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Dyadic models of turbulence on trees

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Lest men suspect your tale untrue,
Keep probability in view.

The Painter who pleased Nobody and Everybody
John Gay

Contents

Contents **v**

1 Introduction **1**
1.1 The model . . . 1
1.2 Outlook on the problems and results . . . 5
1.3 Existence of solutions . . . 5
1.4 Uniqueness . . . 8
1.5 Energy dissipation . . . 13
1.6 Structure function . . . 15

2 Tree dyadic model **17**
2.1 Model and main results . . . 17
2.2 Choice of coefficients . . . 19
2.3 Elementary properties . . . 22
2.4 Relationship with classic dyadic model . . . 26
2.5 Partial superposition principle . . . 27
2.6 Generator . . . 28
2.7 Regular solutions . . . 29

3 Uniqueness through noise **31**
3.1 Self-similar solutions . . . 31
3.2 Non-uniqueness in the deterministic case . . . 32
3.3 Stochastic model . . . 33
3.4 Girsanov transform . . . 38
3.5 Closed equation for $E_{\widehat{P}}\left[X_j^2(t)\right]$ . . . 41
3.6 Uniqueness . . . 55

4 Anomalous dissipation **57**
4.1 Anomalous dissipation . . . 58

5 Stationary solutions **65**
5.1 Existence in the inviscid case 67
5.2 Existence in the viscous case 67
5.3 Uniqueness 71

6 The structure function and turbulence **75**
6.1 Existence and uniqueness of the stationary solution 76
6.2 Structure function 79
6.3 Rescaling of the X_js 87
6.4 Energy dissipation 90

A "Classic" dyadic model **95**

B Excerpts of the code **99**

Bibliography **103**

Chapter 1

Introduction

1.1 The model

The understanding of turbulent fluids is far from being reached. We know the equations that drive this kind of phenomenon, but we still have many questions left without an answer both at the level of the foundations of these equations and about the derivation of turbulence laws from them. If we consider the Euler equations

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

we have the open questions of global in time existence and uniqueness of solutions. If we consider the viscous version of this equation, the Navier-Stokes equation, the existence is settled, thanks to the viscosity term, but uniqueness is still an open problem.

Deducing the laws of turbulence, which are statistical laws and deal with very complex solutions, straight from these equations is very difficult. It is natural, then, to develop simplified models in order to perform these investigations.

Many models have been developed in order to study different features of turbulent fluids. Among those models there is the family of the dyadic models. The original dyadic model was introduced by Desnianskii and Novikov in 1974 ([25], [24]). A different version had been proposed in 1971 by Obukhov ([36]). Since then the dyadic model has been extensively studied, and many variations have been presented, deterministic and stochastic, for example by Waleffe [39], Katz and Pavlović [31], Friedlander and Pavlović [28], Barbato, Flandoli and Morandin [4], Cheskidov, Friedlander and Pavlović [18] and Cheskidov [16]. Other similar models include for example the GOY models introduced by Ohkitani and Yamada [37], and studied afterwards (see, for example, [3], [7]) and SABRA models. While the dyadic models are discrete ones, there are also continuous models, like the one recently presented by Cheskidov, Friedlander and Shvydkoy in [20].

This thesis concerns a model for the energy cascade phenomenon in turbulent fluids. It is based on the picture of the fluid as composed of eddies of various sizes.

Larger eddies split into smaller ones because of dynamical instabilities and transfer their kinetic energy from their scale to the one of the smaller eddies. One can think of a tree-like structure where nodes are eddies; any substructure father-offspring, where we denote the father by $j \in J$ (J the set of nodes) and the set of offspring by $\mathcal{O}_j$, corresponds to an eddy j and the set $\mathcal{O}_j$ of smaller eddies produced by j by instability. In the simplest possible picture, eddies belong to specified discrete levels, *generations*: level 0 is made of the largest eddy, level 1 of the eddies produced by level zero, and so on. The generation of eddy j may be denoted by $|j|$. Denote also the father of eddy j by $\bar{\jmath}$.

Phenomenologically, we associate to any eddy j an *intensity* $X_j(t)$, at time t, such that the kinetic energy of eddy j is $X_j^2(t)$. We relate intensities by a differential rule, which prescribes that the intensity of eddy j increases because of a flux of energy from $\bar{\jmath}$ to j and decreases because of a flux of energy from j to its set of offspring $\mathcal{O}_j$. We choose the rule

$$\frac{d}{dt}X_j = c_j X_{\bar{\jmath}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k, \tag{1.1}$$

where the coefficients c_j are positive.

This model has been introduced by Katz and Pavlović [31] as a simplified wavelet description of Euler equations, suitable for understanding the energy cascade. It is a phenomenological model, but mimics some features of the Euler equation: they are both infinite dimensional systems, with a quadratic nonlinearity and the energy is preserved, as we will see, for sufficiently regular solutions. The wavelet setting can be stated as follows: we consider a velocity field $u = u(x,t)$, and a family of orthonormal l^2 wavelets ψ_j, generated by a single mother wavelet ψ and with support in dyadic cubes Q_js. We can write

$$u(x,t) = \sum_{j \in J} X_j(t)\psi_j(x). \tag{1.2}$$

If we write the true equations satisfied by the coefficients of this wavelet expansion, they are much more complex than (1.1); in particular each X_j is coupled to each X_i (we would say that, in the true equation, there is an “infinite range interaction” between the X_js, in the language of particle systems). Moreover (1.1) is not even a proper “finite range interaction” approximation of the true system, as such an approximation would contain many more terms. Thus (1.1) has to be taken just as a phenomenological model where we keep only some nearest neighbour interactions and only those which move energy from larger to smaller scales, namely those which, from a physical or intuitive viewpoint, seem to be the most relevant for the cascade picture. Let us emphasize that in 2-dimensional fluids it has been observed that “inverse” cascade, from small to large eddies, plays an important role; this is not taken into account by (1.1). On the contrary, in 3D the major role is attributed only to the direct cascade; thus (1.1), while usually expressed in more generality, is physically speaking a model of 3D fluids.

The tree dyadic model (1.1) arises also as a more structured version of the so called dyadic model of turbulence, that we will call *linear* or *classic* dyadic model from now onwards, in order to distinguish it from the *tree* dyadic model. The latter is based on variables Y_n which represent a cumulative intensity of shell n (shell in Fourier or wavelet space) $n = 0, 1, 2, \dots$ In the tree dyadic model, on the contrary, shell n is described by a set of variables, all X_j's with $|j| = n$, the different intensities of eddies of generation n. The equations for Y_n have the form

$$\frac{d}{dt}Y_n = k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1}. \tag{1.3}$$

An overview of the main results on the classic dyadic model can be found in Chapter A.

Since the dyadic model can be seen as a decomposition of the velocity field in Fourier modes, we can think of a similar interpretation of the tree dyadic model.

Model (1.1) is a little bit more realistic than (1.3), although it is still extremely idealized with respect to the true wavelet description of Euler equations. Yet it is enough to get nontrivial results regarding the structure function, which we can compare to the Kolmogorov K41 theory and more recent experimental results, as we will see later on.

1.1.1 Choice of coefficients

If we focus back on the model (1.1), the coefficients c_j represent the speed of the energy flow from an eddy to its children. While most of the properties and results proven in Chapter 2 hold for any choice of positive coefficients c_j, the anomalous dissipation result holds only in a smaller class, that is $c_j = 2^{\alpha|j|}$. This is a physically natural choice for the coefficients. The parameter α is an approximation, averaged in time and space, of the rate of this speed. The right order of magnitude of c_j in the three dimensional setting is actually inside this class of coefficients:

$$c_j \sim 2^{\frac{5}{2}|j|}. \tag{1.4}$$

This choice of c_js, pointed out also by Katz and Pavlović as the correct one, can be heuristically justified in the following way. We take $u = \sum X_j w_j$, and we consider, in the three dimensional setting, $u \cdot \nabla u$. We have

$$\|u \cdot \nabla u\| \leq |u|_\infty \|\nabla u\|$$

We restrict to the wavelet components of u, w_j. When we consider a wavelet w_j of l^2 norm 1 with support in the cube Q_j, its l^∞ norm will be $2^{3/2|j|}$. At the same time $\|\nabla w_j\| \sim 2^{|j|}$, so

$$\|w_j \cdot \nabla w_j\| \lesssim 2^{\frac{3}{2}|j|} 2^{|j|} = 2^{\frac{5}{2}|j|}.$$

This choice for the c_js is the one corresponding to K41, as we will see below. Now we want to understand which is the Kolmogorov spectrum for the solutions X_j

of (1.1). In the case of the classic dyadic model (1.3), Kolmogorov inertial range spectrum reads

$$Y_n \sim k_n^{-1/3}.$$

The exponent is intuitive in such case. For the tree dyadic model (1.1) the Kolmogorov inertial range spectrum corresponds to

$$X_j \sim 2^{-\frac{11}{6}|j|}. \tag{1.5}$$

The correct exponent is not so immediate, so we provide here a heuristic derivation of it.

K41 theory [33] states that, if $u(x)$ is the velocity of the turbulent fluid at position x and the expected value E is suitably understood (for instance if we analyze a time-stationary regime), one has

$$E\left[|u(x) - u(y)|^2\right] \sim |x - y|^{2/3},$$

when x and y are very close each other (but not too close). Very vaguely this means

$$|u(x) - u(y)| \sim |x - y|^{1/3}.$$

Following the wavelet paradigm introduced above, let us think that $u(x)$ may be written in a basis (w_j) of orthonormal wavelets as

$$u(x) = \sum_j X_j w_j(x).$$

The vector field $w_j(x)$ corresponds to the velocity field of eddy j. Let us assume that eddy j has a support Q_j of the order of a cube of side $2^{-|j|}$. Given j, take $x, y \in Q_j$. When we compute $u(x) - u(y)$ we use the approximation $u(x) = X_j w_j(x)$, $u(y) = X_j w_j(y)$. Then

$$|u(x) - u(y)| = |X_j|\,|w_j(x) - w_j(y)|,$$

namely

$$|X_j|\,|w_j(x) - w_j(y)| \sim |x - y|^{1/3}, \qquad x, y \in Q_j.$$

We consider reasonably correct this approximation when $x, y \in Q_j$ have a distance of the order of $2^{-|j|}$, otherwise we should use smaller eddies in this approximation. Thus we have

$$|X_j|\,|w_j(x) - w_j(y)| \sim 2^{-\frac{1}{3}|j|}, \qquad x, y \in Q_j, |x - y| \sim 2^{-|j|}. \tag{1.6}$$

Moreover, we have

$$|w_j(x) - w_j(y)| = |\nabla w_j(\xi)|\,|x - y|, \tag{1.7}$$

for some point ξ between x and y (to be precise, the mean value theorem must be applied to each component of the vector valued function w_j). Recall that

$\int w_j(x)^2 dx = 1$, hence the typical size s_j of w_j in Q_j can be guessed from $s_j^2 2^{-3|j|} \sim 1$, namely $s_j \sim 2^{3/2|j|}$. Since w_j has variations of order s_j at distance $2^{-|j|}$, we deduce that the typical values of ∇w_j in Q_j have the order $2^{3/2|j|}/2^{-|j|} = 2^{5/2|j|}$. Thus, from (1.7),

$$|w_j(x) - w_j(y)| \sim 2^{\frac{5}{2}|j|} 2^{-|j|}.$$

Along with (1.6) this gives us

$$|X_j| 2^{\frac{5}{2}|j|} 2^{-|j|} \sim 2^{-\frac{1}{3}|j|},$$

namely

$$|X_j| \sim 2^{(-\frac{1}{3}+1-\frac{5}{2})|j|} = 2^{-\frac{11}{6}|j|}.$$

We have established (1.5), on a heuristic ground of course.

1.2 Outlook on the problems and results

We will give here a brief outlook on the rest of the Introduction and the thesis, to serve as a guide through the results.

From a mathematical point of view, the first question to ask is whether the tree model has solutions or not. The (positive) answer to such question is given in Chapter 2 and in the following Section 1.3 here in the Introduction.

The next question is on the uniqueness of the solution. In this case the answer is given in Chapter 3 and in Section 1.4 of the Introduction.

The last two questions have a more physical flavour. We show that the energy, while formally preserved, is actually dissipated, if the coefficients chosen are big enough, in Chapter 4 and Section 1.5. We get then to the most interesting result: in Chapter 6 and Section 1.6 we are able to show that the tree dyadic model has a structure function whose exponents follow a concave function instead of a line as in K41. We show some multifractality results for the tree model, too.

1.3 Existence of solutions

The first point we address, in Chapter 2, is an existence result. We define a solution of (1.1) in the following way: given $X^0 \in \mathbb{R}^J$,a componentwise solution of system (1.1) with initial condition X^0 is any family $X = (X_j)_{j\in J}$ of continuously differentiable functions $X_j : [0, \infty) \to \mathbb{R}$ such that $X(0) = X^0$ and all equations in system (1.1) are satisfied. If moreover $X(t) \in l^2$ for all $t \geq 0$, we call it an l^2 solution. We say that a solution is positive if $X_j(t) \geq 0$ for all $j \in J$ and $t \geq 0$.

Theorem 1.1. *Let $X^0 \in l^2$. There exists at least a solution of* (1.1) *in l^2 with initial condition X^0.*

The result presented in Chapter 2 is more general, as it is not restricted to the inviscid and unforced case, but the idea of the proof is the same given here.

We consider, once fixed $N \geq 1$, the finite dimensional system of Galerkin approximations

$$\begin{cases} X_{\bar{0}}(t) \equiv 0 \\ \frac{d}{dt} X_j = c_j X_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k & j \in J,\ 0 \leq |j| \leq N \\ X_k(t) \equiv 0 & k \in J,\ |k| = N+1 \\ X_j(0) = X_j^0 & j \in J,\ 0 \leq |j| \leq N, \end{cases}$$

for all $t \geq 0$. This system has a unique local solution by the Cauchy-Lipschitz theorem. If we compute the derivative of the energy up to generation N, that is

$$\frac{d}{dt} \sum_{|j| \leq N} X_j^2(t) = 2 \sum_{|j| \leq N} X_j \frac{d}{dt} X_j(t) = -2 \sum_{|k|=N+1} c_k X_{\bar{k}}^2(t) X_k(t)$$

we have from the boundary conditions that this derivative is null, so

$$\sum_{|j| \leq N} X_j^2(t) = \sum_{|j| \leq N} X_j^2(0), \qquad \forall t \geq 0,$$

and the existence is global. We call the solution X^N. Now we have

$$\sum_{|j| \leq N} \left(X_j^N(t)\right)^2 \leq \sum_{|j| \leq N} \left(X_j^0\right)^2, \qquad \forall t \geq 0,$$

so, for every j such that $|j| \leq N$,

$$\left|X_j^N(t)\right| \leq \left|X^0\right|_2 \qquad \forall t \geq 0$$

since the initial conditions are in l^2. Once we check the equi-uniform continuity, we can use Ascoli-Arzelà theorem to get a sequence X^{N_k} uniformly convergent to X, which is easily proven to be a solution.

The following proposition shows that solutions of the tree dyadic model (1.1) can be obtained by lifting the solutions of the classic dyadic model (1.3), thus providing a stronger link between the two models, when the choice of coefficients is of the exponential type $c_j = 2^{\alpha|j|}$.

Proposition 1.2. *If Y is a componentwise (resp. l^2) solution of (1.3), then $X_j(t) := 2^{-(|j|+2)\tilde{\alpha}} Y_{|j|}(t)$ is a componentwise (resp. l^2) solution of (1.1) with $\alpha = \beta + \tilde{\alpha}$. If Y is positive, so is X.*

1.3.1 Partial superposition principle

The lifting proposition is a way of constructing solutions. Another one uses a different property of the tree dyadic model: the partial superposition principle. This might sound strange, as the system is nonlinear, and in fact is not an actual superposition principle: we can't just add two solutions and get a new one, but we have to be subtler.

If we consider a positive solution of the inviscid and unforced tree model under some conditions on the coefficients that we will address below, we can traslate and multiply the solution for a suitable coefficient and get a similar solution rooted in another node of the tree. It requires that all the ancestors of such node j have the corresponding X set to 0, but leaves us freedom with regard to disjoint subtrees. In particular we could take the same original solution and move it to another node such that its subtree is disjoint with node j's subtree.

So we go deeper on the conditions on the coefficients. We need the coefficients c_j to be positive, depend only on the generation and be such that $c_{m+1}/c_{n+1} = c_m/c_n$ for any $n \geq m \geq 0$. Actually we can weaken it a little, by asking the coefficients to have a term depending on the generation satisfying the condition just stated and a term of distribution among children of a node, like in Chapter 6 (but then we may have a restriction on the node we can root it to).

If we move the root of the solution, meaning the first node with nonzero intensity from node i in generation m to node j in generation n we have $X_j = \beta X_i$, with $\beta = \sqrt{c_m/c_n}$. It is worth noting that, for the exponential choice of coefficients $c_j = 2^{\alpha|j|}$, the coefficient β is smaller than 1, so we are shrinking the solution, while translating it to the right. On the other hand let us stress that this construction does not need such a strong assumption on coefficients, as the lifting proposition did.

1.3.2 Stationary solutions

In Chapter 5 we consider a particular kind of solutions: the stationary ones, for tree dyadic models with a forcing term, both in the inviscid and viscous case. These solutions are proven to exist and to be unique. Stationary solutions will be fundamental objects for the computation of the structure function but they pose a critical question of interpretation, as we discuss below.

When we consider solutions that are stationary in time, both the wavelet and Fourier interpretations given above look less convincing: the already quite rigid structure of the model appears now unbelievable from a physical point of view. The eddies, that are requested by the model not to interact outside the father-son relationships, are also fixed in time, which is in stark contrast of our physical perception on eddies in a turbulent fluid. This suggests us another way of looking at our model in the stationary case: as a sequence of pictures taken at different times and kept constant until the next picture is taken. Since we are not assuming any kind of geometrical ordering of the children of each node, we can think of our model as a representation of the phenomenon in a particular coordinate system: the one that keeps the eddies fixed, while in the fluid they move around the unit cube.

The results for stationary solutions of the tree dyadic model are summarized by the following theorem.

Theorem 1.3. *Let* $\sharp\mathcal{O}_j = 2^{2\tilde{\alpha}}$ *for all* j. *Suppose* $\tilde{\alpha} < \alpha$ *and* $f > 0$ *in the system*

$$\begin{cases} X_{\bar{0}}(t) \equiv f \\ \dfrac{d}{dt} X_j = -\nu d_j X_j + c_j X_{\bar{j}}^2 - \displaystyle\sum_{k \in \mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J. \end{cases} \tag{1.8}$$

Then there exists a unique l^2 *positive solution* X *which is* stationary. *Moreover*

if $\nu = 0$ *then* $X_j(t) := f 2^{-(2\tilde{\alpha}+\alpha)(|j|+1)/3}$ *for all* $j \in J$;

if $\nu > 0$ *and* $0 < \alpha - \tilde{\alpha} \leq \gamma 3/2$, *the stationary solution is conservative and* regular, *in that for all real* s, $\sum_{j \in J} [2^{s|j|} X_j(t)]^2 < \infty$;

if $\nu > 0$ *and* $\alpha - \tilde{\alpha} > \gamma 3/2$, *there exists* $C > 0$ *such that for all* $f > C$ *the invariant solution of* (1.8) *is not regular and exhibits anomalous dissipation, that is the energy of the solution is dissipated.*

For the inviscid case we can explicitly provide the stationary solutions, while in the viscous case we'll prove that the stationary solutions are regular if and only if N_* is big enough, $N_* \geq 2^{2\alpha - 3\gamma}$ or the forcing term f is small. So the regularity disappears, and dissipation shows up, if the we have a strong enough forcing term and not enough children for every node in order to distribute the surplus energy forced into the system. It is very interesting that there is a threshold for the number of children that guarantees regularity however strong is the forcing term.

1.4 Uniqueness

For stationary solutions we have a strong uniqueness result, but we have a counterexample to uniqueness, in the space of solutions of any sign. If we restrict ourselves to the positive solutions, uniqueness is still an open question. The uniqueness of stationary solutions and the uniqueness for positive solutions for the dyadic model let us think that there is uniqueness in this class of solutions for the tree dyadic model, too, but the proofs are not easily adapted. The estimates are much more difficult to tune due to the presence of more than one child for every node. We could quite easily overcome this difficulty for the stationary solutions due to the absence of the time dependency, but for the general case of positive solutions the problem cannot be solved so easily.

1.4.1 Self-similar solutions, a counterexample

As mentioned, we are able to provide another special kind of solutions, that allow us to give a counterexample to uniqueness: the self-similar solutions. These are solutions of the form

$$X_j(t) = \frac{a_j}{t - t_0},$$

for some $t_0 < 0$, and where the coefficients a_j need to satisfy the following conditions:

$$\begin{cases} a_{\bar{0}} = 0 \\ a_j + c_j a_{\bar{j}}^2 = \sum_{k \in \mathcal{O}_j} c_k a_j a_k, \qquad \forall j \in J. \end{cases}$$

Once we have these self-similar solutions, we can construct a counterexample to uniqueness by using the time reversing technique. We may consider the system (1.1) for $t \leq 0$: given a solution $X(t)$ of this system for $t \geq 0$, we can define $\widehat{X}(t) = -X(-t)$, which is a solution for $t \leq 0$, since

$$\begin{aligned} \frac{d}{dt}\widehat{X}_j(t) = \frac{d}{dt}X_j(-t) &= c_j X_{\bar{j}}^2(-t) - \sum_{k \in \mathcal{O}_j} c_k X_j(-t) X_k(-t) \\ &= c_j \widehat{X}_{\bar{j}}^2(-t) - \sum_{k \in \mathcal{O}_j} c_k \widehat{X}_j(-t) \widehat{X}_k(-t). \end{aligned}$$

We can consider in particular the self-similar solutions for the tree dyadic model, $X_j(t) = \dfrac{a_j}{t - t_0}$, defined for $t > t_0$ and time-reverse them. This way we define

$$\widehat{X}_j(t) = -X_j(-t) \qquad \forall j \in J \qquad t < -t_0,$$

which is a solution of (1.1) in $(-\infty, -t_0)$, con $-t_0 > 0$. Since

$$\lim_{t \to +\infty} |X_j(t)| = 0 \quad \text{and} \quad \lim_{t \to t_0^+} |X_j(t)| = +\infty, \ \forall j \in J$$

we have

$$\lim_{t \to -\infty} \left|\widehat{X}_j(t)\right| = 0 \quad \text{and} \quad \lim_{t \to -t_0^-} \left|\widehat{X}_j(t)\right| = +\infty, \ \forall j \in J.$$

Thanks to the existence theorem seen above there is a solution $\widetilde{X}$, with initial conditions $x = \widehat{X}(0)$, and this solution is a finite energy one, so, in particular, doesn't blow up in $-t_0$. Yet it has the same initial conditions of $\widehat{X}$, so we can conclude that there is no uniqueness of solutions in the deterministic case.

1.4.2 Stochastic model

In order to restore uniqueness we add to our model a suitable random noise:

$$dX_j = \left(c_j X_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k \right) dt + \sigma c_j X_{\bar{j}} \circ dW_j - \sigma \sum_{k \in \mathcal{O}_j} c_k X_k \circ dW_k, \tag{1.9}$$

with $(W_j)_{j \in J}$ a sequence of independent Brownian motions, assuming deterministic initial conditions for (1.9): $X(0) = x = (x_j)_{j \in J} \in l^2$.

Introducing the noise

The idea of a stochastic perturbation of a deterministic model is well established in the literature, see [5] for the classical dyadic model, but also [15], [13] for different models.

The form of the noise may seem unexpected: one could think that the stochastic part would mirror the deterministic one, which is not the case here, since there is a j-indexed Brownian component where we'd expect a $\bar{\jmath}$ one, and there is a k-indexed one instead of a j one. At the same time one could argue that this is not the only possible random perturbation, but a very specific one. It is true: on one hand we chose a multiplicative noise, instead of an additive one, mainly for technical reasons (see for example [27]), on the other hand this specific form of the noise is the one we need to maintain a formal conservation of the energy, as we have in the deterministic case.

1.4.3 Weak uniqueness

Under these assumptions, a weak solution of (1.9) in l^2 is a filtered probability space $(\Omega, \mathcal{F}_t, P)$, together with a J-indexed sequence of independent Brownian motions $(W_j)_{j\in J}$ on $(\Omega, \mathcal{F}_t, P)$ and an l^2-valued process $(X_j)_{j\in J}$ on $(\Omega, \mathcal{F}_t, P)$ with continuous adapted components X_j such that

$$\begin{aligned} X_j = x_j + \int_0^t \Big[c_j X_{\bar{\jmath}}^2(s) - \sum_{k\in\mathcal{O}_j} c_k X_j(s) X_k(s)\Big] ds \\ + \int_0^t \sigma c_j X_{\bar{\jmath}}(s) dW_j(s) - \sum_{k\in\mathcal{O}_j} \int_0^t \sigma c_k X_k(s) dW_k(s) \\ - \frac{\sigma^2}{2}\int_0^t \Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big) X_j(s) ds, \end{aligned}$$

for every $j \in J$, with $c_0 = 0$ and $X_0(t) = 0$. We will denote this solution by

$$(\Omega, \mathcal{F}_t, P, W, X),$$

or simply by X, whenever the context provides information on the other terms.

A weak solution is an energy controlled solution if it satisfies

$$P\Big(\sum_{j\in J} X_j^2(t) \le \sum_{j\in J} x_j^2\Big) = 1,$$

for all $t \ge 0$, that is to say that its energy is bounded by the initial one almost surely.

Once we have established what a solution is for the stochastic model, we have the following existence result.

Theorem 1.4. *There exists an energy controlled solution to*

$$dX_j = \left(c_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k\right)dt + \sigma c_j X_{\bar{j}} dW_j$$
$$- \sigma \sum_{k\in\mathcal{O}_j} c_k X_k dW_k - \frac{\sigma^2}{2}\left(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\right)X_j dt, \quad (1.10)$$

which is an equivalent formulation of (1.9)*, in* $L^\infty(\Omega \times [0,T], l^2)$ *for* $(x_j) \in l^2$*.*

The Itô formulation (1.10) of the stochastic model is easier to handle, so we rely more on it, but the two formulations are equivalent and all results could be stated in terms of (1.9) instead.

We can also state the weak uniqueness result that is the aim of Chapter 3 and of this section.

Theorem 1.5. *There is uniqueness in law for the nonlinear system* (1.10) *in the class of energy controlled* $L^\infty(\Omega \times [0,T], l^2)$ *solutions.*

Both weak existence and weak uniqueness results are achieved through the Girsanov theorem, that transforms our nonlinear SDEs (1.10) into linear ones.

We can rewrite (1.10) as

$$dX_j = c_j X_{\bar{j}}(X_{\bar{j}}dt + \sigma dW_j) -$$
$$\sum_{k\in\mathcal{O}_j} c_k X_k (X_j dt + \sigma dW_k) - \frac{\sigma^2}{2}\left(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\right)X_j dt,$$

to isolate $X_{\bar{j}}dt + \sigma dW_j$ and prove through Girsanov theorem that they are Brownian motions with respect to a new measure $\widehat{P}$ in $(\Omega, \mathcal{F}_\infty)$, simultaneously for every $j \in J$. This way (1.10) becomes a system of linear SDEs under the new measure $\widehat{P}$. This can be formally stated as the following theorem.

Theorem 1.6. *If* $(\Omega, \mathcal{F}_t, P, W, X)$ *is an energy controlled solution of the nonlinear equation* (1.10)*, then* $(\Omega, \mathcal{F}_t, \widehat{P}, B, X)$ *satisfies the linear equation*

$$dX_j = \sigma c_j X_{\bar{j}} dB_j(t) - \sigma \sum_{k\in\mathcal{O}_j} c_k X_k dB_k(t) - \frac{\sigma^2}{2}\left(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\right)X_j dt, \qquad (1.11)$$

where the processes

$$B_j(t) = W_j(t) + \int_0^t \frac{1}{\sigma} X_{\bar{j}}(s)ds$$

are a sequence of independent Brownian motions on $(\Omega, \mathcal{F}_t, \widehat{P})$*, with* $\widehat{P}$ *defined by*

$$\frac{d\widehat{P}}{dP}\Big|_{\mathcal{F}_t} = \exp\left(-\frac{1}{\sigma}\sum_{j\in J}\int_0^t X_{\bar{j}}(s)dW_j(s) - \frac{1}{2\sigma^2}\int_0^t \sum_{j\in J} X_{\bar{j}}^2(s)ds\right).$$

The linear model is easily proven to have a (strong) solution, by means of Galerkin approximations. In order to prove weak uniqueness for the nonlinear model, we need strong uniqueness for the linear one. This is achieved by considering the $\widehat{P}$-moments of order two for the X_js: for every energy controlled solution X of the nonlinear equation (1.10), we have that $E_{\widehat{P}}[X_j^2(t)]$ is finite for every $j \in J$ and satisfies

$$\frac{d}{dt}E_{\widehat{P}}\big[X_j^2(t)\big] = -\sigma^2\bigg(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\bigg)E_{\widehat{P}}\big[X_j^2(t)\big] + \sigma^2 c_j^2 E_{\widehat{P}}\big[X_{\bar{j}}^2(t)\big] + \sigma^2 \sum_{k\in\mathcal{O}_j} c_k^2 E_{\widehat{P}}\big[X_k^2(t)\big].$$

This is very interesting, as it is a system of closed equations. Moreover we can write such system in matricial form:

$$E_j'(t) = \sum_{h\in J} E_h(t) q_{h,j} \tag{1.12}$$

where

$$q_{j,j} = -\sigma^2\bigg(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\bigg) \quad q_{\bar{j},j} = \sigma^2 c_j^2 \quad q_{k,j} = \mathbb{1}_{\{k\in\mathcal{O}_j\}}\sigma^2 c_k^2 \text{ for } k \neq j, \bar{j},$$

and we used the notation $E_j(t) = E_{\widehat{P}}\Big[X_j^2(t)\Big]$.

When we write the system of second moments in such a matricial form, it is strongly reminiscent of the forward equations of a Markov chain, but we should not get the wrong idea that our second moments are transition functions of a Markov chain themselves. At the same time there is a strong link between solutions of the forward equation for a Markov chain with matrix Q and our equation, as shown in the next proposition.

Proposition 1.7. *Given an initial condition $(v_l^0)_{l\in J} \in l^1$, if $(p_{i,j})$ is the minimal solution of the forward equations*

$$\begin{cases} p_{i,j}'(t) = \sum_{k\in J} p_{i,k}(t) q_{k,j} \\ p_{i,j}(0) = \delta_{i,j}, \end{cases}$$

which is the transition function of a continuous time Markov chain, then the family of functions defined by

$$v_j(t) = \sum_{k\in J} v_k^0 p_{k,j}(t),$$

is the minimal solution for the system

$$y_j'(t) = \sum_{k\in J} y_k(t) q_{k,j}$$

with initial conditions $y_j(0) = v_j^0$.

The existence of such a minimal solution is a key ingredient of the proof of the following proposition, which in turn plays a prominent role in the proof of uniqueness of the solution for the linear stochastic model.

Proposition 1.8. *Given the stable, conservative and symmetric q-matrix Q defined above, then the unique nonnegative solution of the equations* (1.12) *in* $L^\infty([0,\infty), l^1)$*, given a null initial condition* $y(0) = 0$*, is* $y(t) = 0$*.*

Moreover, the uniqueness holds in the same class with any nonnegative initial condition in l^1*.*

While we have a strong uniqueness result for the linear stochastic model, it translates to the uniqueness in law theorem stated above for the nonlinear model. The reason for this discrepancy lies in the fact that the two measures P and $\widehat{P}$ are equivalent on each $\mathcal{F}_t$, but not on $\mathcal{F}_\infty$.

1.5 Energy dissipation

The second question, once we set the one on uniqueness up, regards energy. If we consider the inviscid and unforced model (1.1) and we formally compute the derivative of the energy up to generation n, that is

$$\frac{d}{dt}\sum_{|j|\leq n} X_j^2(t) = 2\sum_{|j|\leq n} X_j(t)\frac{d}{dt}X_j(t) = -2\sum_{|k|=n+1} c_k X_{\bar{k}}^2(t) X_k(t),$$

we have that for sufficiently regular solutions the derivative of the energy, that is the limit of this quantity for $n \to \infty$, is 0, so the energy is formally preserved. At the same time, in Section 2.7 we prove that, for the exponential choice of coefficients, there cannot be regular solutions. What happens, once we assume this choice of coefficients, is that the model, while formally preserving energy, dissipates it. This phenomenon is called *anomalous dissipation*.

