N)})_{N\geq 0}$ *satisfies the following central limit theorem*

$$(|\Lambda_N|)^{\frac{-d-2}{2d}}\left(L^{(N)}-|\Lambda_N|\bar{l}\right)\longrightarrow\mathcal{N}(0,(l_{-1}-l_1)^2v^2) \tag{2.11}$$

where $\bar{l}=\bar{m}(l_{-1}-l_1)+l_1$ *and* v^2 *denotes the limiting variance (see [41] for an expression of* v*).*

See [40] for the proofs of these results and possible generalizations. What we would like to stress is that this is a *non standard* CLT since the rate of convergence $(|\Lambda_N|)^{\frac{-d-2}{2d}}$ is not the usual square root. This is due to the interaction between the firms.

2.4.3 A conditionally Markov dynamic model

A generalization to dynamic models that allows for direct contagion between obligors is due to Frey and Backhaus [36]. They consider the case of conditionally Markov default indicators (see Example 2.3.1 for details).
We analyze here only the case in which the default intensity $\lambda^{(N)}(\Psi,M^{(N)})$ is supposed to depend on the default indicator vector $\mathbf{Y}_t$ (describing the state of the system) only via an averaged statistic $M_t^{(N)}$ (the mean-field assumption).
Recall that $M_t^{(N)}$ follows a continuous time Markov process with generator $G_{[\psi]}^{(N)}$ defined in Formula (2.9).

Proposition 2.4.7 *Suppose that* $\lambda^{(N)}$ *converges uniformly on compacts to some Lipschitz function* $\lambda^{(\infty)}$. *Suppose moreover that* $\lim_N M_0^{(N)}=l$, *then for all* $T>0$

$$\lim_{N\to\infty}\sup_{t\leq T,l\in[0,1]}\left\{\left|G_{[\Psi]}^{(N)}f(l)-G_{[\Psi]}^{(\infty)}f(l)\right|\right\}=0;$$

where

$$G_{[\Psi]}^{(\infty)}f(l)=(1-l)\lambda^{(\infty)}(\psi,l)\frac{\partial}{\partial l}f(l).$$

Moreover the pair of processes $(\Psi,M^{(N)})$ *converges in distribution to* $(\Psi,M^{(\infty)})$, *where* $M^{(\infty)}$ *solves*

$$\frac{d}{dt}M_t^\infty(\omega_1)=(1-M_t^\infty(\omega_1))\lambda^\infty(\Psi_t(\omega_1),M_t^\infty(\omega_1))$$

with the initial condition $M_0^\infty(\omega_1)=l$.

See [36] for a proof.

Remark 2.4.8 *Proposition 2.4.7 provides a dynamic version of a* weak law of large number *for the variables* $(M^{(N)})_{N\geq 0}$. *In fact it quantifies the number of defaulted firms at a given time in a portfolio of size* N *going to* ∞. *The result is only* weak *in the sense that the convergence holds in distribution.*
We shall provide in the next chapter a different methodology in order to describe a strong *law of large numbers (where convergence is proved almost surely) and a* central limit theorem *that allows for characterization of fluctuations around the limit.*

2.4.4 Dynamic models with random environment

The latter two models take the contagion into account. The local interaction case provides a LLN (and CLT) at the equilibrium and therefore it is in some sense static: it is not possible to describe how, when, (and whether) the system (once started from a given initial configuration) reaches the equilibrium.
The model by Frey and Backhaus is a dynamic model and provides a (weak) LLN for conditionally Markov processes.

The framework we are going to introduce in the next chapters involves features of both these approaches. We shall develop a mean-field framework that in the first formulation (Chapter 3) has a conditional independence structure. What distinguishes our model from that by Frey and Backhaus is the presence of a *random environment* that is *local* in the sense that it may take different values in different sites.
Moreover we are going to study a strong law of large numbers, meaning that we provide convergence results that hold almost surely. We are also able to characterize the speed of convergence to the equilibrium using large deviations techniques and the fluctuations around the limits (CLT). Moreover our treatment is dynamic: indeed we characterize the paths towards the equilibrium configuration of the asymptotic loss, in particular we are able to exhibit situations in which it is possible to describe credit crises. All these concepts will be developed in the next chapters.

Chapter 3

Exogenous (static) random environment

In this chapter we shall describe our approach for modeling credit contagion and for the study of the losses that a bank may suffer due to deterioration of credit quality of the obligors (firms).
In Section 3.1, we collect some technical results that shall be used in this and also in the next chapter in order to state a large deviation principle. Although most of them are classical results, we prefer to propose these theorems in the form that is more convenient to our proposes. In Section 3.2 we specify the model into details. In Section 3.3 we define one of the main objects of this thesis: the so called *empirical measure* (see formula (3.26)). Suppose that N firms are acting in a market and their default indicators $\underline{\sigma}(t) = (\sigma_i(t);\ i = 1, \dots, N)$, where $\sigma_i(t) \in \{-1, 1\}$, evolve in time. We denote by $\sigma[0, T] \in \mathcal{D}[0, T]$ a trajectory on the interval $[0, T]$ of such a rating indicator. $\mathcal{D}[0, T]$ denotes the space of càdlàg functions endowed with the Skorohod topology. The empirical measure ρ_N is a random measure taking values in $\mathcal{M}_1(\mathcal{D}[0, T])$, the space of probabilities on $\mathcal{D}[0, T]$. ρ_N weights the realizations of the N dimensional process $\underline{\sigma}[0, T]$. Put differently the empirical measure can be thought of the physical (historical) measure of the market. Most of the mathematical results of this thesis are concerned with the sequence of measures $(\rho_N)_N$. In this section we state in particular a large deviation principle for this sequence of measures (Theorem 3.3.3) and then we prove a suitable law of large numbers (Theorem 3.3.6), finding a unique Q_* such that $\rho_N \to Q_*$ almost surely. Significant and very useful for applications is also the characterization of Q_* provided in Proposition 3.3.5.
Section 3.4 applies the previous results to the study of the time evolution of the empirical mean of $\underline{\sigma}(t)$ (the expectations computed under the empirical measure) (Proposition 3.4.1). Simple examples of portfolio losses are provided even though a more involved treatment of the applications is discussed in Chapter 5. In Section 3.5 we finally state and prove a functional central limit theorem (CLT) characterizing the fluctuations of ρ_N around Q_* (Theorem 3.5.6). The proof of this theorem is inspired by the seminal work of E. Bolthausen (see [4]). In [4] a rather general framework is proposed in order to derive central limit theorems in Banach spaces. Since $\mathcal{M}_1$ is not a Banach space we are forced to construct an auxiliary space where a suitable large deviation principle is inherited and a CLT can be proved accordingly. This procedure is summarized in Theorems 3.5.4 and 3.5.5.

3.1 Some general results on large deviations

Let $\mathcal{X}$ be a Polish space (complete separable metric space) and $(P_N)_{N\geq 1}$ be a sequence of probability measures on the Borel subsets of $\mathcal{X}$.

Definition 3.1.1 *The sequence of probabilities $(P_N)_{N\geq 1}$ on $\mathcal{X}$ is said to satisfy a Large Deviations Principle (LDP) with rate N and rate function $I : \mathcal{X} \to [0, +\infty]$ if the following conditions hold true:*

1. *$I \not\equiv +\infty$;*
2. *I is lower semicontinuous;*
3. *for every $C \subset \mathcal{X}$, closed,*

$$\limsup_{N\to+\infty} \frac{1}{N} \log P_N(C) \leq -\inf_{x\in C} I(x); \tag{3.1}$$

4. *for every $G \subset \mathcal{X}$, open,*

$$\liminf_{N\to+\infty} \frac{1}{N} \log P_N(G) \geq -\inf_{x\in G} I(x). \tag{3.2}$$

Moreover if the level sets of I are compact, that is, if for every $k < +\infty$ the set $\{x \in \mathcal{X} : I(x) \leq k\}$ is compact in $\mathcal{X}$, then we say that I is a "good" rate function.

Definition 3.1.2 *The sequence of probabilities $(P_N)_{N\geq 1}$ on $\mathcal{X}$ is said to satisfy a* weak *Large Deviations Principle with rate N and (weak) rate function $I : \mathcal{X} \to [0, +\infty]$ if the following conditions hold true:*

1. *$I \not\equiv +\infty$;*
2. *I is lower semicontinuous;*
3. *for every $K \subset \mathcal{X}$, compact,* $\limsup_{N\to+\infty} \frac{1}{N} \log P_N(K) \leq -\inf_{x\in K} I(x)$;
4. *for every $G \subset \mathcal{X}$, open,* $\liminf_{N\to+\infty} \frac{1}{N} \log P_N(G) \geq -\inf_{x\in G} I(x)$.

Lemma 3.1.3 *Let us denote by $\mathcal{M}_1(\mathcal{X})$ the space of all the probability measures on $\mathcal{X}$. Then for every sequence of probability measures $(P_N)_{N\geq 1} \subset \mathcal{M}_1(\mathcal{X})$ which satisfies a weak LDP, the associated rate function is unique.*

Proof. See Lemma 4.1.4. in [24].

Proposition 3.1.4 *Let $(P_N)_{N\geq 1}$ satisfy a LDP with a good rate function I. Then there exists $x \in \mathcal{X}$ such that $I(x) = 0$, i.e. I attains at least one zero.*

Proof. Take $C = \mathcal{X}$ in (3.1). Necessarily we have $\inf_{x\in\mathcal{X}} I(x) = 0$. Moreover, being I a good rate function, there exists at least one point for which $I(x) = 0$, since the infimum is achieved on closed sets. ∎

Definition 3.1.5 *If $P, Q \in \mathcal{M}_1(\mathcal{X})$, define*

$$H(Q|P) := \begin{cases} \int_{\mathcal{X}} dQ \log \frac{dQ}{dP} & \text{if } Q \ll P \text{ and } \log \frac{dQ}{dP} \in L^1(Q) \\ +\infty & \text{otherwise} \end{cases} \tag{3.3}$$

the "relative entropy of Q with respect to P".

Remark 3.1.6 *Let us consider P fixed. Then the relative entropy $H(\cdot|P)$ is a non-negative convex function on $\mathcal{M}_1(\mathcal{X})$. $H(Q|P) = 0$ if and only if $Q = P$. Besides, it is lower semicontinuous on $\mathcal{M}_1(\mathcal{X})$ endowed with the weak topology.*

Theorem 3.1.7 (Sanov's Theorem) *Let $P \in \mathcal{M}_1(\mathcal{X})$ and $\tilde{P}_N \in \mathcal{M}_1(\mathcal{M}_1(\mathcal{X}))$ be the law under P of the random probability*

$$\rho_N(\underline{y}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{y_i} \tag{3.4}$$

with $\underline{y} = (y_1, \ldots, y_N) \in \mathcal{X}^N$, where $\mathcal{X}^N$ denotes the Cartesian product of N copies of $\mathcal{X}$. Then $H(\cdot|P)$ is a convex good rate function on $\mathcal{M}_1(\mathcal{X})$ and $(\tilde{P}_N)_{N\geq 1}$ satisfies a LDP with rate function $H(\cdot|P)$.

Proof. See Theorem 6.2.10. in [24]. ∎

The following theorem is a version of Cramer's Theorem for i.i.d. random variables taking values on a suitable vector space.

Theorem 3.1.8 *Let $\mathcal{Y}$ be a locally convex Hausdorff topological vector space. Let Y_N be a sequence of i.i.d. $\mathcal{Y}-$valued random variables with law $w \in \mathcal{M}_1(\mathcal{Y})$. For all $N \in \mathbb{N}$, denote by P_N the law of the $\mathcal{Y}-$valued random variable*

$$X_N = \frac{1}{N}\sum_{i=1}^{N} Y_i. \tag{3.5}$$

Then the sequence $(P_N)_{N\geq 1}$ satisfies a weak LDP with rate N and rate function

$$\Lambda^*(y) = \sup_{\varphi\in\mathcal{Y}'} \{\varphi(y) - \Lambda(\varphi)\}; \tag{3.6}$$

where $\Lambda(\varphi) := \log \int_{\mathcal{Y}} e^{\varphi(y)} w(dy)$ and where $\mathcal{Y}'$ is the topological dual of $\mathcal{Y}$.

Proof. The result is a consequence of Theorem 6.1.3. in [24]. ∎

Proposition 3.1.9 (Contraction principle) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces and $(P_N)_{N>0} \subset \mathcal{M}_1(\mathcal{X})$ satisfy a LDP with good rate function I. Moreover, let $T : \mathcal{X} \to \mathcal{Y}$ be a continuous function. Then $(P_N \circ T^{-1})_{N>0}$ satisfies a LDP with good rate function*

$$J(y) = \inf_{x=T^{-1}(y)} I(x). \tag{3.7}$$

Proof. See Theorem 4.2.1. in [24]. ∎

We now provide a *"relaxed"* version the so called *Varadhan Lemma.* The statement is relaxed in the sense that we assume that a suitable function $F : \mathcal{X} \to \mathbb{R}$, instead of being continuous on all its domain, is continuous only on a subset $\mathcal{X}_H \subsetneq \mathcal{X}$ in the following sense:
For any sequence $(x_n)_n \in \mathcal{X}$ such that $x_n \to x$, where $x \in \mathcal{X}_H$ we have $F(x_n) \to F(x)$.
We point out that this is a stronger assumption than assuming continuity of the restriction of F on the subset $\mathcal{X}_H$.

Proposition 3.1.10 (Varadhan's Lemma) *Let $\mathcal{X}$ be a Polish space. Let $(P_N)_N$ satisfy the LDP with rate N and good rate function H. Let $F : \mathcal{X} \to \mathbb{R}$ be measurable, bounded from above and continuous on the set $\mathcal{X}_H := \{x : H(x) < \infty\}$. Then the sequence of probability measures $(P_N^F)_N$ defined by*

$$\frac{dP_N^F}{dP_N}(\cdot) = \frac{\exp(NF(\cdot))}{\int_{\mathcal{X}} \exp(NF(y))P_N(dy)} \tag{3.8}$$

satisfies the LDP with the good rate function

$$I(x) = H(x) - F(x) - \inf_{y \in \mathcal{X}}[H(y) - F(y)]. \tag{3.9}$$

In particular

$$\lim_{N\to+\infty} \frac{1}{N} \log\left[\int_{\mathcal{X}} \exp(NF(y))\, P_N(dy)\right] = -\inf_{y\in\mathcal{X}}\left[H(y) - F(y)\right]. \tag{3.10}$$

Proof.
We have to show the validity of $1., ..., 4.$ of Definition 3.1.1.
Point 1. Being H a good rate function, $\{x : H(x) < \infty\}$ is not empty. Thus $\{x : I(x) < \infty\}$ is not empty as well.
Point 2. We have to show that $\lim x_N \to x$ implies that

$$\liminf_{n\to\infty}[H(x_N) - F(x_N)] \geq H(x) - F(x). \tag{3.11}$$

In the case when $H(x) = +\infty$ then $\liminf H(x_N) = +\infty$ since H is lower semicontinuous. So the thesis follows since F is bounded from above.
If $H(x) < +\infty$ we distinguish two cases.
(a) There exists $\bar{N}$ such that $H(x_N) = +\infty$ for all $N > \bar{N}$. Then (3.11) is trivial.
(b) If instead there exists a subsequence x_{N_k} such that $H(x_{N_k}) < \infty$ for all k, then by lower semicontinuity of H we have that $\liminf H(x_{N_k}) \geq H(x)$ and by the continuity of F on $\mathcal{X}_H$ we have that $F(x_{N_k}) \to F(x)$. So (3.11) follows.
The "goodness" of I can be seen as follows. Without loss of generality, we assume that $\sup_{y\in\mathcal{X}}[F(y)-H(y)] = 0$. For any $l \in \mathbb{R}$, consider the set $\{x : H(x) \leq l+\sup_{\mathcal{X}} F(x)\}$. Notice that $l+\sup F(x)$ is finite, being F bounded from above; thus this set is a level set for H then it is compact. We denote by $C_I(l)$ the level l set of the function I. By definition we have $C_I(l) := \{x : H(x) - F(x) \leq l\} \subset \{x : H(x) \leq l + \sup_{\mathcal{X}} F(x)\}$. The function F is continuous on $C_I(l)$, thus $C_I(l)$ is closed. Indeed, it is compact since it is a closed subset of a compact set. This shows the goodness of I.
Concerning points 3. and 4., we need to prove an upper and a lower bound.

Using the fact that *sup -f=-inf f*, after some algebra, the thesis can be reduced to the validity of the following two inequalities.

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^F(C) \le \sup_{y\in C}[F(y) - H(y)] - \sup_{y\in\mathcal{X}}[F(y) - H(y)]; \tag{3.12}$$

$$\liminf_{N\to\infty} \frac{1}{N} \log P_N^F(O) \ge \sup_{y\in O}[F(y) - H(y)] - \sup_{y\in\mathcal{X}}[F(y) - H(y)]. \tag{3.13}$$

for all C closed and O open. We split the remaining part of the proof in three steps.

Step 1). Take $C \subset \mathcal{X}$ closed. First we show that

$$\limsup_{N\to+\infty} \frac{1}{N} \log \left[\int_C \exp\left(NF(y)\right) P_N(dy)\right] \le \sup_{y\,\in\, C} \left[F(y) - H(y)\right]. \tag{3.14}$$

By assumption, F is bounded from above, thus there exists $K < \infty$ such that $\sup_{y\in C} F(y) \le K$. We fix now $\alpha < \infty$ and $\delta > 0$ and we denote by $C_H(\alpha) := \{y : H(y) \le \alpha\}$ the compact level sets of the good rate function H. We then look at $D := C_H(\alpha) \cap C$. Notice that D is still compact and F is continuous on D, since $C_H(\alpha) \subset \mathcal{X}_H$.
We now can apply the same reasoning as in the proof of Lemma 4.3.6. in [24]. The fact that H is lower semicontinuous and F is continuous imply that for every $x \in D$ there exists an open neighborhood A_x of x such that

$$\inf_{y\in \overline{A_x}} H(y) \ge H(x) - \delta\ ; \quad \sup_{y\in \overline{A_x}} F(y) \le H(x) + \delta. \tag{3.15}$$

By compactness, from the open covering $\bigcup_{x\in D} A_x$ of D, we extract a finite covering of D, e.g., $\bigcup_{i=1}^m A_{x_i}$. Therefore,

$$\int_C e^{NF(y)} P_N(dy) \le \sum_{i=1}^m \int_{A_{x_i}} e^{NF(y)} P_N(dy) + \int_{C\cap\left(\cup_{i=1}^m A_{x_i}\right)^c} e^{NF(y)} P_N(dy) \le$$

$$\le \sum_{i=1}^m \int_{\overline{A_{x_i}}} e^{NF(y)} P_N(dy) + e^{NK} P_N\left\{C \cap (\cup_{i=1}^m A_{x_i})^c\right\} \le$$

$$\le \sum_{i=1}^m e^{N(F(x_i)+\delta)} P_N(\overline{A_{x_i}}) + e^{NK} P_N\left\{C \cap (\cup_{i=1}^m A_{x_i})^c\right\}. \qquad (*)$$

We now define $\bar{x} := \text{argmax}\{e^{NF(x_i)} P_N(\overline{A_{x_i}})\ ;\ \ x_i = 1, ..., m\}$, hence

$$(*) \quad \le m\ e^{N(F(\bar{x})+\delta)} P_N(\overline{A_{\bar{x}}}) + e^{NK} P_N\left\{C \cap (\cup_{i=1}^m A_{x_i})^c\right\} \le$$

$$\le (m+1) \max\left\{e^{N(F(\bar{x})+\delta)} P_N(\overline{A_{\bar{x}}})\ ;\quad e^{NK} P_N\left\{C \cap (\cup_{i=1}^m A_{x_i})^c\right\}\right\}.$$

Being $\overline{A_{\bar{x}}}$ and $\{C \cap (\cup_{i=1}^m A_{x_i})^c\}$ closed we use the fact that for A closed

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N(A) \le -\inf_{y\in A} H(y). \qquad (**)$$

We now take the *limsup* in the above expression

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_C e^{NF(y)} P_N(dy) \leq$$

$$\leq \limsup_{N\to\infty} \frac{1}{N} \log \left[(m+1) \max \left\{ e^{N(F(\bar{x})+\delta)} P_N(\overline{A_{\bar{x}}}) \ ; \ \ e^{NK} P_N \left\{ C \cap (\cup_{i=1}^m A_{x_i})^c \right\} \right\} \right].$$

After some algebra and using (∗∗) we obtain the following upper bound

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_C e^{NF(y)} P_N(dy) \leq$$

$$\max \left\{ F(\bar{x}) + \delta - \inf_{y\in\overline{A_{\bar{x}}}} H(y) \ ; \ \ K - \inf_{y\in\left\{C\cap\left(\cup_{i=1}^m A_{x_i}\right)^c\right\}} H(y) \right\}.$$

We finally use (3.15) and the fact that $(\cup_{i=1}^m A_{x_i})^c \subset C_H(\alpha)^c$, to see that

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_C e^{NF(y)} P_N(dy) \leq \max \left\{ F(\bar{x}) - H(\bar{x}) + 2\delta \ ; \ \ K - \alpha \right\}.$$

Formula (3.14) thus follows taking the limits $\delta \to 0$ and $\alpha \to \infty$.

Step 2). Take $G \subset \mathcal{X}$ open. We show that

$$\liminf_{N\to+\infty} \frac{1}{N} \log \left[\int_G \exp\left(NF(y)\right) P_N(dy) \right] \geq \sup_{y\in G} \left[F(y) - H(y) \right]. \tag{3.16}$$

Fix $x \in G\cap\mathcal{X}_H$. F is continuous in x, we then take an open neighborhood $O \in G\cap\mathcal{X}_H$ of x such that $\inf_{y\in O} F(y) \geq F(x) - \delta$, for $\delta > 0$. Hence,

$$\liminf_{N\to\infty} \frac{1}{N} \log \left[\int_G \exp\left(nF(y)\right) P_N(dy) \right] \geq \liminf_{N\to\infty} \frac{1}{N} \log \left[\int_O \exp\left(nF(y)\right) P_n(dy) \right] \geq$$

$$\geq \inf_{y\in O} F(y) + \liminf_{N\to\infty} \frac{1}{N} \log \left[P_n(O) \right]$$

By the large deviation lower bound applied to the good rate function H (see 4. in Definition 3.1.1) and by the choice of O, we have that

$$\inf_{y\in O} F(y) + \liminf_{N\to\infty} \frac{1}{N} \log \left[P_N(O) \right] \geq \inf_{y\in O} F(y) - \inf_{y\in O} H(y)) \geq$$

$$\geq F(x) - H(x) - \delta.$$

As a consequence, the inequality

$$\liminf_{N\to+\infty} \frac{1}{N} \log \left[\int_G \exp\left(NF(y)\right) P_N(dy) \right] \geq \left[F(x) - H(x) \right] \tag{3.17}$$

holds for any $x \in G \cap \mathcal{X}_H$ since δ was arbitrarily chosen.
Concerning $x \in G \setminus \mathcal{X}_H$, notice that in this case $H(x) = +\infty$ and thus (3.17) still holds true. We have thus shown that (3.17) is valid for each $x \in G$, hence (3.16) is

proved.

Step 3). We notice that (3.14) and (3.16) computed with $C = O = \mathcal{X}$ give

$$\lim_{N\to+\infty} \frac{1}{N} \log \left[\int_{\mathcal{X}} \exp\left(NF(y)\right) P_N(dy) \right] = \sup_{y\in\mathcal{X}} \left[F(y) - H(y)\right]. \tag{3.18}$$

So that Equation (3.10) is proved once we observe that *sup -f=-inf f.*

To conclude the proof, we have to show the validity of (3.12) and (3.13). By definition of P_N^F,

$$\frac{1}{N} \log P_N^F(S) = \frac{1}{N} \left[\log \int_S \exp\left(NF(y)\right) P_N(dy) - \log \int_{\mathcal{X}} \exp\left(NF(y)\right) P_N(dy) \right], \tag{3.19}$$

so that

$$\limsup_{N\to\infty} \frac{1}{N} \log P_N^F(C) =$$

$$= \limsup_{N\to\infty} \frac{1}{N} \left[\log \int_C \exp\left(NF(y)\right) P_N(dy) - \log \int_{\mathcal{X}} \exp\left(NF(y)\right) P_N(dy) \right].$$

As a consequence, (3.12) follows from (3.18) and (3.14). Using the same argument for the *liminf*, it is easy to see that (3.13) follows from (3.18) and (3.16). So the thesis follows. ∎

Remark 3.1.11 *An analogous result of Proposition 3.1.10 holds when the sequence $(P_N)_N$ satisfies a weak LDP. In this case it can be shown that the sequence $(P_N^F)_N$, defined as in (3.8), satisfies a weak LDP with good rate function I as defined in (3.9).*

3.2 The model for contagion

In this section we describe our approach for modeling credit contagion. In order to keep the treatment easy, we prefer to rely on a simplifying assumption: we describe a mean field interaction model. What characterizes a mean-field model -within the large class of particle systems- is the absence of a "geometry" in the configuration space, meaning that each particle interacts with all the others in the same way.

Other approaches, different from the mean-field one, have also been proposed in the literature: Giesecke and Weber have chosen a local-interaction model (the voter model[1], see Example 2.3.3 for details) assuming that each particle interacts with a fixed number d of neighbors; it may be argued that the hypothesis that each firm has the same (constant) number of partners is rather unrealistic. Cont and Bouchaud (see [5]) suggest a *random graph approach*, meaning that the connections are randomly generated with some distribution functions.

Despite this homogeneity assumption we attach to each site (firm) a local *random environment* which plays the role of an idiosyncratic term that influences the credit

[1] The Voter model assumes -roughly speaking- that the variable $\sigma_i \in \{-1, 1\}$ is more likely to take a positive value if the majority of the nearest neighbors of i are in a positive state and vice-versa.

worthiness of the firms. The random environment introduces *heterogeneity* in the contagion process.

We consider a network of N firms. The state of each firm is identified by two variables, that will be denoted by σ and ω ((σ_i, ω_i) is the state of the i-th firm). The variable σ may be interpreted as a credit quality indicator[2]: a low value basically reflects an higher probability of not being able to pay back obligations.
The variable ω represents the economic random environment in which a firm is operating. The simplest interpretation could be the *size* (or the market volume) of the firm itself. It summarizes the capacity of the firm of buffering news coming from the market. In this first model the fundamental variable ω_i will be stochastic but not time varying. Hence its distribution is assigned at time zero and remains constant.
To keep the model simple we consider the case in which $\sigma_i \in \{-1; 1\}$, hence it can take only two values. Being this model based on interacting intensities (as described in Section 2.3), we have to assign directly the rates/intensities (inverse of the average waiting time) at which the transitions $\sigma_i \mapsto -\sigma_i$ take place. In order to ensure the Markovianity of the system we choose intensities that depend on the state of the economy at time t (represented by $\underline{\sigma} = (\sigma_1, ..., \sigma_N)$ and $\underline{\omega} = (\omega_1, ..., \omega_N)$ where the dependence on time is implicitly assumed), so that

$$\sigma_i \mapsto -\sigma_i \quad \text{with intensity} \quad \lambda_i \equiv \lambda_i^N(\theta, \underline{\sigma}, \underline{\omega}) \tag{3.20}$$

where θ represents a vector of parameters that will be defined case by case in different specifications. In financial terms these parameters should represents a measure of the interaction (hence of the dependence) in the financial system.
What distinguishes the random environment ω from the Ψ defined in Example 2.3.1 is that ω may take different values in different sites. It should be stressed that Ψ was assumed to be a *time varying* exogenous parameter, whereas ω is fixed. We could also look at a dynamic (exogenous) ω. Nevertheless, we prefer to keep the treatment as simple as possible. In Chapter 4 we shall see extensions to a (endogenous) dynamically varying random environment.

The variable $\underline{\sigma} \in \{-1; 1\}^N$, for a given realization $\underline{\omega}$ of the random environment, evolves as a continuous time Markov chain with infinitesimal generator acting on functions $f : \{-1; 1\}^N \to \mathbb{R}$ as follows

$$\mathcal{G}_{[\underline{\omega}]} f(\underline{\sigma}) = \sum_{i=1}^{N} \lambda_i^N(\theta, \underline{\sigma}, \underline{\omega})[f(\underline{\sigma}^i) - f(\underline{\sigma})] \tag{3.21}$$

where $\underline{\sigma}^i$ represents the vector $\underline{\sigma}$ where the $i-$th component has been flipped.

Notice that the trajectories of this process, restricted to a time interval $[0, T]$, belong to the space $(\mathcal{D}[0, T])^N$ where $\mathcal{D}[0, T]$ denotes the space of right continuous, piecewise constant functions $[0, T] \to \{-1; 1\}$, endowed with the Skorohod topology (see [32]). We shall denote by $\sigma[0, T]$ a generic trajectory in $\mathcal{D}[0, T]$.

Remark 3.2.1 *We would like to stress the fact that from the mathematical viewpoint we are going to provide versions of the law of large numbers and central limit theorems*

[2] We shall specify in the following what σ represents in different models (a default indicator or a rating class indicator in credit migration models).

relying on large deviation techniques.
As already argued in Section 2.4.4, these results have the advantage to be dynamically consistent and to ensure closed form solutions for the asymptotic dynamics of aggregate variables (as the number of firms in financial distress at a given time) providing useful tools for the computation of large portfolio losses. Moreover they give a contribution to the development of the large deviation techniques applied to Finance.

We shall see that this model allows to describe dynamically credit contagion in a very natural way.
In this first model, the indicator σ represents a *default indicator*: $\sigma_i(t) = 1$ if the firm is still operating and $\sigma_i(t) = -1$ when default happens[3].

We consider only one parameter β indicating the level of interaction in the network, so that in this case $\theta \equiv \beta$.
An important statistic which collects the information coming from the individual firms is the so called *global health indicator*

$$m_N = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \tag{3.22}$$

This aggregate variable takes values in $[-1, 1]$ and gives a picture of the market since it counts the number of defaulted firms. The number of defaulted firms is exactly $N(1 - m_N)/2$. We are ready to specify our model.

Assumption 3.2.2 *We assume that*

(C.1) $\omega_i,\ i = 1, ..., N$ *are real i.i.d. random variables such that*

$$\omega_i \sim \eta$$

where η *has bounded support.*

(C.2) The rate of flipping for σ *at time* t *is given by*

$$\lambda_i^N(\beta, \underline{\sigma}, \underline{\omega}) = \mathbb{I}_{\{\sigma_i=1\}} \exp\{-\beta(\omega_i + m_N)\} \tag{3.23}$$

(C.3) $\sigma_i(0) = +1$ *for all* $i = 1, ..., N$.

$(C.1)$ says that in this model the environment is fixed at time zero in each site. Moreover we assume that the support of its distribution is bounded in the real line.
$(C.2)$ characterizes the marginal "intensity of default" for firm i. It depends on the random environment ω_i and on the global health of the system (via m^N). Notice that a good economic environment and a good health in the system decrease the probability of default.
$(C.3)$ simply says that at time zero all the obligors are alive.

Remark 3.2.3 *We give here two more remarks on Assumption 3.2.2.*

[3] The slightly different notation from the default indicator Y defined in 2.1.1 is simply for computational reasons. In this context it is in fact more useful to work with variables taking values in $\{-1; 1\}$ instead of $\{0; 1\}$.

1. *According to Assumption* $(C.1)$ *the variable* ω *has been exogenously specified. Moreover the dynamics of the variable* σ *are Markovian once a distribution for* ω *is given. In Chapter 4 we shall propose a different model where the behavior of* σ *and* ω *are closely linked and in particular the pair* $(\sigma(t), \omega(t))_{t\in[0,T]}$ *will define a Markov process.*

2. *By Assumption* $(C.2)$*, each component of* $\underline{\sigma}$ *is allowed to jump at most once. Thus the model may be described in terms of the default times* $\tau_1, ..., \tau_N$*, where* $\tau_i := \inf\{t > 0 : \sigma_i(t) = -1\}$*, avoiding the use of the path space* $\mathcal{D}[0,T]$*. We shall use this fact later. However we keep our approach and notations at the level of full generality for two reasons: most arguments will be readily extendible to the model in Chapter 4; the "finite dimensional" description in terms of* $(\tau_1, ..., \tau_N)$ *does not appear to allow relevant simplifications from the technical viewpoint.*

We are interested in describing the evolution over a time period $[0,T]$ of the system $(\sigma_i[0,T])_{i=1}^N$ as well as of the global health indicator m_N for large N. Our approach proceeds according to the following steps

i) Look for the limit dynamics of the system ($N \to \infty$);

ii) Describe the "finite volume approximations" (for large but finite N) via a central limit-type result.

Substituting (3.23) into (3.21) we obtain

$$\mathcal{G}_{[\underline{\omega}]} f(\underline{\sigma}) = \sum_{i=1}^{N} \mathbb{I}_{\{\sigma_i=1\}} \exp\{-\beta(\omega_i + m_N)\}[f(\underline{\sigma}^i) - f(\underline{\sigma})] \tag{3.24}$$

The operator $\mathcal{G}$ given in (3.24) defines an irreducible, finite-state Markov chain once a configuration for $\underline{\omega}$ has been specified.
It is easy to see that its unique stationary distribution has to be $\mu_N = \delta_{-1}^{\otimes N}$. This is because the configuration $\underline{\sigma} = (-1, ..., -1)$ is a coffin state.

3.3 Implementation of a Large Deviation Principle

We are going to obtain a large deviation principle for this model for contagious default, that will be stated in Theorem 3.3.3. We first collect some notations and definitions.

For a generic $T > 0$, we denote by $\tilde{\mathcal{D}}[0,T]$ the subspace od $\mathcal{D}[0,T]$ of those trajectories $\sigma[0,T]$ such that $\sigma(t) = 1$, for all $t < \tau \wedge T$ and $\sigma(t) = -1$ for $t \geq \tau \wedge T$ where τ is defined as $\inf\{t > 0 | \sigma(t) = -1\}$.
Let $(\sigma_i[0,T])_{i=1}^N \in \tilde{\mathcal{D}}([0,T])^N$ denote a path of the Markov process induced by (3.24), with initial condition $\sigma_i(0) = 1$ for all $i = 1, ..., N$.
We recall that $\mathcal{M}_1(\tilde{\mathcal{D}}[0,T] \times \mathbb{R})$ represents the space of probability measures on $\tilde{\mathcal{D}}[0,T] \times \mathbb{R}$, endowed with the topology induced by weak convergence. Moreover, with a slight abuse of notation we shall often write $\mathcal{M}_1$ without specifying the underlying trajectories space.
Any measure $Q \in \mathcal{M}_1$ can be written as

$$Q(d\sigma[0,T], d\omega) = Q^{\omega}(d\sigma[0,T]) \cdot \nu^{Q}(d\omega)$$

where ν^Q is the projection of Q on the field variable ω and Q^ω is a stochastic kernel (depending on ω) which is a measure on $\tilde{\mathcal{D}}[0,T]$.
We define $q_t^\omega = \Pi_t Q^\omega \in \mathcal{M}_1(\{-1;1\})$ as the projection at time t of the measure Q^ω.
Notice that $\Pi_t Q(d\sigma, d\omega) = q_t(d\sigma, d\omega) = q_t^\omega(d\sigma) \cdot \nu^Q(d\omega)$.
For any $q \in \mathcal{M}_1(\{-1,1\} \times \mathbb{R})$ we define

$$m_q = \int \sigma q(d\sigma, d\omega). \tag{3.25}$$

We denote by P_N^ω the law of $\underline{\sigma}[0,T] = (\underline{\sigma}_t)_{t\in[0,T]}$ given $\underline{\omega}$ under the dynamics induced by (3.24).
Its unconditional version will be denoted by $P_N(\cdot) = \int P_N^\omega(\cdot)\eta^{\otimes N}(d\underline{\omega})$, where $\eta^{\otimes N}$ denotes the product of N copies of the law η defined in (C.1).
We consider moreover the law W of the $\{-1;1\}$ valued process $\sigma(t)$ such that $\sigma(0) = 1$ and the rate of transition is equal to one in survival and zero otherwise so that $\lambda_i^W = \mathbb{I}_{\{\sigma_i=1\}}$.
We define the empirical measure

$$\rho_N(\underline{\sigma}[0,T], \underline{\omega}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{(\sigma_i[0,T],\omega_i)}. \tag{3.26}$$

Notice that $\rho_N \in \mathcal{M}_1(\tilde{\mathcal{D}}[0,T] \times \mathbb{R})$ and its projection at time t is denoted by $\rho_N(t)$. $\rho_N(t)$ is a measure on $\{-1;1\} \times \mathbb{R}$; following (3.25) we have

$$m_{\rho_N(t)} = \int \sigma \rho_N(t)(d\sigma, d\omega); \tag{3.27}$$

in particular we see that the global health indicator, computed at time t, as defined in (3.22), can be represented as an *empirical mean*, that is

$$m_N(t) = m_{\rho_N(t)} \qquad \forall t \in [0,T]. \tag{3.28}$$

Finally we define $\mathcal{P}_N$ (resp. $\mathcal{W}_N$) to be the law of ρ_N under the joint distribution of $(\underline{\sigma}, \underline{\omega})$, i.e., $\mathcal{P}_N(\cdot) = \int P_N^\omega(\rho_N \in \cdot\,)\eta^{\otimes N}(d\underline{\omega})$ (resp. under $(W \otimes \eta)^{\otimes N}$).

Having introduced many definitions we summarize in table 3.1 the main objects and notations that we are going to use frequently in what follows. Consider now the function $F : \mathcal{M}_1(\tilde{\mathcal{D}}[0,T] \times \mathbb{R}) \to \mathbb{R}$ defined as follows

$$F(Q) := \int Q(d\sigma[0,T], d\omega) \left\{ \int_0^{T\wedge\tau} \left(1 - e^{-\beta(\omega + m_{q_t})}\right) dt + \mathbb{I}_{\{\tau \le T\}} \left(-\beta(\omega + m_{q_{\tau^-}})\right) \right\} \tag{3.29}$$

where $\tau := \inf\{t > 0 : \sigma(t) = -1\}$.

We shall follow [18] in order to state a large deviation principle for the law $\mathcal{P}_N$ of the empirical measure and infer a strong limiting result for the system and the global health indicator. In this case we want to obtain a LDP as in Definition 3.1.1 where the Polish space to be considered is $\mathcal{M}_1(\tilde{\mathcal{D}}[0,T] \times \mathbb{R})$ and the sequence of probabilities is $\mathcal{P}_N$. We thus have to identify a suitable rate function I. We shall see that the relative entropy H (see Definition 3.1.5) and the function F will play a major role. Before stating the main theorem, we need to prove some technical lemmas.

