Large deviations for differential stochastic equations with additive noise

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Introduction

In this thesis we are concerned with large deviation principle for the laws of solutions of a family of stochastic differential equations.

The large deviation principle is a very important tool in probabilistic research. It is used to investigate the behavior of a family of probability laws μ_ε, on a complete separable metric space (F, ρ), depending on a parameter ε, when the parameter tends to 0.

In general μ_ε will converge weakly to a Dirac measure μ_0 for $\varepsilon \to 0$. We want to estimate the rate of convergence to 0 of the measure of Borel subsets of F: if the large deviation principle is satisfied (if the family μ_ε has the large deviation property), the rate for a Borel subset is the infimum on the subset of a given, lower semicontinuous, non-negative, function I, named the *rate functional*.

The large deviation principle is used in many situations: from the behavior of dynamical systems in a small noise environment for better understanding of real systems in physics, to establishing the standard form of growing crystals.

We give the definitions about the large deviation principle in Chapter 1, where we also establish (Theorem 1.9) that a family of Gaussian measures with covariance εQ, where Q is a given trace class operator, has the large deviation property.

In this work we focus on some particular families of measures, those given by solutions of stochastic differential equations on infinite dimensional spaces.

This matter was studied by many authors in various settings: Varadhan [57] formulated the large deviation principle, and in the finite dimensional case it was established by Freĭdlin & Wentzell [35] and Azencott [3], and studied, later, by Doss & Dozzi [28] and Tudor [56].

We remind some known results to give the background setting: in Chapter 2, we are concerned with the following well-known family of linear problems, that have solution in the space of pathwise continuous and adapted processes from $[0,T]$ into a Hilbert space H:

$$u_\varepsilon(t) = x + \int_0^t Au_\varepsilon(\vartheta)\,\mathrm{d}\vartheta + \sqrt{\varepsilon}B\,W(t), \quad x \in H, \quad t \in [0,T].$$

Here $A : \mathrm{D}(A) \subset H \to H$ is an unbounded linear operator generating a strongly continuous semigroup S on H, $B : U \to H$ is a bounded linear operator, B maps a separable Hilbert space U in H, $\ker(B) = \{0\}$, and W is a U-cylindrical Wiener process defined in a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

We prove that the laws of solutions are Gaussian measures and we give an explicit formulation of the covariance operator. Therefore we establish the large deviation property for the linear case. This property holds, with the same rate functional, also in the case of a problem defined in a separable Banach space $E \subset H$, densely and as Borel subset embedded in H.

In Section 2.4, we also extend these results allowing the base time interval for the solutions to be infinite, as in the work from the author [31].

In Chapter 3 we show results for the following semilinear problem the non-linear part G is of locally Lipschitz or dissipative type, both on the space E and H.

$$\begin{cases} du_\varepsilon(t) = (Au_\varepsilon(t) + G(u_\varepsilon(t)))dt + \sqrt{\varepsilon}BdW(t), & \varepsilon > 0, \\ u_\varepsilon(0) = x. \end{cases}$$

We recall some results about existence and uniqueness of this problem in the various setting, then we establish the large deviation principle for the laws of solutions. Our method is based on the use of the contraction principle (see Remark 1.5), to transport the large deviation principle for the linear problem to the non linear, via an application Ψ, that we prove to be continuous. Ψ goes from the space of solutions of the linear equation to the space of solutions of the non linear one, and, given the uniqueness of solutions, it is bijective. These results extend preceding works, giving an explicit formula for the rate function and providing the large deviation principle for equations with dissipative non-linear terms on H, see the author [29, 30]

The case of stochastic differential equations with additive perturbation in a Banach subspace E of H is not a new subject and was studied also by Smoleński *et al.* [54], by applying the contraction principle. There the problem is solved assuming that the semilinear part G is Lipschitz in E. Peszat [48] generalized this result still assuming that G is Lipschitz in E. Chenal and Millet [16], proved a more general large deviation result assuming E to be the space of $\alpha-$Hölder continuous function on $[0,1]$, and that the non linear terms are Lipschitz and sublinear on E. The theory can be also applied to systems of reaction-diffusion equations with additive noise considered in Cerrai [13, 14, 15].

In Chapter 4, we arrive to the new part of this thesis concerning the Volterra problems; the results presented in this chapter are mainly proved in the papers from the author and Bonaccorsi [8, 7, 9], and here we give a systematic exposition from the point of view of large deviations.

We are concerned about the following Volterra semilinear problem:

$$u_\varepsilon(t) = x + \int_0^t a(t-\vartheta)[Au_\varepsilon(\vartheta) + G(u_\varepsilon(\vartheta))]\,\mathrm{d}\vartheta + \sqrt{\varepsilon}B\,W(t),$$

where $a :]0,+\infty[\to]0,+\infty[$ is a continuous locally integrable kernel. This problem is a generalization of the previous one (it is sufficient to consider $a \equiv 1$), and arises from the analogous deterministic problem related to visco-elasticity and population dynamics.

The linear version of this problem was first introduced in Clément & Da Prato [19, 20] and further analyzed by Clément *et al.* [22]. The idea developed in these papers is to extend the semigroup approach of Da Prato & Zabczyk [26]. Rovira & Sanz-Solé [51, 52] affronted the problem when the stochastic term is a Brownian sheet and the other terms are globally Lipschitz; they proved also a Large deviation property for the laws of solutions of the problem.

In Subsection 2.5.1 we proved that the laws of solutions of the Volterra linear problem are Gaussian and we give an explicit representation of the covariance operator. Thus we establish the large deviation principle for those laws.

The case of the non-linear equation is studied in the case of Lipschitz and of dissipative semilinear part. Here are established existence and uniqueness of the Volterra problems.

Then, as in the former chapter, we establish large deviation principle for the Volterra problems, proving that the functional Ψ, from solution space of the linear equation to the solution space of the non linear one, is continuous.

Finally, in Chapter 5, we give a short introduction to the problem of the exit time from bounded domains of solution of differential stochastic equations in the small noise asymptotics. There will not be results about the Volterra case since its solutions has no longer the Markovian property, which is the fundamental toll to obtain the results therein. We hope to give results in this direction in future papers.

Chapter 1

Large deviations

In this chapter we shall give an abstract formulation for a class of large deviation problems. We will follow the exposition of Varadhan [58, Section 2].

Then we give a simple result in which the principle is satisfied: the Gaussian case.

1.1 The large deviation principle

Let (F, ρ) be a complete separable metric space. We set for all Borel subsets M of F and for all $\delta > 0$

$$B(M, \delta) = \bigcup_{x \in M} B(x, \delta) = \bigcup_{x \in M} \{y : \rho(x, y) < \delta\}.$$

Let $\{\mu_\varepsilon\}_{\varepsilon > 0}$ be a family of Borel probability measures on F; typically, as $\varepsilon \downarrow 0$, μ_ε weakly converges to a probability measure which is degenerate, i.e., has unit mass, at some point x_0 in F. For several sets G, then, $\mu_\varepsilon(G) \to 0$ as $\varepsilon \downarrow 0$. In the examples we look at, $\mu_\varepsilon(G)$ will tend exponentially rapidly to zero as $\varepsilon \downarrow 0$, with an exponential constant depending on the set G and the relevant situation. We describe the situation in the following context.

Definition 1.1. We shall say that a function $I : F \to [0, +\infty]$ is a *rate function* if I is lower semi-continuous and if, for arbitrary $r > 0$, the *level* set $K(r) = \{x \in F : I(x) \leq r\}$ is compact.

Definition 1.2. We say that $\{\mu_\varepsilon\}_{\varepsilon > 0}$ obeys the *large deviation principle* or has the *large deviation property* with rate function $I(\cdot)$, if there exists a function $I(\cdot) : F \to [0, +\infty]$, satisfying the Definition 1.1, such that for each closed set $\Gamma \subset F$

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(\Gamma) \leq - \inf_{x \in \Gamma} I(x), \tag{1.1.1}$$

and for each open set $G \subset F$

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(G) \geq - \inf_{x \in G} I(x). \tag{1.1.2}$$

It follows that $I(x_0)$ must be be zero and typically $I(x) > 0$ for $x \neq x_0$. We shall see several examples of this situation in the next chapters. One can establish easily that if A is a Borel set such that

$$\inf_{x \in \mathring{A}} I(x) = \inf_{x \in A} I(x) = \inf_{x \in \bar{A}} I(x)^\dagger,$$

then

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(A) = - \inf_{x \in A} I(x).$$

We give now the properties a rate function must fulfill:

There is an interesting consequence of this definition that we now state and prove as a proposition

Proposition 1.3. *Let μ_ε satisfy the large deviation principle with a rate function $I(\cdot)$. Then for any bounded continuous function $G(x)$ on F*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \left[\int_F e^{\frac{G(x)}{\varepsilon}} \, \mu_\varepsilon(\mathrm{d}x) \right] = \sup_{x \in F} [G(x) - I(x)] \tag{1.1.3}$$

Proof. Upper bound. Given $\varepsilon > 0$, there exists a finite number n of closed sets C_i covering F such that the oscillation of $G(\cdot)$ on these sets is at most $\delta > 0$. Then we have

$$\int_F e^{\frac{G(x)}{\varepsilon}} \, \mu_\varepsilon(\mathrm{d}x) \leq \sum_{i=1}^n \int_{C_i} e^{\frac{G(x)}{\varepsilon}} \, \mu_\varepsilon(\mathrm{d}x) \leq \sum_{i=1}^n \int_{C_i} e^{\frac{G_i + \delta}{\varepsilon}} \, \mu_\varepsilon(\mathrm{d}x),$$

where G_i is $\min_{x \in C_i} \{G(x)\}$. Therefore

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \; \varepsilon \log \left[\int_F e^{\frac{G(x)}{\varepsilon}} \, \mu_\varepsilon(\mathrm{d}x) \right] &\leq \sup_{1 \leq i \leq n} [G_i + \delta - \inf_{x \in C_i} I(x)] \\ &\leq \sup_{1 \leq i \leq n} \sup_{x \in C_i} [G(x) - I(x)] + \delta \\ &= \sup_{x \in F} [G(x) - I(x)] + \delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary we have done.

Lower bound. Given $\delta > 0$ there exists a point $y \in F$ such that

$$G(y) - G(y) \geq \sup_{x \in F} [G(x) - I(x)] - \delta/2.$$

†Here $\mathring{A}$ and $\bar{A}$ are respectively the interior and the closure of the Borel set A.

We can also find a neighborhood U of y such that

$$G(x) \geq G(y) - \delta/2 \qquad \text{for } x \in U.$$

Then we have

$$\begin{aligned}\liminf_{\varepsilon \downarrow 0} \varepsilon \log \left[\int_F e^{\frac{G(x)}{\varepsilon}} \mu_\varepsilon(\mathrm{d}x) \right] &\geq \liminf_{\varepsilon \downarrow 0} \varepsilon \log \left[\int_U e^{\frac{G(x)}{\varepsilon}} \mu_\varepsilon(\mathrm{d}x) \right] \\ &\geq G(y) - \delta/2 - \inf_{x \in U} I(x) \geq G(y) - I(y) - \delta/2 \\ &\geq \sup_{x \in F} [G(x) - I(x)] - \delta.\end{aligned}$$

Since δ is arbitrary we have done □

Sometimes is useful a slight variation of the above theorem, which we state and prove as another proposition.

Proposition 1.4. *Let μ_ε satisfy the large deviation principle with a rate function $I(\cdot)$. Let $G_\varepsilon(\cdot)$ a family of non-negative functions such that for some lower semi-continuous non negative function $G(\cdot)$ one has*

$$\liminf_{\substack{\varepsilon \downarrow 0 \\ y \to x}} G_\varepsilon(y) \geq G(x) \quad \textit{for all } x \in F. \tag{1.1.4}$$

Then

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \left[\int_F e^{-\frac{G_\varepsilon(x)}{\varepsilon}} \mu_\varepsilon(\mathrm{d}x) \right] \leq - \inf_{x \in F} [G(x) + I(x)] \tag{1.1.5}$$

Proof. Let $l = \inf_{x \in F} [G(x) + I(x)]$. For any $\delta > 0$ and $x \in F$ there is a open neighborhood $U_{x,\delta}$ of x such that

$$\inf_{y \in \bar{U}_{x,\delta}} I(y) \geq I(x) - \delta, \qquad \liminf_{\varepsilon \downarrow 0} \inf_{y \in \bar{U}_{x,\delta}} G_\varepsilon(y) \geq G(x) - \delta.$$

Therefore as $\varepsilon \downarrow 0$

$$\int_{U_{x,\delta}} e^{-\frac{G_\varepsilon(z)}{\varepsilon}} \mu_\varepsilon(\mathrm{d}z) \leq e^{-\frac{l-2\delta}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)}.$$

We choose an r much greater than l. We have that $K(r)$ is a compact set, so there exist a finite number $n(r)$ of $U_{x,\delta}$ that cover $K(r)$; this number does not depend on ε. We call U the union of those $n(r)$ sets. Since $G_\varepsilon \geq 0$, we have

$$\int\limits_{F\setminus U} e^{-\frac{G_\varepsilon(z)}{\varepsilon}}\,\mu_\varepsilon(\mathrm{d}z) \leq \mu_\varepsilon(F\setminus U) \leq e^{-\frac{1}{\varepsilon}\inf_{z\in F\setminus U} I(z)+o\left(\frac{1}{\varepsilon}\right)}$$

$$\leq e^{-\frac{1}{\varepsilon}\inf_{z\in F\setminus K(r)} I(z)+o\left(\frac{1}{\varepsilon}\right)}$$

$$\leq e^{-\frac{r}{\varepsilon}+o\left(\frac{1}{\varepsilon}\right)}.$$

On the other hand, as $\varepsilon \downarrow 0$

$$\int\limits_{U} e^{-\frac{G_\varepsilon(z)}{\varepsilon}}\,\mu_\varepsilon(\mathrm{d}z) \leq n(r)e^{-\frac{l-2\delta}{\varepsilon}+o\left(\frac{1}{\varepsilon}\right)}.$$

If we chose k large enough the term $e^{-\frac{k}{\varepsilon}}$ is negligible compared to $e^{-\frac{l-2\delta}{\varepsilon}}$ for $\varepsilon \downarrow 0$. Since δ is arbitrary, the proof is complete. □

Remark 1.5. Let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on a Polish space F satisfying the large deviation principle with a rate function $I(\cdot)$. Let Ψ be a continuous mapping from F to another Polish space $\hat{F}$. Then the family of image measures $\{\nu_\varepsilon\}_{\varepsilon>0}$ on $\hat{F}$ defined by Ψ

$$\nu_\varepsilon = \mu_\varepsilon \circ \Psi^{-1}$$

also satisfies the large deviation principle with rate function $J(\cdot)$ given by

$$J(y) = \inf_{x\in\Psi^{-1}(y)} I(x). \tag{1.1.6}$$

We will refer to this as the "**contraction principle**".