This anomalous dissipation for the tree dyadic model is not unexpected, as it shows up for the classic dyadic model too. At the same time it is not clear that the tree dyadic model should behave in the same way: even if the global flux from a generation to the next one behaves similarly to the shell case (1.3), energy may split between eddies of the same generation, which increase exponentially in number, so that the energy coming from ancestors could spread around. The result we prove in Chapter 4 is the following.

Theorem 1.9. *Assume that* $\alpha > \tilde{\alpha}$*, where* $2^{2\tilde{\alpha}} = N_* = \sharp\mathcal{O}_j$ *is the constant number of children for every node. Then for every* $\varepsilon > 0$ *and* $\eta > 0$ *there exists some* $T > 0$ *such that for all positive* l^2 *solution of* (1.1) *with initial energy* $\mathcal{E}(0) \leq \eta$ *one has* $\mathcal{E}(T) \leq \varepsilon$*. In particular*

$$\lim_{t\to\infty} \mathcal{E}(t) = 0,$$

i.e. there is anomalous dissipation.

The proof of this result relies on some bounds on the energy and on the monotonicity of the energy itself.

The condition on the on the number of children can be easily relaxed to $1 \leq \sharp\mathcal{O}_j \leq N_*$, but the interesting point, that reconciles the two intuitions stated above, is the condition $\alpha > \widetilde{\alpha}$, which postulates that the flow has to be quick enough with respect to the (maximum) number of children for each node in order for dissipation to occur. We are also able to provide an upper bound on the rate of decay of the energy. Under the hypotheses of the energy dissipation theorem, X be any positive l^2 solution with initial condition X^0. Then there exists $C > 0$, depending only on $\|X^0\|$, such that for all $t > 0$

$$\mathcal{E}(t) := \|X(t)\|^2 := \sum_{j \in J} X_j^2(t) < \frac{C}{t^2} \ .$$

Moreover this estimate cannot be improved much, as the self-similar solutions introduced above have energy that goes to 0 exactly as t^{-2}.

1.5.1 Decay of X_j with anomalous dissipation

Let us now give a heuristic explanation of the fact that, when anomalous dissipation occurs, the decay (1.5) appears. In a sense, this may be seen as a confirmation that (1.5) is the correct decay corresponding to K41. Let us start from equations (1.1) with $c_j \sim 2^{5/2|j|}$, the choice previously stated as the physically relevant one. Let $\mathcal{E}_n$ be the energy up to generation n:

$$\mathcal{E}_n = \sum_{|j| \leq n} X_j^2.$$

Then, by a simple computation, we have

$$\frac{d\mathcal{E}_n}{dt} = -2^{\frac{5}{2}(n+1)} \sum_{|k|=n+1} X_{\bar{k}}^2 X_k.$$

In order to have anomalous dissipation, we should have

$$\frac{d\mathcal{E}_n}{dt} \overset{n}{\sim} -C \neq 0.$$

If we assume a power decay,

$$X_j \sim 2^{-\eta|j|}.$$

Then, since the cardinality of $\{Q_j : |j| = n\}$ should be of the order of 2^{3n},

$$2^{\frac{5}{2}(n+1)} \sum_{|k|=n+1} X_{\bar{k}}^2 X_k \sim 2^{\frac{5}{2}n} 2^{3n} 2^{-3\eta n} = 2^{(\frac{11}{2} - 3\eta)n},$$

and thus $\eta = 11/6$.

1.6 Structure function

In Chapter 6 we slightly change perspective towards a more physical and quantitative one. We are interested in understanding how our model behaves when compared to other models for turbulence, to the theoretical results of Kolmogorov's K41 theory and to the recent experimental results.

We consider only the 3D case, while we make the following assumptions on the coefficients: each coefficient has a factor that depends only on the generation and a factor depending on the child it links to, more specifically

$$c_j = 2^{\frac{5}{2}|j|} d_j \,, \qquad \{d_k : \; k \in \mathcal{O}_j\} = \{\widetilde{d}_h : \; h = 1 \dots 2^3\},$$

where the two sets are equal taking multiplicity into account. Moreover we consider the stationary solution for this system.

As it was the case for the purely exponential choice of coefficients, we have uniqueness of the stationary solution. Again, as in Chapter 5 being in the inviscid case, we are able to show the actual form of the solution. The choice of a stationary solution might look like a poor one from a physical point of view, since as already argued the fluids are anything but stationary. Once more we stress that when considering a stationary solution, the most convincing interpretation of the model is a statistical one, not a strict interpretation as wavelet coefficients. With our model we are trying to capture some of the features of the turbulent fluid once it has reached a stationary regime.

In the study of turbulence the structure function of order p is defined, for a fluid with a stationary velocity field u as

$$S_p(\rho) = E_x[|(u(x + \rho e) - u(x)) \cdot e|^p]. \tag{1.13}$$

We can consider the unit cube where the dynamics takes place with a different coordinate system: we consider $x \in \mathbb{X}$ as an infinite branch on the tree, $x = (j_0^x, j_1^x, \dots)$, that is to say as the limit of the dyadic cubes $Q_{j_n^x}$ in which it is contained. Of course as soon as we consider this coordinate system we lose any geometric property of the actual physical space in which the fluid lives, as we have no information as where each eddy actually is in the space. This allows us to disregard the versor e in the definition of S_p, while the expectation with respect to x is still the integral on the whole unit cube Q.

In the literature the structure function for dyadic models has been defined as (see for example [14])

$$S_p(2^{-n}) = E_x\left[\sum_{|j|=n} |u_j|^p\right]$$

that is the sum of all the terms with scale of order 2^{-n}. While studying the structure function, one usually focuses more on its exponents, the so called

$$\zeta_p = \lim_{n\to\infty} -\frac{1}{n} \log_2 S_p(2^{-n}).$$

In Kolmogorov K41 theory such coefficients are claimed to behave as $\frac{p}{3}$, but more recent experimental results and models argue that the right behaviour for these coefficients is along a concave function, lying below $\frac{p}{3}$ for $p > 3$. We prove that when interested in calculating the ζ_p function, the generations greater than n do not contribute. For the generations from 0 to $n-1$ we are only able to provide a heuristic argument stating that their contribution is negligible, thus justifying the approximate version of the structure function.

Using this approximated version of the structure function, we get

$$\zeta_p = \frac{p}{3} + \frac{p}{2}\left(\log_2\left(\frac{1}{8}\sum_{i=1}^{8}\tilde{d}_i^{3/2}\right)^{2/3} - \log_2\left(\frac{1}{8}\sum_{i=1}^{8}\tilde{d}_i^{p/2}\right)^{2/p}\right).$$

This function is concave, passes through the point $(1,3)$, and lies below the Kolmogorov straight line afterwards. Under reasonable assumptions we have that it has a positive oblique asymptote.

Then we try to understand where is the energy dissipated, in our eddies. Because of the lack of geometry in the formulation of our model we are only able to calculate the Hausdorff dimensions of the sets of points dissipating the same amount of energy and to establish a connection between this multifractality and the one originated by the different scaling exponents in different points. We can compute those dimensions by interpreting the coefficients in any node as chosen uniformly among the eight possible ones and then relying on the Large Deviation Principle.

Results in Chapters 2, 4 and 5 were published in:
David Barbato, Luigi Amedeo Bianchi, Franco Flandoli, and Francesco Morandin, *A dyadic model on a tree*, Journal of Mathematical Physics, 54:021507, 2013.

Results in Chapter 3 were published in:
Luigi Amedeo Bianchi, *Uniqueness for an inviscid stochastic dyadic model on a tree*, Electronic Communications in Probability, 18:1–12, 2013.

Results in Chapter 6 are the subject of a paper in preparation together with Francesco Morandin.

Chapter **2**

Tree dyadic model

In this chapter we introduce the tree dyadic model, main subject of this thesis. We start from the description of the model and the definition of a solution for such a model, presented in Section 2.1. We continue with some heuristic considerations on the coefficients of the tree dyadic model and on its behaviour in the Kolmogorov inertial range, in Section 2.2, in anticipation to the more quantitative study of this behaviour in Chapter 6. We then prove some interesting properties of this model and an existence result for solutions, in Section 2.3. Since the tree dyadic model is a natural generalization of the classic dyadic model, we prove a lifting theorem that connects the two and some other considerations on the similarities and differences between them, in Section 2.4. To conclude the chapter we show a partial superposition principle for solutions of the tree dyadic model, in Section 2.5, we present a generator formulation of the model in Section 2.6 and we discuss the case of regular solutions, in Section 2.7, a topic strongly related to the discussion on anomalous dissipation presented in Chapter 4.

2.1 Model and main results

The eddies in our turbulent fluid split in smaller eddies, because of instabilities, and transfer their kinetic energy to their offspring. So a simple way, maybe the simplest, to look at this phenomenon is to think of the eddies as nodes of a tree. If we consider the nodes belonging to discrete levels, which we'll call *generations*, we can draw an edge from a node to another of a younger generation whenever the second is generated from the splitting of the first one. The set of eddies generated by an eddy will be called its *offspring*.

Inside the set of nodes J we identify one special node, called *root* or ancestor of the tree, denoted by 0. For all $j \in J$ we define the generation number $|j| \in \mathbb{N}$ (such that $|0| = 0$), the set of offsprings of j, denoted by $\mathcal{O}_j \subset J$, such that $|k| = |j| + 1$ for all $k \in \mathcal{O}_j$ and a unique parent $\bar{\jmath}$ with $j \in \mathcal{O}_{\bar{\jmath}}$. The root 0 has no parent inside J, but with slight notation abuse we will nevertheless use the symbol $\bar{0}$ when needed.

Phenomenologically, we associate to any eddy j an *intensity* $X_j(t)$, at time t, such that the kinetic energy of eddy j is $X_j^2(t)$. We relate intensities by a differential rule, which prescribes that the intensity of eddy j increases because of a flux of energy from $\bar{\jmath}$ to j and decreases because of a flux of energy from j to its set of offspring $\mathcal{O}_j$. We choose the rule

$$\frac{d}{dt}X_j = c_j X_{\bar{\jmath}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k, \tag{2.1}$$

where the coefficients c_j are positive.

One way to interpret this, as seen in the Introduction, is to write the velocity field of the turbulent fluid $u(t,x)$ in a suitable wavelet basis $(w_j)_j$ as in Katz-Pavlović [31]:

$$u(t,x) = \sum_j X_j(t) w_j(x),$$

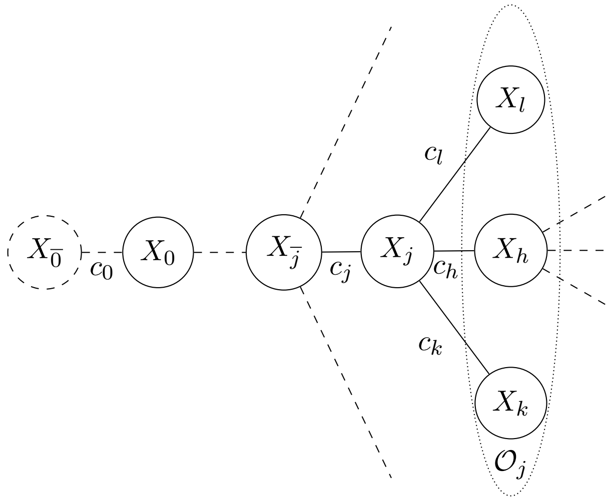
where every eddy j has an associated velocity field given by the wavelet w_j. So we have associated to every eddy j an intensity $X_j(t)$, at time t; the kinetic energy of eddy j is $X_j^2(t)$. Of course this is just an interpretation: if we were to consider this form, the differential rule satisfied by the coefficients of the wavelet formulations would be different, in particular each X_j would be coupled to each other X_i. So (2.1) has to be considered just a phenomenological model where we take into account only some nearest neighbour interactions, those that from an intuitive point of view seem to be the most relevant for the cascade picture.

We will write the dynamics in (2.1) in a more general and formal way as a family $(X_j)_{j\in J}$ of functions $X_j : [0,\infty) \to \mathbb{R}$ satisfying the equations below:

$$\begin{cases} X_{\bar{0}}(t) \equiv f \\ \dfrac{d}{dt}X_j = -\nu d_j X_j + c_j X_{\bar{\jmath}}^2 - \displaystyle\sum_{k\in\mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J. \end{cases} \tag{2.2}$$

Here we suppose that $f \geq 0$, $\nu \geq 0$; if $f = 0$ we call the system *unforced*, if $\nu = 0$ we call it *inviscid*, so the equations in (2.1) are those of the inviscid and unforced dyadic model. It's worth stressing that $X_{\bar{0}}$ does not belong to the family and is merely a convenient symbolic alias for the constant forcing term.

This system will usually come with an initial condition which will be denoted by $X^0 = (X_j^0)_{j\in J}$. One natural space for $X(t)$ to live is $l^2(J;\mathbb{R})$, which we will simply denote by l^2, the setting being understood. The l^2 norm will be simply denoted by $\|\cdot\|$.



For sake of simplicity we will suppose throughout this thesis that the cardinality of $\mathcal{O}_j$ is constant, $\sharp\mathcal{O}_j =: N_*$ for all $j \in J$, unless otherwise noted, but some results can be easily generalized at least to the case where $\sharp\mathcal{O}_j$ is positive and uniformly bounded. It will turn out to be very important to compare N_* to some coefficients of the model. To this end we set also $\tilde{\alpha} := 1/2 \log_2 N_*$ so that $N_* = 2^{2\tilde{\alpha}}$

Definition 2.1. Given $X^0 \in \mathbb{R}^J$, we call componentwise solution of system (2.2) with initial condition X^0 any family $X = (X_j)_{j\in J}$ of continuously differentiable functions $X_j : [0, \infty) \to \mathbb{R}$ such that $X(0) = X^0$ and all equations in system (2.2) are satisfied. If moreover $X(t) \in l^2$ for all $t \geq 0$, we call it an l^2 solution.

We say that a solution is positive if $X_j(t) \geq 0$ for all $j \in J$ and $t \geq 0$.

2.2 Choice of coefficients and decay of X_j

Let us spend some word on the choice of the coefficients c_j (and d_j, consequently). One point of interest is the sign of those coefficients. If all the coefficients share the same sign, if we take initial conditions of the same sign as the coefficients we keep the sign, energy flows to the right and everything behaves as in the all positive setting. If the initial conditions are of a different sign than that of the coefficients, then we have back-propagation for a while, then all the sign go back to be the same and the usual behaviour is restored. Suppose that we have positive coefficients and X_js of both signs, and let's consider a particular X_j that starts with a negative value and a father $X_{\bar{j}}$ with positive value. Then $X_{\bar{j}}$ will increase its value, and X_j' will grow and become positive, so X_j itself will become positive in some time.

As soon as we consider a system with positive coefficients and components X_js of any sign, we are considering all the possible cases. If we assign c_js of any sign, we can consider the auxiliary system

$$Y_j' = |c_j| Y_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} |c_k| Y_j Y_k,$$

where Y is a solution if and only if $Y_j = X_j\, sgn(c_j)$ and X is a solution of the original system. Thus we can only consider the case of positive coefficients.

Once we settle the question of the sign, there is another matter to take care of: is there any particular choice of the coefficients that is more natural than the others? One possible assumption is that the coefficients are constant in each generation, that is $c_j = c(|j|)$. This provides us with a more tractable form of the model. Another option, slightly more general, but still more tractable than the most general case, is to have two contributes to each c_j: one depending from the generation and one depending on the specific node j. These are helpful properties to lay onto the coefficients, but are not yet a choice.

One choice is to take the c_js exponential in the generation, that is $c_j = 2^{\alpha|j|}$. As we will see in Chapter 4, this is the natural choice if we want to investigate the anomalous dissipation phenomenon, and as we will heuristically justify below, is also the one arising from "physical" considerations. In Chapter 6 we will consider a slight modification of this choice, by taking $c_j = 2^{\alpha|j|}e_j$, where the e_js are the same in every offspring, that is

$$\{e_j \,:\, j \in \mathcal{O}_h\} = \{\widetilde{e}_1, \dots, \widetilde{e}_{N_*}\} \qquad \forall h \in J,$$

where the equality has to be considered with multiplicity and the $\widetilde{e}_j$ need not to be different from each other.

If we focus back on the model (1.1), the coefficients c_j represent the speed of the energy flow from an eddy to its children. While most of the properties and results proven in Chapter 2 hold for any choice of positive coefficients c_j, the anomalous dissipation result holds only in a smaller class, that is $c_j = 2^{\alpha|j|}$. This is a physically natural choice for the coefficients. The parameter α is an approximation, averaged in time and space, of the rate of this speed. The right order of magnitude of c_j in the three dimensional setting is actually inside this class of coefficients:

$$c_j \sim 2^{\frac{5}{2}|j|}. \tag{2.3}$$

This choice of c_js, pointed out also by Katz and Pavlović as the correct one, can be heuristically justified in the following way. We take $u = \sum X_j w_j$, and we consider, in the three dimensional setting, $u \cdot \nabla u$. We have

$$\|u \cdot \nabla u\| \leq |u|_\infty \|\nabla u\|$$

We restrict to the wavelet components of u, w_j. When we consider a wavelet w_j of l^2 norm 1 with support in the cube Q_j, its l^∞ norm will be $2^{3/2|j|}$. At the same time $\|\nabla w_j\| \sim 2^{|j|}$, so

$$\|w_j \cdot \nabla w_j\| \lesssim 2^{\frac{3}{2}|j|}2^{|j|} = 2^{\frac{5}{2}|j|}.$$

This choice for the c_js is the one corresponding to K41, as we will see below. Now we want to understand which is the Kolmogorov spectrum for the solutions X_j of (2.1). In the case of the classic dyadic model, Kolmogorov inertial range spectrum reads

$$Y_n \sim k_n^{-1/3}.$$

The exponent is intuitive in such case. For the tree dyadic model (2.1) the Kolmogorov inertial range spectrum corresponds to

$$X_j \sim 2^{-\frac{11}{6}|j|}. \tag{2.4}$$

The correct exponent is not so immediate, so we provide here a heuristic derivation of it.

K41 theory [33] states that, if $u(x)$ is the velocity of the turbulent fluid at position x and the expected value E is suitably understood (for instance if we analyze a time-stationary regime), one has

$$E\left[|u(x) - u(y)|^2\right] \sim |x - y|^{2/3},$$

when x and y are very close each other (but not too close). Very vaguely this means

$$|u(x) - u(y)| \sim |x - y|^{1/3}.$$

Following the wavelet paradigm introduced above, let us think that $u(x)$ may be written in a basis (w_j) of orthonormal wavelets as

$$u(x) = \sum_j X_j w_j(x).$$

The vector field $w_j(x)$ corresponds to the velocity field of eddy j. Let us assume that eddy j has a support Q_j of the order of a cube of side $2^{-|j|}$. Given j, take $x, y \in Q_j$. When we compute $u(x) - u(y)$ we use the approximation $u(x) = X_j w_j(x)$, $u(y) = X_j w_j(y)$. Then

$$|u(x) - u(y)| = |X_j|\,|w_j(x) - w_j(y)|,$$

namely

$$|X_j|\,|w_j(x) - w_j(y)| \sim |x - y|^{1/3}, \qquad x, y \in Q_j.$$

We consider reasonably correct this approximation when $x, y \in Q_j$ have a distance of the order of $2^{-|j|}$, otherwise we should use smaller eddies in this approximation. Thus we have

$$|X_j|\,|w_j(x) - w_j(y)| \sim 2^{-\frac{1}{3}|j|}, \qquad x, y \in Q_j, |x - y| \sim 2^{-|j|}. \tag{2.5}$$

Moreover, we have

$$|w_j(x) - w_j(y)| = |\nabla w_j(\xi)|\,|x - y|, \tag{2.6}$$

for some point ξ between x and y (to be precise, the mean value theorem must be applied to each component of the vector valued function w_j). Recall that $\int w_j(x)^2 dx = 1$, hence the typical size s_j of w_j in Q_j can be guessed from $s_j^2 2^{-3|j|} \sim 1$, namely $s_j \sim 2^{3/2|j|}$. Since w_j has variations of order s_j at distance $2^{-|j|}$, we deduce that the typical values of ∇w_j in Q_j have the order $2^{3/2|j|}/2^{-|j|} = 2^{5/2|j|}$. Thus, from (2.6),

$$|w_j(x) - w_j(y)| \sim 2^{\frac{5}{2}|j|}2^{-|j|}.$$

Along with (2.5) this gives us

$$|X_j|2^{\frac{5}{2}|j|}2^{-|j|} \sim 2^{-\frac{1}{3}|j|},$$

namely

$$|X_j| \sim 2^{(-\frac{1}{3}+1-\frac{5}{2})|j|} = 2^{-\frac{11}{6}|j|}.$$

We have established (2.4), on a heuristic ground of course.

2.3 Elementary properties

We will provide, in this section, some basic results on the tree dyadic model. The results are analogous to those provided for the dyadic model in [6] and [27] and recapped in Chapter A, but the proofs require some new ideas to cope with the more general structure.

We will suppose throughout the chapter that the initial condition X^0 is in l^2 for all $j \in J$. We will consider generic positive coefficients c_j, not necessarily of the exponential form argued in Section 2.2.

Definition 2.2. For $n \geq -1$, we denote by $\mathcal{E}_n(t)$ the total energy on nodes j with $|j| \leq n$ at time t and $\mathcal{E}(t)$ the energy of all nodes at time t (which is possibly infinite):

$$\mathcal{E}_n(t) := \sum_{|j|\leq n} X_j^2(t), \qquad \mathcal{E}(t) := \sum_{j\in J} X_j^2(t).$$

Note in particular that $\mathcal{E}_{-1} \equiv 0$.

We will use very often the derivative of $\mathcal{E}_n$, for $n \geq 0$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_n(t) &= 2\sum_{|j|\leq n} X_j \frac{d}{dt}X_j(t) \\
&= -2\nu \sum_{|j|\leq n} d_j X_j^2 + 2\sum_{|j|\leq n} c_j X_{\bar{j}}^2 X_j - 2\sum_{|j|\leq n}\sum_{k\in\mathcal{O}_j} c_k X_j^2 X_k \\
&= -2\nu \sum_{|j|\leq n} d_j X_j^2 + 2c_0 X_{\bar{0}}^2 X_0 - 2\sum_{|k|=n+1} c_k X_{\bar{k}}^2 X_k.
\end{aligned}$$

so we get for all $n \geq 0$

$$\frac{d}{dt}\mathcal{E}_n(t) = -2\nu \sum_{|j|\leq n} d_j X_j^2(t) + 2f^2 X_0(t) - 2\sum_{|k|=n+1} c_k X_{\bar{k}}^2(t) X_k(t). \tag{2.7}$$

Proposition 2.1. *If $X_j^0 \geq 0$ for all j, then any componentwise solution is positive. If X^0 is in l^2, any positive componentwise solution is a positive l^2 solution, in particular for all $t \geq 0$*

$$\mathcal{E}(t) \leq (\mathcal{E}(0)+1)e^{2f^2t}. \tag{2.8}$$

Proof. From the definition of componentwise solution we get that for all $j \in J$

$$X_j(t) = X_j^0 e^{-\int_0^t(\nu d_j+\sum_k c_k X_k(r))dr} + \int_0^t c_j X_{\bar{j}}^2(s) e^{-\int_s^t(\nu d_j+\sum_k c_k X_k(r))dr} ds, \tag{2.9}$$

yielding $X_j(t) \geq 0$ for all $t > 0$ and all $j \in J$.

Now we turn to the estimates of $\mathcal{E}(t)$. In (2.7), since $X_k(t) \geq 0$ we have two negative contribution which we drop and we use the bound $X_0(t) \leq X_0^2(t) + 1 \leq \mathcal{E}_n(t) + 1$ to get that for all $n \geq 0$,

$$\frac{d}{dt}\mathcal{E}_n(t) \leq 2f^2(\mathcal{E}_n(t)+1),$$

so by Grönwall lemma $\mathcal{E}_n(t) + 1 \leq (\mathcal{E}_n(0)+1)e^{2f^2t}$. Letting $n \to \infty$ we obtain (2.8). □

Proposition 2.2. *For any positive l^2 solution X, the following energy balance principle holds, for all $0 \leq s < t$.*

$$\mathcal{E}(t) = \mathcal{E}(s) + 2f^2\int_s^t X_0(u)du - 2\nu\sum_{j\in J} d_j \int_s^t X_j^2(u)du \\ - 2\lim_{n\to\infty}\int_s^t \sum_{|k|=n} c_k X_{\bar{k}}^2(u) X_k(u)du, \tag{2.10}$$

where the limit always exists and is non-negative. In particular, for the unforced, inviscid ($f = \nu = 0$) tree dyadic model, $\mathcal{E}$ is non-increasing.

Proof. Let $0 \leq s < t$, then by (2.7) for all $n \geq 0$,

$$\mathcal{E}_n(t) = \mathcal{E}_n(s) - 2\nu\sum_{|j|\leq n} d_j \int_s^t X_j^2(u)du + 2f^2\int_s^t X_0(u)du \\ - 2\int_s^t \sum_{|k|=n+1} c_k X_{\bar{k}}^2(u) X_k(u)du.$$

As $n \to \infty$, since the solution is in l^2, $\mathcal{E}_n(s) \uparrow \mathcal{E}(s) < \infty$ and the same holds for t. The viscosity term is a non-decreasing sequence bounded by

$$2\nu\sum_{|j|\leq n} d_j \int_s^t X_j^2(u)du \leq \mathcal{E}(s) + 2f^2\int_s^t X_0(u)du < \infty,$$

so it converges too. Then the border term converges being the sum of converging sequences. □

Definition 2.3. We say that a positive l^2 solution X is *conservative* in $[s,t]$ if the limit in (2.10) is equal to zero that is if

$$\mathcal{E}(t) = \mathcal{E}(s) + 2f^2 \int_s^t X_0(u)du - 2\nu \sum_{j\in J} d_j \int_s^t X_j^2(u)du.$$

Otherwise we say that X has *anomalous dissipation* in $[s,t]$.

We will get back to anomalous dissipation in Chapter 4.

Theorem 2.3. *Let $X^0 \in l^2$ with $X_j^0 \geq 0$ for all $j \in J$. Then there exists at least a positive l^2 solution with initial condition X^0.*

Proof. The proof by finite dimensional approximates is completely classic. Fix $N \geq 1$ and consider the finite dimensional system

$$\begin{cases} X_{\bar{0}}(t) \equiv f \\ \frac{d}{dt}X_j = -\nu d_j X_j + c_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k & j \in J,\ 0 \leq |j| \leq N \\ X_k(t) \equiv 0 & k \in J,\ |k| = N+1 \\ X_j(0) = X_j^0 & j \in J,\ 0 \leq |j| \leq N, \end{cases} \tag{2.11}$$

for all $t \geq 0$. Notice that proposition 2.1 is true also for this truncated system (with unchanged proof), so there is a unique global solution. (Local existence and uniqueness follow from the local Lipschitz continuity of the vector field and global existence comes from the bound in (2.8).) We'll denote such unique solution by X^N.

Now fix $j \in J$ and consider on a bounded interval $[0,T]$ the family $(X_j^N)_{N>|j|}$. By (2.8) we have a strong bound that does not depend on t and N

$$|X_j^N(t)| \leq (\mathcal{E}(0)+1)^{\frac{1}{2}} e^{\frac{1}{2}Tf^2} \qquad \forall N \geq 1 \quad \forall t \in [0,T] \ ,$$

thus the family $(X_j^N)_{N>|j|}$ is uniformly bounded, and by applying the same bound to (2.11), equicontinuous. From Ascoli-Arzelà theorem, for every $j \in J$ there exists a sequence $(N_{j,k})_{k\geq 1}$ such that $(X_j^{N_{j,k}})_k$ converges uniformly to a continuous function X_j. By a diagonal procedure we can modify the extraction procedure and get a single sequence $(N_k)_{k\geq 1}$ such that for all $j \in J$, $X_j^{N_k} \to X_j$ uniformly. Now we can pass to the limit as $k \to \infty$ in the equation

$$X_j^{N_k} = X_j^0 + \int_0^t \Big[-\nu d_j X_j^{N_k}(r) + c_j \Big(X_{\bar{j}}^{N_k}(r) \Big)^2 - \sum_{i\in\mathcal{O}_j} c_i X_j^{N_k}(r) X_i^{N_k}(r) \Big] dr,$$

and prove that the functions X_j are continuously differentiable and satisfy system (2.2) with initial condition X_j^0. Continuation from an arbitrary bounded time interval to all $t \geq 0$ is obvious. Finally, X is a positive l^2 solution by Proposition 2.1. $\square$

We can also state a stronger version of the same result, holding not only for positive solutions, when we are in the inviscid $\nu = 0$ and unforced $f = 0$ case.

Theorem 2.4. *For every $X^0 \in l^2$ there exists at least one finite energy solution of* (2.1)*, with initial conditions $X(0) = X^0$ and such that*

$$\sum_{j \in J} X_j^2(t) \leq \sum_{j \in J} X_j^2(s) \qquad \forall\, 0 \leq s \leq t.$$

The proof of this theorem is classical, via Galerkin approximations (analogue to the previous one). We provide just a sketch of the proof.

Proof. We introduce the same Galerkin approximations system as in the proof of Theorem 2.3. We have that, as above,

$$\sum_{|j| \leq N} X_j^2(t) = \sum_{|j| \leq N} X_j^2(0), \qquad \forall t \geq 0,\ \forall N \geq 0,$$

and since the initial condition is in l^2, we have for the solution X^N

$$\sum_{|j| \leq N} (X_j^N(t))^2 \leq |X^0|^2, \qquad \forall t \geq 0,\ \forall N \geq 0,$$

so we can estimate each component as follows:

$$\|X_j^N(t)\| \leq |X^0| \forall t \geq 0,\ \forall N \geq 0,\ \forall |j| \leq N.$$

Now we can proceed as in the previous proof, and extract a uniformly convergent subsequence, which will provide us with a solution. □

We conclude the section on elementary results by collecting a useful estimate on the energy transfer and a statement clarifying that all components are strictly positive for $t > 0$.

Proposition 2.5. *Let X be a solution of* (2.2)*. The following properties hold:*

1. *If $f = 0$, for all $n \geq -1$*

$$2 \int_0^{+\infty} \sum_{|k|=n+1} c_k X_{\bar{k}}^2(s) X_k(s) ds \leq \mathcal{E}_n(0); \tag{2.12}$$

2. *if $X_j^0 > 0$ for all j s.t. $|j| = M$ for some $M \geq 0$, then $X_j(t) > 0$ for every j s.t. $|j| \geq M$ and all $t > 0$.*

Proof. 1. If $n = -1$ the inequality is trivially true. If $n \geq 0$, by integrating equation (2.7) with $f = 0$, we find that

$$\mathcal{E}_n(t) + 2\nu \int_0^t \sum_{|j| \leq n} d_j X_j^2(s) ds = \mathcal{E}_n(0) - 2 \int_0^t \sum_{|k|=n+1} c_k X_{\bar{k}}^2(s) X_k(s) ds.$$

The left hand side is non-negative for all t, so taking the limit for $t \to \infty$ in the right hand side completes the proof.

2. For $|j| = M$ we have from (2.9)

$$X_j(t) \geq X_j^0 e^{-\int_0^t (\nu d_j + \sum_k c_k X_k(r))dr} > 0.$$

Now suppose that for some $j \in J \setminus \{0\}$, $X_{\bar{j}}(t) > 0$ for every $t > 0$. Then again by (2.9),

$$X_j(t) \geq \int_0^t c_j X_{\bar{j}}^2(s) e^{-\int_s^t (\nu d_j + \sum_{k \in \mathcal{O}_j} c_k X_k(r))dr} ds > 0.$$

By induction on $|j| \geq M$ we have our thesis. □

2.4 Relationship with classic dyadic model

We stated that the tree dyadic model is a generalization of the classic dyadic model presented in Chapter A. We want now to address the question of how the two models are related to each other.

Let's recall the differential equations for the tree and classic dyadic models.

$$\begin{cases} X_{\bar{0}}(t) \equiv f \\ \dfrac{d}{dt} X_j = -\nu d_j X_j + c_j X_{\bar{j}}^2 - \displaystyle\sum_{k \in \mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J, \end{cases} \tag{2.13}$$

$$\begin{cases} Y_{-1}(t) \equiv f \\ \dfrac{d}{dt} Y_n = -\nu l_n Y_n + k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1}, \quad \forall n \geq 0, \end{cases} \tag{2.14}$$

where $f \geq 0$, $\nu \geq 0$ and for all $n \in \mathbb{N}$ and $j \in J$,

$$c_j = 2^{\alpha|j|}, \quad k_n = 2^{\beta n}, \quad d_j = 2^{\gamma|j|}, \quad l_n = 2^{\gamma n}.$$

Again we assume that $\sharp \mathcal{O}_j = N_* = 2^{2\tilde{\alpha}}$ for all $j \in J$, but we stress that for this section this is a fundamental hypothesis and not a technical one. Also the restriction to the physical exponential coefficients is relevant.