$\sigma[0,T]$	The trajectory of the variable σ on the interval $[0,T]$
τ	$\inf\{t>0\|\sigma(t)=-1\}$
$\tilde{\mathcal{D}}[0,T]$	The space of trajectories such that: $\sigma(t)=1$, for all $t<\tau\wedge T$ and $\sigma(t)=-1$ for $t\geq\tau\wedge T$; endowed with the Skorohod topology
W	The law of $\sigma[0,T]$ under independence
η	The law of the ω component (see Assumption 3.2.2)
P_N^{ω}	The law of $\underline{\sigma}[0,T]$, induced by (3.24), given $\underline{\omega}$
$P_N(\cdot)$	The law of $(\underline{\sigma}[0,T],\underline{\omega})$ induced by (3.24)
$W^{\otimes N}\otimes\eta^{\otimes N}$	The law of $(\underline{\sigma}[0,T],\underline{\omega})$ under independence
$\rho_N(\underline{\sigma}[0,T],\underline{\omega})$	$\frac{1}{N}\sum_{i=1}^N\delta_{(\sigma_i[0,T],\omega_i)}$: The empirical measure
$\mathcal{P}_N$	The law of ρ_N under P_N
$\mathcal{W}_N$	The law of ρ_N under $W^{\otimes N}\otimes\eta^{\otimes N}$
$\mathcal{M}(E)$	The set of signed measures on the Polish space E
$\mathcal{M}_0(E)$	The set of signed measures with zero total mass on E
$\mathcal{M}_1(E)$	The set of probability measures on the Polish space E
$\mathcal{M}_A$	$\{Q\in\mathcal{M}_1(\tilde{\mathcal{D}}[0,T]\times\mathbb{R}):Q\ll W\otimes\eta\}$
$\mathcal{M}_H$	$\{Q\in\mathcal{M}_1(\tilde{\mathcal{D}}[0,T]\times\mathbb{R}):H(Q\|W\otimes\eta)<\infty\}$ where $H(\cdot\ \|W\otimes\eta)$ denotes the relative entropy w.r.t. $W\otimes\eta$

Table 3.1: Main notations and definitions of Chapter 3.

Lemma 3.3.1 *For given* $\underline{\omega}$

$$\frac{dP_N^{\omega}}{dW^{\otimes N}}(\underline{\sigma}[0,T])=\exp\{NF(\rho_N(\underline{\sigma}[0,T],\underline{\omega}))\}. \tag{3.30}$$

Proof. It basically follows from the Girsanov formula for point processes (See [7]).

$$\frac{dP_N^{\omega}}{dW^{\otimes N}}=\exp\left\{\sum_{i=1}^N\int_0^T\left[\mathbb{I}_{\{\tau_i>t\}}-\mathbb{I}_{\{\tau_i>t\}}e^{-\beta(\omega_i+m_{\rho_N(t)})}\right]dt+\right.$$

$$\left.+\sum_{i=1}^N\mathbb{I}_{\{\tau_i\leq T\}}\left(-\beta(\omega_i+m_{\rho_N(\tau_i^-)})\right)\right\};$$

where, as usual, $\tau_i=\inf\{t>0:\sigma_i(t)=-1\}$.
On the other hand we compute $F(\rho_N)$ as

$$F(\rho_N)=\frac{1}{N}\sum_{i=1}^N\int_0^{T\wedge\tau_i}\left(1-e^{-\beta(\omega_i+m_{\rho_N(t)})}\right)dt+$$

$$+\frac{1}{N}\sum_{i=1}^N\mathbb{I}_{\{\tau_i\leq T\}}(-\beta(\omega_i+m_{\rho_N(t^-)})),$$

hence the thesis follows. ∎

Lemma 3.3.2 *F(Q) is bounded on* $\mathcal{M}_1$ *and continuous on* $\mathcal{M}_A:=\{Q:Q\ll W\otimes\eta\}$.

Proof. We rewrite $F(Q)$ as given in Equation (3.29).

$$F(Q) = \int Q(d\sigma[0,T], d\omega)\left\{\int_0^T \mathbb{I}_{\{\tau>t\}}\left(1 - e^{-\beta(\omega+m_{q_t})}\right) dt + \right.$$

$$\left. + \mathbb{I}_{\{\tau\leq T\}}\ \left(-\beta(\omega + m_{q_{\tau^-}})\right)\right\} =$$

$$= E^Q\left[\int_0^T \frac{1+\sigma(t)}{2}\ \left(1 - e^{-\beta(\omega+m_{q_t})}\right) dt - \beta(\omega + m_{q_{\tau^-}})\mathbb{I}_{\{\tau\leq T\}}\right] \tag{3.31}$$

where we have used the fact that $\int_0^T I_{\{\tau>t\}}(\cdot)dt = \int_0^T \frac{1+\sigma(t)}{2}\ (\cdot)dt$.
We show first that F can be rewritten in the following form

$$F(Q) = \int_0^T \frac{1+m_{q_t}}{2}\ dt - \int_0^T E^Q\left[\frac{1+\sigma(t)}{2}\ e^{-\beta\omega}\right] e^{-\beta m_{q_t}}\ dt -$$

$$-\beta E^Q\left[\omega\mathbb{I}_{\{\tau\leq T\}}\right] - \beta[Q(\tau\leq T) - Q^2(\tau\leq T)] - \beta\sum_{t\in[0,T]}(\Delta Q(\tau\leq t))^2; \tag{3.32}$$

where $\Delta f(t) := f(t) - f(t^-)$.

First of all, notice that the argument of the integral in (3.31) is bounded, thus we are allowed to interchange the expectation with respect to Q and the time integral. In particular

$$E^Q\left[\int_0^T \frac{1+\sigma(t)}{2}\ dt\right] = \int_0^T \frac{1+m_{q_t}}{2}\ dt.$$

and

$$E^Q\left[\int_0^T \frac{1+\sigma(t)}{2}\left(-e^{-\beta(\omega+m_{q_t})}\right)\ dt\right] = -\int_0^T E^Q\left[\frac{1+\sigma(t)}{2}\ e^{-\beta\omega}\right] e^{-\beta m_{q_t}}\ dt.$$

The term $-\beta E^Q[\omega\mathbb{I}_{\{\tau\leq T\}}]$ appears in both (3.31) and (3.32). It remains to show that

$$E^Q\left[m_{q_{\tau^-}}\mathbb{I}_{\{\tau\leq T\}}\right] = Q(\tau\leq T) - Q^2(\tau\leq T) + \sum_{t\in[0,T]}(\Delta Q(\tau\leq t))^2. \tag{3.33}$$

The l.h.s. can be written as

$$E^Q\left[m_{q_{\tau^-}}\mathbb{I}_{\{\tau\leq T\}}\right] = \int_0^T m_{q_{t^-}} dQ(\tau\leq t) = \int_0^T [1 - 2Q(\tau<t)]dQ(\tau\leq t),$$

where we use the fact that $m_{q_{t^-}} = Q(\tau\geq t) - Q(\tau<t) = 1 - 2Q(\tau<t)$ and where we denote by $dQ(\tau\leq t)$ the Lebesgue-Stieltjes measure associated to the distribution function of τ. Furthermore,

$$\int_0^T [1 - 2Q(\tau<t)]dQ(\tau\leq t) = Q(\tau\leq T) - 2\int_0^T Q(\tau<t)dQ(\tau\leq t). \tag{3.34}$$

Integrating the latter term by parts[4], we see that

$$\int_0^T Q(\tau < t)dQ(\tau \leq t) = -\int_0^T Q(\tau \leq t)dQ(\tau \leq t) + Q^2(\tau \leq T) - Q^2(\tau \leq 0).$$

We now add $\int_0^T Q(\tau < t)dQ(\tau \leq t)$ to both terms, obtaining

$$2\int_0^T Q(\tau < t)dQ(\tau \leq t) = -\int_0^T [Q(\tau \leq t) - Q(\tau < t)]dQ(\tau \leq t) + Q^2(\tau \leq T).$$

Notice that $\int_0^T [Q(\tau \leq t) - Q(\tau < t)]dQ(\tau \leq t) = \sum_{t\in[0,T]}(\Delta Q(\tau \leq t))^2$. Hence

$$-2\int_0^T Q(\tau < t)dQ(\tau \leq t) = \sum_{t\in[0,T]} (\Delta Q(\tau \leq t))^2 - Q^2(\tau \leq T). \tag{3.35}$$

Substituting the r.h.s. of (3.35) into (3.34) we obtain

$$E^Q\left[m_{q_{\tau^-}} \mathbb{I}_{\{\tau\leq T\}}\right] = Q(\tau \leq T) - Q^2(\tau \leq T) + \sum_{t\in[0,T]} (\Delta Q(\tau \leq t))^2,$$

and Equation (3.32) follows.

We show now boundedness and continuity of F. The boundedness is easily proved since $\sigma \in \{-1; 1\}$, $\omega \sim \eta$ and η has bounded support and $\sum_{t\in[0,T]}(\Delta Q(\tau \leq t))^2 \leq 1$. In order to prove the continuity on $\mathcal{M}_A$, we consider a sequence of probabilities $(Q_n)_{n\geq 0} \in \mathcal{M}_1$ converging weakly to $Q \in \mathcal{M}_A$. We want to show that

$$\lim_n E^{Q_n}[f(\omega)\sigma(t)] = E^Q[f(\omega)\sigma(t)]$$

for all $t \in [0, T]$ and for any measurable and continuous function $f : \mathbb{R} \to \mathbb{R}$ bounded on the support of η. This statement is not trivial, since the projection $\sigma[0, T] \to \sigma(t)$ is not continuous in $\tilde{\mathcal{D}}[0, T]$. However, define for any $\varepsilon > 0$ the functions

$$g_t^-(\varepsilon; f, \sigma) := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(\omega)\sigma(s)ds\ , \qquad g_t^+(\varepsilon; f, \sigma) := \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} f(\omega)\sigma(s)ds;$$

where we suppose that the trajectory $\sigma_{[0,T]}$ can be extended to the larger interval $[0 - \varepsilon,\ T + \varepsilon]$ by continuity.
These functions are continuous in $\tilde{\mathcal{D}}[0, T]$, bounded by $\|f\|_\infty$ for any ε and such that $g_t^-(\varepsilon; f, \sigma) \leq f(\omega)\sigma(t) \leq g_t^+(\varepsilon; f, \sigma)$ a.s. for any t. Thus, by the Lebesgue convergence theorem,

$$\limsup_n E^{Q_n}[f(\omega)\sigma(t)] \leq \lim_n E^{Q_n}[g^+(\varepsilon; f, \sigma)] = E^Q[g^+(\varepsilon; f, \sigma)], \quad \forall\varepsilon > 0.$$

[4]We use the generalized integration by part formula for functions with bounded variation. Indeed for f and g functions with bounded variation we have

$$\int_0^T f(t^-)dg(t) = -\int_0^T g(t)df(t) + f(T)g(T) - f(0)g(0).$$

We apply this formula with $f(t) = g(t) = Q(\tau \leq t)$.

Letting $\varepsilon \to 0$ and noticing that $\lim_{\varepsilon\to 0} g_t^+(\varepsilon; f, \sigma) = f(\omega)\sigma(t^-)$ we get

$$\limsup_n E^{Q_n}[f(\omega)\sigma(t)] \leq E^Q[f(\omega)\sigma(t^-)].$$

The same argument holds for $g_t^-(\varepsilon; f, \sigma)$; here $\lim_{\varepsilon\to 0} g_t^-(\varepsilon; f, \sigma) = f(\omega)\sigma(t)$. Thus

$$E^Q[f(\omega)\sigma(t)] \leq \liminf_n E^{Q_n}[f(\omega)\sigma(t)] \leq \limsup_n E^{Q_n}[f(\omega)\sigma(t)] \leq E^Q[f(\omega)\sigma(t^-)].$$

Notice that $f(\omega)\sigma(t)$ and $f(\omega)\sigma(t^-)$ may differ only on the event $\{\sigma(t^-) \neq \sigma(t)\}$. But this event has measure zero for any $Q \in \mathcal{M}_A$, since $(W \otimes \eta)(\{\sigma(t^-) \neq \sigma(t)\}) = 0$. This implies that the corresponding expected values must coincide; as a consequence $E^Q[f(\omega)\sigma(t)] - E^Q[f(\omega)\sigma(t^-)] = 0$. We have thus proved that [5]

$$\lim_n E^{Q_n}[f(\omega)\sigma(t)] = E^Q[f(\omega)\sigma(t)] \quad \text{for all } t. \tag{3.36}$$

Taking $f(\omega) \equiv 1$ we simply have that for all t, $m_{q_t} = E^Q[\sigma(t)]$ is a continuous mapping in Q on $\mathcal{M}_A$. Choosing instead $f(\omega) = \omega$ and $f(\omega) = e^{-\beta\omega}$, we prove the thesis for $E^Q\left[\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right]$ and $E^Q\left[\omega \mathbb{I}_{\{\tau \leq T\}}\right]$, where the latter follows from the fact that $\mathbb{I}_{\{\tau \leq T\}} = \frac{1-\sigma(T)}{2}$. The same argument ensures that $Q(\tau \leq T) = E^Q[\mathbb{I}_{\{\tau \leq T\}}]$ is continuous in Q.
The next step is to show that $Q_n \to Q$ implies

$$\left| E^Q\left[\int_0^T \left(\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right) e^{-\beta m_{q_t}} dt\right] - E^{Q_n}\left[\int_0^T \left(\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right) e^{-\beta m_{q_t^n}} dt\right]\right| \to 0 \tag{3.37}$$

where $q_t^n := \Pi_t Q_n$.
We add and subtract $E^{Q_n}\left[\int_0^T \left(\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right) e^{-\beta m_{q_t}}\, dt\right]$, to the expression in $|\cdot|$:

$$\left| E^Q\left[\int_0^T \left(\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right) e^{-\beta m_{q_t}}\, dt\right] - E^{Q_n}\left[\int_0^T \left(\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right) e^{-\beta m_{q_t}}\, dt\right] + \right.$$

$$\left. + E^{Q_n}\left[\int_0^T \left(\frac{1+\sigma(t)}{2} e^{-\beta\omega}\right)\left(e^{-\beta m_{q_t}} - e^{-\beta m_{q_t^n}}\right)\, dt\right]\right| \leq |a_n| + |b_n|$$

where

$$a_n = E^Q\left[\int_0^T \left(\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right) e^{-\beta m_{q_t}} dt\right] - E^{Q_n}\left[\int_0^T \left(\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right) e^{-\beta m_{q_t}} dt\right];$$

$$b_n = E^{Q_n}\left[\int_0^T \left(\frac{1+\sigma(t)}{2} e^{-\beta\omega}\right)\left(e^{-\beta m_{q_t}} - e^{-\beta m_{q_t^n}}\right)\, dt\right].$$

$|a_n|$ goes to zero by weak convergence. Concerning b_n we see that

$$|b_n| \leq \int_0^T \left| E^{Q_n}\left[\left(\frac{1+\sigma(t)}{2} e^{-\beta\omega}\right)\left(e^{-\beta m_{q_t}} - e^{-\beta m_{q_t^n}}\right)\right]\right| dt \leq$$

[5] In saying that $(W \otimes \eta)(\{\sigma(t^-) \neq \sigma(t)\}) = 0$ we have used the fact that the distribution function of τ under $W \otimes \eta$ is an exponential distribution with parameter 1. In particular, it is absolutely continuous. This observation suggests that $E^{Q_n}[\sigma(t)]$ converges to $E^Q[\sigma(t)]$ pointwise in t, for all those t such that $Q(\tau = t) = 0$ even if Q does not belong to $\mathcal{M}_A$. We shall use this fact in what follows.

$$\leq \int_0^T \left| e^{-\beta m_{q_t}} - e^{-\beta m_{q_t^n}} \right| \cdot \left| E^{Q_n} \left[\left(\frac{1+\sigma(t)}{2} e^{-\beta\omega} \right) \right] \right| dt.$$

Notice that, being $E^{Q_n}\left[\left(\frac{1+\sigma(t)}{2}e^{-\beta\omega}\right)\right]$ uniformly bounded in n, and since by what shown in (3.36), $m_{q_{t_n}} \to m_{q_t}$ we have that

$$\lim_{n\to\infty} \left| e^{-\beta m_{q_t}} - e^{-\beta m_{q_t^n}} \right| \cdot \left| E^{Q_n} \left[\left(\frac{1+\sigma(t)}{2} e^{-\beta\omega} \right) \right] \right| = 0.$$

As a consequence, $|b_n|$ goes to zero as well, since we are allowed to interchange the limit and the time integral, by dominated convergence.

It remains to show that $Q_n \to Q$ implies

$$\sum_{t\in[0,T]} (\Delta Q_n(\tau \leq t))^2 \to \sum_{t\in[0,T]} (\Delta Q(\tau \leq t))^2 = 0;$$

where the last equality is due to the fact that $Q \ll (W \otimes \eta)$, that is, the distribution of τ under Q has no jumps.
For all n, we take enumerations $\{t_1^{(n)}, t_2^{(n)}, ...\}$ of the jumps of $Q_n(\tau \leq t)$.
Define $s_n := \sup_k \left| \Delta Q_n(\tau \leq t_k^{(n)}) \right|$. We claim that

$$\lim_{n\to\infty} s_n = 0. \tag{3.38}$$

In this case we have

$$\sum_{t\in[0,T]} (\Delta Q_n(\tau \leq t))^2 \leq s_n \cdot \sum_{t\in[0,T]} (\Delta Q_n(\tau \leq t)) \leq s_n;$$

where the second inequality follows since $\sum_{t\in[0,T]} (\Delta Q_n(\tau \leq t)) \leq 1$.
Thus we see that the thesis follows once we prove the validity of (3.38).
Suppose, by way of contradiction, that there exists $t_{k_n}^{(n)}$ such that

$$\left| \Delta Q_n(\tau \leq t_{k_n}^{(n)}) \right| \geq \varepsilon > 0,$$

that is

$$Q_n\left(\tau \in (t_{k_n}^{(n)} - \delta, t_{k_n}^{(n)} + \delta] \right) \geq \varepsilon \ ; \qquad \forall \delta > 0.$$

On the other hand, being $[0,T]$ compact, $t_{k_n}^{(n)}$ admits a convergent subsequence. We denote by $\bar{t}$ its limit. Hence, along this subsequence and for n large enough

$$Q_n\left(\tau \in (\bar{t} - \delta, \bar{t} + \delta] \right) \geq \varepsilon \ ; \qquad \forall \delta > 0.$$

This implies that $Q\left(\tau \in (\bar{t} - \delta, \bar{t} + \delta] \right) \geq \varepsilon$ for all $\delta > 0$.
This fact gives a contradiction, since $Q(\tau \leq t)$ has no jumps. We have thus proved the continuity of all the summands of the function F. ∎

As already said, we want to state a LDP for the sequence of distributions $\mathcal{P}_N$. Thanks to Lemma 3.3.1, we have identified the Radon Nikodym derivative that relates $W^{\otimes N}$ and P_N^ω (where $W^{\otimes N}$ plays the role of the reference measure). The most natural way to develop a large deviation principle is now to rely on the Varadhan Lemma (see Proposition 3.1.10).
We are ready to prove the main result of this section.

Theorem 3.3.3 *For each* $Q \in \mathcal{M}_1(\tilde{\mathcal{D}}[0,T] \times \mathbb{R})$ *define*

$$I(Q) = H(Q|W \otimes \eta) - F(Q), \tag{3.39}$$

then the sequence $(\mathcal{P}_N)_N$ *obeys a* Large Deviation Principle *(LDP) with* good rate function $I(\cdot)$.

Proof.
Since $(\sigma_i[0,T];\omega_i)$ are i.i.d. random variables under $(W \otimes \eta)^{\otimes N}$, we can apply Sanov's Theorem (see Theorem 3.1.7) to the sequence of measures $(\mathcal{W}_N)_N$, where $\mathcal{W}_N$ represents the law of the empirical measure in the case of independence (i.e. under $(W \otimes \eta)^{\otimes N}$). Hence $(\mathcal{W}_N)_N$ obeys a large deviation principle with rate function $H(Q|W \otimes \eta)$.
Being $F(Q)$ bounded in the weak topology and continuous on $\mathcal{M}_A \supset \mathcal{M}_H = \{Q \in \mathcal{M}_1 : H(Q|W \otimes \eta) < \infty\}$, we can rely on Proposition 3.1.10 to conclude that the sequence $(\mathcal{P}_N)_N$ obeys a large deviation principle with good rate function

$$I(Q) = H(Q|W \otimes \eta) - F(Q) - \inf_{R \in \mathcal{M}_1} [H(R|W \otimes \eta) - F(R)].$$

We finish the proof by showing that

$$\inf_{R \in \mathcal{M}_1} [H(R|W \otimes \eta) - F(R)] = \lim_{N \to \infty} \frac{1}{N} \log \left[\int_{\mathcal{M}_1} e^{NF(Q)} \mathcal{W}_N(dQ) \right] = 0. \tag{3.40}$$

The first equality is simply a consequence of Equation (3.10).
The thesis thus follows if $\int_{\mathcal{M}_1} e^{NF(Q)} \mathcal{W}_N(dQ) = 1$. This fact is a consequence of Lemma 3.3.1, indeed

$$\mathcal{P}_N(\,\cdot\,) = \int \eta^{\otimes N}(d\underline{\omega}) P_N^{\omega}(\rho_N \in \cdot\,) =$$

$$= \int \eta^{\otimes N}(d\underline{\omega}) \int \mathbb{I}_{\{\rho_N \in \cdot\,\}} \frac{dP_N^{\omega}}{dW^{\otimes N}} dW^{\otimes N} = \int \mathbb{I}_{\{\rho_N \in \cdot\,\}} e^{NF(\rho_N)}\, d(W^{\otimes N} \otimes \eta^{\otimes N}) =$$

$$= \int \mathbb{I}_{\{Q \in \cdot\,\}} e^{NF(Q)} \mathcal{W}_N(dQ).$$

Being $\mathcal{P}_N(\mathcal{M}_1) = 1$, the thesis follows. ∎

Let now $Q \in \mathcal{M}_1(\tilde{\mathcal{D}}[0,T] \times \mathbb{R})$. We associate with Q the law of a time inhomogeneous Markov process on $\{-1;1\}$ which evolves according to the following rules:

$\sigma = +1 \to \sigma = -1$	with intensity	$e^{-\beta(\omega + m_{q_t})}$
$\sigma = -1 \to \sigma = +1$	with intensity	0

and with $\sigma_i(0) = 1$ for all $i = 1, ..., N$.
We recall that $q_t = \Pi_t Q \in \mathcal{M}_1(\{-1;1\})$ represents the projection at time t of the measure Q and $m_{q_t} = \int \sigma dq_t$. We denote by $P^{\omega,Q}$ the law of this process and by $P^Q = P^{\omega,Q} \otimes \eta$. In other words, $P^{\omega,Q}$ is the law of the Markov process on $\{-1;1\}$ with initial distribution δ_1 and time-dependent generator $\mathcal{L}_t^Q$ defined as

$$\mathcal{L}_t^Q f(\sigma) = \mathbb{I}_{\{\sigma=1\}} e^{-\beta(\omega + m_{q_t})} (f(-\sigma) - f(\sigma)). \tag{3.41}$$

We show now an important property of P^Q.

Proposition 3.3.4 *For every $Q \in \mathcal{M}_1(\tilde{D}[0,T] \times \mathbb{R})$, we have*

$$I(Q) = H(Q|P^Q).$$

Proof.
We distinguish two cases:
<u>*Case 1*</u>. $Q : H(Q|W \otimes \eta) < \infty$. We have (see (3.39))

$$I(Q) = \int \log \frac{dQ}{d(W \otimes \eta)} dQ - F(Q).$$

By Girsanov's formula for continuous time Markov chains, we obtain

$$\log \frac{dP^{\omega,Q}}{dW} = \int_0^{T \wedge \tau} \left(1 - e^{-\beta(\omega + m_{q_t})}\right) dt + \left[\mathbb{I}_{\{\tau \leq T\}} \left(-\beta(\omega + m_{q_{\tau-}})\right)\right];$$

hence, by definition of F given in (3.29) we have

$$F(Q) = \int \log \frac{dP^{\omega,Q}}{dW} \, dQ$$

so that

$$I(Q) = \int \log \frac{dQ}{d(W \otimes \eta)} dQ - \int \log \frac{dP^{\omega,Q}}{dW} dQ = \int \log \frac{dQ}{dP^Q} dQ \qquad (3.42)$$

where the last equality follows from

$$\frac{dQ}{d(W \otimes \eta)} \frac{dW}{dP^{\omega,Q}} = \frac{dQ}{d(W \otimes \eta)} \frac{d(W \otimes \eta)}{dP^Q} = \frac{dQ}{dP^Q}.$$

Being $\int \log \frac{dQ}{dP^Q} dQ = H(Q|P^Q)$, the thesis follows.

<u>*Case 2*</u>. $Q : H(Q|W \otimes \eta) = +\infty$. In this case $I(Q) = +\infty$.
Thus we have to check that $H(Q|P^Q) = +\infty$ as well. Being

$$H(Q|P^Q) = \int \log \frac{dQ}{d(W \otimes \eta)} dQ + \int \log \frac{dW}{dP^{\omega,Q}} dQ,$$

the thesis follows since $W \sim P^{w,Q}$ whereas $\int \log \frac{dQ}{d(W \otimes \eta)} dQ = +\infty$ since $H(Q|W \otimes \eta) = +\infty$. ∎

Furthermore we have

Proposition 3.3.5 *The equation $I(Q) = 0$ has a unique solution Q_*. Moreover Q_* has the following properties*

i) $\nu^{Q_*} = \eta$, *with η defined in Assumption 3.2.2;*

ii) Q_*^ω *is $\eta -$ a.s. the law of a Markov process;*

iii) $q_t^{*,\omega} := \Pi_t Q_*^\omega$ *solves the so called McKean-Vlasov equation*

$$\begin{cases} \frac{\partial}{\partial t} q_t^{*,\omega} &= \mathcal{G}_t^\omega q_t^{*,\omega} \\ q_0^{*,\omega} &= \delta_1 \end{cases} \tag{3.43}$$

where

$$(\mathcal{G}_t^\omega q_t^{*,\omega})(x) = -xe^{-\beta\left(\omega + m_{q_t^*}\right)} q_t^{*,\omega}(1) \tag{3.44}$$

and where $Q_*(d\sigma[0,T], d\omega) = Q_*^\omega(d\sigma[0,T]) \cdot \nu^{Q_*}(d\omega)$.

Proof. By properness of the relative entropy ($H(\mu|\nu) = 0 \Rightarrow \mu = \nu$), from Proposition 3.3.4 we have that the equation $I(Q) = 0$ is equivalent to $Q = P^Q$. Suppose Q_* is a solution of this last equation. In this case (i) and (ii) easily follow. Moreover, $q_t^* := \Pi_t Q_* = \Pi_t P^{Q_*}$. The marginals of a Markov process are solutions of the corresponding *forward equation* that, in this case, leads to the fact that q_t^* is a solution of equation (3.43) where $\mathcal{G}_t^\omega$ is the adjoint of $\mathcal{L}_t^Q$. This differential equation, being an equation in finite dimension with locally Lipschitz coefficients, has at most one solution in $[0,T]$, for a given initial condition. Since P^{Q_*} is totally determined by the flow q_t^*, it follows that equation $Q = P^Q$ has at most one solution. The existence of a solution follows from the fact that $I(Q)$ is the rate function of a LDP, and therefore *must* have at least one zero.
We now show that $\mathcal{G}_t^\omega$ has the form given in (3.44). Being the adjoint of $\mathcal{L}_t^Q$, it must satisfy

$$(\mathcal{G}_t^\omega q)(x) = \mathbb{I}_{\{\sigma=1\}}(-x)e^{-\beta(\omega+m_q)}q(-x) - \mathbb{I}_{\{\sigma=1\}}(x)e^{-\beta(\omega+m_q)}q(x).$$

When $x = 1$ we have

$$(\mathcal{G}_t^\omega q)(1) = \mathbb{I}_{\{\sigma=1\}}(-1)e^{-\beta(\omega+m_q)}q(-1) - \mathbb{I}_{\{\sigma=1\}}(1)e^{-\beta(\omega+m_q)}q(1) =$$

$$= -e^{-\beta(\omega+m_q)}q(1)$$

and when $x = -1$

$$(\mathcal{G}_t^\omega q)(-1) = \mathbb{I}_{\{\sigma=1\}}(1)e^{-\beta(\omega+m_q)}q(1) - \mathbb{I}_{\{\sigma=1\}}(-1)e^{-\beta(\omega+m_q)}q(-1) =$$

$$= e^{-\beta(\omega+m_q)}q(1).$$

We have thus proved Formula (3.44). ∎

In the following theorem we derive a law of large numbers for the sequence of empirical measures.

Theorem 3.3.6 *Consider the Markov process* $(\underline{\sigma}(t))_{t\geq 0}$ *with generator (3.24) and such that the random variables* $\sigma_i(0) = 1$ *and* $\omega_i \sim \eta$, $i = 1, \ldots, N$. *Then*

$$\rho_N \;\rightarrow\; Q_* \qquad \textit{almost surely}$$

in the weak topology, where Q_* *has been defined in Proposition 3.3.5. Moreover*

$$\rho_N(t) \rightarrow q_t^* \qquad \textit{almost surely} \tag{3.45}$$

in the weak topology, where $q_t^* = q_t^{*,\omega} \cdot \nu^{Q_*}$ *and* $q_t^{*,\omega}$ *satisfies (3.43).*

Proof. Let Q_* be the unique zero of the good rate function $I(\cdot)$ as given by Proposition 3.3.5. Let B_{Q_*} be an arbitrary open neighborhood of Q_*. By the upper bound in Formula (3.1), we have

$$\limsup_N \frac{1}{N} \log \mathcal{P}_N(B^c_{Q_*}) \leq - \inf_{Q \notin B_{Q_*}} I(Q) < 0, \tag{3.46}$$

where the last inequality comes from lower-semicontinuity of $I(\cdot)$, compactness of its level sets and the fact that $I(Q) > 0$ for every $Q \neq Q_*$. Indeed, if $\inf_{Q \notin B_{Q_*}} I(Q) = 0$, then there exists a sequence $Q_n \notin B_{Q_*}$ such that $I(Q_n) \to 0$. By the compactness of the level sets there exists then a subsequence $Q_{n_k} \to \bar{Q} \notin B_{Q_*}$. By lower- semicontinuity it then follows $I(\bar{Q}) \leq \liminf I(Q_{n_k}) = 0$ which contradicts $I(Q) > 0$ for $q \neq Q_*$. By the above inequality we thus have that $\mathcal{P}_N(B^c_{Q_*})$ decays to 0 exponentially fast. By a standard application of the Borel-Cantelli Lemma, we obtain that $\rho_n \to Q_*$ almost surely and this leads to the first assertion of the theorem.
The convergence of the projections as in Formula (3.45) is not trivial since the projection maps are not a priori continuous under the weak topology.
Nevertheless they are continuous in Q_*. This follows by the argument used in the proof of Lemma 3.3.2, that is, the projections are continuous in all the measures Q such that the distribution of τ under Q is continuous. In particular Q_* belongs to this set, hence the convergence is ensured also on its projections. ∎

3.4 A law of large numbers for portfolio losses

In this section we derive a (dynamic) law of large numbers for the global health indicator $m_N(t) = \sum_i \sigma_i(t)/N$. Recall that $m_N(t) = \int \sigma \rho_N(t)(d\sigma, d\omega)$. This implies that the law of large numbers for the empirical law ρ_N, proved in Theorem 3.3.6, is the starting point for deriving limiting results for the global health indicator and eventually for the distribution of losses in large portfolios.
In this section, in order to simplify the notation, we use the following expressions for the asymptotic health indicators

$$m(t) \text{ instead of } m_{q^*_t},$$

$$m(t;\omega) \text{ instead of } m_{q^{*,\omega}_t};$$

where q^* and $q^{*,\omega}$ have been defined in Theorem 3.3.6.

Proposition 3.4.1 *Under Assumption 3.2.2 we have that the sequence of global health indicators satisfies, for any $t \in [0,T]$ fixed,*

$$\lim_{N\to\infty} m_N(t) = m(t), \quad a.s. \tag{3.47}$$

Moreover $m(t) = \int m(t;\omega)\eta(d\omega)$, where $m(t;\omega)$ solves

$$\begin{cases} \dot{m}(t;\omega) &= -(1 + m(t;\omega))e^{-\beta(\omega+m(t))} \\ m(0;\omega) &= 1 \end{cases} \tag{3.48}$$

Proof. We recall that $m_N(t) = m_{\rho_N(t)}$. By Theorem 3.3.6, $\rho_N(t) \to q_t^*$ a.s. in the weak topology, that is

$$\int F(\sigma,\omega)\rho_N(t)(d\sigma,d\omega) \to \int F(\sigma,\omega)q_t^*(d\sigma,d\omega)$$

for continuous and bounded functions F. Taking $F(\sigma,\omega) = \sigma$ for $\sigma \in \{-1;1\}$, we have exactly (3.47).
The fact that $m(t) = \int m(t;\omega)\eta(d\omega)$ follows from the decomposition $q_t^* = q_t^{*,\omega} \cdot \nu^{Q_*}$, as provided in Theorem 3.3.6 and noticing that, by Proposition 3.3.5, $\nu^{Q_*} = \eta$. Concerning Equation (3.48) we can write

$$m(t;\omega) = q_t^{*,\omega}(1) - q_t^{*,\omega}(-1)$$

then $m(0;\omega) = 1$ and

$$\dot{m}(t;\omega) = \dot{q}_t^{*,\omega}(1) - \dot{q}_t^{*,\omega}(-1) = -2q_t^{*,\omega}(1)e^{-\beta(\omega+m(t))} \tag{3.49}$$

where we have used (3.43). Since $m(t;\omega) = q_t^{*,\omega}(1) - q_t^{*,\omega}(-1) = -1 + 2q_t^{*,\omega}(1)$ we have $q_t^{*,\omega}(1) = (1+m(t;\omega))/2$ hence

$$\dot{m}(t;\omega) = -(1+m(t;\omega))e^{-\beta(\omega+m(t))}$$

■

We now use these results, in order to compute the losses that a bank may suffer in a credit portfolio, due to the fact that the firms involved may default before paying back their obligations.
We recall the definition of portfolio losses as given in Definition 2.1.4. $L^{(N)}(t)$ stands for the aggregate loss at time t and more precisely

$$L^{(N)}(t) = \sum_{i=1}^{N} L_i(t),$$

where $L_i(t)$ are the marginal losses due to the single firms.
As an example, we consider here the very basic case in which $L_i(t) = 1$ in case of default and zero otherwise[6].

Corollary 3.4.2 *Let $L^{(N)}(t)$ denote the aggregate loss at time t of a N-dimensional portfolio as given in Definition 2.1.4. Suppose moreover that $L_i(t) = \mathbb{I}_{\{\sigma(t)=-1\}}$. Then*

$$L^{(N)}(t) \to \frac{1-m(t)}{2}$$

Proof. We have

$$L^{(N)}(t) = \sum_{i=1}^{N} L_i(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{1-\sigma_i(t)}{2} = \frac{1-m_N(t)}{2}.$$

[6]It is possible to consider generalizations of this case assuming a more complex recovery structure in case of default (see Examples 5.0.12 and 5.0.14 in Chapter 5).

Thus the thesis immediately follows from Proposition 3.4.1. ∎

These limiting results are in line with the literature and propose a different approach to the study of large portfolio losses as we already stressed in Section 2.4.4. From the technical viewpoint we have thus developed a large deviation approach to the study of large portfolios, obtaining strong limiting results for the empirical distributions of the Markov chain $(\underline{\sigma}(t))_{t\in[0,T]}$ and its conditional and unconditional empirical means.

In the next section we shall provide some numerical studies of the evolution in time of the global health indicator under different specifications of the model (different values of β and different laws η). We shall then show some interesting features of these trajectories. In particular we shall discuss why this first model is not completely satisfactory in our perspective and how it can be modified in order to capture important real world effects that can not be described by this simple model.

3.4.1 Simulation results

We first explore the easiest case where we have a fixed (non random) environment, meaning that $\omega = \rho$ where $\rho \in \mathbb{R}$.

A: Non random environment

This specification corresponds to the case $\eta = \delta_\rho$ for $\rho \in \mathbb{R}$. Without loss of generality we may assume $\rho = 0$. Notice that in this case $m(t;\omega) = m(t;0) = m(t)$, so that Equation (3.48) can be rewritten as

$$\begin{cases} \dot{m}(t) &= -(1+m(t))e^{-\beta m(t)} \\ m(0) &= 1 \end{cases} \tag{3.50}$$

We can even consider the trivial case in which there is no interaction at all (i.e. $\beta = 0$), then the solution is simply given by

$$m(t) = 2e^{-t} - 1$$

When $\beta \neq 0$ we can only give some numerical descriptions of the solution of the differential equation above. In Figure 3.1 we have plotted the trajectories of $m(t)$ for different values of β. When β is small, the exponential term does not give a significant contribution to the evolution, thus $m(t)$ is not far from the benchmark case in which $\beta = 0$. The situation changes for $\beta > 1$. In this case the global health indicator decreases very slowly for small t but at a certain time it starts decaying fast to the value -1. Notice that the decay to the equilibrium is typically very fast and this effect is more pronounced for higher values of β. Notice also that the concavity/convexity is different for β bigger and lower than one.

B: Introducing a random environment

As a simple example we make Assumption $(C.1)$ more precise by assuming that $\eta = \frac{1}{2}\delta_\rho + \frac{1}{2}\delta_{-\rho}$. This means that ω may take the values $-\rho$ and ρ with the same probability (the asymmetric case $p \neq 1/2$ can be treated in the same way).

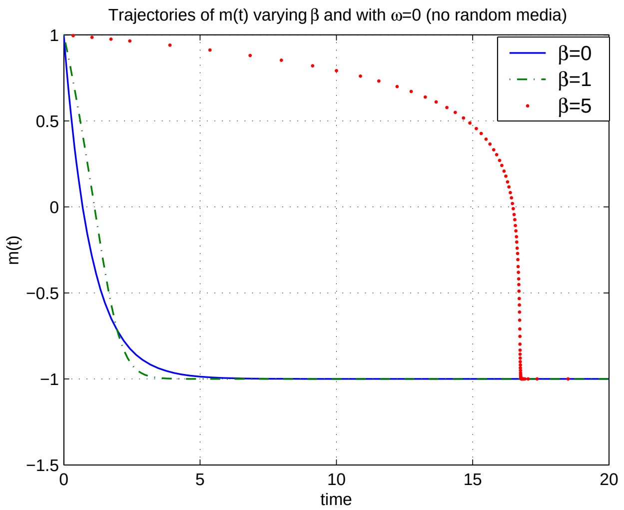

Figure 3.1: In this figure we plot the trajectories of $m(t)$, the solution of Equation (3.50), with $t \in [0, 20]$ for different values of β. Recall that in this case $\omega = 0$ hence we do not consider a random environment.

Having dynamic equations for $m(t; \omega)$ we can look jointly at the evolution of $m(t; \rho)$ and $m(t; -\rho)$. We have

$$\begin{cases} \dot{m}(t; \rho) &= -(1 + m(t; \rho))\ e^{-\beta\rho} \exp\left\{-\beta\left[\frac{m(t;\rho)}{2} + \frac{m(t;-\rho)}{2}\right]\right\}; \quad m(0; \rho) = 1 \\ \dot{m}(t; -\rho) &= -(1 + m(t; -\rho))\ e^{\beta\rho} \exp\left\{-\beta\left[\frac{m(t;\rho)}{2} + \frac{m(t;-\rho)}{2}\right]\right\}; \quad m(0; -\rho) = 1 \end{cases}$$

Without loss of generality we may assume $\rho = 1$. A comparative numerical study gives the following results.
When β is big, the solution of $m(t; 1)$ remains for a long time (when compared to the solution of $m(t; -1)$) near to the value $m = 1$. In Figures 3.2 and 3.3 we show this fact plotting different time scales for the same trajectories.
In Figure 3.4 we compare the random with the non random case. We can see that when $\beta = 1$ the dynamics in the non random case is an average between the two trajectories $m(t; 1)$ and $m(t; -1)$. This is not the case when $\beta = 5$ as in Figure 3.5.