The large deviation principle has some equivalent formulations that sometimes are easier to use compared to those in Definition 1.2.

Proposition 1.6. *Let $I : F \to [0,+\infty]$ be a rate function, then the following statements are equivalent:*

I. *the family $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the upper bound* (1.1.1)

II. $\forall r > 0$, $\forall \delta > 0$, $\forall \gamma > 0$, $\exists \varepsilon_0 > 0$ *such that* $\forall \varepsilon \in]0, \varepsilon_0[$

$$\mu_\varepsilon(B(K(r),\delta)) \geq 1 - e^{-\frac{1}{\varepsilon}(r-\gamma)}. \tag{1.1.7}$$

Proof. I $\Longrightarrow$ II. Let $r > 0$, $\delta > 0$, $\gamma > 0$, $\Gamma = F \setminus B(K(r), \delta)$, such that $\inf_{x \in \Gamma} I(x) \geq r$. From (I) it follows that $\exists \varepsilon_0 > 0$ such that, if $\varepsilon \in]0, \varepsilon_0[$, we have $\varepsilon \log \mu_\varepsilon(\Gamma) < -r + \gamma$, which implies (II).

II $\Longrightarrow$ I. Let us consider a closed set Γ of F. Let $r_0 = \inf_{x \in \Gamma} I(x)$. If $r_0 = 0$ then (I) obviously hold, so we can assume $r_0 > 0$. Let us choose $r \in]0, r_0[$; then, $K(r)$ and Γ are disjoint and there exists δ such that the sets $B(K(r), \delta)$ and Γ are disjoint as well. So, we have that

$$\mu_\varepsilon(\Gamma) \leq 1 - \mu_\varepsilon(B(K(r), \delta)).$$

Now from (II), given $\gamma > 0$, $\exists \varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$

$$\mu_\varepsilon(\Gamma) \leq e^{\frac{1}{\varepsilon}(r-\gamma)}.$$

It follows that

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(\Gamma) \leq -r + \gamma.$$

Since γ and r can be chosen arbitrary near to 0 and r_0, (I) follows. □

Proposition 1.7. *Let $I : F \to [0, +\infty]$ be a rate function. Then, the following statements are equivalent:*

I. *the family $\{\mu_\varepsilon\}_{\varepsilon > 0}$ satisfies the lower bound* (1.1.2)*;*

II. $\forall x \in F$*,* $\forall \delta > 0$*,* $\forall \gamma > 0$*,* $\exists \varepsilon_0 > 0$ *such that* $\forall \varepsilon \in]0, \varepsilon_0[$

$$\mu_\varepsilon(B(x, \delta)) \geq e^{-\frac{1}{\varepsilon}(I(x)+\gamma)}. \tag{1.1.8}$$

Proof. I $\Longrightarrow$ II. Let $x \in F$, $\delta > 0$, $\gamma > 0$, $K = B(x, \delta)$. From (I) it follows that $\exists \varepsilon_0 > 0$ such that, if $\varepsilon \in]0, \varepsilon_0[$, we have

$$\varepsilon \log B(x, \delta) \geq -\inf_{y \in K} I(y) - \gamma \geq -I(x) - \gamma,$$

and (II) follows.

II $\Longrightarrow$ I. Let us consider an open set G of F. Let us consider $x \in G$ and choose a $\delta > 0$ such that $B(x, \delta) \subset G$. Then, given $\gamma > 0$, $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0[$

$$\mu_\varepsilon(G) \geq \mu_\varepsilon(B(x, \delta)) \geq e^{-\frac{1}{\varepsilon}(I(x)+\gamma)}.$$

Then we have that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(G) \geq -(I(x) + \gamma) \qquad \text{for all } x \in G$$

Since γ can be chosen arbitrary small, (I) follows. □

The inequalities (1.1.7) and (1.1.8) are often referred to as *exponential estimates of Freĭdlin-Wentzell.*

1.2 Large deviation principle for a family of Gaussian measures

Let E be a separable Banach space with norm $\|\cdot\|_E$ and μ a symmetric Gaussian measure on E. We define H_μ to be the reproducing kernel of μ. We remind the definition of reproducing kernel in the following definition:

Definition 1.8. Let μ be a symmetric Gaussian measure on separable Banach space E. A linear subspace H_μ, equipped with Hilbert norm $|\cdot|_\mu$ and inner product $\langle\cdot,\cdot\rangle_\mu$, is said to be a *reproducing kernel space* for μ if H_μ is complete, continuously embedded in E, and such that, for arbitrary $\phi \in E^*$, the law of ϕ on $\mathbb{R}$ is a Gaussian measure

$$\mathfrak{L}(\phi) = \mathcal{N}(0, |\phi|_\mu^2),$$

where $|\phi|_\mu = \sup_{\|h\|_E \le 1} |\phi(h)|$ and $\mathcal{N}(a, A)$ denote a Gaussian measure on a Hilbert space with mean a and covariance operator A.

The existence and uniqueness of such space in our case can be found e. g. in Da Prato & Zabczyk [26, Theorem 2.7].

Given the measure μ we can define a family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of Gaussian measures, accumulating near 0, by the formula

$$\mu_\varepsilon(\Gamma) = \mu(\sqrt{\varepsilon}\,\Gamma), \tag{1.2.9}$$

for all Borel subset Γ of E, and $\varepsilon > 0$.

Then we are able to prove the following theorem

Theorem 1.9. *Let the family $\{\mu_\varepsilon\}_{\varepsilon>0}$ be defined by* (1.2.9)*, then $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the Freĭdlin-Wentzell estimates with rate function*

$$I(x) = \begin{cases} \frac{1}{2}|x|_\mu^2 & \text{if } x \in H_\mu \\ +\infty & \text{otherwise.} \end{cases}$$

This is a particular but significant case of Cramér's Theorem, see for instance Varadhan [58, Section 3].

Proof. As before we set $K(r) = \{x \in H_\mu : \frac{1}{2}|x|_\mu^2 \le r\}$. We recall that it is possible to find an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of H_μ such that, for an arbitrary sequence $\{\xi_k\}_{k\in\mathbb{N}}$ of independent normal random variables on a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, the series $\sum_{k=0}^{+\infty} \xi_k e_k$ converges in E, $\mathbb{P}$-a.s., to a random variable X with $\mathfrak{L}(X) = \mu$.

We first establish the estimate in point (II) of Proposition 1.6. Let $r_0 > 0$, $\delta > 0$, $\gamma > 0$, $\varepsilon \in]0,1[$ be fixed, and , for $n \in \mathbb{N}$, let X be a random variable such that $\mathfrak{L}(X) = \mu$. We have

$$\begin{aligned}
\mu_\varepsilon(E\backslash B(K(r),\delta)) =& \mathbb{P}\big(\sqrt{\varepsilon}X \notin B(K(r),\delta)\big) \\
=& \mathbb{P}\big(\{\sqrt{\varepsilon}X \notin B(K(r),\delta)\} \cap \{\sqrt{\varepsilon}\textstyle\sum_{k=0}^n \xi_k e_k \notin K(r)\}\big) \\
&+ \mathbb{P}\big(\{\sqrt{\varepsilon}X \notin B(K(r),\delta)\} \cap \{\sqrt{\varepsilon}\textstyle\sum_{k=0}^n \xi_k e_k \in K(r)\}\big) \\
\leq& \mathbb{P}\big(\sqrt{\varepsilon}\textstyle\sum_{k=0}^n \xi_k e_k \notin K(r)\big) + \mathbb{P}\big(\|\sqrt{\varepsilon}\textstyle\sum_{k=n+1}^\infty \xi_k e_k\|_E \geq \delta\big) \\
=& \mathbb{P}\big(\textstyle\sum_{k=0}^n {\xi_k}^2 > \frac{2r}{\varepsilon}\big) + \mathbb{P}\big(\|\textstyle\sum_{k=n+1}^\infty \xi_k e_k\|_E^2 \geq \frac{\delta^2}{\varepsilon}\big) \\
=& \mathbb{P}\big(\exp(a\textstyle\sum_{k=0}^n {\xi_k}^2) > \exp(2a\frac{r}{\varepsilon})\big) \\
&+ \mathbb{P}\big(\exp(b\|\textstyle\sum_{k=n+1}^\infty \xi_k e_k\|_E^2) \geq \exp(b\frac{\delta^2}{\varepsilon})\big) \\
\leq& (1-2a)^{-\frac{n}{2}} \exp(-2a\tfrac{r}{\varepsilon}) \\
&+ \exp(-b\tfrac{\delta^2}{\varepsilon})\mathbb{E}\left[\exp(b\|\textstyle\sum_{k=n+1}^\infty \xi_k e_k\|_E^2)\right] \\
=& I_1 + I_2.
\end{aligned}$$

We first choose b in such way that

$$e^{-\frac{b\delta^2}{\varepsilon}} \leq \left(\frac{e^2-1}{2e^2}\right)\frac{1}{2}e^{-\frac{1}{\varepsilon}(r-\gamma)}, \qquad \forall r \in]0,r_0[, \ \forall \varepsilon > 0.$$

Then we choose n such that

$$\mathbb{E}\left[\exp(b\|\textstyle\sum_{k=n+1}^\infty \xi_k e_k\|_E^2)\right] \leq \frac{e^2}{e^2-1};$$

this is possible since $\mathbb{E}\left[\exp(b\|\sum_{k=n+1}^\infty \xi_k e_k\|_E^2)\right]$ is bounded for all $n \in \mathbb{N}$ and because $\sum_{k=n+1}^\infty \xi_k e_k \to 0$ in probability for $n \to +\infty$.

Finally we choose $a \in]]0, \frac{1}{2}[$ such that $(1-2a)r_0 < \frac{1}{2}\gamma$. It is easy to see, now, that for sufficiently small $\varepsilon > 0$ and for all $r \in]0, r_0[$

$$I_1 \leq \frac{1}{2}e^{-\frac{1}{\varepsilon}(r-\gamma)} \qquad \text{and} \qquad I_2 \leq \frac{1}{2}e^{-\frac{1}{\varepsilon}(r-\gamma)}$$

as required.

Now to establish the estimate in point (II) of Proposition 1.7, we shall need a stronger version:

$\forall r_0 > 0$, $\forall \delta > 0$, $\forall \gamma > 0$, $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall x \in F$ such that $|x|_\mu^2 \leq r_0$,

$$\mu_\varepsilon(B(x,\delta)) \geq e^{-\frac{1}{\varepsilon}(\frac{1}{2}|x|_\mu^2+\gamma)}. \tag{1.2.10}$$

This estimate is a direct corollary of the following property of Gaussian measures:

$$\mu(B(h,r)) \geq \mu(B(0,r))e^{-\frac{1}{2}|h|_\mu^2},$$

for arbitrary $r > 0$ and $h \in H_\mu$. This property can be proved using the Cameron-Martin formula and symmetry of the Gaussian measure as follows:

$$\begin{aligned}
\mu(B(h,r)) =&\mu(\{x \in E \mid \|x-h\|_E \leq r\}) = \int\limits_{\|x-h\|_E \leq r} \mu(\mathrm{d}x) \\
=& \int\limits_{\|x\|_E \leq r} e^{-\langle x,h\rangle_\mu - \frac{1}{2}|h|_\mu^2}\, \mu(\mathrm{d}x) = e^{-\frac{1}{2}|h|_\mu^2} \int\limits_{\|x\|_E \leq r} e^{-\langle x,h\rangle_\mu}\, \mu(\mathrm{d}x) \\
=&\frac{1}{2} e^{-\frac{1}{2}|h|_\mu^2} \int\limits_{\|x\|_E \leq r} e^{-\langle x,h\rangle_\mu} + e^{\langle x,h\rangle_\mu}\, \mu(\mathrm{d}x) \\
\geq& e^{-\frac{1}{2}|h|_\mu^2} \int\limits_{\|x\|_E \leq r} \mu(\mathrm{d}x) = e^{-\frac{1}{2}|h|_\mu^2}\mu(B(0,r)),
\end{aligned}$$

that end the proof. □

An interesting application of the Proposition 1.9 is the following. Let $\beta(\cdot)$ be a real Brownian motion on $[0,1]$, then its law will be a Wiener measure μ on $C_0([0,1])$.