The following proposition shows that examples of solutions of the tree dyadic model (2.13) can be obtained by lifting the solutions of the classic dyadic model (2.14).

Proposition 2.6. *If Y is a componentwise (resp. l^2) solution of* (2.14)*, then $X_j(t) := 2^{-(|j|+2)\tilde{\alpha}} Y_{|j|}(t)$ is a componentwise (resp. l^2) solution of* (2.13) *with $\alpha = \beta + \tilde{\alpha}$. If Y is positive, so is X.*

Proof. A direct computation shows that X is a componentwise solution. Then observe that, for any $n \geq 0$,

$$\sum_{|j|=n} X_j^2 = 2^{2\tilde{\alpha}n} X_j^2 = 2^{2\tilde{\alpha}n} 2^{-(2n+4)\tilde{\alpha}} Y_n^2 = 2^{4\tilde{\alpha}} Y_n^2,$$

so

$$\mathcal{E}_n = \sum_{|j|\leq n} X_j^2 = \sum_{k\leq n} 2^{4\tilde{\alpha}} Y_k^2 \leq 2^{4\tilde{\alpha}} \|Y\|^2.$$

Positivity is obvious. □

Remark 2.1. If we consider α fixed, since $\beta = \alpha - \tilde{\alpha}$, for small values of N_* we'll have larger values of β, and the other way around. That is to say, the less offspring every node has, the faster the dynamics will be.

Remark 2.2. Let us stress that $\beta > 0$ when $N_* < 2^{2\alpha}$. Since the behavior of the solutions of (2.14) is strongly related to the sign of β, then the behavior of the solutions of (2.13) is strongly connected to the sign of $\alpha - \tilde{\alpha}$. For example, in the classic dyadic there is anomalous dissipation if and only if $\beta > 0$, and hence in the tree dyadic there will be lifted solutions with anomalous dissipation when $\alpha > \tilde{\alpha}$ and lifted solutions which are conservative when $\alpha \leq \tilde{\alpha}$.

2.5 Partial superposition principle

The lifting proposition is a way of constructing solutions. Another one uses a different property of the tree dyadic model: the partial superposition principle. This might sound strange, as the system is nonlinear, and in fact is not an actual superposition principle: we can't just add two solutions and get a new one, but we have to be subtler.

If we consider a positive solution of the inviscid and unforced tree model under some conditions on the coefficients that we will address below, we can traslate and multiply the solution for a suitable coefficient and get a similar solution rooted in another node of the tree. It requires that all the ancestors of such node j have the corresponding X set to 0, but leaves us freedom with regard to disjoint subtrees. In particular we could take the same original solution and move it to another node such that its subtree is disjoint with node j's subtree.

So we go deeper on the conditions on the coefficients. We need the coefficients c_j to be positive, depend only on the generation and be such that $c_{m+1}/c_{n+1} = c_m/c_n$ for any $n \geq m \geq 0$. Actually we can weaken it a little, by asking the coefficients to have a term depending on the generation satisfying the condition just stated and a term of distribution among children of a node, like in Chapter 6 (but then we may have a restriction on the node we can root it to).

If we move the root of the solution, meaning the first node with nonzero intensity from node i in generation m to node j in generation n we have $X_j = \beta X_i$, with $\beta = \sqrt{c_m/c_n}$. It is worth noting that, for the exponential choice of coefficients $c_j = 2^{\alpha|j|}$, the coefficient β is smaller than 1, so we are shrinking the solution, while translating it to the right.

Let us stress that this construction does not need such a strong assumption on coefficients, as the lifting proposition did. This "kind of" superposition is possible in this nonlinear model because of the tree structure and the lack of back-propagation, thanks to the positive coefficients. And the very close range interactions.

2.6 Generator

If we consider the unforced and inviscid model, we can formally compute the generator. We have

$$dX_j = \left(c_j X_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k \right) dt$$

and we consider a function $f : \mathbb{R}^J \to \mathbb{R}$ sufficiently regular and the corresponding semigroup

$$(S_t f)(x) = f(X^x(t)),$$

where $X^x(t)$ is a solution of the model with initial condition $X(0) = x$.

Now our generator will be

$$(\mathcal{L}f)(x) = \frac{d}{dt} f(X^x(t)) \bigg|_{t=0}. \tag{2.15}$$

We have

$$f'(X^x(t)) = \nabla f(X^x(t)) \cdot \frac{d}{dt} X^x(t),$$

so we can substitute in (2.15) and get, component by component,

$$(\mathcal{L}f)(x) = \sum_{j \in J} \partial_j f(x) \cdot \left(c_j x_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k x_j x_k \right). \tag{2.16}$$

Now we can get rid of the f and write just $\mathcal{L}$, from (2.16):

$$\begin{aligned} \mathcal{L} &= \sum_{j \in J} c_j x_{\bar{j}}^2 \partial_j - \sum_{j \in J} \sum_{k \in \mathcal{O}_j} c_k x_j x_k \partial_j \\ &= \sum_{j \in J} c_j x_{\bar{j}}^2 \partial_j - \sum_{k \in J \setminus \{0\}} c_k x_{\bar{k}} x_k \partial_k \\ &= \sum_{j \in J} c_j x_{\bar{j}} (x_{\bar{j}} \partial_j - x_j \partial_{\bar{j}}) + c_0 x_{\bar{0}} x_0 \partial_0 \\ &= \sum_{j \in J} c_j x_{\bar{j}} (x_{\bar{j}} \partial_j - x_j \partial_{\bar{j}}), \end{aligned} \tag{2.17}$$

where in the last equality we used the hypothesis that the model is unforced, hence $x_{\bar{0}} = 0$, otherwise there would be another term.

We can check that the formal preservation of energy can be immediately derived from the generator: it is enough to apply it to the sum of the squares of the intensities.

2.7 Regular solutions

In this section we would like to inquire what happens if we choose solutions that are more regular than those explored previously. This is based on [32], which in turn refine ideas from [28] and [31]. We are here considering the model in the inviscid and unforced (2.1) formulation with the "physical" exponential coefficients presented in Section 2.2.

We introduce a suitable Sobolev space H^s as

$$H^s = \left\{ X \text{ s.t. } \|X\|^2_{H^s} = \sum_{j\in J} 2^{2\alpha|j|s}|X_j|^2 < +\infty \right\}.$$

The first ingredient we need is Picard fixed point theorem.

Theorem 2.7 (Picard)**.** *Let S be a Banach space, $B : S \times S \mapsto S$ a bilinear map such that for every X, Y in S*

$$\|B(X,Y)\|_S \leq c\|X\|_S\|Y\|_S. \tag{2.18}$$

Then for any $X_0 \in S$ satisfying $4c\|X_0\|_S < 1$, the equation

$$X = X_0 + B(X,X)$$

has a unique solution $X \in S$ such that $\|X\|_S \leq \frac{1}{2c}$.

Theorem 2.8. *For every initial condition $X(0) \in H^s$, $s \geq 1$ of the (inviscid and unforced) tree dyadic model (2.1), there exists a time $T = T(\|X(0)\|_{H^s}) > 0$ such that there is a unique solution $X(t)$ of (2.1) in the space $\mathcal{C}([0,T];H^s)$.*

Moreover the H^s norm of this solution satisfies

$$\|X(t)\|_{H^s} \leq \|X(0)\|_{H^s} e^{C\int_0^t \sum_{j\in J}(2^{\alpha|j|}X_j(r))dr}. \tag{2.19}$$

In particular the solution blows up in a finite time τ if

$$\int_0^\tau \sup_{j\in J}(2^{\alpha|j|}X_j(r))dr = +\infty.$$

Proof. Let's define, for every t, $Z_0(t) = X(0)$,

$$b(Z,Y)_j(t) = 2^{\alpha|j|}Z_{\bar{j}}(t)Y_{\bar{j}}(t) - 2^{\alpha(|j|+1)}\sum_{k\in\mathcal{O}_j} Z_j(t)Y_k(t),$$

and

$$B(Z,Y)_j(T) = \int_0^T b(Z,Y)_j(t)dt.$$

All we have to do in order to prove the existence and uniqueness of the solution is to show that the bound (2.18) holds for B, then Picard's Theorem 2.7 will kick in. Recalling that $s \geq 1$ we have:

$$\begin{aligned}
\|b(Z,Y)(t)\|_{H^s}^2 &= \sum_{j\in J} 2^{2\alpha|j|s}\Big(2^{\alpha|j|}Z_{\bar{j}}(t)Y_{\bar{j}}(t) - 2^{\alpha(|j|+1)}Z_j(t)\sum_{k\in\mathcal{O}_j} Y_k(t)\Big)^2 \\
&\leq \sum_{j\in J} 2^{2\alpha|j|s}\big(2^{N_*}+1\big)\Big(2^{2\alpha|j|}Z_{\bar{j}}^2(t)Y_{\bar{j}}^2(t) - 2^{2\alpha(|j|+1)}Z_j^2(t)\sum_{k\in\mathcal{O}_j} Y_k^2(t)\Big) \\
&= \big(2^{N_*}+1\big)\Big(\sum_j \in J 2^{2\alpha(|j|-1)s}Z_{\bar{j}}^2(t)\cdot 2^{2\alpha(|j|-1)}Y_{\bar{j}}^2(t)\cdot 2^{2\alpha s}2^{2\alpha} \\
&\quad + \sum_{j\in J} 2^{2\alpha|j|s}Z_j^2(t)\cdot 2^{2\alpha(|j|+1)}\sum_{k\in\mathcal{O}_j} Y_k^2(t)\Big) \\
&\leq \big(2^{N_*}+1\big)2^{2\alpha s}\Big(2^{2\alpha}\sum_{j\in J} 2^{2\alpha|j|s}Z_j^2(t)\cdot 2^{2\alpha|j|s}Y_j^2(t) \\
&\quad + \sum_{j\in J} 2^{2\alpha|j|s}Z_j^2(t)\cdot\sum_{k\in\mathcal{O}_j} 2^{2\alpha|k|s}Y_k^2(t)\Big) \\
&\leq \big(2^{N_*}+1\big)2^{2\alpha s}\big(2^{2\alpha}\|Z(t)\|_{H^s}^2\|Y(t)\|_{H^s}^2 + \|Z(t)\|_{H^s}^2\|Y(t)\|_{H^s}^2\big) \\
&\leq \big(2^{N_*}+1\big)2^{2\alpha s}\big(2^{2\alpha}+1\big)\|Z(t)\|_{H^s}^2\|Y(t)\|_{H^s}^2.
\end{aligned}$$

Now we can estimate

$$\begin{aligned}
\|B(Z,Y)\|_{\mathcal{C}([0,T];H^s)} &\leq C(N_*,\alpha,s)\int_0^T \|Z(t)\|_{H^s}\|Y(t)\|_{H^s}dt \\
&\leq C(N_*,\alpha,s)\cdot T\cdot\|Z(t)\|_{\mathcal{C}([0,T];H^s)}\|Y(t)\|_{\mathcal{C}([0,T];H^s)},
\end{aligned}$$

which gives existence and uniqueness, if we take T small enough.

Now we have to prove the bound (2.19). We have

$$\begin{aligned}
\frac{d}{dt}\sum_{j\in J} 2^{2\alpha|j|s}X_j^2(t) &= \sum_{j\in J} 2\cdot 2^{2\alpha|j|s}X_j(t)\frac{d}{dt}X_j(t) \\
&= \sum_{j\in J} 2\cdot 2^{2\alpha|j|s}\Big(2^{\alpha|j|}X_{\bar{j}}^2X_j - 2^{\alpha(|j|+1)}X_j^2\sum_{k\in\mathcal{O}_j} X_k\Big) \\
&= 2\sum_{j\in J}\Big(2^{2\alpha|j|s}X_{\bar{j}}^2\cdot 2^{\alpha|j|}X_j - 2^{2\alpha|j|s}X_j^2\sum_{k\in\mathcal{O}_j} 2^{\alpha|k|}X_k\Big) \\
&\leq 2\cdot\sup_{j\in J}\Big(2^{\alpha|j|}X_j(t)\Big)\sum_{j\in J}\Big(2^{2\alpha|j|s}X_{\bar{j}}^2(t) + 2^{N_*}\cdot 2^{2\alpha|j|s}X_j^2(t)\Big) \\
&\leq C\cdot\sup_{j\in J}\Big(2^{\alpha|j|}X_j(t)\Big)\sum_{j\in J} 2^{2\alpha|j|s}X_j^2(t),
\end{aligned}$$

and by Grönwall inequality we have (2.19). □

Chapter 3

Uniqueness through noise

In Chapter 2 we proved that there exist solutions for the tree dyadic model, but we didn't address the problem of uniqueness, which is the subject of this chapter. We will show with a counterexample using self-similar solutions, that there is no uniqueness of the solutions for the inviscid and unforced tree dyadic model, in Section 3.2. To overcome this we will introduce the stochastic tree dyadic model, a perturbation of the deterministic one, in Section 3.3 and prove uniqueness for the solution of this model in a suitable space.

The idea of a stochastic perturbation of a deterministic model is well established in the literature, see [5] for the classical dyadic model, but also [15], [13] for different models. The results in this chapter, in particular, can be seen as a generalization to the dyadic tree model of the results proven for the classic dyadic model in [4] and recapped in Chapter A, but the proof of uniqueness given here relies of a new, different approach based on a general abstract property instead of a trick. The $q-$matrix we rely on is closely related to an infinitesimal generator, so the technique is valid for a larger class of models.

When dealing with uniqueness of solutions in stochastic shell models, the inviscid case we study is more difficult than the viscous one, because of the strong impact that a dissipating term has on the solution, see for example [3] about GOY models. For this reason throughout this chapter we'll consider the tree dyadic model in its unforced ($f = 0$) and inviscid ($\nu = 0$) version:

$$\begin{cases} \dfrac{d}{dt}X_j = c_j X_{\bar{j}}^2 - \displaystyle\sum_{k\in\mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J \\ X_{\bar{0}}(t) \equiv 0. \end{cases} \tag{3.1}$$

3.1 Self-similar solutions

We devote the first section of this chapter to prove the existence of self-similar solutions, which will provide an easy counterexample to the uniqueness of solution. We call *self-similar* any solution X of system (3.1) of the form $X_j(t) = a_j\varphi(t)$, for all j and all $t \geq 0$. By substituting this formula inside (3.1) it is easy to show that any such solution must be of the form

$$X_j(t) = \frac{a_j}{t - t_0},$$

for some $t_0 < 0$. The condition on the coefficients a_j is much more complicated

$$\begin{cases} a_{\bar{0}} = 0 \\ a_j + c_j a_{\bar{j}}^2 = \sum_{k \in \mathcal{O}_j} c_k a_j a_k, \qquad \forall j \in J, \end{cases}$$

so we base instead our argument upon Theorem A.6, where it is proven existence and some kind of uniqueness of self-similar solution. We obtain the following statement.

Proposition 3.1. *Given* $t_0 < 0$ *there exists at least one self-similar positive* l^2 *solution of* (3.1) *with* $a_0 > 0$.

Proof. We use Theorem A.6 which, translated in the notation used here, states that there exists a unique sequence of non-negative real numbers $(b_n)_{n \geq 0}$ such that $b_0 > 0$ and $Y_n := b_n/(t - t_0)$ is a positive l^2 solution of the unforced inviscid classic dyadic. Thanks to Proposition 2.6 this solution may be lifted to a solution of the inviscid tree dyadic (3.1) with the required features. □

Remark 3.1. For the tree dyadic model self-similar solutions are many. In the classic dyadic model (the already cited Theorem A.6) it is shown that given $t_0 < 0$ and $n_0 \geq 1$ there is only one l^2 self-similar solution such that n_0 is the index of the first non-zero coefficient. If $n_0 > 1$, this solution can be lifted on the tree to a self-similar solution which is zero on the first $n_0 - 1$ generations. We can then define a new self-similar solution which is equal to this one on one of the subtrees starting at generation n_0 and zero everywhere else. Finally, we can combine many of these solutions, even with different n_0, as long as t_0 is the same for all and their subtrees do not overlap, as stated more in general in Section 2.5.

3.2 Non-uniqueness in the deterministic case

In [6] it has been proven that there exist examples of non uniqueness of l^2 solutions for the dyadic model if we consider solutions of the form $Y_n(t) = \dfrac{a_n}{t - t_0}$, called self-similar solutions. Thanks to the lifting result (Proposition 2.6) that is enough to obtain two different solutions of the dyadic tree model, with the same initial conditions. Given the construction of self-similar solutions for the tree dyadic model in the previous section, we can also construct a direct counterexample to uniqueness of solutions.

Now we recall the time reversing technique. We may consider the system (3.1) for $t \leq 0$: given a solution $X(t)$ of this system for $t \geq 0$, we can define $\widehat{X}(t) = -X(-t)$, which is a solution for $t \leq 0$, since

$$\frac{d}{dt}\widehat{X}_j(t) = \frac{d}{dt}X_j(-t) = c_j X_{\bar{j}}^2(-t) - \sum_{k\in\mathcal{O}_j} c_k X_j(-t)X_k(-t)$$
$$= c_j \widehat{X}_{\bar{j}}^2(-t) - \sum_{k\in\mathcal{O}_j} c_k \widehat{X}_j(-t)\widehat{X}_k(-t).$$

We can consider in particular the self similar solutions for the tree dyadic model, $X_j(t) = \dfrac{a_j}{t-t_0}$, defined for $t > t_0$ and time-reverse them. This way we define

$$\widehat{X}_j(t) = -X_j(-t) \qquad \forall j \in J \qquad t < -t_0,$$

which, as we pointed out earlier, is a solution of (3.1) in $(-\infty, -t_0)$, con $-t_0 > 0$. Since

$$\lim_{t\to+\infty} |X_j(t)| = 0 \quad \text{and} \quad \lim_{t\to t_0^+} |X_j(t)| = +\infty, \ \forall j \in J$$

we have

$$\lim_{t\to-\infty} \left|\widehat{X}_j(t)\right| = 0 \quad \text{and} \quad \lim_{t\to -t_0^-} \left|\widehat{X}_j(t)\right| = +\infty, \ \forall j \in J.$$

Thanks to theorem 2.4 there is a solution $\widetilde{X}$, with initial conditions $X^0 = \widehat{X}(0)$, and this solution is a finite energy one, so, in particular, doesn't blow up in $-t_0$. Yet it has the same initial conditions of $\widehat{X}$, so we can conclude that there is no uniqueness of solutions in the deterministic case.

3.3 Stochastic model

In order to overcome this lack of uniqueness we rely on the regularization by noise technique, and consider a random perturbation of our model:

$$dX_j = \left(c_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k\right)dt + \sigma c_j X_{\bar{j}} \circ dW_j - \sigma \sum_{k\in\mathcal{O}_j} c_k X_k \circ dW_k, \tag{3.2}$$

with $(W_j)_{j\in J}$ a sequence of independent Brownian motions. We also assume deterministic initial conditions for (3.2): $X(0) = X^0 = \left(X_j^0\right)_{j\in J} \in l^2$.

It is worth noting that the form of the noise may seem unexpected: one could think that the stochastic part would mirror the deterministic one, which is not the case here, since there is a j-indexed Brownian component where we'd expect a $\bar{j}$ one, and there is a k-indexed one instead of a j one. At the same time one could argue that this is not the only possible random perturbation, but a very specific one. It is true: on one hand we chose a multiplicative noise, instead of an additive one, mainly for technical reasons (see for example [27]), on the other hand this specific form of the noise is the one we need to maintain a formal conservation of the energy, as we have in the deterministic case (see Chapter 2). If we use Itô formula to calculate

$$\begin{aligned}\frac{1}{2}dX_j^2 &= X_j \circ dX_j \\ &= \left(c_j X_{\bar{j}}^2 X_j - \sum_{k\in\mathcal{O}_j} c_k X_j^2 X_k\right)dt \\ &\quad + \sigma c_j X_{\bar{j}} X_j \circ dW_j - \sigma \sum_{k\in\mathcal{O}_j} c_k X_j X_k \circ dW_k,\end{aligned}$$

we can sum *formally* on the first $n+1$ generations, taking $X_0(t) = 0$:

$$\begin{aligned}\sum_{|j|=0}^{n} \frac{1}{2}dX_j^2 &= -\sum_{|j|=n}\sum_{k\in\mathcal{O}_j}\left[c_k X_j^2 X_k dt + \sigma c_k X_j X_k \circ dW_k\right] \\ &= -\sum_{|j|=n}\sum_{k\in\mathcal{O}_j} c_k X_j X_k (X_j dt + \sigma \circ dW_k),\end{aligned}$$

since the series is telescoping in both the drift and the diffusion parts independently. That means we have P-a.s. the formal conservation of energy, if we define the energy as

$$\mathcal{E}_n(t) = \sum_{|j|\leq n} X_j^2(t) \qquad \mathcal{E}(t) = \sum_{j\in J} X_j^2(t) = \lim_{n\to\infty} \mathcal{E}_n(t).$$

Remark 3.2. A final note on the noise form, in (3.2) the parameter $\sigma \neq 0$ is inserted just to stress the open problem of the zero noise limit, for $\sigma \to 0$. This has provided an interesting selection result for simple examples of linear transport equations (see [2]), but it is nontrivial in general, and in particular in our nonlinear setting, due to the singularity that arises with the Girsanov transform, for example in (3.12).

We can also write the infinite dimensional system (3.2) in Itô formulation:

$$\begin{aligned}dX_j = &\left(c_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k\right)dt + \sigma c_j X_{\bar{j}} dW_j \\ &- \sigma \sum_{k\in\mathcal{O}_j} c_k X_k dW_k - \frac{\sigma^2}{2}\left(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\right) X_j dt. \qquad (3.3)\end{aligned}$$

We will use this formulation since it's easier to handle the calculations, while all results can also be stated in the Stratonovich formulation.

3.3.1 Generator

We can use the Itô formulation to compute formally the generator for the stochastic model, as we did in 2.6 for the deterministic one. We have, dropping the σ:

$$dX_j = \left(c_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k - \frac{1}{2}\left(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2 \right) X_j \right) dt \\ + c_j X_{\bar{j}} dW_j - \sum_{k\in\mathcal{O}_j} c_k X_k dW_k,$$

and we can consider again a function $f : \mathbb{R}^J \to \mathbb{R}$ sufficiently regular, obtaining the corresponding semigroup

$$(S_t f)(x) = Ef(X^x(t)),$$

assuming deterministic initial conditions $X(0) = X^0$. As we did in the deterministic case, we now write, invoking the Itô formula,

$$df(X^x(t)) = \sum_{j\in J} \partial_j f(X^x(t)) dX_j + \frac{1}{2} \sum_{j,l\in J} \partial_j \partial_l f(X^x(t)) d\langle X_j, X_l \rangle_t. \tag{3.4}$$

We compute the quadratic variation

$$d\langle X_j, X_l \rangle_t = \begin{cases} \left(c_j^2 X_{\bar{j}}^2 + \sum_{k\in\mathcal{O}_j} c_k^2 X_k^2 \right) dt & j = l \\ -c_l^2 X_j X_l dt & j = \bar{l} \\ -c_j^2 X_j X_l dt & j \in \mathcal{O}_l \\ 0 & \text{otherwise} \end{cases}$$

ad we substitute it in (3.4), together with the Itô expression of dX_j:

$$
\begin{aligned}
df(X^x(t)) = &\sum_{j\in J} \partial_j f(X^x(t)) c_j X_{\bar{j}}^2 dt - \sum_{j\in J} \partial_{\bar{j}} f(X^x(t)) c_j X_{\bar{j}} X_j dt \\
&- \frac{1}{2}\sum_{j\in J} \partial_j f(X^x(t)) \Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2 \Big) X_j dt + \sum_{j\in J} \partial_j f(X^x(t)) c_j X_{\bar{j}} dW_j \\
&- \sum_{j\in J} \partial_{\bar{j}} f(X^x(t)) c_j X_j dW_j - \sum_{j\in J} \partial_{\bar{j}}\partial_j f(X^x(t)) c_j^2 X_{\bar{j}} X_j dt \\
&+ \frac{1}{2}\sum_{j\in J} \partial_j^2 f(X^x(t)) \Big(c_j^2 X_{\bar{j}}^2 + \sum_{k\in\mathcal{O}_j} c_k^2 X_k^2 \Big) dt \\
&= \sum_{j\in J} c_j X_{\bar{j}} (X_{\bar{j}}\partial_j f - X_j \partial_{\bar{j}} f) dt + \sum_{j\in J} c_j (X_{\bar{j}}\partial_j f - X_j \partial_{\bar{j}} f) dW_j \\
&+ \frac{1}{2}\sum_{j\in J} c_j^2 (X_{\bar{j}}^2 \partial_j^2 f + X_j^2 \partial_{\bar{j}}^2 f) - \sum_{j\in J} \partial_{\bar{j}}\partial_j f(X^x(t)) c_j^2 X_{\bar{j}} X_j dt \\
&- \frac{1}{2}\sum_{j\in J} c_j^2 (X_j \partial_j f + X_{\bar{j}} \partial_{\bar{j}} f) dt \\
&= \sum_{j\in J} c_j X_{\bar{j}} (X_{\bar{j}}\partial_j - X_j \partial_{\bar{j}}) f dt + \sum_{j\in J} c_j (X_{\bar{j}}\partial_j - X_j \partial_{\bar{j}}) f dW_j \\
&+ \frac{1}{2}\sum_{j\in J} c_j^2 (X_{\bar{j}}\partial_j - X_j \partial_{\bar{j}})^2 f dt.
\end{aligned}
$$

This is already enough to see that the energy is formally preserved, since for $f(x) = \sum x^2$, the terms $(X_{\bar{j}}\partial_j - X_j\partial_{\bar{j}})$ make sure that all the terms cancel out. To complete the calculations we should now consider it at $t = 0$, where $X_j = x_j$, and take the expectation. If we do so and also drop the f to have it in a general form as we did in the deterministic case, we get

$$
\mathcal{L} = \sum_{j\in J} c_j x_{\bar{j}} (x_{\bar{j}}\partial_j - x_j\partial_{\bar{j}}) + \frac{1}{2}\sum_{j\in J} c_j^2 (x_{\bar{j}}\partial_j - x_j\partial_{\bar{j}})^2,
$$

where the terms in dW_j have been canceled out by taking the expectation.

3.3.2 Existence of solutions

Let's now introduce the definition of weak solution. A filtered probability space $(\Omega, \mathcal{F}_t, P)$ is a probability space $(\Omega, \mathcal{F}_\infty, P)$ together with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_\infty$ is the σ-algebra generated by $\bigcup_{t\geq 0} \mathcal{F}_t$.

Definition 3.1. Given $x \in l^2$, a weak solution of (3.2) in l^2 is a filtered probability space $(\Omega, \mathcal{F}_t, P)$, a J-indexed sequence of independent Brownian motions $(W_j)_{j\in J}$ on

$(\Omega, \mathcal{F}_t, P)$ and an l^2-valued process $(X_j)_{j\in J}$ on $(\Omega, \mathcal{F}_t, P)$ with continuous adapted components X_j such that

$$\begin{aligned} X_j = x_j + \int_0^t \Big[c_j X_{\bar{j}}^2(s) - \sum_{k\in\mathcal{O}_j} c_k X_j(s) X_k(s) \Big] ds \\ + \int_0^t \sigma c_j X_{\bar{j}}(s) dW_j(s) - \sum_{k\in\mathcal{O}_j} \int_0^t \sigma c_k X_k(s) dW_k(s) \\ - \frac{\sigma^2}{2} \int_0^t \Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2 \Big) X_j(s) ds, \end{aligned} \tag{3.5}$$

for every $j \in J$, with $c_0 = 0$ and $X_0(t) = 0$. We will denote this solution by

$$(\Omega, \mathcal{F}_t, P, W, X),$$

or simply by X.

Definition 3.2. A weak solution is an energy controlled solution if it is a solution as in Definition 3.1 and it satisfies

$$P\Big(\sum_{j\in J} X_j^2(t) \le \sum_{j\in J} x_j^2 \Big) = 1,$$

for all $t \ge 0$.

Theorem 3.2. *There exists an energy controlled solution to* (3.3) *in* $L^\infty(\Omega\times[0,T], l^2)$ *for deterministic initial conditions* $(X_j^0) \in l^2$.

We will give a proof of this Theorem at the end of Section 3.6. It is a weak existence result and uses the Girsanov transform.

We'll prove in the following result that a process satisfying (3.3) satisfies (3.2) too.

Proposition 3.3. *If X is a weak solution, for every $j \in J$ the process $(X_j(t))_{t\ge 0}$ is a continuous semimartingale, so the following equalities hold:*

$$\int_0^t \sigma c_j X_{\bar{j}}(s) \circ dW_j(s) = \int_0^t \sigma c_j X_{\bar{j}}(s) dW_j(s) - \frac{\sigma^2}{2} \int_0^t c_j^2 X_j(s) ds$$

$$\begin{aligned} \int_0^t \sigma \sum_{k\in\mathcal{O}_j} c_k X_k(s) \circ dW_k(s) = \sum_{k\in\mathcal{O}_j} \int_0^t \sigma c_k X_k(s) dW_k(s) \\ + \frac{\sigma^2}{2} \int_0^t \sum_{k\in\mathcal{O}_j} c_k^2 X_j(s) ds, \end{aligned}$$

where the Stratonovich integrals are well defined. So X satisfies the Stratonovich formulation of the problem (3.2).

Proof. We know that

$$\int_0^t \sigma c_j X_{\bar{j}}(s) \circ dW_j(s) = \int_0^t \sigma c_j X_{\bar{j}}(s) dW_j(s) + \frac{\sigma c_j}{2}[X_{\bar{j}}, W_j]_t,$$

but from (3.2) we have that the only contribution to $[X_{\bar{j}}, W_j]$ is given by the $-\sigma c_j X_j \circ dW_j$ term, so

$$\frac{\sigma c_j}{2}[X_{\bar{j}}, W_j]_t = \frac{\sigma c_j}{2}[-\int_0^t \sigma c_j X_j \circ dW_j, W_j]_t = -\frac{\sigma^2 c_j^2}{2}\int_0^t X_j ds.$$

Now if we consider the other integral, we have

$$\int_0^t \sigma \sum_{k\in\mathcal{O}_j} c_k X_k(s) \circ dW_k(s) = \sum_{k\in\mathcal{O}_j}\int_0^t \sigma c_k X_k(s) dW_k(s) + \sum_{k\in\mathcal{O}_j}\frac{\sigma c_k}{2}[X_k, W_k]_t.$$

For each X_k we get, with the same computations, that the only contribution to $[X_k, W_k]_t$ comes from the term $\sigma c_k X_j \circ dW_k$, so that we get

$$\frac{\sigma c_k}{2}[X_k, W_k]_t = \frac{\sigma c_k}{2}\left[\int_0^t \sigma c_k X_j \circ dW_k, W_k\right]_t = \frac{\sigma^2 c_k^2}{2}\int_0^t X_j ds.$$ □

3.4 Girsanov transform

Let's consider (3.3) and rewrite it as

$$dX_j = c_j X_{\bar{j}}(X_{\bar{j}}dt + \sigma dW_j) - \sum_{k\in\mathcal{O}_j} c_k X_k(X_j dt + \sigma dW_k) - \frac{\sigma^2}{2}\left(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\right)X_j dt. \tag{3.6}$$

The idea is to isolate $X_{\bar{j}}dt + \sigma dW_j$ and prove through Girsanov theorem that they are Brownian motions with respect to a new measure $\widehat{P}$ in $(\Omega, \mathcal{F}_\infty)$, simultaneously for every $j \in J$. This way (3.3) becomes a system of linear SDEs under the new measure $\widehat{P}$. The infinite dimensional version of Girsanov theorem can be found in [23] and [26].

Remark 3.3. We can obtain the same result under Stratonovich formulation.