These remarks are summarized in the following observations:

- *Behavior of $m(t; \rho)$ for small times.* For all β and for all pairs $\rho_i > \rho_j$ there exists a time $\underline{\zeta} = \underline{\zeta}(i, j) > 0$ such that for all $t < \underline{\zeta}$ we have $m(t; \rho_i) > m(t; \rho_j)$. This implies that for small times a favorable environment gives a positive contribution to the health of the system.

- *Behavior of $m(t; \rho)$ for large times.* For all β and for all pairs $\rho_i > \rho_j$ there exists a time $\overline{\zeta} = \overline{\zeta}(i, j) > 0$ such that for all $t > \overline{\zeta}$ we have $m(t; \rho_i) > m(t; \rho_j)$. This implies that for large enough times a favorable environment gives a positive contribution to the health of the system.
  Between $\underline{\zeta}$ and $\overline{\zeta}$ things are different.

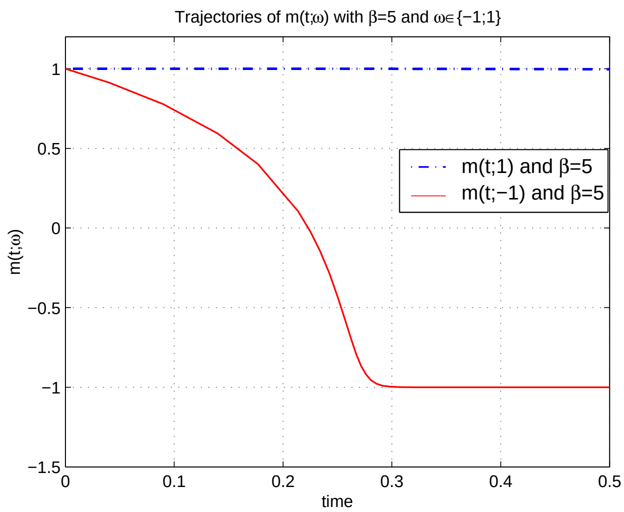

Figure 3.2: In this figure we plot the trajectories of $m(t;\omega)$, $t \in [0, 5 \cdot 10^{-4}]$ and $\beta = 5$. The random field may take values -1 and 1 with probability 1/2. We can see the different behavior of the trajectories in the two cases.

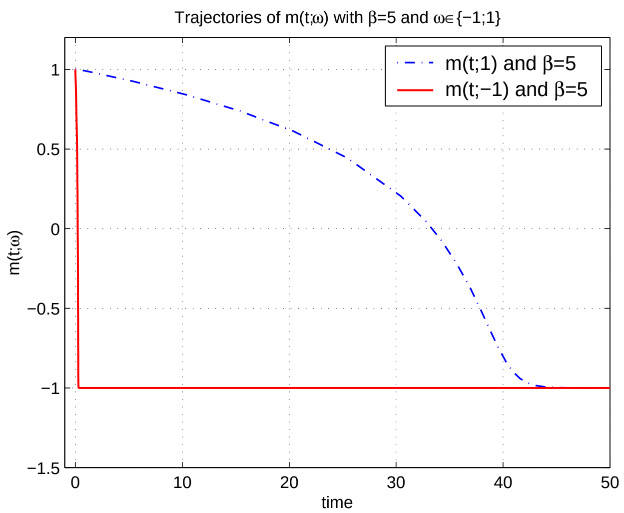

Figure 3.3: In this figure we plot the trajectories of $m(t;\omega)$, $t \in [0, 50]$, $\beta = 5$ and $\rho = 1$.

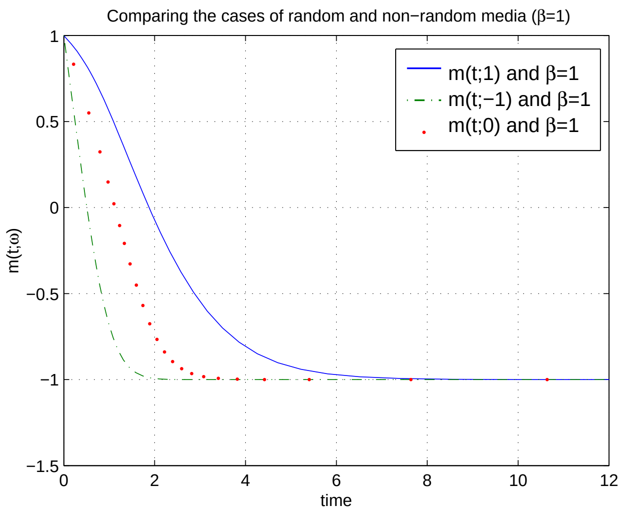

Figure 3.4: In this figure we compare $m(t;\omega)$ in the case of random $(\omega \in \{-\rho;\rho\})$ and non-random $(\omega = 0)$ media. Here $\beta = 1$ and $\rho = 1$.

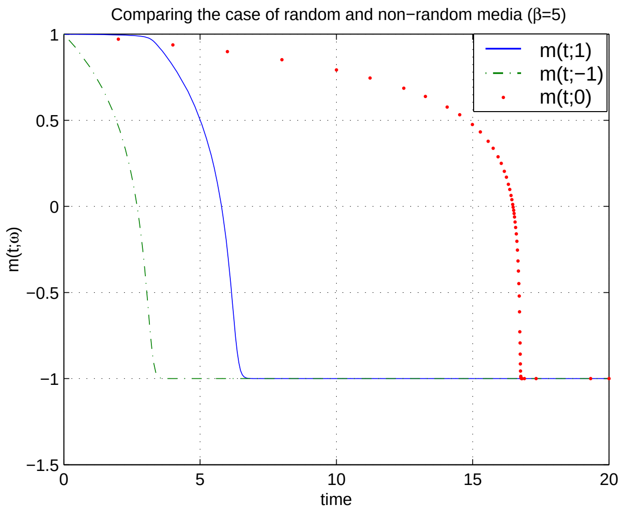

Figure 3.5: In this figure we compare $m(t;\omega)$ in the case of random $(\omega \in \{-\rho;\rho\})$ and non-random $(\omega = 0)$ media. Here $\beta = 5$ and $\rho = 1$.

- For any fixed $\rho \in (0,1)$ there exists a value $\bar{\beta}(\rho)$ such that for $\beta < \bar{\beta}$ we have for all t
$$m^{\beta}(t;\rho) \geq m^{\beta}(t;0).$$
For $\beta \geq \bar{\beta}$ and $t \in (\underline{\zeta};\bar{\zeta})$
$$m^{\beta}(t;\rho) < m^{\beta}(t;0).$$
The last fact suggests that for β large enough it may happen that $m^{\beta}(t;\rho)$ (that is, the proportion of defaulted firms computed *only* on favorable sites) may be smaller (in certain times) than the proportion of defaulted firms in the case in which we are not (a priori) distinguishing between good and bad environment. This latter fact may be explained in financial terms as follows. In presence of direct contagion, if *bad* (and *good*) firms are known to act on the market then even a good firm may have an higher probability of default compared to the case in which we do not have any -a priori- knowledge about good or bad firms.

Looking at Figure 3.5 (the continuous line path for $m(t;1)$) we can see that the global indicator $m(t;1)$ may spend a rather long time near the configuration $m = 1$ before suddenly falling down to the equilibrium $m = -1$.
This fact becomes more and more evident when β increases. However, the behavior of this limiting system depends smoothly on β. We shall see in Chapter 4 a model where there is a breakdown of the smoothness. In other words there exists a *critical value* β_c. This will be referred to as *phase transition*.

3.5 A central limit theorem

Up to now we have described the limiting behavior of a system when the number of sites (firms) tends to infinity, in other words we have characterized a law of large numbers as follows
$$\rho_N \to Q_* \qquad a.s.$$
An important issue is to study fluctuations around this limit, in other words a central limit theorem (CLT).
Consider a vector $\underline{\Phi} = (\Phi_1, ..., \Phi_n)$ of measurable, continuous and bounded functions such that $\Phi_i : \tilde{\mathcal{D}}[0,T] \times \mathbb{R} \to \mathbb{R};\ (\sigma(t),\omega) \mapsto \Phi_i(\sigma(t),\omega)$. In what follows, the dimension n of such a vector will be fixed. We are aiming at obtaining a result of the form
$$\sqrt{N}\left(\int \Phi_i d\rho_N - \int \Phi_i dQ_*\right)_{i=1}^{n} \to Z\ , \quad \text{as} \ \ N \to \infty$$
where Q_* is as defined in Proposition 3.3.5 and where $Z \sim \mathcal{N}_n(0,C)$ is a centered $n-$dimensional Gaussian random variable with a suitable covariance matrix $C = (C)_{ij} = C(\Phi_i, \Phi_j)$.

The spirit of the derivation of such a CLT is borrowed from F.den Hollander and P.Dai Pra [18]. We briefly explain the guideline of their idea.
Looking at the proof of Theorem 3.3.6 we see that the validity of the LLN is ensured by a LDP with a suitable rate function I with the important property that $I(Q) = 0$ if and only if $Q = Q_*$. The idea is to rely on the same LDP in order to describe also

a CLT; the milestone is the following *pseudotheorem.*

Pseudotheorem *Let X_N be a sequence of random variables taking values in a topological space V and let $(P_N)_{N\geq 1}$ denote the corresponding sequence of laws. Assume*

1. *$(P_N)_{N\geq 1}$ satisfies a LDP with rate function I;*
2. *There exists a unique $x^* \in V$ such that $I(x^*) = 0$;*
3. *Denote by $DI(x^*)[y] = \lim_{h\to 0} \frac{I(x^*+hy)-I(x^*)}{h}$ the directional derivative of I at x^* computed in the direction $y \in V$. Assume that the second order derivatives $\mathcal{H}_{y,z} := D^2I(x_*)[y,z] = D(DI(x)[y])[z]|_{x=x^*}$ exist.*

Then $\sqrt{N}(X_N - x_*)$ converges as $N \to \infty$ to a Gaussian random variable Z, whose covariance can be expressed in terms of the second order directional derivatives $\mathcal{H}_{y,z}$.

This *sloppy* statement has been turned into a rigorous theorem in [4].

Theorem 3.5.1 (Bolthausen) *Let $(B, \|\cdot\|)$ be a real separable Banach space. Let $(Y_k)_{k\geq 1}$ be a sequence of B-valued, i.i.d. random variables, defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and denote by w their common law. Define $X_N := \frac{1}{N}\sum_{k=1}^N Y_k$ and consider a continuous map $\Psi : B \to \mathbb{R}$. Suppose that the following conditions are satisfied:*

(B.1) $\int \exp(r|x|)w(dx) < \infty$ *for all $r \in \mathbb{R}$.*

(B.2) For any $x \in B$, $\Psi(x) \leq C_1 + C_2\|x\|$, for some $C_1, C_2 > 0$. Moreover, Ψ is three times continuously Fréchet differentiable.

(B.3) Define, for $h \in B'$ (the topological dual of B), $\Lambda(h) := \int e^{h(y)}w(dy)$, and for $x \in B$, $\Lambda^*(x) := \sup_{h\in B'}[h(x) - \Lambda(h)]$. Assume that there exists a unique $y^* \in B$ such that $\Lambda^*(y^*) - \Psi(y^*) = \inf_{y\in B}[\Lambda^*(y) - \Psi(y)]$.

(B.4) Define the probability p on B by $\frac{dp}{dw} = \frac{e^{D\Psi(y_*)}}{z}$ for a suitable normalizing factor z. This probability is well defined and $\int yp(dy) = y^*$. Let p_* denote the centered version of p, i.e., $p_* = p \circ \theta_{x^*}^{-1}$, where $\theta_a : B \to B$ is defined by $\theta_a(x) = x - a$. For $\lambda \in B'$ define $\tilde{\lambda} \in B$ by $\tilde{\lambda} = \int y\lambda(y)p_*(dy)$. Then we assume that for every $\lambda \in B'$ such that $\tilde{\lambda} \neq 0$

$$\int \lambda^2(y)p_*(dy) - D^2\Psi(y^*)[\tilde{\lambda}, \tilde{\lambda}] > 0.$$

(B.5) B is a Banach space of type 2. [7]

Now, letting π_N be the probability on $(\Omega, \mathcal{A})$ given by

$$\frac{d\pi_N}{d\mathbb{P}} = \frac{e^{N\Psi(X_N)}}{E^{\mathbb{P}}\left[e^{N\Psi(X_N)}\right]}, \tag{3.51}$$

[7] A Banach space B is said to be of type 2 if $\ell^2(B) \subseteq C(B)$. Here $\ell^2(B) = \{(x_n) \in B^\infty : \sum_i \|x_i\|^2 < \infty\}$ and $C(B) = \{(x_n) \in B^\infty : \sum_j \varepsilon_j x_j$ converges in probability$\}$ where (ε_n) is a Bernoulli sequence, i.e., a sequence of independent random variables such that $P(\varepsilon_n = \pm 1) = \frac{1}{2}$. For more details see [45] and [4].

then, for every $\lambda_1, ..., \lambda_n \in B'$, the π_N-law of the $n-$dimensional vector

$$\sqrt{N}\left(\lambda_i(X_N) - \lambda_i(y_*)\right)_{i=1}^n$$

converges weakly, as $N \to \infty$, to the law of a centered Gaussian vector with covariance matrix $\mathcal{C} \in \mathbb{R}^{n \times n}$, such that for $i, j = 1, ..., n$

$$(\mathcal{C})_{i,j} = \int \lambda_i(y)\lambda_j(y)p_*(dy) - D^2\Psi(y_*)[\tilde{\lambda}_i, \tilde{\lambda}_j]. \tag{3.52}$$

Proof. See Theorem 2 in [4]. ∎

Remark 3.5.2 *For our purposes it will be necessary to use a slightly modified version of this statement, where Equation (3.51) is changed into*

$$\frac{d\pi_N}{d\,\mathbb{P}} = \frac{e^{N\left(\Psi(X_N)+\frac{\Sigma(X_N)}{N}\right)}}{E^{\mathbb{P}}\left[e^{N\left(\Psi(X_N)+\frac{\Sigma(X_N)}{N}\right)}\right]}\ , \tag{3.53}$$

where Σ is linear and continuous.
Bolthausen's proof is essentially insensitive to this generalization.

We notice that Assumption $(B.5)$ is clearly not satisfied by the space of measures $\mathcal{M}$ (which is not a Banach space either). For this reason we map $\mathcal{M}$ to a Banach space by a linear map T, obtaining first a CLT for $T(\rho_N)$ via Theorem 3.5.1. Finally we derive a CLT for ρ_N in $\mathcal{M}$.
This proceeding is summarized into the following three steps:

i) Define a linear mapping $T : \mathcal{M}(\tilde{\mathcal{D}}[0,T] \times \mathbb{R}) \to B$ from the set of measures into a Banach space of type 2.

ii) Ensure the validity of $(B.1)$, ..., $(B.5)$ and prove a CLT in B.

iii) Obtain a corresponding CLT in $\mathcal{M}$.

The three steps correspond to the following three theorems. The proofs are postponed to the appendix.

Definition 3.5.3 *We define ν_* as the law, induced by Q_*, of the $\mathcal{M}_0$-valued random variable $(\delta_{\{\sigma_{[0,T]},\omega\}} - Q_*)$.*
Let $\mathcal{C}_b$ be the space of bounded, continuous, measurable functions $\Phi : \tilde{\mathcal{D}}[0,T] \times \mathbb{R} \to \mathbb{R}$. We write $\hat{\Phi} \in \mathcal{M}_0(\tilde{\mathcal{D}}[0,T] \times \mathbb{R})$ for the signed measure of zero total mass induced by Φ and defined as

$$\hat{\Phi}(A) = \int \left(R(A) \cdot \textstyle\int \Phi dR\right) \nu_*(dR). \tag{3.54}$$

for any measurable subset $A \subset \tilde{\mathcal{D}}[0,T] \times \mathbb{R}$. Finally, we define Φ^* as follows

$$\Phi^* := \int \Phi dQ_*.$$

Theorem 3.5.4 *The following properties hold true* *i) There exists a Banach space of type 2 $(B, \|\cdot\|)$, a linear map $T : \mathcal{M}(\tilde{\mathcal{D}}[0,T] \times \mathbb{R}) \to B$, continuous on the set $\{Q : Q(\tau = T) = 0\}$. Moreover there exist two continuous maps $\Psi, \Sigma : B \to \mathbb{R}$, where Ψ is bounded and three times Fréchet differentiable and Σ is linear, such that*

$$\frac{dP_N^\omega}{dW^{\otimes N}} = \exp\left\{ N \left[\Psi(T(\rho_N)) + \frac{\Sigma(T(\rho_N))}{N} \right] \right\} , \qquad a.s. \tag{3.55}$$

ii) For any vector $\underline{\Phi} = (\Phi_1, ..., \Phi_n) \subset \mathcal{C}_b$ there exist $\underline{\lambda} = (\lambda_1, ..., \lambda_n) \subset B'$ such that $(\lambda_i \circ T)(Q) = \int \Phi_i dQ$, where B' stands for the topological dual of B.

Proof. The proof is postponed to Appendix A.1.

Theorem 3.5.5 *The sequence of $B-$valued random variables $Y_i := T(\delta_{\{\sigma_i[0,T],\omega_i\}})$ satisfies a CLT.*

Proof. The proof is postponed to Appendix A.2.

Theorem 3.5.6 *Let ρ_N be as defined in (3.26) and Q_* as defined in Proposition 3.3.5. For any vector $\underline{\Phi} = (\Phi_1, ..., \Phi_n) \subset \mathcal{C}_b$, as $N \to \infty$*

$$\sqrt{N} \left(\int \Phi_i d\rho_N - \int \Phi_i dQ_* \right)_{i=1}^n \tag{3.56}$$

converges weakly under P_N to a $n-$dimensional Gaussian random variable with covariance matrix $C \in \mathbb{R}^{n \times n}$, such that for $i, j = 1, ..., n$

$$(C)_{ij} = \int (\Phi_i - \Phi_i^*)(\Phi_j - \Phi_j^*) dQ_* - D^2 F(Q_*)[\hat{\Phi}_i, \hat{\Phi}_j] = \tag{3.57}$$

$$= E^{Q_*}\left[\left(\Phi_i - \Phi_i^* + \beta \int_0^{T \wedge \tau} m_{\hat{\Phi}_i(s)} dM_s^\sigma \right) \left(\Phi_j - \Phi_j^* + \beta \int_0^{T \wedge \tau} m_{\hat{\Phi}_j(s)} dM_s^\sigma \right) \right];$$

where $m_{\hat{\Phi}_i(s)} = \int \sigma(s)(\Phi - \Phi^*) dQ_* =$ and

$$M_t^\sigma := \mathbb{I}_{\{t \geq \tau\}} - \int_0^{t \wedge \tau} e^{-\beta(\omega + m_{q_s^*})} ds \tag{3.58}$$

is the compensated Q_*-martingale associated with the jump process of $\sigma(t)$.

Proof. The proof is postponed to Appendix A.3. ∎

As an application of Theorem 3.5.6, we show in the following corollary how to explicitly compute the asymptotic limit, in a particular case of function Φ. Indeed, we shall specialize this general result in order to infer information on the fluctuations of the global health indicator $m_{\rho_N(T)}$ around its limit $m_{q_T^*}$.

Corollary 3.5.7 *As $N \to \infty$ we have that*

$$\sqrt{N} \left[m_{\rho_N(T)} - m_{q_T^*} \right]$$

converges weakly to a centered Gaussian random variable with variance

$$V(T) = E^{Q_*}\left[\left(\sigma(T) - m_{q_T^*} + \beta \int_0^{T \wedge \tau} Cov_{Q_*}(\sigma(s), \sigma(T)) \, dM_s^\sigma \right)^2 \right], \tag{3.59}$$

where q_T^* has been defined in Theorem 3.3.6.

Proof. Let us define $\varphi_T : \tilde{\mathcal{D}}[0,T] \times \mathbb{R} \to \mathbb{R}$ as $\varphi_T(\sigma[0,T],\omega) = \sigma_T$, namely the projection on the first component at time T. This is a particular choice of $(\Phi_1, ..., \Phi_n)$ with $n = 1$. We can thus apply Theorem 3.5.6.
First of all we notice that

$$\int \varphi_T(x,y)\rho_{N(T)}(dx,dy) = m_{\rho_{N(T)}} , \qquad \int \varphi_T(x,y)Q_*(dx,dy) = m_{q_T^*}.$$

Concerning the variance, we have that $\Phi = \varphi_T = \sigma(T)$ and $\Phi^* = \int \varphi_T dQ_* = m_{q_T^*}$. Thus

$$m_{\hat{\Phi}(s)} = \int \sigma(s)(\Phi - \Phi^*)dQ_* =$$

$$= \int \sigma(s)(\sigma(T) - m_{q_T^*})dQ_* = Cov_{Q_*}(\sigma(s),\sigma(T)), \quad \forall s \in [0,T].$$

Finally

$$V(T) = E^{Q_*}\left[\left((\sigma(T) - m_{q_T^*}) + \beta\int_0^{T\wedge\tau} Cov_{Q_*}(\sigma(s),\sigma(T))\ dM_s^\sigma\right)^2\right].$$

∎

We shall see how to apply this kind of results to the study of large portfolio losses in Chapter 5.
Concerning Equation (3.59), notice that in the case of no interaction, (i.e., $\beta = 0$) we have

$$V(T) = E^{Q_*}\left[(\sigma(T) - m_{q_T^*})^2\right] = Var_{Q_*}(\sigma(T)).$$

In the case of $\beta > 0$ there is a suppletive noise given by the interaction. It depends on the *autocovariance* function of $\sigma[0,T]$, that is, on $Cov_{Q_*}(\sigma(t),\sigma(T))$. This term introduces dependence on the past history of the process, hence a sort of "*memory*" of the variance $V(T)$.

Remark 3.5.8 *Corollary 3.5.7 can be generalized to the study of more complex portfolios as follows. Relying on Equation (2.2), consider the case in which for all t, $e_i(t)\delta_i(t) = \phi(\omega_i)$ for a suitable function ϕ. In doing so, we are reasonably assuming that the randomness of the loss given default and the exposure at default is due to the idiosyncratic characteristics of the firms, summarized in our model by the random environment ω. Then*

$$L^{(N)}(t) = \sum_{i=1}^{N} \frac{1}{2}\phi(\omega_i)(1 - \sigma_i(t)),$$

where we have used the fact that $Y_i(t) = \frac{1-\sigma_i(t)}{2}$, being Y_i the default indicator of the i-th firm. If we define $L(\sigma(t),\omega) := \frac{1}{2}\phi(\omega)(1 - \sigma(t))$ then

$$L^{(N)}(t) = N \int L(\sigma(t),\omega)\rho_N(d\sigma[0,T],\omega) \tag{3.60}$$

This straightforward representation of $L^{(N)}$ in terms of the empirical measure can be used to compute portfolio losses relying on Theorem 3.5.6.

This fact matches the point of view of the so called top-down approaches (see [16], [39] or [60]). In those models one "forgets" the underlying marginal processes and describes directly the statistical properties of the random process $L^N(t)$. Via Equation (3.60) we obtain a functional description of the aggregate losses in terms of the empirical measure that makes the approximation of the loss distribution feasible, without adding any computational effort. In particular we do not need to simulate the full N-dimensional underlying Markov process, avoiding for instance heavy Monte Carlo methods.

The vicinity between homogeneous *Markov models and top-down models has been already captured by Frey and Backhaus (see [35]). We would like to notice that our representation via the empirical measure let us to introduce a certain degree of* heterogeneity *(represented by the random field ω), without compromising the tractability.*

Indeed it can be shown (see Corollary 2 in [20]) that

$$\sqrt{N}\left[\frac{L^{(N)}(t)}{N} - E^{Q_*}[L(t)]\right]$$

converges weakly to a centered Gaussian random variable with variance $\hat{V}(t)$ that generalizes Eqaution (3.59) and that can be explicitly computed.
In [20] simulations are also provided in order to show that the CLT approximation is reasonable for portfolios of $100-150$ *obligors, this volume is rather standard for CDO's contracts. Nevertheless an actual validation of this model has not jet been pursued. For an example of calibration under the top-down approach, see the algorithm proposed in [16] for the pricing of portfolio credit derivatives.*

In the next chapter we propose a different model that allows us to describe in details the formation of a credit crises.

Chapter 4

Endogenous (dynamic) random environment

This chapter is devoted to the analysis of a new model characterized by interacting intensities where a different transition mechanism is in place. As before, our system is described by variables σ and ω where the former are intended to reflect the market perception of the financial health of the firm (e.g. a rating class indicator) whereas the latter one should capture the fundamental situation of the firm.
In this new framework we shall see that the indicators ω will evolve in time, moreover, their dynamics will result *coupled* with the dynamics of the state variables σ. Eventually we are dealing with a $2N$ dimensional continuous time Markov chain $(\underline{\sigma}(t), \underline{\omega}(t))_{t\in[0,T]}$. Before entering into details on the modeling issues we would like to stress the philosophy behind this new model.
What we would like to capture is a more complex structure, in particular the fact that the fundamental health of a firm (summarized by ω) determines basically the perception of the market concerning its capacity of honoring financial obligations (summarized by σ). Gathering all this information we construct (endogenously) a picture of the health of the market summarized by the so called global health indicator $m_N(t) = \frac{1}{N}\sum_i \sigma_i$. Finally this aggregate factor may have an impact on the single firm, by influencing the dynamics of ω. Summarizing we obtain the following "chain" of contagion

$$\ldots \ \omega_i \dashrightarrow \sigma_i \dashrightarrow m_N \dashrightarrow \omega_j \ \ldots$$

We anticipate here what is probably the main implication of this modeling idea. The intensities that we are going to define make the model "complex" enough to see what is called *phase transition* in the literature of Statistical Mechanics. Phase transition means that for different values of the parameters we see different limiting behaviors of the Markov chain. In particular, when N goes to infinity (the asymptotic model), the dynamics may have multiple stable equilibria. The effects of phase transition for the system with finite N can be seen on different time-scales. On a long time-scale we observe what it is usually meant by metastability in Statistical Mechanics: the system may spend a very long time in a small region of the state space around a stable equilibrium of the limiting dynamics and then switch relatively fast to another region around a different stable equilibrium. This switch, of which the rigorous analysis will be postponed to future work, occurs on a time-scale proportional to e^{kN} for

a suitable $k > 0$, that could be unrealistic for financial applications.
The model we propose exhibits, however, a different feature that can be interpreted as a *credit crisis*. A credit crisis is a concentration of many defaults (or downgrades in a rating systems) in a short time caused by an high level of dependence between the obligors. The crisis is the consequence of a cascade or domino effect (a run in the banking sector) caused often by the deterioration of financial quality and the loss of trust by the market participants.
In our model this crisis may be explained as follows: for certain values of the initial condition the system is initially driven towards an unstable equilibrium. After a certain time that depends on the initial state, the system is "captured" by an unstable direction of this equilibrium, and moves towards a stable one; during the transition to the stable equilibrium, the volatility of the system increases sharply, before decaying to a stationary value. All this occurs at a time-scale of order $O(1)$ (i.e. the time scale does not depend on N).
Notice that we explain the formation of credit crises as a purely microeconomic phenomenon. Recall in fact that the variable $m(t)$, describing the financial health of the system, is an aggregate factor (a possible indicator of the business cycle) that is endogenously generated and not a priori assigned. All these issues will be exploited in details in what follows.

Besides the explanation of the credit crises we have to deal also with the risk management issues and in particular we want to provide tools for the computation of large portfolio losses (possibly taking crises into account). Hence we are going to quantify the impact of contagion on the losses suffered by a financial institution holding a large portfolio with positions issued by the firms. In particular, we aim at obtaining a dynamic description of a risky portfolio in the context of our contagion model. The standard literature on risk management usually focuses on static models allowing to compute the distribution of a risky portfolio over a given fixed time horizon T. For a recent paper that introduces a discussion relating to static and dynamic models see the one by Deuschel, Duffie and Dembo (2004) [23].

As already mentioned in the previous sections, we shall consider large homogeneous portfolios. Attention to large homogeneous portfolios becomes crucial when looking at portfolios with many small entries. If the firms are supposed to be exchangeable, in the sense that the losses that they may cause to the bank in case of financial distress depend on the single firm only via its financial state indicator, it is worth to evaluate an homogeneous model where N goes to infinity and then to look for "large-N" approximations.[1] This apparently restrictive assumption may be easily relaxed by considering many *homogeneous groups* within the network (in this context see also [36]).

Although we shall provide only formulas to compute Var-like probabilities for excess losses in the context of our contagion model, we shall in fact determine the entire portfolio loss distribution; furthermore, these Var-like probabilities will in turn allow to compute other credit risk related quantities as we shall briefly mention at the end of the chapter.

[1] As already discussed in Section 2.4.1 this simplifications is also assumed in the Basel II Accord, where the internal based models for computing the Risk Capital are based on an asymptotic model where the N issuers are modeled as in a one-factor Gaussian model.

Outline of Chapter 4

Section 4.1 contains a punctual exposition of the model and highlights the differences compared to the model analyzed in the previous chapter. In Section 4.2 we discuss the non-reversibility of the model and the consequences of this fact. In particular we show how to implement a dynamic study in Section 4.3. Similarly to Chapter 3, the first main result is a Law of Large Numbers (Theorem 4.3.2) based on a Large Deviation Principle (Proposition 4.3.4).
Subsection 4.3.2 describes the equilibria of the *asymptotic* ($N \to \infty$) dynamics. In particular we shall prove in Theorem 4.3.11 that different asymptotic configurations can be found, depending on the values of the parameters. This phenomenon (called *phase transition*) has implications for the description of a credit crisis as we shall explain in more details in Chapter 5.
The last section of this chapter is devoted to the study of the fluctuations of the empirical measure around its limit. Two different approaches are described, the former (corresponding to Section 4.4.1) is based on uniform convergence of generators (Theorem 4.4.1). The latter (in Section 4.4.2) mimics the functional approach of Section 3.5.

4.1 The model in details

In this section we describe a mean-field interaction model different from that of the previous chapter.[2]

The philosophy behind our model can be summarized as follows.

- We introduce only a small number of variables that however have a simple economic interpretation.
- We define *dynamic* rules that describe interaction between the variables.
- We keep the model as simple as possible; on one side this may make the model less adherent to reality, on the other it leads to exact computations, and allows to show what basic features of the model produce phenomena such as clustering of defaults, phase transition, etc. More generally, it allows to show how, contrary to most models relying on macroeconomic factors, the "health" of the system can here be described by endogenous financial indicators so that a credit crisis can be viewed as a microeconomic phenomenon.

As before we consider a network of N firms. The state of each firm is identified by two variables, σ and ω ((σ_i, ω_i) is the state of the i-th firm).
Compared to the model presented in Chapter 3, there are two main differences.

1. The variable σ takes values in $\{-1; 1\}$ as before, but it is allowed to return from state -1 to the safe state 1. The bad state is associated with a financial distress position, that is not necessarily a default state.

[2] We would like to notice that mean-field models are used also in the literature of Social Sciences in order to capture the interaction of agents when facing any kind of decision problems. We refer the reader to the paper by Brock and Durlauf [8] for an example in this area.

2. The variable ω takes values in $\{-1; 1\}$ as before but its value is not constant in time and moreover the dynamics of σ and ω are coupled. We shall see how they are related.

In this context σ may be interpreted as the *rating class indicator*: a low value reflects a bad rating class, i.e. a higher probability of not being able to pay back obligations. The variable ω represents (as before) an indicator of the financial health of the firm and is typically not directly observable. It could e.g. be a *liquidity indicator* as in Giesecke and Weber [40]. The important fact is that, while there is usually a strong interaction between σ_i and ω_i, the non-observability of ω makes it reasonable to assume that ω_i cannot *directly* influence the rating indicators σ_j for $j \neq i$.

Again we assume that the two indicators σ_i, ω_i can only take two values, that we label by 1 ("good" financial state) and -1 (financial distress). In the case of portfolios consisting of defaultable bonds, we may then refer to the rating class corresponding to $\sigma = -1$ also as "speculative grade" and corresponding to $\sigma = +1$ as "investment grade".

Contrary to other rating class models (see Section 2.2.3 for details), we have not explicitly introduced a default state. We may do so, and this becomes more natural in the case when σ may take more than just the two values $-1, +1$. In our binary variable case we would obviously assign the default state to -1 as done in the previous chapter. What we always need however is that there is a positive probability that the system can exit from a state where σ takes its lowest value. This is the main reason why we have not explicitly termed -1 as the default state.

As said in the introduction we assume that the interaction between different firms only depends on the value of the *global financial health indicator*

$$m_N^\sigma := \frac{1}{N}\sum_{i=1}^{N}\sigma_i,$$

where we have preferred to explicitly indicate that the average is taken on the component σ. We shall also speak about m^ω and even $m^{\sigma\omega}$ when referring to the expected mean of the product of the two[3]. We are now going to specify the rate of transition for each component of the Markov chain. Under the assumption made before we are led to consider intensities of the form:

$$\begin{array}{lll} \sigma_i \mapsto -\sigma_i & \text{with intensity} & a(\theta_\sigma, \sigma_i, \omega_i, m_N^\sigma) \\ \omega_i \mapsto -\omega_i & \text{with intensity} & b(\theta_\omega, \sigma_i, \omega_i, m_N^\sigma), \end{array} \tag{4.1}$$

where $a(\cdot,\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot,\cdot)$ are given functions and θ_σ, θ_ω are parameters. Since both financial health and distress tend to propagate, we assume that $a(\theta_\sigma, -1, \omega_i, m_N^\sigma)$ is increasing in both ω_i and m_N^σ, and $a(\theta_\sigma, 1, \omega_i, m_N^\sigma)$ is decreasing.

Similarly, $b(\theta_\omega, \sigma_i, -1, m_N^\sigma)$ and $b(\theta_\omega, \sigma_i, 1, m_N^\sigma)$ should be respectively increasing and decreasing in their variables.

The next assumption is that the intensity $a(\theta_\sigma, \sigma_i, \omega_i, m_N^\sigma)$ is actually independent of m_N^σ, i.e. of the form $a(\theta_\sigma, \sigma_i, \omega_i)$. Although this assumption amounts to a rather

[3]Notice that m^ω has nothing to do with the (conditional) health indicator $m(t;\omega)$ encountered in Corollary 3.4.1. The latter one indicates in fact an average on the σ component once conditioned on a realization of ω.

mild computational simplification, it allows to show that aggregate behavior (phase transition, etc.) may occur even in absence of a direct interaction between rating indicators.

Although a model of this generality could be fully analyzed, we make the following choice of the intensities, inspired by spin-glass systems, to make the model depend on only two parameters:

$$\begin{array}{lll} \sigma_i \mapsto -\sigma_i & \text{with intensity} & e^{-\beta\sigma_i\omega_i} \\ \omega_i \mapsto -\omega_i & \text{with intensity} & e^{-\gamma\omega_i\, m_N^{\sigma}}. \end{array} \tag{4.2}$$

Here $\theta_\sigma = \beta$ and $\theta_\omega = \gamma$ are positive parameters which indicate the strength of the corresponding interaction. Put differently, we are considering a continuous-time Markov chain on $\{-1;1\}^{2N}$ with the following infinitesimal generator:

$$Lf(\underline{\sigma},\underline{\omega}) = \sum_{i=1}^{N} e^{-\beta\sigma_i\omega_i}\nabla_i^{\sigma} f(\underline{\sigma},\underline{\omega}) + \sum_{j=1}^{N} e^{-\gamma\omega_j m_N^{\sigma}}\nabla_j^{\omega} f(\underline{\sigma},\underline{\omega}) \tag{4.3}$$

where $\nabla_i^{\sigma} f(\underline{\sigma},\underline{\omega}) = f(\underline{\sigma}^i,\underline{\omega}) - f(\underline{\sigma},\underline{\omega})$ (analogously for ∇_i^{ω}), and where the $j-$th component of $\underline{\sigma}^i$ is

$$\sigma_j^i = \begin{cases} \sigma_j & \text{for } j \neq i \\ -\sigma_i & \text{for } j = i. \end{cases}$$

The rest of this chapter is devoted to a detailed analysis of the above model. We conclude this subsection with some further considerations.

- The techniques that we use in this chapter apply, in principle, to modifications of the above model in which variables take values in larger or even continuous spaces. For instance, one could assume the fundamental values ω_i to be $\mathbb{R}^+$-valued, and evolving according to the stochastic differential equation

$$d\omega_i(t) = \omega_i(t)[f(m_N^{\sigma}(t))dt + g(m_N^{\sigma}(t))dB_i(t)] + dJ_i(t),$$

  where f and g are given functions, the $B_i(\cdot)$ are independent Brownian motions, and $J_i(\cdot)$ is a pure jump process whose intensity is a function of $\omega_i(t)$ and $m_N^{\sigma}(t)$.

- An interesting extension of the above model consists in letting the functions $a(\cdot,\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot,\cdot)$ in (4.1) to be random rather than deterministic; in particular they may depend on (possibly time-dependent) exogenous macro-economic variables.

- The mean-field assumption may be weakened by assuming that the rate at which ω_i changes depends on an i-dependent weighted global health of the form

$$m_{N,i}^{\sigma} := \frac{1}{N}\sum_{j=1}^{N} J\left(\frac{i}{N},\frac{j}{N}\right)\sigma_j,$$

  where $J : [0,1]^2 \to \mathbb{R}$ is a function describing the interaction between pairs of firms. In other words, the i-th firm "feels" the information given by the rating of the other firms in a non-uniform way.

- We have viewed the variable σ as a rating class indicator. Contrary to the standard models for rating class transitions, our rating indicator σ is not Markov by itself, but it is Markov only if paired with ω. This property is in line with empirical data and with recent research in the field of credit migration models. It is in fact well documented that real data of credit migration between rating classes exhibit a "non Markovian" behavior. For a discussion on this topic see e.g. Christensen et al. (2004) [11] [4].

- With a choice of the intensities as in (4.2) we introduce a form of symmetry in our model, whereby the values $\sigma = -1$ and $\sigma = +1$ for the rating indicator turn out to be equally likely. One could however modify the model in order to make the value $\sigma = -1$ less (more) likely than the value $\sigma = +1$ and this could e.g. be achieved by letting the intensity for ω_i be of the form $e^{\omega_i \phi(m_N^\sigma)}$, where ϕ is an increasing, nonlinear and non even function. A possible "prototype" choice would be $\phi(x) = \gamma(x-K)^+ + \delta$ with $\gamma, \delta > 0$ and $K \in (0,1)$. Note that with this latter choice we have $\phi \geq 0$ so that the value $\omega_i = +1$ (and hence also $\sigma_i = +1$) becomes more likely. Such an asymmetric setup might be more realistic in financial applications but, besides leading to more complicated derivations, it depends also on the specific application at hand. Since, as already mentioned, we want to study a model that is as simple as possible and yet capable of producing the basic features of interest, in this paper we concentrate on the "symmetric choice" in (4.2), possibly commenting in the text below the situation whenever a major difference may arise with respect to a non symmetric setup (see Remark 5.0.10).