Proposition 1.10. *If we define $\mu_\varepsilon = \mathfrak{L}\left(\sqrt{\varepsilon}\beta(\cdot)\right)$, then μ_ε will satisfy a large deviation principle with rate functional*

$$I(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |f'(\vartheta)|^2\,\mathrm{d}\vartheta & \text{for } f \text{ absolutely continuous with a square integrable derivative} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2.11}$$

Here we give an alternative, more direct, proof that does not use the general theorem for the Gaussian measures.

Proof. Lower bound. Given $f \in C_0([0,1])$ with $I(f) < +\infty$ and a neighborhood U of f, for any $\delta > 0$ we can find a $g \in U$ and a neighborhood $V \subset U$ of g, such that $I(g) \leq I(f) + \delta$, and g is twice continuously differentiable. Then

$$\mu_\varepsilon(U) \geq \mu_\varepsilon(V),$$

and it is sufficient to show that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(V) \geq -I(g) \geq -I(f) - \delta.$$

Let us denote with ν_ε the measure corresponding to $x - g$, where x is distributed according to μ_ε. If we denote by G the translate $V - g$, then G is a neighborhood of the zero function. By the definition of ν_ε,

$$\mu_\varepsilon(V) = \nu_\varepsilon(G) = \int\limits_G \frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\mu_\varepsilon}\,\mathrm{d}\mu_\varepsilon.$$

By the Cameron-Martin formula, ν_ε is absolutely continuous with respect to μ_ε, and the Radon-Nikodým derivative $\frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\mu_\varepsilon}$ is given by

$$\begin{aligned}\frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\mu_\varepsilon} &= \exp\left[-\frac{1}{\varepsilon}\int\limits_0^1 g'(\vartheta)\,\mathrm{d}x(\vartheta) - \frac{1}{2\varepsilon}\int\limits_0^1 g'(\vartheta)^2\,\mathrm{d}\vartheta\right] \\ &= \exp\left[\frac{1}{\varepsilon}\int\limits_0^1 x(\vartheta)g''(\vartheta)\,\mathrm{d}\vartheta - \frac{1}{\varepsilon}g'(1)x(1) - \frac{1}{\varepsilon}I(g)\right].\end{aligned}$$

For $r > 0$ sufficiently small we have that $B(0,r) \subset G$, then we have

$$\begin{aligned}&\mu_\varepsilon(V) \geq \nu_\varepsilon(G) \geq \nu_\varepsilon(B(0,r)) = \int\limits_{B(0,r)} \frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\mu_\varepsilon}\,\mathrm{d}\mu_\varepsilon \\ &\geq \mu_\varepsilon(B(0,r)) \inf_{x\in B(0,r)} \frac{\mathrm{d}\nu_\varepsilon}{\mathrm{d}\mu_\varepsilon}(x) \geq \mu_\varepsilon(B(0,r)) \exp\left[-\frac{1}{\varepsilon}I(g) - \frac{1}{\varepsilon}r\|g''\|_{C([0,1])}\right].\end{aligned}$$

Since $\mu_\varepsilon(B(0,r)) \to 1$ for $\varepsilon \downarrow 0$ for any $r > 0$, we have

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log \mu_\varepsilon(V) \geq -I(g) - r\|g''\|_{C([0,1])}.$$

But since g is twice differentiable, we have that $\|g''\|_{C([0,1])}$ is finite, and letting $r \to 0$ we obtain our lower bound.

Upper bound. Let Γ be any closed set in $C_0[0,1]$. Choose $\delta > 0$ then if π is any map from $C_0[0,1]$ into itself, we have

$$\mu_\varepsilon(\Gamma) \leq \mu_\varepsilon(f \mid \pi f \in B(\Gamma,\delta)) + \mu_\varepsilon(f \mid \|\pi f - f\|_{C_0([0,1])} \geq \delta).$$

Take $\pi = \pi_n$, the polygonalisation of any function x in $C_0[0,1]$ with step size $\frac{1}{n}$. Then

$$\mu_\varepsilon(f \mid \pi_n f \in B(\Gamma,\delta)) \leq \mu_\varepsilon(I(\pi_n f) \geq l_\delta),$$

where $l_\delta = \inf_{f\in B(\Gamma,\delta)} I(f)$. $I(\pi_n f)$ is a finite random variable under μ_ε and $\frac{2}{\varepsilon}I(\pi_n f)$ has a χ^2 distribution with n degrees of freedom. Therefore

$$\begin{aligned}\mu_\varepsilon(I(\pi_n f)\geq l_\delta) =& \frac{1}{\Gamma\left(\frac{n}{2}\right)}\int_{l_\delta}^{\infty} e^{-\frac{y}{\varepsilon}}\left(\frac{y}{\varepsilon}\right)^{\frac{n}{2}-1}\mathrm{d}\frac{y}{\varepsilon}\\ \leq& \exp\left[-\frac{l_\delta}{\varepsilon}+o\left(\frac{1}{\varepsilon}\right)\right] \qquad \text{as } \varepsilon\downarrow 0.\end{aligned}$$

It follows that $\limsup_{\varepsilon\downarrow 0}\varepsilon\log\mu_\varepsilon(I(\pi_n f)\geq l_\delta)\leq -l_\delta$.

On the other hand,

$$\begin{aligned}\mu_\varepsilon(f \mid \|\pi f - f\|_{C_0([0,1])}\geq\delta) \leq& n\,\mu_\varepsilon\left(\sup_{0\leq t\leq 1/n}|x(t)-x(0)|\geq\frac{\delta}{2}\right)\\ \leq& 2n\,\mu_1\left(\sup_{0\leq t\leq 1/n}|x(t)-x(0)|\geq\frac{n\delta}{2\sqrt{\varepsilon}}\right)\\ =& 4n\frac{1}{\sqrt{2\pi}}\int_{\frac{n\delta}{2\sqrt{\varepsilon}}}^{\infty} e^{-\frac{1}{2}y^2}\,\mathrm{d}y\\ =& \exp\left[-\frac{n\delta}{8\varepsilon}+o\left(\frac{1}{\varepsilon}\right)\right] \quad \text{as } \varepsilon\downarrow 0.\end{aligned}$$

It follows that $\limsup_{\varepsilon\downarrow 0}\varepsilon\log\mu_\varepsilon(f \mid \|\pi f - f\|_{C_0([0,1])}\geq\delta)\leq -\frac{n\delta}{8}$.

Combining the above calculations we obtain

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log\mu_\varepsilon(\Gamma)\leq\min\left[l_\delta,\frac{n\delta}{8}\right] = l_\delta \quad \text{for } n\to\infty.$$

For any closed set Γ its easy to verify that

$$\lim_{\delta\downarrow 0} l_\delta = \inf_{f\in\Gamma} I(f).$$

We now let $\delta\downarrow 0$ and we have the upper bound. □

Chapter 2

Large deviation for the stochastic convolution

2.1 The linear stochastic equation

Let H and U be separable Hilbert spaces, and $\{e_k\}_{k\in\mathbb{N}}$ and $\{f_h\}_{k\in\mathbb{N}}$ complete orthonormal systems in H and in U respectively. We are concerned with the following family of linear stochastic equations in H of the form

$$u_\varepsilon(t) = x + \int_0^t Au_\varepsilon(\vartheta)\,\mathrm{d}\vartheta + \sqrt{\varepsilon}B\,W(t), \quad x \in H, \quad t \in [0,T], \tag{2.1.1}$$

where $A : \mathrm{D}(A) \subset H \to H$ is a self-adjoint (generally unbounded) operator in H, negative definite; $B : U \to H$ is a bounded linear operator, withker$(B) = \{0\}$; W is a U-cylindrical Wiener process defined in a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$; ε is a positive number. We assume the following:

Hypothesis 2.1. *The operator A is diagonal in the basis $\{e_k\}_{k\in\mathbb{N}}$:*

$$Ae_k = -\mu_k e_k, \qquad \mu_k \geq \omega \in \mathbb{R}.$$

Definition 2.2. We define $S(\cdot)$ as the semi-group generated by A:

$$S(t) = e^{tA}, \qquad t \geq 0.$$

We define a *mild* solution to (2.1.1) to be a mean square continuous stochastic process $u(t)$, $t \in [0,T]$, such that

$$u_\varepsilon(t) = S(t)x + \sqrt{\varepsilon}W_S(t), \tag{2.1.2}$$

where $W_S(t)$, called the stochastic convolution. It is defined by:

$$W_S(t) = \int_0^t S(t-\vartheta)B\,\mathrm{d}W(\vartheta).$$

Definition 2.3. Let $L_2 = L_2(U, H)$ be the space of Hilbert-Schmidt operators from U into H, equipped with the norm

$$\|\Psi\|_{L^2}^2 = \sum_{h,k=0}^{+\infty} |\langle \Psi f_h, e_k\rangle|^2.$$

Then, in order to prove that the stochastic convolution process is well defined, we impose the following condition.

Hypothesis 2.4. *We require*

$$\int_0^T \|S(\vartheta)B\|_{L_2}^2\,\mathrm{d}\vartheta < +\infty.$$

We know, see Da Prato & Zabczyk [26, Theorem 5.4], that if Hypotheses 2.1 and 2.4 hold, there exists a unique mild solution of (2.1.1) on $[0, T]$.

In this chapter we shall discuss large deviations for the laws of the family of solutions u_ε.

2.2 Stochastic convolution as random variable

Let us consider the stochastic convolution process $W_S(\cdot)$. Assume that $W(t)$ has the form

$$W(t) = \sum_{h=0}^{+\infty} f_h \beta_h(t), \qquad t \geq 0,$$

where $\{\beta_h(t)\}_{h\in\mathbb{N}}$ is a family of real standard independent Brownian motions.

Proposition 2.5. *Let us assume that Hypotheses 2.1 and 2.4 hold; then, for any $t \geq 0$, $W_S(t)$ is a Gaussian random variable on U, with zero mean and covariance operator*

$$Q = \int_0^t S(\vartheta)BB^*S^*(\vartheta)\,\mathrm{d}\vartheta.$$

Proof. We may write

$$W_S(t) = \int_0^t S(t-r)B\,\mathrm{d}W(r) = \sum_{h=0}^{+\infty} \int_0^t S(t-r)Bf_h\,\mathrm{d}\beta_h(r);$$

it follows that

$$\mathbb{E}\left[|W_S(t)|^2\right] = \sum_{k=0}^{+\infty} \int_0^t |S(t-r)Bf_h|^2\,\mathrm{d}r = \int_0^t \|S(\vartheta)B\|_{L_2}^2\,\mathrm{d}\vartheta < +\infty.$$

Set

$$W_S^n(t) = \sum_{h=0}^{n} \int_0^t S(t-r)Bf_h\,\mathrm{d}\beta_h(r);$$

then $W_S^n(t)$ is a centered Gaussian random variable on H, and the limit for n going to infinity converges to $W_S(t)$. Hence, $W_S(t)$ is a centered Gaussian random variable.

Finally, the covariance operator Q is defined by

$$\begin{aligned}
\langle Qx, y\rangle &= \mathbb{E}\left[\langle W_S(t), x\rangle\ \langle W_S(t), y\rangle\right] \\
&= \mathbb{E}\left[\sum_{h=0}^{+\infty} \int_0^t \langle S(t-r)Bf_h, x\rangle\,\mathrm{d}\beta_h(r) \int_0^t \langle S(t-r)Bf_h, y\rangle\,\mathrm{d}\beta_h(r)\right] \\
&= \sum_{h=0}^{+\infty} \int_0^t \langle S(t-r)Bf_h, x\rangle\ \langle S(t-r)Bf_h, y\rangle\,\mathrm{d}r \\
&= \int_0^t \langle S(t-r)BB^*S^*(t-r)x, y\rangle\,\mathrm{d}r
\end{aligned}$$

as required. □

Now we consider the stochastic convolution as a random variable on the space $L^2(0,T;H)$. In the next theorem we shall show that $W_S(\cdot)$ is a centered Gaussian random variable, and we shall give an explicit formulation for the covariance operator.

Theorem 2.6. *Let us assume that Hypotheses 2.1 and 2.4 hold. Then the law μ of the stochastic convolution $W_S(\cdot)$ is a symmetric Gaussian measure* *on* $\mathcal{H} = L^2(0,T;H)$ *with covariance operator*

$$\mathcal{Q}\varphi(t) = \int_0^T g(t,\vartheta)\varphi(\vartheta)\,\mathrm{d}\vartheta,$$

where

$$g(t,\vartheta) = \int_0^{t\wedge s} S(t-r)BB^*S^*(t-r)\,\mathrm{d}r.$$

Proof. First of all, we prove that the trajectories of stochastic convolution belong to $\mathcal{H}$ a.s. For a measurable version of stochastic convolution we have, by the Fubini Theorem,

$$\mathbb{E}\left[\|W_S(\cdot)\|^2_{\mathcal{H}}\right] = \int_0^T \mathbb{E}\left[|W_S(\vartheta)|^2_H\right]\mathrm{d}\vartheta = \int_0^T\int_0^{\vartheta} \|S(\sigma)B\|^2_{L_2}\,\mathrm{d}\sigma\,\mathrm{d}\vartheta < +\infty,$$

so $W_S(\cdot)$ takes values in $\mathcal{H}$.