Let X be an energy controlled solution: its energy $\mathcal{E}(t)$ is bounded, so we can define the process

$$M_t = -\frac{1}{\sigma}\sum_{j\in J}\int_0^t X_{\bar{j}}(s) dW_j(s) \tag{3.7}$$

which is a martingale. Its quadratic variation is

$$[M, M]_t = \frac{1}{\sigma^2}\int_0^t \sum_{j\in J} X_{\bar{j}}^2(s) ds.$$

Because of the same boundedness of $\mathcal{E}(t)$ stated above, by the Novikov criterion $\exp(M_t - \frac{1}{2}[M,M]_t)$ is a (strictly) positive martingale. We now define $\widehat{P}$ on $(\Omega, \mathcal{F}_t)$ as

$$\begin{aligned}\frac{d\widehat{P}}{dP}\Big|_{\mathcal{F}_t} &= \exp\Big(M_t - \frac{1}{2}[M,M]_t\Big)\\ &= \exp\Big(-\frac{1}{\sigma}\sum_{j\in J}\int_0^t X_{\overline{j}}(s)dW_j(s) - \frac{1}{2\sigma^2}\int_0^t \sum_{j\in J} X_{\overline{j}}^2(s)ds\Big),\end{aligned} \tag{3.8}$$

for every $t \geq 0$. P and $\widehat{P}$ are equivalent on each $\mathcal{F}_t$, because of the strict positivity of the exponential.

We can now prove the following:

Theorem 3.4. *If $(\Omega, \mathcal{F}_t, P, W, X)$ is an energy controlled solution of the nonlinear equation (3.3), then $(\Omega, \mathcal{F}_t, \widehat{P}, B, X)$ satisfies the linear equation*

$$dX_j = \sigma c_j X_{\overline{j}} dB_j(t) - \sigma \sum_{k\in\mathcal{O}_j} c_k X_k dB_k(t) - \frac{\sigma^2}{2}\Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big) X_j dt, \tag{3.9}$$

where the processes

$$B_j(t) = W_j(t) + \int_0^t \frac{1}{\sigma} X_{\overline{j}}(s)ds$$

are a sequence of independent Brownian motions on $(\Omega, \mathcal{F}_t, \widehat{P})$, with $\widehat{P}$ defined by (3.8).

Proof. Now let's define

$$B_j(t) = W_j(t) + \int_0^t \frac{1}{\sigma} X_{\overline{j}}(s)ds.$$

Under $\widehat{P}$, $(B_j(t))_{j\in J, t\in[0,T]}$ is a sequence of independent Brownian motions. Since

$$\begin{aligned}\sigma\int_0^t c_j X_{\overline{j}}(s)dB_j(s) &= \sigma\int_0^t c_j X_{\overline{j}}(s)dW_j(s) + \int_0^t c_j {X_{\overline{j}}}^2(s)ds\\ \sigma\int_0^t c_k X_k(s)dB_k(s) &= \sigma\int_0^t c_k X_k(s)dW_k(s) + \int_0^t c_k X_j(s)X_k(s)ds \quad k\in\mathcal{O}_j.\end{aligned}$$

Then (3.6) can be rewritten in integral form as

$$\begin{aligned}X_j(t) = x_j + \sigma\int_0^t c_j X_{\overline{j}}(s)dB_j(s) - \sigma\sum_{k\in\mathcal{O}_j}\int_0^t c_k X_k(s)dB_k(s)\\ - \frac{\sigma^2}{2}\int_0^t\Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big)X_j(s)ds,\end{aligned} \tag{3.10}$$

which is a linear stochastic equation. □

Remark 3.4. We can write our linear equation (3.9) also in Stratonovich form:

$$dX_j = \sigma c_j X_{\overline{j}} \circ dB_j(t) - \sum_{k\in\mathcal{O}_j} \sigma c_k X_k \circ dB_k(t).$$

Remark 3.5. If we look at (3.9) we can see that it is possible to drop the σ, considering it a part of the coefficients c_j.

We can use Itô formula to calculate

$$\begin{aligned}\frac{1}{2}dX_j^2 &= X_j dX_j + \frac{1}{2}d[X_j]_t \\ &= \sigma c_j X_{\overline{j}} X_j dB_j - \sigma \sum_{k\in\mathcal{O}_j} c_k X_j X_k dB_k \\ &\quad - \frac{\sigma^2}{2}\Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big) X_j^2 dt + \frac{\sigma^2}{2}\Big(c_j^2 X_{\overline{j}}^2 + \sum_{k\in\mathcal{O}_j} c_k^2 X_k^2\Big) dt \\ &= -\frac{\sigma^2}{2}\Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big) X_j^2 dt + dN_j + \frac{\sigma^2}{2}\Big(c_j^2 X_{\overline{j}}^2 + \sum_{k\in\mathcal{O}_j} c_k^2 X_k^2\Big) dt,\end{aligned} \tag{3.11}$$

with

$$N_j(t) = \sigma \int_0^t c_j X_{\overline{j}} X_j dB_j - \sigma \sum_{k\in\mathcal{O}_j} \int_0^t c_k X_j X_k dB_k. \tag{3.12}$$

This equality will be useful in the following.

We now present an existence result also for system (3.9).

Proposition 3.5. *There exists a solution of* (3.9) *in* $L^\infty(\Omega\times[0,T], l^2)$ *with continuous components, with initial conditions* $X^0 \in l^2$.

Proof. Fix $N \geq 1$ and consider the finite dimensional stochastic linear system

$$dX_j^N = \sigma c_j X_{\overline{j}}^N dB_j(t) - \sigma \sum_{k\in\mathcal{O}_j} c_k X_k^N dB_k(t) - \frac{\sigma^2}{2}\Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big) X_j^N dt, \tag{3.13}$$

for $j \in J$ with $0 \leq |j| \leq N$, and the following boundary and initial conditions:

$$\begin{cases} X_k^N(t) \equiv 0 & k \in J,\ |k| = N+1 \\ X_j^N(0) = x_j & j \in J,\ 0 \leq |j| \leq N. \end{cases}$$

This system has a unique global strong solution $(X_j^N)_{j\in J}$. We can compute, using (3.11) and the definition of N_j in (3.12),

$$\begin{aligned}\frac{1}{2}d\Big(\sum_{|j|\le N}\big(X_j^N(t)\big)^2\Big) &= \sum_{|j|\le N}\Big(-\frac{\sigma^2}{2}\Big(c_j^2+\sum_{k\in\mathcal{O}_j}c_k^2\Big)\big(X_j^N\big)^2dt + dN_j^N \\ &\quad + \frac{\sigma^2}{2}\Big(c_j^2(X_j^N)^2+\sum_{k\in\mathcal{O}_j}c_k^2\big(X_k^N\big)^2\Big)dt\Big) \\ &= -\sum_{|j|=N}\frac{\sigma^2}{2}\sum_{k\in\mathcal{O}_j}c_k^2\big(X_j^N\big)^2dt \\ &= -\frac{\sigma^2}{2}\sum_{|k|=N+1}c_k^2\Big(X_k^N\Big)^2\le 0.\end{aligned}$$

Hence

$$\sum_{|j|\le N}\big(X_j^N(t)\big)^2\le\sum_{|j|\le N}x_j^2\le\sum_{j\in J}x_j^2 \qquad \widehat{P}-\text{a.s.}\quad \forall t\ge 0.$$

This implies that there exists a sequence $N_m\uparrow\infty$ such that $(X_j^{N_m})_{j\in J}$ converges weakly to some $(X_j)_{j\in J}$ in $L^2(\Omega\times[0,T],l^2)$ and also weakly star in $L^\infty(\Omega\times[0,T],l^2)$, so $(X_j)_{j\in J}$ is in $L^\infty(\Omega\times[0,T],l^2)$.

Now for every $N\in\mathbb{N}$, $(X_j^N)_{j\in J}$ is inProg, the subspace of progressively measurable processes in $L^2(\Omega\times[0,T],l^2)$. But Prog is strongly closed, hence weakly closed, so $(X_j)_{j\in J}\in$ Prog.

We just have to prove that $(X_j)_{j\in J}$ solves (3.9). All the one dimensional stochastic integrals that appear in each equation in (3.10) are linear strongly continuous operators Prog $\to L^2(\Omega)$, hence weakly continuous. Then we can pass to the weak limit in (3.13). Moreover from the integral equations (3.10) we have that there is a modification of the solution which is continuous in all the components. □

3.5 Closed equation for $E_{\widehat{P}}\big[X_j^2(t)\big]$

Proposition 3.6. *For every energy controlled solution X of the nonlinear equation (3.3), $E_{\widehat{P}}\Big[X_j^2(t)\Big]$ is finite for every $j\in J$ and satisfies*

$$\begin{aligned}\frac{d}{dt}E_{\widehat{P}}\big[X_j^2(t)\big] &= -\sigma^2\Big(c_j^2+\sum_{k\in\mathcal{O}_j}c_k^2\Big)E_{\widehat{P}}\big[X_j^2(t)\big] \\ &\quad + \sigma^2c_j^2E_{\widehat{P}}\big[X_{\bar{j}}^2(t)\big]+\sigma^2\sum_{k\in\mathcal{O}_j}c_k^2E_{\widehat{P}}\big[X_k^2(t)\big]. \qquad (3.14)\end{aligned}$$

Proof. Let $(\Omega,\mathcal{F}_t,P,W,X)$ be an energy controlled solution of the nonlinear equation (3.3), with initial condition $X\in l^2$ and let $\widehat{P}$ be the measure given by Theorem 3.4. Denote by $E_{\widehat{P}}$ the expectation with respect to $\widehat{P}$ in $(\Omega,\mathcal{F}_t)$.

Notice that

$$E_{\widehat{P}}\left[\int_0^T X_j^4(t)dt\right] < \infty \qquad \forall j \in J. \tag{3.15}$$

For energy controlled solutions from the definition we have that P-a.s.

$$\sum_{j\in J} X_j^4(t) \le \max_{j\in J} X_j^2(t) \sum_{j\in J} X_j^2(t) \le \Big(\sum_{j\in J} x_j^2\Big)^2,$$

because of the behavior of the energy we showed. But on every $\mathcal{F}_t$, $P \sim \widehat{P}$, so

$$\widehat{P}\bigg(\sum_{j\in J} X_j^4(t) \le \Big(\sum_{j\in J} x_j^2\Big)^2\bigg) = 1,$$

and (3.15) holds.

From (3.15) it follows that $N_j(t)$ is a martingale for every $j \in J$. Moreover

$$E_{\widehat{P}}\left[\sum_{j\in J} X_j^2(t)\right] < \infty,$$

since $X_j(t)$ is an energy controlled solution and the condition is invariant under the change of measure $P \leftrightarrow \widehat{P}$ on $\mathcal{F}_t$ and, in particular,

$$E_{\widehat{P}}\big[X_j^2(t)\big] < \infty \qquad \forall j \in J.$$

Now let's write (3.11) in integral form:

$$X_j^2(t) - x_j^2 = -\sigma^2 \int_0^t \Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big) X_j^2(s)ds \\ + 2\int_0^t dN_j(s) + \sigma^2 \int_0^t \Big(c_j^2 X_{\bar{j}}^2(s) + \sum_{k\in\mathcal{O}_j} c_k^2 X_{\bar{k}}^2(s)\Big)ds.$$

We can take the $\widehat{P}$ expectation,

$$E_{\widehat{P}}[X_j^2(t)] - x_j^2 = -\sigma^2 \int_0^t \Big(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\Big) E_{\widehat{P}}[X_j^2(s)]ds \\ + \sigma^2 \int_0^t c_j^2 E_{\widehat{P}}[X_{\bar{j}}^2(s)]ds + \sigma^2 \sum_{k\in\mathcal{O}_j} \int_0^t c_k^2 E_{\widehat{P}}[X_{\bar{k}}^2(s)]ds,$$

where the N_j term vanishes, since it's a $\widehat{P}$-martingale. Now we can derive and the proposition is established. □

It's worth stressing that $E_{\widehat{P}}[X_j^2(t)]$ satisfies a closed equation, and its coefficients have a very peculiar shape.

We can write (3.14) in matrix form; let Q be the infinite dimensional matrix which entries are defined as

$$q_{j,j} = -\sigma^2\left(c_j^2 + \sum_{k\in\mathcal{O}_j} c_k^2\right) \quad q_{\bar{\jmath},j} = \sigma^2 c_j^2 \quad q_{k,j} = \mathbb{1}_{\{k\in\mathcal{O}_j\}}\sigma^2 c_k^2 \text{ for } k \neq j, \bar{\jmath}. \tag{3.16}$$

Proposition 3.7. *The infinite matrix $Q = (q_{jl})_{j,l\in J}$ just defined is symmetric and such that*

$$0 \le q_{l,j} < +\infty \qquad \forall j \neq l \in J,$$
$$\sum_{l\neq j} q_{l,j} = q_j := -q_{j,j} < +\infty \qquad \forall j \in J.$$

Proof. First of all $q_{j,j} < 0$ for all $j \in J$ and $q_{j,l} \ge 0$ for all $j \neq l$. Then

$$q_j = \sum_{l\neq j} q_{l,j} = q_{\bar{\jmath},j} + \sum_{k\in\mathcal{O}_j} q_{k,j} = \sigma^2 c_j^2 + \sum_{k\in\mathcal{O}_j} \sigma^2 c_k^2 = -q_{j,j}.$$

Moreover it is very easy to check that the matrix is symmetric:

$$q_{l,j} = \left\{\begin{matrix} \sigma^2 c_j^2 & l = \bar{\jmath} & \Leftrightarrow & j \in \mathcal{O}_l & \sigma^2 c_j^2 \\ \sigma^2 c_l^2 & l \in \mathcal{O}_j & \Leftrightarrow & j = \bar{l} & \sigma^2 c_l^2 \end{matrix}\right\} = q_{j,l}.$$

□

Now that we have the matrix we want to write the infinite system (3.14) in matricial form:

$$E_j'(t) = \sum_{h\in J} E_h(t) q_{h,j} \tag{3.17}$$

where we used the notation $E_j(t) = E_{\hat{P}}\left[X_j^2(t)\right]$.

To prove that the solution of (3.17) is unique we need to make a little detour in the field of Markov chains in continuous time.

Remark 3.6. Matrices with properties such as the matrix Q above play a role in the field of continuous time Markov chains. They are called q-matrices and are closely related to infinitesimal generators. We can consider for a q-matrix Q the forward and backward equations, as follows

$$\begin{cases} P'(t) = P(t)Q \\ P(0) = \mathrm{Id} \end{cases} \qquad \begin{cases} P'(t) = QP(t) \\ P(0) = \mathrm{Id}. \end{cases}$$

We can write them also as infinite systems of differential equations:

$$\begin{cases} p_{i,j}'(t) = \sum_{k\in J} p_{i,k}(t) q_{k,j} \\ p_{i,j}(0) = \delta_{i,j} \end{cases} \qquad \begin{cases} p_{i,j}'(t) = \sum_{k\in J} q_{i,k} p_{k,j}(t) \\ p_{i,j}(0) = \delta_{i,j}. \end{cases} \tag{3.18}$$

Definition 3.3. A non-negative function $f_{j,l}(t)$ with $j, l \in J$ and $t \geq 0$ is a *transition function* on J if $f_{jl}(0) = \delta_{jl}$, $\lim_{t\to 0} f_{i,j}(t) = \delta_{i,j}$,

$$\sum_{l\in J} f_{jl}(t) \leq 1 \qquad \forall j \in J,\ \forall t \geq 0,$$

and it satisfies the semigroup property (or Chapman-Kolmogorov equation)

$$f_{jl}(t+s) = \sum_{h\in J} f_{jh}(t) f_{hl}(s) \qquad \forall j, l \in J,\ \forall t, s \geq 0.$$

Definition 3.4. A *q-matrix* $Q = (q_{jl})_{j,l\in J}$ is a square matrix such that

$$0 \leq q_{jl} < +\infty \qquad \forall j \neq l \in J,$$
$$\sum_{l\neq j} q_{jl} \leq -q_{jj} =: q_j \leq +\infty \qquad \forall j \in J.$$

A q-matrix is called *stable* if all q_j's are finite, and *conservative* if

$$q_j = \sum_{l\neq j} q_{jl} \qquad \forall j \in J.$$

If $Q = (q_{jl})_{j,l\in J}$ is a q-matrix, a Q-function is a transition function $f_{jl}(t)$ such that $f'_{jl}(0) = q_{jl}$.

In light of this definition, we can rephrase Proposition 3.7 as follows:

Proposition 3.8. *The infinite matrix Q defined in* (3.16) *is a stable and conservative q-matrix. Moreover Q is symmetric.*

We want now to present a very special solution of the forward and backward equations, the minimal one, which is the jump and hold process. In order to do that we need the following lemmas.

Lemma 3.9. *The following are equivalent formulations of the forward equations with generic initial conditions $y(0)$, with $y(t) \in l^1$:*

$$y'_{i,j}(t) = \sum_{k\in J} y_{i,k}(t) q_{k,j} \tag{3.19}$$

$$y_{i,j}(t) = y_{i,j}(0)e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k\neq j} y_{i,k}(t-s) q_{k,j} ds. \tag{3.20}$$

Proof. We can rearrange (3.26) and multiply it by $e^{q_j t}$:

$$y'_{i,j}(t) + q_j y_{i,j}(t) = \sum_{k\neq j} y_{i,k}(t) q_{k,j}$$

$$(e^{q_j t} y_{i,j}(t))' = e^{q_j t}(y'_{i,j}(t) + q_j y_{i,j}(t)) = e^{q_j t} \sum_{k\neq j} y_{i,k}(t) q_{k,j}$$

and we can integrate in t and change variable:

$$e^{q_j t} y_{i,j}(t) y_{i,j}(0) = \int_0^t e^{q_j s} \sum_{k \neq j} y_{i,k}(s) q_{k,j} ds$$

$$y_{i,j}(t) = y_{i,j}(0) e^{-q_j t} + \int_0^t e^{-q_j (t-s)} \sum_{k \neq j} y_{i,k}(s) q_{k.j} ds$$

$$y_{i,j}(t) = y_{i,j}(0) e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} y_{i,k}(t-s) q_{k.j} ds.$$

Conversely let $y_{i,j}(t)$ solve (3.27); we can backtrace some of the last passages above and write $y_{i,j}(t)$ as

$$y_{i,j}(t) = y_{i,j}(0) e^{-q_j t} + e^{-q_j t} \int_0^t e^{q_j s} \sum_{k \neq j} y_{i,k}(t-s) q_{k,j} ds$$

which proves the continuity in t of the $y_{i,j}$s. To conclude the equivalence, we need to differentiate in t, hence we need to prove that the sum $\sum y_{i,k}(t) q_{k,j}$ is continuous in time. We can consider for every j an increasing sequence of sets $(B_n^j)_{n \geq 1}$ such that $B_n^j \subseteq J \setminus \{j\}$ and is finite for all n, and $B_n^j \uparrow J \setminus \{j\}$. Now for every B_n^j

$$\sum_{k \in B_n^j} y_{i,k}(t) q_{k,j}$$

is continuous and

$$|\sum_{k \neq j} y_{i,k}(t) q_{k,j} - \sum_{k \in B_n^j} y_{i,k}(t) q_{k,j}| = |\sum_{\substack{k \in J \setminus B_n^j \\ k \neq j}} y_{i,k}(t) q_{k,j}| \leq \sum_{\substack{k \in J \setminus B_n^j \\ k \neq j}} q_{k,j} \to 0$$

where we used the fact $y(t) \in l^1$ and that the

$$\sum_{\substack{k \in J \setminus B_n^j \\ k \neq j}} q_{k,j} \qquad n \geq 1$$

are tail sums of $\sum_{k \neq j} q_{k,j} = q_j < +\infty$.

This allows us to conclude that the sums on the finite sets B_n^j converge uniformly to the one on the whole $J \setminus \{j\}$, so this sum is continuous too, and the two formulations are equivalent. □

With some slight adjustments the arguments in this proof work also for the backward equations.

Lemma 3.10. *The following are equivalent formulations of the backward equations with generic initial conditions $y(0)$, with $y(t) \in l^1$:*

$$y'_{i,j}(t) = \sum_{k \in J} q_{i,k} y_{k,j}(t) \tag{3.21}$$

$$y_{i,j}(t) = y_{i,j}(0) e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} q_{i,k} y_{k,j}(t-s) ds. \tag{3.22}$$

The following result will be crucial in proving the uniqueness of the solution for (3.14).

Proposition 3.11. *There is a minimal non negative Q-function that is a solution of both the forward and backward equations* (3.18)*. This solution is called* minimal semigroup of Q *or just* semigroup of Q *and is the transition function of a continuous time Markov chain $(Z_t)_{t \geq 0}$ on J, given an initial distribution $\mu = Z_0$.*

Proof. To prove this result we will use the forward and backward integral recursion, introduced by Feller and used by Anderson in this very context. First of all we give a solution of the forward equations in the (3.27) formulation with initial conditions $p_{i,j}(0) = \delta_{i,j}$:

$$p_{i,j}^{(n)}(t) = \begin{cases} \delta_{i,j} e^{-q_j t} & n = 0 \\ p_{i,j}^{(0)}(t) + \int_0^t e^{-q_j s} \sum_{k \neq j} p_{i,k}^{(n-1)}(t-s) q_{k,j} ds & n \geq 1. \end{cases} \tag{3.23}$$

We need to show by induction on n that

$$p_{i,j}^{(n+1)}(t) \geq p_{i,j}^{(n)}(t) \geq 0 \qquad \forall n \geq 0 \quad \forall t \geq 0 :$$

the first step is obvious

$$\begin{aligned} p_{i,j}^{(1)}(t) &= p_{i,j}^{(0)}(t) + \int_0^t e^{-q_j s} \sum_{k \neq j} p_{i,k}^{(0)}(t-s) q_{k,j} ds \\ &\geq p_{i,j}^{(0)}(t) \geq 0. \end{aligned}$$

Now, supposing that the inequality holds for $p_{i,j}^{(n)}$, we can show that the difference between two consecutive terms is positive:

$$p_{i,j}^{(n+1)}(t) - p_{i,j}^{(n)}(t) = \int_0^t e^{-q_j s} \sum_{k \neq j} (p_{i,k}^{(n)}(t-s) - p_{i,k}^{(n-1)}(t-s)) q_{k,j} ds \geq 0.$$

Now that this property is established, we can deduce the existence of

$$p_{i,j}(t) = \lim_{n \to \infty} p_{i,j}^{(n)}(t)$$

which satisfy the forward equations (3.26), by the monotone convergence theorem.

Now we turn our attention to the backward equations (3.22), with the same initial conditions. We construct $\pi_{i,j}^{(n)}$ as

$$\pi_{i,j}^{(n)}(t) = \begin{cases} \delta_{i,j} e^{-q_i t} & n = 0 \\ \pi_{i,j}^{(0)}(t) + \int_0^t e^{-q_i s} \sum_{k \neq i} q_{i,k} \pi_{k,j}^{(n-1)}(t-s) ds & n \geq 1. \end{cases} \tag{3.24}$$

We can check, as we did above with the forward recursion, that

$$\pi_{i,j}^{(n+1)}(t) \geq \pi_{i,j}^{(n)}(t) \geq 0 \qquad \forall n \geq 0 \quad \forall t \geq 0,$$

hence the limit

$$\pi_{i,j}(t) = \lim_{n \to \infty} \pi_{i,j}^{(n)}(t)$$

is well defined and, by monotone convergence, satisfies the backward equation.

We want to show that the two solutions $p_{i,j}(t)$ and $\pi_{i,j}(t)$ are actually the same. We will do that by induction on n, proving that $p_{i,j}^{(n)}(t) = \pi_{i,j}^{(n)}(t)$. This equality holds trivially for $n = 0$, while for $n = 1$ we have:

$$\begin{aligned} p_{i,j}^{(1)}(t) &= p_{i,j}^{(0)}(t) + \int_0^t e^{-q_j s} \sum_{k \neq j} \delta_{i,k} e^{-q_i(t-s)} q_{k,j} ds \\ &= p_{i,j}^{(0)}(t) + (1 - \delta_{i,j}) \int_0^t e^{-q_j s - q_i(t-s)} q_{i,j} ds \\ &= \pi_{i,j}^{(0)}(t) + \int_0^t e^{-q_i s} \sum_{k \neq i} q_{i,k} \delta_{k,j} e^{-q_j(t-s)} ds \\ &= \pi_{i,j}^{(1)}(t). \end{aligned}$$

Now suppose we have proven $p_{i,j}^{(m)}(t) = \pi_{i,j}^{(m)}(t)$ for all $m \leq n$ for $n \geq 1$. We can write

$$
\begin{aligned}
p_{i,j}^{(n+1)}(t) &= \pi_{i,j}^{(0)}(t) + \int_0^t e^{-q_j s} \sum_{k\neq j} \pi_{i,k}^{(n)}(t-s) q_{k,j} ds \\
&= \pi_{i,j}^{(0)}(t) + \int_0^t e^{-q_j s} \sum_{k\neq j} \Bigg(\delta_{i,k} e^{-q_i(t-s)} \\
&\quad + \int_0^{t-s} e^{-q_i u} \sum_{l\neq i} q_{i,l} \pi_{l,k}^{(n-1)}(t-s-u) du \Bigg) q_{k,j} ds \\
&= \pi_{i,j}^{(0)}(t) + (1-\delta_{i,j}) q_{i,j} \int_0^t e^{-q_j s - q_i(t-s)} ds \\
&\quad + \sum_{k\neq j} \sum_{l\neq i} q_{i,l} q_{k,j} \int_0^t e^{-q_j s} \int_0^{t-s} e^{-q_i u} \pi_{l,k}^{(n-1)}(t-s-u) du ds \\
&= \pi_{i,j}^{(0)}(t) + (1-\delta_{i,j}) q_{i,j} \int_0^t e^{-q_j s - q_i(t-s)} ds + \sum_{k\neq j} \sum_{l\neq i} q_{i,l} q_{k,j} I_{k,l}^{(p)}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\pi_{i,j}^{(n+1)}(t) &= \pi_{i,j}^{(0)}(t) + \int_0^t e^{-q_i(t-s)} \sum_{l\neq i} q_{i,l} \pi_{l,j}^{(n)}(s) ds \\
&= \pi_{i,j}^{(0)}(t) + \int_0^t e^{-q_i(t-s)} \sum_{l\neq i} \Bigg(\delta_{l,j} e^{-q_j s} \\
&\quad + \int_0^s e^{-q_j(s-u)} \sum_{k\neq j} \pi_{l,k}^{(n-1)}(u) du \Bigg) q_{k,j} ds \\
&= \pi_{i,j}^{(0)}(t) + (1-\delta_{i,j}) q_{i,j} \int_0^t e^{-q_i(t-s) - q_j s} ds \\
&\quad + \sum_{k\neq j} \sum_{l\neq i} q_{i,l} q_{k,j} \int_0^t e^{-q_i(t-s)} \int_0^s e^{-q_j(s-u)} \pi_{l,k}^{(n-1)}(u) du ds \\
&= \pi_{i,j}^{(0)}(t) + (1-\delta_{i,j}) q_{i,j} \int_0^t e^{-q_i(t-s) - q_j s} ds + \sum_{k\neq j} \sum_{l\neq i} q_{i,l} q_{k,j} I_{k,l}^{(\pi)}(t)
\end{aligned}
$$

The first two terms are equal, so we just have to prove that the last sums are equal, and in particular it is enough to show that all the terms in the sums are equal, which we can do by changes of variables:

$$\begin{aligned} I_{k,l}^{(\pi)}(t) &= \int_0^t e^{-q_i(t-s)} \int_0^s e^{-q_j(s-u)} \pi_{l,k}^{(n-1)}(u) du ds \\ &= \int_0^t \int_0^s e^{-q_i(t-s)} e^{-q_j v} \pi_{l,k}^{(n-1)}(s-v) dv ds \\ &= \int_0^t \int_0^{t-r} e^{-q_i r} e^{-q_j v} \pi_{l,k}^{(n-1)}(t-r-v) dv dr \\ &= I_{k,l}^{(p)}(t). \end{aligned}$$

So the two constructed functions are actually the same. We will now prove that it is a Q-function and is the minimal solution and the minimal Q-function.

We begin with proving that

$$\sum_{j\in J} p_{i,j}^{(n)}(t) \le 1 \qquad \forall n \ge 0 \quad \forall t \ge 0.$$

We have, by the definition of π,

$$\sum_{j\in J} p_{i,j}^{(n)}(t) = \begin{cases} \sum_{j\in J} \delta_{i,j} e^{-q_i t} & n = 0 \\ \sum_{j\in J} \delta_{i,j} e^{-q_i t} + \int_0^t e^{-q_i s} \sum_{k\ne i} q_{i,k} \sum_{j\in J} p_{k,j}^{(n-1)}(t-s) ds & n \ge 1. \end{cases}$$

We proceed by induction: the case $n = 0$ is trivially true; supposing the result holds for $n - 1$, then we have

$$\begin{aligned} \sum_{j\in J} p_{i,j}^{(n)}(t) &\le e^{-q_i t} + \sum_{k\ne i} q_{i,k} \int_0^t e^{-q_i s} ds \\ &= e^{-q_i t}(1 - q_i^{-1}) + q_i^{-1} \le 1. \end{aligned}$$

The property that the limit for $t \to 0$ is $\delta_{i,j}$ follows from

$$0 \le \lim_{t\to 0} \int_0^t e^{-q_i s} \sum_{k\ne i} q_{i,k} p_{k,j}^{(n-1)}(t-s) ds \le \sum_{k\ne i} q_{i,k} \lim_{t\to 0} \int_0^t e^{-q_i s} ds = 0.$$

The last property we have to show in order for p to be a Q-function is the Chapman-Kolmogorov one. To do so, we begin by defining

$$d_{i,j}^{(n)}(t) = p_{i,j}^{(n)}(t) - p_{i,j}^{(n-1)}(t),$$

considering $p_{i,j}^{(-1)}(t) = 0$. By its definition $d_{i,j}$ is such that

$$d_{i,j}^{(n+1)}(t) = \int_0^t e^{-q_i s} \sum_{k\ne i} q_{i,k} d_{k,j}^{(n)}(t-s) ds \qquad n \ge 1.$$

We prove, again by induction on n, that

$$p_{i,j}^{(n)}(t+s) = \sum_{k\in J}\sum_{m=0}^{n} d_{i,j}^{(m)}(s)p_{k,j}^{(n-m)}(t). \tag{3.25}$$

The case $n=0$ is trivially satisfied, then, if the property holds for n, we have

$$\begin{aligned}
\sum_{k\in J}\sum_{m=0}^{n+1} d_{i,j}^{(m)}(s)p_{k,j}^{(n+1-m)}(t) &= e^{-q_i s}p_{i,j}^{(n+1)}(t) + \sum_{k\in J}\sum_{m=0}^{n} d_{i,j}^{(m+1)}(s)p_{k,j}^{(n-m)}(t)\\
&= e^{-q_i s}\left(e^{-q_i t} + \int_0^t e^{-q_i(t-u)}\sum_{l\neq i} q_{i,l}p_{l,j}^{(n)}(u)du\right)\\
&\quad + \sum_{k\in J}\sum_{m=0}^{n}\int_0^s e^{-q_i u}\sum_{l\neq i} q_{i,l}d_{l,k}^{(m)}(s-u)du\, p_{k,j}^{(n-r)}(t)\\
&= e^{-q_i(t+s)} + \int_0^t e^{-q_i(t+s-u)}\sum_{l\neq i} q_{i,l}p_{l,j}^{(n)}(u)du\\
&\quad + \int_0^s e^{-q_i u}\sum_{l\neq i} q_{i,l}\sum_{k\in J}\sum_{m=0}^{n} d_{l,k}^{(m)}(s-u)p_{k,j}^{(n-r)}(t)du\\
&= e^{-q_i(t+s)} + \int_0^t e^{-q_i(t+s-u)}\sum_{l\neq i} q_{i,l}p_{l,j}^{(n)}(u)du\\
&\quad + \int_0^s e^{-q_i u}\sum_{l\neq i} q_{i,l}p_{l,j}^{(n)}(t+s-u)du\\
&= e^{-q_i(t+s)} + \int_0^t +s e^{-q_i(t+s-u)}\sum_{l\neq i} q_{i,l}p_{l,j}^{(n)}(u)du\\
&= p_{i,j}^{(n+1)}(t+s).
\end{aligned}$$

If we now pass to the limit for $n\to\infty$ we have that the same result holds for $p_{i,j}$, thanks to the monotone convergence theorem. We have then proven that $p_{i,j}$ is a transition function and, more precisely, a Q-function.

Let now $y_{i,j}(t)$ be a generic nonnegative solution of the forward equations (3.27). Let's stress that this is not necessarily a Q-function. We have that

$$y_{i,j}(t) \geq \delta_{i,j}e^{-q_j t} = p_{i,j}^{(0)}(t).$$

Now suppose that

$$y_{i,j}(t) \geq p_{i,j}^{(n)}(t) \qquad \forall j\in J \quad \forall t\geq 0,$$

holds for some $n \geq 0$: we have

$$\begin{aligned} y_{i,j}(t) &= \delta_{i,j} e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} y_{i,k}(t-s) q_{k,j} ds \\ &\geq \delta_{i,j} e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} p_{i,j}^{(n)}(t-s) q_{k,j} ds \\ &= p_{i,j}^{(n+1)}(t), \end{aligned}$$

so $y_j(t) \geq v_j^{(n)}(t)$ for all $n \geq 0$ by induction, and the inequality passes to the limit over n, hence the minimality of $p_{i,j}(t)$ among the solutions of the forward equations. An analogous argument allows us to prove its minimality among solutions of the backward equations.