4.2 Invariant measures and non-reversibility

Mean field models as the one we propose in this paper have already appeared, mostly in the Statistical mechanics literature (see in particular [18] and [13], from which we borrow many of the mathematical tools). However, unlike what happens for the models in the cited references, we now show that our model is non-reversible. This implies that an explicit formula for the stationary in time distribution and its $N \to \infty$ asymptotic is not available. It is thus appropriate to follow a more specifically dynamic approach to understand the long time behavior of the system. As already mentioned, we shall thus first study the $N \to \infty$ limit of the dynamics of the system, obtaining limit evolution equations. Then we study the equilibria of these equations. This is not necessarily equivalent to studying the $N \to \infty$ properties of the stationary distribution μ_N. However, as we will show later in this paper, this provides rather sharp information on how the system behaves for t and N large.

The operator L given in (4.3) defines an irreducible, finite-state Markov chain. It follows that the process admits a unique stationary distribution μ_N, i.e. a distribution

[4] In this paper the authors propose an *hidden Markov process* to model credit migration. The basic criticism that these authors move to the standard Markov models is that they cannot capture a real effect, seen in market data, that the probability of being downgraded is higher for firms that have been just downgraded. In order to capture this issue, the authors consider an *"excited"* rating state (for example B^* from which there is an higher probability to be downgraded compared to the standard state B). This point of view is not far from ours, even though the mechanism of the transition is different. The downgrade to $\sigma = -1$ is higher when $(\sigma = 1, \omega = -1)$ compared to $(\sigma = 1, \omega = 1)$.

such that, for each function f on the configuration space of $(\underline{\sigma}, \underline{\omega})$,

$$\sum_{\underline{\sigma},\underline{\omega}} \mu_N(\underline{\sigma}, \underline{\omega}) Lf(\underline{\sigma}, \underline{\omega}) = 0. \tag{4.4}$$

This distribution reflects the long-time behavior of the system, in the sense that, for each f and any initial distribution,

$$\lim_{t \to +\infty} E[f(\underline{\sigma}(t), \underline{\omega}(t))] = \sum_{\underline{\sigma},\underline{\omega}} \mu_N(\underline{\sigma}, \underline{\omega}) f(\underline{\sigma}, \underline{\omega}).$$

The stationarity condition (4.4) can be rewritten in the form

$$\sum_{i=1}^{N} \left[\mu_N(\underline{\sigma}^i, \underline{\omega}) e^{\beta \sigma_i \omega_i} - \mu_N(\underline{\sigma}, \underline{\omega}) e^{-\beta \sigma_i \omega_i} \right] + \sum_{i=1}^{N} \left[\mu_N(\underline{\sigma}, \underline{\omega}^i) e^{\gamma \omega_i m_N^{\sigma}} - \mu_N(\underline{\sigma}, \underline{\omega}) e^{-\gamma \omega_i m_N^{\sigma}} \right] = 0 \tag{4.5}$$

for every $\underline{\sigma}, \underline{\omega} \in \{-1; 1\}^N$.

Simpler sufficient conditions for stationarity are the so-called *detailed balance* conditions. We say that a probability ν on $\{-1, ; 1\}^{2N}$ satisfies the detailed balance condition for the generator L if

$$\nu(\underline{\sigma}^i, \underline{\omega}) e^{\beta \sigma_i \omega_i} = \nu(\underline{\sigma}, \underline{\omega}) e^{-\beta \sigma_i \omega_i} \quad \text{and} \quad \nu(\underline{\sigma}, \underline{\omega}^i) e^{\gamma \omega_i m_N^{\sigma}} = \nu(\underline{\sigma}, \underline{\omega}) e^{-\gamma \omega_i m_N^{\sigma}} \tag{4.6}$$

for every $\underline{\sigma}, \underline{\omega}$. When the detailed balance conditions (4.6) hold, we say the system is reversible: the stationary Markov chain with generator L and marginal law ν has a distribution which is left invariant by *time-reversal*. In the case (4.6) admit a solution, they usually allow to derive the stationary distribution explicitly. This is not the case in our model. We have in fact

Proposition 4.2.1 *The detailed balance equations (4.6) admit no solution.*

Proof. By way of contradiction, assume a solution ν of (4.6) exists. Then one easily obtains

$$\begin{aligned} \nabla_i^{\sigma} \log \nu(\underline{\sigma}, \underline{\omega}) &= -2\beta \sigma_i \omega_i \\ \nabla_i^{\omega} \log \nu(\underline{\sigma}, \underline{\omega}) &= -2\gamma \omega_i m_N^{\sigma}, \end{aligned}$$

which implies

$$\begin{aligned} \nabla_i^{\omega} \nabla_i^{\sigma} \log \nu(\underline{\sigma}, \underline{\omega}) &= 4\beta \sigma_i \omega_i \\ \nabla_i^{\sigma} \nabla_i^{\omega} \log \nu(\underline{\sigma}, \underline{\omega}) &= 4N^{-1} \gamma \omega_i \sigma_i. \end{aligned}$$

This is not possible since $\nabla_i^{\omega} \nabla_i^{\sigma} \log \nu(\underline{\sigma}, \underline{\omega}) \equiv \nabla_i^{\sigma} \nabla_i^{\omega} \log \nu(\underline{\sigma}, \underline{\omega})$. ■

4.3 Studying the dynamics of the system

Our results in this section concern the dynamics of the system $(\sigma_i[0,T], \omega_i[0,T])_{i=1}^N$ as well as of the global financial health indicator m_N^{σ} for large N.

Remark 4.3.1 *In this section we are going to study the dynamic of the system using techniques similar to the ones implemented in Chapter 3. For this reason, some statements and proofs shall appear similar to what seen in the previous chapter. On the other hand, the different specifications of the model make some technical steps of the derivation of the large deviation principle more involved.*
We have decided to use the same notations of Chapter 3 to indicate corresponding objects, this is done in order to maintain a sort of symmetry *between the two models. As an example, when speaking of $Q \in \mathcal{M}_1$ we have to keep in mind that the underlying path space on which the measure is referred to, is now different (ω evolves dynamically whereas σ may jump more than once in $[0, T]$, see also Remark 3.2.3).*

Our approach proceeds according to the following three steps

i) Look for the limit dynamics of the system ($N \to \infty$);

ii) Study the equilibria of the limiting dynamics;

iii) Describe the "finite volume approximations" (for large but finite N) via a central limit-type result.

The first two items are treated in the next two subsections. Concerning the finite volume approximation we shall treat it in Section 4.4 where two different approaches are proposed and analyzed.

4.3.1 Deterministic limit: Law of large numbers

Let $(\sigma_i[0,T], \omega_i[0,T])_{i=1}^N \in \mathcal{D}([0,T])^{2N}$ denote a path of the system process in the time-interval $[0, T]$ for a generic $T > 0$. If $f(\sigma_i[0,T], \omega_i[0,T])$ is a function of the trajectory of the variables related to a single firm, one is interested in the asymptotic behavior of empirical averages of the form

$$\frac{1}{N}\sum_{i=1}^{N} f(\sigma_i[0,T], \omega_i[0,T]) =: \int f d\rho_N,$$

where ρ_N is the sequence of empirical measures

$$\rho_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(\sigma_i[0,T], \omega_i[0,T])}. \tag{4.7}$$

We may think of ρ_N as a (random) element of $\mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$, the space of probability measures on $\mathcal{D}([0,T]) \times \mathcal{D}([0,T])$ endowed with the weak convergence topology.

Our first aim is to determine the limit of $\int f d\rho_N$ as $N \to \infty$, for f continuous and bounded; in other words we look for the weak limit $\lim_N \rho_N$ in $\mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$. This corresponds to a Law of Large Numbers with the limit being a deterministic measure. This limit, being an element of $\mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$, can be viewed as a stochastic process, and represents the dynamics of the system in the limit $N \to \infty$. The fluctuations of ρ_N around this deterministic limit will be studied in subsection 4.4 below, and this turns out to be particularly relevant in the risk analysis of a portfolio (Chapter 5).

Let now $q \in \mathcal{M}_1(\{-1;1\}^2)$ be a probability on $\{-1;1\}^2$. Define

$$m_q^\sigma := \sum_{\sigma,\omega=\pm 1} \sigma q(\sigma,\omega),$$

that can be interpreted as the expected rating under q. The main result of this subsection is the following.

Theorem 4.3.2 *Suppose that the distribution at time $t=0$ of the Markov process $(\underline{\sigma}(t),\underline{\omega}(t))_{t\geq 0}$ with generator (4.3) is such that the random variables $(\sigma_i(0),\omega_i(0))$, $i=1,\ldots,N$, are independent and identically distributed with law λ. Then there exists a probability $Q^* \in \mathcal{M}_1(\mathcal{D}([0,T])\times\mathcal{D}([0,T]))$ such that*

$$\rho_N \;\rightarrow\; Q^* \quad \textit{almost surely}$$

in the weak topology. Moreover, if $q_t \in \mathcal{M}_1(\{-1;1\}^2)$ denotes the marginal distribution of Q^* at time t, then q_t is the unique solution of the nonlinear (McKean-Vlasov) equation

$$\begin{cases} \frac{\partial q_t}{\partial t} = \mathcal{L}q_t, t \in [0,T] \\ q_0 = \lambda \end{cases} \tag{4.8}$$

where

$$\mathcal{L}q(\sigma,\omega) = \nabla^\sigma\left[e^{-\beta\sigma\omega}q(\sigma,\omega)\right] + \nabla^\omega\left[e^{-\gamma\omega m_q^\sigma}q(\sigma,\omega)\right] \tag{4.9}$$

with $(\sigma,\omega)\in\{-1,;1\}^2$.

Remark 4.3.3 *The strong assumption on the initial distribution in Theorem 4.3.2 could be weakened at the cost of more technical assumptions; this point is not really relevant for the purpose of understanding the limiting dynamics and will thus not be considered further.*

The rest of this subsection will be devoted to the proof of this theorem. The proof, based fundamentally on a Large Deviations Principle, proceeds along three steps corresponding to the three propositions below. We start with some preliminary notions letting, in what follows, $W \in \mathcal{M}_1(\mathcal{D}([0,T])\times\mathcal{D}([0,T]))$ denote the law of the $\{-1;1\}^2$-valued process $(\sigma(t),\omega(t))$ such that $(\sigma(0),\omega(0))$ has distribution λ, and both $\sigma(\cdot)$ and $\omega(\cdot)$ change sign with constant intensity 1.
For $Q \in \mathcal{M}_1(\mathcal{D}([0,T])\times\mathcal{D}([0,T]))$, $\Pi_t Q$ denotes its marginal law at time t, and

$$\gamma_t^Q := \gamma\int \sigma\Pi_t Q(d\sigma,d\tau). \tag{4.10}$$

For a given path $(\sigma[0,T],\omega[0,T]) \in \mathcal{D}([0,T])\times\mathcal{D}([0,T])$, let N_t^σ (resp. N_t^ω) be the process counting the jumps of $\sigma(\cdot)$ (resp. $\omega(\cdot)$). Define

$$F(Q) = \int\left[\int_0^T\left(1-e^{-\beta\sigma(t)\omega(t)}\right)dt + \int_0^T\left(1-e^{-\omega(t)\gamma_t^Q}\right)dt \right.$$
$$\left. +\beta\int_0^T \sigma(t)\omega(t^-)dN_t^\sigma + \int_0^T \omega(t)\gamma_{t^-}^Q dN_t^\omega\right]dQ, \tag{4.11}$$

whenever

$$\int (N_T^\sigma + N_T^\omega)\,dQ < +\infty,$$

and $F(Q) = 0$ otherwise. Finally let

$$I(Q) := H(Q|W) - F(Q).$$

where $H(\cdot|W)$ denotes the relative entropy with respect to W (see Definition 3.1.5). We remark that, if $\int (N_T^\sigma + N_T^\omega)\, dQ = +\infty$, then $H(Q|W) = +\infty$ (this will be shown in Appendix A.4, Lemma A.4.3) and thus also $I(Q) = +\infty$.

Proposition 4.3.4 *For each $Q \in \mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$ we have that $I(Q) \geq 0$, and $I(\cdot)$ is a lower-semicontinuous function with compact level-sets (i.e. for each $k > 0$ one has that $\{Q : I(Q) \leq k\}$ is compact in the weak topology). Moreover, for $A, C \subseteq \mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$ respectively open and closed for the weak topology, we have*

$$\liminf_N \frac{1}{N} \log P(\rho_N \in A) \quad \geq \quad -\inf_{Q \in A} I(Q) \tag{4.12}$$

$$\limsup_N \frac{1}{N} \log P(\rho_N \in C) \quad \leq \quad -\inf_{Q \in C} I(Q). \tag{4.13}$$

This means that the distribution of ρ_N obeys a Large Deviation Principle *(LDP) with* rate function $I(\cdot)$ *(see Section 3.1 for the definition and fundamental facts on LDP).*

Proof. The proof of Proposition 4.3.4 is given in Appendix A.4 and follows from rather standard arguments, similar to those in [18]. Some technical difficulties are due to unboundedness of F, which is related to the non-reversibility of the model. ∎

Let now $Q \in \mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$. Arguing similarly as in Section 3.3, we associate with Q the law of a Markov process on $\{-1;1\}^2$ which evolves according to the following rules:

$$\begin{array}{lll} \sigma \to -\sigma & \text{with intensity} & e^{-\beta\sigma\omega} \\ \omega \to -\omega & \text{with intensity} & \exp\left[-\gamma\omega \sum_{\sigma,\tau\in\{-1;1\}} \sigma \Pi_t Q(\sigma,\tau)\right] = e^{-\gamma\omega m^\sigma_{\Pi_t Q}} \end{array}$$

and with initial distribution λ. We denote by P^Q the law of this process. In other words, P^Q is the law of the Markov process on $\{-1;1\}^2$ with initial distribution λ and time-dependent generator

$$\mathcal{L}_t^Q f(\sigma,\omega) = e^{-\beta\sigma\omega} \nabla^\sigma f(\sigma,\omega) + e^{-\gamma\omega m^\sigma_{\Pi_t Q}} \nabla^\omega f(\sigma,\omega).$$

We show now an important property of P^Q.

Proposition 4.3.5 *For every $Q \in \mathcal{M}_1(\mathcal{D}([0,T]) \times \mathcal{D}([0,T]))$ such that $I(Q) < +\infty$, we have*

$$I(Q) = H(Q|P^Q).$$

Proof. The proof is given in Appendix A.5. ∎

Finally, we have

Proposition 4.3.6 *The equation $I(Q) = 0$ has a unique solution Q^*, of which the marginals $q_t := \Pi_t Q^*$ solve equation (4.8).*

Proof. The proof is essentially the same as the proof of Proposition 3.3.5, with only minor modifications in the notations. ∎

Proof of Theorem 4.3.2 Let Q^* be the unique zero of the rate function $I(\cdot)$ as given by Proposition 4.3.6. Let B_{Q^*} be an arbitrary open neighborhood of Q^*. By the upper bound in Proposition 4.3.4, we have

$$\limsup_N \frac{1}{N} \log P(\rho_N \notin B_{Q^*}) \leq - \inf_{Q \notin B_{Q^*}} I(Q) < 0,$$

where the last inequality comes from lower semicontinuity of $I(\cdot)$, compactness of its level sets and the fact that $I(Q) > 0$ for every $Q \neq Q^*$. The rest of the proof follows from Proposition 4.3.6 and arguing in the same way as in the proof of theorem 3.3.6. ∎

Before studying in detail the phenomenon of phase transition we prove a couple of propositions that give an explicit form to the asymptotic marginal measure $\bar{q}$ satisfying $\mathcal{L}\bar{q} = 0$, in other terms, the invariant solution of Equation (4.8).

Remark 4.3.7 *Why is it so important to describe $\bar{q}$? It is related to the concept of "asymptotic" behavior of the system in the limit when $N \to \infty$. Remember that Q^* is actually the limiting law described in Theorem 4.3.2, and that $q_t = \Pi_t Q^*$ represents its projection at time t; thus $\bar{q}$, such that $\mathcal{L}\bar{q} = 0$, describes the asymptotic behavior of Q^* (when $t \to \infty$).*

Now it should be clear that the "expected global health" $m^\sigma_{\bar{q}} = \sum_{\sigma,\omega} \sigma\bar{q}(\sigma,\omega)$ computed under $\bar{q}$, is the asymptotic value of the liquidity in the system where we let N go to infinity. If different values of $m^\sigma_{\bar{q}}$ are found, suppose $\{m^\sigma_*, 0, -m^\sigma_*\}$[5], the finite volume system will spend a long time near the unstable equilibrium $m^\sigma = 0$ before decaying to one of the stable ones, for instance $m^\sigma_{\bar{q}} = -m^\sigma_*$. This fact will be supported by numerical simulations in Section 5.1 and is the basic concept for explaining default clustering and credit crises.

Proposition 4.3.8 *Fix $m \in [-1,1]$, and consider the probability q_m on $\{-1;1\}^2$ given by*

$$\begin{cases} q_m(1,1) &= K\left[\frac{e^{\beta+\gamma m}(e^\beta+e^{\gamma m})+e^{-\beta+\gamma m}(e^\beta+e^{-\gamma m})}{-(e^{-\beta}-e^\beta)+e^{-\beta}(e^\beta+e^{-\gamma m})(e^{-\beta}+e^{-\gamma m})}\right] \\ q_m(1,-1) &= K \\ q_m(-1,1) &= K\left[\frac{e^{\beta+\gamma m}(e^\beta+e^{\gamma m})+e^{-\beta+\gamma m}(e^\beta+e^{-\gamma m})}{-(e^{-\beta}-e^\beta)+e^{-\beta}(e^\beta+e^{-\gamma m})(e^{-\beta}+e^{-\gamma m})}\frac{e^{-\beta}+e^{-\gamma m}}{e^\beta} + e^{-\beta+\gamma m}\right] \\ q_m(-1,-1) &= K\left[\frac{e^{\beta+\gamma m}(e^\beta+e^{\gamma m})+e^{-\beta+\gamma m}(e^\beta+e^{-\gamma m})}{(e^{-\beta}-e^\beta)-e^{-\beta}(e^\beta+e^{-\gamma m})(e^{-\beta}+e^{-\gamma m})}e^{\beta-\gamma m} + e^\beta(e^\beta+e^{\gamma m})\right] \end{cases}$$

where

$$K = \left[\frac{e^{\beta+\gamma m}(e^\beta+e^{\gamma m})+e^{-\beta+\gamma m}(e^\beta+e^{-\gamma m})}{-(e^{-\beta}-e^\beta)+e^{-\beta}(e^\beta+e^{-\gamma m})(e^{-\beta}+e^{-\gamma m})}\left(1+\frac{e^{-\beta}+e^{-\gamma m}}{e^\beta} - e^{\beta-\gamma m}\right) - \right.$$
$$\left. - e^{-\beta+\gamma m} + e^\beta(e^\beta+e^{\gamma m}) + 1\right]^{-1}.$$

[5] This will actually be the case for certain values of the parameters. We shall characterize such values in the following results.

Then $\mathcal{L}q_m = 0$ if and only if

$$m = m(q_m) := \sum_{\sigma,\omega=\pm 1} \sigma q_m(\sigma,\omega). \tag{4.14}$$

Moreover all solutions of $\mathcal{L}q = 0$ are of this form.

Proof. The aim of the proof is to show that all the solutions of $\mathcal{L}q = 0$ are of the form of q_m, with m defined in Equation (4.14).

Firstly we fix m and we solve an auxiliary problem

$$\mathcal{L}_m q(\sigma,\omega) = 0, \quad \sum_{\omega,\sigma} q(\sigma,\omega) = 1$$

where

$$\mathcal{L}_m q(\sigma,\omega) := \nabla^\sigma \left[e^{-\beta\sigma\omega} q(\sigma,\omega) \right] + \nabla^\omega \left[e^{-\gamma\omega m} q(\sigma,\omega) \right]. \tag{4.15}$$

Writing explicitly the operator for each entry $q(1,1), q(-1,1), ...$, we obtain a linear system in four unknowns. A long and tedious inspection of this linear system shows that there exists a unique solution for each m and this is exactly q_m. We omit the details of this computation.

Secondly we have to ensure that q_m solves $\mathcal{L}q = 0$. Notice that $\mathcal{L}_m$ is different from $\mathcal{L}$ (we have fixed m to get linearity in q).
Choosing $m \equiv m(q_m)$ as in Equation (4.14) we describe all the solutions q_m of $\mathcal{L}q = 0$, so that the thesis follows. ∎

Proposition 4.3.9 *The value $m = 0$ is a solution of equation (4.14) for all values of γ and β. Moreover, define*

$$\bar{\gamma}(\beta) = \frac{1}{\tanh \beta}, \tag{4.16}$$

then the solution is unique if and only if $\gamma \leq \bar{\gamma}(\beta)$. If $\gamma > \bar{\gamma}(\beta)$ there exists $m_*^\sigma > 0$ such that the set of solutions of equation (4.14) is given by $\{-m_*^\sigma, 0, m_*^\sigma\}$.

Proof. Equation (4.14) is of the form $m = f(m)$ for a suitable continuous and odd function $f : [-1,1] \to [-1,1]$. In particular one can show (we omit the details) that $f(m) = 2\frac{A(m)+1}{B(m)} - 1$ where

$$A(m) = \frac{e^{\beta+2\gamma m} + e^{2\beta+\gamma m} + e^{\gamma m} + e^{-\beta}}{e^{-\beta-2\gamma m} + e^{-2\beta-\gamma m} + e^{-\gamma m} - e^{\beta}}$$

$$B(m) = A(m) \left[1 - e^{\beta-\gamma m} + e^{-\beta-\gamma m} + e^{-2\beta} \right] + e^{\beta+\gamma m} - e^{-\beta+\gamma m} + e^{2\beta} + 1.$$

We have to deal with a fixed point problem. In particular it can be shown that

$$A(0) = e^\beta \; ; \quad B(0) = 2(e^\beta + 1)$$

so that

$$f(0) = 2\frac{e^\beta + 1}{2(e^\beta + 1)} - 1 = 0$$

and the first part of the thesis follows.

Concerning the second part. Notice that, being the function f odd, we can restrict the research to the positive m. Moreover, it can be shown that $f''(m) < 0$ for $m \geq 0$. As a consequence of these facts, there exists a unique $m_*^\sigma > 0$ such that $f(m_*^\sigma) = m_*^\sigma$ if and only if $f'(0) > 1$.
Consequently if $f'(0) \leq 1$, $m = 0$ is the unique solution of the fixed point problem. If $f'(0) > 1$, $\{0, +m_*^\sigma, -m_*^\sigma\}$ are the solutions of the fixed point problem. We are thus left to compute $f'(0)$.

$$f'(0) = -2\frac{B'(0)}{B^2(0)}(e^\beta + 1) + 2\gamma B^{-1}(0)\frac{e^\beta(e^\beta+1)^2}{1+e^{2\beta}}$$

and finally some basic algebra gives

$$f'(0) > 1 \quad \Leftrightarrow \quad \gamma > \frac{e^{2\beta}+1}{e^{2\beta}-1} = \frac{1}{\tanh\beta}$$

and this completes the proof. ∎

We shall see in the next section how the result of Proposition 4.3.9 may be generalized and rephrased in a more profitable framework. In particular we shall translate the McKean-Vlasov Equation (4.8) into a differential system involving the expected values m^σ, m^ω and $m^{\sigma\omega}$. This will be useful in order to discuss the stability of the equilibria.

4.3.2 Equilibria of the limiting dynamics: Phase transition

Equation (4.8) describes the dynamics of the system with generator (4.3) in the limit as $N \to +\infty$. In this section we determine the equilibrium points, or stationary (in t) solutions of Equation (4.8), i.e. solutions of $\mathcal{L}q_t = 0$ and, more generally, the large time behavior of its solutions. First of all, we provide a result that shows how to re-parameterize the unknown q_t in Equation (4.8) in terms of the expected values m^σ, m^ω and $m^{\sigma\omega}$.

Lemma 4.3.10 *Let μ be a probability on $\{-1;1\}^2$, then μ is completely identified by the expectations:*

$$m_\mu^\sigma := \sum_{\sigma,\omega=\pm1} \sigma\mu(\sigma,\omega), \quad m_\mu^\omega := \sum_{\sigma,\omega=\pm1} \omega\mu(\sigma,\omega), \quad m_\mu^{\sigma\omega} := \sum_{\sigma,\omega=\pm1} \sigma\omega\mu(\sigma,\omega). \tag{4.17}$$

In particular, if $\mu = q_t$, the marginal of Q^* appearing in Theorem 4.3.2, then we write m_t^σ for $m_{q_t}^\sigma$, and similarly for $m_t^\omega, m_t^{\sigma\omega}$. Equation (4.8) can be rewritten in the following form:

$$\begin{cases} \dot{m}_t^\sigma &= 2\sinh(\beta)m_t^\omega - 2\cosh(\beta)m_t^\sigma \\ \dot{m}_t^\omega &= 2\sinh(\gamma m_t^\sigma) - 2\cosh(\gamma m_t^\sigma)m_t^\omega \\ \dot{m}_t^{\sigma\omega} &= 2\sinh(\beta) + 2\sinh(\gamma m_t^\sigma)m_t^\sigma - 2(\cosh(\beta) + \cosh(\gamma m_t^\sigma))m_t^{\sigma\omega} \end{cases} \tag{4.18}$$

with initial condition $m_0^\sigma = m_\lambda^\sigma$, $m_0^{\sigma\omega} = m_\lambda^{\sigma\omega}$, $m_0^\omega = m_\lambda^\omega$.

Proof. Take μ probability on $\{-1;1\}^2$. Each $f : \{-1;1\}^2 \to \mathbb{R}$ can be written in the form $f(\sigma,\omega) = a\sigma + b\omega + c\sigma\omega + d$. It follows that μ is completely identified by

the three expectations m_μ^σ, m_μ^ω and $m_\mu^{\sigma\omega}$ defined above.
The second part can be proved arguing as follows.
By definition $m_t^\sigma = \sum_{\sigma,\omega} \sigma q_t(\sigma,\omega)$, hence $\dot{m}_t^\sigma = \sum_{\sigma,\omega} \sigma \dot{q}_t(\sigma,\omega)$. Relying on (4.15) we then have

$$\begin{aligned}\dot{m}_t^\sigma &= \sum_{\sigma,\omega} \sigma \left(\nabla^\sigma [e^{-\beta\sigma\omega} q_t(\sigma,\omega)] + \nabla^\omega \left[e^{-\gamma\omega m_t^\sigma} q_t(\sigma,\omega) \right] \right) = \\ &= \sum_{\sigma,\omega} \sigma \left(e^{\beta\sigma\omega} q_t(-\sigma,\omega) - e^{-\beta\sigma\omega} q_t(\sigma,\omega) \right) + \\ &\qquad + \sum_{\sigma,\omega} \sigma \left(e^{\gamma\omega m_t^\sigma} q_t(\sigma,-\omega) - e^{-\gamma\omega m_t^\sigma} q_t(\sigma,\omega) \right).\end{aligned}$$

We now use the following facts

$$\sum_{\sigma,\omega} \sigma e^{\beta\sigma\omega} q_t(-\sigma,\omega) = -\sum_{\sigma,\omega} \sigma e^{-\beta\sigma\omega} q_t(\sigma,\omega)$$

$$\sum_{\sigma,\omega} \sigma e^{\gamma\omega m_t^\sigma} q_t(\sigma,-\omega) = \sum_{\sigma,\omega} \sigma e^{-\gamma\omega m_t^\sigma} q_t(\sigma,\omega)$$

So that

$$\dot{m}_t^\sigma = -2 \sum_{\sigma,\omega} \sigma e^{-\beta\sigma\omega} q_t(\sigma,\omega).$$

Moreover, it is easy to check that for $\sigma,\omega \in \{-1;1\}$, it holds

$$e^{-\beta\sigma\omega} = -\sigma\omega \frac{e^\beta - e^{-\beta}}{2} + \frac{e^\beta + e^{-\beta}}{2}$$

Thus, using the definition of $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$, we have

$$\begin{aligned}\dot{m}_t^\sigma &= -2 \sum_{\sigma,\omega} \sigma[-\sigma\omega \sinh(\beta) + \cosh(\beta)] q_t(\sigma,\omega) = \\ &= 2\sinh(\beta) m_t^\omega - 2\cosh(\beta) m_t^\sigma\end{aligned}$$

where the last equality follows since $\sigma^2 = 1$ and $\sum_{\sigma,\omega} \omega q_t(\sigma,\omega) = m_t^\omega$.
Equations for m^ω and $m^{\sigma\omega}$ are found using similar techniques. ∎

Note that $m_t^{\sigma\omega}$ does not appear in the first and in the second equation in (4.18); this means that the differential system (4.18) is essentially two-dimensional: first one solves the two-dimensional system (on $[-1,1]^2$)

$$(\dot{m}_t^\sigma, \dot{m}_t^\omega) = V(m_t^\sigma, m_t^\omega), \tag{4.19}$$

with $V(x,y) = (2\sinh(\beta)y - 2\cosh(\beta)x, 2\sinh(\gamma x) - 2y\cosh(\gamma x))$, and then one solves the third equation in (4.18), which is linear in $m_t^{\sigma\omega}$. Note also that to any (m_*^σ, m_*^ω) satisfying $V(m_*^\sigma, m_*^\omega) = 0$, there corresponds a unique $m_*^{\sigma\omega} := \frac{\sinh(\beta) + m_*^\sigma \sinh(\gamma m_*^\sigma)}{\cosh(\beta) + \cosh(\gamma m_*^\sigma)}$ such that $(m_*^\sigma, m_*^\omega, m_*^{\sigma\omega})$ is an equilibrium (stable solution) of (4.18). Moreover, if $m_t^\sigma \to m_*^\sigma$ as $t \to +\infty$, then $m_t^{\sigma\omega} \to m_*^{\sigma\omega}$. Thus, to discuss the equilibria of (4.18) and their stability, it is enough to analyze (4.19) and for this we have the following Proposition, where by "linearly stable equilibrium" we mean a pair $(\bar{x},\bar{y})$ such that $V(\bar{x},\bar{y}) = 0$, and the linearized system $(\dot{x},\dot{y}) = DV(\bar{x},\bar{y})(x-\bar{x}, y-\bar{y})$ is stable, i.e. the eigenvalues of the Jacobian matrix $DV(\bar{x},\bar{y})$ have all negative real parts.

Theorem 4.3.11

i) Suppose $\gamma \leq \frac{1}{\tanh(\beta)}$. *Then equation (4.19) has* $(0,0)$ *as a unique equilibrium solution, which is globally asymptotically stable, i.e. for every initial condition* (m_0^σ, m_0^ω), *we have*

$$\lim_{t \to +\infty} (m_t^\sigma, m_t^\omega) = (0,0).$$

ii) For $\gamma < \frac{1}{\tanh(\beta)}$ *the equilibrium* $(0,0)$ *is linearly stable. For* $\gamma = \frac{1}{\tanh(\beta)}$ *the linearized system has a neutral direction, i.e.* $DV(0,0)$ *has one zero eigenvalue.*

iii) For $\gamma > \frac{1}{\tanh(\beta)}$ *the point* $(0,0)$ *is still an equilibrium for (4.19), but it is a saddle point for the linearized system, i.e. the matrix* $DV(0,0)$ *has two nonzero real eigenvalues of opposite sign. Moreover (4.19) has two linearly stable solutions* (m_*^σ, m_*^ω), $(-m_*^\sigma, -m_*^\omega)$, *where* m_*^σ *is the unique strictly positive solution of the equation*

$$x = \tanh(\beta)\tanh(\gamma x),$$

and

$$m_*^\omega = \frac{1}{\tanh(\beta)} m_*^\sigma.$$

iv) For $\gamma > \frac{1}{\tanh(\beta)}$, *the phase space* $[-1,1]^2$ *is bi-partitioned by a smooth curve* Γ *containing* $(0,0)$ *such that* $[-1,1]^2 \setminus \Gamma$ *is the union of two disjoint sets* Γ^+, Γ^- *that are open in the induced topology of* $[-1,1]^2$. *Moreover*

$$\lim_{t \to +\infty} (m_t^\sigma, m_t^\omega) = \begin{cases} (m_*^\sigma, m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^+ \\ (-m_*^\sigma, -m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^- \\ (0,0) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma. \end{cases}$$

Proof. We first observe that the square $[-1,1]^2$ is stable for the flow of Equation (4.19), since the vector field $V(x,y)$ points inward at the boundary of $[-1,1]^2$. It is also immediately seen that the equation $V(x,y) = 0$ holds if and only if $x = \tanh(\beta)\tanh(\gamma x)$ and $y = \frac{1}{\tanh(\beta)} x$. Moreover a simple convexity argument shows that $x = \tanh(\beta)\tanh(\gamma x)$ has $x = 0$ as unique solution for $\gamma \leq \frac{1}{\tanh(\beta)}$, while for $\gamma > \frac{1}{\tanh(\beta)}$ a strictly positive solution, and its opposite, bifurcate from the null solution. We have therefore found all equilibria of (4.19).
We now remark that Equation (4.19) has no cycles (periodic solutions). Indeed, suppose (x_t, y_t) is a cycle of period T. Then by the Divergence Theorem

$$0 \leq \int_0^T [V_1(x_t,y_t)\dot{x}_t + V_2(x_t,y_t)\dot{y}_t]dt = \int_C \mathrm{div}V(x,y)dxdy, \tag{4.20}$$

where V_1, V_2 are the components of V and C is the open set enclosed by the cycle. But

$$\mathrm{div}V(x,y) = -2\cosh(\beta) - 2\cosh(\gamma x) < 0$$

in all of $[-1,1]^2$, so that (4.20) cannot hold.
It follows by the Poincaré-Bendixon Theorem that every solution must converge to an equilibrium as $t \to +\infty$. This completes the proof of i).

Now let us denote by $(\bar{x},\bar{y})$ a pair such that $V(\bar{x},\bar{y})=0$. Then the matrix of the linearized system in $(\bar{x},\bar{y})$ is

$$DV(\bar{x},\bar{y}) = 2\begin{pmatrix} -\cosh(\beta) & \sinh(\beta) \\ \gamma\left(\cosh(\gamma\bar{x})-\bar{y}\sinh(\gamma\bar{x})\right) & -\cosh(\gamma\bar{x}) \end{pmatrix}.$$

Let λ_1,λ_2 be its eigenvalues. Then

$$\lambda_1 = \frac{-(\cosh(\beta)+1)-\sqrt{(\cosh(\beta)-\cosh(\gamma\bar{x}))^2+4\gamma\left[\sinh(\beta)\cosh(\gamma\bar{x})-\bar{y}\sinh(\gamma\bar{x})\right]}}{2},$$

$$\lambda_2 = \frac{-(\cosh(\beta)+1)+\sqrt{(\cosh(\beta)-\cosh(\gamma\bar{x}))^2+4\gamma\left[\sinh(\beta)\cosh(\gamma\bar{x})-\bar{y}\sinh(\gamma\bar{x})\right]}}{2}.$$

When $(\bar{x},\bar{y})=(0,0)$, it is easy to see that $\lambda_{1,2}\in\mathbb{R}$, $\lambda_1<0$, while the sign of λ_2 depends on the value of γ. But we know that $\det DV(0,0)=\lambda_1\lambda_2$. Then $\text{sign}\left(\det DV(0,0)\right) = -\text{sign}(\lambda_2)$, with $\det DV(0,0) = 2\left[\cosh(\beta)-\gamma\sinh(\beta)\right]$. So it follows that:

- if $\gamma<\frac{1}{\tanh(\beta)}$, then $\lambda_2<0$, i.e. $(0,0)$ is linearly stable;
- if $\gamma=\frac{1}{\tanh(\beta)}$, then $\lambda_2=0$, i.e. $DV(0,0)$ has a neutral direction;
- if $\gamma>\frac{1}{\tanh(\beta)}$, then $\lambda_2>0$, i.e. $(0,0)$ is a saddle point for the linearized system.

So ii) and the first part of iii) are shown.
Now, when $\gamma>\frac{1}{\tanh(\beta)}=:\gamma_c$, also $(\bar{x},\bar{y})=(m_*^\sigma,m_*^\omega)$ and $(\bar{x},\bar{y})=(-m_*^\sigma,-m_*^\omega)$, solve $V(\bar{x},\bar{y})=0$.
If $\lambda_{1,2}\in\mathbb{C}\setminus\mathbb{R}$, i.e. if the argument of the square root of $\lambda_{1,2}$ is negative, then $\text{Re }\lambda_1=\text{Re }\lambda_2=-(\cosh(\beta)+1)/2<0$ and so (m_*^σ,m_*^ω), $(-m_*^\sigma,-m_*^\omega)$ are linearly stable.
Instead, if $\lambda_{1,2}\in\mathbb{R}$, then it is immediate to see that $\lambda_1<0$ and, as before, $\text{sign}(\det DV(m_*^\sigma,m_*^\omega)) = -\text{sign}(\lambda_2)$. Notice that

$$\begin{aligned}\det DV(m_*^\sigma,m_*^\omega) &= 2\left[\cosh(\beta)\cosh(\gamma m_*^\sigma)-\gamma\sinh(\beta)\left(\cosh(\gamma m_*^\sigma)-m_*^\omega\sinh(\gamma m_*^\sigma)\right)\right]\\ &= 2\left[\cosh(\beta)\cosh(-\gamma m_*^\sigma)-\gamma\sinh(\beta)\left(\cosh(-\gamma m_*^\sigma)+m_*^\omega\sinh(-\gamma m_*^\sigma)\right)\right]\\ &= \det DV(-m_*^\sigma,-m_*^\omega).\end{aligned}$$

Then in both cases we have to study $\text{sign}(\det DV(m_*^\sigma,m_*^\omega))$. We have

$$\det DV(m_*^\sigma,m_*^\omega)>0 \Leftrightarrow \cosh(\beta)\cosh(\gamma m_*^\sigma)-\gamma\sinh(\beta)\left(\cosh(\gamma m_*^\sigma)-m_*^\omega\sinh(\gamma m_*^\sigma)\right)>0 \Leftrightarrow$$

$$\Leftrightarrow \gamma\, m_*^\omega\,\tanh(\gamma m_*^\sigma)>\gamma-\gamma_c. \tag{4.21}$$

But $m_*^\sigma=\tanh(\beta)\tanh(\gamma m_*^\sigma)$, $m_*^\omega=\frac{1}{\tanh(\beta)}\,m_*^\sigma$, yield

$$\tanh(\gamma m_*^\sigma)=\frac{1}{\tanh(\beta)}\,m_*^\sigma=m_*^\omega. \tag{4.22}$$

So, if we substitute (4.22) in (4.21), we obtain

$$\gamma\,\tanh^2(\gamma m_*^\sigma)>\gamma-\gamma_c \Leftrightarrow 1-\frac{1}{\cosh^2(\gamma m_*^\sigma)}>1-\frac{\gamma}{\gamma_c},$$

which is equivalent to

$$\frac{\gamma}{\gamma_c} < \cosh^2(\gamma m_*^\sigma)\,. \tag{4.23}$$

Then, set $y = \gamma m_*^\sigma$,

$$m_*^\sigma = \frac{1}{\gamma_c}\tanh(\gamma m_*^\sigma) \quad \Leftrightarrow \quad y = \frac{\gamma}{\gamma_c}\tanh(y)\,. \tag{4.24}$$

So det $DV(m_*^\sigma, m_*^\omega) > 0$ is equivalent to

$$\frac{\gamma}{\gamma_c} < \cosh^2(y). \tag{4.25}$$

On the other hand, from (4.24), we obtain

$$\frac{\gamma}{\gamma_c} = \frac{y}{\tanh(y)} = \frac{y}{\sinh(y)}\cosh(y) < \cosh(y) < \cosh^2(y),$$

because $y/\sinh(y) < 1$ and $\cosh(y) < \cosh^2(y)$, since $y = \gamma m_*^\sigma > 0$ if $\gamma > \gamma_c$. This shows that det $DV(m_*^\sigma, m_*^\omega) > 0$.
Then, if $\gamma > \gamma_c = \frac{1}{\tanh(\beta)}$, $\lambda_{1,2} < 0$, i.e. (m_*^σ, m_*^ω), $(-m_*^\sigma, -m_*^\omega)$ are linearly stable, from which also the second part of iii) is shown.
It remains to show iv). For $\gamma > \frac{1}{\tanh(\beta)}$, we let v_s be an eigenvector of the negative eigenvalue of $DV(0,0)$. By the Stable Manifold Theorem (see Section 2.7 in [55]), the set of initial conditions that are asymptotically driven to $(0,0)$ form a one-dimensional manifold Γ that is tangent to v_s at $(0,0)$. Since any solution converges to an equilibrium point, and solutions starting in Γ^c cannot cross Γ (otherwise uniqueness would be violated), the remaining part of statement iv) follows. ∎

In the following section we are going to provide the technical tools useful to apply this limiting results to the study of large portfolio losses and to the description of a credit crisis. In particular we are going to state a central limit theorem that measures the dispersion around the limit when the volume N of the system is large but finite.