We must check that $\mathfrak{L}(W_S(\cdot))$ is Gaussian. We recall the following result, see for instance Da Prato & Zabczyk [26, Proposition 2.9]

Proposition 2.7. *Let* μ *be a measure on a separable Banach space* E *and* F *a linear subspace of* E^* *separating points*† *in* E *and generating the Borel* σ*-field of* E*. Then, if every* $\varphi \in F$ *has symmetric Gaussian law then* μ *is symmetric Gaussian.*

Now let us consider the following family F of functionals $(h\otimes a) \in \mathcal{H}^*$; $h : [0,T] \to \mathbb{R}$, $a \in H$,

$$(h\otimes a)(\varphi) = \int_0^T h(\vartheta)\,\langle a, \varphi(\vartheta)\rangle_H\,\mathrm{d}\vartheta, \quad \varphi \in \mathcal{H}.$$

Since

$$(h\otimes a)(W_S(\cdot)) = \int_0^T h(\vartheta)\,\langle a, W_S(\vartheta)\rangle_H\,\mathrm{d}\vartheta,$$

and $\langle a, W_S(\cdot)\rangle_H$ is a real Gaussian process, mean square continuous, with mean 0, by Proposition 2.7 we have that $\mathfrak{L}(W_S(\cdot))$ is symmetric Gaussian distribution $\mathcal{N}(0,\mathcal{Q})$ on $\mathcal{H}$.

†We mean a linear subspace of E^* such that for each point x of E there exist at least two points y_1, y_2 in the subspace such that $y_1(x) \neq y_2(x)$.

We want to give a representation formula for $\mathcal{Q}$, in order to have an explicit description of $\mathcal{Q}^{-\frac{1}{2}}$, which shall be necessary for the definition of the rate functional $I(\cdot)$.

Given two functions φ, ψ in $\mathcal{H}$, we have

$$\begin{aligned}
\langle \mathcal{Q}\varphi, \psi\rangle_{\mathcal{H}} =& \mathbb{E}\left[\langle \varphi, W_S(\cdot)\rangle_{\mathcal{H}} \ \langle \psi, W_S(\cdot)\rangle_{\mathcal{H}}\right] \\
=& \mathbb{E}\left[\int_0^T \langle \varphi(\vartheta), W_S(\vartheta)\rangle \,\mathrm{d}\vartheta \int_0^T \langle \psi(t), W_S(t)\rangle \,\mathrm{d}t\right] \\
=& \int_0^T \int_0^T \mathbb{E}\left[\langle \varphi(\vartheta), W_S(\vartheta)\rangle \ \langle \psi(t), W_S(t)\rangle\right] \mathrm{d}t\,\mathrm{d}\vartheta. \qquad (2.2.3)
\end{aligned}$$

Since for $t > \vartheta$

$$\begin{aligned}
&\mathbb{E}[\langle \varphi(\vartheta), W_S(\vartheta)\rangle \ \langle \psi(t), W_S(t)\rangle] \\
&\qquad = \mathbb{E}\left[\mathbb{E}\left[\langle \varphi(\vartheta), W_S(\vartheta)\rangle \langle \psi(t), W_S(t)\rangle \middle| \mathcal{F}_\vartheta\right]\right] \\
&\qquad = \mathbb{E}\left[\langle \varphi(\vartheta), W_S(\vartheta)\rangle \ \mathbb{E}\left[\langle \psi(t), W_S(t)\rangle \middle| \mathcal{F}_\vartheta\right]\right] \\
&\qquad = \mathbb{E}\left[\langle \varphi(\vartheta), W_S(\vartheta)\rangle \ \langle \psi(t), \mathbb{E}\left[W_S(t) \middle| \mathcal{F}_\vartheta\right]\rangle\right],
\end{aligned}$$

and by independence from the past

$$\begin{aligned}
\mathbb{E}\left[W_S(t) \middle| \mathcal{F}_\vartheta\right] &= \mathbb{E}\left[\int_0^\vartheta S(t-\sigma)B\,\mathrm{d}W(\sigma) + \int_\vartheta^t S(t-\sigma)B\,\mathrm{d}W(\sigma) \middle| \mathcal{F}_\vartheta\right] \\
&= \int_0^\vartheta S(t-\sigma)B\,\mathrm{d}W(\sigma),
\end{aligned}$$

we have

$$\begin{aligned}
&\mathbb{E}[\langle \varphi(\vartheta), W_S(\vartheta)\rangle \ \langle \psi(t), W_S(t)\rangle] \\
&\qquad = \mathbb{E}\left[\langle \varphi(\vartheta), W_S(\vartheta)\rangle \ \langle \psi(t), \mathbb{E}\left[W_S(t) \middle| \mathcal{F}_\vartheta\right]\rangle\right] \\
&\qquad = \left\langle \left[\int_0^{\vartheta\wedge t} S(t-\sigma)BB^*S^*(\vartheta-\sigma)\,\mathrm{d}\sigma\right] \varphi(\vartheta), \psi(t)\right\rangle.
\end{aligned}$$

If we define

$$g(t,\vartheta) = \int_0^{\vartheta\wedge t} S(t-\sigma)BB^*S^*(\vartheta-\sigma)\,\mathrm{d}\sigma,$$

(2.2.3) becomes

$$\langle \mathcal{Q}\varphi, \psi\rangle_{\mathcal{H}} = \int_0^T \int_0^T \langle g(t,\vartheta)\varphi(\vartheta), \psi(t)\rangle \,\mathrm{d}\vartheta \,\mathrm{d}t$$
$$= \int_0^T \left\langle \int_0^T g(t,\vartheta)\varphi(\vartheta)\,\mathrm{d}\vartheta, \psi(t)\right\rangle \mathrm{d}t,$$

that is

$$\mathcal{Q}\varphi(t) = \int_0^T g(t,\vartheta)\varphi(\vartheta)\,\mathrm{d}\vartheta.$$

□

Remark 2.8. We see that the relation

$$\mathcal{Q} = \mathcal{L}\mathcal{L}^*,$$

is satisfied with $\mathcal{L}$ defined in the following way:

$$\mathcal{L}\psi(t) = \int_0^t S(t-\vartheta)B\psi(\vartheta)\,\mathrm{d}\vartheta, \qquad \psi(\cdot) : [0,T] \to U, \tag{2.2.4}$$

and consequently

$$\mathcal{L}^*\varphi(t) = \int_t^T B^*S^*(\vartheta - t)\varphi(\vartheta)\,\mathrm{d}\vartheta, \qquad \varphi(\cdot) : [0,T] \to H, \tag{2.2.5}$$

with φ, ψ square integrable functions.

We have then the following corollary, see Da Prato & Zabczyk [26, Corollary B.5]:

Corollary 2.9. *Let $\mathcal{Q}$ the covariance operator of the stochastic convolution $W_S(\cdot)$. If we define $\mathcal{L}$ as in* (2.2.4) *we have*

$$\mathrm{Im}(\mathcal{Q}^{\frac{1}{2}}) = \mathrm{Im}(\mathcal{L})$$

and

$$\| \mathcal{Q}^{-\frac{1}{2}}\varphi\| = \| \mathcal{L}^{-1}\varphi\| \qquad \textit{for all } \varphi \in \mathrm{Im}(\mathcal{L}).$$

2.3 Large deviations for the stochastic convolution

For any $\varepsilon > 0$, we consider the laws of the processes $\sqrt{\varepsilon} W_S(\cdot)$ on the space $\mathcal{H}$.

Theorem 2.10. *Suppose that Hypotheses 2.1 and 2.4 hold, and let μ be the law of the stochastic convolution process $W_S(\cdot)$. Then, the family of laws $\{\mu_\varepsilon\}_{\varepsilon>0}$*

$$\mu_\varepsilon = \mathfrak{L}\left(\sqrt{\varepsilon} W_S(\cdot)\right)$$

satisfies a large deviation principle with respect to the rate functional $I(\cdot)$ given by

$$I(u) = \begin{cases} \frac{1}{2} \int_0^T \left| B^{-1}[u'(\vartheta) - Au(\vartheta)] \right|_U^2 \mathrm{d}\vartheta & \textit{for } u \in \mathrm{Im}(\mathcal{L}) \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. We proved in Theorem 2.6 that the process $W_S(\cdot)$ has a Gaussian law on the space $\mathcal{H}$, so from Theorem 1.9, the family μ_ε satisfy a large deviation principle with rate functional

$$I(u) = \begin{cases} \frac{1}{2} \int_0^T |\mathcal{Q}^{-\frac{1}{2}} u(\vartheta)|^2 \mathrm{d}\vartheta & u \in \mathrm{Im}(\mathcal{Q}^{-\frac{1}{2}}) \\ +\infty & \text{otherwise.} \end{cases}$$

To obtain an explicit formula for the rate $I(\cdot)$, we recall the Corollary 2.9, so it is sufficient to find an explicit formula for the inverse of the operator $\mathcal{L}$ introduced in (2.2.4).

Given $z \in \mathcal{H}$, we set $u = \mathcal{Q}z = \mathcal{L}\mathcal{L}^* z$. Then we have

$$u(t) = \int_0^t g(t,\vartheta) z(\vartheta)\,\mathrm{d}\vartheta + \int_t^T g(t,\vartheta) z(\vartheta)\,\mathrm{d}\vartheta. \tag{2.3.6}$$

Differentiating the previous identity in t, we have:

$$\begin{aligned} u'(t) =& g(t,t)z(t) + \int_0^t \left(A \int_0^\vartheta S(t-\sigma)BB^*S^*(\vartheta-\sigma)\,\mathrm{d}\sigma \right) z(\vartheta)\,\mathrm{d}\vartheta \\ & - g(t,t)z(t) + \int_t^T \left(A \int_0^\vartheta S(t-\sigma)BB^*S^*(\vartheta-\sigma)\,\mathrm{d}\sigma \right) z(\vartheta)\,\mathrm{d}\vartheta \\ & + \int_t^T BB^*S^*(\vartheta-t)z(\vartheta)\,\mathrm{d}\vartheta. \end{aligned}$$

Therefore, from (2.3.6) we have

$$u'(t) = Au(t) + \int_t^T BB^*S^*(\vartheta-t)z(\vartheta)\,\mathrm{d}\vartheta.$$

Hence the following expression is meaningful

$$B^{-1}\left[u'(t) - Au(t)\right] = \int_t^T B^*S^*(\vartheta-t)z(\vartheta)\,\mathrm{d}\vartheta.$$

Recalling (2.2.5), we see that

$$B^{-1}\left[u'(t) - Au(t)\right] = \mathcal{L}^*z(t).$$

Since by definition $\mathcal{L}^{-1}u = \mathcal{L}^*z$, we have the following expression for the rate functional:

$$I(u) = \begin{cases} \frac{1}{2}\int_0^T \left|B^{-1}[u'(\vartheta) - Au(\vartheta)]\right|_U^2 \mathrm{d}\vartheta & \text{for } u \in \mathrm{Im}(\mathcal{L}) \\ +\infty & \text{otherwise.} \end{cases}$$

□

Remark 2.11. The rate functional can be defined in another way. Assume that $z \in L^2(0,T;U)$ and consider the following integral control system

$$u^z(t) = \int_0^t Au^z(\vartheta)\,\mathrm{d}\vartheta + \int_0^t Bz(\vartheta)\,\mathrm{d}\vartheta, \qquad t \in [0,T],$$

which is solved by

$$u^z(t) = \int_0^t S(t-\vartheta) B z(\vartheta)\, \mathrm{d}\vartheta.$$

It follows that the rate functional can be expressed in terms of z:

$$I(u) = \inf \left\{ \frac{1}{2} \int_0^T |z(\vartheta)|_U^2 \,\mathrm{d}\vartheta \;\middle|\; u^z = u \right\}$$

This formulation has the following interpretation: $I(\cdot)$ can be viewed as the minimal "energy", given by the forcing term z, to allow the system to remain in u.

2.4 Stochastic convolution on infinite time

In this section we will assume:

Hypothesis 2.12.

$$\int_0^{+\infty} \|S(\vartheta)B\|_{L_2} \,\mathrm{d}\vartheta < +\infty.$$

This hypothesis implies that the parameter ω in Hypotheses 2.1 is positive, i.e. A is strictly monotone.

As before, we know, see Da Prato and Zabczyk [26, Theorem 5.4], that if Hypotheses 2.1 and 2.12 hold, there exists a unique mild solution of (2.1.1) for all T and then on $[0, +\infty[$.

We want to prove the following

Theorem 2.13. *Suppose that Hypotheses 2.1 and 2.12 hold. Given a function* $\rho \in C([0, +\infty[;]0, +\infty[) \cap L^1(0, +\infty)$*, the stochastic convolution* $W_S(\cdot)$ *is a centered Gaussian random variable on* $\mathcal{H}_\infty = L^2([0, +\infty), \rho; H)$*, where* $\mathcal{H}_\infty$ *is the Hilbert space of measurable functions from* $(0, +\infty)$ *to* H *endowed with the following weighted scalar product*

$$\langle \varphi, \psi \rangle_{\mathcal{H}_\infty} = \int_0^{+\infty} \langle \varphi(\vartheta), \psi(\vartheta) \rangle \; \rho(\vartheta) \,\mathrm{d}\vartheta.$$

Proof. As in Theorem 2.6, we prove that the trajectories of a stochastic convolution belongs to $\mathcal{H}_\infty$ a.s.. For a measurable version of stochastic convolution we have, by the Fubini Theorem,

$$\mathbb{E}\left[\|W_S(\cdot)\|^2_{\mathcal{H}_\infty}\right] = \int_0^{+\infty} \mathbb{E}\left[|W_S(\vartheta)|^2_H\right] \rho(\vartheta)\,\mathrm{d}\vartheta$$
$$= \int_0^{+\infty}\int_0^{\vartheta} \|S(\sigma)BB^*S^*(\sigma)\|_{L_2}\,\mathrm{d}\sigma\,\rho(\vartheta)\,\mathrm{d}\vartheta \leq \int_0^{+\infty} M\rho(\vartheta)\,\mathrm{d}\vartheta < +\infty,$$

so $W_S(\cdot)$ takes values in $\mathcal{H}_\infty$.