We want to conclude the proof by showing that given any Q-function $f_{i,j}(t)$, not necessarily solving the forward or backward equations, we have $f_{i,j}(t) \geq p_{i,j}(t)$. Any Q-function $f_{i,j}(t)$ satisfies the following inequality, since $q_{i,j} = f'_{i,j}(0)$:

$$f'_{i,j}(t) \geq \sum_{l \in J} f_{i,l}(t) q_{l,j}$$

hence, proceeding as in the proof of Lemma 3.9, we have

$$f_{i,j}(t) \geq \delta_{i,j} e^{-q_j t} + e^{-q_j t} \int_0^t e^{q_j s} \sum_{k \neq j} f_{i,k}(t-s) q_{k,j} ds$$

and in particular

$$f_{i,j}(t) \geq \delta_{i,j} e^{-q_j t} = p_{i,j}^{(0)}(t).$$

Suppose now that $f_{i,j}(t) \geq p_{i,j}^{(n)}(t)$ for some $n \geq 0$, then

$$\begin{aligned} f_{i,j}(t) &\geq \delta_{i,j} e^{-q_j t} + e^{-q_j t} \int_0^t e^{q_j s} \sum_{k \neq j} f_{i,k}(t-s) q_{k,j} ds \\ f_{i,j}(t) &\geq \delta_{i,j} e^{-q_j t} + e^{-q_j t} \int_0^t e^{q_j s} \sum_{k \neq j} p_{i,k}^{(n)}(t-s) q_{k,j} ds \\ &= p_{i,j}^{(n+1)}(t), \end{aligned}$$

so the inequality hold by induction for all $n \geq 0$, and it passes to the limit proving the minimality of $p_{i,j}(t)$ as a Q-function. □

Remark 3.7. While the relationship between our equations for the second moments and the forward equations for a Markov chain shown above in Proposition 3.8 certainly looks interesting and worth digging into, we should not get the wrong idea that our second moments are transition functions of a Markov chain themselves. In fact, were we to interpret our E_js in a form like the $p_{i,j}$s above, we would have $p_{i,j} = E_j$ for all $i \in J$, and, given that or matrix is symmetric and

$$p_{i,j}(t) = p_{j,i}(t) = \mathbb{P}(Z_t = i | Z_0 = j),$$

we would have a contradiction, since

$$1 = \sum_{i \in J} p_{i,j}(t) = E_j(t) \sum_{i \in J} 1.$$

At the same time there is a strong link between solutions of the forward equation for a Markov chain with matrix Q and our equation, as shown in the next proposition.

Proposition 3.12. *Given an initial condition $(v_l^0)_{l \in J} \in l^1$ the family of functions defined by*

$$v_j(t) = \sum_{k \in J} v_k^0 p_{k,j}(t),$$

is the minimal solution for the system

$$y_j'(t) = \sum_{k \in J} y_k(t) q_{k,j} \tag{3.26}$$

with initial conditions $y_j(0) = v_j^0$.

First of all we can prove in the same way as Lemma 3.9 the following equivalent formulation.

Lemma 3.13. *The following formulations of the forward equations* (3.26) *are equivalent, given generic initial conditions $y(0)$ in l^1:*

$$y_j'(t) = \sum_{k \in J} y_k(t) q_{k,j}$$

$$y_j(t) = y_j(0) e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} y_k(t-s) q_{k,j} ds. \tag{3.27}$$

Proof of Proposition 3.12. We have immediately that

$$v_j(0) = \sum_{k \in J} v_k^0 p_{k,j}(0) = \sum_{k \in J} v_k^0 \delta_{k,j} = v_j^0$$

and the initial conditions are satisfied. If we derive $v_j(t)$ in time we have

$$\begin{aligned} v_j'(t) = \frac{d}{dt} \sum_{k \in J} v_k^0 p_{k,j}(t) = \sum_{k \in J} v_k^0 p_{k,j}'(t) &= \sum_{k \in J} v_k^0 \sum_{i \in J} p_{k,i}(t) q_{i,j} \\ &= \sum_{i \in J} \sum_{k \in J} v_k^0 p_{k,i}(t) q_{i,j} = \sum_{i \in J} v_i(t) q_{i,j} \end{aligned} \tag{3.28}$$

and $v_j(t)$ satisfies (3.26). We want to show that the solution $v_j(t)$ is the minimal one, that is if $y_j(t)$ is another nonnegative solution of the forward equation (3.26) with initial conditions v_j^0, than

$$y_j(t) \geq v_j(t) \qquad \forall j \in J \quad \forall t \geq 0. \tag{3.29}$$

The idea is to show that this property is inherited from $p_{i,j}(t)$, so we define

$$v_j^{(n)}(t) = \sum_{i \in J} v_i^0 p_{i,j}^{(n)}(t) \qquad \forall n \geq 0$$

and we prove (3.29) by induction over n.

We have, by the equivalent formulation (3.27),

$$y_j(t) \geq v_j^0 e^{-q_j t} = v_j^{(0)}(t).$$

Then, if we assume that for some $n \geq 0$

$$y_j(t) \geq v_j^{(n)}(t) \qquad \forall j \in J \quad \forall t \geq 0,$$

we have

$$\begin{aligned}
y_j(t) &= v_j^0 e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} y_k(t-s) q_{k,j} ds \\
&\geq v_j^0 e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} v_k^{(n)}(t-s) q_{k,j} ds \\
&= v_j^0 e^{-q_j t} + \int_0^t e^{-q_j s} \sum_{k \neq j} \sum_{i \in J} v_i^0 p_{i,k}^{(n)}(t-s) q_{k,j} ds \\
&= v_j^0 e^{-q_j t} + \sum_{i \in J} v_i^0 \int_0^t e^{-q_j s} \sum_{k \neq j} p_{i,k}^{(n)}(t-s) q_{k,j} ds \\
&= v_j^0 e^{-q_j t} + v_j^{(n+1)}(t) - v_j^0 e^{-q_j t} \\
&= v_j^{(n+1)}(t),
\end{aligned}$$

so $y_j(t) \geq v_j^{(n)}(t)$ for all $n \geq 0$ and it passes to the limit over n, hence the thesis. □

We can present the uniqueness result given above for the forward equations with Id initial conditions in the case of

$$v_j = \sum_{i \in J} v_i^0 p_{i,j}$$

solution of (3.17).

Proposition 3.14. *Given the stable, conservative and symmetric q-matrix Q defined in* (3.16)*, then the unique nonnegative solution of the equations* (3.17) *in* $L^\infty([0,\infty), l^1)$*, given a null initial condition* $y(0) = 0$*, is* $y(t) = 0$*.*

Moreover, the uniqueness holds in the same class with any nonnegative initial condition in l^1*.*

Proof. Let's start with the first part, following the classical Feller approach, via Laplace transform. Let y be a generic solution, then

$$\begin{cases} \dfrac{d}{dt} y_j(t) = \displaystyle\sum_{i \in J} y_i(t) q_{ij} \\ y_j(t) \geq 0 \quad j \in J \\ y_j(0) = 0 \quad j \in J \\ \displaystyle\sum_{j \in J} y_j(t) < +\infty. \end{cases} \tag{3.30}$$

We can consider for every node $\hat{y}_j = \int_0^{+\infty} e^{-t} y_j(t) dt$, the Laplace transform. From the last equation of the system above, we have $\sum_j \hat{y}_j \leq M$, for some constant $M > 0$, so in particular we can consider $k \in J$ such that $\hat{y}_k \geq \hat{y}_j$, for all $j \in J$.

Now we want to show that $y_k'(t)$ is limited: thanks to the symmetry of Q and the fact that all its entries are finite, we have

$$|y_k'(t)| \leq |-q_k y_k(t)| + \left| \sum_{l \neq k} y_l(t) q_{lk} \right| \leq q_k M + q_k M < +\infty.$$

We can integrate by parts

$$\begin{aligned} \hat{y}_k = \int_0^{+\infty} e^{-t} y_k'(t) dt = \int_0^{+\infty} e^{-t} \sum_{l \in J} y_l(t) q_{lk} dt = \sum_{l \in J} \hat{y}_l q_{lk} \\ = -\hat{y}_k q_k + \sum_{l \neq k} \hat{y}_l q_{lk} \leq \hat{y}_k (-q_k + \sum_{l \neq k} q_{kl}) = 0, \end{aligned} \tag{3.31}$$

where the last equality follows from the null sum on every row (and column) of Q, and we used the symmetry and finiteness of all the entries. Now we have $\hat{y}_k = 0$ and so all $\hat{y}_j = 0$, hence $y_j(t) = 0$ for all $j \in J$, for all $t \geq 0$.

For the second part, let $p_{i,j}(t)$ be the minimal semigroup of Q and let $(v_j^0)_j \in l^1$ be a nonnegative initial condition. Then, as we pointed out before in Proposition 3.12,

$$v_j(t) = \sum_{i \in J} v_i^0 p_{i,j}(t)$$

is a solution of the forward equation (3.26) with initial conditions v_j^0. Thanks to the minimality of such v proven in the same Proposition 3.12, we can, given another solution u of the same equation, consider the difference $y = u - v$, which satisfies all the hypotheses of the first part of the proposition, so the second part holds too. □

3.6 Uniqueness

Now we can use the results of the previous section to prove the main results of this chapter.

Theorem 3.15. *There is strong uniqueness for the linear system* (3.9) *in the class of energy controlled* $L^\infty(\Omega \times [0,T], l^2)$ *solutions.*

Proof. By linearity of (3.9) it is enough to prove that for null initial conditions there is no nontrivial solution. Since we have (3.14) and Propositions 3.14, then $E_{\widehat{P}}[X_j^2(t)] = 0$ for all j and t, hence $X = 0$ a.s. □

Let's recall that we already proved an existence result for (3.9) with proposition 3.5.

Theorem 3.16. *There is uniqueness in law for the nonlinear system* (3.3) *in the class of energy controlled* $L^\infty(\Omega \times [0,T], l^2)$ *solutions.*

Proof. Assume that $(\Omega^{(i)}, \mathcal{F}_t^{(i)}, P^{(i)}, W^{(i)}, X^{(i)})$, for $i = 1, 2$, are two solutions of (3.3) with the same initial conditions $x \in l^2$. Given $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0,T]$ and a measurable and bounded function $f : (l^2)^n \to \mathbb{R}$, we want to prove that

$$E_{P^{(1)}}\Big[f(X^{(1)}(t_1), \dots, X^{(1)}(t_n))\Big] = E_{P^{(2)}}\Big[f(X^{(2)}(t_1), \dots, X^{(2)}(t_n))\Big]. \qquad (3.32)$$

By theorem 3.4 and the definition of $\widehat{P}$ given in (3.8) we have that, for $i = 1, 2$,

$$E_{P^{(i)}}\Big[f(X^{(i)}(t_1), \dots, X^{(i)}(t_n))\Big] = \\ E_{\widehat{P}^{(i)}}\Big[\exp\Big\{-M_T^{(i)} + \frac{1}{2}\Big[M^i, M^{(i)}\Big]_T\Big\} f\Big(X^{(i)}(t_1), \dots, X^{(i)}(t_n)\Big)\Big], \qquad (3.33)$$

where $M^{(i)}$is defined as in (3.7). We have proven in proposition 3.5 and theorem 3.15 that the linear system (3.9) has a unique strong solution. Thus it has uniqueness in law on $\mathcal{C}([0,T], \mathbb{R})^{\mathbb{N}}$ by Yamada-Watanabe theorem, that is under the measures $\widehat{P}^{(i)}$, the processes $X^{(i)}$ have the same laws. For a detailed proof of this theorem in infinite dimension see [23].

Now we can also include $M^{(i)}$ in the system and conclude that $(X^{(i)}, M^{(i)})$ under $\widehat{P}^{(i)}$ have laws independent of $i = 1, 2$, hence, by (3.33), we have (3.32). □

We can now conclude with the proof of Theorem 3.2.

Proof of Theorem 3.2. Let $(\Omega, \mathcal{F}_t, \widehat{P}, B, X)$ be the solution of (3.9) in $L^\infty(\Omega \times [0,T], l^2)$ provided by theorem 3.15. We follow the same argument as in Section 3.4, only from $\widehat{P}$ to P. We construct P as a measure on $(\Omega, \mathcal{F}_T)$ satisfying

$$\frac{dP}{d\widehat{P}}\Big|_{\mathcal{F}_T} = \exp\Big(\widehat{M}_T - \frac{1}{2}\Big[\widehat{M}, \widehat{M}\Big]_T\Big),$$

where $\widehat{M}_t = \frac{1}{\sigma}\sum_{j\in J}\int_0^t X_{\overline{j}}(s)dB_j(s)$. Under P the processes

$$W_j(t) = B_j(t) - \int_0^t \frac{1}{\sigma} X_{\overline{j}}(s)ds,$$

are a sequence of independent Brownian motions. Hence $(\Omega, \mathcal{F}_t, P, W, X)$ is a solution of (3.3) and it is in L^∞, since P and $\widehat{P}$ are equivalent on $\mathcal{F}_T$. $\square$

Chapter **4**

Anomalous dissipation

In the previous chapter we have answered the question about uniqueness. Now we would like to clarify the topic of energy. If we consider the inviscid and unforced model

$$\frac{d}{dt}X_j = c_j X_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k, \tag{4.1}$$

and we formally compute the derivative of the energy up to generation n, that is

$$\frac{d}{dt}\sum_{|j|\leq n} X_j^2(t) = 2\sum_{|j|\leq n} X_j(t)\frac{d}{dt}X_j(t) = -2\sum_{|k|=n+1} c_k X_{\bar{k}}^2(t) X_k(t),$$

we have that for sufficiently regular solutions the derivative of the energy, that is the limit of this quantity for $n \to \infty$, is 0, so the energy is formally preserved. At the same time, in Section 2.7 we have proven that, for the exponential choice of coefficients, there cannot be regular solutions. What happens, once we assume this choice of coefficients, is that the model, while formally preserving energy, dissipates it. This phenomenon is called *anomalous dissipation*, and is proven in Theorem 4.6.

This anomalous dissipation for the tree dyadic model is not unexpected, as it shows up for the classic dyadic model too, see Theorem A.8. At the same time it is not clear that the tree dyadic model should behave in the same way: even if the global flux from a generation to the next one behaves similarly to the classic case, energy may split between eddies of the same generation, which increase exponentially in number, so that the energy coming from ancestors could spread around.

While the result is similar to the anomalous dissipation result for the classic dyadic model, the proof requires new ideas and ingredients, because of the different structure of the skeleton. Another important result in this chapter, that follows naturally from Theorem 4.6, is Theorem 4.7, where we provide an upper bound to the energy decay. Since it is exactly the decay shown by the energy of self-similar solution introduced in Section 3.1, we think that this estimate cannot be improved much.

4.1 Anomalous dissipation in the inviscid and unforced case.

Throughout this section we'll consider the tree dyadic model in its unforced ($f = 0$) and inviscid ($\nu = 0$) version:

$$\begin{cases} X_{\bar{0}}(t) \equiv 0 \\ \dfrac{d}{dt} X_j = c_j X_{\bar{j}}^2 - \displaystyle\sum_{k \in \mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J. \end{cases} \tag{4.2}$$

The derivative of energy up to the n-th generation, described in the general case by equation (2.7), becomes

$$\frac{d}{dt}\mathcal{E}_n(t) = -2 \sum_{|k|=n+1} c_k X_{\bar{k}}^2(t) X_k(t), \qquad n \geq 0. \tag{4.3}$$

Since only the border term survives, one would expect it to vanish in the limit $n \to \infty$. This can be rigorously proven only if the solution lives in a sufficiently regular space, that is to say that X_j^2 goes fast to zero as $|j| \to \infty$. But in Section 2.7 we proved that the solutions we are interested in are not regular, as those that are regular in the beginning, stay regular for some time but then lose regularity in finite time.

Let us give some definitions. Let us denote by γ_j the energy at time 0 in the subtree T_j rooted in j plus all the energy flowing in j from the upper generations,

$$\gamma_j := \sum_{k \in T_j} X_k^2(0) + \int_0^\infty 2 c_j X_j X_{\bar{j}}^2 ds.$$

Let $0 \leq s < t$ and define for all $j \in J$

$$m_j := \inf_{r \in [s,t]} X_j(r).$$

Let us also restate two results prove in Chapter 2 that will be needed in the following proofs.

Proposition 4.1 (Proposition 2.2)**.** *For any positive l^2 solution X, the following energy balance principle holds, for all $0 \leq s < t$.*

$$\mathcal{E}(t) = \mathcal{E}(s) - 2 \lim_{n \to \infty} \int_s^t \sum_{|k|=n} c_k X_{\bar{k}}^2(u) X_k(u) du,$$

where the limit always exists and is non-negative. In particular we have that $\mathcal{E}$ is non-increasing.

Proposition 4.2 (Proposition 2.5)**.** *Let X be a solution of* (4.2)*. The following properties hold:*

1. If $f = 0$, for all $n \geq -1$

$$2\int_0^{+\infty} \sum_{|k|=n+1} c_k X_{\bar{k}}^2(s) X_k(s) ds \leq \mathcal{E}_n(0); \tag{4.4}$$

2. if $X_j^0 > 0$ for all j s.t. $|j| = M$ for some $M \geq 0$, then $X_j(t) > 0$ for every j s.t. $|j| \geq M$ and all $t > 0$.

We start with a few lemmata that will provide us with useful energy estimates to prove the dissipation result.

Lemma 4.3. *Let X be a positive l^2 solution of system* (4.2)*. The following inequalities hold for all $n \geq 0$.*

$$\mathcal{E}_n(t) - \mathcal{E}_{n-1}(s) \leq \sum_{|j|=n} m_j^2 \leq \mathcal{E}(0),$$

$$\sum_{|j|=n} \gamma_j \leq \mathcal{E}(0),$$

$$\sum_{k \in T_j} X_k(r)^2 \leq \gamma_j, \qquad \forall r \geq 0.$$

Proof. The upper bound is obvious, since

$$\sum_{|j|=n} m_j^2 \leq \sum_{|j|=n} X_j(s)^2 \leq \mathcal{E}_n(s) \leq \mathcal{E}(0),$$

where we used Proposition 4.1. Now let $j \in J$. From (4.2) we have for the differential of X_j^2

$$\frac{d}{dt} X_j^2 = 2c_j X_{\bar{j}}^2 X_j - \sum_{k \in \mathcal{O}_j} 2c_k X_j^2 X_k \ ,$$

Let $r \in [s,t]$ and integrate on $[s,r]$, yielding

$$X_j^2(r) = X_j^2(s) + \int_s^r 2c_j X_{\bar{j}}^2(\tau) X_j(\tau) d\tau - \sum_{k \in \mathcal{O}_j} \int_s^r 2c_k X_j^2(\tau) X_k(\tau) d\tau.$$

Choosing now $r \in \operatorname{argmin}_{[s,t]} X_j$, we get

$$m_j^2 \geq X_j^2(s) - \sum_{k \in \mathcal{O}_j} \int_s^t 2c_k X_{\bar{k}}^2(\tau) X_k(\tau) d\tau.$$

By summation over all nodes j with $|j| = n$ we have

$$\sum_{|j|=n} m_j^2 \geq \sum_{|j|=n} X_j^2(s) - \int_s^t \sum_{|k|=n+1} 2c_k X_{\bar{k}}^2(\tau) X_k(\tau) d\tau.$$

Finally, we apply for $m = n-1$, n the following integral form of (4.3) to get the first part of the thesis. (Even if $n = 0$ and $m = -1$ this is true, trivially.)

$$\mathcal{E}_m(t) - \mathcal{E}_m(s) = -\int_s^t \sum_{|j|=m+1} 2c_j X_{\bar{j}}^2(\tau) X_j(\tau) d\tau.$$

We turn to the second part. Sum γ_j on every j with $|j| = n$ to get

$$\sum_{|j|=n} \gamma_j = \sum_{|k|\geq n} X_k^2(0) + \int_0^\infty 2 \sum_{|j|=n} c_j X_{\bar{j}}^2 X_j ds,$$

by (4.4) the integral term is bounded above by $\mathcal{E}_{n-1}(0)$, so

$$\sum_{|j|=n} \gamma_j \leq \sum_{|k|\geq n} X_k^2(0) + \mathcal{E}_{n-1}(0) = \sum_{k\in J} X_k^2(0) = \mathcal{E}(0).$$

Finally, the third part. Let $r \geq 0$. By computing the time derivative of

$$\sum_{\substack{k\in T_j \\ |k|\leq n}} X_k^2$$

which is analogous to (4.3), dropping the border term and integrating on $[0, r]$, we have,

$$\sum_{\substack{k\in T_j \\ |k|\leq n}} X_k(r)^2 \leq \sum_{\substack{k\in T_j \\ |k|\leq n}} X_k(0)^2 + 2\int_0^r 2c_j X_j X_{\bar{j}}^2 du \leq \gamma_j.$$

Now, let $n \to \infty$ to conclude. □

The following statement will be used in the proof of Lemma 4.5.

Lemma 4.4. *For every $h > 0$ and $\lambda > 0$ the following inequality holds:*

$$\int_0^h \int_0^s e^{-\lambda(s-r)} dr\, ds \geq \frac{h}{2\lambda}\left(1 - e^{-\lambda\frac{h}{2}}\right).$$

Proof.

$$\int_0^h \int_0^s e^{-\lambda(s-r)} dr\, ds \geq \int_{\frac{h}{2}}^h \int_{s-\frac{h}{2}}^s e^{-\lambda(s-r)} dr\, ds = \frac{h}{2\lambda}\left(1 - e^{-\lambda\frac{h}{2}}\right).$$ □

Lemma 4.5. *Assume that $\alpha > \tilde{\alpha}$, where $2^{2\tilde{\alpha}} = N_* = \sharp\mathcal{O}_j$ is the constant number of children for every node. Let X be a positive l^2 solution of (4.2). Let $(\delta_n)_{n\geq 0}$ be a sequence of positive numbers such that $\sum_n \delta_n$ and $\sum_n \delta_n^{-2} 2^{-(\alpha-\tilde{\alpha})n}$ are both finite. Then there exists a sequence of positive numbers $(h_n)_{n\geq 0}$ such that $\sum_n h_n < \infty$ and for all $n \geq 0$ for all $t > 0$*

$$\mathcal{E}_n(t + h_n) - \mathcal{E}_{n-1}(t) \leq \delta_n. \tag{4.5}$$

In particular, for every $M \geq 0$,

$$\mathcal{E}\Big(\sum_{n=M}^{\infty} h_n\Big) \leq \mathcal{E}_{M-1}(0) + \sum_{n=M}^{\infty} \delta_n. \tag{4.6}$$

The sequence

$$h_n = \frac{\mathcal{E}(0)^{3/2}}{\delta_n^2} 2^{-(\alpha-\tilde{\alpha})n+3/2}, \tag{4.7}$$

satisfies (4.5) *and* (4.6).

Proof. Fix $n \geq 0$ and positive real numbers t, h_n. For all j of generation n, let $m_j := \inf_{r\in[t,t+h_n]} X_j(r)$. We claim that if h_n is defined by (4.7), then $\sum_{|j|=n} m_j^2 \leq \delta_n$, which together with Lemma 4.3 completes the proof of (4.5).

We prove the claim by contradiction: suppose that $\sum_{|j|=n} m_j^2 > \delta_n$. We will find a contradiction in the estimates on $\mathcal{E}(0)$. By Proposition 4.2

$$\mathcal{E}(0) \geq 2 \int_0^{h_n} \sum_{|j|=n} \sum_{k\in\mathcal{O}_j} c_k X_k(t+s) X_j^2(t+s) ds.$$

We have a lower bound for X_j, namely m_j, but we need one also for X_k.

For all $j \in J$, let $\Gamma_j := \max(\gamma_j, \mathcal{E}(0) N_*^{-|j|})$. From Lemma 4.3 we have $\sum_{|j|=n} \gamma_j \leq \mathcal{E}(0)$ and hence $\sum_{|j|=n} \Gamma_j \leq 2\mathcal{E}(0)$; by the same lemma, for all $i \in T_j$ we have $X_i^2 \leq \gamma_j \leq \Gamma_j$ uniformly in time, so for all $k \in \mathcal{O}_j$,

$$\dot{X}_k = c_k X_j^2 - \sum_{i\in\mathcal{O}_k} c_i X_i X_k \geq c_k m_j^2 - \lambda_j X_k,$$

where $\lambda_j = N_* 2^{n\alpha+2\alpha}\sqrt{\Gamma_j}$. This gives

$$X_k(t+s) \geq c_k m_j^2 \int_0^s e^{-\lambda_j(s-r)} dr.$$

We can write

$$\mathcal{E}(0) \geq 2 \sum_{|j|=n} m_j^4 \int_0^{h_n} \int_0^s e^{-\lambda_j(s-r)} dr ds \sum_{k\in\mathcal{O}_j} c_k^2,$$

and by lemma 4.4 we have

$$\mathcal{E}(0) \geq 2 \sum_{|j|=n} m_j^4 \frac{h_n}{2\lambda_j}\Big(1 - e^{-\lambda_j h_n/2}\Big) \sum_{k\in\mathcal{O}_j} c_k^2.$$

Let us focus on the exponential. We substitute (4.7) and make use of the inequality $\Gamma_j \geq \mathcal{E}(0) N_*^{-n} = \mathcal{E}(0) 2^{-2\tilde{\alpha}n}$,

$$\frac{\lambda_j h_n}{2} = N_* 2^{n\alpha+2\alpha}\sqrt{\Gamma_j}\frac{\sqrt{2}\mathcal{E}(0)^{3/2}}{2^{(\alpha-\tilde{\alpha})n}\delta_n^2} \geq \frac{\mathcal{E}(0)^2}{\delta_n^2}\sqrt{2}.$$

By the hypothesis that $\sum_{|j|=n} m_j^2 > \delta_n$ and Lemma 4.3, we know that $\delta_n < \mathcal{E}(0)$ we get $1 - e^{-\lambda_j h_n/2} > 1/2$. We obtain

$$\mathcal{E}(0) > \sum_{|j|=n} m_j^4 \frac{h_n}{2\lambda_j} \sum_{k\in\mathcal{O}_j} c_k^2 = \frac{\sqrt{2}\mathcal{E}(0)^{3/2}}{2^{-\tilde{\alpha}n}\delta_n^2} \sum_{|j|=n} \frac{m_j^4}{\sqrt{\Gamma_j}}. \tag{4.8}$$

Now we can use Cauchy-Schwarz and the AM-QM inequalities to get

$$\sum_{|j|=n} \frac{m_j^4}{\sqrt{\Gamma_j}} \geq \frac{\left(\sum_{|j|=n} m_j^2\right)^2}{\sum_{|j|=n} \sqrt{\Gamma_j}} \geq \frac{\left(\sum_{|j|=n} m_j^2\right)^2}{\sqrt{N_*^n \sum_{|j|=n} \Gamma_j}},$$

again by the hypothesis that $\sum_{|j|=n} m_j^2 > \delta_n$ and thanks to $\sum_{|j|=n} \Gamma_j \leq 2\mathcal{E}(0)$,

$$\sum_{|j|=n} \frac{m_j^4}{\sqrt{\Gamma_j}} > \frac{\delta_n^2}{\sqrt{2\mathcal{E}(0)}2^{\tilde{\alpha}n}},$$

so that the right-hand side of (4.8) becomes larger than $\mathcal{E}(0)$, which is impossible.

We turn to the second part. Let $M \geq 0$ and define the following sequence $(t_n)_{n\geq M-1}$ by $t_{M-1} = 0$ and $t_n = t_{n-1} + h_n$. By (4.5) with $t = t_{n-1}$ we get

$$\mathcal{E}_n(t_n) - \mathcal{E}_{n-1}(t_{n-1}) \leq \delta_n.$$

We sum for n from M to N, yielding

$$\mathcal{E}_N(t_N) - \mathcal{E}_{M-1}(0) \leq \sum_{n=M}^{N} \delta_n,$$

which, due to monotonicity of $\mathcal{E}_N$, yields

$$\mathcal{E}_N\left(\sum_{n=M}^{\infty} h_n\right) \leq \mathcal{E}_N(t_N) \leq \mathcal{E}_{M-1}(0) + \sum_{n=M}^{N} \delta_n.$$

Now we let N go to infinity to get the thesis. □

Remark 4.1. It is easy to prove this result also if relaxing the condition on the number of children from constant number to $1 \leq \sharp\mathcal{O}_j \leq N_*$. One has to change slightly the definition of h_n, which becomes

$$h_n = \frac{\mathcal{E}(0)^{3/2}}{\delta_n^2} 2^{-(\alpha-\tilde{\alpha})n+2\tilde{\alpha}+3/2}.$$

Theorem 4.6. *Assume that $\alpha > \tilde{\alpha}$, where $2^{2\tilde{\alpha}} = N_* = \sharp\mathcal{O}_j$ is the constant number of children for every node. Then for every $\varepsilon > 0$ and $\eta > 0$ there exists some $T > 0$ such that for all positive l^2 solution of (4.2) with initial energy $\mathcal{E}(0) \leq \eta$ one has $\mathcal{E}(T) \leq \varepsilon$. In particular*

$$\lim_{t\to\infty} \mathcal{E}(t) = 0,$$

i.e. there is anomalous dissipation.

Proof. Given $\varepsilon > 0$ let us take a sequence of positive numbers $(\delta_n)_{n\geq 0}$ such that

$$\sum_{n=o}^{\infty} \delta_n = \varepsilon \qquad \text{and} \qquad \sum_{n=o}^{\infty} \frac{1}{2^{(\alpha-\tilde{\alpha})n}\delta_n^2} < +\infty.$$

This is possible, for example, taking $\delta_n = \varepsilon\big(1 - 2^{-(\alpha-\tilde{\alpha})/3}\big)2^{-(\alpha-\tilde{\alpha})n/3}$. Now we can apply Lemma 4.5, so by the definition of h_n given in (4.7)

$$h_n \leq \frac{2\sqrt{2}\eta^{3/2}}{2^{(\alpha-\tilde{\alpha})n}\delta_n^2} \qquad \text{and} \qquad \sum_{n=0}^{\infty} h_n \leq \frac{2\sqrt{2}\eta^{3/2}}{(1 - 2^{-(\alpha-\tilde{\alpha})/3})^3} =: T.$$

Take $M = 0$ in (4.6) and by monotonicity of energy $\mathcal{E}(T) \leq \varepsilon$. □

We are now able to prove the following theorem, which is a consequence of Theorem 4.6 with a rescaling argument based on the fact that the non-linearity is homogeneous of degree two.

Theorem 4.7. *Let $\sharp\mathcal{O}_j = 2^{2\tilde{\alpha}}$ for all j. Suppose $\tilde{\alpha} < \alpha$ in equations (4.2). Let X be any positive l^2 solution with initial condition X^0. Then there exists $C > 0$, depending only on $\|X^0\|$, such that for all $t > 0$*

$$\mathcal{E}(t) := \|X(t)\|^2 := \sum_{j\in J} X_j^2(t) < \frac{C}{t^2}\ .$$

As the previous Theorem 4.6, this theorem also holds also if we use the weaker hypothesis $1 \leq \sharp\mathcal{O}_j \leq 2^{2\tilde{\alpha}}$ for all j. The statement tells us that the energy of the system goes to zero at least as fast as t^{-2}. In Section 3.1 we show that for this model there are some self-similar solutions and that their energy goes to zero exactly like t^{-2}. So the estimate of Theorem 4.7 cannot be improved much.

Proof of Theorem 4.7. By Theorem 4.6 for every $0 < \rho < 1$ there exists $\tau > 0$ depending only on ρ and $\mathcal{E}(0)$, such that $\mathcal{E}(\tau) \leq \rho^2\mathcal{E}(0)$. We will apply this bound to many different solutions, all of which have energy at time zero not above $\mathcal{E}(0)$.