4.4 Fluctuations: A central limit theorem

Having established a law of large numbers $\rho_N \to Q^*$, it is natural to analyze fluctuations around the limit, i.e. the rate at which ρ_N converges to Q^* and the asymptotic distribution of $\rho_N - Q^*$.

To study the asymptotic distribution of $\rho_N - Q^*$ there are at least the following two possible approaches :

i. A weak convergence-type approach based on uniform convergence of the generators (see [32]).

ii. An approach based on a functional Central Limit Theorem using a result in [4] that relates Large Deviations with the Central Limit Theorem. (as already seen in Chapter 3).

In the following two subsections we are going to discuss these two methods and their applicability to financial problems.

4.4.1 Convergence of generators approach

In this section we shall provide a dynamical interpretation of the Law of Large Numbers discussed in Theorem 4.3.2. Let $\psi : \{-1;1\}^2 \to \mathbb{R}$, and define $\rho_N(t)$ by

$$\int \psi d\rho_N(t) := \frac{1}{N}\sum_{i=1}^{N} \psi(\sigma_i(t), \omega_i(t)). \tag{4.26}$$

In other words, $\rho_N(t)$ is the marginal of ρ_N at time t and we also have $m_N^\sigma(t) = m^\sigma_{\rho_N(t)}$. Note that, for each fixed t, $\rho_N(t)$ is a probability on $\{-1;1\}^2$, and so, by the considerations leading to (4.17), it can be viewed as a three-dimensional object. Thus $(\rho_N(t))_{t\in[0,T]}$ is a three-dimensional flow. A simple consequence of Theorem 4.3.2 is the following convergence of flows:

$$(\rho_N(t))_{t\in[0,T]} \;\to\; (q_t)_{t\in[0,T]} \;\; a.s., \tag{4.27}$$

where the convergence of flows is meant in the uniform topology. Since the flow of marginals contains less information than the full measure of paths, the law of large numbers in (4.27) is weaker than the one in Theorem 4.3.2. However, the corresponding fluctuation flow

$$(\sqrt{N}(\rho_N(t) - q_t))_{t\in[0,T]}$$

is also a finite-dimensional flow, and it allows for a very explicit characterization of the limiting distribution.

Theorem 4.4.1 *Consider the following three dimensional fluctuation process*

$$\begin{aligned}
x_N(t) &:= \sqrt{N}\left(m^\sigma_{\rho_N(t)} - m^\sigma_t\right)\\
y_N(t) &:= \sqrt{N}\left(m^\omega_{\rho_N(t)} - m^\omega_t\right)\\
z_N(t) &:= \sqrt{N}\left(m^{\sigma\omega}_{\rho_N(t)} - m^{\sigma\omega}_t\right).
\end{aligned}$$

Then $(x_N(t), y_N(t), z_N(t))$ converges as $N \to \infty$, in the sense of weak convergence of stochastic processes, to a limiting three-dimensional Gaussian process $(x(t), y(t), z(t))$ which is the unique solution of the following linear stochastic differential equation

$$\begin{pmatrix} dx(t)\\ dy(t)\\ dz(t)\end{pmatrix} = A(t)\begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix} dt + D(t)\begin{pmatrix} dB_1(t)\\ dB_2(t)\\ dB_3(t)\end{pmatrix} \tag{4.28}$$

where B_1, B_2, B_3 are independent, standard Brownian motions,

$$A(t) = 2\begin{pmatrix} -\cosh(\beta) & \sinh(\beta) & 0\\ -\gamma m^\omega_t \sinh(\gamma m^\sigma_t) + \gamma\cosh(\gamma m^\sigma_t) & -\cosh(\gamma m^\sigma_t) & 0\\ \sinh(\gamma m^\sigma_t) + \gamma m^\sigma_t\cosh(\gamma m^\sigma_t) + \gamma m^{\sigma\omega}_t \sinh(\gamma m^\sigma_t) & 0 & -(\cosh(\beta) + \cosh(\gamma m^\sigma_t))\end{pmatrix}$$

$$\frac{D(t)D^*(t)}{2} =$$
$$\begin{pmatrix} -m^{\sigma\omega}_t \sinh(\beta) + \cosh(\beta) & 0 & -m^\sigma_t \sinh(\beta) + m^\omega_t\cosh(\beta)\\ 0 & -m^\omega_t \sinh(\gamma m^\sigma_t) + \cosh(\gamma m^\sigma_t) & m^\sigma_t\cosh(\gamma m^\sigma_t) - m^{\sigma\omega}_t \sinh(\gamma m^\sigma_t)\\ -m^\sigma_t\sinh(\beta) + m^\omega_t\cosh(\beta) & m^\sigma_t\cosh(\gamma m^\sigma_t) - m^{\sigma\omega}_t\sinh(\gamma m^\sigma_t) & -m^{\sigma\omega}_t\sinh(\beta) + \cosh(\beta) - m^\omega\sinh(\gamma m^\sigma_t) + \cosh(\gamma m^\sigma_t)\end{pmatrix}$$

and $(x(0), y(0), z(0))$ *have a centered Gaussian distribution with covariance matrix*

$$\begin{pmatrix} 1-(m_\lambda^\sigma)^2 & m_\lambda^{\sigma\omega} - m_\lambda^\sigma m_\lambda^\omega & m_\lambda^\omega - m_\lambda^\sigma m_\lambda^{\sigma\omega} \\ m_\lambda^{\sigma\omega} - m_\lambda^\sigma m_\lambda^\omega & 1-(m_\lambda^\omega)^2 & m_\lambda^\sigma - m_\lambda^{\sigma\omega} m_\lambda^\omega \\ m_\lambda^\omega - m_\lambda^\sigma m_\lambda^{\sigma\omega} & m_\lambda^\sigma - m_\lambda^{\sigma\omega} m_\lambda^\omega & 1-(m_\lambda^{\sigma\omega})^2 \end{pmatrix} \tag{4.29}$$

Proof.
One key remark is the fact that the stochastic process $(m^\sigma_{\rho_N(t)}, m^\omega_{\rho_N(t)}, m^{\sigma\omega}_{\rho_N(t)})$ is a *sufficient statistics* for our model: in this context this means that its evolution is Markovian.
Notice that $(x_N(t), y_N(t), z_N(t))$ is obtained from $(m^\sigma_{\rho_N(t)}, m^\omega_{\rho_N(t)}, m^{\sigma\omega}_{\rho_N(t)})$ through a time dependent, linear invertible transformation. Hence $(x_N(t), y_N(t), z_N(t))$ is itself a (time inhomogeneous) Markov process, whose infinitesimal generator $\mathcal{H}_{N,t}$ can be explicitly obtained. A punctual derivation is given in [57], we obtain

$$\begin{aligned} \mathcal{H}_{N,t} f(x,y,z) &= \frac{N}{4} \sum_{j,k \in \{-1;1\}} \left(\frac{x}{\sqrt{N}} j + \frac{y}{\sqrt{N}} k + \frac{z}{\sqrt{N}} jk + m_t^\sigma j + m_t^\omega k + m_t^{\sigma\omega} jk + 1 \right) \cdot \\ &\quad \cdot \left\{ e^{-\beta jk} \left[f\left(x - \frac{2}{\sqrt{N}} j, y, z - \frac{2}{\sqrt{N}} jk \right) - f(x,y,z) \right] \right. \\ &\quad \left. + e^{-\gamma\left(\frac{x}{\sqrt{N}} + m_t^\sigma\right) k} \left[f\left(x, y - \frac{2}{\sqrt{N}} k, z - \frac{2}{\sqrt{N}} jk \right) - f(x,y,z) \right] \right\} \\ &\quad - \sqrt{N} \dot{m}_t^\sigma \frac{\partial f}{\partial x}(x,y,z) - \sqrt{N} \dot{m}_t^\omega \frac{\partial f}{\partial y}(x,y,z) - \sqrt{N} \dot{m}_t^{\sigma\omega} \frac{\partial f}{\partial z}(x,y,z). \end{aligned} \tag{4.30}$$

If we now take $f : \mathbb{R}^3 \to \mathbb{R}$ a $\mathcal{C}^3$ function with compact support, an exercise in Taylor expansion yields

$$\lim_{N \to \infty} \sup_{x,y,z \in \mathbb{R}^3} |\mathcal{H}_{N,t} f(x,y,z) - \mathcal{H}_t f(x,y,z)| = 0,$$

where $\mathcal{H}_t$ is the infinitesimal generator of the linear diffusion process (4.28). Using Theorem 1.6.1 in [32], the proof is completed if we show that $(x_N(0), y_N(0), z_N(0))$ converges in distribution to $(x(0), y(0), z(0))$. This last statement follows by the standard central limit theorem for i.i.d. random variables: indeed, by assumption, $(\sigma_i(0), \omega_i(0))$ are independent with law λ, and (4.29) is just the covariance matrix under λ of $(\sigma(0), \omega(0), \sigma(0)\omega(0))$. ∎

Theorem 4.4.1 ensures that, for each $t > 0$, the distribution of $(x_N(t), y_N(t), z_N(t))$ is asymptotically Gaussian, and provides a method to compute the limiting covariance matrix. Indeed, denote by Σ_t the covariance matrix of $(x(t), y(t), z(t))$. A simple application of Ito's rule to (4.28) shows that Σ_t solves the Lyapunov equation

$$\frac{d\Sigma_t}{dt} = A(t)\Sigma_t + \Sigma_t A(t)^* + D(t)D^*(t). \tag{4.31}$$

In order to solve Equation (4.31), it is convenient to interpret Σ as a vector in $\mathbb{R}^{3\times 3} = \mathbb{R}^3 \otimes \mathbb{R}^3$. To avoid ambiguities, for a 3×3 matrix C we write $\mathrm{vec}(C)$ whenever we interpret it as a vector. It is easy to check that Equation (4.31) can be rewritten as follows

$$\frac{d(\mathrm{vec}(\Sigma_t))}{dt} = (A(t) \otimes I + I \otimes A(t))\mathrm{vec}(\Sigma_t) + \mathrm{vec}(D(t)D^*(t)), \tag{4.32}$$

where "$\otimes$" denotes the tensor product of matrices. Equation (4.32) is linear, so its solution can be given an explicit expression and can be computed after having solved (4.18). More importantly, the behavior of Σ_t for large t can be obtained explicitly as follows.

A. *Case* $\gamma < \frac{1}{\tanh(\beta)}$. In this case we have shown in Theorem 4.3.11 that the solution $(m_t^\sigma, m_t^\omega, m_t^{\sigma\omega})$ of Equation (4.18) converges to $(0, 0, \tanh(\beta))$ as $t \to +\infty$. In particular, one immediately obtains the limits

$$A := \lim_{t\to+\infty} A(t), \quad DD^* := \lim_{t\to+\infty} D(t)D^*(t). \tag{4.33}$$

A direct inspection (see Appendix A.6) shows that A has three real strictly negative eigenvalues. Moreover, the eigenvalues of the matrix $A \times I + I \times A^*$ are all of the form $\lambda_i + \lambda_j$ where λ_i and λ_j are eigenvalues of A, and therefore they are all strictly negative. It follows from (4.32) that $\lim_{t\to+\infty} \Sigma_t = \Sigma$ where

$$\mathrm{vec}(\Sigma) = -(A \otimes I + I \otimes A)^{-1}\mathrm{vec}(DD^*). \tag{4.34}$$

B. *Case* $\gamma > \frac{1}{\tanh(\beta)}$. Also in this case, by Theorem 4.3.11, the limit

$$\lim_{t\to+\infty} (m_t^\sigma, m_t^\omega, m_t^{\sigma\omega})$$

exists. Disregarding the exceptional case in which the initial condition of (4.18) belongs to the stable manifold Γ introduced in Theorem 4.3.11 iv), the limit above equals either $(m_*^\sigma, m_*^\omega, m_*^{\sigma\omega})$, or $(-m_*^\sigma, -m_*^\omega, m_*^{\sigma\omega})$, depending on the initial condition, where $(m_*^\sigma, m_*^\omega, m_*^{\sigma\omega})$ are obtained by Theorem 4.3.11 (*iii*). In both cases one obtains as in (4.33) the limits A and DD^*, and we show in Appendix A.6 that also in this case the eigenvalues of A are real and strictly negative, so that $\lim_{t\to+\infty} \Sigma_t = \Sigma$ is obtained as in (4.34).

C. *Case* $\gamma = \frac{1}{\tanh(\beta)}$. In this case, as shown in Appendix A.6, the limiting matrix A is singular: it follows that the limit $\lim_{t\to+\infty} \Sigma_t$ does not exists, as one eigenvalue of Σ_t grows polynomially in t. This means that, for *critical* values of the parameters, the size of normal fluctuations around the deterministic limit grows in time. Similarly to what is done in [13] for reversible models, we can determine the critical long-time behavior of the fluctuation by a suitable space-time scaling in the model, giving rise to non-normal fluctuations. More precisely, one can show the following convergence in distribution

$$N^{\frac{1}{4}}\left(m^{\cdot}_{\rho_N}(\sqrt{N}t) - m^{\cdot}(\sqrt{N}t)\right) \overset{N\to\infty}{\longrightarrow} Z$$

where Z is non-Gaussian.

In this thesis we shall not study any further the critical case, we shall rather specialize the result of Theorem 4.4.1 in order to infer information on the evolution of the finite volume global health indicator and this will be used in the next section on large portfolio losses. We have in fact the immediate

Corollary 4.4.2 *As $N \to \infty$ we have that*

$$\sqrt{N} \ \left[m^{\sigma}_{\rho_N(t)} - m^{\sigma}_t\right]$$

converges in law to a centered Gaussian random variable Z with variance

$$V(t) = \Sigma_{11}(t) \tag{4.35}$$

where $\Sigma(t)$ solves Equation (4.31) and m^{σ}_t solves Equation (4.18).
Moreover for $\alpha \in \mathbb{R}$ we have

$$P(m^{\sigma}_N(t) \geq \alpha) \approx \mathcal{N}\left(\frac{\sqrt{N}m^{\sigma}_t - \sqrt{N}\alpha}{\sqrt{V(t)}}\right). \tag{4.36}$$

■

We conclude this section with the following

Remark 4.4.3 *The evolution equation (4.31) for the covariance matrix Σ_t is coupled with the McKean-Vlasov Equation (4.18), and their joint behavior exhibits interesting aspects even before the system gets close to the stable fixed point. In particular, in the case $\gamma > \frac{1}{\tanh(\beta)}$, if the initial condition is sufficiently close to the stable manifold Γ, the system (4.18) spends some time close to the symmetric equilibrium before drifting to one of the stable equilibria. A closer look at Equation (4.31) shows that when the system is close to the symmetric equilibrium, the covariance matrix Σ grows exponentially fast in time, causing sharp peaks in the variances. This is related to the credit crisis mentioned in the introduction. A more detailed discussion on this point is given in the next chapter, in relation to applications to portfolio losses.*

4.4.2 A functional approach

We are now going to propose a second (and more general) approach to the study of the fluctuation of the empirical measure ρ_N around its limit Q^*.
What do we mean by "more general"? Recall Equation (4.26)

$$\int \psi(\sigma_i(t), \omega_i(t)) d\rho_N(t) := \frac{1}{N}\sum_{i=1}^{N} \psi(\sigma_i(t), \omega_i(t))$$

and the convergence result in Equation (4.27)

$$(\rho_N(t))_{t\in[0,T]} \ \to \ (q_t)_{t\in[0,T]} \ \ a.s.$$

Notice that the convergence is made on the flow $(\rho_N(t))_{t\in[0,T]}$ of the projection at given times, instead of on the trajectory measure ρ_N. This implies that with the methods explored before, we are able to characterize the fluctuation only on functionals that are linear in the measure ρ (e.g. the expected means of the components or functions of expected means).
In the next chapter we are going to illustrate an example of marginal and aggregate

losses (see Example 5.0.14) where the method of the convergence of generators, described in Section 4.4.1, is not powerful enough to explain the fluctuations.
A more general result must involve directly the empirical measure ρ_N on the trajectory space.
As in Section 3.5, we are aiming at describing a limiting result of the form:

$$\sqrt{N}\left(\int \Phi_i d\rho_N - \int \Phi_i dQ^*\right)_{i=1}^n \to \tilde{Z} \ , \qquad N \to \infty$$

where $\Phi_i, \ i = 1, ..., n$ is a fixed vector of functions in $\mathcal{C}_b(\mathcal{D}[0,T] \times \mathcal{D}[0,T]; \mathbb{R})$ and $\tilde{Z} \sim \mathcal{N}_n(0, \tilde{C})$, for a well defined covariance matrix $\tilde{C}(\Phi_i, \Phi_j)$.

The spirit of the derivation below follows the approach already seen in Section 3.5. We have shown there, see Theorem 3.5.4, that one crucial point is to define a linear map $T : \mathcal{M} \to B$ where B is a suitable Banach space in which it is possible to prove a central limit theorem.

Conjecture: *The space $\mathcal{M}(\mathcal{D}[0,T] \times \mathcal{D}[0,T])$ can be mapped into a suitable Banach space, where Theorem 3.5.1 can be applied. (cfr Section 3.5)*

Remark 4.4.4 *Concerning the setting of this chapter we are still not able to ensure the validity of the Conjecture. The existence of a linear map T is in fact related to the "smoothness" of the function $F(Q)$ defined in Formula (4.11).*
We are quite confident that this technical problem can be solved and leave this issue to future research. In spite of this incomplete result, we have decided to state the main theorem, in order to let the reader appreciate the generality of this methodology.

Assuming the validity of the Conjecture we have

Theorem 4.4.5 *Let ν_* be the law, induced by Q^*, of the $\mathcal{M}_0$-valued random variable $(\delta_{\{\sigma_{[0,T]}, \omega_{[0,T]}\}} - Q^*)$. Let $(\Phi_i)_{i=1}^n \in \mathcal{C}_b(\mathcal{D}[0,T] \times \mathcal{D}[0,T]; \mathbb{R})$ be a fixed vector.*
Define moreover $\hat{\Phi}_i(A) = \int R(A) \int \Phi_i dR \ \nu_*(dR)$, for all $i = 1, ..., n$.
Finally, let ρ_N be as defined in (4.7) and Q^* as defined in Theorem 4.3.2. As $N \to \infty$

$$\sqrt{N} \ \left(\int \Phi_i d\rho_N - \int \Phi_i dQ^*\right)_{i=1}^n \tag{4.37}$$

converges under P_N to a multivariate Gaussian random variable with covariance

$$(\tilde{C})_{ij} = \int (\Phi_i - \Phi_i^*)(\Phi_j - \Phi_j^*) dQ^* - D^2 F(Q^*)[\hat{\Phi}_i, \hat{\Phi}_j] = \tag{4.38}$$

$$= E^{Q^*}\left[\left((\Phi_i - \Phi_i^*) + \gamma \int_0^T \omega(t) m_{\hat{\Phi}_i(t)} dM_t^\omega\right)\left((\Phi_j - \Phi_j^*) + \gamma \int_0^T \omega(t) m_{\hat{\Phi}_j(t)} dM_t^\omega\right)\right];$$

where $m_{\hat{\Phi}_i(t)} = \int \sigma(t) \ \hat{\Phi}_i(d\sigma[0,T], d\omega)$ for $i = 1, ..., n$ and where

$$M_t^\omega := N_t^\omega - \int_0^t e^{-\gamma\omega(s) m_{q_s}} ds \tag{4.39}$$

denotes the Q^*−martingale associated with the jump process of the ω component.

Proof. The proof of this result is similar to the one given for Theorem 3.5.6. We would like to notice that the analogous of Theorem 3.5.4 and 3.5.5, have to be proved as well. The latter one can be proved with minor modifications in the notations. Concerning the former, it is basically the argument of the Conjecture. ∎

We shall provide in the next chapter an example (see Example 5.0.14), in which, in order to obtain a CLT for the global health indicator, we need to use Theorem 4.4.5.

Remark 4.4.6 *Compare Equation (3.57) and Equation (4.38), that is, the expressions for the covariances C and $\tilde{C}$ of the central limit theorems in the two model proposed respectively in Chapter 3 and in Chapter 4. One sees that in the former it is involved the martingale M^σ, whereas in the latter we see M^ω, corresponding respectively to the jump process of the σ and ω components of the Markov chain.*
This discrepancy is due to the different specifications of the corresponding jump intensities. Indeed, compare the second expression in (4.2) with Assumption (C.2) in 3.2.2. In Chapter 3, the term involving e^{m_N} is related to the jump of σ, whereas in the model proposed in Section 4.1 it is related to ω. This fact has an impact on the second order derivatives of the functionals involved and eventually on the covariances found in the corresponding central limit theorems.

Chapter 5

Applications to portfolio losses

We now address the problem of computing losses in a portfolio of positions issued by the N firms. The main result is Theorem 5.0.8 where an approximation for the distribution of losses suffered via large portfolios is provided. Examples with different specifications of portfolios and a qualitative description of a credit crisis based on simulations follow.

A rather general modeling framework is to consider the total loss that a bank may suffer due to a risky portfolio at time t as a random variable defined by $L^N(t) = \sum_i L_i(t)$. Different specifications for the single (marginal) losses $L_i(t)$ can be chosen accounting for heterogeneity, time dependence, interaction, macroeconomic factors and so on[1].

In this work we adopt the point of view of Giesecke and Weber (2005) (see [40]). The idea is to compute the aggregate losses as a sum of marginal losses $L_i(t)$, of which the distribution is supposed to depend on the realization of the variable σ_i, i.e. on the rating class. In particular, conditioned on the realization of σ, the marginal losses will be assumed to be independent and identically distributed (the independence condition can be weakened, see Example 5.0.12 below). More precisely, we assume given a suitable conditional distribution function G_x, $x \in \{-1; 1\}$ namely

$$G_x(u) := P(L_i(t) \leq u | \sigma_i(t) = x) \tag{5.1}$$

where the first and second moments are well defined, namely

$$l_1 := E(L_i(t)|\sigma_i(t) = 1) < E(L_i(t)|\sigma_i(t) = -1) =: l_{-1} \tag{5.2}$$

and

$$v_1 := Var(L_i(t)|\sigma_i(t) = 1); \quad v_{-1} := Var(L_i(t)|\sigma_i(t) = -1) \tag{5.3}$$

The inequality in (5.2) specifies that we expect to loose more when in financial distress.

The aggregate loss of a portfolio of size N at time t is then defined as

$$L^N(t) = \sum_{i=1}^{N} L_i(t)$$

[1] A punctual treatment of this general modeling framework can be found in the book by Embrechts, Frey and McNeil (2005)[31]. For a comparison with the most widely used industry examples of credit risk models see Frey and McNeil (2002)[38], Crouhy, Galai and Mark (2000)[17] or Gordy (2000)[42]. The same modeling insights are also developed in the more recent literature on risk management and large portfolio losses analysis, see [40], [36], [23] and [46] for different specifications.

We recall the definition of the global health indicators $m_N^\sigma(t) := \frac{1}{N}\sum_{i=1}^N \sigma_i(t)$, and $m_t^\sigma := \int \sigma dq_t$ where q_t solves the McKean-Vlasov Equation (see Equation (4.8)).

We also introduce a deterministic time function, which will be seen to represent an "asymptotic" loss when the number of firms goes to infinity, namely

$$L(t) = \frac{(l_1 - l_{-1})}{2} m_t^\sigma + \frac{(l_1 + l_{-1})}{2} \tag{5.4}$$

We state now a technical lemma and the main result of this section.

Lemma 5.0.7 *Define $f : \{-1;1\} \to \{l_{-1}, l_1\}$ as $f(-1) = l_{-1}, f(1) = l_1$ with l_{-1}, l_1 as in (5.2). Then for $t \in [0,T]$ we have the convergence in distribution*

$$\sqrt{N}\left(\frac{\sum_j f(\sigma_j(t))}{N} - L(t)\right) \to X \sim N\left(0, \frac{(l_1 - l_{-1})^2 V(t)}{4}\right).$$

where $L(t)$ is defined in Equation (5.4) and $V(t)$ in (4.35).

Proof. Define, for $x \in \{-1;1\}$, the quantity $A_x^N(t)$ as the number of σ_i that, at a given time t, are equal to x. We may then write $\frac{1+m_N^\sigma(t)}{2} = \frac{A_1^N(t)}{N}$ and $\frac{1-m_N^\sigma(t)}{2} = \frac{A_{-1}^N(t)}{N}$. Recall moreover that for $N \to \infty$, $m_N^\sigma(t) \to m_t^\sigma$. We then have

$$\sqrt{N}\left(\frac{\sum_i f(\sigma_i(t))}{N} - L(t)\right) = \sqrt{N}\left(\frac{\sum\limits_{i:\sigma_i=1} f(\sigma_i(t)) + \sum\limits_{i:\sigma_i=-1} f(\sigma_i(t))}{N} - L(t)\right) =$$

$$= \sqrt{N}\left(\frac{l_1 A_1^N(t) + l_{-1}A_{-1}^N(t)}{N} - L(t)\right) = \sqrt{N}\left(l_1 \frac{1+m_N^\sigma(t)}{2} + l_{-1}\frac{1-m_N^\sigma(t)}{2} - L(t)\right) =$$

$$= \sqrt{N}\left(\frac{(l_1+l_{-1})}{2} + \frac{(l_1-l_{-1})}{2} m_N^\sigma(t) - \frac{(l_1-l_{-1})}{2} m_t^\sigma - \frac{(l_1+l_{-1})}{2}\right) =$$

$$= \sqrt{N}\left(\frac{(l_1-l_{-1})}{2}\left(m_N^\sigma(t) - m_t^\sigma\right)\right) \to X \sim N\left(0, \frac{(l_1-l_{-1})^2 V(t)}{4}\right)$$

where the convergence follows from Corollary 4.4.2, since $m_N^\sigma(t) = m^\sigma_{\rho_N(t)}$. ∎

Theorem 5.0.8 *Assume $L_i(t)$ has a distribution of the form (5.1). Then for any $t \in [0,T]$ and for any value of the parameters $\beta > 0$ and $\gamma > 0$, we have*

$$\sqrt{N}\left(\frac{L^N(t)}{N} - L(t)\right) \to Y \sim N\left(0, \hat{V}(t)\right)$$

in distribution, where $L(t)$ has been defined in Equation (5.4) and

$$\hat{V}(t) = \frac{(l_1 - l_{-1})^2 V(t)}{4} + \frac{(1+m_t^\sigma)\, v_1}{2} + \frac{(1-m_t^\sigma)\, v_{-1}}{2}. \tag{5.5}$$

with $V(t)$ as defined in (4.35).

Proof. We have to check that

$$\sqrt{N}\left(\frac{L^N(t)}{N} - L(t)\right) \to Y \sim N\left(0, \hat{V}(t)\right)$$

where $\hat{V}(t)$ is defined in (5.5).

Separating the firms that belong to different rating classes,

$$\sqrt{N}\left(\frac{\sum_j L(\sigma_j(t))}{N} - L(t)\right) = \sqrt{N}\left(\frac{\sum_{j:\sigma_j=1} L(\sigma_j(t)) + \sum_{j:\sigma_j=-1} L(\sigma_j(t))}{N} - L(t)\right)$$

Since $L(\sigma_j(t))$ conditioned on the realization of $\sigma_j(t)$ are i.i.d. random variables, we construct $X_{1,j}$ for $j = 1, .., A_1^N(t)$, as $A_1^N(t)$ independent copies of $L(\sigma_j(t) = 1)$ and $X_{-1,j}$, for $j = 1, .., A_{-1}^N(t)$, as copies of $L(\sigma_j(t) = -1)$. We then add and subtract $\sum_j f(\sigma_j(t))$, obtaining

$$\sqrt{N}\left(\frac{\sum_{j:\sigma_j=1}(X_{1,j} - l_1)}{N} + \frac{\sum_{j:\sigma_j=-1}(X_{-1,j} - l_{-1})}{N} + \frac{\sum_j f(\sigma_j(t))}{N} - L(t)\right) \tag{5.6}$$

Since we have only independence conditionally on σ, we need to check whether the CLT still applies. Let us show the convergence of the corresponding characteristic functions. Define for $r \in \mathbb{R}$, $\varphi_1(r) = E\left(e^{ir(X_{1,j}-l_1)}\right)$ for all j (respectively $\varphi_{-1}(t)$), then we have

$$E\left[\exp\left\{ir\frac{L^N(t) - NL(t)}{\sqrt{N}}\right\}\right] = \tag{5.7}$$

$$E\left[E\left[\exp\left\{ir\frac{\sum_{j:\sigma_j=1}(X_{1,j} - l_1)}{\sqrt{N}} + \frac{\sum_{j:\sigma_j=-1}(X_{-1,j} - l_{-1})}{\sqrt{N}} + \frac{\sum_j f(\sigma_j(t)) - NL(t)}{\sqrt{N}}\right\}\middle|\sigma(t)\right]\right]$$

The last of the three terms is measurable with respect to the sigma algebra generated by $\sigma(t)$ so that we can take it out from the inner expectation. Because of the conditional independence we can separate the remaining terms in the expectations:

$$E\left[\exp\left\{ir\frac{\sum_{j:\sigma_j=1}(X_{1,j} - l_1)}{\sqrt{N}}\right\}\middle|\sigma(t)\right] \cdot E\left[\exp\left\{ir\frac{\sum_{j:\sigma_j=-1}(X_{-1,j} - l_{-1})}{\sqrt{N}}\right\}\middle|\sigma(t)\right]$$

We now prove that a CLT holds for each term:

$$E\left[\exp\left\{ir\frac{\sum_{j:\sigma_j=1}(X_{1,j} - l_1)}{\sqrt{N}}\right\}\middle|\sigma(t)\right] = E\left[\prod_{j=1}^{A_1^N(t)} \exp\left\{ir\frac{(X_{1,j} - l_1)}{\sqrt{N}}\right\}\middle|\sigma(t)\right] =$$

$$= \prod_{j=1}^{A_1^N(t)} E\left[\exp\left\{ir\frac{(X_{1,j} - l_1)}{\sqrt{N}}\right\}\middle|\sigma(t)\right] = \left[1 - \frac{v_1}{2}\frac{r^2}{N} + o\left(\frac{1}{N}\right)\right]^{A_1^N(t)}$$

where the last equality follows because l_1 and v_1 are the first two moments of $X_{1,j}$.

Recalling that $\frac{A_1^N(t)}{N} = \frac{1+m_N^\sigma(t)}{2}$ converges almost surely to $\frac{1+m_t^\sigma}{2}$ we obtain

$$\lim_{N\to\infty}\left[1-\frac{v_1}{2}\frac{r^2}{N}+o\left(\frac{1}{N}\right)\right]^{A_1^N(t)} =$$
$$= \lim_{N\to\infty}\left[1-\frac{v_1}{2}\frac{r^2}{A_{-1}^N(t)}\frac{A_{-1}^N(t)}{N}+o\left(\frac{1}{N}\right)\right]^{A_1^N(t)} = e^{-\frac{r^2}{2}\frac{1+m_t^\sigma}{2}v_1}$$

The same argument is used to prove the convergence for the terms where $\sigma_j = -1$. Since $\frac{A_{-1}^N(t)}{N} \to \frac{1-m_t^\sigma}{2}$, we have

$$\lim_{N\to\infty}\left[1-\frac{v_{-1}}{2}\frac{r^2}{A_{-1}^N(t)}\frac{A_{-1}^N(t)}{N}+o\left(\frac{1}{N}\right)\right]^{A_{-1}^N(t)} = e^{-\frac{r^2}{2}\frac{1-m_t^\sigma}{2}v_{-1}}.$$

We now take the limit in Equation (5.7); by dominated convergence we can interchange lim and expected value so that

$$\lim_{N\to\infty} E\left[E\left[...|\sigma(t)\right]\right] = E\left[\lim_{N\to\infty} E\left[...|\sigma(t)\right]\right]$$

Moreover recall from Lemma 5.0.7 that $\frac{\sum_j f(\sigma_j(t))-NL(t)}{\sqrt{N}}$ converges to $X \sim N\left(0, \frac{(l_1-l_{-1})^2 V(t)}{4}\right)$, so that

$$\lim_{N\to\infty} E\left[\exp\left\{ir\frac{\sum_j f(\sigma_j(t))-NL(t)}{\sqrt{N}}\right\}\right] = e^{-\frac{r^2}{2}\frac{(l_1-l_{-1})^2V(t)}{4}}$$

Finally we find

$$\lim_{N\to\infty} E\left[...\ |\sigma(t)\right] = e^{-\frac{r^2}{2}\frac{(l_1-l_{-1})^2V(t)}{4}} e^{-\frac{r^2}{2}\frac{1+m_t^\sigma}{2}v_1} e^{-\frac{r^2}{2}\frac{1-m_t^\sigma}{2}v_{-1}} = e^{-\frac{r^2}{2}\hat{V}(t)}$$

and this completes the proof. ∎

From Theorem 5.0.8 we immediately have the following

Corollary 5.0.9 *We have*

$$P(L^N(t)\geq\alpha) \approx \mathcal{N}\left(\frac{N\,L(t)-\alpha}{\sqrt{N}\sqrt{\hat{V}(t)}}\right).$$

∎

Remark 5.0.10 *By the symmetry of the model, the above Gaussian approximation for the losses is appropriate for a wide (depending on N) range of values of α. If we modify the model to become asymmetric as discussed at the end of Section 4.1 and, more precisely, we modify it so that $\sigma = -1$ becomes much less likely than $\sigma = +1$, then for a "realistic" value of N, the number of firms with $\sigma_i = -1$ could be too small for the Gaussian approximation to be sufficiently precise. One could then rather consider a Poisson-type approximation.*

We shall now provide examples illustrating possible specifications for the marginal loss distributions where, without loss of generality, we assume a unitary loss (e.g. loss due to a corporate bond) when a firm is in the bad state.

We start with a very basic example where we assume that the marginal losses (when conditioned on the value of σ) are deterministic. This means that the riskiness of the loss portfolio is related only to the number of firms in financial distress and so we can use directly the results of section 4.3, in particular of Corollary 4.4.2.

Example 5.0.11 *Suppose that marginal losses are described as follows:*

$$L_i(t) = \begin{cases} 1 & if \quad \sigma_i(t) = -1 \\ 0 & if \quad \sigma_i(t) = 1 \end{cases}$$

On the other hand

$$L^N(t) = \sum_{i=1}^{N} \frac{1 - \sigma_i(t)}{2}.$$

Recalling that $m_N^\sigma(t) = \frac{1}{N}\sum_i \sigma_i(t)$, *by Corollary 4.4.2 (that here becomes a particular case of Corollary 5.0.9), we can compute various risk measures related to the portfolio losses such as the following Var-type measure*

$$P(L^N(t) \geq \alpha) = P\left(\frac{N - Nm_N^\sigma(t)}{2} \geq \alpha\right) = P\left(m_N^\sigma(t) \leq \frac{N - 2\alpha}{N}\right) \approx$$

$$\approx \mathcal{N}\left(\frac{-2\alpha + (1 - m_t^\sigma)N}{\sqrt{N}\sqrt{V(t)}}\right) = \mathcal{N}\left(\frac{-2\alpha + 2L^\infty(t)N}{\sqrt{N}\sqrt{V(t)}}\right) \tag{5.8}$$

where $L^\infty(t) := \lim_{N\to\infty} \frac{L^N(t)}{N} = \lim_{N\to\infty} \sum_i \frac{1-\sigma_i(t)}{2N} = \frac{1-m_t^\sigma}{2}$.

Looking at a portfolio of $N = 10000$ *small firms, we compute the excess loss probability for different values of the parameters* β, γ *comparing them with the benchmark case where there is no interaction at all, i.e. where* $\beta = \gamma = 0$ *("independence case"). In Figure 5.1 we show the cumulative probability of having excess losses for the same portfolios. In this figure we see that, when the dependence increases, the corresponding risk measures increase. The distributions become in fact more spread out.*

More general specifications are already suggested in the existing literature. In particular in the following example we apply our approach to a very tractable class of models, the "Bernoulli mixture models". This kind of modeling has been used in the context of cyclical correlations, i.e. in models where exogenous factors are supposed to characterize the evolution of the indicator of defaults (the classical factor models). In the context of contagion based models this class was first introduced by Giesecke and Weber in [40].

Example 5.0.12 (Bernoulli mixture models) *Assume that the marginal losses* $L_i(t)$ *are Bernoulli mixtures, i.e.*

$$L_i(t) = \begin{cases} 1 & with\ probability \quad P(\sigma_i(t), \Psi) \\ 0 & with\ probability \quad 1 - P(\sigma_i(t), \Psi) \end{cases} \tag{5.9}$$

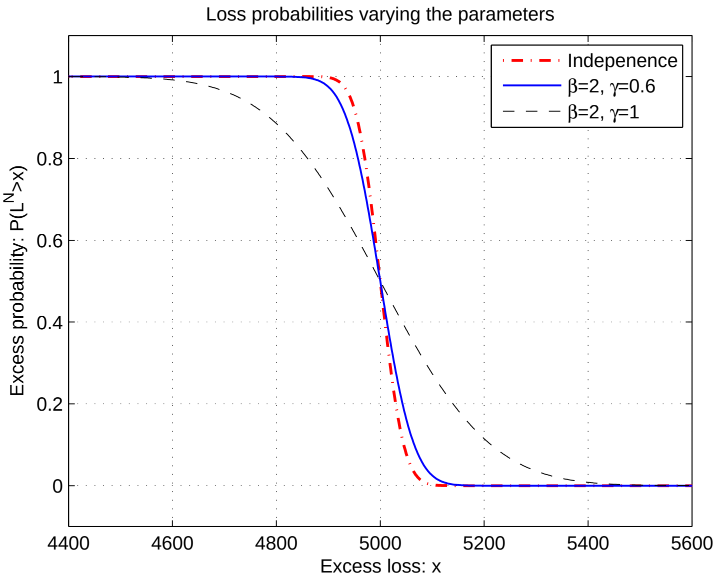

Figure 5.1: Excess loss in a large portfolio ($N = 10000$) for different values of the parameters γ and β compared with the independence case.

where the mixing derives not only from the rating class indicator $\sigma_i(t)$ of firm i, but also from an exogenous factor $\Psi \in \mathbb{R}^p$ that represents macroeconomic variables that reflect the business cycle and thus allow for both contagion and cyclical effects on the rating probabilities.