It easy to see as in Theorem 2.6 that $\mathfrak{L}\left(W_S(\cdot)\right)$ is Gaussian.

To explicit the covariance operator $\mathcal{Q}_\infty$ we take $\varphi, \psi \in \mathcal{H}_\infty$. Then, we follow the same calculation of the Theorem 2.6, so we have:

$$\langle \mathcal{Q}_\infty \varphi, \psi\rangle_{\mathcal{H}_\infty} = \mathbb{E}\left[\langle \varphi, W_S\rangle_{\mathcal{H}_\infty}\langle \psi, W_S\rangle_{\mathcal{H}_\infty}\right]$$
$$= \int_0^{+\infty}\int_0^{+\infty} \mathbb{E}\left[\langle \varphi(\vartheta), W_S(\vartheta)\rangle_H \langle \psi(\sigma), W_S(\sigma)\rangle_H\right] \rho(\sigma)\,\mathrm{d}\sigma\,\rho(\vartheta)\,\mathrm{d}\vartheta$$

Since for $\vartheta > \sigma$

$$\mathbb{E}\left[\langle \varphi(\vartheta), W_S(\vartheta)\rangle_H \langle \psi(\sigma), W_S(\sigma)\rangle_H\right]$$
$$= \left\langle \left[\int_0^\sigma S(\sigma)BB^*S^*(\sigma)\,\mathrm{d}\sigma\right] S^*(\vartheta-\sigma)\varphi(\vartheta), \psi(\sigma)\right\rangle_H,$$

defining again

$$g(t,\vartheta) = \int_0^{t\wedge\vartheta} S(t-\sigma)BB^*S^*(\vartheta-\sigma)\,\mathrm{d}\sigma.$$

we obtain

$$\langle \mathcal{Q}_\infty \varphi, \psi\rangle_{\mathcal{H}_\infty} = \int_0^{+\infty}\int_0^{+\infty} \langle g(\sigma,\vartheta)\varphi(\vartheta), \psi(\sigma)\rangle_H\, \rho(\vartheta)\,\mathrm{d}\vartheta\,\rho(\sigma)\,\mathrm{d}\sigma$$
$$= \int_0^{+\infty} \left\langle \int_0^{+\infty} g(\sigma,\vartheta)\varphi(\vartheta)\,\rho(\vartheta)\,\mathrm{d}\vartheta, \psi(\sigma)\right\rangle_H \rho(\sigma)\,\mathrm{d}\sigma.$$

Then we have that $W_S(\cdot)$ is a symmetric Gaussian random variable on $\mathcal{H}_\infty$ with covariance operator $\mathcal{Q}_\infty$, where $\mathcal{Q}_\infty : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is defined by

$$\mathcal{Q}_\infty h(t) = \int_0^{+\infty} g(t,\vartheta)h(\vartheta)\,\rho(\vartheta)\,\mathrm{d}\vartheta, \qquad \text{for } t \in [0,+\infty[.$$

□

If we denote with U_∞ the Hilbert space of measurable functions from $[0,+\infty[$ to U endowed with the standard scalar product

$$\langle f,g\rangle_{U_\infty} = \int_0^{+\infty} \langle f(\vartheta), g(\vartheta)\rangle_U \,\mathrm{d}\vartheta,$$

then, as before, we see that is satisfied the relation

$$\mathcal{Q}_\infty = \mathcal{L}\mathcal{L}^*,$$

with $\mathcal{L} : U_\infty \to \mathcal{H}_\infty$ defined as

$$\mathcal{L}\psi(t) = \int_0^{t} S(t-\vartheta)B\psi(\vartheta)\,\mathrm{d}\vartheta, \qquad \psi(\cdot) : [0,+\infty[\to U, \tag{2.4.7}$$

with ψ square integrable function. Consequently, by the chosen scalar products on $\mathcal{H}_\infty$ and U_∞, we have

$$\mathcal{L}^*\varphi(t) = \int_t^{\infty} B^*S^*(\vartheta-t)\varphi(\vartheta)\,\rho(\vartheta)\,\mathrm{d}\vartheta, \qquad \varphi(\cdot) : [0,+\infty[\to H, \tag{2.4.8}$$

with φ square integrable too.

Remark 2.14. We point out that the standard (not weighted) scalar product on U_∞ is the unique one that allows such a decomposition of $\mathcal{Q}_\infty$.

2.4.1 Large deviation in infinite time

For any $\varepsilon > 0$, we consider the family of measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ defined by

$$\mu_\varepsilon = \mathfrak{L}\left(\sqrt{\varepsilon}W_S(\cdot)\right)$$

on the space $\mathcal{H}_\infty$.

Theorem 2.15. *Suppose that Hypotheses 2.1 and 2.12 hold. Then the family μ_ε satisfies a large deviation principle with respect to the rate functional $I(\cdot)$ given by*

$$I(u) = \begin{cases} \frac{1}{2} \int\limits_0^{+\infty} \left| B^{-1}[u'(\vartheta) - Au(\vartheta)] \right|_U^2 \mathrm{d}\vartheta & \text{for } u \in \mathrm{Im}(\mathcal{L}) \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. We proved in Theorem 2.13 that the Gaussian process $W_S(\cdot)$ has a Gaussian law μ on the space $\mathcal{H}_\infty$, so we have automatically that a large deviation principle is fulfilled, with the following rate functional:

$$I(u) = \begin{cases} \frac{1}{2} \| \mathcal{Q}_\infty^{-\frac{1}{2}} u \|^2_{\mathcal{H}_\infty}, & u \in \mathrm{Im}\left(\mathcal{Q}_\infty^{\frac{1}{2}} \right), \\ +\infty & \text{otherwise.} \end{cases}$$

As in Theorem 2.10 we can give a more explicit formulation for the rate functional $I(\cdot)$.

Given $z \in \mathcal{H}_\infty$, we set $u = \mathcal{Q}_\infty z = \mathcal{L}\mathcal{L}^* z$. Then we have

$$u(t) = \int\limits_0^t g(t,\vartheta) z(\vartheta)\, \rho(\vartheta)\, \mathrm{d}\vartheta + \int\limits_t^{+\infty} g(t,\vartheta) z(\vartheta)\, \rho(\vartheta)\, \mathrm{d}\vartheta.$$

Differentiating in t we have:

$$\begin{aligned} u'(t) =& g(t,t)z(t)\rho(t) + \int\limits_0^t \left(A \int\limits_0^\vartheta S(t-\sigma) BB^* S^*(\vartheta - \sigma)\, \mathrm{d}\sigma \right) z(\vartheta)\, \rho(\vartheta)\, \mathrm{d}\vartheta \\ & - g(t,t)z(t)\rho(t) + \int\limits_t^{+\infty} \left(A \int\limits_0^\vartheta S(t-\sigma) BB^* S^*(\vartheta - \sigma)\, \mathrm{d}\sigma \right) z(\vartheta)\, \rho(\vartheta)\, \mathrm{d}\vartheta \\ & + \int\limits_t^{+\infty} BB^* S^*(\vartheta - t) z(\vartheta)\, \rho(\vartheta)\, \mathrm{d}\vartheta, \end{aligned}$$

thus we obtain

$$u'(t) = Au(t) + \int\limits_t^{+\infty} BB^* S^*(\vartheta - t) z(\vartheta)\, \rho(\vartheta)\, \mathrm{d}\vartheta.$$

Hence the following expression is meaningful

$$B^{-1}\left[u'(t) - Au(t)\right] = \int_t^{+\infty} B^* S^*(\vartheta - t) z(\vartheta)\, \rho(\vartheta)\, \mathrm{d}\vartheta,$$

Recalling (2.4.8) we see that

$$B^{-1}\left[u'(t) - Au(t)\right] = \mathcal{L}^* z(t).$$

since, by definition $\mathcal{L}^* z = \mathcal{L}^{-1} u$, we have, in accordance with the scalar product on $\mathcal{U}_\infty$, that

$$\|\mathcal{L}^{-1} u\|^2_{U_\infty} = \int_0^{+\infty} \left|B^{-1}[u'(\vartheta) - Au(\vartheta)]\right|^2_U \mathrm{d}\vartheta;$$

Then the expression for the rate functional, that does not depend on the weight ρ, follows:

$$I(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^{+\infty} \left|B^{-1}[u'(\vartheta) - Au(\vartheta)]\right|^2_U \mathrm{d}\vartheta & \text{for } u \in \mathrm{Im}(\mathcal{L}) \\ +\infty & \text{otherwise.} \end{cases}$$

□

2.5 The linear Volterra equation

There is another interesting extension of problem (2.1.1), which involves the use of convolution kernels as in the following problem.

$$u_\varepsilon(t) = x + \int_0^t a(t-\vartheta) A u_\varepsilon(\vartheta)\, \mathrm{d}\vartheta + \sqrt{\varepsilon} B\, W(t), \quad x \in H, \quad t \in [0,T], \tag{2.5.9}$$

where, as in section 2.1, $A : \mathrm{D}(A) \subset H \to H$ is a self-adjoint (generally unbounded) operator in H, negative definite, that satisfies Hypothesis 2.1; $B : U \to H$ is a bounded linear operator, with $\ker(B) = \{0\}$; W is a U-cylindrical Wiener process defined in a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$; ε is a positive number; moreover we impose the following:

Hypothesis 2.16. *The kernel* $a :]0,T] \to]0,+\infty[$ *satisfies:*

$$a(t) \in C(]0,T]) \cap L^1_{\mathrm{loc}}(0,T).$$

That this problem is a generalization of (2.1.1) is straightforward posing $a \equiv 1$.

This problem is the stochastic version of the following Volterra problem

$$u(t) = x + \int_0^t a(t-\vartheta) A u(\vartheta)\,\mathrm{d}\vartheta \qquad x \in H, \quad t \in [0,T]. \tag{2.5.10}$$

The solution to this problem can be given as follows

$$u(t,x) = S(t)x,$$

where the resolvent $\{S(t)\}_{t>0}$ is a family of bounded linear operators:

Definition 2.17. A family $\{S(t),\ t \in [0,T]\}$ of bounded linear operators in a Banach space X is called a resolvent of (2.5.10) if the following conditions are satisfied:

I. $S(t)$ is strongly continuous on $[0,T]$ and $S(0) = I$;

II. $S(t)$ commutes with A;

III. the resolvent equation holds:

$$S(t)x = x + \int_0^t a(t-\vartheta) A S(\vartheta) x\,\mathrm{d}\vartheta, \tag{2.5.11}$$

for all $x \in \mathrm{D}(A)$, $t \in [0,T]$.

Notice that the Volterra equation (2.5.10) has a unique solution if and only if it admits a resolvent. Moreover it worts mention that here we restrict ourselves to consider solutions on a finite interval $[0,T]$ while normally it is aasumed that time goes throught all $\mathbb{R}_+$, see Prüss [50] for an exaustive reference on Volterra equations.

Hypothesis 2.18. *The Volterra equation* (2.5.10) *admits an unique solution, and we denote by $S(t)$, $t \in [0,T]$, the associated resolvent.*

It is possible to show, in view of Hypothesys 2.1, that the resolvent is diagonal in the basis $\{e_k\}_{k\in\mathbb{N}}$ of H. We introduce now the solution $s_\alpha(\cdot)$, $\alpha \in \mathbb{R}$, of the scalar integral equation

$$s_\alpha(t) + \alpha \int_0^t a(t-\vartheta) s_\alpha(\vartheta)\,\mathrm{d}\vartheta = 1, \qquad t \in [0,T]. \tag{2.5.12}$$

Let $-\mu_k$ be an eigenvalue of A with eigenvector e_k. Then,

$$S(t)e_k = s_{\mu_k}(t)e_k, \qquad t \in [0,T].$$

We now consider the stochastic convolution process $W_S(t)$:

$$W_S(t) = \int_0^T S(t-\vartheta)B \, \mathrm{d}W(\vartheta). \tag{2.5.13}$$

where, as in the standard case, we assume that the process $W(t)$ has the form

$$W(t) = \sum_{h=0}^{+\infty} f_h \beta_h(t), \qquad t \in [0,T],$$

where $\{f_h\}_{h\in\mathbb{N}}$ is an orthonormal system in U and $\{\beta_h(t)\}_{h\in\mathbb{N}}$ is a family of real standard independent Brownian motions.