Let $\vartheta = 1/\rho > 1$. We can define the sequence

$$X^{(0)} = X$$
$$X^{(n)}(t) = \vartheta X^{(n-1)}(\vartheta t + \tau) = \vartheta^n X\left(\vartheta^n t + \frac{\vartheta^n - 1}{\vartheta - 1}\tau\right), \qquad n \geq 1.$$

It is immediate to verify that all of these satisfy the system of equations (4.2), but with possibly different initial conditions. We have

$$\sum_{j\in J}\Big(X_j^{(n)}(0)\Big)^2 = \vartheta^2 \sum_{j\in J}\Big(X_j^{(n-1)}(\tau)\Big)^2.$$

Recalling the definition of τ, the above equation allows to prove by induction on n that for all $n \geq 0$ one has $\sum_{j\in J}\left(X_j^{(n)}(0)\right)^2 \leq \mathcal{E}(0)$. For all $n \geq 0$, let

$$t_n = \frac{\vartheta^n - 1}{\vartheta - 1}\tau.$$

Then by the definition of $X^{(n)}$, we have proved $\mathcal{E}(t_n)^2 \leq \vartheta^{-2n}\mathcal{E}(0)$. Since $\vartheta > 1$, $t_n \uparrow \infty$, hence given $t > 0$ there is n such that $t_n \leq t < t_{n+1}$. That means we have by monotonicity

$$\mathcal{E}(t) \leq \vartheta^{-2n}\mathcal{E}(0) \qquad \text{and} \qquad \frac{1}{t_{n+1}^2} < \frac{1}{t^2}.$$

Finally, by definition

$$t_{n+1} < \tau\vartheta^{n+1}/(\vartheta - 1) = \vartheta^n\tau/(1-\rho),$$

so for $C = \mathcal{E}(0)\big(\tau/(1-\rho)\big)^2$ we get

$$\mathcal{E}(t) \leq \vartheta^{-2n}\mathcal{E}(0) < \frac{C}{t_{n+1}^2} < \frac{C}{t^2}. \qquad \square$$

Remark 4.2. For the classic dyadic model, we have an anomalous dissipation result also in the stochastic version of the model (see Theorem A.10). For the tree dyadic model it is still an open question. The main difficulty in translating the result to the tree formulation lies once more, as it was the case with the uniqueness of positive solutions, in the difficulty of providing sharp bounds for the branches of the tree.

Chapter 5

Stationary solutions

In this chapter we consider a particular kind of solutions: the stationary ones, for tree and classic dyadic models with a forcing term, both in the inviscid and viscous case. These solutions are proven to exist and to be unique. Stationary solutions will be fundamental objects for the computation of the structure function in Chapter 6, but they pose a critical question of interpretation, as we discuss below.

When we consider solutions that are stationary in time, both the wavelet and Fourier interpretations look less convincing: the already quite rigid structure of the model appears now unbelievable from a physical point of view. The eddies, that are requested by the model not to interact outside the father-son relationships, are also fixed in time, which is in stark contrast of our physical perception on eddies in a turbulent fluid. This suggests us another way of looking at our model in the stationary case: as a sequence of pictures taken at different times and kept constant until the next picture is taken. Since we are not assuming any kind of geometrical ordering of the children of each node, we can think of our model as a representation of the phenomenon in a particular coordinate system: the one that keeps the eddies fixed, while in the fluid they move around the unit cube.

Our purpose is to prove the existence and uniqueness of stationary solution on the tree dyadic model and extend existence and uniqueness results given in [18] and [17] for the classic dyadic model. In [18] it is proven that the inviscid dyadic model with $\beta = 5/2$ has a unique stationary solution, while in the companion paper [19] it is proven that such a solution is a global attractor. One should notice that these results on the dyadic model are stronger than what can be proven for more realistic shell models such as SABRA (see for example [22]).

The viscous dyadic model is studied in [17], where it is proven that for $\beta \in (3/2, 5/2]$ the stationary solution is unique and is a global attractor. In [8] it is proven that for the viscous case it is possible, dropping the $Y_n \geq 0$ condition, to explicitly provide examples of non-uniqueness of the stationary solution. In this chapter we prove the existence and uniqueness of stationary solutions in l^2 for every positive value of the β and γ parameters both in viscous and inviscid dyadic models. This will provide a corresponding result of existence and uniqueness for $\alpha > \tilde{\alpha}$ and

$\gamma > 0$ in the tree dyadic model. Furthermore in the inviscid case we will explicitly provide those solutions (Proposition 5.3), while in the viscous case we'll prove that the stationary solutions are regular if and only if N_* is big enough, $N_* \geq 2^{2\alpha-3\gamma}$ or the forcing term f is small. For $f = 0$ the unique (non-negative) stationary solution is trivially the null one, so in this section we assume $f > 0$. This is the only interesting case, since we have a stationary regime, while energy keeps flowing along the tree, due to the forcing term.

We recall the equations we are interested in: the tree dyadic model

$$\begin{cases} X_{\bar{0}}(t) \equiv f \\ \dfrac{d}{dt}X_j = -\nu d_j X_j + c_j X_{\bar{j}}^2 - \displaystyle\sum_{k\in\mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J, \end{cases} \tag{5.1}$$

and the classic one, where J is simply the set of non-negative integers with $\mathcal{O}_j := \{j+1\}$ for all j, whose solution we will denote by Y_n and whose equations are

$$\begin{cases} Y_{-1}(t) \equiv f \\ \dfrac{d}{dt}Y_n = -\nu l_n Y_n + k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1}, \quad \forall n \geq 0. \end{cases} \tag{5.2}$$

We are considering $f \geq 0$, $\nu \geq 0$, $c_j = 2^{\alpha|j|}$, $d_j = 2^{\gamma|j|}$, $k_n = 2^{\beta n}$, $l_n = 2^{\gamma n}$, with $\alpha > 0$, $\beta > 0$ and $\gamma > 0$.

When the classic model (5.2) is interpreted as a special case of (5.1), we will have $N_* = 1$, that is $\tilde{\alpha} = 0$, and $\beta = \alpha$. Observe that the definitions of solutions given on the tree model extend easily to this one, but notice that in this setting l^2 will correspond to the standard space of sequences.

The following theorems sum up the results of this chapter on stationary solutions of classic and tree dyadic model respectively.

Theorem 5.1. *If $f > 0$, then there exists a unique l^2 positive solution Y of system* (5.2) *which is stationary. Moreover*

if $\nu = 0$ then $Y_n(t) := f2^{-\beta/3(n+1)}$;

if $\nu > 0$ and $3\gamma \geq 2\beta$, the stationary solution is conservative and regular*, in that for all real s, $\sum_n [2^{sn} Y_n(t)]^2 < \infty$;*

if $\nu > 0$ and $3\gamma < 2\beta$, there exists $C > 0$ such that for all $f > C$ the invariant solution of (5.2) *is not regular and exhibits anomalous dissipation.*

In the inviscid case, this theorem extends an analogue result of [18] where it is proved for $\beta = 5/2$. In the viscous case it extends a result of [17], in which existence and uniqueness of stationary solutions are proved for $\gamma = 2$ and $\beta \in (3/2, 5/2]$.

We wont detail the proof, since, by what we said above, it is a special case of the following result.

Theorem 5.2. *Let $\sharp\mathcal{O}_j = 2^{2\tilde{\alpha}}$ for all j. Suppose $\tilde{\alpha} < \alpha$ and $f > 0$ in equations* (5.1)*. Then there exists a unique l^2 positive solution X which is* stationary*. Moreover*

if $\nu = 0$ then $X_j(t) := f 2^{-(2\tilde{\alpha}+\alpha)(|j|+1)/3}$ for all $j \in J$;

if $\nu > 0$ and $0 < \alpha - \tilde{\alpha} \leq \gamma 3/2$, the stationary solution is conservative and regular*, in that for all real s, $\sum_{j\in J}[2^{s|j|}X_j(t)]^2 < \infty$;*

if $\nu > 0$ and $\alpha - \tilde{\alpha} > \gamma 3/2$, there exists $C > 0$ such that for all $f > C$ the invariant solution of (5.1) *is not regular and exhibits anomalous dissipation.*

We will give the proof of this result at the and of the chapter, building it up with intermediate results.

5.1 Existence in the inviscid case

In the inviscid case, the differential equation is very simple, so it is easy to find stationary solutions in the class of exponential functions. One immediately finds the following result.

Proposition 5.3. *Consider the tree dyadic model* (5.1) *and the classic dyadic model* (5.2)*, both inviscid ($\nu = 0$). Let $2^{2\tilde{\alpha}} = N_* = \sharp\mathcal{O}_j$ be constant for all $j \in J$. Then:*

1. *the sequence of constant functions $Y_n(t) := f 2^{-(n+1)\beta/3}$ is a positive l^2 solution of the system* (5.2)*.*
2. *the family of constant functions $X_j(t) := f 2^{-(2\tilde{\alpha}+\alpha)(|j|+1)/3}$ for $j \in J$ is a positive componentwise solution of system* (5.1)*; it is also an l^2 solution if and only if $\alpha > \tilde{\alpha}$;*

Proof. A direct computation shows that X and Y are componentwise solutions. To show that Y is l^2 observe that, since $\beta > 0$, $\|Y\| < \infty$. To check whether X is l^2 compute the energy by generations; we have for $n \geq 0$,

$$\mathcal{E}_n - \mathcal{E}_{n-1} = \sum_{|j|=n} X_j^2 = 2^{2\tilde{\alpha}n} f^2 2^{-\frac{n+1}{3}(4\tilde{\alpha}+2\alpha)} = C 2^{\frac{2}{3}(\tilde{\alpha}-\alpha)n},$$

with C not depending on n. Hence X is l^2 if and only if $\alpha - \tilde{\alpha} > 0$. □

5.2 Existence in the viscous case

In the viscous case, the recurrence relation coming from the definition of stationary solution is more complex, and has no solutions in the class of exponential functions. Anyway, by careful control of the recurrence behavior, we are able to prove that a stationary solution exists, and also to distinguish if it is conservative or has anomalous dissipation.

Definition 5.1. We say that a stationary positive l^2 solution X is regular if for all $h \in \mathbb{R}$

$$\sum_{j \in J} \left[2^{h|j|} X_j \right]^2 < \infty. \tag{5.3}$$

Theorem 5.4. *There exists a stationary positive l^2 solution of the classic dyadic model* (5.2) *when $\nu > 0$.*

Theorem 5.5. *Consider any stationary positive l^2 solution of the classic dyadic model* (5.2) *with $\nu > 0$.*

1. *If $3\gamma \geq 2\beta$ then it is regular and conservative.*

2. *If $3\gamma < 2\beta$ then there exists some $C > 0$ such that if $f > C$ the stationary solution is not regular and there is anomalous dissipation.*

Before we go into the proofs of these theorems, let us introduce a useful change of variables, that will come handy in both proofs. If Y is a stationary solution of (5.2) then, for every $n \geq 0$, we have

$$-\nu 2^{\gamma n} Y_n + 2^{\beta n} Y_{n-1}^2 - 2^{\beta n + \beta} Y_n Y_{n+1} = 0.$$

This equation can be made into a recurrence, and the change of variables that best simplifies its form is

$$Z_n := \nu^{-1} 2^{\frac{\beta}{3}(n+2)} Y_n. \tag{5.4}$$

Since the stationary solution in the inviscid case decreases like $2^{-n\beta/3}$, the exponent's rate $n\beta/3$ is in some sense expected. The system of differential equations for Z becomes

$$\begin{cases} Z_{-1} = \nu^{-1} 2^{\frac{\beta}{3}} f =: g \\ Z_{n+1} = \dfrac{Z_{n-1}^2}{Z_n} - 2^{(\gamma - \frac{2}{3}\beta)n} & \forall n \geq 0. \end{cases} \tag{5.5}$$

Proof of theorem 5.4. Let us consider the change of variable (5.4), we have to show that the system (5.5) has a positive solution for which Y is l^2. System (5.5) gives a recursion which, given $Z_{-1} = g$ and Z_0 allows to construct the sequence $(Z_n)_{n \geq -1}$ in a unique way. Any such sequence will give a stationary componentwise solution. What we want to prove is that there is some value of Z_0 such that this turn out to be a positive l^2 solution. Let we exploit the dependence from Z_0 by defining a sequence of real functions

$$\begin{aligned} Z_{-1}(a) &= g, \\ Z_0(a) &= a, \\ Z_{n+1}(a) &= \tfrac{Z_{n-1}^2(a)}{Z_n(a)} - 2^{(\gamma - \frac{2}{3}\beta)n}, \qquad n \geq 0. \end{aligned} \tag{5.6}$$

Now we construct a descending sequence of open real intervals $(I_n)_{n\geq 0}$ such that $(0,\infty) = I_0 \supset I_1 \supset I_2 \supset \dots$ and such that Z_n is continuous and bijective from I_n to $(0,\infty)$, with Z_n strictly increasing for even n and strictly decreasing for odd n.

Let $I_0 = (0,\infty)$. $Z_0(a)$ is monotone increasing, continuous and bijective from I_0 to $(0,\infty)$.

By (5.6) we have that $Z_1(a) = g/a^2 - 2^{(\gamma-2\beta/3)}$ is monotone decreasing, continuous and bijective from I_0 to $(-2^{(\gamma-2\beta/3)},\infty)$ so there exists a limited interval $(b_1,c_1) := I_1 \subset I_0$ such that $Z_1(a)$ is monotone decreasing, continuous and bijective from I_1 to $(0,\infty)$.

Now suppose we already proved for $m \leq n$ that $Z_m(a)$ is continuous and bijective from I_m to $(0,\infty)$, with Z_m strictly increasing for even m and strictly decreasing for odd m.

Suppose that n is odd (resp. even). Then by (5.6) $Z_{n+1}(a)$ is monotone increasing (resp. decreasing), continuous and bijective from I_n to $(-2^{(\gamma-2\beta/3)n},\infty)$ so there exists an interval $(b_{n+1},c_{n+1}) := I_{n+1} \subset I_n$ such that $Z_{n+1}(a)$ is monotone increasing (resp. decreasing), continuous and bijective from I_{n+1} to $(0,\infty)$.

Observe moreover that the borders of these intervals are not definitively constant, since for all n, $b_{n+2} \neq b_n$ and $c_{n+2} \neq c_n$. Hence if we define $b = \lim_n b_n$ and $c = \lim_n c_n$, it is clear that for all n, $b_n < b \leq c < c_n$, that is the closed interval (possibly degenerate) $[b,c]$ is contained in every I_n.

Now we choose any $\bar{a} \in [b,c]$ and we know that the sequence $Z_n(\bar{a})$ is strictly positive. We are left to prove that it is also l^2. To this end let Y_n be any stationary, positive componentwise solution. Let $\mathcal{E}_n = \sum_{k=0}^n Y_k^2$ in analogy with the definition for the tree model. We compute the derivative

$$0 = \frac{d}{dt}\mathcal{E}_n(t) = -\nu \sum_{k\leq n} l_k Y_k^2 + f^2 Y_0 - k_{n+1} Y_n^2 Y_{n+1},$$

hence, since $l_k \geq 1$, $\mathcal{E}_n \leq \sum_{k\leq n} l_k Y_k^2 \leq \nu^{-1} f^2 Y_0$ for all n. □

Proof of theorem 5.5. Let us consider again system (5.5) and let $\mu := \gamma - 2\beta/3$. If $\mu > 0$ the corrective term goes to infinity, while if $\mu < 0$ it goes to zero, so we expect two different behaviors in the two cases. We'll show that in the first case Z_n goes to zero super-exponentially for $n \to \infty$, while in the second one $Z_n \downarrow z$ and $z > 0$ if g is large enough.

Case $\mu := \gamma - 2\beta/3 \geq 0$. From (5.5) we get

$$2^{\mu n} Z_n^2 = Z_{n-1}^2 Z_n - Z_n^2 Z_{n+1}.$$

Sum over n to get

$$\sum_{k\leq n} 2^{\mu k} Z_k^2 = g^2 Z_0 - Z_n^2 Z_{n+1}. \tag{5.7}$$

Since $\mu \geq 0$, by positivity of Z, we have

$$\lim_{n\to\infty} Z_n = 0. \tag{5.8}$$

From (5.5) and $Z_{n+1} > 0$ we get $Z_n < Z_{n-1}^2$ and since by (5.8) $Z_{\bar{n}} =: \lambda < 1$ for some $\bar{n}$, by iterating the above equation we get for all $m \geq 0$

$$Z_{\bar{n}+m} \leq \lambda^{2^m},$$

that is to say that Z_n goes to zero for n going to infinity like the exponential of an exponential, so for every $s > 0$ we have

$$\sum_n (2^{sn} Z_n)^2 < +\infty \qquad \text{and} \qquad \sum_n (2^{sn} Y_n)^2 < +\infty.$$

It is now clear that $\lim_n k_{n+1} Y_n^2 Y_{n+1} = 0$, so Y is conservative by Definition 2.3.

Case $\mu := \gamma - 2\beta/3 < 0$. The first step is to prove that Z_n is non-increasing in n. Suppose by contradiction that for some n we have $Z_n/Z_{n-1} = \lambda > 1$, then we claim that $Z_{n+2}/Z_{n+1} > \lambda^4 > 1$ and hence by induction $Z_{n+2m}/Z_{n+2m-1} > \lambda^{4^m}$. By (5.5) for all $k \geq 0$

$$Z_{k+1} < \frac{Z_{k-1}^2}{Z_k} = \frac{Z_{k-1}}{Z_k} Z_{k-1}.$$

This can be used iteratively together with the claim to show that

$$\begin{aligned} Z_{n+2m+1} = & \frac{Z_{n+2m-1}^2}{Z_{n+2m}} - 2^{\mu(n+2m)} \\ & < \frac{Z_{n+2m-1}}{Z_{n+2m}} \frac{Z_{n+2m-3}}{Z_{n+2m-2}} \cdots \frac{Z_{n-1}}{Z_n} Z_{n-1} - 2^{\mu(n+2m)} \\ & \qquad < Z_{n-1} \lambda^{-4^m} - 2^{\mu(n+2m)}, \end{aligned}$$

so we get a contradiction because $Z_{n+2m+1} < 0$ for some m.

We prove the claim. Let $x = 2^{\mu n} Z_n / Z_{n-1}^2 = 2^{\mu n} \lambda^2 / Z_n$. Observe that

$$Z_{n+1} = \frac{Z_{n-1}^2}{Z_n} - 2^{\mu n} = \frac{2^{\mu n}}{x}(1 - x). \tag{5.9}$$

We divide by Z_n (and we notice that $x < 1$),

$$\frac{Z_{n+1}}{Z_n} = \lambda^{-2}(1 - x). \tag{5.10}$$

Now

$$Z_{n+2} = \frac{Z_n^2}{Z_{n+1}} - 2^{\mu(n+1)} > \frac{Z_n^2}{Z_{n+1}} - 2^{\mu n},$$

so dividing by Z_{n+1} and substituting (5.10) and (5.9), we get

$$\frac{Z_{n+2}}{Z_{n+1}} > \lambda^4 (1 - x)^{-2} - \frac{2^{\mu n}}{Z_{n+1}} > \frac{\lambda^4}{1 - x} - \frac{x}{1 - x}.$$

Since $\lambda > 1 > x > 0$, it is now clear that $(\lambda^4 - x)/(1 - x) > \lambda^4$. So we have proven the claim and showed that $\{Z_n\}_{n \geq 0}$ is non-increasing in n.

The last step is to show that for g large enough $Z_n \downarrow z > 0$. By rearranging (5.7) and recalling what we proved above,

$$Z_n^3 \geq Z_n^2 Z_{n+1} = g^2 Z_0 - \sum_{k=0}^{n} 2^{\mu k} Z_k^2 \geq g^2 Z_0 - g Z_0 \sum_{k=0}^{n} 2^{\mu k} > g Z_0 \Big(g - \frac{1}{1-2^\mu} \Big),$$

so if $g > 1/(1-2^\mu)$ then Z_n converges to a strictly positive constant z.

To prove anomalous dissipation we compute the limit

$$\lim_{n\to\infty} k_{n+1} Y_n^2 Y_{n+1} = \lim_{n\to\infty} 2^{\beta n+\beta} \nu^3 2^{-\beta n - 7\beta/3} Z_n^2 Z_{n+1} = 2^{-4\beta/3} \nu^3 z^3 > 0.$$

So by Definition 2.3 there is anomalous dissipation. □

5.3 Uniqueness in the inviscid and viscous case

We prove uniqueness in the class of stationary positive l^2 solutions for the tree dyadic model. The result also holds for the classic dyadic, because it is a particular case of the former, or by virtue of the lifting Proposition 2.6.

Theorem 5.6. *Consider the tree dyadic model* (5.1) *and assume that* $\alpha > \tilde{\alpha}$, *where* $2^{2\tilde{\alpha}} = N_* = \sharp \mathcal{O}_j$ *is the constant number of children for every node. Then there exists a unique stationary positive* l^2 *solution.*

Proof. Existence is a consequence of Proposition 5.3 in the inviscid case ($\nu = 0$) and Proposition 2.6 and Theorem 5.4 in the viscous case.

To prove uniqueness we apply a change of variables similar to (5.4)

$$Z_j := 2^{\frac{(2+|j|)\alpha}{3}} X_j, \qquad \forall j \in J. \tag{5.11}$$

Then from (5.1) we have

$$\frac{d}{dt} Z_j = -\nu 2^{\gamma |j|} Z_j + 2^{\frac{2}{3}\alpha |j|} Z_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} 2^{\frac{2}{3}\alpha |j|} Z_j Z_k, \tag{5.12}$$

so if X is a stationary solution, Z must satisfy

$$\begin{cases} Z_{\bar{0}} = f 2^{\alpha/3} \\ \displaystyle\sum_{k \in \mathcal{O}_j} Z_k = \frac{Z_{\bar{j}}^2}{Z_j} - \nu 2^{(\gamma - \frac{2}{3}\alpha)|j|}. \end{cases} \tag{5.13}$$

Moreover observe that the condition $X \in l^2$ is equivalent to

$$\sum_{j \in J} \big(2^{-\frac{\alpha}{3}|j|} Z_j \big)^2 < \infty. \tag{5.14}$$

Assume by contradiction that there are two different stationary solutions of (5.13) which we denote by $W = \{W_j\}_{j\in J}$ and $Z = \{Z_j\}_{j\in J}$. Let n be the smallest integer such that there exist $j_1 \in J$ with $|j_1| = n$ and $W_{j_1} \neq Z_{j_1}$. Without loss of generality we can take $W_{j_1}/Z_{j_1} =: \lambda > 1$.

Let $j_0 = k_0 = \bar{j_1}$ and $k_1 = j_1$. Extend these to two sequences of indices $(j_m)_{m\geq 0}$ and $(k_m)_{m\geq 0}$ with $j_m \in \mathcal{O}_{j_{m-1}}$ and $k_m \in \mathcal{O}_{k_{m-1}}$, picking alternatively among those that maximize or minimize W_{j_m} and Z_{k_m}.

More precisely for $m \geq 2$ choose $j_m \in \mathcal{O}_{j_{m-1}}$ and $k_m \in \mathcal{O}_{k_{m-1}}$ in such a way that if m is even

$$W_{j_m} = \min\{W_i : i \in \mathcal{O}_{j_{m-1}}\} \qquad\qquad Z_{k_m} = \max\{Z_i : i \in \mathcal{O}_{k_{m-1}}\},$$

and if m is odd

$$W_{j_m} = \max\{W_i : i \in \mathcal{O}_{j_{m-1}}\} \qquad\qquad Z_{k_m} = \min\{Z_i : i \in \mathcal{O}_{k_{m-1}}\}.$$

The idea supporting the definition of these sequences is to choose the indices so that

$$W_{j_1} < Z_{k_1}, \qquad W_{j_2} > Z_{k_2}, \qquad W_{j_3} < Z_{k_3}, \qquad \ldots$$

We will now prove that, with our construction, those inequalities hold and, moreover, the ratio between W_m and Z_m grows according to

$$\frac{Z_{k_m}}{W_{j_m}} \geq \frac{W_{j_{m-1}}}{Z_{k_{m-1}}} \frac{Z^2_{k_{m-2}}}{W^2_{j_{m-2}}} > \lambda^{2^{m-2}} \qquad \forall\, m \geq 2 \text{ even} \tag{5.15}$$

$$\frac{W_{j_m}}{Z_{k_m}} \geq \frac{Z_{k_{m-1}}}{W_{j_{m-1}}} \frac{W^2_{j_{m-2}}}{Z^2_{k_{m-2}}} > \lambda^{2^{m-2}} \qquad \forall\, m \geq 3 \text{ odd.} \tag{5.16}$$

We prove inequalities (5.15) and (5.16) by induction on $m \geq 2$. First note that for $m = 0$ and $m = 1$,

$$\frac{Z_{k_0}}{W_{j_0}} = 1 \qquad \text{and} \qquad \frac{W_{j_1}}{Z_{k_1}} = \lambda. \tag{5.17}$$

Now we proceed by induction. Let $m \geq 2$ even. By the definition of j_m, k_m and by (5.13) we get

$$W_{j_m} = \min_{i\in\mathcal{O}_{j_{m-1}}} W_i \leq N_*^{-1} \sum_{i\in\mathcal{O}_{j_{m-1}}} W_i = N_*^{-1}\left[\frac{W^2_{j_{m-2}}}{W_{j_{m-1}}} - \nu 2^{(\gamma-\frac{2}{3}\alpha)(n+m-2)}\right],$$

$$Z_{k_m} = \max_{i\in\mathcal{O}_{k_{m-1}}} Z_i \geq N_*^{-1} \sum_{i\in\mathcal{O}_{k_{m-1}}} Z_i = N_*^{-1}\left[\frac{Z^2_{k_{m-2}}}{Z_{k_{m-1}}} - \nu 2^{(\gamma-\frac{2}{3}\alpha)(n+m-2)}\right]. \tag{5.18}$$

By (5.17) when $m = 2$ or by inductive hypothesis (5.15) and (5.16) when $m \geq 4$,

$$\frac{Z^2_{k_{m-2}}}{Z_{k_{m-1}}} \Bigg/ \frac{W^2_{j_{m-2}}}{W_{j_{m-1}}} = \frac{Z^2_{k_{m-2}}}{W^2_{j_{m-2}}} \frac{W_{j_{m-1}}}{Z_{k_{m-1}}} > \begin{cases} \lambda & m = 2 \\ (\lambda^{2^{m-4}})^2 \lambda^{2^{m-3}} = \lambda^{2^{m-2}} & m \geq 4, \end{cases}$$

so in particular the ratio is above 1 and, since for every $a > b > c \geq 0$ we have $(a-c)/(b-c) \geq a/b$, for $m \geq 2$ even

$$\frac{Z_{k_m}}{W_{j_m}} \geq \frac{\frac{Z^2_{k_{m-2}}}{Z_{k_{m-1}}} - \nu 2^{(\gamma-\frac{2}{3}\alpha)(n+m-2)}}{\frac{W^2_{j_{m-2}}}{W_{j_{m-1}}} - \nu 2^{(\gamma-\frac{2}{3}\alpha)(n+m-2)}} \geq \frac{Z^2_{k_{m-2}}}{Z_{k_{m-1}}} \Bigg/ \frac{W^2_{j_{m-2}}}{W_{j_{m-1}}} > \lambda^{2^{m-2}}.$$

This concludes the inductive step for m even; for m odd the reasoning is analogous. We now want to use inequalities (5.15) and (5.16) to get a contradiction. We will consider separately the cases $\nu > 0$ and $\nu = 0$.
Case $\nu > 0$. Let m be even; by (5.15)

$$\frac{Z^2_{k_{m-2}}}{Z_{k_{m-1}}} > \lambda^{2^{m-2}} \frac{W^2_{j_{m-2}}}{W_{j_{m-1}}},$$

applying (5.13) to $W_{j_{m-1}}$ we have

$$\frac{W^2_{j_{m-2}}}{W_{j_{m-1}}} \geq \nu 2^{(\gamma-\frac{2}{3}\alpha)(n+m-2)},$$

so from (5.18), putting everything together, we get

$$Z_{k_m} \geq N_*^{-1}\left[\frac{Z^2_{k_{m-2}}}{Z_{k_{m-1}}} - \nu 2^{(\gamma-\frac{2}{3}\alpha)(n+m-2)}\right] \geq N_*^{-1}\nu 2^{(\gamma-\frac{2}{3}\alpha)(n+m-2)}\big(\lambda^{2^{m-2}} - 1\big).$$

For m even going to infinity we have obviously that Z_{k_m} grows as the exponential of an exponential, which is in contradiction with (5.14).
Case $\nu = 0$. If $\nu = 0$ we already know one explicit stationary solution, by Proposition 5.3, namely $X_j = f2^{-(2\tilde{\alpha}+\alpha)(|j|+1)/3}$. By the usual change of variables (5.11) $V_j = f2^{2(\alpha-\tilde{\alpha}(|j|-1))/3}$ is a solution of (5.13) satisfying the regularity condition (5.14). Without loss of generality we can suppose that $W_j = V_j$ or $Z_j = V_j$. In the first case, for m even

$$Z_{k_m} > W_{j_m}\lambda^{2^{m-2}} = f2^{\frac{2}{3}(\alpha-\tilde{\alpha}(n+m-2)}\lambda^{2^{m-2}},$$

in the second case for m odd

$$W_{j_m} > Z_{k_m}\lambda^{2^{m-2}} = f2^{\frac{2}{3}(\alpha-\tilde{\alpha}(n+m-2)}\lambda^{2^{m-2}}.$$

In both cases the right-hand side grows super-exponentially as $m \to \infty$ and this is in contradiction with (5.14). □

Proof of Theorem 5.2. Existence and uniqueness are given by Theorem 5.6.

If $\nu = 0$ the solution is identified by Proposition 5.3. If $\nu > 0$, by uniqueness, the solution is the lift of the stationary solution of the classic dyadic with $\beta = \alpha - \tilde{\alpha}$, as per Proposition 2.6. Then the two regimes are proven in Theorem 5.5. □

Chapter **6**

The structure function and turbulence

In this chapter we slightly change perspective towards a more physical and quantitative one. We are interested in understanding how our model behaves when compared to other models for turbulence, to the theoretical results of Kolmogorov's K41 theory and to the recent experimental results.

We are consider the inviscid and unforced system:

$$dX_j = \Big(c_j X_{\bar{j}}^2 - \sum_{k\in\mathcal{O}_j} c_k X_j X_k\Big)dt,$$

with coefficients of the form

$$c_j = 2^{\alpha|j|}d_j\,, \qquad \{d_k:\ k\in\mathcal{O}_j\} = \{\widetilde{d}_h:\ h=1\dots N_*\}$$

where we are assuming that $\sharp\mathcal{O}_j = N_* = 2^{2\widetilde{\alpha}}$ for all $j\in J$ and that the two sets are equal taking multiplicity into account. This choice of coefficients, while still in the family of the exponential coefficients considered before, in particular for anomalous dissipation, is slightly different: each coefficient has an exponential factor that depends only on the generation and a factor depending on the child it links to. It is an interesting choice: we are assuming that apart from some rescaling due to the generation, the structure of the model is the same if we consider the tree rooted in any node, that is to say there is some sort of structure invariant of the scale.

We look into stationary solutions of this system. As it was the case for the purely exponential choice of coefficients seen in Chapter 5, we have uniqueness of the stationary solution. Again, being in the inviscid case, we are able to show the actual form of the solution. The choice of a stationary solution might look like a poor one from a physical point of view, since as already argued the fluids are anything but stationary. Once more we stress that when considering a stationary solution, the most convincing interpretation of the model is a statistical one, not a strict interpretation as wavelet coefficients. With our model we are trying to capture some of the features of the turbulent fluid once it has reached a stationary regime. Also having an explicit form for the solution allows us to compute numerically some interesting quantities in the 3D setting, namely the exponents of the structure function and the multifractality of the scaling of the X_js and the energy.

6.1 Existence and uniqueness of the stationary solution

Let's consider now a stationary (in time) solution for

$$\begin{cases} dX_j = \Big(c_j X_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k \Big) dt, \\ X_{\bar{0}} = f. \end{cases} \tag{6.1}$$

where we are considering a forcing term to make the stationary solutions relevant, as argued in Chapter 5.

Theorem 6.1. *If $\alpha > \widetilde{\alpha}$, there exists a unique stationary solution X in l^2 to* (6.1) *with forcing term f.*

This theorem will come from Proposition 6.4 (existence) and Proposition 6.2 (uniqueness).