Notice that, with the above specification, the quantities defined in (5.2) and (5.3) now depend on the random factor Ψ, i.e.

$$l_1 = P(1, \Psi), \;\; v_1 = P(1, \Psi)(1 - P(1, \Psi)) \quad \textit{and analogously for} \quad l_{-1}, v_{-1}.$$

As a consequence, the asymptotic loss $L(t)$ as well as the variance of the Gaussian approximation $\hat{V}(t)$ defined in Equations (5.4) and (5.5) are also functions of Ψ. With a slight abuse of notation we shall write $L_\psi(t)$ (respectively $\hat{V}_\psi(t)$) for the asymptotic loss (variance) at time t given that $\Psi = \psi$.

Next we give a possible expression for the mixing distribution for $P(\sigma, \Psi)$ that is in line with existing models on contagion[2]. Let a and b_i, $i = 1, 2$, be non negative weight factors. Let us assume for simplicity that $\Psi \in \mathbb{R}$ is a Gamma distributed random variable. Define then

$$P(\sigma, \Psi) = 1 - \exp\left\{-a\Psi - b_1\left(\frac{1-\sigma}{2}\right) - b_2\right\}$$

This specification follows the CreditRisk+ *modeling structure, even though in the standard industry examples direct contagion is not taken into account. Notice that*

[2] We want to stress the fact that this modeling framework has been introduced by Giesecke and Weber in [40]. What is different from their approach is the microscopic interaction between the firms. In particular our framework makes it possible to quantify the time varying fluctuations of $m^{\sigma}_{\rho_N}(t)$ for any $t \in [0, T]$, i.e. in a dynamically consistent way.

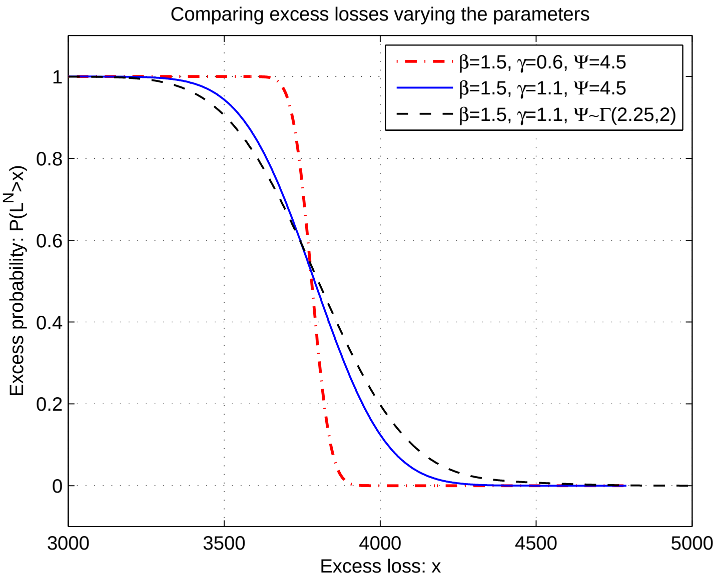

Figure 5.2: Loss amount in a large portfolio ($N = 10000$) in the case of marginal losses which (depending on the rating class) are distributed as Bernoulli random variables for which the parameter depends on Ψ.

the factor $\frac{1-\sigma}{2}$ *increases the probability of default for the firms in the bad rating class* ($\sigma = -1$)*. Applying Corollary 5.0.9 to this case we have that*

$$P(L_N(t) \geq \alpha) \approx \int \mathcal{N}\left(\frac{NL_\psi(t) - \alpha}{\sqrt{N\hat{V}_\psi(t)}}\right) df_\Psi(\psi).$$

where f_Ψ *is the density function of the Gamma random variable* Ψ*.*

In Figure 5.2 we plot the excess loss probability in the case where $a = 0.1$*,* $b_1 = 1$*,* $b_2 = 0.5$ *and* $\beta = 1.5$ *is supposed to be fixed. We compare different specifications for* Ψ *and* γ*. In particular we consider the following cases:*

$$\Psi = 4.5,\ \ \gamma = 0.6; \quad \Psi = 4.5,\ \ \gamma = 1.1; \quad \Psi \sim \Gamma(2.25, 2),\ \ \gamma = 1.1$$

The shape of the excess losses suggests that the loss may be sensibly higher in the case of high uncertainty about the value of the macroeconomic factor ($\Psi \sim \Gamma(2.25; 2)$) *and in the case of high level of contagion* ($\gamma = 1.1$)*. Notice that in all three situations we are in the subcritical case, since the critical value for* γ *is* $\gamma_c = 1/\tanh(\beta) \simeq 1.105$*. This also implies that the equilibrium value is the same in the three situations and depends only on* Ψ*.*

Remark 5.0.13 *Notice that the asymptotic loss distribution in the above Bernoulli mixture model does not only depend on a mixing parameter as in standard Bernoulli mixture models but, via* $L(t)$*, it depends also on the value* m_t^σ *of the asymptotic average global health indicator. Moreover, compared to Giesecke and Weber [40], we are able to quantify the time varying fluctuations of the global indicator* $m^\sigma_{\rho_N(t)}$*. We shall see* *that this may sensibly influence the distribution of losses in particular when looking at two different time horizons T_1 and T_2 before and after a credit crisis.*

Example 5.0.14 *(*A further example*) Further examples may be considered, in particular when the distribution of the marginal losses $L_i(t)$ depends on the entire past trajectory of the rating indicator σ_i, taking e.g. into account how long the firm has been in the bad state. Instead of depending simply on $\sigma_i(t)$, the distribution of $L_i(t)$ could then be made dependent on $S_i(t) := \mathbb{I}_{\left\{\int_0^t\left(\frac{1-\sigma_i(s)}{2}\right)ds \geq \delta t\right\}}$ with $\delta \in (0,1)$, which is equal to one if firm i has spent at least a fraction δ of time in the bad state. Corresponding to (5.9) we would then have*

$$L_i(t) = \begin{cases} 1 & \textit{with probability} \quad P(S_i(t), \Psi) \\ 0 & \textit{with probability} \quad 1 - P(S_i(t), \Psi) \end{cases}$$

Although this last example appears as a straightforward extension of Example 5.0.12, the Central Limit Theorem result in Section 4.4.1, is not sufficient to handle it. For this we need the approach based on the functional Central Limit Theorem explained in Section 4.4.2.

Let us point out that in the examples above we have considered only the problem of computing large portfolio losses which led to computing (approximately) the Var-like probabilities $P(L^N(t) \geq \alpha)$ where α is a (large) integer. From here, passing to expressions of the form $P(\frac{L^N(t)}{N} \geq \frac{\alpha}{N})$, one could then compute the probability that the loss ratio belongs to a given interval and this would then allow to compute (approximately) for our contagion model also other quantities in a risk sensitive environment. In any case notice that Theorem 5.0.8 provides the entire distribution for the portfolio losses.

In the previous examples we have described large portfolio losses at a predetermined time horizon T for different specifications of the conditional loss distribution. In what follows, we shall (implementing different numerical simulations) how the phenomenon of a *credit crisis* may be explained in our setting and how this issue may influence the quantification of losses. This dynamic point of view on risk management that accounts for the possibility of a credit crisis in the market, is one of the main contributions of this work.

As one could expect, the possibility of having a credit crisis is related to the existence of particular conditions on the market, more precisely to certain levels of interaction between the obligors (i.e. the parameters β and γ) and certain values of the state variables describing the rating classes and the fundamentals (i.e. σ and ω).

5.1 Simulation results

To illustrate the situation we shall now present some simulation results. We shall proceed along two steps: the first one relates more specifically to the particle system, the second to the portfolio losses.

Step 1.: (*Domains of attraction*)

In Section 4.3.2 we have characterized all the equilibria of the system depending on the values of the parameters. In particular we have shown that for supercritical

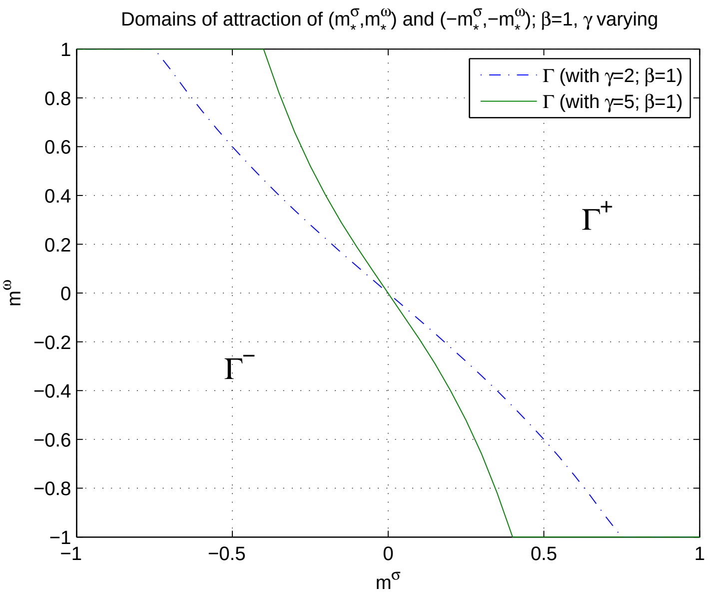

Figure 5.3: Domains of attraction Γ^+ for (m_*^σ, m_*^ω) and Γ^- for $(-m_*^\sigma, -m_*^\omega)$ and their boundary Γ for $\beta = 1$ and varying γ. Here the critical value for γ is $\gamma_c := 1/\tanh(\beta) \simeq 1.313$.

values, by which we mean $\gamma > \frac{1}{\tanh(\beta)}$, there are two asymmetric equilibrium configurations in the space (m^σ, m^ω) that are symmetric to one another and are defined as (m_*^σ, m_*^ω) and $(-m_*^\sigma, -m_*^\omega)$.
In particular, Theorem 4.3.11 allows to characterize their *domains of attraction*, i.e. the sets of initial conditions that lead the trajectory to one of the equilibria. Numerical simulations provide diagrams as in Figure 5.3.

Step 2.: (*Credit crises*)

We show results from numerical simulations that detect the crises when the values of the parameters are supercritical and the initial conditions are “near” the boundary of the domains of attraction, i.e. near Γ.

In Figure 5.4 we have plotted a trajectory starting in $(m_0^\sigma, m_0^\omega) \in \Gamma^-$ but near the boundary. It can be seen that the path moves towards $(m^\sigma, m^\omega) = (0, 0)$ and then leaves it decaying to the stable equilibrium.

Concerning the time evolution of this trajectory we see in Figure 5.5 that, for an initial condition near the boundary, the variable m_t^σ (the same would happen also with m_t^ω that for clarity is not plotted) is first attracted to the unstable value zero, around which it spends a long time before moving to the stable equilibrium. This can be explained, in financial terms, as follows:
Suppose that at a certain moment the market conditions are such that the parameters are subcritical implying that $(m^\sigma, m^\omega) = (0, 0)$ is a stable equilibrium. When the market conditions change so that the interaction between the firms increases, the parameters may become supercritical. In this new situation $(m^\sigma, m^\omega) = (0, 0)$ is no

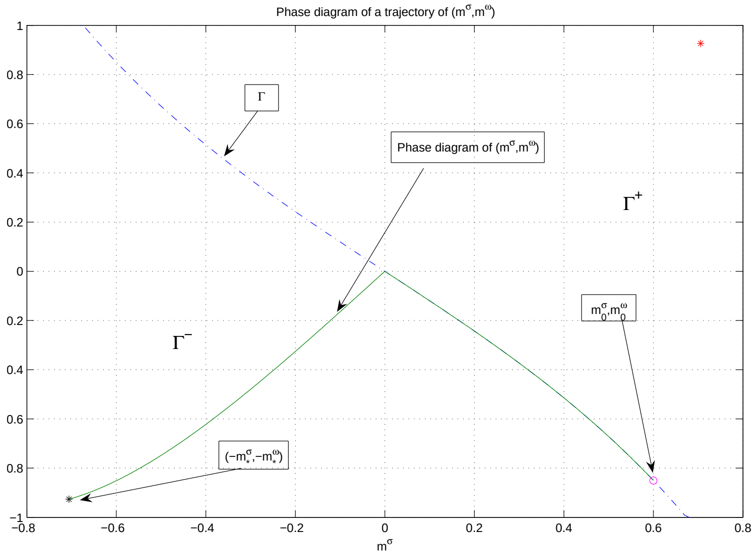

Figure 5.4: Domains of attraction Γ^+ for (m_*^σ, m_*^ω) and Γ^- for $(-m_*^\sigma, -m_*^\omega)$ and phase diagram of (m_t^σ, m_t^ω) with initial conditions $(m_0^\sigma, m_0^\omega) = (0.8, 0.74)$ when $\beta = 1$ and $\gamma = 2.3$ (here $\gamma_c = 1/\tanh(\beta) \simeq 1.313$)

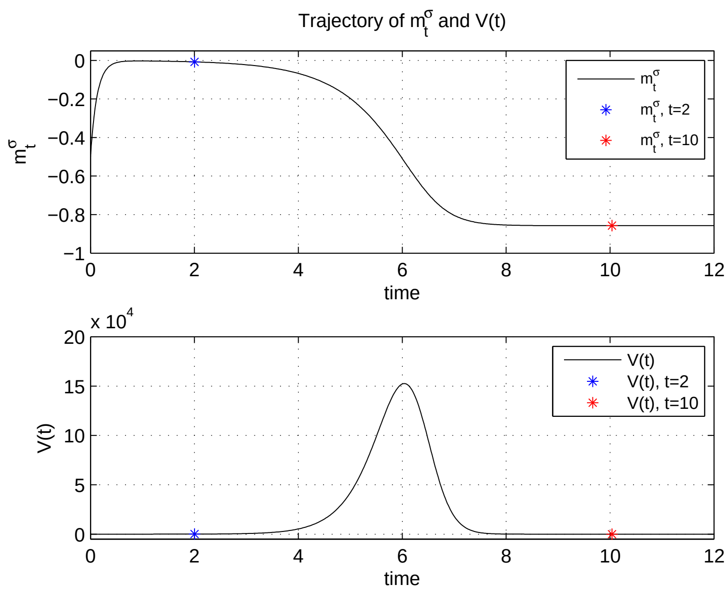

Figure 5.5: Trajectory of m_t^σ and $V(t)$ with initial conditions $m_0^\sigma = -0.5$ when $\beta = 1.5$ and $\gamma = 2.1$ (here $\gamma_c = 1/\tanh(\beta) \simeq 1.105$) . We have marked by $(*)$ the time horizons $T_1 = 2$ and $T_2 = 10$ before and after the crisis where in the next Figure 5.6 we shall compute the excess loss probabilities.

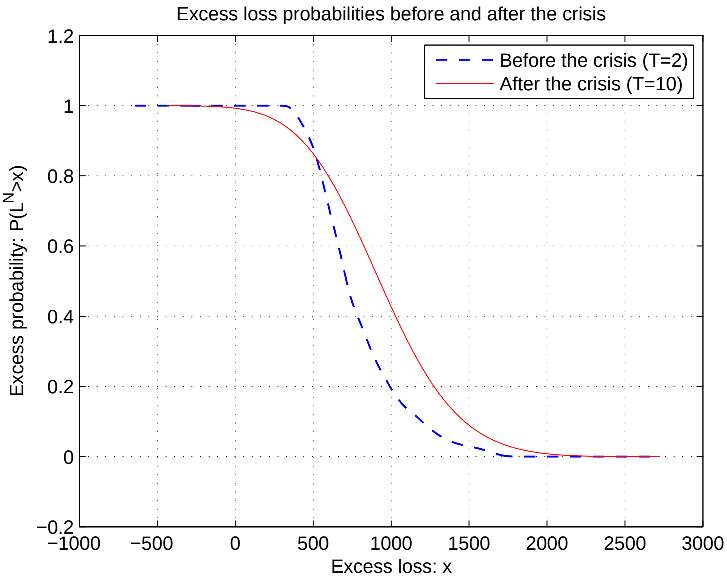

Figure 5.6: Excess probability of losses in a portfolio of $N = 10000$ obligors, $\beta = 1.5$ and $\gamma = 2.1$ computed in $T_1 = 2$ and $T_2 = 10$, namely before and after the crisis in the case of Example 5.0.12 with $\Psi \sim \Gamma(2.25; 2)$ (here $\gamma_c = 1/\tanh(\beta) \simeq 1.105$).

longer a stable equilibrium and the system starts moving towards a new equilibrium configuration. If the system configuration belongs to Γ^-, the new stable equilibrium that the system is attracted to is given by $(-m_*^\sigma, -m_*^\omega)$. As soon as the system moves away from $(0,0)$, the uncertainty (volatility) increases fast and the credit quality indicators move to the *stable* configuration changing completely the picture of the market (the speed of the convergence depends on the level of interaction).
This situation is also well illustrated by the loss probability computed before and after the crisis (i.e. in certain time instants T_1 and T_2). In Figure 5.6 we see the excess probability of suffering a loss of value x for the case of Example 5.0.12 with an exogenous parameter $\Psi \sim \Gamma(2.25; 2)$. One can see that before the crisis both the expected loss and the dispersion around the mean (as shown also in Figure 5.5) may be underestimated as well as the corresponding risk measures.

Finally we mention the fact that for different levels of interaction we can distinguish between a smoothly varying business cycle and a crisis. When β and γ, the parameters describing the level of interaction are sufficiently small, the business cycle (described in our simple model by the proportion of firms in the rating classes) evolves smoothly and the induced variance (level of uncertainty about the number of bad rated firms) is lower compared to the crisis case. In Figure 5.7 we show this fact for two levels of β and γ, both supercritical.

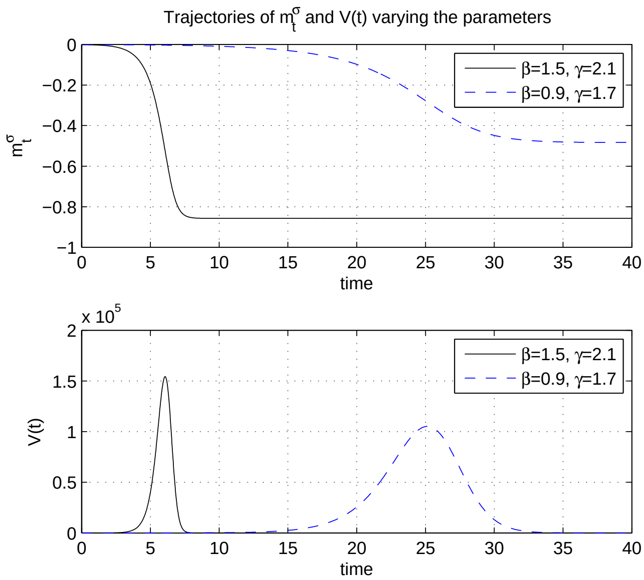

Figure 5.7: Trajectories of m_t^σ and $V(t)$ for different levels of interaction i.e. letting β and γ vary. In the case of higher values we really see a crisis and a corresponding peak in the uncertainty in the market. In the case of smaller values the number of badly rated firms decreases smoothly to a new equilibrium, i.e. towards a bad business cycle. The critical values for γ are, respectively, $1/\tanh(1.5) \simeq 1.105$ and $1/\tanh(0.9) \simeq 1.396$.

Conclusions

In this thesis we have described a new framework useful to study the propagation of *financial distress* in a network of firms linked by business relationships.

In particular we have proposed two models for *credit contagion*, based on interacting particle systems and we have quantified the impact of contagion on the losses suffered by a financial institution holding a large portfolio with positions issued by the firms.

Compared to the existing literature on credit contagion, we have proposed a *dynamic* model where it is possible to describe the evolution of the indicators of financial distress. In this way we are able to compute the distribution of the losses in a large portfolio for any time horizon T, via a suitable version of the central limit theorem.

The peculiarity of our models is the fact that the financial health indicators (the σ variables) are related to the degree of health of the system (the global indicator m^{σ}). There is a further characteristic of the firms that is summarized by a second variable ω (a liquidity indicator) that describes the ability of the firm to act as a buffer against adverse news coming from the market.

The behavior of the pair (σ, ω) is different in the two models and depends only on few parameters: β in the first one-parametric model and γ and β in the second one. These parameters indicate the strength of the interaction.

In the second model described in Chapter 4 we see phase transition and the presence of multiple equilibria, leading to the characterization of a possible credit crisis on the credit markets.

The fact that our model leads to endogenous financial indicators that describe the general health of the systems has allowed us to view a credit crisis as a microeconomic phenomenon. This has also been exemplified through simulation results.

As already stressed in the introduction the proposed models are rather simple and may not allow for a punctual calibration to real time series of default data.
To this aim, the future research may go in two directions: firstly the development of more comprehensive models. The aspects that we would like to take into account have already been proposed throughout the exposition of the model in Chapter 4. We summarize them:

- Develop a realistic credit migration model, i.e., let the variable σ take more than only two values. Moreover -given our setup- it could be a non-Markovian migration model in line with the current literature.

- Let the intensity of transition be random rather then deterministic. In particular add the dependence on some macroeconomic factors.
- Weaken the mean field assumption and the symmetry of the system where the sites $+1$ and -1 are perfectly symmetric.
- Implement a calibration of the models to real data.

Secondly we would like to pursue the study of some technical aspects:

- Prove the Conjecture of Chapter 4.
- Analyze in more detail the critical case of the parameters (see point **C** in Section 4.4.1), where the standard central limit theorem does not apply since the variance Σ_t of the Gaussian approximation (see Equation (4.31)) grows polynomially in t.
- Prove a *law of small numbers* for the asymptotic system (with infinite obligors), providing a *Poissonian* limiting distribution. This might better reflect real data when looking at credit events.

Appendix A

Technical proofs (Chapters 3-4)

Before proving the remaining theorems, we collect in table A.1 the notations that we are going to use in the first three sections of this appendix.
In what follows, in order to simplify the notations, we shall often write, with a slight abuse of notation, Φ instead of $\Phi(\sigma[0,T],\omega)$.

$\mathcal{C}_b$	The space of bounded continuous and measurable functions $\Phi : \tilde{D}[0,T]\times\mathbb{R}\to\mathbb{R}$
ν_*	The law of $(\delta_{\{\sigma_{[0,T]},\omega\}} - Q_*)$ induced by Q_*
Φ^*	$\int \Phi dQ_*$
$\hat{\Phi}(S)$	$\int\ (R(S)\cdot\int\Phi dR)\ \nu_*(dR)$; for all measurable $S\subset\tilde{D}[0,T]\times\mathbb{R}$
$(A;B_t^Q)$	$A=\beta\omega$; $B_t^Q=\beta\int\sigma\Pi_t Q(d\sigma[0,T],d\omega)$
$(B;B')$	$B=L^2[0,T]\times L^2[0,T]\times\mathbb{R}^{2+n}$; B' its topological dual
T	$T:\mathcal{M}\to B$
$p_N(\cdot)$	$\mathcal{P}_N\ \circ T^{-1}(\cdot)$: The measure induced by $\mathcal{P}_N$ on B
$w_N(\cdot)$	$\mathcal{W}_N\circ T^{-1}(\cdot)$: The measure induced by $\mathcal{W}_N$ on B
p_*	The law of $T(\delta_{\{\sigma_{[0,T]},\omega\}})-T(Q_*)$ induced by Q_*
λ	The map in B' such that $(\lambda\circ T)(Q)=\int\Phi dQ$, for $\Phi\in\mathcal{X}$
$\tilde{\lambda}$	$\int y\lambda(y)p_*(dy)$

Table A.1: Main notations and definitions of Appendix $A.1$, $A.2$ and $A.3$.

A.1 Proof of Theorem 3.5.4

We begin this section with some preliminary definitions.

Definition A.1.1 *Let B be the Banach space defined by $B=B^I\times B^{II}$ where B^I and B^{II} are respectively*

$$B^I := L^2[0,T]\times L^2[0,T]\times\mathbb{R}^2\ ;\qquad B^{II}:=\mathbb{R}^n.$$

Let T be the operator $T:\mathcal{M}(\tilde{\mathcal{D}}[0,T]\times\mathbb{R})\to B$,

$$T := (T_1,T_2,T_3,T_4,S_1,S_2,...,S_n)$$

such that

$$\begin{aligned} T_1(Q) &= E^Q[\sigma(t)] \ \in \ L^2[0,T] \\ T_2(Q) &= E^Q\left[\tfrac{1+\sigma(t)}{2}\ e^{-\beta\omega}\right] \ \in \ L^2[0,T] \\ T_3(Q) &= E^Q[\mathbb{I}_{\{\tau\le T\}}] \ \in \ \mathbb{R} \\ T_4(Q) &= E^Q[\omega\mathbb{I}_{\{\tau\le T\}}] \ \in \ \mathbb{R} \end{aligned}$$

where $\tau = \inf\{t > 0 : \sigma(t) = -1\}$ *and such that*

$$S_i(Q) = \int \Phi_i dQ \ \in \ \mathbb{R}, \quad \forall i = 1, ..., n; \tag{A.1}$$

for a given vector $\underline{\Phi} = (\Phi_1, ..., \Phi_n) \in \mathcal{C}_b$ *as defined in Theorem 3.5.4.*

Lemma A.1.2 *The map* T *is linear on* $\mathcal{M}$ *and continuous on* $\{Q : Q(\tau = T) = 0\}$.

Proof. The linearity is obviously satisfied in $\mathcal{M}$ for each component of T. Concerning the continuity, we first show that T_1 is continuous. We have to ensure that, given a sequence of measures Q_n converging towards Q such that $Q(\tau = T) = 0$, it follows

$$\int_0^T \left(E^{Q_n}[\sigma(t)] - E^Q[\sigma(t)] \right)^2 dt \to 0.$$

We have already noticed in the proof of Lemma 3.3.2 (see footnote 5) that $E^{Q_n}[\sigma(t)]$ converges to $E^Q[\sigma(t)]$ for those t such that $Q(\tau = t) = 0$. Notice that the set of t such that $Q(\tau = t) > 0$ is at most countable; in particular it has Lebesgue measure zero. Thus $E^{Q_n}[\sigma(t)] \to E^Q[\sigma(t)]$ almost everywhere w.r.t. the Lebesgue measure. Moreover, being $E^{Q_n}[\sigma(t)]$ and $E^Q[\sigma(t)]$ bounded uniformly in n and t, the convergence holds also in the L^2 sense, by Dominated Convergence. The same argument is used to show the continuity of T_2.
Concerning T_3 and T_4, notice that both $E^Q[\mathbb{I}_{\{\tau\le T\}}]$ and $E^Q[\omega\mathbb{I}_{\{\tau\le T\}}]$ are continuous on $\{Q : Q(\tau = T) = 0\}$. This can be seen by the same argument used to prove (3.36) (see again footnote 5). Here we are dealing with signed measures, but the argument can be applied to the positive and the negative part separately.
Finally, the S_i's, for $i = 1, ..., n$, are continuous by definition of weak topology. ∎

Remark A.1.3 *We would like to notice that for our purposes the fact that* T *fails to be continuous in* $A = \{Q : Q(\tau = T) > 0\}$, *plays no role at all in our dissertation. Indeed,* $\mathcal{P}_N(A) = \mathcal{W}_N(A) = 0$ *by definitions of* $\mathcal{W}_N$ *and* $\mathcal{P}_N$ *given in Section 3.3. For this reason we can ignore the set* A. *All the proofs and the arguments used in this chapter hold true if we restrict our analysis to* $\tilde{\mathcal{M}} := \mathcal{M} \setminus A$.

Proof of Theorem 3.5.4.
Point (i). The Banach space B proposed in Definition A.1.1 is of type 2 since it is an Hilbert space and Hilbert spaces are Banach spaces of type 2.
In order to define properly the map Ψ needed to prove Theorem 3.5.4, we introduce an auxiliary function $g : \mathbb{R} \to \mathbb{R}$ defined as follows

$$g(x) = \begin{cases} g_-(x) & if \quad x < -1 \\ x & if \quad -1 \le x \le e^\beta \\ g_+(x) & if \quad x > e^\beta \end{cases} \tag{A.2}$$

where g_- and g_+ are real function smooth enough in order to make $g(\cdot)$ bounded, continuous and three times continuously differentiable.

Consider now the maps $\Psi : B \to \mathbb{R}$ and $\Sigma : B \to \mathbb{R}$ such that

$$\begin{aligned} \Psi(y) &= \int_0^T \frac{1+g(y_1(t))}{2} dt - \int_0^T g(y_2(t))\, e^{-\beta g(y_1(t))} dt - \beta[g(y_4)+g(y_3)-(g(y_3))^2], \\ \Sigma(y) &= -\beta\, y_3; \end{aligned} \tag{A.3}$$

where $y = (y_1, ..., y_4, z_1, ..., z_n) \in B$.

Ψ is bounded and continuous; moreover, it can be shown by an explicit computation, that Ψ is at least three times Fréchet differentiable. Σ is clearly linear and continuous. We recall the expression for F as given in Equation (3.32), namely

$$F(Q) = \int_0^T \frac{1+m_{q_t}}{2}\, dt - \int_0^T E^Q \left[\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right] e^{-\beta m_{q_t}}\, dt -$$

$$-\beta E^Q \left[\omega \mathbb{I}_{\{\tau \le T\}}\right] - \beta[Q(\tau \le T) - Q^2(\tau \le T)] - \beta \sum_{t\in[0,T]} (\Delta Q(\tau \le t))^2; \tag{A.4}$$

For our purposes, it is more convenient to introduce a new function F_1:

$$F_1(Q) := \int_0^T \frac{1+m_{q_t}}{2}\, dt - \int_0^T E^Q \left[\frac{1+\sigma(t)}{2}\, e^{-\beta\omega}\right] e^{-\beta m_{q_t}}\, dt -$$

$$-\beta E^Q \left[\omega \mathbb{I}_{\{\tau \le T\}}\right] - \beta[Q(\tau \le T) - Q^2(\tau \le T)]. \tag{A.5}$$

It is clear that $F(Q) = F_1(Q) - \beta \sum_{t\in[0,T]}(\Delta Q(\tau \le t))^2$. We now claim that

$$F(\rho_N) = \Psi(T(\rho_N)) + \frac{\Sigma(T(\rho_N))}{N}\, , \qquad (W \otimes \eta)^{\otimes N} - a.s. \tag{A.6}$$

We show first that

$$\Psi(T(Q)) = F_1(Q). \tag{A.7}$$

To this aim, notice that $g(\cdot)$ defined in (A.2) is such that $g(T(Q)) = T(Q)$ for all Q in $\mathcal{M}_1$. In fact, for all $t \in [0,T]$, we see that $T_1(Q)(t) \in [-1; 1]$ and $T_2(Q)(t) \in [-1; e^\beta]$. Moreover $T_3(Q) \in [0, 1]$ and finally $T_4(Q) \in [-1; e^\beta]$.

As a consequence we have that $g(T(Q)) \equiv T(Q)$ for all $Q \in \mathcal{M}_1$.

Look now at the definition of $\Psi : B \to \mathbb{R}$ as given in (A.3). We compute it in $T(Q) \in B$:

$$\Psi(T(Q)) =$$

$$= \int_0^T \frac{1+T_1(Q)(t)}{2} dt - \int_0^T T_2(Q)(t)\, e^{-\beta T_1(Q)(t)} dt - \beta[T_4(Q) + T_3(Q) - (T_3(Q))^2].$$

Looking at the definition of F_1 given in (A.5) and recalling the definition of $T_1, ..., T_4$, we easily see that $\Psi(T(Q)) = F_1(Q)$. In order to prove (A.6), it remains to show that

$$\frac{\Sigma(T(\rho_N))}{N} = -\beta \sum_{t\in[0,T]} (\Delta\rho_N(\tau \le t))^2\, , \qquad (W \otimes \eta)^{\otimes N} - a.s.$$

To this aim, notice that $\rho_N(\tau \le t) = \frac{J(t)}{N}$, where $J(t)$ denotes the number of defaulted firms up to time t. In fact, being $\rho_N = \frac{1}{N}\sum_i \delta_{\{\sigma_i[0,T],\omega_i\}}$, we are putting mass $1/N$

on each jump occurred up to time t.
From this fact, we easily see that $\Delta\rho_N(\tau \leq t) = \frac{1}{N}\sum_i \mathbb{I}_{\{\sigma_i(t)\neq\sigma_i(t^-)\}}$. Since simultaneous jumps may happen only with zero $(W\otimes\eta)^{\otimes N}$-probability, we have that

$$\sum_{t\in[0,T]} (\Delta\rho_N(\tau\leq t))^2 = \frac{1}{N^2}J(T) \ , \qquad (W\otimes\eta)^{\otimes N} - a.s.$$

On the other hand $J(T) = N\rho_N(\tau \leq T)$, then

$$\sum_{t\in[0,T]} (\Delta\rho_N(\tau\leq t))^2 = \frac{\rho_N(\tau\leq T)}{N} \ , \qquad (W\otimes\eta)^{\otimes N} - a.s.$$

Now, by definition of Σ and T_3, we have that $\Sigma(T(\rho_N)) = -\beta T_3(\rho_N) = -\beta\rho_N(\tau\leq T)$. Hence

$$-\beta\sum_{t\in[0,T]} (\Delta\rho_N(\tau\leq t))^2 = \frac{\Sigma(T(\rho_N))}{N} \ , \qquad (W\otimes\eta)^{\otimes N} - a.s.$$

This ensures the validity of (A.6).
Equation (3.55) now follows by the fact that $\frac{dP_N^\omega}{dW^{\otimes N}} = \exp\{NF(\rho_N)\}$ (see Lemma 3.3.1) and applying (A.6).

Point (ii). It is an immediate consequence of the definition of T. For any $\Phi_i \in \mathcal{C}_b$ we can define $\lambda_i \in B'$, where B' stands for the topological dual of B, such that $\lambda_i(y) = \lambda_i(y_1,...,y_4,z_1,...,z_n) := z_i$. Thus, by definition, $\lambda_i \circ T(Q) = S_i(Q) = \int \Phi_i dQ$. ∎

A.2 Proof of Theorem 3.5.5

We apply Theorem 3.5.1 to the sequence $Y_i = T(\delta_{\{\sigma_i[0,T],\omega_i\}})$ taking values on B as defined in Definition A.1.1. Notice that in our setting $\Omega = (\tilde{\mathcal{D}}[0,T]\times\mathbb{R})^N$ and $\mathbb{P} = (W\otimes\eta)^N$.
We need to check that the assumptions $(B.1), ..., (B.5)$ of Theorem 3.5.1 hold true.

(B.1):
We recall that w is the law of $T(\delta_{\{\sigma_i[0,T],\omega_i\}})$ induced by $W\otimes\eta$. Thus

$$\int \exp(r|x|)w(dx) = E^{W\otimes\eta}\left[e^{r|T(\delta_{\{\sigma_i[0,T],\omega_i\}})|}\right].$$

The expected value is necessarily finite, since the image of T is bounded in the $\|\cdot\|_B$ norm. This ensures the validity of $(B.1)$.

(B.2):
Consider the function Ψ as defined in (A.3). Being bounded, Ψ clearly grows less than linearly, and, as stated in Theorem 3.5.4, it is three times Fréchet differentiable. Thus it satisfies $(B.2)$.

(B.3):
In order to ensure the validity of $(B.3)$, we define a suitable sequence of measures

$(p_N)_N$ on B and we show that it satisfies a LDP with rate function $\Lambda^*(y) - \Psi(y)$ as defined in Theorem 3.5.1. We shall see that $(B.3)$ follows as a corollary of this LDP. Let $p_N \in \mathcal{M}_1(B)$ (respectively w_N) be the probability measure induced by $\mathcal{P}_N$ (resp. $\mathcal{W}_N$) on B defined as

$$p_N(\cdot) = \mathcal{P}_N \circ T^{-1}(\cdot) \qquad (resp.\ w_N(\cdot) = \mathcal{W}_N \circ T^{-1}(\cdot)\ ).$$

Lemma A.2.1 *The following property holds true*

$$\frac{dp_N}{dw_N} = e^{N\left(\Psi + \frac{\Sigma}{N}\right)}, \tag{A.8}$$

where Ψ and Σ have been defined in (A.3).

Proof. For any $S \subset B$, we have

$$p_N(S) = \mathcal{P}_N(T^{-1}(S)) = \int \mathbb{I}_S(T(\rho_N(\underline{\sigma}[0,T],\underline{\omega})))\eta^{\otimes N}(d\underline{\omega})P_N^{\omega}(d\underline{\sigma}[0,T]) =$$

$$= \int \mathbb{I}_S(T(\rho_N(\underline{\sigma}[0,T],\underline{\omega})))\eta^{\otimes N}(d\underline{\omega})e^{N\left(\Psi(T(\rho_N(\underline{\sigma}[0,T],\underline{\omega}))) + \frac{\Sigma(T(\rho_N(\underline{\sigma}[0,T],\underline{\omega})))}{N}\right)}W^{\otimes N}(d\underline{\sigma}[0,T]) =$$

$$= \int_S e^{N\left(\Psi(y) + \frac{\Sigma(y)}{N}\right)} w_N(dy).$$

■

where the third equality is a consequence of Equation (3.55).