Then the stochastic convolution $W_S(\cdot)$ is well defined for all $t \in [0,T]$, if the following hypothesis holds

Hypothesis 2.19. *We require*

$$\int_0^T \|S(\vartheta)B\|_{L_2}^2 \, \mathrm{d}\vartheta < +\infty.$$

Theorem 2.20. *Under Hypotheses 2.1, 2.16, 2.18 and 2.19, for any* $t \geq 0$*,* $W_S(t)$ *is a real, Gaussian random variable, with mean* 0 *and covariance operator*

$$\mathcal{Q} = \int_0^t S(\vartheta)BB^*S^*(\vartheta) \, \mathrm{d}\vartheta.$$

Proof. We may write

$$W_S(t) = \int_0^t S(t-\vartheta)B \, \mathrm{d}W(\vartheta) = \sum_{h=0}^{+\infty} \int_0^t S(t-\vartheta)Bf_h \, \mathrm{d}\beta_h(\vartheta);$$

it follows that

$$\mathbb{E}\left[W_S(t)\right]^2 = \sum_{h=0}^{+\infty} \int_0^t |S(t-\vartheta)Bf_h|^2 \, \mathrm{d}\vartheta = \int_0^t \|S(\vartheta)B\|_{L_2}^2 \, \mathrm{d}\vartheta < +\infty.$$

Set

$$W_S^n(t) = \sum_{h=0}^{n} \int_0^t S(t-\vartheta)Bf_h \, \mathrm{d}\beta_h(\vartheta);$$

then $W_S^n(t)$ is a centered Gaussian random variable on H, and the limit for $n \to +\infty$ converges to $W_S(t)$. Hence, $W_S(t)$ is a centered Gaussian random variable.

Finally, the covariance operator Q is defined by

$$\begin{aligned}\langle Qx, y\rangle &= \mathbb{E}\left[\langle W_S(t), x\rangle \ \langle W_S(t), y\rangle\right] \\ &= \int_0^t \langle S(t-\vartheta)BB^*S^*(t-\vartheta)x, y\rangle \, \mathrm{d}\vartheta,\end{aligned}$$

as required. □

As in the standard case ($a \equiv 1$) we can show that the stochastic convolution is a Gaussian process although it is not Markovian anymore.

Theorem 2.21. *Suppose that Hypotheses 2.1, 2.16, 2.18 and 2.19 hold, then the law μ of the stochastic convolution $W_S(\cdot)$ is a symmetric Gaussian measure on $\mathcal{H} = L^2(0,T;H)$ with covariance operator*

$$\mathcal{Q}\varphi(t) = \int_0^T g(t,\vartheta)\varphi(\vartheta) \, \mathrm{d}\vartheta,$$

where

$$g(t,\vartheta) = \int_0^{t\wedge\vartheta} S(t-r)BB^*S^*(\vartheta-r) \, \mathrm{d}r.$$

We omit the proof of this theorem since it is exactly the same proof as in Theorem 2.6. We point out that, in the proof of Theorem 2.6, we don't need to suppose the stochastic convolution to be Markovian.

As in the standard case, we see that is satisfied the relation

$$\mathcal{Q} = \mathcal{L}\mathcal{L}^*,$$

with $\mathcal{L}$ defined:

$$\mathcal{L}\psi(t) = \int_0^t S(t-\vartheta)B\psi(\vartheta) \, \mathrm{d}\vartheta, \qquad \psi(\cdot) : [0,T] \to U \tag{2.5.14}$$

and consequently

$$\mathcal{L}^*\varphi(t) = \int_t^T B^* S^*(\vartheta - t)\varphi(\vartheta)\,\mathrm{d}\vartheta, \qquad \varphi(\cdot) : [0,T] \to H, \tag{2.5.15}$$

with ψ, φ square integrable functions.

2.5.1 Large deviation for the stochastic convolution in the Volterra case

For any $\varepsilon > 0$, we shall consider the laws of the processes $\sqrt{\varepsilon} W_S(\cdot)$ on the space $L^2(0,T;H)$.

We will use a stronger assumption on the kernel a:

Hypothesis 2.22. *The kernel* $a : [0,T] \to]0,+\infty[$ *satisfies:*

$$a(t) \in C([0,T]) \cap W^{1,1}(0,T).$$

Theorem 2.23. *Suppose that Hypotheses 2.1, 2.18, 2.22 and 2.19 hold. Then the family* μ_ε *satisfies a large deviation principle with respect to the rate functional* $I(\cdot)$ *given by*

$$I(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \big|B^{-1}[u'(\vartheta) - a(0)Au(\vartheta) \\ \qquad\qquad - (a' * Au(\cdot))(\vartheta)]\big|_U^2 \,\mathrm{d}\vartheta \quad \textit{for } u \in \mathrm{Im}\,(\mathcal{L}), \\ +\infty \quad \textit{otherwise.} \end{cases}$$

Here $(a * b)(t)$ stands for $\int_0^t a(t-\vartheta)b(\vartheta)\,\mathrm{d}\vartheta$.

Proof. From Theorem 2.21 it follows that the family μ_ε satisfy a large deviation principle with rate functional

$$I(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T |\mathcal{L}^{-1}(u)(\vartheta)|_U^2 \,\mathrm{d}\vartheta \quad u \in \mathrm{Im}\,(\mathcal{L}) \\ +\infty \quad \text{otherwise.} \end{cases}$$

We give now an explicit formulation for the operator $\mathcal{L}^{-1}$ introduced above. Let $u = \mathcal{L}(z)$, with $z \in L^2(0,T;U)$; then we have:

$$u(t) = (S * Bz(\cdot))(t).$$

From the resolvent equation

$$S(t) - (a * AS(\cdot))(t) = I,$$

we have

$$\begin{aligned}(a * Au)(t) = (a * AS * Bz)(t) &= -(I * Bz(\cdot))(t) + (S * Bz(\cdot))(t) \\ &= -(I * Bz(\cdot))(t) + u(t).\end{aligned}$$

Differentiating, we have

$$Bz(t) = u'(t) - \frac{\mathrm{d}}{\mathrm{d}t}(a * Au)(t),$$

thus

$$\begin{aligned}z(t) = \mathcal{L}^{-1}u(t) =& B^{-1}[u'(t) - \frac{\mathrm{d}}{\mathrm{d}t}(a * Au)(t)] \\ =& B^{-1}[u'(t) - a(0)Au(t) - (a' * Au)(t)].\end{aligned} \tag{2.5.16}$$

This is sufficient to obtain the explicit formula of the rate functional $I(\cdot)$. □

Remark 2.24. As before we can write another definition of the rate functional. Assume that $z \in L^2(0, T; U)$ and consider the following integral control system

$$u^z(t) = \int_0^t a(t - \vartheta)Au^z(\vartheta)\,\mathrm{d}\vartheta + \int_0^t Bz(\vartheta)\,\mathrm{d}\vartheta, \qquad t \in [0, T],$$

which is solved by

$$u^z(t) = \int_0^t S(t - \vartheta)Bz(\vartheta)\,\mathrm{d}\vartheta.$$

It follows that the rate functional can be expressed in terms of z:

$$I(u) = \inf \left\{ \frac{1}{2}\int_0^T |z(\vartheta)|_U^2\,\mathrm{d}\vartheta \;\middle|\; u^z = u \right\} \tag{2.5.17}$$

Again this formulation has the following interpretation: $I(\cdot)$ can be viewed as the minimal "energy", given by the forcing term z, to allow the system to remain in u.

Remark 2.25. The rate functional defined in (2.23) coincides with the Onsager-Machlup functional for equation (2.5.9), compare Bonaccorsi [5, Theorem 1].

Let us recall that the Onsager-Machlup functional, for a diffusion X, answers the following question: given two smooth curves arising from the same point, which one is more probable for the evolution of the system? In the finite dimensional case, the problem has been analyzed by many authors, starting from the work of Onsager & Machlup [47], see for instance the book of Ikeda & Watanabe [39] and the bibliography therein.

In the infinite dimensional case, it was proposed in Bardina *et al.* [4] to consider the limiting behavior of ratios of the form

$$\gamma_\varepsilon(\varphi) = \frac{\Pr(\|X(\cdot) - \varphi(\cdot)\|_{L^2(0,T;H)} \le \varepsilon)}{\Pr(\|W_S(\cdot)\|_{L^2(0,T;H)} \le \varepsilon)}$$

when $\varepsilon \to 0$: if the limit $\lim_{\varepsilon\to 0} \gamma_\varepsilon(\varphi) = \exp(J_0(\varphi))$ exists for all φ in a suitable class of deterministic function, then the functional J_0 is called the Onsager-Machlup functional associated to the diffusion X.

In Bonaccorsi [5] the Onsager-Machlup functional is defined similarly; then, it is proved that the limit exists for all functions $\varphi = u^z$, when $z \in L^2(0,T;U)$, and it coincides with the functional $I(\cdot)$ defined in (2.5.17).

2.6 Stochastic convolution on Banach space

In this section we denote with $E \subset H$ a separable Banach space, densely embedded in H as Borel subset. Moreover, E will be equipped with the norm $\|\cdot\|_E$.

We assume that Hypotheses 2.1 and 2.4 hold.

We are interested in studying the following problem on the space E:

$$\begin{cases} du_\varepsilon(t) = Au_\varepsilon(t)dt + \sqrt{\varepsilon}BdW(t), & \varepsilon > 0, \\ u_\varepsilon(0) = x \in E. \end{cases} \tag{2.6.18}$$

We shall denote by A_E the part of A in E ‡.

We will assume:

Hypothesis 2.26. *A_E generates a strongly continuous semigroup of operators $S_E(\cdot)$ on E, and there exists $\omega \in \mathbb{R}$ such that $\|S_E(t)\|_E \le e^{\omega t}$ for all $t \in [0,T]$,*

‡Given a subspace $E \subset H$ and an operator $S : H \to H$, we define by $S_E(\cdot)$ the part of $S(\cdot)$ in E as the operator $S(\cdot)$ restricted to the subset $\{x \in \mathrm{D}(S(\cdot)) \cap E : S(\cdot)x \in E\}$.

Hypothesis 2.27. *W_S has an E-continuous version W_{S_E}.*

By a *mild* solution of (2.6.18) we mean a mean square continuous stochastic process $u_\varepsilon(t,x) \in C([0,T];E)$, such that:

$$u_\varepsilon(t) = S_E(t)x + \sqrt{\varepsilon} W_{S_E}(t).$$

2.6.1 Large deviation for the stochastic convolution on the Banach space E

We want, here, to show that the laws $\mu_\varepsilon = (\mathfrak{L}(u_\varepsilon))_{\varepsilon>0}$ of u_ε fulfill a large deviation principle with respect to a suitable functional $I(\cdot)$ (the following result is known see Peszat [48]).

Theorem 2.28. *Let us assume that Hypotheses 2.1, 2.4, 2.26 and 2.27 hold. Then $\{\mu_\varepsilon\}_{\varepsilon>0}$ fulfills a large deviation principle with respect to the functional $I(\cdot)$, defined as follows:*

$$I(u) = \begin{cases} \frac{1}{2}\int_0^T |B^{-1}(u'(\vartheta) - A_E u(\vartheta))|_U^2 \, \mathrm{d}\vartheta, & u \in \mathrm{Im}(\mathcal{L}_E), \\ +\infty & \textit{otherwise}, \end{cases}$$

where $\mathcal{L}_E$ is the part of $\mathcal{L}$ in $C([0,T];E)$:

$$\mathcal{L}_E\psi(t) = \int_0^t S_E(t-\vartheta)B\psi(\vartheta)\,\mathrm{d}\vartheta, \qquad \psi(\cdot) : [0,T] \to U$$

Proof. Since E is dense in H, the spaces $L^2(0,T;E)$ and $C([0,T];E)$ are both densely embedded in $L^2(0,T;H)$.

Since the laws of u_ε are Gaussian both on the space $L^2(0,T;H)$ and the space $C([0,T];E)$, from the uniqueness of the reproducing kernel, we have that the reproducing kernels H_{μ_ε} are the same in both spaces, see Da Prato & Zabczyk [26, Proposition 2.8]. From this we have that the family of Gaussian measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle with respect to the functional $I(\cdot)$ defined as:

$$I(u) = \begin{cases} \frac{1}{2}|u|^2_{H_{\mu_1}}, & u \in H_{\mu_1}, \\ +\infty & \text{otherwise}, \end{cases}$$

Since, on the Hilbert space $L^2(0,T;H)$, it can be shown that

$$I(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|B^{-1}[u'(\vartheta) - Au(\vartheta)]\right|_U^2 \mathrm{d}\vartheta & \text{for } u \in \mathrm{Im}(\mathcal{L}), \\ +\infty \quad \text{otherwise}, \end{cases}$$

the same statement holds for the sub-space $C([0,T];E)$. $\square$

Chapter 3

Large deviation for the semilinear problem

In this chapter we shall study the following semilinear problem:

$$u_\varepsilon(t) = x + \int_0^t Au_\varepsilon(\vartheta) + G(u_\varepsilon(\vartheta))\,\mathrm{d}\vartheta + \sqrt{\varepsilon}B\,W(t),$$

with $x \in H$, $t \in [0, T]$. Or, in the equivalent formulation,

$$\begin{cases} du_\varepsilon(t) = Au_\varepsilon(t)dt + G(u_\varepsilon(t))dt + \sqrt{\varepsilon}BdW(t), & \varepsilon > 0, \\ u_\varepsilon(0) = x \end{cases}$$

Here G is a non linear perturbation.

This problem has been studied in many settings: we shall analyze some of them, proving that it has an unique solution in some suitable sense. Then, we shall show a Large Deviation Principle for the laws of the solutions $u_\varepsilon(\cdot)$, via a continuous correspondence between solutions of the linear equation and solutions of the non-linear equation, using the contraction principle.

In the Section 3.1 we shall study this problem on the space H in the case of a non linear term G Lipschitz, then in Section 3.2 we shall study the case of dissipative G on a Banach space E densely embedded in H.