Proposition 6.2. *Suppose that there exists a stationary solution of* (6.1) *in l^2 with forcing term f, then it is unique.*

Proof. For every $j \in J$, we define $v_j := X_j / X_{\bar{j}}$. We also set $c_0 = 1$. From (6.1) we have

$$0 = c_j v_j^{-2} X_j^2 - \sum_{k \in \mathcal{O}_j} c_k v_k X_j X_j$$

and, by substituting the c_j's and simplifying, we may define

$$M_j := \frac{d_j}{v_j^2} = 2^\alpha \sum_{k \in \mathcal{O}_j} d_k v_k, \tag{6.2}$$

so that, by some rearranging

$$X_j = f \prod_{k \le j} v_k = f \left(\frac{\prod_{k \le j} d_k}{\prod_{k \le j} M_k} \right)^{1/2}. \tag{6.3}$$

By equation (6.2), $v_j = \sqrt{d_j / M_j}$, so the same equation can be rewritten as

$$M_j = 2^\alpha \sum_{k \in \mathcal{O}_j} d_k^{3/2} M_k^{-1/2}.$$

If we suppose for a moment that M_j is constant, than we have

$$M_j \equiv M := \left(2^\alpha \sum_{i=1}^{N_*} \widetilde{d}_i^{3/2}\right)^{2/3}, \qquad \text{for all } j \in J, \tag{6.4}$$

so that (6.3) rewrites as

$$X_j = f M^{-|j|/2} \prod_{k \leq j} \sqrt{d_k}. \tag{6.5}$$

To prove that M_j is constant indeed, we may introduce the new variables $\delta_j := M_j/M$ for all j, so that the recursion can be written in yet another form,

$$\delta_j = \frac{\sum_{k \in \mathcal{O}_j} d_k^{3/2} \delta_k^{-1/2}}{\sum_{i=1}^{N_*} \widetilde{d}_i^{3/2}}.$$

This means in particular that $\delta_j \leq \min_{k \in \mathcal{O}_j} \delta_k^{-1/2}$ and hence, considering also the analogous inequality for the maximum we get

$$\min_{k \in \mathcal{O}_j} \delta_k \leq \delta_j^{-2} \leq \max_{k \in \mathcal{O}_j} \delta_k.$$

Suppose now by contradiction that for some $j_1 \in J$ and $a > 1$ we have $\delta_j = a$. (The case $a < 1$ being analogous.)

Define a sequence $(j_n)_{n \geq 1}$ in J by

$$j_{n+1} := \begin{cases} \operatorname{argmin}_{k \in \mathcal{O}_{j_n}} \delta_k & \text{if } n+1 \text{ even} \\ \operatorname{argmax}_{k \in \mathcal{O}_{j_n}} \delta_k & \text{if } n+1 \text{ odd} \end{cases} \qquad n \geq 1$$

then

$$\delta_{j_{2n-1}} \geq \delta_{j_{2n-2}}^{-2}, \qquad \delta_{j_{2n}} \leq \delta_{j_{2n-1}}^{-2}$$

so that in particular

$$\delta_{j_{2n}} \leq \delta_{j_{2n-2}}^4 \qquad \text{and} \qquad \delta_{j_{2n-1}} \delta_{j_{2n}} \leq \delta_{j_{2n-1}}^{-1} \leq \delta_{j_{2n-2}}^2,$$

yielding

$$\delta_{j_{2n}} \leq a^{-2^{2n-1}} \qquad \text{and} \qquad \prod_{k=1}^{2n} \delta_{j_k} \leq \delta_{j_{2n-1}} \delta_{j_{2n}} \leq a^{-2^{2n-3}}.$$

This means that $X_{j_n}^2$ grows super-exponentially fast for n even, since, by (6.3),

$$X_{j_n}^2 = X_{j_1}^2 M^{-n} \frac{\prod_{k=1}^n d_{j_k}}{\prod_{k=1}^n \delta_{j_k}} \geq X_{j_1}^2 \left(\frac{\min_i \widetilde{d}_i}{M}\right)^n a^{2^{n-3}}, \qquad n \text{ even}$$

yielding that $X \notin l^2$, which is a contradiction. □

Before we move on to the existence result, let us prove the following identity, which will be used frequently in the following.

Lemma 6.3. *For any real function φ and positive integer n*

$$\sum_{|j|=n}\prod_{k\leq j}\varphi(d_k)=\Big(\sum_{i=1}^{N_*}\varphi(\widetilde{d}_i)\Big)^n.$$

Proof. Let $Z=\{1,2,\dots,N_*\}^n$. Then both sides of the identity are equal to

$$\sum_{z\in Z}\varphi(\widetilde{d}_{z_1})\varphi(\widetilde{d}_{z_2})\dots\varphi(\widetilde{d}_{z_n}).$$ □

In particular, if X is the stationary solution in l^2

$$\sum_{j\in J}X_j^2=\frac{f^2}{1-M^{-1}\sum_{i=1}^{N_*}\widetilde{d}_i}.$$

We are now able to prove that the solution given by (6.5) is indeed in l^2 when $\alpha>\widetilde{\alpha}$, which was the condition for anomalous dissipation of Theorem 4.6.

Proposition 6.4. *There exists an l^2 stationary solution if and only if*

$$\sum_{i=1}^{N_*}\widetilde{d}_i<M:=\Big(2^\alpha\sum_{i=1}^{N_*}\widetilde{d}_i^{3/2}\Big)^{2/3}. \tag{6.6}$$

A sufficient condition for this inequality to hold is $\alpha>\widetilde{\alpha}$.

Proof. Consider the unique solution given in Theorem 6.1 and apply Lemma 6.3 to compute

$$\sum_{j\in J}X_j^2=f^2\sum_{n=0}^{\infty}M^{-n}\sum_{|j|=n}\prod_{k\leq j}d_k=f^2\sum_{n=0}^{\infty}M^{-n}\Big(\sum_{i=1}^{N_*}\widetilde{d}_i\Big)^n.$$

The solution is l^2 if and only if the geometric sum converges.

The inequality (6.6) is easily proven to be true if $\alpha>\widetilde{\alpha}=\frac{1}{2}\log_2 N_*$, since for any choice of the $\widetilde{d}_i$'s

$$\frac{1}{N_*}\sum_{i=1}^{N_*}\widetilde{d}_i\leq\Big(\frac{1}{N_*}\sum_{i=1}^{N_*}\widetilde{d}_i^{3/2}\Big)^{2/3}$$

by power means inequality (or Hölder inequality on a finite space). □

Now the proof of the existence and uniqueness Theorem 6.1 is complete. Moreover we have that the solution is of the form (6.5).

It would be interesting to understand if this solution is stable and attractor. The existence of a global attractor has been studied for the classic dyadic case in [19]. This is still an open problem for the tree dyadic model. While some preliminary results show that the linearized system is stable in a suitable topology, the main formulation is still without an answer. The difficulties lie in calculating the Lyapunov function and in getting suitable bounds or estimates.

6.2 Structure function

In the remainder of this chapter we will consider only the case $d = 3$ and $\alpha = 5/2 > 3/2 = \widetilde{\alpha} = d/2$, since we are mainly interested in the physical 3D case. However all the following computations could be don in a generic d-dimensional space and with different values of α.

Since 1941 and the founding results of Kolmogorov, turbulent velocity fields have been studied by means of the moments of the velocity increments, also called *structure function*. The structure function of order p, that is to say the p-th statistical moment, is defined for a fluid with velocity field u as

$$S_p(r) = E_x[|u(x+r) - u(x)|^p],$$

when considered at scale r. Due to the scaling properties of the tree dyadic model we can consider only some scales and take

$$S_p(2^{-n}) = E_x\left[|(u(x+2^{-n}e) - u(x)) \cdot e|^p\right] \tag{6.7}$$

We can consider the unit cube where the dynamics take place with a different coordinate system: we consider $x \in \mathbb{X}$ as an infinite branch on the tree, $x = (j_0^x, j_1^x, \ldots)$, that is to say as the limit of the dyadic cubes $Q_{j_n^x}$ in which it is contained. Of course as soon as we consider this coordinate system we lose any geometric property of the actual physical space in which the fluid lives, as we have no information as where each eddy actually is in the space. This allows us to disregard the versor (or direction) e in the definition of S_p above, while the expectation with respect to x is still the integral on the whole unit cube Q.

On $\mathbb{X}$ we introduce the distance d defined by

$$d(x, y) := 2^{-\sup\{n \geq 0 : j_n^x = j_n^y\}}$$

with the convention that $2^{-\infty} = 0$. With this distance, a ball centered in x of radius 2^{-n} will be the set

$$B(x, 2^{-n}) := \{y \in \mathbb{X} : \ j_i^y = j_i^x, \ \forall\, 0 \leq i \leq n\}. \tag{6.8}$$

in particular $\mathbb{X} = B(x, 1) = Q_{j_0^x}$ for any x. The supports of the wavelets of the nth generation are exactly all the balls $B(x, 2^{-n})$.

The Kolmogorov K41 theory states that the structure function scales as

$$S_p(2^{-n}) \sim (2^{-n})^{p/3},$$

so we are interested in the exponents of the structure function, more than the structure function itself, as we want to compare them to the predicted value of $p/3$, and the experimentally measured ones. Hence we will be mainly interested, given our choice of scales, in the following quantities, called *exponents of the structure function*:

$$\zeta_p = \lim_{n \to \infty} -\frac{1}{n} \log_2 S_p(2^{-n}) \tag{6.9}$$

In the literature of shell models for Euler and Navier-Stokes equations (see for example [14]), the structure function is defined for scale 2^{-n} as

$$S_p(2^{-n}) = E_x\left[\sum_{|j|=n} |u_j|^p\right]$$

that is considering only the contributes of the terms with scale of order 2^{-n}. We will try to obtain a structure function in the case of the unique stationary solution using the definition of structure function (6.7).

We have, considering u written in wavelet basis as $u = \sum X_j \psi_j$

$$\begin{aligned}
S_p(2^{-n}) &= \int_Q |u(x+2^{-n}e) - u(x)|^p dx \\
&= \int_Q \left| \sum_{k=0}^{+\infty} \sum_{|j|=k} \langle u(\cdot + 2^{-n}e) - u, \psi_j \rangle \psi_j(x) \right|^p dx \\
&= \sum_{k=0}^{+\infty} \sum_{|j|=k} \int_Q |\langle u(\cdot + 2^{-n}e) - u, \psi_j \rangle|^p |\psi_j(x)|^p dx
\end{aligned}$$

For $|j| \geq n$ we have that for every x, x and $x + 2^{-n}e$ are always in the support of two different wavelets, orthogonal with one another, so we can write the contribution to $S_p(2^{-n})$ of the generations $m \geq n$ as

$$\begin{aligned}
S_p^{\geq}(2^{-n}) &= \sum_{|j|\geq n} X_j^p \left(\int_Q |\psi_j(x+2^{-n}e)|^p dx + \int_Q |\psi_j(x)|^p dx \right) \\
&= \sum_{|j|\geq n} X_j^p \left(\int_{Q_j} 2^{|j|\frac{3}{2}p} \psi^p(2^{|j|}x + 2^{|j|-n}e + \sigma_j) dx \right. \\
&\quad \left. + \int_{Q_j} 2^{|j|\frac{3}{2}p} \psi^p(2^{|j|}x + \sigma_j) dx \right) \\
&= \sum_{|j|\geq n} X_j^p \left(\int_Q 2^{3|j|p/2} |\psi(y)|^p 2^{-3|j|} dy + \int_Q 2^{3|j|p/2} |\psi(y)|^p 2^{-3|j|} dy \right),
\end{aligned}$$

where we used the properties of the wavelet, generated by a single mother wavelet ψ. Now we use the fact that the integral of ψ depends only on p and not on j:

$$\begin{aligned}
S_p^{\geq}(2^{-n}) &= \sum_{|j|\geq n} X_j^p 2^{|j|3(\frac{p}{2}-1)} 2 \int_Q |\psi(y)|^p dy \\
&= \sum_{k=n}^{+\infty} 2^{k3(\frac{p}{2}-1)} 2I(p) \sum_{|j|=k} X_j^p,
\end{aligned}$$

and we also put together the contribution given by any term of each generation $k \geq n$. Now we use the fact that we have an explicit unique stationary solution, so

$$
\begin{aligned}
S_p^{\geq}(2^{-n}) &= \sum_{k=n}^{+\infty} 2^{3k(\frac{p}{2}-1)} 2I(p) \sum_{|j|=k} f^p M^{-pk/2} \prod_{l\leq j} d_l^{p/2} \\
&= \sum_{k=n}^{+\infty} 2I(p) f^p 2^{kp\cdot 3/2 - 3k - kp\cdot 5/6} \left(\frac{1}{8}\sum_{i=1}^{8} \tilde{d}_i^{3/2}\cdot 8\right)^{-\frac{kp}{3}} \left(\frac{1}{8}\sum_{i=1}^{N_*} \tilde{d}_i^{p/2}\cdot 8\right)^{k}
\end{aligned}
$$

We define

$$
\ell_p = \log_2 \left(1/8 \sum_{i=1}^{8} \tilde{d}_i^p\right)^{1/p},
$$

the logarithm in base 2 of the p-th mean of the $\tilde{d}_i$s, and we can use it in the previous expression to get

$$
\begin{aligned}
S_p^{\geq}(2^{-n}) &= \sum_{k=n}^{+\infty} 2I(p) f^p 2^{kp\cdot 3/2 - 3k - kp\cdot 5/6 - kp/3 - kp/2\ell_{3/2} - kp + k\ell_{p/2} + 3k} \\
&= \sum_{k=n}^{+\infty} 2I(p) f^p 2^{-k(p/3 + p/2(\ell_{3/2} - \ell_{p/2}))}.
\end{aligned}
$$

Finally we can define

$$
\zeta_p = \frac{p}{3} + \frac{p}{2}(\ell_{3/2} - \ell_{p/2}), \tag{6.10}
$$

which is independent of n and k. We can put this quantity in evidence in the previous equation and get

$$
\begin{aligned}
S_p^{\geq}(2^{-n}) &= \sum_{k=n}^{+\infty} 2I(p) f^p 2^{-k\zeta_p} \\
&= 2I(p) f^p 2^{-n\zeta_p} \sum_{k=0}^{+\infty} 2^{-k\zeta_p}.
\end{aligned} \tag{6.11}
$$

The last sum converges independently of n for $\zeta_p > 0$. We will get back to this condition.

Now we need to assess the contribution of the first n generations. We are not able to provide a proof of this fact, but we will give an heuristic argument suggesting that its contribute won't affect the exponent of the structure function, when we consider the limit in (6.9).

We have that to estimate

$$
S_p^{<}(2^{-n}) = \sum_{k=0}^{n-1} \sum_{|j|=k} \int_Q |\langle u(\cdot + 2^{-n}e) - u, \psi_j \rangle|^p |\psi_j(x)|^p dx. \tag{6.12}
$$

In order to do that we start with the following chain of equalities

$$\begin{aligned}
\langle u(\cdot + 2^{-n}e) - u, \psi_j\rangle &= \langle u(\cdot + 2^{-n}e), \psi_j\rangle - \langle u, \psi_j\rangle \\
&= \langle u, \psi_j(\cdot - 2^{-n}e) - \psi_j\rangle \\
&= \Big\langle \sum_{i\in J} X_i\psi_i, \psi_j(\cdot - 2^{-n}e) - \psi_j \Big\rangle \\
&= \sum_{i\in J} X_i\langle \psi_i, \psi_j(\cdot - 2^{-n}e) - \psi_j\rangle \\
&= \sum_{i\neq j} X_i\langle \psi_i, \psi_j(\cdot - 2^{-n}e)\rangle + X_j\langle \psi_j, \psi_j(\cdot - 2^{-n}e) - \psi_j\rangle
\end{aligned}$$

So we can now write the following inequality:

$$\begin{aligned}
|\langle u(\cdot + 2^{-n}e) - u, \psi_j\rangle| &\leq |X_j||\langle \psi_j, \psi_j(\cdot - 2^{-n}e) - \psi_j\rangle| \\
&\leq |X_j|\|\psi_j(\cdot - 2^{-n}e) - \psi_j\|.
\end{aligned}$$

We go on estimating the second term, but here we are requiring that the wavelets are Lipschitz functions. Moreover we can't just restrict ourselves to the dyadic cubes Q_j, since if we are near the boundary of the cube, a translation of $2^{-n}e$ would send us out, hence we consider the enlarged cube Q_j^e.

$$\begin{aligned}
\|\psi_j(\cdot - 2^{-n}e) - \psi_j\|^2 &= \int_Q (\psi_j(x - 2^{-n}e) - \psi_j(x))^2 dx \\
&= \int_{Q_j^e} 2^{3|j|}(\psi(2^{|j|}x - 2^{-n+|j|}e + \sigma_j) - \psi(2^{|j|}x + \sigma_j))^2 dx \\
&\leq \mathrm{vol}(Q_j^e) 2^{3|j|} L^2 2^{-2(n-|j|)} \\
&= 2^{-2|j|}(2^{-|j|} - 2^{-n}) 2^{3|j|} L^2 2^{-2(n-|j|)} \\
&= (1 - 2^{-(n-|j|)}) L^2 2^{-2(n-|j|)} \\
&\leq L^2 2^{-2(n-|j|)},
\end{aligned}$$

where L is the Lipschitz constant for ψ.

We can put all together now, and we have that the contribution to $S_p(2^{-n})$ given by the first n generations can be (roughly) estimated as follows:

$$\begin{aligned}
S_p^<(2^{-n}) &= \sum_{k=0}^{n-1}\sum_{|j|=k}\int_Q |\langle u(\cdot+2^{-n}e)-u,\psi_j\rangle|^p|\psi_j(x)|^p dx \\
&\leq \sum_{k=0}^{n-1}\sum_{|j|=k}|X_j|^p L^p 2^{-p(n-k)}\int_Q|\psi_j(x)|^p dx \\
&= \sum_{k=0}^{n-1} L^p 2^{-p(n-k)} 2^{-3k+3kp/2}\int_Q|\psi(x)|^p dx \sum_{|j|=k}|X_j|^p \\
&= L^p I(p) f^p \sum_{k=0}^{n-1} 2^{-p(n-k)} 2^{-k\zeta_p} \\
&= L^p I(p) f^p 2^{-n\zeta_p}\sum_{k=0}^{n-1} 2^{-p(n-k)} 2^{(n-k)\zeta_p} \\
&\leq L^p I(p) f^p 2^{-n\zeta_p}\sum_{k=0}^{+\infty} 2^{-(p-\zeta_p)k}
\end{aligned}$$

and since, definitely in p, $\zeta_p < p$ the last sum converges independently of n to a quantity itself independent of n. Then the contribution of the first generation is bounded by

$$S_p^<(2^{-n}) \leq C(p)2^{-n\zeta_p},$$

so we can approximately consider the structure function as made only of the generations from n onwards, for the sake of computing the exponents via (6.9).

So we have the exponents of the structure function, ζ_p, given by (6.10). We can write it in a more explicit way, dropping the ℓ_ps, and we get

$$\zeta_p = -\frac{2}{3}p + 3 + \frac{p}{3}\log_2\Big(\sum_{i=1}^{8}\tilde{d}_i^{3/2}\Big) - \log_2\Big(\sum_{i=1}^{8}\tilde{d}_i^{p/2}\Big). \tag{6.13}$$

So, as already mentioned, we have that the exponents depend only on p and the coefficients $\tilde{d}_i$. But, whatever the choice of the coefficients, we have

$$\zeta_0 = 0 \qquad \zeta_3 = 1,$$

the second one being a requirement in turbulence theory arising from the (non-phenomenological) Kolmogorov four-fifths law. With these two points fixed, the actual form of the exponents as a continuous function of p change depending on the coefficients $\tilde{d}_i$. With the following lemma we prove a useful property of ζ_p, regarding the way it depends on the coefficients chosen.

Lemma 6.5. *Function ζ_p is invariant with respect to a rescaling of all coefficients $\tilde{d}_i$ by a common factor $\lambda > 0$.*

Proof. It's just an easy computation:

$$\begin{aligned}
\zeta_p\Big(\lambda\tilde{d}_i\Big) &= -3\Big(\frac{p}{2}-1\Big)+\frac{5}{6}p+\frac{p}{3}\log_2\Big(\lambda^{3/2}\sum_{i=1}^{8}\tilde{d}_i^{3/2}\Big)-\log_2\Big(\lambda^{p/2}\sum_{i=1}^{8}\tilde{d}_i^{p/2}\Big)\\
&= -3\Big(\frac{p}{2}-1\Big)+\frac{5}{6}p+\frac{p}{3}\log_2\Big(\sum_{i=1}^{8}\tilde{d}_i^{3/2}\Big)+\frac{p}{2}\log_2\lambda\\
&\quad -\log_2\Big(\sum_{i=1}^{8}\tilde{d}_i^{p/2}\Big)-\frac{p}{2}\log_2\lambda,
\end{aligned}$$

which is $\zeta_p\Big(\tilde{d}_i\Big)$. □

We can check that the function ζ_p has an oblique asymptote, that is to say its behaviour is eventually linear. Such asymptote is

$$\begin{aligned}
\lim_{p\to\infty}\frac{\zeta_p}{p} &= \frac{1}{3}\log_2\sum_{i=1}^{8}\tilde{d}_i^{3/2}-\frac{3}{2}+\frac{5}{6}-\lim_{p\to\infty}\log_2\Big(\sum_{i=1}^{8}\tilde{d}_i^{p/2}\Big)^{1/p}\\
&= \frac{1}{3}+\frac{1}{2}(\ell_{3/2}-\ell_\infty),
\end{aligned} \tag{6.14}$$

where $\ell_\infty = \log_2\Big(\max\{\tilde{d}_i\}\Big)$. From this equation we can see that the eventual behaviour is along a line of coefficient less or equal than 1/3, the Kolmogorov prescription, but greater than 0.

This is a consequence of the positivity condition on ζ_p, which we needed to define it: from (6.10) we need to have

$$p\Big(\frac{1}{3}+\frac{1}{2}\big(\ell_{3/2}-\ell_{p/2}\big)\Big)>0\,\forall p>0 \qquad \text{that is} \qquad \frac{1}{3}+\frac{1}{2}\big(\ell_{3/2}-\ell_\infty\big), \tag{6.15}$$

so the asymptote has to be positive.

It is intuitive that the exponents have to be positive, the lack of this property is considered the biggest drawback of the lognormal model (see [29]). Moreover in the literature (see [29] and [21]) it is also proven that the function ζ_{2p}, that is the restriction of ζ_p to the positive even integers is concave and non-decreasing. Moreover if we go point by point in the cube or on $\mathbb{X}$, we can write the energy, where we use that $\int\psi_j^2=1$

$$\begin{aligned}
\mathcal{E}(x) &= \sum_{k=0}^{+\infty}X_{j_k^x}^2\psi_{j_k^x}^2(x)=\sum_{k=0}^{+\infty}f^2M^{-k}\prod_{l=0}^{k}d_{j_l^x}\cdot 2^{3k}\\
&= f^2\sum_{k=0}^{+\infty}\prod_{l=1}^{k}\frac{d_{j_l^x}}{2^{\alpha 2/3-3}(\sum\tilde{d}_i^{3/2})^{2/3}}\le\sum_{k=0}^{+\infty}\Big(\frac{\tilde{d}_{max}}{2^{\alpha 2/3-3}(\sum\tilde{d}_i^{3/2})^{2/3}}\Big)^k
\end{aligned}$$

So, in order to have finite energy in every point it is enough to have convergence for this geometric sum, that is

$$\frac{\tilde{d}_{max}}{2^{-4/3}(\sum \tilde{d}_i^{3/2})^{2/3}} < 1,$$

or equivalently, taking the logarithm,

$$\ell_\infty < -\frac{4}{3} + \ell_{3/2} + 2 \qquad \text{that is} \qquad \frac{2}{3} > \ell_\infty - \ell_{3/2},$$

which is (6.15).

6.2.1 Comparison with other models

We want to give now a brief comparison of the behaviour of our model with respect to the Kolmogorov $p/3$ line, the She-Lévêque model and some experimental results retrieved from [35] and [29].

In his 1962 paper [34], Kolmogorov stated a refined version of his scaling principle, suggesting that the scaling behaviour of the structure function would be related to the scaling with respect to the distance r considered and to the scaling of the energy dissipation for a ball of size r: so

$$S_p(r) \sim r^{\zeta_p} \qquad \text{became} \qquad S_p(r) \sim \mathcal{E}_r^{p/3} \cdot r^{p/3}.$$

Setting $\tau(p)$ the scaling exponent of the energy with respect to the distance r, this would give

$$\zeta_p = \frac{p}{3} + \tau\Big(\frac{p}{3}\Big).$$

Under the Kolmogorov K41 assumptions, the scaling of the energy is independent of r, hence the prediction $\zeta_p = p/3$.

The appearing of the term τ is however very interesting, because it suggests that the discrepancies between the scaling predicted by Kolmogorov and the ones observed in experiments are due to the energy dissipated by the fluid. This discrepancy has taken the name of *intermittency*. Such intermittency phenomena have indeed been observed, see, for example, [35], [29], [11], [10].

Some models have been developed to capture this intermittency phenomenon. Among them the already cited lognormal model

$$\zeta_p = \frac{p}{3} + \frac{\mu}{18}\big(3p - p^2\big),$$

which has the disadvantage of being decreasing for

$$p > \frac{3(\mu + 2)}{2\mu},$$

and definitely negative, and the acclaimed She-Lévêque model:

$$\zeta_p = \frac{p}{9} + 2\left(1 - \left(\frac{2}{3}\right)^{\frac{p}{3}}\right),$$

Now we plot in Figure 6.1 some of the possible functions, obtained with different choices of d_i against the experimental values for ζ_p for $p = 1 \ldots 8$ shown in [29] and [35]. We also add a plot of the She-Lèvêque model. The plot has been realized with the package `ggplot2` for R ([40], [38]).

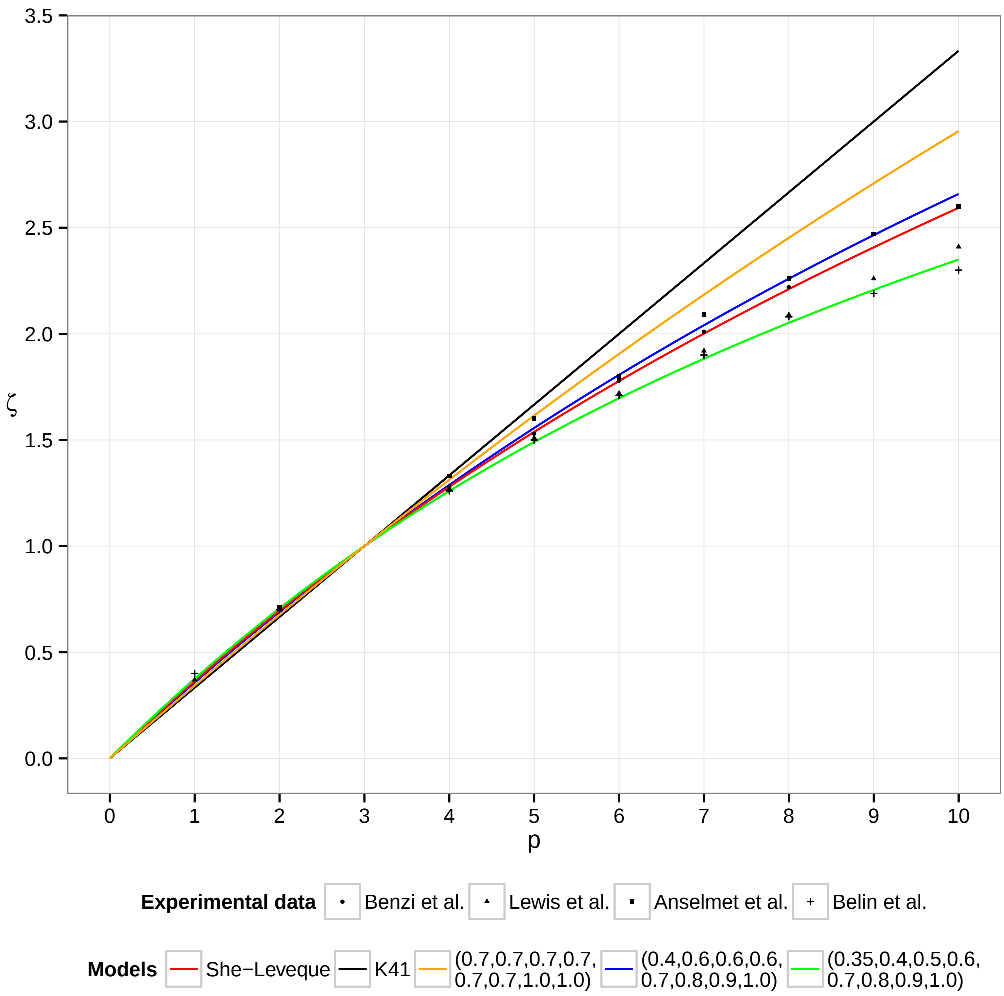


Figure 6.1: ζ_p in terms of p for some choices of coefficients, including the K41 limit result when the 8 coefficients are all equal, in the She-Lévêque model and in some experimental results, from Benzi et al. [12], Lewis and Swinney [35], Anselmet et al. [1] and Belin et al. [9].

6.3 Rescaling of the X_js

There are some questions that rise naturally: where does the dissipation of energy occur and where is the rescaling of the X_js bigger and smaller? Due to the lack of geometry assumptions for our model we cannot provide precise statements about these questions, but we are still able to obtain some dimensional results.

We can see our model as some kind of multifractal model, as in [29], since with different choices of our coefficients we get different kind of curves for ζ_p, which would require a continuous infinity of scaling exponents.

What we want to do is to address the question of the rescaling of X_j in generation n with respect to X_0:

$$X_{j_n^x} = X_0(2^{-n})^{h_n(x)},$$

hence

$$\begin{aligned} h_n(x) = -\frac{\log_2 X_{j_n^x}/X_0}{n} &= \frac{1}{2}\log_2 M - \frac{1}{n}\log_2 \prod_{k=1}^{n} \sqrt{d_{j_k^x}} \\ &= \frac{5}{6} + \frac{1}{3}\log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2} - \frac{1}{n}\log_2 \prod_{k=1}^{n} \sqrt{d_{j_k^x}} \\ &= \frac{5}{6} + \frac{1}{3}\log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2} + \frac{1}{n}\sum_{k=1}^{n} \frac{1}{2}\log_2(\frac{1}{d_{j_k^x}}) \\ &= \frac{5}{6} + \frac{1}{3}\log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2} + \frac{1}{n}\sum_{k=1}^{n} Z_k(x) \\ &= \frac{5}{6} + \frac{1}{3}\log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2} + L_n(x). \end{aligned} \tag{6.16}$$

The last two formulations are just to put in evidence the following: we can see the $Z_k(x)$ as i.i.d. random variables on the set

$$\left\{ \frac{1}{2}\log_2 \frac{1}{\tilde{d}_i}, \ i = 1 \dots 8 \right\},$$

with respect to the Lebesgue (probability) measure on the cube. In fact we can pick x as a random point and that is equivalent to pick all the coefficients d_j in its past at random. By interpreting $h_n(x)$ in this way we can state that, by the Law of Large Numbers,

$$L_n(x) \xrightarrow[n\to\infty]{\text{a.s.}} E[Z_1] = \frac{1}{2\cdot 8}\sum_{i=1}^{8} \log_2 \frac{1}{\tilde{d}_i}$$

so, combining this with (6.16), we have

$$h_n(x) \xrightarrow[n\to\infty]{\text{a.s.}} \nu = \frac{5}{6} + \frac{1}{3}\log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2} + \frac{1}{16}\sum_{i=1}^{8} \log_2 \frac{1}{\tilde{d}_i}.$$

Remark 6.1. We could also consider the rescaling of the n-th generation component of $u(x)$, that is $X_{j_n}\psi_{j_n}(x)$. With this choice we'd obtain the same result for $h_n(x)$, only translated by $-3/2$.

More generally we can say that the limit function $h(x)$, defined as the pointwise limit of the $h_n(x)$, takes value in an interval, $[h_{min}, h_{max}]$, corresponding to the points that have all the coefficients d_i equal to the maximum and minimum respectively. It would be interesting to understand at least how many points in the cube scale with a certain exponent h. The Lebesgue measure on the cube is not a good way to quantify this, since the set that scales with the limit exponent ν has Lebesgue measure 1. A suitable candidate is the Hausdorff dimension.

So, as the next step in understanding the behaviour of the rescaling of our model we consider, for each h in the interval $[h_{min}, h_{max}]$, the set $\mathcal{S}_h$ where the (limit) scaling exponent is $h(x) = h$ and its Hausdorff dimension

$$D(h) = \dim_H(\mathcal{S}_h).$$

Our plan is to compute the Hausdorff dimension by exploiting the fact that the $Z_k(x)$ are i.i.d. discrete uniform random variables and $L_n(x)$ satisfies a large deviation principle. In particular we will use the formulation that states that there exists a non-negative, convex function $I : \mathbb{R} \to [0, \infty]$ such that

$$\begin{cases} \lim_n \frac{1}{n} \log \mathcal{L}(L_n > a) = -I(a) & \text{if } a > \nu \\ \lim_n \frac{1}{n} \log \mathcal{L}(L_n < a) = -I(a) & \text{if } a < \nu. \end{cases}$$

We will consider the case $a > \nu$. The approach is completely symmetric if we are in the case $a < \nu$, even if the resulting function is not.