Lemma A.2.2 *There exists a functional $J : B \to \mathbb{R}^+$ such that the sequence of measures $(p_N)_N$ satisfies a LDP with rate function J. Moreover J is a good rate function with a unique zero at $y_* = T(Q_*)$.*

Proof. The first part of the statement follows as a corollary of Proposition 3.1.9. In particular

$$J(y) = \inf_{Q = T^{-1}(y)} I(Q). \tag{A.9}$$

Concerning the uniqueness of y_*, it follows since Q_* is the unique point such that $I(Q) = 0$. ■

Lemma A.2.3 *The good rate function J defined in Lemma A.2.2 is of the form*

$$J(y) = [\Lambda^*(y) - \Psi(y)];$$

where

$$\Lambda^*(y) := \sup_{\varphi \in B'} \{\varphi(y) - \Lambda(\varphi)\}\ ; \qquad \Lambda(\varphi) := \ln \int e^{\varphi(y)}\ w(dy).$$

Proof. In this proof we show that the sequence $(p_N)_N$ satisfies a weak LDP with rate function $\Lambda^* - \Psi$. As a consequence, by virtue of Lemma 3.1.3, concerning the uniqueness of the rate function, we conclude that $J(y) \equiv \Lambda^*(y) - \Psi(y)$ for all $y \in B$. To this aim, we apply Theorem 3.1.8 to the sequence of measures $w_N \in \mathcal{M}_1(B)$. B

is in fact a locally convex Hausdorff topological vector space and w_N are the laws of the random variables

$$X_N = \frac{1}{N}\sum_{i=1}^{N} Y_i \ \in B,$$

where Y_i are i.i.d. $B-$valued random variables with law w. Thus we have that w_N satisfies a weak LDP with rate function Λ^*.
We would like to rely on Proposition 3.1.10 (more precisely on Remark 3.1.11) in order to derive a weak LDP for the sequence $p_N \in \mathcal{M}_1(B)$. Notice that, in our setting, $\Psi(\cdot) + \Sigma(\cdot)/N$ plays the role of the function $F(\cdot)$ of Proposition 3.1.10. The presence of the term involving $\Sigma(\cdot)$ does not influence the proof of this proposition. In fact Σ is bounded when computed on the support of the measures p_N (that is on the image of T). Thus this term gives no contribution when taking the limit for N going to infinity.
We thus can apply Proposition 3.1.10 and Remark 3.1.11 to the sequence p_N such that $\frac{dp_N}{dw_N} = e^{N\left(\Psi + \frac{\Sigma}{N}\right)}$, concluding that p_N satisfies a weak LDP with rate function

$$\tilde{J}(y) = [\Lambda^*(y) - \Psi(y)] - \inf_{z\in B}[\Lambda^*(z) - \Psi(z)].$$

Moreover, arguing as in the proof of Theorem 3.3.3 (see Equation (3.40)), we see that

$$\lim_{N\to\infty} \frac{1}{N} \log E\left[e^{\left\{N\left(\Psi(X_N) + \frac{\Sigma(X_N)}{N}\right)\right\}}\right] = -\inf_{z\in B}[\Lambda^*(z) - \Psi(z)] = 0.$$

Hence $\tilde{J}(y) = [\Lambda^*(y) - \Psi(y)]$.
By virtue of Lemma 3.1.3, we conclude that $\tilde{J}(y) \equiv J(y)$ for all $y \in B$. ■

As a corollary of Lemmas A.2.2 and A.2.3, we have that there exists a unique $y^* \in B$ such that $\Lambda^*(y^*) - \Psi(y^*) = \inf_z[\Lambda^*(z) - \Psi(z)]$. So $(B.3)$ follows.

(B.4):
We have to ensure that for each $\lambda \in B'$ such that $\tilde{\lambda} = \int y\lambda(y)p_*(dy) \neq 0$ we have

$$\int \lambda^2(y)p_*(dy) - D^2\Psi(y^*)[\tilde{\lambda}, \tilde{\lambda}] > 0; \qquad (A.10)$$

where p and p_* are defined in Theorem 3.5.1.
This proof is rather technical and long. We divide it into three steps. We first show that the measure p such that $\frac{dp}{dw} = \frac{e^{D\Psi(y_*)}}{z}$, for a suitable normalizing factor z, is exactly the law of the random variable $T(\delta_{\{\sigma[0,T],\omega\}})$ induced by Q_*. This argument is then used in the second step to ensure the positivity of a suitable functional $\mathcal{H} : \mathcal{C}_b \times \mathcal{C}_b \to \mathbb{R}$. In the last part we see how to relate $\mathcal{H}$ to assumption $(B.4)$.

Step 1:

The key result of this first step is given in Lemma A.2.5 below. We look at the measure p on B, defined by putting

$$\frac{dp}{dw}(y) = e^{D\Psi(y_*)[y]} \ , \quad \text{being } y_* = T(Q_*)$$

where, as already seen, w represents the law of $T(\delta_{\{\sigma[0,T],\omega\}})$ induced by $W \otimes \eta$. We shall prove in Lemma A.2.5 that p is the law of $T(\delta_{\{\sigma[0,T],\omega\}})$ induced by Q_*. We first need to prove a technical lemma.

Lemma A.2.4 *For any $P \in \mathcal{M}$, we call $F_2(P)$ the quantity*

$$F_2(P) := \sum_{t\in[0,T]} (\Delta P(\tau \le t))^2. \tag{A.11}$$

Then for Q such that $F_2(Q) = 0$ and any $r \in \mathcal{M}$ we have

$$D\Psi(T(Q))[T(r)] = DF(Q)[r], \tag{A.12}$$

where F has been defined in (3.29).

Proof. First of all we show that $DF(Q)[r]$ is well defined. Indeed, we explicitly compute the Fréchet derivative on the function F.
To simplify the notation, we define

$$A := \beta\omega \ ; \qquad B_t^r = \beta \int \sigma \Pi_t \ r(d\sigma, d\omega) = \beta m_{r_t}, \quad r \in \mathcal{M} \tag{A.13}$$

We rewrite (3.29) substituting A and B_t^r computed under $Q \in \mathcal{M}_1$.

$$F(Q) = \int dQ \left\{ \int_0^{T\wedge\tau} (1 - \exp\{-A - B_t^Q\})dt + \mathbb{I}_{\{\tau\le T\}}(-A - B_{\tau^-}^Q) \right\}.$$

Being B_t^Q linear in Q, we have

$$\begin{aligned} F(Q+hr) &= \int d(Q+hr) \int_0^{T\wedge\tau} (1 - e^{-A-B_t^{Q+hr}})dt + \\ &+ \int d(Q+hr) \ \left[\mathbb{I}_{\{\tau\le T\}}(-A - B_{\tau^-}^{Q+hr})\right] \end{aligned}$$

the linearity of the integrals (as well as of the differentials) allows us to split the integrals, so that we can write $F(Q + hr) - F(Q)$ as

$$\begin{aligned} F(Q+hr) &- F(Q) = \\ &= \int dQ \int_0^{T\wedge\tau} [(1 - e^{-A-B_t^Q - hB_t^r}) - (1 - e^{-A-B_t^Q})]dt + \\ &+ h\int dr \int_0^{T\wedge\tau} (1 - e^{-A-B_t^Q-hB_t^r})dt + \\ &+ h\int dr \ \left[-\mathbb{I}_{\{\tau\le T\}}(A + B_{\tau^-}^Q + hB_{\tau^-}^r)\right] + \\ &+ h\int dQ \ \left[-\mathbb{I}_{\{\tau\le T\}}B_{\tau^-}^r\right]. \end{aligned} \tag{A.14}$$

We have now to divide by h and let h go to zero. Notice that there exists $C < \infty$ such that

$$\left| \frac{(1 - e^{-A-B_t^Q-hB_t^r}) - (1 - e^{-A-B_t^Q})}{h} \right| \le C \tag{A.15}$$

for h small enough. We can thus use the dominated convergence Theorem, taking the limit under the integral in the first term of (A.14). We obtain

$$DF(Q)[r] = \lim_{h\to 0} \frac{F(Q+hr)-F(Q)}{h} = \tag{A.16}$$
$$= \int dQ \int_0^{T\wedge\tau} B_t^r \; e^{-A-B_t^Q} dt + \int dr \int_0^{T\wedge\tau} (1-e^{-A-B_t^Q}) dt +$$
$$+ \int dr \; \left[-\mathbb{I}_{\{\tau\le T\}}(A+B_{\tau^-}^Q)\right] + \int dQ \; \left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^r\right].$$

This shows that $DF(Q)[r]$ is well defined.
Back to the statement, recall that $\Psi(T(\cdot)) = F_1(\cdot)$ (see (A.5) and (A.7)).
We now show that under the assumptions of the lemma

$$DF_1(Q)[r] = DF(Q)[r]. \tag{A.17}$$

By (A.5) we immediately see that $F_2(Q) = \sum_{t\in[0,T]}(\Delta Q(\tau\le t))^2 = F(Q) - F_1(Q)$. We have to show that $DF_2(Q)[r] = 0$. Indeed

$$DF_2(Q)[r] = \lim_{h\to 0} \frac{1}{h}\left[F_2(Q+hr) - F_2(Q)\right].$$

By assumption $F_2(Q) = 0$, moreover

$$F_2(Q+hr) = \sum_{t\in[0,T]} (\Delta(Q+hr)(\tau\le t))^2 = h^2 F_2(r). \tag{A.18}$$

On the other hand, for any measure $r \in \mathcal{M}$, $F_2(r)$ is bounded. Indeed, for $r \in \mathcal{M}$ we have that $\sum_{t\in[0,T]} |\Delta r(\tau\le t)| \le |r|_{TV} < \infty$, where $|r|_{TV}$ denotes the total variation of r. Hence

$$0 \le F_2(r) = \sum_{t\in[0,T]} (\Delta r(\tau\le t))^2 < \infty.$$

As a consequence $DF_2(Q)[r] = \lim_{h\to 0} \frac{h^2 F_2(r)}{h} = 0$.
This proves (A.17). We now show the validity of (A.12).

$$D\Psi(T(Q))[T(r)] = \lim_{h\to 0} \frac{\Psi(T(Q+hr)) - \Psi(T(Q))}{h} =$$
$$= \lim_{h\to 0} \frac{F_1(Q+hr) - F_1(Q)}{h} = DF_1(Q)[r] = DF(Q)[r].$$

Where we have used (A.17) and the fact that $\Psi(T(\cdot)) = F_1(\cdot)$ (see (A.7)). ■

Lemma A.2.5 *The measure p is the law of $T(\delta_{\{\sigma[0,T],\omega\}})$ induced by Q_*.*

Proof. We first prove the following claim

$$DF(Q_*)[\delta_{\{\sigma[0,T],\omega\}}] = \log \frac{dQ_*}{d(W\otimes\eta)}(\sigma[0,T],\omega), \tag{A.19}$$

for $W\otimes\eta-$almost all $(\sigma[0,T],\omega)$.

To this aim, we compute $DF(Q)[r]$ in Q_*

$$\begin{aligned} &DF(Q_*)[r] = \\ = &\int dQ_* \int_0^{T\wedge\tau} \beta m_{r_t}\, e^{-\beta(\omega+m_{q_t^*})} dt + \int dr \int_0^{T\wedge\tau} (1-e^{-\beta(\omega+m_{q_t^*})}) dt + \\ + &\int dr\, \left[-\mathbb{I}_{\{\tau\le T\}}(\beta\omega + \beta m_{q^*_{\tau-}})\right] + \int dQ_*\, \left[-\mathbb{I}_{\{\tau\le T\}}\beta m_{r_{\tau-}}\right]. \end{aligned}$$

In the case when $r = \delta_{\{\sigma[0,T],\omega\}}$, we see that the second equation in (A.13) can be rewritten as

$$\beta m_{\delta_{\{\sigma[0,T],\omega\}}} = \beta \int \sigma \Pi_t(\delta_{\{\sigma[0,T],\omega\}})(d\sigma, d\omega) = \beta\sigma(t) = \beta\ , \quad \forall t \in [0,\tau].$$

Then

$$\begin{aligned} &DF(Q_*)[\delta_{\{\sigma[0,T],\omega\}}] = \\ = &\int dQ_*\, \beta \int_0^{T\wedge\tau} e^{-\beta(\omega+m_{q_t^*})} dt + \int d\delta_{\{\sigma[0,T],\omega\}} \int_0^{T\wedge\tau} \left(1-e^{-\beta(\omega+m_{q_t^*})}\right) dt + \\ + &\int d\delta_{\{\sigma[0,T],\omega\}}\, \left[\mathbb{I}_{\{\tau\le T\}} \left\{-\beta(\omega + m_{q^*_{\tau-}})\right\}\right] - \int dQ_*\, \beta \int_0^{T\wedge\tau} dN_t^\sigma. \end{aligned}$$

where in writing the last integral we have used the fact that $\mathbb{I}_{\{\tau\le T\}} = \int_0^{T\wedge\tau} dN_t^\sigma$, where $(N_t^\sigma)_t$ is defined by

$$N_t^\sigma := \mathbb{I}_{\{\tau\ge t\}}. \tag{A.20}$$

Notice that $(N_t^\sigma)_t$ is a Poisson process with intensity $\mathbb{I}_{\{\tau\ge t\}} e^{-\beta(\omega+m_{q_t^*})}$. Thus its *compensated* Q_*−martingale is exactly M_t^σ defined in (3.58). Hence

$$\int_0^{T\wedge\tau} dN_t^\sigma - \int_0^{T\wedge\tau} e^{-\beta(\omega+m_{q_t^*})} dt = \int_0^{T\wedge\tau} dM_t^\sigma.$$

Moreover, being $\int_0^{T\wedge\tau} dM_t^\sigma$ a Q_*− martingale we have $\int dQ_* \int_0^{T\wedge\tau} dM_t^\sigma = 0$. Hence

$$\begin{aligned} &DF(Q_*)[\delta_{\{\sigma[0,T],\omega\}}] = \\ = &\int d\delta_{\{\sigma[0,T],\omega\}} \left\{ \int_0^{T\wedge\tau} \left(1-e^{-\beta(\omega+m_{q_t^*})}\right) dt + \left[\mathbb{I}_{\{\tau\le T\}} \left\{-\beta(\omega + m_{q^*_{\tau-}})\right\}\right] \right\} = \\ = &\int_0^{T\wedge\tau} \left(1-e^{-\beta(\omega+m_{q_t^*})}\right) dt + \left[\mathbb{I}_{\{\tau\le T\}} \left\{-\beta(\omega + m_{q^*_{\tau-}})\right\}\right]. \end{aligned}$$

By virtue of Girsanov's Formula for Markov chains it can be seen that

$$\int_0^{T\wedge\tau} \left(1-e^{-\beta(\omega+m_{q_t^*})}\right) dt + \left[\mathbb{I}_{\{\tau\le T\}} \left\{-\beta(\omega + m_{q^*_{\tau-}})\right\}\right] = \log \frac{dP^{Q_*}}{d(W\otimes\eta)},$$

where P^Q is the law of the Markov process with generator given in (3.41).
(A.19) thus follows since $P^{Q_*} = Q_*$ as shown in the proof of Proposition 3.3.5.

Back to the statement of the lemma, we see that for h measurable and bounded

$$\int h(y)p(dy) = \int h(y) \frac{e^{D\Psi(y_*)[y]}}{z} w(dy) =$$

$$= \int h(T(\delta_{\{\sigma[0,T],\omega\}}))\frac{e^{D\Psi(y_*)[T(\delta_{\{\sigma[0,T],\omega\}})]}}{z}\,(W\otimes\eta)(d\sigma[0,T],d\omega) =$$

$$= \int h(T(\delta_{\{\sigma[0,T],\omega\}}))\frac{e^{DF(Q_*)[\delta_{\{\sigma[0,T],\omega\}}]}}{z}\,(W\otimes\eta)(d\sigma[0,T],d\omega) = \int h(T(\delta_{\{\sigma[0,T],\omega\}}))dQ_*$$

where we have used (A.19) in the last equality and Lemma A.2.4 in the next to last. Notice moreover that, thanks to (A.19), $z \equiv 1$.
The fact that $D\Psi(y_*)[T(\delta_{\{\sigma[0,T],\omega\}})] = DF(Q_*)[\delta_{\{\sigma[0,T],\omega\}}]$ follows since $Q_* \ll W\otimes\eta$ and $F_2(W\otimes\eta) = 0$, being the distribution of τ under $W\otimes\eta$ of exponential type. We can thus apply Lemma A.2.4. ∎

Step 2:

In this second step we prove positivity of the functional $\mathcal{H}$ defined in (A.26) below. Its positivity will imply (A.10), as shown in *Step 3*. We start proving a technical lemma.

Lemma A.2.6 *The following properties hold true*

i) $\hat{\Phi}$*, defined by (3.54), is absolutely continuous w.r.t.* Q_* *and in particular*

$$\frac{d\hat{\Phi}}{dQ_*} = \Phi - \Phi^*. \tag{A.21}$$

ii) The second order Fréchet derivative of the function $F(Q)$ *computed in* Q_* *can be written as*

$$D^2F(Q_*)[\hat{\Phi}_i,\hat{\Phi}_j] = E^{Q_*}\Bigg[\int_0^{T\wedge\tau} -\beta^2 m_{\hat{\Phi}_i(t)}m_{\hat{\Phi}_j(t)}e^{-\beta(\omega+m_{q_t})}dt + \tag{A.22}$$

$$+(\Phi_j-\Phi_j^*)\int_0^{T\wedge\tau}\beta m_{\hat{\Phi}_i(t)}e^{-\beta(\omega+m_{q_t})}dt + (\Phi_i-\Phi_i^*)\int_0^{T\wedge\tau}\beta m_{\hat{\Phi}_j(t)}e^{-\beta(\omega+m_{q_t})}dt\Bigg] +$$

$$-\beta E^{\hat{\Phi}_i}\left[\mathbb{I}_{\{\tau\le T\}}m_{\hat{\Phi}_j(\tau^-)}\right] - \beta E^{\hat{\Phi}_j}\left[\mathbb{I}_{\{\tau\le T\}}m_{\hat{\Phi}_i(\tau^-)}\right].$$

Proof.
Point (i). We observe that, given $\hat{\Phi}$ as in Definition 3.5.3, we have

$$\hat{\Phi}(S) = \int_{\mathcal{M}_0(E)}\left[R(S)\int\Phi dR\right]\nu_*(dR) =$$

$$= \int_E(\mathbb{I}_{\{(\sigma[0,T],\omega)\in S\}} - Q_*(S))\cdot\left(\int\Phi\; d\delta_{\{\sigma[0,T],\omega\}} - \int\Phi dQ_*\right)dQ_* =$$

$$= \int_E\left[(\mathbb{I}_{\{(\sigma[0,T],\omega)\in S\}} - Q_*(S))\cdot(\Phi(\sigma[0,T],\omega)-\Phi^*)\right]dQ_*$$

for any $S \subset E := \tilde{\mathcal{D}}[0,T]\times\mathbb{R}$. The second equality follows since ν_* is the law of the random variable $\delta_{\{\sigma[0,T],\omega\}} - Q_*$ induced by Q_*.
Notice that $Q_*(S)\int_E(\Phi-\Phi^*)dQ_* = 0$, being Φ^* the expectation under Q_* of $\Phi(\cdot)$. Hence

$$\hat{\Phi}(S) = \int_S(\Phi-\Phi^*)dQ_*$$

and point (i) follows.

Point (ii). We have to compute the second order derivatives of the function F. We first compute the first order derivative as given in (A.16), substituting $r = \hat{\Phi}_i$.

$$DF(Q)[\hat{\Phi}_i] = \lim_{h\to 0} \frac{F(Q + h\hat{\Phi}_i) - F(Q)}{h} = \tag{A.23}$$

$$= \int dQ \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_i}\, e^{-A-B_t^Q} dt + \int d\hat{\Phi}_i \int_0^{T\wedge\tau} (1 - e^{-A-B_t^Q}) dt +$$

$$+ \int d\hat{\Phi}_i \left[-\mathbb{I}_{\{\tau\le T\}}(A + B_{\tau^-}^Q)\right] + \int dQ \left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_i}\right].$$

We now compute the second order derivatives, where

$$D^2F(Q)[\hat{\Phi}_i, \hat{\Phi}_j] = \lim_{h\to 0} \frac{1}{h}\left(DF(Q + h\hat{\Phi}_j)[\hat{\Phi}_i] - DF(Q)[\hat{\Phi}_i]\right).$$

To this aim, we see that

$$DF(Q + h\hat{\Phi}_j)[\hat{\Phi}_i] =$$

$$= \int d(Q + h\hat{\Phi}_j) \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_i} e^{-A-B_t^Q - hB_t^{\hat{\Phi}_j}} dt + \int d\hat{\Phi}_i \int_0^{T\wedge\tau} (1 - e^{-A-B_t^Q - hB_t^{\hat{\Phi}_j}}) dt +$$

$$+ \int d\hat{\Phi}_i \left[-\mathbb{I}_{\{\tau\le T\}}(A + B_{\tau^-}^Q) + hB_{\tau^-}^{\hat{\Phi}_j}\right] + \int d(Q + h\hat{\Phi}_j) \left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_i}\right].$$

hence

$$DF(Q + h\hat{\Phi}_j)[\hat{\Phi}_i] - DF(Q)[\hat{\Phi}_i] =$$

$$= \int dQ \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_i}[(e^{-A-B_t^Q - hB_t^{\hat{\Phi}_j}}) - (e^{-A-B_t^Q})] dt +$$

$$+ h\int d\hat{\Phi}_j \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_i}(e^{-A-B_t^Q - hB_t^{\hat{\Phi}_j}}) dt +$$

$$+ \int d\hat{\Phi}_i \int_0^{T\wedge\tau} [(1 - e^{-A-B_t^Q - hB_t^{\hat{\Phi}_j}}) - (1 - e^{-A-B_t^Q})] dt +$$

$$+ h\int d\hat{\Phi}_i \left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_j}\right] + h\int d\hat{\Phi}_j \left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_i}\right].$$

Dividing by h, letting h go to zero and relying on a boundedness argument, similar to (A.15), we see that

$$D^2F(Q)[\hat{\Phi}_i, \hat{\Phi}_j] = \int dQ \int_0^{T\wedge\tau} -B_t^{\hat{\Phi}_i} B_t^{\hat{\Phi}_j} e^{-A-B_t^Q} dt +$$

$$+ \int d\hat{\Phi}_j \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_i} e^{-A-B_t^Q} dt + \int d\hat{\Phi}_i \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_j} e^{-A-B_t^Q} dt +$$

$$+ \int d\hat{\Phi}_i \left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_j}\right] + \int d\hat{\Phi}_j \left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_i}\right].$$

Taking $Q = Q_*$ and using point (i), we can write

$$D^2F(Q_*)[\hat{\Phi}_i, \hat{\Phi}_j] = E^{Q_*}\left[\int_0^{T\wedge\tau} -B_t^{\hat{\Phi}_i} B_t^{\hat{\Phi}_j} e^{-A-B_t^{Q_*}}\, dt + \right. \tag{A.24}$$
$$+ \quad \left.(\Phi_j - \Phi_j^*) \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_i} e^{-A-B_t^{Q_*}}\, dt + (\Phi_i - \Phi_i^*) \int_0^{T\wedge\tau} B_t^{\hat{\Phi}_j} e^{-A-B_t^{Q_*}}\, dt\right] +$$
$$+ \quad E^{\hat{\Phi}_i}\left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_j}\right] + E^{\hat{\Phi}_j}\left[-\mathbb{I}_{\{\tau\le T\}} B_{\tau^-}^{\hat{\Phi}_i}\right].$$

Using the definitions of A and B (see (A.13)), we have

$$D^2F(Q_*)[\hat{\Phi}_i, \hat{\Phi}_j] = E^{Q_*}\left[\int_0^{T\wedge\tau} -\beta^2 m_{\hat{\Phi}_i(t)} m_{\hat{\Phi}_j(t)} e^{-\beta(\omega+m_{q_t})} dt + \right. \tag{A.25}$$
$$+ (\Phi_j - \Phi_j^*) \int_0^{T\wedge\tau} \beta m_{\hat{\Phi}_i(t)} e^{-\beta(\omega+m_{q_t})} dt + \left.(\Phi_i - \Phi_i^*) \int_0^{T\wedge\tau} \beta m_{\hat{\Phi}_j(t)} e^{-\beta(\omega+m_{q_t})} dt\right] +$$
$$-\beta E^{\hat{\Phi}_i}\left[\mathbb{I}_{\{\tau\le T\}} m_{\hat{\Phi}_j(\tau^-)}\right] - \beta E^{\hat{\Phi}_j}\left[\mathbb{I}_{\{\tau\le T\}} m_{\hat{\Phi}_i(\tau^-)}\right].$$

So the conclusion follows. ∎

Proposition A.2.7 *Given Φ_1 and Φ_2 in $\mathcal{C}_b$, let*

$$\mathcal{H}(\Phi_1, \Phi_2) := Cov_{Q_*}(\Phi_1, \Phi_2) - D^2F(Q_*)[\hat{\Phi}_1, \hat{\Phi}_2]; \tag{A.26}$$

where $Cov_{Q_*}(\Phi_1, \Phi_2) := \int (\Phi_1 - \Phi_1^*)(\Phi_2 - \Phi_2^*) dQ_*$. Then

$$\mathcal{H}(\Phi, \Phi) > 0, \text{ for all } \Phi \text{ such that } \hat{\Phi} \neq 0.$$

Proof. We first show that $\mathcal{H}(\Phi, \Phi)$ is the expected value of a square. Indeed

$$\mathcal{H}(\Phi, \Phi) = Cov_{Q_*}(\Phi, \Phi) - D^2F(Q_*)[\hat{\Phi}, \hat{\Phi}] = \tag{A.27}$$
$$= \quad E^{Q_*}[(\Phi - \Phi^*)^2] + E^{Q_*}\left[\int_0^{T\wedge\tau} \beta^2 m^2_{\hat{\Phi}(t)} e^{-\beta(\omega+m_{q_t})} dt - \right.$$
$$- \quad \left. 2(\Phi - \Phi^*) \int_0^{T\wedge\tau} \beta m_{\hat{\Phi}(t)} e^{-\beta(\omega+m_{q_t})} dt\right] + 2\beta E^{\hat{\Phi}}\left[\mathbb{I}_{\{\tau\le T\}} m_{\hat{\Phi}(\tau^-)}\right].$$

We rewrite the last term using the definition of (N_t^σ) given in (A.20).
As a consequence we see that $\beta \mathbb{I}_{\{\tau\le T\}} m_{\hat{\Phi}(\tau^-)} = \beta \int_0^{T\wedge\tau} m_{\hat{\Phi}(t^-)} dN_t^\sigma$. Thus

$$2\beta E^{\hat{\Phi}}\left[\mathbb{I}_{\{\tau\le T\}} m_{\hat{\Phi}(\tau^-)}\right] = 2E^{Q_*}\left[(\Phi - \Phi^*) \int_0^{T\wedge\tau} \beta m_{\hat{\Phi}(t^-)} dN_t^\sigma\right].$$

We rewrite (A.27) as follows

$$\mathcal{H}(\Phi, \Phi) = E^{Q_*}[(\Phi - \Phi^*)^2] + E^{Q_*}\left[\int_0^{T\wedge\tau} \beta^2 m^2_{\hat{\Phi}(t)} e^{-\beta(\omega+m_{q_t})} dt\right] +$$
$$+ \quad E^{Q_*}\left[2(\Phi - \Phi^*)\left\{-\int_0^{T\wedge\tau} \beta m_{\hat{\Phi}(t)} e^{-\beta(\omega+m_{q_t})} dt + \int_0^{T\wedge\tau} \beta m_{\hat{\Phi}(t^-)} dN_t^\sigma\right\}\right].$$

We now use the Q_*−martingale $(M_t^s)_t$ defined in (3.58), in order to rewrite the latter term in brackets as a martingale.

$$\mathcal{H}(\Phi,\Phi) = E^{Q_*}[(\Phi-\Phi^*)^2] + E^{Q_*}\left[\int_0^{T\wedge\tau} \beta^2 m^2_{\hat\Phi(t)} e^{-\beta(\omega+m_{q_t^*})}dt\right] +$$
$$+ \quad E^{Q_*}\left[2(\Phi-\Phi^*)\left\{\int_0^{T\wedge\tau} \beta m_{\hat\Phi(t)} dM_t^\sigma\right\}\right].$$

Moreover by the isometry property of square integrable martingales (notice that the argument of the integral: $(\beta m_{\hat\Phi(t)})$ is bounded) we have

$$E^{Q_*}\left[\int_0^{T\wedge\tau} \beta^2 m^2_{\hat\Phi(t)} e^{-\beta(\omega+m_{q_t^*})}dt\right] = E^{Q_*}\left[\left(\int_0^{T\wedge\tau} \beta m_{\hat\Phi(t)} dM_t^\sigma\right)^2\right].$$

Hence

$$\begin{aligned}
&\mathcal{H}(\Phi,\Phi) = \\
&= E^{Q_*}\left[(\Phi-\Phi^*)^2 + \left(\int_0^{T\wedge\tau} \beta m_{\hat\Phi(t)} dM_t^\sigma\right)^2 + 2(\Phi-\Phi^*)\left(\int_0^{T\wedge\tau} \beta m_{\hat\Phi(t)} dM_t^\sigma\right)\right] = \\
&= E^{Q_*}\left[\left((\Phi-\Phi^*) + \int_0^{T\wedge\tau} \beta m_{\hat\Phi(t)} dM_t^\sigma\right)^2\right]. \qquad \text{(A.28)}
\end{aligned}$$

$\mathcal{H}(\Phi,\Phi)$ is thus the expected value of a square, hence it cannot be negative. For this reason, we simply need to prove that it is non-zero. Without loss of generality we take $\Phi^* = 0$.
Suppose by way of contradiction that $\mathcal{H}(\Phi,\Phi) = 0$. Then necessarily

$$\left(\Phi(\sigma_{[0,T]},\omega) + \int_0^{T\wedge\tau} \beta m_{\hat\Phi(s)} dM_s^\sigma\right) = 0, \quad Q_* \ a.s.$$

Using the fact that

$$m_{\hat\Phi(s)} = \int \sigma(s)\ \hat\Phi(d\sigma,d\omega) = \int \sigma(s)\ \Phi(\sigma[0,T],\omega)\ Q_*(d\sigma,d\omega),$$

where the last equality follows since $\frac{d\hat\Phi}{dQ_*} = \Phi$. We rewrite the expression above as

$$\left(\Phi(\sigma_{[0,T]},\omega) + \int_0^{T\wedge\tau} \beta\left[\int \sigma(s)\Phi(\sigma_{[0,T]},\omega)dQ_*\right]dM_s^\sigma\right) = 0, \quad Q_* - a.s.$$

hence

$$\Phi(\sigma_{[0,T]},\omega) = -\int_0^{T\wedge\tau} \beta\left[\int \sigma(s)\Phi(\sigma_{[0,T]},\omega)dQ_*\right]dM_s^\sigma\ , \quad Q_* - a.s. \qquad \text{(A.29)}$$

On the other hand, define $\Phi_t = E^{Q_*}[\Phi|\mathcal{F}_t]$, where

$$\mathcal{F}_t = \sigma\{\sigma_s : 0 \le s \le t;\ \omega\}.$$

Notice that

$$\int \sigma(t)\Phi(\cdot)dQ_* = E^{Q_*}[\sigma(t)\Phi(\cdot)] = E^{Q_*}[\sigma(t)E^{Q_*}[\Phi(\cdot)|\mathcal{F}_t]] = \int \sigma(t)\Phi_t dQ_*.$$

Taking the conditional expectation in (A.29), we obtain

$$\begin{aligned} \Phi_t &= E^{Q_*}\left[-\int_0^{T\wedge\tau} \beta\left(\int \sigma(s)\Phi_s dQ_*\right) dM_s^\sigma \middle| \mathcal{F}_t\right], \quad Q_* \ a.s. \\ &= -\int_0^{t\wedge\tau} \beta\left(\int \sigma(s)\Phi_s dQ_*\right) dM_s^\sigma, \quad Q_* \ a.s. \end{aligned}$$

We now take the L^2-norm in both sides. For all $t \in [0,T]$ we have

$$\begin{aligned} ||\Phi_t||^2_{L^2(Q_*)} &= \left\| \int_0^{t\wedge\tau} \beta\left(\int \sigma(s)\Phi_s dQ_*\right) dM_s^\sigma \right\|^2_{L^2(Q_*)} = \\ &= E^{Q_*}\left[\int_0^{t\wedge\tau} \beta^2 \left(\int \sigma(s)\Phi_s dQ_*\right)^2 e^{-\beta(\omega+m_{q_s^*})} ds\right]. \end{aligned}$$

Notice that $\left(\int \sigma(s)\Phi_s dQ_*\right)^2 \le \left(\int \Phi_s dQ_*\right)^2 \le \int \Phi_s^2 dQ_* \le \int \Phi_t^2 dQ_* = \|\Phi_t\|^2_{L^2(Q_*)}$, where $t \ge s$. The first inequality follows since $\sigma \in \{-1;1\}$; the second one is trivial and the latter one is due to the fact that $(\Phi_s^2)_s$ is a submartingale and thus its expected value is an increasing function of time. Then

$$\begin{aligned} ||\Phi_t||^2_{L^2(Q_*)} &\le E^{Q_*}\left[\int_0^{t\wedge\tau} \beta^2 \|\Phi_t\|^2_{L^2(Q_*)} e^{-\beta(\omega+m_{q_s^*})} ds\right] \le \\ &\le (t\wedge\tau)\,\varepsilon^{-1}\,||\Phi_t||^2_{L^2(Q_*)} \le t\,\varepsilon^{-1}\,||\Phi_t||^2_{L^2(Q_*)} \end{aligned}$$

where $0 < \varepsilon < \infty$ is a constant such that $\beta^2 e^{-\beta(\omega+m_{q_s^*})} \le \varepsilon^{-1}$. As a consequence, $\Phi_s = 0$, Q_* $a.s.$ for $s \in [0,\varepsilon)$.
This argument can be iterated defining $\Phi_t^{(2)} := \Phi_{t+\varepsilon}$. The same argument shows that $\Phi_s^{(2)} = 0$, Q_* $a.s.$ for $s \in [0,\varepsilon)$; hence $\Phi_t = 0$, Q_* $a.s.$ for $s \in [0,2\varepsilon)$. Eventually we extend the statement to $s \in [0,T]$. Being $\Phi_T = \Phi$, we would have $\hat{\Phi} = 0$ and this gives a contradiction. Hence the thesis follows. ∎

Step 3:

Consider $\lambda_1, \lambda_2 \in B'$. Since $\lambda_i \circ T$, for $i = 1,2$, are in the topological dual of $\mathcal{M}$, there exist $\Phi_1, \Phi_2 \in \mathcal{C}_b$ such that $\lambda_i \circ T(Q) = \int \Phi_i dQ$. We define

$$Cov_{p_*}(\lambda_1,\lambda_2) = \int \lambda_1(y)\lambda_2(y)p_*(dy) \quad \text{and} \quad \tilde{\lambda}_i = \int y\lambda_i(y)p_*(dy); \ i = 1,2$$

where we recall that p_*, defined in $(B.4)$ of Theorem 3.5.1, is the centered version of the law of $T(\delta_{\{\sigma[0,T],\omega\}})$ induced by Q_*. Then the following result holds true

Lemma A.2.8

i)

$$Cov_{p_*}(\lambda_1, \lambda_2) = Cov_{Q_*}(\Phi_1, \Phi_2);$$
$$D^2\Psi(y_*)[\tilde{\lambda}_1, \tilde{\lambda}_2] = D^2F(Q_*)[\hat{\Phi}_1, \hat{\Phi}_2].$$

ii) For $\lambda_i,\ i = 1, 2$ we have

$$Cov_{p_*}(\lambda_i, \lambda_i) - D^2\Psi(y_*)[\tilde{\lambda}_i, \tilde{\lambda}_i] > 0.$$

Proof. Point (i). By the definition of p_* and $\lambda_i,\ i = 1, 2$ we see that

$$Cov_{p_*}(\lambda_1, \lambda_2) = \int [S_1(\delta_{\{\sigma[0,T],\omega\}}) - S_1(Q_*)][S_2(\delta_{\{\sigma[0,T],\omega\}}) - S_2(Q_*)]dQ_* =$$

$$= \int [\Phi_1 - \Phi_1^*][\Phi_2 - \Phi_2^*]dQ_*,$$

where we have used the fact that $\lambda_i \circ T(Q) = S_i(Q) = \int \Phi_i dQ$.
Concerning the second statement, we first prove the following claim.

$$\tilde{\lambda}_i = T(\hat{\Phi}_i)\ ; \quad i = 1, 2. \tag{A.30}$$

To show the validity of (A.30), we use the following two facts:

$$\begin{aligned} \tilde{\lambda}_i &= E^{Q_*}\left\{[T(\delta_{\{\sigma[0,T],\omega\}}) - T(Q_*)][\Phi_i(\sigma[0,T], \omega) - \Phi_i^*]\right\}; \\ \hat{\Phi}_i &= E^{Q_*}\left\{[\delta_{\{\sigma[0,T],\omega\}} - Q_*][\Phi_i(\sigma[0,T], \omega) - \Phi_i^*]\right\}. \end{aligned}$$

The former follows by definition of p_*, λ and $S_i(Q)$, whereas the latter is a consequence of the definition of $\hat{\Phi}$ given in (3.54).
(A.30) is a consequence of the fact that T is both linear and continuous, hence we are allowed to interchange the T operator with the expectation.
Having proved (A.30), we compute the second order Fréchet derivatives on the function Ψ as follows.

$$D^2\Psi(y_*)[\tilde{\lambda}_1, \tilde{\lambda}_2] = \lim_{k\to 0} \frac{D\Psi(y_* + k\tilde{\lambda}_2)[\tilde{\lambda}_1] - D\Psi(y_*)[\tilde{\lambda}_1]}{k}. \tag{A.31}$$

Notice that, by the linearity of T and by (A.30), we have that

$$y_* + k\tilde{\lambda}_2 = T(Q_* + k\hat{\Phi}_2)\ , \qquad y_* = T(Q_*).$$

Thus

$$\lim_{k\to 0} \frac{D\Psi(y_* + k\tilde{\lambda}_2)[\tilde{\lambda}_1] - D\Psi(y_*)[\tilde{\lambda}_1]}{k} = \lim_{k\to 0} \frac{D\Psi(T(Q_* + k\hat{\Phi}_2))[\tilde{\lambda}_1] - D\Psi(T(Q_*))[\tilde{\lambda}_1]}{k}.$$

We now claim that

$$\lim_{k\to 0} \frac{D\Psi(T(Q_* + k\hat{\Phi}_2))[\tilde{\lambda}_1] - D\Psi(T(Q_*))[\tilde{\lambda}_1]}{k} = \lim_{k\to 0} \frac{DF(Q_* + k\hat{\Phi}_2)[\hat{\Phi}_1] - DF(Q_*)[\hat{\Phi}_1]}{k}. \tag{A.32}$$

To show this equality we rely on Lemma A.2.4. Recall that this lemma guarantees that $D\Psi(T(Q))[T(r)] = DF(Q)[r]$, if $F_2(Q) = 0$ where $F_2(P) := \sum_{t\in[0,T]}(\Delta P(\tau \le t))^2$. We are in this situation: in fact $F_2(Q_*) = 0$ since $Q_* \ll (W \otimes \eta)$ and $\hat{\Phi}_i$ is absolutely continuous with respect to Q_*. Hence both $F_2(\hat{\Phi}_i)$ and $F_2(Q_* + k\hat{\Phi}_i) = 0$. This proves (A.32). Finally we use the fact that F is Fréchet differentiable

$$\lim_{k\to 0} \frac{DF(Q_* + k\hat{\Phi}_2)[\hat{\Phi}_1] - DF(Q_*)[\hat{\Phi}_1]}{k} = D^2F(Q_*)[\hat{\Phi}_1, \hat{\Phi}_2].$$

We have thus proved that $D^2\Psi(y_*)[\tilde{\lambda}_1, \tilde{\lambda}_2] = D^2F(Q_*)[\hat{\Phi}_1, \hat{\Phi}_2]$.

Concerning point (ii), we notice that

$$Cov_{p_*}(\lambda_i, \lambda_i) - D^2\Psi(y_*)[\tilde{\lambda}_i, \tilde{\lambda}_i] = Cov_{Q_*}(\Phi_i, \Phi_i) - D^2F(Q_*)[\hat{\Phi}_i, \hat{\Phi}_i] = \mathcal{H}(\Phi_i, \Phi_i),$$

where $\mathcal{H}$ has been defined in Equation (A.26). Hence by Proposition A.2.7 the positivity condition is ensured and the thesis follows. ∎

By virtue of Lemma A.2.8, for any $\lambda \in B'$ such that $\tilde{\lambda} \neq 0$, (A.10) holds true. As a consequence, assumption $(B.4)$ is ensured.