3.1 The case with Lipschitz non-linearity

As in Chapter 2, let H and U be separable Hilbert spaces and let $\{e_k\}_{k\in\mathbb{N}}$ and $\{f_h\}_{h\in\mathbb{N}}$ be complete orthonormal systems in H and in U respectively.

$$u(t) = x + \int_0^t Au(\vartheta) + G(u(\vartheta))\,\mathrm{d}\vartheta + B\,W(t), \qquad (3.1.1)$$

with $x \in H$, $t \in [0,T]$.

We assume that the Hypotheses 2.1 and 2.4 hold and, moreover, that the non-linear term G satisfies the following

Hypothesis 3.1. *The non linear functional $G : \mathrm{D}\,(G) \subset H \to H$ is locally Lipschitz continuous and sublinear.*

Note that, under Hypothesis 3.1 on the nonlinear coefficient G, it is possible to define the following constant

$$C(|x|) = C_0(1 + |x|),$$

such that

$$C(|x|) \geq \sup_{|z|\leq|x|} G(z).$$

By definition, an *implicit* mild solution to (3.1.1) is a mean square continuous stochastic process $u(t)$, $t \in [0,T]$, adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, such that

$$u(t) = S(t)x + \int_0^t S(t-\vartheta)\,G(u(\vartheta))\,\mathrm{d}\vartheta + W_S(t),$$

where $S(\cdot)$ and $W_S(\cdot)$ are the the semigroup generated by A and the stochastic convolution respectively.

Problem (3.1.1) is well known and it is studied by many authors; here we remind the main result. Define $C_{\mathcal{F}}([0,T];H)$ to be the space of processes $u(t), t \in [0,T]$, $\mathcal{F}_t$-adapted, such that there exists a version of u with $u(\cdot) \in C([0,T];H)$. We can state the following

Theorem 3.2. *Let us assume Hypotheses 2.1, 2.4 and 3.1. Then* (3.1.1) *has a unique mild solution in $C_{\mathcal{F}}([0,T];H)$.*

The proof follows easily from a fixed point argument; see the proof of the more general Theorem 4.4 that applies here with $a \equiv 1$.

3.1.1 Large deviations

To obtain a large deviation property for the family of laws of $u_\varepsilon(\cdot)$, we cannot apply directly the contraction principle through (3.1.1), since we have not a large deviation property for the cylindrical Wiener process $W(\cdot)$.

Instead we define $\Psi : Z_T \to Z_T$ as the solution functional that maps each given trajectory $z(\cdot)$ of $W_S(\cdot)$ to the corresponding solution $u(\cdot)$, as follows

$$u(t) = S(t)x + \int_0^t S(t-\vartheta)\, G(u(\vartheta))\, \mathrm{d}\vartheta + z(t). \tag{3.1.2}$$

We see that Ψ maps solutions of the linear problem (2.1.1) on to solutions of the non-linear problem (3.1.1).

We can, now, state the large deviation property for the family of laws of $u_\varepsilon(\cdot)$.

Theorem 3.3. *Suppose that the assumptions of Theorem 3.2 hold; then the family of laws of $u_\varepsilon(\cdot)$ satisfies a large deviation principle with respect to the rate functional*

$$J(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|B^{-1}[u'(\vartheta) - Au(\vartheta) + G(u(\vartheta))]\right|^2 \mathrm{d}\vartheta & \text{for } u \in R \\ +\infty & \text{otherwise.} \end{cases}$$

where R is the the subspace of $C([0,T];H)$ given by

$$R = \Big\{ u \in C([0,T];H)) \Big| \exists z \in L^2(0,T;U) : \\ u(t) = S(t)x + \int_0^t S(t-\vartheta)G(u(\vartheta))\,\mathrm{d}\vartheta + \int_0^t S(t-\vartheta)Bz(\vartheta)\,\mathrm{d}\vartheta \Big\}.$$

We want to strength that the functional J is:

$$J(\cdot) = I(\Psi^{-1}(\cdot)).$$

Proof. We have that $\mathfrak{L}\,(u_\varepsilon(\cdot)) = \Psi \circ \mathfrak{L}\,(\sqrt{\varepsilon} z(\cdot))$, so if we show that the bijection Ψ is continuous, the result follows from Theorem 2.10 and Remark 1.5.

Consider $z_1(\cdot), z_2(\cdot)$ in Z_T and the corresponding solutions $u_1(\cdot), u_2(\cdot)$ of (3.1.2). Suppose $\|z_i\|_{Z_T} \le K_0$, then from Theorem 3.2 we have $\|u_i\|_{Z_T} \le K$, so without loss of generality we can consider G globally Lipschitz.

Then from (3.1.2) there exists a constant C such that

$$\|u_1 - u_2\|_{Z_T} \le C\|z_1 - z_2\|_{Z_T},$$

and this ends the proof. □

3.2 The case with dissipative non-linearity

Let H, U be separable Hilbert spaces, and $E \subset H$ a separable Banach space, densely embedded in H as Borel subset. Let $|\cdot|$ and $\|\cdot\|_E$ be the norms respectively on H and E.

We assume the Hypotheses 2.1, 2.4, 2.26 and 2.27 hold. Now we assume that $G : \mathrm{D}(G) \subset H \to H$ is a nonlinear operator defined on a subset of the Hilbert space H. We shall consider then (3.1.1) in the smaller state space E, on which the operator G is well defined and continuous. This method requires also that the initial condition takes values in the space E.

We are concerned with the problem

$$\begin{cases} du_\varepsilon(t) = (Au_\varepsilon(t) + G(u_\varepsilon(t)))dt + \sqrt{\varepsilon} B dW(t), \quad \varepsilon > 0, \\ u_\varepsilon(0) = x \in E. \end{cases} \tag{3.2.3}$$

Let us first give the definition of mild solution for (3.2.3): a predictable H-valued process $u(t)$, $t \in [0, T]$, is said to be a mild solution of (3.2.3) if

$$\mathbb{P}\left(\int_0^T |u(\vartheta)|_H^2 \,\mathrm{d}\vartheta < +\infty \right) = 1, \quad \mathbb{P}\text{-a.s.}$$

and, for arbitrary $t \in [0, T]$, we have

$$u(t) = S(t)x + \int_0^t S(t-\vartheta) G(u(\vartheta)) \,\mathrm{d}\vartheta + W_S(t), \quad \mathbb{P}\text{-a.s.}.$$

We want to show that the laws $\mu_\varepsilon = \mathfrak{L}(u_\varepsilon)$ for $\varepsilon > 0$ of u_ε fulfill a large deviation principle with respect to a suitable functional I.

Dissipativity of the non-linear operator

We have that G is a dissipative functional, i.e. the following Hypotheses hold:

Hypothesis 3.4. *$G : \mathrm{D}(G) \to H$ is a functional such that $E \subset \mathrm{D}(G) \subset H$ and the part G_E of G in E is dissipative and uniformly continuous on bounded subsets of E.*

Let us recall some properties of dissipative mappings.

$G : E \to E$ is said to be dissipative if

$$\|x - y\|_E \le \|x - y - \alpha(G(x) - G(y))\|_E, \quad \forall x, y \in E, \forall \alpha > 0.$$

To obtain equivalent definitions of the previous one, we introduce the notion sub-differential of the norm in a Banach space E: a mapping (possibly multivalued) $\sigma : E \to E^*$ such that

$$\sigma(x) = \{y \in E^* : \langle x, y\rangle_{E,E^*} = \|x\|_E, \|y\|_{E^*} = 1\}.$$

The following equivalence holds: G is dissipative if and only if for any $x, y \in E$ there exists $z^* \in \sigma(x-y)$ such that

$$\langle G(x) - G(y), z^*\rangle_{E,E^*} \leq 0. \tag{3.2.4}$$

In a Hilbert space (3.2.4) becomes

$$\langle G(x) - G(y), x - y\rangle_H \leq 0, \quad \forall x, y \in \mathrm{D}(G).$$

Under Hypothesis 3.4, G is maximal dissipative i.e. the following result holds, see e.g. Da Prato & Zabczyk [26, Appendix D]:

Proposition 3.5. *If Hypothesis 3.4 holds, then for any $\alpha > 0$ and any $y \in E$ there exists a unique $x = J_\alpha(y)$ such that*

$$x - \alpha G(x) = y.$$

We define the Yosida approximations G_α, $\alpha > 0$, of G by setting

$$G_\alpha(x) = G(J_\alpha(x)) = \frac{1}{\alpha}(J_\alpha(x) - x), \quad x \in E, \tag{3.2.5}$$

where

$$J_\alpha(x) = (I - \alpha G)^{-1}(x), \quad x \in E. \tag{3.2.6}$$

The following proposition describes the main properties of J_α and G_α.

Proposition 3.6. *Let $G : E \to E$ be a continuous dissipative mapping, and let J_α and G_α be defined by* (3.2.6) *and* (3.2.5) *respectively. Then we have:*

I. *J_α is a contraction:*

$$\|J_\alpha(x) - J_\alpha(y)\|_E \leq \|x - y\|_E, \quad \forall x, y \in E,$$

and

$$\lim_{\alpha \to 0} J_\alpha(x) = x, \quad \forall x \in E,$$

II. *G_α is dissipative and Lipschitz continuous:*

$$\|G_\alpha(x) - G_\alpha(y)\|_E \leq \frac{2}{\alpha}\|x - y\|_E, \quad \forall x, y \in E$$

and

$$\|G_\alpha(x)\| \leq \|G(x)\|_E, \quad \forall x \in E \subset \mathrm{D}(G).$$

3.2.1 Solution of dissipative stochastic problem on E

Here we give the following well known result:

Theorem 3.7. *Assume Hypotheses 2.1, 2.4, 2.26, 2.27 and 3.4 hold; then the problem* (3.2.3) *has a unique solution in* $C([0,T];E)$.

Proof. Here we follow Da Prato & Zabczyk [26, Theorem 7.13]. First we set

$$v(t) = u(t) - z(t),$$

where $z(\cdot)$ is a fixed trajectory of $W_{S_E}(\cdot)$. Consider the following equation

$$v(t) = S_E(t)x + \int_0^t S_E(t-\vartheta)G(v(\vartheta)+z(\vartheta))\,\mathrm{d}\vartheta, \quad x \in E. \tag{3.2.7}$$

Let us introduce, for any $\alpha > 0$, the following approximating equation:

$$v_\alpha(t) = S_E(t)x + \int_0^t S_E(t-\vartheta)G_\alpha(v_\alpha(\vartheta)+z(\vartheta))\,\mathrm{d}\vartheta, \quad x \in E.$$

Since G_α is dissipative and Lipschitz continuous, the previous equation has a unique solution $v \in C([0,T];E)$ for all $z \in C([0,T];E)$. Moreover, it is easy to check that

$$\|v_\alpha(t)\|_E \le e^{\omega t}\|x\|_E + \int_0^t e^{\omega(t-\vartheta)}\|G(z(\vartheta))\|_E\,\mathrm{d}\vartheta \tag{3.2.8}$$

This shows that, z being fixed, the sequence $\{v_\alpha(\cdot)\}_{\alpha>0}$ is uniformly bounded.

To show the convergence of the approximating sequence, we set, for any $\alpha, \beta > 0$,

$$g_{\alpha,\beta} = v_\alpha - v_\beta, \quad u_\alpha = v_\alpha + z, \quad u_\beta = v_\beta + z.$$

Then $g_{\alpha,\beta}$ is the solution of the following problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} g_{\alpha,\beta}(t) = A_E g_{\alpha,\beta}(t) + G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t)), \\ g_{\alpha,\beta}(0) = 0 \end{cases}$$

Let $y^*_{\alpha,\beta,t} \in \sigma(g_{\alpha,\beta}(t))$, then we have

$$\begin{aligned}
\frac{\mathrm{d}^-\|g_{\alpha,\beta}(t)\|_E}{\mathrm{d}t} &\leq \omega\|g_{\alpha,\beta}(t)\|_E + \left\langle G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t)), y^*_{\alpha,\beta,t}\right\rangle_{E,E^*} \\
&\leq \omega\|g_{\alpha,\beta}(t)\|_E + \left\langle G(u_\alpha(t)) - G(u_\beta(t)), y^*_{\alpha,\beta,t}\right\rangle_{E,E^*} \\
&\quad + \left\langle G(J_\alpha(u_\alpha(t))) - G(u_\alpha(t)), y^*_{\alpha,\beta,t}\right\rangle_{E,E^*} \\
&\quad - \left\langle G(J_\beta(u_\beta(t))) - G(u_\beta(t)), y^*_{\alpha,\beta,t}\right\rangle_{E,E^*} \\
&\leq \omega\|g_{\alpha,\beta}(t)\|_E + \|G(J_\alpha(u_\alpha(t))) - G(u_\alpha(t))\|_E \\
&\quad + \|G(J_\beta(u_\beta(t))) - G(u_\beta(t))\|_E.
\end{aligned}$$

Now by (3.2.8) and recalling that G is bounded on bounded subsets of E, there exists $R > 0$ such that

$$\|u_\alpha(t)\|_E \leq R, \quad \|G(u_\alpha(t))\|_E \leq R, \quad \forall t \in [0,T], \forall \alpha \in (0,1].$$

Moreover

$$\|J_\alpha(u_\alpha(t)) - u_\alpha(t)\|_E \leq \alpha\|G(u_\alpha(t))\|_E \leq \alpha R,$$

and so

$$\begin{aligned}
\|G(J_\alpha(u_\alpha(t))) - G(u_\alpha(t))\|_E + \|G(J_\beta(u_\beta(t))) - G(u_\beta(t))\|_E \\
\leq K(\alpha R) + K(\beta R),
\end{aligned}$$

where $K(\cdot)$ is a modulus of continuity[†] of G restricted to $B(0,R)$. From the previous estimate we have

$$\|g_{\alpha,\beta}(t)\|_E \leq (K(\alpha R) + K(\beta R)) \int_0^t e^{ws}\, \mathrm{d}\vartheta, \tag{3.2.9}$$

that yields the convergence of the sequence $\{v_\alpha\}_{\alpha>0}$ in $C([0,T];E)$ to a function v. It is easily seen that v solves (3.2.7). The uniqueness follows taking the difference of two solutions, and using the Gronwall lemma. □

From (3.2.8) it follows that the bound in (3.2.9) also holds when z is not fixed but ranges into a bounded subset of $C([0,T];E)$. From this we have the following.