First of all we introduce some sets:

$$\begin{aligned} E_n(a) &:= \{x \in \mathbb{X} : h_n(x) > a\} \\ E(a) &:= \{x \in \mathbb{X} : \liminf_n h_n(x) > a\}. \end{aligned}$$

We can rewrite

$$E(a) = \liminf_{n\to\infty} E_n(a) = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} E_m(a) \tag{6.17}$$

and denote by

$$D_n(a) = \bigcap_{m\geq n} E_m(a).$$

For every n we can consider a finite set C_n of points in $\mathbb{X}$ such that

$$\mathbb{X} = \dot{\bigcup_{c_n \in C_n}} B(c_n, 2^{-n})$$

where the union is a disjoint union, and for every $y \in B(x, 2^{-n})$ we have $h_n(y) = h_n(x)$. So we can write

$$E_n(a) = \dot{\bigcup_{x \in C_n \cap E_n(a)}} B(x, 2^{-n}),$$

that is a disjoint union of a finite number af balls and in particular we have that for $m \geq n$, $E_m(a)$ is a covering of $D_n(a)$ with a finite number of balls.

with

$$|C_n \cap E_n(a)| = 2^{3n} \cdot \mathcal{L}(E_n(a)),$$

which is just the number of balls in generation n times the ratio of volumes.

By definition the Hausdorff dimension of $D_n(a)$ is

$$\begin{aligned}
\dim_H(D_n(a)) &= \inf\{\delta \geq 0 : \inf_{m \geq n} (|C_m \cap E_m(a)| \cdot 2^{-\delta m}) = 0\} \\
&= \inf\{\delta \geq 0 : \inf_{m \geq n} (2^{3m - m\delta} \mathcal{L}(E_m(a))) = 0\} \\
&= \inf\{\delta \geq 0 : \inf_{m \geq n} (3m - m\delta + \log_2 \mathcal{L}(E_m(a))) = -\infty\} \\
&= \inf\{\delta \geq 0 : \inf_{m \geq n} (m(3 - \delta + \frac{1}{m} \log_2 \mathcal{L}(E_m(a)))) = -\infty\}.
\end{aligned}$$

In order for the infimum to be $-\infty$ we need a subsequence to go to $-\infty$, and for this it is enough that for every $\varepsilon > 0$

$$3 - \delta + \frac{1}{m} \log_2 \mathcal{L}(E_m(a)) < -\varepsilon \qquad \text{frequently.}$$

This happens for

$$\begin{aligned}
\delta &> \lim_{m \to \infty} 3 - \frac{1}{m} \log_2 \mathcal{L}(E_m(a)) \\
&= 3 - \lim_{m \to \infty} \frac{1}{m} \log_2 \mathcal{L}(x : h_m(x) > a) \\
&= 3 - \lim_{m \to \infty} \frac{1}{m} \log_2 \mathcal{L}\left(x : L_m(x) > a - \frac{5}{6} - \frac{1}{3} \log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2}\right) \\
&= 3 - \frac{I\left(a - \frac{5}{6} - \frac{1}{3} \log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2}\right)}{\log(2)},
\end{aligned}$$

by the large deviation principle, then

$$\dim_H(D_n(a)) = 3 - \frac{I\left(a - \frac{5}{6} - \frac{1}{3} \log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2}\right)}{\log(2)}. \tag{6.18}$$

From (6.17) and (6.18) we have that, being $E(a)$ a countable union,

$$\dim_H(E(a)) = 3 - \frac{I\left(a - \frac{5}{6} - \frac{1}{3}\log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2}\right)}{\log(2)}.$$

To conclude we have to compute the rate function $I(a)$, which is quite straightforward in the case of discrete uniform i.i.d. random variables. By definition

$$I(a) = \sup_{\xi\in\mathbb{R}}(\xi a - \lambda(xi)),$$

where λ is the cumulant generating function of the random variable Z_1:

$$\begin{aligned}\lambda(\xi) &= \log E(\exp(\xi Z_1)) \\ &= \log\left(\frac{1}{8}\cdot\sum_{i=1}^{8}\exp\left(\xi\frac{1}{2}\log_2\left(\frac{1}{\tilde{d}_i}\right)\right)\right) \\ &\log\frac{1}{8} + \log\sum_{i=1}^{8}\tilde{d}_i^{-\frac{\xi}{2\log 2}}.\end{aligned}$$

The function $I(a)$ is strictly monotone in $[\nu, \max Z_1]$, and a similar result occurs in the symmetric interval $[\min Z_1, \nu]$.

Hence we are able to conclude the following:

$$\begin{aligned}D(h) = \dim_H \mathcal{S}_h &= \dim_H\{x \in \mathbb{X} \,:\, \liminf_{n\to\infty} h_n(x) = h\} \\ &= 3 - \frac{I\left(h - \frac{5}{6} - \frac{1}{3}\log_2 \sum_{i=1}^{8} \tilde{d}_i^{3/2}\right)}{\log(2)}.\end{aligned}$$

We can compute explicitly the values of the Hausdorff dimension in terms of the scaling exponents of the X_js, by numerically calculating the rate function $I(\cdot)$ with the help of the software suite SciPy[30]. We can then plot the resulting dimensions in terms of the scaling exponents h using the package `ggplot2` in the R framework ([40], [38]). The results are in Figure 6.2.

6.4 Energy dissipation

Now we change perspective. As in the previous section we focused on the X_js and their scaling exponents, now we turn our attention to the energy. We will introduce a measure on the unit cube depending on the solution X and related to the flow of the energy on subtrees.

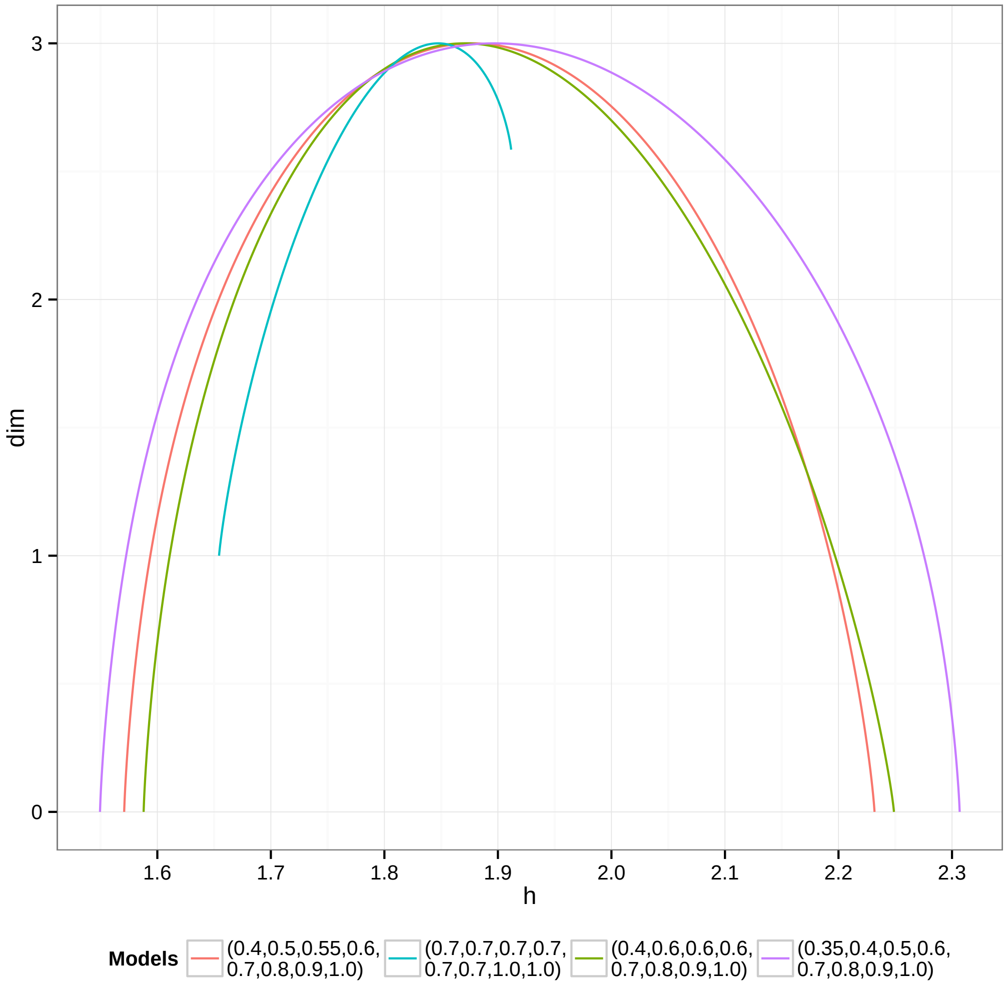


Figure 6.2: Hausdorff dimension of the sets $\mathcal{S}_h$ in terms of the X_js scaling exponent h, calculated for some choices of the coefficients $\tilde{d}_i$.

We define $\mu = \mu_X$ on the balls defined in (6.8) as follows. For all $x \in \mathbb{X}$ and $n \geq 0$,

$$\mu(B(x, 2^{-n})) = 2c_{j_n^x} X_{j_n^x} X_{\overline{j_n^x}}^2 = 2c_{j_n^x} X_{j_n^x} X_{j_{n-1}^x}^2.$$

We can see it as the energy that flows through j_n^x into the subtree $B(x, 2^{-n})$. Since we are considering the flow in a stationary regime, it might seem a little far-fetched, but we shouldn't interpret it in a strict way. An interesting picture to better understand this measure is to view the limit measure on the points x as the heat dissipated in the point, photographed in a particular moment.

Since we have an explicit form of the unique stationary solution, we can write

$$\begin{aligned}\mu(B(x,2^{-n})) &= 2c_{j_n^x} f M^{-n/2} \prod_{k=1}^{n} \sqrt{d_{j_k^x}} f^2 M^{-(n-1)} \prod_{k=1}^{n-1} d_{j_k^x} \\ &= 2f^3 M^{\frac{-3n+2}{2}} 2^{\frac{5}{2}n} \prod_{k=1}^{n} d_{j_k^x}^{(3/2)} \\ &= 2f^3 (M^{3/2})^{-n} M 2^{\frac{5}{2}n} \prod_{k=1}^{n} d_{j_k^x}^{3/2} \\ &= 2f^3 M \prod_{k=1}^{n} \frac{d_{j_k^x}^{3/2}}{\sum_{i=1}^{8} \widetilde{d}_i^{3/2}}\end{aligned}$$

and the whole space has measure

$$\mu(\mathbb{X}) = \mu(B(x,1)) = 2f^3 M = 2f^3 \left(2^{\frac{5}{2}} \sum_{i=1}^{8} \widetilde{d}_i^{3/2} \right)^{2/3}.$$

That μ can be extended in a unique way to a finite measure on the Borel σ-algebra is straightforward, since the algebra of finite unions of balls does not contain sequences of disjoint sets whose union is again a ball.

If we identify $\mathbb{X}$ with the unit cube, we see that for the Lebesgue measure

$$\mathcal{L}(B(x,2^{-n})) = 8^{-n} = 2^{-3n}$$

For $x \in \mathbb{X}$ and $n \geq 1$, let $R_n(x) := -\frac{1}{n} \log_2 \mu(B(x,2^{-n}))$. Then ,

$$\begin{aligned}R_n(x) &= -\frac{1}{n} \log_2 \left(2f^3 M^{3/2} \prod_{k=1}^{n} \frac{d_{j_k^x}^{3/2}}{\sum_{i=1}^{8} \widetilde{d}_i^{3/2}} \right) = -\frac{C}{n} - \frac{1}{n} \sum_{k=1}^{n} \log_2 \left(\frac{d_{j_k^x}^{3/2}}{\sum_{i=1}^{8} \widetilde{d}_i^{3/2}} \right) \\ &=: -\frac{C}{n} + L_n(x)\end{aligned}$$

Where $L_n(x) := \frac{1}{n} \sum_{k=1}^{n} Y_k(x)$ and

$$Y_k(x) := \log_2 \left(d_{j_k^x}^{-3/2} \sum_{i=1}^{8} \widetilde{d}_i^{3/2} \right)$$

If we interpret $\mathcal{L}$ as a probability measure on the unit cube, then Y_k are i.i.d. random variables with discrete uniform distribution on the set

$$\left\{ \log_2 \sum_{i=1}^{8} \widetilde{d}_i^{3/2} - \log_2 \widetilde{d}_i^{3/2} : 1 \leq i \leq 8 \right\}$$

taken with multiplicity.

This means that by LLN there is $\mathcal{L}$-a.s. convergence as $n \to \infty$

$$R_n(x) \to E_{\mathcal{L}}[Y_1]$$

where

$$E_{\mathcal{L}}[Y_1] = \frac{1}{8}\sum_{i=1}^{8}\left(\log_2 \sum_{i=1}^{8} \widetilde{d}_i^{3/2} - \log_2 \widetilde{d}_i^{3/2}\right)$$
$$= -\frac{1}{8}\sum_{i=1}^{8} \log_2 \frac{\widetilde{d}_i^{3/2}}{\sum_{i=1}^{8} \widetilde{d}_i^{3/2}} \geq -\log_2 \frac{1}{8}\sum_{i=1}^{8} \frac{\widetilde{d}_i^{3/2}}{\sum_{i=1}^{8} \widetilde{d}_i^{3/2}} = 3$$

by Jensen inequality, in which the equality is obtained if the coefficients $\widetilde{d}_i$ are all equal, that is the case when our model behaves exactly like K41.

Now the tune might sound familiar, as in fact it is the same path walked earlier with the multifractality that rises from the scaling of the X_js. We can write

$$h_n(x) = \frac{1}{3}R_n(x) + \frac{5}{6} + \frac{C'}{n} \tag{6.19}$$

since we have

$$R_n(x) = -\frac{C}{n} - \frac{1}{n}\sum_{k=1}^{n} \log_2 \left(\frac{\widetilde{d}_{j_k^x}^{3/2}}{\sum_{i=1}^{8} \widetilde{d}_i^{3/2}}\right)$$
$$-\frac{C}{n} + \log_2 \left(\sum_{i=1}^{8} \widetilde{d}_i^{3/2}\right) + 3 \cdot \frac{1}{n}\sum_{k=1}^{n} \frac{1}{2} \log_2 \frac{1}{d_{j_k^x}^{3/2}},$$

and in particular when we consider $h(x)$ and $R(x)$ we have, since C' does not depend on n,

$$h(x) = \frac{1}{3}R(x) + \frac{5}{6}.$$

We can now proceed for $R_n(x)$ in the same way we did for $h_n(x)$ and define for every r in the interval $[r_{min}, r_{max}]$, where the extremes of this interval are the minimum and maximum value of the random variable Y_1, the sets S_r, where the limit logarithm measure $R(x) = r$. Then we can compute its Hausdorff dimension, exactly in the same way as we obtained it for the scaling exponents

$$D(r) = \dim_H(\mathcal{S}_r).$$

By numerically computing the rate function, we can plot the Hausdorff dimension against the possible values of $R(x)$, as shown in Figure 6.3

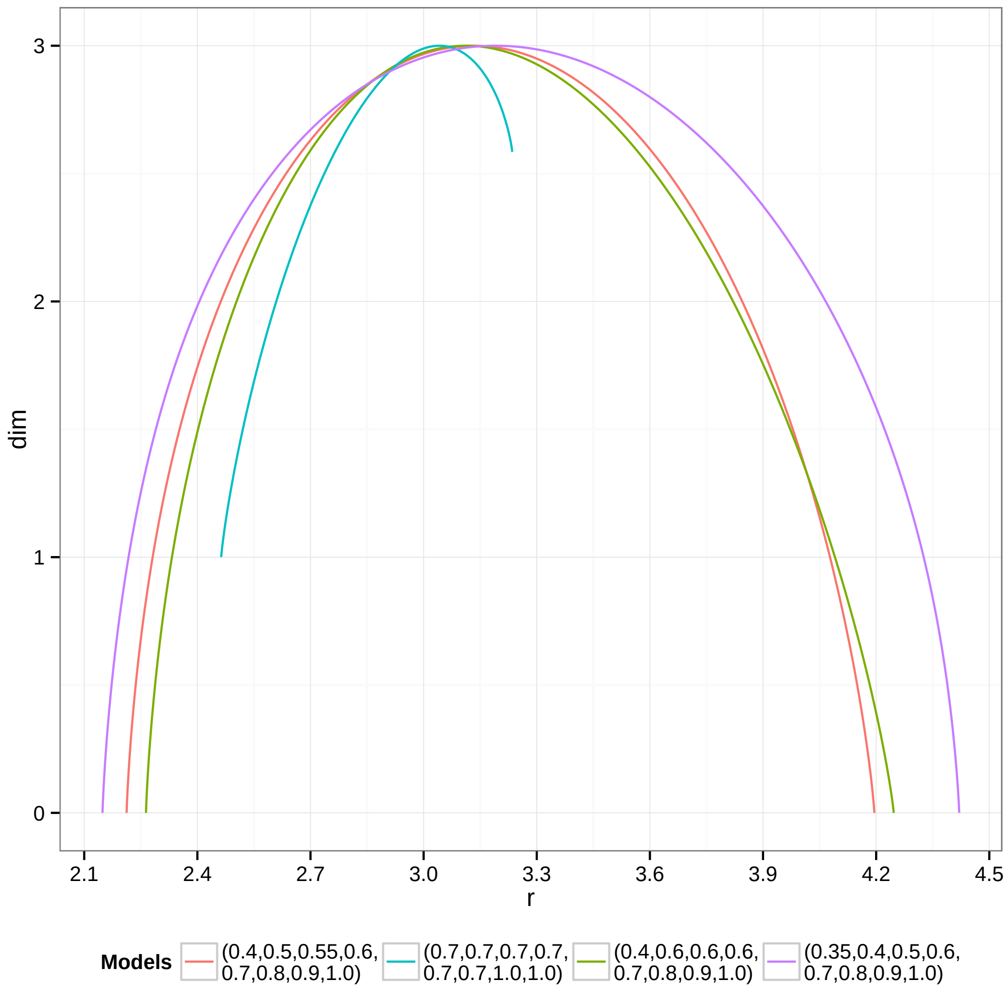


Figure 6.3: Hausdorff dimension of the sets $\mathcal{S}_r$ in terms of the energy dissipation rate r, calculated for some choices of the coefficients $\tilde{d}_i$.

Appendix **A**

"Classic" dyadic model

We present in this chapter some results regarding the classic dyadic model. We will consider only the inviscid and unforced case, even if there are some results in the literature concerning the viscous and forced case, some of which are presented in this thesis.

$$\begin{cases} Y_{-1}(t) \equiv 0 \\ \dfrac{d}{dt} Y_n = k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1}, \quad \forall n \geq 0, \end{cases} \tag{A.1}$$

Definition A.1. Given $Y^0 \in \mathbb{R}^{\mathbb{N}}$, we call the *componentwise solution* of system (A.1) with initial condition Y^0 any sequence $Y(\cdot) = (Y_n(\cdot))_{n\in\mathbb{N}}$ of continuously differentiable functions $Y_n : [0, \infty) \to \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $Y_n(0) = Y_n^0$ and all equations in system (A.1) are satisfied.

If $Y(t) \in \mathbb{R}_+^{\mathbb{N}}$ for all $t \in [0, \infty)$, we call it a *positive componentwise solution.*

If $Y(t) \in l^2$, we call it a *finite energy solution.*

In fact we will call $\mathcal{E}(t) = \sum_n \in \mathbb{N} Y_n^2(t)$ the energy of Y at time t.

Theorem A.1 (Existence)**.** *Given $Y^0 \in \mathbb{R}_+^{\mathbb{N}}$, any componentwise solution of system* (A.1) *with initial condition Y^0 is positive. At least one such solution exists. Moreover, any such solution has the following properties:*

1. *for every $n \geq 1$ and $t \geq 0$ we have*

$$\frac{d}{dt} \sum_{n=1}^{m} Y_n^2(t) = -k_{m+1}^2 Y_m^2(t) Y_{m+1}(t)$$

and hence

$$\sum_{n=1}^{m} Y_n^2(t) \leq \sum_{n=1}^{m} (Y_n^0)^2, \tag{A.2}$$

2. *if $Y_n^0 > 0$ for some $n > 0$, then $Y_m(t) > 0$ for all $m \geq n$ and $t > 0$.*

More generally we can state the following

Theorem A.2. *For every $Y^0 \in l^2$, there exists at least one finite energy solution of system* (A.1) *with initial condition Y^0, with the property*

$$|Y(t)|_{l^2} \leq |Y(s)|_{l^2} \qquad \forall 0 \leq s \leq t. \tag{A.3}$$

Moreover if $Y^0 \in l^2 \cap \mathbb{R}_+^{\mathbb{N}}$, then all componentwise solutions are finite energy and satisfy (A.3)*.*

For the classic model with physical coefficients $k_n = 2^{\beta n}$ it holds a result of regularity.

Theorem A.3. *Let $Y^0 = (Y_n^0)_{n\in\mathbb{N}} \in l^2$ with $Y_n^0 > 0$ for all n; let $Y = (Y_n)_{n\in\mathbb{N}}$ be a componentwise solution of* (A.1) *with initial condition Y^0. Then there exists a constant c depending only on $\|Y^0\|_{l^2}$ and β such that for any positive, non-increasing sequence $(a_n)_{n\in\mathbb{N}}$ the following inequality holds*

$$\mathcal{L}\{t > 0 \mid Y_n(t) > a_n \textit{ for some } n\} \leq c \sum_{n\in\mathbb{N}} \frac{1}{k_n a_n^3}, \tag{A.4}$$

where $\mathcal{L}$ denotes the Lebesgue measure. The quantity $c = 2^{7+\beta}\|Y^0\|_{l^2}^2$ satisfies this theorem.

From the previous theorem it stems the following corollary:

Corollary A.4. *There exists a constant $c = c(\beta)$ such that, if $Y^0 = (Y_n^0)_{n\in\mathbb{N}} \in l^2$ with $Y_n^0 \geq 0$ for all n, the following inequality holds for all n and $M > 0$:*

$$\mathcal{L}\{Y_n > M\} = \{t \geq 0 \mid Y_n(t) > M\} \leq \frac{c\|Y^0\|_{l^2}^2}{k_n M^3}.$$

As a consequence of the previous corollary we can establish the following uniqueness result, for positive solutions.

Theorem A.5. *Let $Y^0 = (Y_n^0)_{n\in\mathbb{N}} \in l^2$ with $Y_n^0 \geq 0$ for all n.For all $\beta > 1$ there exists a unique componentwise solution Y of* (A.1) *with initial condition Y^0.*

Theorem A.6. *Given $t_0 < 0$ and $n_0 > 0$, there exists a unique finite energy self-similar solution, that is a solution Y such that $Y_n = a_n \cdot \varphi(t)$ for all $n \in \mathbb{N}$, with $a_1 = \ldots = a_{n_0} = 0$ and $a_{n_0+1} \neq 0$ (where the first conditions are to be considered only for $n_0 > 0$).*

Such solutions, through time reversal, provide a counterexample to uniqueness of solutions when solutions are not necessarily positive. Uniqueness is weakly restored by means of a stochastic perturbation of the model.

We consider the infinite dimensional system

$$dY_n = (k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1})dt + \sigma k_n Y_{n-1} \circ dW_{n-1} - \sigma k_{n+1} X_{n+1} \circ dW_n, \tag{A.5}$$

where the W_n are a sequence of independent Brownian motions, $Y_0(t) = 0$ and $\sigma \neq 0$.

Theorem A.7. *There is uniqueness in law for* (A.5) *in the space* $L^\infty([0,T];l^2)$*, also called the space of energy controlled solutions.*

Theorem A.8 (Anomalous dissipation)**.** *Let* Y *be a positive componentwise solution of system* (A.1) *with initial condition* $Y^0 \in l^\infty \cap \mathbb{R}_+^{\mathbb{N}}$*. Then* Y *has finite energy for positive times.*

If Y *is a positive finite energy solution, then*

$$\lim_{t\to\infty} \|Y(t)\|^2 = 0.$$

Moreover, given $L > 0$ *and* $\varepsilon > 0$*, there exists* $\bar{t} > 0$ *depending only on* L *and* ε *such that for all positive finite energy solutions* Y *with* $\|Y(0)\| \leq L$ *we have* $\|Y(\bar{t})\| \leq \varepsilon$*.*

Theorem A.9 (Bound on the decay of energy)**.** *Let* Y *be a positive componentwise solution with initial condition* $Y^0 \in l^\infty \cap \mathbb{R}_+^{\mathbb{N}}$*. Then there exists* $C > 0$ *such that*

$$\|Y(t)\|^2 \leq \frac{C}{t^2}, \qquad \text{for } t \geq 1.$$

Theorem A.10. *Assume* $k_n = \lambda^n$ *for some* $\lambda > 1$*. Given deterministic initial conditions* $Y^0 \in l^2$*, let* $Y(t)$ *be the (unique) energy controlled solution of system* (A.5)*. Then, for all positive times* $t > 0$*,*

$$P(\mathcal{E}(t) = \mathcal{E}(0)) < 1.$$

Moreover for all $\varepsilon > 0$ *there exists* $\bar{t}$ *such that*

$$P(\mathcal{E}(\bar{t}) \leq \varepsilon) > 0.$$

Finally, if the initial energy $\mathcal{E}(0)$ *is sufficiently small, then* $\mathcal{E}(t)$ *decays to zero at least exponentially fast both almost surely and in* L^1*.*

This result, which is still open for the tree dyadic model, shows that while there is anomalous dissipation also for the stochastically perturbed classic dyadic model, the decay of the energy is completely different, decaying exponentially fast, where in the deterministic formulation the decay was of the order of t^{-2}. Even more, there is a gap in between the two dissipating regimes: for $\sigma = 0$, that is in the deterministic case, we have anomalous dissipation, as we have for big enough values of σ (see [7]).

Appendix B

Excerpts of the code

In this chapter we provide some of the code used to calculate and print the multifractality results in Chapter 6. The main calculations, namely the computation of the rate function, are coded in Python, taking advance of the Scipy library [30]. The graphical representation is realized in R [38], via the `ggplot2` package [40] that uses the grammar of graphics.

Listing B.1: Code for multifractality in rescaling

```
from __future__ import division
import scipy
import scipy.optimize
import numpy as np
import math

# The following specifies parameter alpha
# and the choice of coefficients
alpha = 5/2
d = np.array([0.35,0.4,0.5,0.6,0.7,0.8,0.9,1.0])

# We choose here the length of the answer array
lung = 2000

# Do not change the following
epsi = 0.0000000001

# The next three functions just compute the mean value nu
def ch(d):
        y = 1/3*(alpha+ math.log((d**(3/2)).sum())/math.log(2))
        return y

def ez1(d):
        y = 1/(2*len(d)*math.log(2))*(np.log(1/d)).sum()
        return y

```

```
def nu(d):
        y = ch(d)+ez1(d)
        return y

# Now we compute the upper limit of the interval, and the lower
# only ofr the random part, so we will add ch later
def maxh(d):
        y = np.max(1/(2*math.log(2))*np.log(1/d))
        return y

def minh(d):
        y = np.min(1/(2*math.log(2))*np.log(1/d))
        return y

def llambda(xi):
        y = math.log(1/8) + math.log(np.exp(-xi/(2*math.log(2))*np.log(d)).sum())
        return y

# This is the definition of the rate function, through scipy.optimize.fmin
def ratef(h):
        xopt = scipy.optimize.fmin(lambda x: -(x*h-llambda(x)),1.5)
        y = (xopt*h-llambda(xopt))
        return y

# This computes the dimension
def dimmh(h):
        y = 3 - ratef(h)/math.log(2)
        return y

hvar = np.linspace(minh(d)+epsi,maxh(d)-epsi,lung)
dimh = np.zeros(len(hvar))

for i in range(len(hvar)):
        h = hvar[i]
        dimh[i] = dimmh(h)

hvar = hvar + ch(d)

# Finally we export the data to csv, to treat it in R
np.savetxt("~/path/dimacca.csv",dimh,delimiter=",")
np.savetxt("~/path/acca.csv",hvar,delimiter=",")
```

Listing B.2: Code for multifractality in energy dissipation

```

def m(d):
        y = (2**alpha*(d**(3/2)).sum())**(2/3)
        return y

def nu(d):
```

```python
        y = math.log((d**(3/2)).sum())/math.log(2) -
            1/(len(d)*math.log(2))*((np.log(d**(3/2))).sum())
        return y

def mass(d):
        y = math.log((d**(3/2)).sum())/math.log(2) -
            math.log(np.min(d)**(3/2))/math.log(2)
        return y

def minn(d):
        y = math.log((d**(3/2)).sum())/math.log(2) -
            math.log(np.max(d)**(3/2))
        return y


def llambda(xi):
        y = math.log(1/8) + xi/math.log(2)*math.log((d**(3/2)).sum()) +
            math.log((d**(-3*xi/(2*math.log(2)))).sum())
        return y

def ratef(r):
        xopt = scipy.optimize.fmin(lambda x: -(x*r-llambda(x)), 1.5*math.log(2))
        y = (xopt*r-llambda(xopt))
        return y


def dimh(r):
        y = 3 - ratef(r)/math.log(2)
        return y

rvar = np.linspace(minn(d)+epsi,mass(d)-epsi,lung)
dimr = np.zeros(len(rvar))

for i in range(len(rvar)):
        r = rvar[i]
        dimr[i] = dimh(r)

np.savetxt("~/path/dimr.csv",dimr,delimiter=",")
np.savetxt("~/path/r.csv",rvar,delimiter=",")
```

Listing B.3: Importing data from previous code in R and plotting throug ggplot2

```r
# First we define some suitable dataframe (only at the beginning)
multfracsca <- data.frame(h=numeric(),dim=numeric(),coeff=character())

# The following code is for one choice of coefficients.
# It could be automated to iterate among different choices.
multfracsca <- rbind(multfracsca, data.frame(h = read.csv("h.csv", header = FALSE)$V1,
+ dim = read.csv("dim.csv", header = FALSE)$V1,
+ coeff = rep("d10", length(read.csv("h.csv", header = FALSE)$V1))))
```

```
multfracen <- data.frame(r=numeric(),dimr=numeric(),coeff=character())

multfracen <- rbind(multfracen, data.frame(a = read.csv("r.csv", header = FALSE)$V1,
+ dima = read.csv("dimr.csv", header = FALSE)$V1,
+ coeff = rep("d10", length(read.csv("r.csv", header = FALSE)$V1))))

# Now we plot the data obtained via ggplot2

# First the rescaling multifractality
plotvel <- ggplot(data = subset(multfracsca, (coeff == "d7" | coeff == "d9" |
+ coeff == "d10" | coeff == "d4")),
+ aes(x = h, y = dim, group = coeff, colour = coeff)) + geom_line()
multvelplot <- plotvel + theme_bw() + scale_x_continuous(breaks = seq(1.5,2.3,0.1)) +
+ theme(legend.position = "bottom") +
+ scale_colour_discrete(name="Models", labels=c("(0.4,0.5,0.55,0.6,\n0.7,0.8,0.9,1.0)",
+ "(0.7,0.7,0.7,0.7,\n0.7,0.7,1.0,1.0)", "(0.4,0.6,0.6,0.6,\n0.7,0.8,0.9,1.0)",
+ "(0.35,0.4,0.5,0.6,\n0.7,0.8,0.9,1.0)"), breaks = c("d4","d9","d7","d10"))

# Now the energy multifractality
plotdiss <- ggplot(data = subset(multfracen, (coeff == "d7" | coeff == "d9" |
+ coeff == "d10" | coeff == "d4")),
+ aes(x = a, y = dim, group = coeff, colour = coeff)) + geom_line()
multdissplot <- plotdiss + theme_bw() +
+ scale_x_continuous(breaks = seq(2.1,4.5,0.3)) +
+ theme(legend.position = "bottom") +
+ scale_colour_discrete(name= "Models", labels =
+ c("(0.4,0.5,0.55,0.6,\n0.7,0.8,0.9,1.0)", "(0.7,0.7,0.7,0.7,\n0.7,0.7,1.0,1.0)",
+ "(0.4,0.6,0.6,0.6,\n0.7,0.8,0.9,1.0)", "(0.35,0.4,0.5,0.6,\n0.7,0.8,0.9,1.0)"),
+ breaks=c("d4","d9","d7","d10"))+xlab("r")
```

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