(B.5):
It follows by Theorem 3.5.4.

Having ensured the validity of assumptions $(B.1), ..., (B.5)$, we are allowed to apply Theorem 3.5.1, so Theorem 3.5.5 is proved. Notice that in our setting we have $\Omega = (\tilde{\mathcal{D}}[0,T] \times \mathbb{R})^N$ and $\mathbb{P} = (W \otimes \eta)^{\otimes N}$, then (3.51) can be written as

$$\frac{d\pi_N}{d(W \otimes \eta)^{\otimes N}} = \frac{e^{N\left(\Psi(X_N) + \frac{\Sigma(X_N)}{N}\right)}}{E^{(W\otimes\eta)^{\otimes N}}\left[e^{N\left(\Psi(X_N) + \frac{\Sigma(X_N)}{N}\right)}\right]}. \tag{A.33}$$

A.3 Proof of Theorem 3.5.6

Recalling that $y_* = T(Q_*)$ and $\lambda_i \circ T(Q_*) = \int \Phi_i dQ_*$, by virtue of Theorem 3.5.4 point (ii), we have

$$\sqrt{N}\left(\lambda_i(T(\rho_N)) - \lambda_i(y_*)\right)_{i=1}^n = \sqrt{N}\left(\int \Phi_i d\rho_N - \int \Phi_i dQ_*\right)_{i=1}^n;$$

Concerning the covariance matrix, $(C)_{ij} \equiv \mathcal{H}(\Phi_i, \Phi_j)$ for all $i, j = 1, ..., n$, as shown in (A.26). Moreover it follows immediately by Lemma A.2.8, that

$$\mathcal{H}(\Phi_i, \Phi_j) = Cov_{p_*}(\lambda_i, \lambda_j) - D^2\Psi(y_*)[\tilde{\lambda}_i, \tilde{\lambda}_j].$$

The r.h.s. in the above expression is indeed equal to $(\mathcal{C})_{ij}$ defined in (3.52).
The fact that

$$(C)_{ij} = E^{Q_*}\left[\left(\Phi_i - \Phi_i^* + \beta\int_0^{T\wedge\tau} m_{\hat{\Phi}_i(s)} dM_s^\sigma\right)\left(\Phi_j - \Phi_j^* + \beta\int_0^{T\wedge\tau} m_{\hat{\Phi}_j(s)} dM_s^\sigma\right)\right],$$

follows easily from (A.28), where in this case we need to compute $\mathcal{H}(\Phi_i, \Phi_j) \equiv (C)_{ij}$ also on the terms outside the diagonal. We omit this straightforward computation.

It remains to show that π_N defined in Theorem 3.5.1 is nothing but P_N, mentioned in the statement of Theorem 3.5.6, where P_N is the law of $(\underline{\sigma}[0,T], \omega)$ induced by (3.24); for more details see Section 3.3.
We first notice that, by the linearity of T, X_N as defined in Theorem 3.5.1, can be written as

$$X_N = \frac{1}{N}\sum_{i=1}^N Y_i = \frac{1}{N}\sum_{i=1}^N T(\delta_{\{\sigma_i[0,T],\omega\}}) = T\left(\frac{1}{N}\sum_{i=1}^N \delta_{\{\sigma_i[0,T],\omega\}}\right) = T(\rho_N).$$

Then

$$\frac{d\pi_N}{d(W\otimes\eta)^{\otimes N}} = \frac{e^{N\left(\Psi(X_N)+\frac{\Sigma(X_N)}{N}\right)}}{E^{(W\otimes\eta)^{\otimes N}}\left[e^{N\left(\Psi(X_N)+\frac{\Sigma(X_N)}{N}\right)}\right]} = \frac{e^{N\left(\Psi(T(\rho_N))+\frac{\Sigma(T(\rho_N))}{N}\right)}}{E^{(W\otimes\eta)^{\otimes N}}\left[e^{N\left(\Psi(T(\rho_N))+\frac{\Sigma(T(\rho_N))}{N}\right)}\right]}.$$

On the other hand

$$\frac{dP_N}{d(W\otimes\eta)^{\otimes N}} = \frac{d(P_N^\omega\otimes\eta^{\otimes N})}{d(W^{\otimes N}\otimes\eta^{\otimes N})} = \frac{dP_N^\omega}{dW^{\otimes N}} = \frac{e^{NF(\rho_N)}}{E^{W^{\otimes N}}\left[e^{NF(\rho_N)}\right]};$$

where the last equality follows from Lemma 3.3.1. Notice moreover that as a consequence of (3.30), $E^{W^{\otimes N}}\left[e^{NF(\rho_N)}\right] = 1$.
The thesis thus follows from the fact that almost surely, $F(\rho_N) = \Psi(\rho_N) + \frac{\Sigma(\rho_N)}{N}$, as shown in (A.6). ∎

A.4 Proof of Proposition 4.3.4

In what follows we denote by P_N the law on the path space of $(\underline{\sigma}[0,T], \underline{\omega}[0,T]) \in (\mathcal{D}[0,T])^{2N}$ under the interacting dynamics, with initial conditions satisfying the assumptions in Theorem 4.3.2. As in Section 4.3.1 we let $W \in \mathcal{M}_1(\mathcal{D}([0,T])\times\mathcal{D}([0,T]))$ denote the law of the $\{-1;1\}^2$-valued process $\sigma(t), \omega(t)$ such that $(\sigma(0), \omega(0))$ has distribution λ, and both $\sigma(\cdot)$ and $\omega(\cdot)$ change sign with constant rate 1. By $W^{\otimes N}$ we mean the product of N copies of W. We begin with some preliminary lemmas.

Lemma A.4.1

$$\frac{dP_N}{dW^{\otimes N}}\,(\underline{\sigma}[0,T],\, \underline{\omega}[0,T]) = \exp\left[NF\left(\rho_N(\underline{\sigma}[0,T],\, \underline{\omega}[0,T])\right)\right], \tag{A.34}$$

where F is the function defined in (4.11).

Proof. This lemma is the analogous of Lemma 3.3.1 of Chapter 3. Nevertheless it requires a separate proof. We rely on Girsanov's Formula for Markov chains (see [7]). Let $\left(N_t^\sigma(i)\right)_{i=1}^N$ be the multivariate Poisson process which counts the jumps of

σ_i for $i = 1, \dots, N$, and $\left(N_t^{\omega}(i)\right)_{i=1}^N$ be the multivariate Poisson process which counts the jumps of ω_i for $i = 1, \dots, N$. Girsanov's Formula yields

$$\begin{aligned}
&\frac{dP_N}{dW^{\otimes N}} \left(\underline{\sigma}[0,T],\, \underline{\omega}[0,T]\right) \\
&\quad = \exp\left[\sum_{i=1}^N \int_0^T \left(1 - e^{-\beta\,\sigma_i(t)\,\omega_i(t)}\right) dt + \sum_{i=1}^N \int_0^T \log e^{-\beta\,\sigma_i(t^-)\,\omega_i(t^-)}\, dN_t^{\sigma}(i)\right. \\
&\qquad \left. + \sum_{i=1}^N \int_0^T \left(1 - e^{-\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}}\right) dt + \sum_{i=1}^N \int_0^T \log e^{-\gamma\,\omega_i(t^-)\,m^{\sigma}_{\rho_N(t^-)}}\, dN_t^{\omega}(i)\right]
\end{aligned}$$

where $\sigma_i\left(t^-\right) = \lim_{\,s\to t^-} \sigma_i\left(s\right)$, analogously for $\omega_i(t^-)$. Since, with probability 1, there are no simultaneous jumps, we have

$$\sum_{i=1}^N \int_0^T \log e^{-\beta\,\sigma_i(t^-)\,\omega_i(t^-)}\, dN_t^{\sigma}(i) = \sum_{i=1}^N \int_0^T -\beta\left(-\sigma_i(t)\right)\omega_i(t)\, dN_t^{\sigma}(i)$$

and

$$\sum_{i=1}^N \int_0^T \log e^{-\gamma\,\omega_i(t^-)\,m^{\sigma}_{\rho_N(t^-)}}\, dN_t^{\omega}(i) = \sum_{i=1}^N \int_0^T -\gamma\left(-\omega_i(t)\right) m^{\sigma}_{\rho_N(t)}\, dN_t^{\omega}(i)$$

from which (A.34) follows easily after having observed that, $W^{\otimes N}$ almost surely,

$$\int \left(N_T^{\sigma} + N_T^{\omega}\right) d\rho_N < +\infty,$$

and that simultaneous jumps of σ and ω do not occur.

■

The main problem in the proof of Proposition 4.3.4 is related to the fact that the function F in (4.11) is neither continuous nor bounded. The following technical lemmas have the purpose of circumventing this problem. In what follows, we let

$$\mathcal{I} := \left\{Q \in \mathcal{M}_1\left(\mathcal{D}[0,T]^2\right) : \int \left(N_T^{\sigma} + N_T^{\omega}\right) dQ < +\infty\right\}. \tag{A.35}$$

We first define, for $r > 0$ and $Q \in \mathcal{I}$,

$$\begin{aligned}
F_r(Q) = \int \Bigg[&\int_0^T \left(r - e^{-\beta\sigma(t)\omega(t)}\right) dt + \int_0^T \left(r - e^{-\omega(t)\gamma_t^Q}\right) dt \\
&+ \int_0^T (\beta\sigma(t)\omega(t^-) - \log r) dN_t^{\sigma} + \int_0^T (\omega(t)\gamma_{t^-}^Q - \log r) dN_t^{\omega}\Bigg] dQ.
\end{aligned} \tag{A.36}$$

Note that $F = F_1$. Moreover, Lemma A.4.1 can be easily extended to show that

$$\frac{dP_N}{dW_r^{\otimes N}} \left(\underline{\sigma}[0,T],\, \underline{\omega}[0,T]\right) = \exp\left[NF_r\left(\rho_N(\underline{\sigma}[0,T],\, \underline{\omega}[0,T])\right)\right], \tag{A.37}$$

where W_r is the law of the $\{-1;1\}^2$-valued process $\sigma(t), \omega(t)$ such that $(\sigma(0), \omega(0))$ has distribution λ, and both $\sigma(\cdot)$ and $\omega(\cdot)$ change sign with constant rate r.

Lemma A.4.2 *For* $0 < r \leq \min(e^{-\beta}, e^{-\gamma})$*,* F_r *is lower semicontinuous on* $\mathcal{I}$*. For* $r \geq \max(e^{\beta}, e^{\gamma})$*,* F_r *is upper semicontinuous.*

Proof. By definition of weak topology the fact that the map

$$Q \mapsto \int \left[\int_0^T \left(r - e^{-\beta\sigma(t)\omega(t)} \right) dt + \int_0^T \left(r - e^{-\omega(t)\gamma_t^Q} \right) dt \right] dQ$$

is continuous is rather straightforward (since Q-expectations of bounded continuous functions in $\mathcal{D}[0,T]$ are continuous in Q). Thus we only have to deal with the term

$$\int \left[\int_0^T (\beta\sigma(t)\omega(t^-) - \log r) dN_t^\sigma \right] dQ + \int \left[\int_0^T (\omega(t)\gamma_{t^-}^Q - \log r) dN_t^\omega \right] dQ. \quad \text{(A.38)}$$

We show that for $0 < r \leq \min(e^{-\beta}, e^{-\gamma})$ the expression in (A.38) is lower semicontinuous in $Q \in \mathcal{I}$. This shows that F_r is lower semicontinuous. The case $r \geq \max(e^{\beta}, e^{\gamma})$ is treated similarly.

For $\varepsilon > 0$ consider the function $\varphi_\varepsilon : \mathcal{D}[0,T] \to \mathbb{R}$ defined by

$$\varphi_\varepsilon(\eta) := \begin{cases} \frac{1}{\varepsilon} & \text{if } \eta(t) \text{ jumps for some } t \in (0,\varepsilon] \\ 0 & \text{otherwise.} \end{cases}$$

Given $\eta \in \mathcal{D}[0,T]$ we define $\eta(s)$ for $s > T$ by letting $\eta(s) \equiv \eta(T)$. Then, letting θ_t denote the shift operator, we have that, for $t \in [0,T]$, $\theta_t\eta$ is the element of $\mathcal{D}[0,T]$ given by $\theta_t\eta(s) := \eta(t+s)$. Consider now two functions $f, g : \{-1;1\}^2 \to \mathbb{R}$, and define $f_\varepsilon, g_\varepsilon : \mathcal{D}[0,T]^2 \to \mathbb{R}$ by

$$f_\varepsilon(\sigma_{[0,T]}, \omega_{[0,T]}) := \inf\{f(\sigma(t), \omega(t)) : t \in (0,\varepsilon)\},$$

and similarly for g_ε. Then define

$$\Phi_\varepsilon(\sigma_{[0,T]}, \omega_{[0,T]}) := \int_0^T f_\varepsilon(\theta_t\sigma, \theta_t\omega)\varphi_\varepsilon(\theta_t\sigma) dt + \int_0^T g_\varepsilon(\theta_t\sigma, \theta_t\omega)\varphi_\varepsilon(\theta_t\omega) dt.$$

The key to the continuation of the proof below are the following two properties of Φ_ε. These properties are essentially straightforward, and their proofs are omitted.

- Φ_ε is continuous and bounded on $\{(\sigma_{[0,T]}, \omega_{[0,T]}) : N_T^\sigma + N_T^\omega < +\infty\}$.
- Suppose $f, g \geq 0$. Then, for $\delta_{(\sigma_{[0,T]}, \omega_{[0,T]})} \in \mathcal{I}$, $\Phi_\varepsilon(\sigma_{[0,T]}, \omega_{[0,T]})$ increases when $\varepsilon \downarrow 0$ to
$$\int_0^T f(\sigma_{t^-}, \omega_{t^-}) dN_t^\sigma + \int_0^T g(\sigma_{t^-}, \omega_{t^-}) dN_t^\omega.$$
Therefore by monotone convergence
$$\int \left[\int_0^T f(\sigma_{t^-}, \omega_{t^-}) dN_t^\sigma + \int_0^T g(\sigma_{t^-}, \omega_{t^-}) dN_t^\omega \right] dQ = \sup_{\varepsilon > 0} \int \Phi_\varepsilon(\sigma_{[0,T]}, \omega_{[0,T]}) dQ,$$
In particular, the map
$$Q \mapsto \int \left[\int_0^T f(\sigma_{t^-}, \omega_{t^-}) dN_t^\sigma + \int_0^T g(\sigma_{t^-}, \omega_{t^-}) dN_t^\omega \right] dQ$$
is lower semicontinuous on $\mathcal{I}$.

Now, for $r \leq \min(e^{-\beta}, e^{-\gamma})$, the function $f(\sigma,\omega) = -\beta\sigma\omega - \log r$ is nonnegative. As for the function g, that should be $-\omega(t)\gamma_t^Q - \log r$, we notice that it is not a function of (σ,ω), but rather a function of $(\sigma, \Pi_t Q)$, thus depending explicitly on t and Q. However, due to its boundedness and the fact that γ_t^Q is continuous in Q uniformly in t, σ, the argument above applies with minor modifications thus leading to the conclusion of the proof. ∎

Lemma A.4.3 *Let $Q \in \mathcal{M}_1\left(\mathcal{D}[0,T]^2\right)$ be such that $H(Q|W) < +\infty$. Then $Q \in \mathcal{I}$. The same result applies if W_r replaces W.*

Proof. By the entropy inequality (see (6.2.14) in [24])

$$\int N_T^\sigma dQ \leq \int e^{N_T^\sigma} dW + H(Q|W).$$

But N_T^σ has Poisson distribution under W, so $\int e^{N_T^\sigma} dW < +\infty$. By applying the same argument to N_T^ω, the proof is completed. This proof extends with no modifications to the case $r \neq 1$. ∎

Lemma A.4.4 *The function*

$$I(Q) := H(Q|W) - F(Q)$$

is lower semicontinuous on $\mathcal{M}_1\left(\mathcal{D}[0,T]^2\right)$.

Proof. It is well known (see [24], Lemma 6.2.13) that the entropy $H(Q|W)$ is lower semicontinuous in Q in all of $\mathcal{M}_1\left(\mathcal{D}[0,T]^2\right)$. Moreover, by Lemma A.4.3 and the definition of $F(Q)$ we have that $H(Q|W) = I(Q)$ if $H(Q|W) = +\infty$. Thus we are left to prove the following two statements:

i) $I(Q)$ is lower semicontinuous in $\mathcal{I}$.

ii) If $H(Q|W) = +\infty$ and $Q_n \to Q$ weakly, then $I(Q_n) \to +\infty$.

The following key identity, which holds for $r > 0$ is a simple consequence of the definition of relative entropy and of the Girsanov formula for Markov Chains.

$$H(Q|W_r) = H(Q|W) + \int \frac{dW}{dW_r} dQ = H(Q|W) + 2T(r-1) + \log r \int \left(N_T^\sigma + N_T^\omega\right) dQ. \tag{A.39}$$

In particular, by Lemma A.4.3, we have that $H(Q|W) < +\infty \iff H(Q|W_r) < +\infty$. A simple consequence of (A.39) is then the following:

$$I(Q) = H(Q|W_r) - F_r(Q), \tag{A.40}$$

where the difference in (A.40) is meant to be $+\infty$ whenever $H(Q|W_r) = +\infty$ (which is equivalent to $H(Q|W) = +\infty$).

We are now ready to prove (i) and (ii).
To prove (i) it is enough to choose $r \geq \max(e^\beta, e^\gamma)$ and use Lemma A.4.2. Moreover,

for the same choice of r, the stochastic integrals in (A.36) are non-positive, hence $F_r(Q) \leq 2Tr$. Therefore, if $H(Q|W) = +\infty$ and $Q_n \to Q$,

$$\liminf I(Q_n) \geq \liminf H(Q_n|W_r) - 2Tr = +\infty,$$

where last equality follows from lower semicontinuity of $H(\cdot|W_r)$ and $H(Q|W_r) = +\infty$. Thus (ii) is proved. ∎

Lemma A.4.5 *The function $I(Q)$ has compact level sets, i.e. for every $k > 0$ the set $\{Q : I(Q) \leq k\}$ is compact.*

Proof. Choosing, as above, $r \geq \max(e^{\beta}, e^{\gamma})$, we have that $F_r(Q) \leq 2Tr$ for every Q. Thus, by (A.40),

$$\{Q : I(Q) \leq k\} \subseteq \{Q : H(Q|W_r) \leq k + 2Tr\}.$$

Since (see [24], Lemma 6.2.13) the relative entropy has compact level sets, $\{Q : I(Q) \leq k\}$ is contained in a compact set. Moreover, by lower semicontinuity of I, $\{Q : I(Q) \leq k\}$ is closed, and this completes the proof. ∎

Lemma A.4.6 *There exists $\delta > 1$ such that*

$$\limsup_{N \to +\infty} \frac{1}{N} \log E\{\exp[\delta N F(\rho_N)]\} < +\infty$$

Proof. The proof consists of the following manipulations:

$$\begin{aligned}
&\frac{1}{N} \log E\{\exp[\delta N F(\rho_N)]\} \\
&= \frac{1}{N} \log E\Bigg\{\exp\Bigg[\sum_{i=1}^{N} \int_0^T (\delta - \delta e^{-\beta\sigma_i(t)\,\omega_i(t)})\,dt + \sum_{i=1}^{N} \int_0^T \delta \log e^{-\beta\sigma_i(t^-)\,\omega_i(t^-)} dN_t^{\underline{\sigma}}(i) \\
&\quad + \sum_{i=1}^{N} \int_0^T (\delta - \delta e^{-\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}})\,dt + \sum_{i=1}^{N} \int_0^T \delta \log e^{-\gamma\,\omega_i(t^-)\,m^{\sigma}_{\rho_N(t^-)}} dN_t^{\underline{\omega}}(i)\Bigg]\Bigg\} \\
&= \frac{1}{N} \log E\Bigg\{\exp\Bigg[\sum_{i=1}^{N} \int_0^T (1 - e^{-\delta\beta\sigma_i(t)\,\omega_i(t)})\,dt + \sum_{i=1}^{N} \int_0^T \delta\beta\sigma_i(t)\,\omega_i(t)\,dN_t^{\underline{\sigma}}(i) \\
&\quad + \sum_{i=1}^{N} \int_0^T (1 - e^{-\delta\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}})\,dt + \sum_{i=1}^{N} \int_0^T \delta\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}\,dN_t^{\underline{\omega}}(i)\Bigg] \cdot \\
&\quad \cdot \exp\Bigg[\sum_{i=1}^{N} \int_0^T (\delta - \delta e^{-\beta\sigma_i(t)\,\omega_i(t)} - (1 - e^{-\delta\beta\sigma(t)\,\omega(t)}))\,dt\Bigg] \cdot \\
&\quad \cdot \exp\Bigg[\sum_{i=1}^{N} \int_0^T (\delta - \delta e^{-\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}} - (1 - e^{-\delta\gamma\,\omega(t)\,m^{\sigma}_{\rho_N(t)}}))\,dt\Bigg]\Bigg\}
\end{aligned}$$

$$\leq \frac{1}{N} \log \Bigg(E\Bigg\{ \exp\Bigg[\sum_{i=1}^{N} \int_0^T (1 - e^{-\delta\beta\sigma_i(t)\,\omega_i(t)})\,dt + \sum_{i=1}^{N} \int_0^T \delta\beta\sigma_i(t)\,\omega_i(t)\,dN_t^{\sigma}(i)$$
$$+ \sum_{i=1}^{N} \int_0^T (1 - e^{-\delta\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}})\,dt + \sum_{i=1}^{N} \int_0^T \delta\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}\,dN_t^{\omega}(i) \Bigg] \Bigg\} \cdot$$
$$\cdot \exp\Bigg[\sum_{i=1}^{N} \int_0^T (\delta - \delta e^{-\beta} - 1 + e^{\delta\beta})\,dt + \sum_{i=1}^{N} \int_0^T (\delta - \delta e^{-\gamma} - 1 + e^{\delta\gamma})\,dt \Bigg] \Bigg)$$
$$= \frac{1}{N} \log \exp\left[NT(\delta - \delta e^{-\beta} - 1 + e^{\delta\beta} + \delta - \delta e^{-\gamma} - 1 + e^{\delta\gamma}) \right]$$
$$= T(2\delta - \delta(e^{-\beta} + e^{-\gamma}) - 2 + e^{\delta\beta} + e^{\delta\gamma}) < +\infty$$

where the next-to-last equality holds because

$$\exp\Bigg[\sum_{i=1}^{N} \int_0^T (1 - e^{-\delta\beta\sigma_i(t)\,\omega_i(t)})\,dt + \sum_{i=1}^{N} \int_0^T \delta\beta\sigma_i(t)\,\omega_i(t)\,dN_t^{\sigma}(i) +$$
$$+ \sum_{i=1}^{N} \int_0^T (1 - e^{-\delta\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}})\,dt + \sum_{i=1}^{N} \int_0^T \delta\gamma\,\omega_i(t)\,m^{\sigma}_{\rho_N(t)}\,dN_t^{\omega}(i) \Bigg]$$

is a Radon-Nikodym derivative with expected value = 1. ∎

Completing the proof of Proposition 4.3.4.

It remains to show the upper and the lower bounds (3.1) and (3.2). We prove them separately; our main tool is the Varadhan Lemma in the version in [24], Lemmas 4.3.4 and 4.3.6.

We deal first with the upper bound (3.1). Take $r \geq \max(e^{\beta}, e^{\gamma})$, so that the function F_r in (A.36) is upper semicontinuous. Denote by $\mathcal{P}_N$ the distribution of ρ_N under P_N, and by $\mathcal{W}_N^{(r)}$ its distribution under $W_r^{\otimes N}$. By (A.37)

$$\frac{d\mathcal{P}_N}{d\mathcal{W}_N^{(r)}}(Q) = \exp\left[NF_r(Q)\right] \tag{A.41}$$

Since F_r is upper semicontinuous and satisfies the superexponential estimate in Lemma A.4.6, we can apply Lemma 4.3.6 in [24] to obtain the upper bound (3.1).The lower bound (3.2) is proved similarly, by taking $0 < r \leq \min(e^{-\beta}, e^{-\gamma})$, so that F_r becomes lower semicontinuous, using (A.41) again and Lemma 4.3.4 in [24].

A.5 Proof of Proposition 4.3.5

We begin by observing that, since by assumption $I(Q) < \infty$, we have $H(Q|W) < +\infty$ and so by Lemma A.4.3 it follows that $Q \in \mathcal{I}$, which implies that the integrals below are well defined. By Girsanov's Formula for Markov Chains

$$
\begin{aligned}
&\int \log \frac{dP^Q}{dW}\, (\sigma[0,T],\omega[0,T])\, dQ = \\
&\quad = \int \Big[\int_0^T \left(1 - e^{-\beta\,\sigma(t)\omega(t)}\right) dt + \int_0^T \left(1 - e^{-\gamma\,\omega(t)\int \sigma\Pi_t Q(\,d\sigma,\,d\omega)}\right) dt \\
&\quad + \int_0^T \left(-\beta\,\sigma(t^-)\,\omega(t^-)\right) dN_t^\sigma + \int_0^T -\gamma\,\omega(t^-) \left[\int \sigma\Pi_{t^-} Q\,(\,d\sigma,\,d\omega)\right] dN_t^\omega \Big]\, dQ \\
&\quad = \int \Big[\int_0^T \left(1 - e^{-\beta\,\sigma(t)\omega(t)}\right) dt + \int_0^T \left(1 - e^{-\gamma\,\omega(t)\int \sigma\Pi_t Q(\,d\sigma,\,d\omega)}\right) dt \\
&\quad +\beta \int_0^T \sigma(t)\,\omega(t)\; dN_t^\sigma + \gamma \int_0^T \omega(t) \left[\int \sigma\Pi_t Q\,(\,d\sigma,\,d\omega)\right] dN_t^\omega \Big]\, dQ \\
&\quad = \int \Big[\int_0^T \left(1 - e^{-\beta\sigma(t)\omega(t)}\right) dt + \int_0^T \left(1 - e^{-\omega(t)\gamma_t^Q}\right) dt \\
&\qquad +\beta \int_0^T \sigma(t)\omega(t) dN_t^\sigma + \int_0^T \omega(t)\gamma_{t^-}^Q dN_t^\omega \Big]\, dQ = F(Q),
\end{aligned}
$$

where last equality comes from Fubini's Theorem. Finally, just observe that

$$
I(Q) = \int dQ \,\log \frac{dQ}{dW} - \int dQ\, \log \frac{dP^Q}{dW} = \int dQ\, \log \frac{dQ}{dP^Q} = H(Q \mid P^Q).
$$

A.6 The eigenvalues of the matrix A

We begin by writing down explicitly the limit matrix A:

$$
A = 2 \begin{pmatrix}
-\cosh(\beta) & \sinh(\beta) & 0 \\
-\gamma \frac{\sinh(\gamma m_*^\sigma)}{\cosh(\gamma m_*^\sigma)} \sinh(\gamma m_*^\sigma) + \gamma \cosh(\gamma m_*^\sigma) & -\cosh(\gamma m_*^\sigma) & 0 \\
\sinh(\gamma m_*^\sigma) + \gamma m_*^\sigma \cosh(\gamma m_*^\sigma) + \gamma \frac{\sinh(\beta) + m_*^\sigma \sinh(\gamma m_*^\sigma)}{\cosh(\beta) + \cosh(\gamma m_*^\sigma)} \sinh(\gamma m_*^\sigma) & 0 & -(\cosh(\beta) + \cosh(\gamma m_*^\sigma))
\end{pmatrix}
$$

where for the first term in the second row we have used the relations in iii) of Theorem 4.3.11. By direct computation, we see that their eigenvalues are:

$$
\begin{aligned}
\lambda_1 &= -\left(\cosh(\beta) + \cosh(\gamma m_*^\sigma)\right) \\
\lambda_2 &= -\frac{1}{2}\Big\{ \cosh(\beta) + \cosh(\gamma m_*^\sigma)) + \sqrt{(\cosh(\beta) - \cosh(\gamma m_*^\sigma)))^2 + 4\gamma \frac{\sinh(\beta)}{\cosh(\gamma m_*^\sigma)}} \Big\} \\
\lambda_3 &= -\frac{1}{2}\Big\{ \cosh(\beta) + \cosh(\gamma m_*^\sigma)) - \sqrt{(\cosh(\beta) - \cosh(\gamma m_*^\sigma)))^2 + 4\gamma \frac{\sinh(\beta)}{\cosh(\gamma m_*^\sigma)}} \Big\}
\end{aligned}
\tag{A.42}
$$

The eigenvalues are all real, and $\lambda_1, \lambda_2 < 0$. Moreover $\lambda_3 < 0$ if and only if

$$
\frac{\gamma}{\gamma_c} < \cosh^2(\gamma m_*^\sigma) \tag{A.43}
$$

where $\gamma_c = \frac{1}{\tanh(\beta)}$.

a) If $\gamma < \gamma_c$, which implies $m_*^\sigma = 0$, $(A.43)$ holds, because

$$\frac{\gamma}{\gamma_c} < 1 = \cosh^2(\gamma \cdot 0).$$

In this case the matrix A has three different real eigenvalues, all strictly negative.

b) If $\gamma = \gamma_c$, we still have $m_*^\sigma = 0$, but it is immediately seen that $\lambda_3 = 0$.

c) Finally, if $\gamma > \gamma_c$, set $y = \gamma m_*^\sigma$,

$$m_*^\sigma = \frac{1}{\gamma_c} \tanh(\gamma m_*^\sigma) \quad \Leftrightarrow \quad y = \frac{\gamma}{\gamma_c} \tanh(y)\ . \tag{A.44}$$

Then (A.43) is equivalent to showing that

$$\frac{\gamma}{\gamma_c} < \cosh^2(y) \tag{A.45}$$

and from (A.44) we obtain

$$\frac{\gamma}{\gamma_c} = \frac{y}{\tanh(y)} = \frac{y}{\sinh(y)} \cosh(y) < \cosh(y) < \cosh^2(y)$$

because $y/\sinh(y) < 1$ and $\cosh(y) < \cosh^2(y)$, since $y = \gamma m_*^\sigma > 0$ if $\gamma > \gamma_c$. Then, in this case too, the matrix A has three different real eigenvalues, all strictly negative.

Bibliography

[1] Allen, F., Gale, D. *Financial Contagion,* Journal of Political Economy, 108: 1-33, 2000.

[2] Basel Commettee on Banking Supervision (BCBS), 2004.

[3] Bielecki, T., Rutkowski, M. *Credit Risk: Modeling, Valuation and Hedging*, Springer Finance, 2002.

[4] Bolthausen, E. *Laplace approximations for sums of independent random vectors*, Probability Theory and Related Fields, 72: 305-318, 1986.

[5] Bouchaud, J., Cont, R. *Herd behavior and aggregate fluctuations in financial markets,* Macroeconomic Dynamics, 4: 170-196, 2000.

[6] Bovier, A., Cerny, J., Hryniv, O. *The Option Game: Stock price evolution from microscopic market modeling*, Int. J. Theor. Appl. Finance 9:91–111, 2006.

[7] Bremaud, P., *Markov Chains. Gibbs Fields, Monte Carlo Simulation, and Queues*, Texts in Applied Mathematics , 31: Springer Verlag 2001.

[8] Brock, W., Durlauf, S. *Discrete choice with social interactions*, Review of economic studies, 68: 235-260, 2001.

[9] Çetin, U., Campi, L., *Insider trading in an equilibrium model with default: a passage from reduced-form to structural modelling*, Finance and Stochastics, Vol. 11, No. 4, 2007.

[10] Çetin, U., Jarrow, R., Protter, P., Yildrim, Y. *Modeling credit risk with partial information*, The Annals of Applied Probability, Vol. 14, No. 3, 1167-1178, 2004.

[11] Christensen, J., Hansen, E., Lando, D. *Confidence sets for continuous-time rating transition probabilities* Journal of Banking and Finance, 28: 2575-2602, 2004.

[12] Collin-Dufresne, P., Goldstein, R., Helwege, J. *Is credit event risk priced? Modeling contagion via updating of beliefs.* Working paper, University of California at Berkeley, 2003.

[13] Comets, F. *Nucleation for a long range magnetic model*, Ann. Inst. Henri Poincaré, Probabilités et Statistiques 23: 135-178,1987.

[14] Cont, R. *Modeling economic randomness: Statistical mechanics of market phenomena*, Working paper, 1999.

[15] Cont, R., Löwe, M. *Discrete choices in an heterogeneous economy with interacting agents*, Working paper, 1998

[16] Cont, R., Minca, A. *"Recovering Portfolio Default Intensities Implied by CDO Quotes"*, Columbia University Center for Financial Engineering, Financial Engineering Report No. 2008-01

[17] Crouhy, M., Galai, D., Mark, R. *A comparative analysis of current credit risk models,* Journal of Banking and Finance, 24: 59-117, 2000.

[18] Dai Pra, P., Den Hollander, F. *McKean-Vlasov Limit for Interacting Random Processes in Random Media*, Journal of Statistical Physics, vol. 84, no3-4:735-772, 1996.

[19] Dai Pra, P., Runggaldier, W.J., Sartori, E., Tolotti, M. *Large portfolio losses, a dynamic contagion model*, forthcoming on Annals of Applied Probability, arXiv:0704.1348v2 , 2008.

[20] Dai Pra, P., Tolotti, M. *Heterogeneous credit portfolios and the dynamics of the aggregate losses*, ArXiv:0806.3399, 2008.

[21] Davis, M., Lo, V. *Modeling Default Correlation in Bond Portfolios*, in Alexander,C., Editor, Mastering risk, Harlow, Financial Times Prentice Hall, 141-151, 2001.

[22] Davis, M., Lo, V. *Infectious default*, Quantitative Finance 1(4):382-387, 2001.

[23] Dembo, D., Deuschel, J.D., Duffie, D. *Large portfolio losses*, Finance and Stochastics, 8:3-16, 2004.

[24] Dembo, A., Zeitouni O. *Large Deviations Techniques*, Jones and Bartlett Publishers, Boston, London, 1993.

[25] Den Hollander, F. *Metastability under stochastic dynamics*, Stochastic Processes and their Applications, 114:1-26, 2004.

[26] Deuschel, J.D., Stroock, D.W. *Large deviations*, Academic Press, Boston, 1989.

[27] Duffie, D., Saita, L., Wang, K. *Multiperiod corporate default prediction with stochastic covariates,* The Journal of Financial Economics, forthcoming.

[28] Duffie, D., Eckner, A., Horel, G., Saita, L. *Frailty correlated default,* Working paper, Stanford University, 2006.

[29] Duffie, D., Singleton, K. *Credit Risk: Pricing, Measurement and Management,* Princeton University Press, 2003.

[30] Egloff, D., Leippold, M., Vanini, P. *A simple model of credit contagion*, Journal of Banking and Finance, 31:2475-2492, 2007.

[31] Embrechts, P., Frey, R., McNeil, A. *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton Series in Finance, 2005.

[32] Ethier, S.N., Kurtz, T.G. *Markov Processes, Characterization and Convergence*, John Wiley and Sons, 1986.

[33] Föllmer, H. *Stock price fluctuation as a diffusion in random environment*, Philosophical Transaction of the Royal Society of London, Series A 347:471-483, 1994.

[34] Föllmer, H., Horst, U., Kirman, A. *Equilibria in Financial Markets with Heterogeneous Agents: A Probabilistic Perspective*, Journal of Mathematical Economics 41(1-2): 123-155, 2005.

[35] Frey, R., Backhaus, J. *Credit Derivatives in Models with interacting default intensities: A Markovian approach*, Preprint, 2006.

[36] Frey, R., Backhaus, J. *Dynamic hedging of syntentic CDO tranches with spread risk and default contagion*, Working paper, 2007.

[37] Frey, R., McNeil, A. *Dependent defaults in models of portfolio credit risk*, Journal of Risk 6(1):59-92, 2003.

[38] Frey, R., McNeil, A. *VaR and Expected shortfall in portfolios of dependent credit risks: Conceptual and practical insights,* Working paper, 2002.

[39] Giesecke, K., Goldberg, L., *A top-down approach to multi-name credit*, Working paper, 2008.

[40] Giesecke, K., Weber, S. *Cyclical correlations, credit contagion and portfolio losses*, Journal of Banking and Finance 28(12): 3009-3036, 2005.

[41] Giesecke, K, Weber, S. *Credit contagion and aggregated losses*, Journal of Economic Dynamics and Control 30(5): 741-767, 2006.

[42] Gordy, M.B. *A comparative anatomy of credit risk models,* Journal of Banking and Finance, 24, 119-149, 2000.

[43] Guo X., Jarrow R., Menn C. *A generalized Lando's forlmula: A filtration expansion perspective*, Working Paper, March 2006.

[44] Guo X., Zeng Y. *Intensity process and compensator: A new filtration expansion approach and the Jeulin-Yor formula*, Working Paper, March 2006.

[45] Hoffmann-JØrgensen, J., Pisier, G. *The law of large number and the central limit theorem in Banach spaces*, Annals of Probability, Vol.4 No.4 587-599, 1976.

[46] Horst, U. *Stochastic cascades, contagion and large portfolio losses*, Journal of Economic Behaviour and Organization, 63:25-54, 2007.

[47] Jarrow, R.A., Yu, F. *Counterparty risk and the pricing of defaultable securities*, Journal of Finance 53:2225-2243, 2001.

[48] Kiyotaki, N., Moore, J., *Credit Chains,* Working Paper, LSE, 1997.

[49] Lando, D. *Credit risk modeling, Theory and Applications*, Princeton Series in Finance, 2004.

[50] Lando, D., Skødeberg, T.M. *Analyzing rating transitions and rating drift with continuous observations*, Journal of Banking and Finance 26: 423-444, 2002.

[51] Li, D. *On default correlation: A copula function approach*, Journal of Fixed Income 9:43-54, 2001.

[52] Ligget, T. *Interacting particle systems*, Springer Verlag, Berlin, 1995.

[53] Merton, R.C., *On the pricing of corporate debt: the risk structure of interest rates*, Journal of Finance, 29:449-470, 1974.

[54] McNeil A., Wendin, J. *Dependent Credit Migrations*, Journal of Credit Risk, 2(3):87-114.

[55] Perko, L. *Differential Equations and Dynamical Systems*, Springer-Verlag, New York, 1991.

[56] Pham, H. *Some applications and methods of large deviations in finance and insurance*, ArXiv:math.PR/0702473v2, 2007.

[57] Sartori, E. *Some aspects of spin systems with mean-field interaction*, Ph.D. thesis, University of Padova, 2007.

[58] Schönbucher, P. *Credit derivatives pricing models*, Wiley Finance, 2003.

[59] Schönbucher, P. *Information driven default*, Working Paper, 2003.

[60] Schönbucher, P., *Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives*, Working paper, 2006.

[61] Varadhan, S.R.S. *Large deviations and applications*, Society for Industrial and Applied Mathematics, Philadelphia, 1984.

[62] Vasicek, O.A., *The distribution of loan portfolio value*, Risk, 2002.

[63] Yosida, K. *Functional Analysis*, Springer, 1980.

[64] Zhou, C. *An analysis of default correlations and multiple defaults*, Review of Financial Studies, 14(2): 555-576, 2001.