[†]We recall that any function $K : \mathbb{R}^+ \to \mathbb{R}^+$ such that

I. $\lim_{r\to 0} K(r) = 0$

II. $|G(x) - G(y)| \leq K(|x-y|), \quad \forall x, y \in D$

is called modulus of continuity of G restricted to D.

Corollary 3.8. *Suppose the hypotheses of the Theorem 3.7 hold, and suppose* $\|z(t)\|_E \leq M$, *for all* $t \in [0,T]$, *for a fixed* $M \in \mathbb{R}$. *Then* $\|v - v_\alpha\|_{C([0,T];E)}$ *does not depend on* z, *and we have*

$$\|v(t) - v_\alpha(t)\|_E \leq 2K(\alpha)\frac{1}{\omega}(e^{wT} - 1) = K'(\alpha).$$

with

$$K'(\alpha) \xrightarrow[\alpha \to 0]{} 0.$$

Large deviations for the semilinear dissipative equation

Let us consider the following functional $\Psi : C([0,T];E) \to C([0,T];E)$, defined as

$$\Psi(z) = u, \quad z \in C([0,T];E),$$

where u is the solution of the following control problem associated to (3.2.3):

$$u(t) = S(t)x + \int_0^t S(t-\vartheta)G(u(\vartheta))\,\mathrm{d}\vartheta + z(t), \quad t \in [0,T]. \tag{3.2.10}$$

We want to prove that Ψ is an homeomorphism from $C([0,T];E)$ into itself.

We give a preliminary result

Lemma 3.9. *Assume, besides the Hypotheses 2.1, 2.26 and 3.4, that* G *is Lipschitz. Then* Ψ *is continuous.*

Proof. Given two functions z, z_1 in $C([0,T];E)$, we set $u = \Psi(z)$ and $u_1 = \Psi(z_1)$. From (3.2.10) we have:

$$u(t) - u_1(t) = z(t) - z_1(t) + \int_0^t S_E(t-\vartheta)(G(u(\vartheta)) - G(u_1(\vartheta)))\,\mathrm{d}\vartheta$$

From Hypotheses 2.1 and 2.26, we have that $\|S_E(t)\| \leq M, \forall t \in [0,T]$; so denoting with L the Lipschitz constant of G, we find

$$\|u(t) - u_1(t)\|_E \leq \|z - z_1\|_{C([0,T];E)} + ML\int_0^t \|u(\vartheta) - u_1(\vartheta)\|_E\,\mathrm{d}\vartheta,$$

for all $t \in [0,T]$. Now the proof follows from the Gronwall Lemma. □

Theorem 3.10. *Assume Hypotheses 2.1, 2.26 and 3.4 hold. Then, Ψ is an homeomorphism from $C([0,T];E)$ into itself.*

Proof. From the proof of Theorem 3.7, we have that Ψ is a bijection from $C([0,T];E)$ into itself. We first prove the continuity of Ψ, then the continuity of its inverse.

As before, let $u = \Psi(z)$ and $u_1 = \Psi(z_1)$. Since we must prove continuity, we can suppose, without loss of generality, that

$$\max\{\|z - z_1\|_{C([0,T];E)},\ \|z\|_{C([0,T];E)},\ \|z_1\|_{C([0,T];E)}\} \leq C,$$

so the hypotheses of Corollary 3.8 are satisfied.

Denoting by u_α and $u_{\alpha,1}$ the solutions of the approximated problem, with G replaced by its Yosida approximation G_α, corresponding to z and z_1 respectively, we have

$$\|u - u_\alpha\|_{C([0,T];E)} \leq K'(\alpha), \qquad \|u_1 - u_{\alpha,1}\|_{C([0,T];E)} \leq K'(\alpha),$$

where $K'(\cdot)$ is the same as in the Corollary 3.8.

Since G_α is Lipschitz, from Lemma 3.9 the theorem follows.

We can now prove the inverse. Let $z = \Psi^{-1}(u)$ and $z_1 = \Psi^{-1}u_1$; again, without loss of generality, we can suppose that

$$\max\{\|u - u_1\|_{C([0,T];E)},\ \|u\|_{C([0,T];E)},\ \|u_1\|_{C([0,T];E)}\} \leq C.$$

So, from the continuity of u, u_1, we have that the values of $u_1(t)$ and $u(t)$ belong to a bounded subset D of E.

From (3.2.10) we have

$$\|z_1(t) - z(t)\|_E \leq \|u_1 - u\|_{C([0,T];E)} + M \int_0^t \|G(u_1(\vartheta)) - G(u(\vartheta))\|_E \, \mathrm{d}\vartheta.$$

Since the values of u, u_1 belong to D, we have that $\|G(u_1(\vartheta)) - G(u(\vartheta))\|_E$ is uniformly continuous on $[0,T]$ and the claim follows. So we have proved that Ψ is an homeomorphism. □

From Theorem 3.10, we can now prove a large deviation principle for the solutions of the semilinear stochastic equation (3.2.3).

We denote with $\nu_\varepsilon = \mathfrak{L}(u_\varepsilon(\cdot))$ on the space $C([0,T];E)$.

Theorem 3.11. *The family of measure* $\{\nu_\varepsilon\}_{\varepsilon>0}$ *fulfills the large deviation principle with respect to the following functional* J*:*

$$J(u) = \begin{cases} \frac{1}{2}\int_0^T |B^{-1}(u'(\vartheta) - A_E u(\vartheta) - G(u(\vartheta)))|_U^2 \, \mathrm{d}\vartheta, & u \in R, \\ +\infty & \textit{otherwise}, \end{cases}$$

where R *is the the subspace of* $C([0,T];E)$ *given by*

$$R = \Bigg\{ u \in C([0,T];E)) \Bigg| \exists z \in L^2(0,T;U) : \\ u(t) = S_E(t)x + \int_0^t S_E(t-\vartheta)G(u(\vartheta))\,\mathrm{d}\vartheta + \int_0^t S_E(t-\vartheta)Bz(\vartheta)\,\mathrm{d}\vartheta \Bigg\}.$$

Proof. From the very definition of Ψ it follows $\nu_\varepsilon = \Psi \circ \mu_\varepsilon$.

Since we have that $J(u) = I(\Psi^{-1}(u))$, from Theorem 2.28, recalling that Ψ is an homeomorphism, we can conclude that the large deviation property for ν_ε holds. □

3.2.2 Solution of dissipative stochastic problem on H

We are concerned with equation (3.2.3), on the space H under suitable assumptions.

First of all let us recall some definitions.

Definition 3.12. $G : \mathrm{D}(G) \subset E \to E^*$ is said to be hemicontinuous if and only if G is continuous on the straight lines i.e.

$$G((1-t)x + ty) \xrightarrow[t\to 0]{} G(x),$$

for all $x, y \in \mathrm{D}(G)$ such that $((1-t)x + ty) \in \mathrm{D}(G)$ for all $t \in [0,1]$

Definition 3.13. $G : \mathrm{D}(G) \subset E \to E^*$ is said to be coercive if and only if

$$\lim_{|x|_E \to \infty} \frac{\langle G(x), x\rangle_{E,E^*}}{|x|_E} = -\infty.$$

Here we give two hypotheses that guarantee that G is maximal monotone.

Hypothesis 3.14. $G : \mathrm{D}(G) = E \to H$ *is dissipative, hemicontinuous, coercive.*

Hypothesis 3.15. $G : \mathrm{D}(G) = E \to H$ *is dissipative, hemicontinuous and it satisfies*

$$\langle G(x), x \rangle_H \le 0, \quad x \in \mathrm{D}(G).$$

Theorem 3.16. *If Hypothesis 3.14 or 3.15 holds, then G is maximal dissipative in H*

For the proof see Minty [44], Browder [12].

We will also assume:

Hypothesis 3.17. *There exists* $0 < k < 1$ *and* $\psi : \mathbb{R} \to \mathbb{R}$*, bounded on bounded sets, such that*

$$|G(x)| \le k|Ax| + \psi(|x|), \quad x \in \mathrm{D}(A).$$

Let us first give the definition of generalized solution for (3.2.3): a predictable H-valued process $u_\varepsilon(t)$, $t \in [0,T]$, is said to be a generalized solution of (3.2.3) if and only if

$$\mathbb{P}\left(\int_0^T |u_\varepsilon(\vartheta)|_H^2 \, \mathrm{d}\vartheta < +\infty \right) = 1,$$

and for an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}} \in E$ such that $|x_n - x|_H \to 0$ the corresponding sequence of solutions u_n of (3.2.3) tends to u in $C([0,T];H)$, $\mathbb{P}$-a.s.

Define:

$$v_\varepsilon(t) = u_\varepsilon(t) - z_\varepsilon(t),$$

where $z_\varepsilon(\cdot)$ is the solution of the linear problem (2.1.1). Equation (3.2.3) can now be written as

$$v_\varepsilon(t) = \int_0^t S(t-\vartheta) G(v_\varepsilon(\vartheta) + z_\varepsilon(\vartheta)) \, \mathrm{d}\vartheta. \tag{3.2.11}$$

Let us prove the following

Theorem 3.18. *Let us assume that Hypotheses 2.1, 3.17, and 3.14 or 3.15 hold then* (3.2.3) *has a unique generalized solution.*

We are going to solve (3.2.11) pathwise, i.e. we claim that the following equation has a generalized solution for all $z \in C([0,T];H)$

$$u(t) = S(t)x + \int_0^t S(t-\vartheta) G(u(\vartheta)) \, \mathrm{d}\vartheta + z(t), \quad x \in H. \tag{3.2.12}$$

The deterministic semilinear equation

We are going to prove that (3.2.12) has a unique solution $u \in C([0,T];H)$ for all $z \in C([0,T];H)$.

Let us consider the following approximating equation

$$u_\alpha(t) = S(t)x + \int_0^t S(t-\vartheta)G_\alpha(u_\alpha(\vartheta))\,\mathrm{d}\vartheta + z(t). \tag{3.2.13}$$

Since G_α is differentiable and Lipschitz, (3.2.13) has strict a solution u_α, with $u_\alpha \in C([0,T];\mathrm{D}\,(A))$, see Da Prato and Zabczyk [26, Appendix A].

First of all we prove the following Lemma.

Lemma 3.19. *Let $\alpha > 0$, $z_1, z_2 \in C([0,T];H)$, and u_1, u_2 the corresponding solutions of* (3.2.13)*. We have*

$$|u_1(t) - u_2(t)| \le e^{(t-\vartheta)\omega}|u_1(\vartheta) - u_2(\vartheta)| + \omega^{-1}(e^{(t-\vartheta)\omega} - 1)\|z_1 - z_2\|_{C([0,T];H)}, \quad \forall 0 \le \vartheta \le t \le T. \tag{3.2.14}$$

In particular we have

$$\|u_1 - u_2\|_{C([0,T];H)} \le \omega^{-1}(e^{\omega T} - 1)\|z_1 - z_2\|_{C([0,T];H)}.$$

Proof. Since $G_\alpha = \frac{1}{\alpha}(J_\alpha - I)$, from (3.2.13) we have

$$\begin{aligned} u_1(t) - u_2(t) &= S(t-\vartheta)(u_1(\vartheta) - u_2(\vartheta)) \\ &\quad - \frac{1}{\alpha}\int_\vartheta^t S(t-\sigma)(u_1(\sigma) - u_2(\sigma))\,\mathrm{d}\sigma \\ &\quad + \frac{1}{\alpha}\int_\vartheta^t S(t-\sigma)(J_\alpha(u_1(\sigma)) - J_\alpha(u_2(\sigma)))\,\mathrm{d}\sigma \\ &\quad + z_1(t) - z_2(t). \end{aligned}$$

From the variation of constant formula, we have

$$\begin{aligned} u_1(t) - u_2(t) &= e^{-\frac{t-\vartheta}{\alpha}}S(t-\vartheta)(u_1(\vartheta) - u_2(\vartheta)) \\ &\quad + \frac{1}{\alpha}\int_\vartheta^t e^{-\frac{t-\sigma}{\alpha}}S(t-\sigma)(J_\alpha(u_1(\sigma)) - J_\alpha(u_2(\sigma)))\,\mathrm{d}\sigma \\ &\quad + \int_\vartheta^t e^{-\frac{t-\sigma}{\alpha}}S(t-\sigma)\big(\mathrm{d}z_1(\sigma) - \mathrm{d}z_2(\sigma) - A(z_1(\sigma) - z_2(\sigma))\,\mathrm{d}\sigma\big). \end{aligned}$$

So, since J_α is 1-Lipschitz

$$|u_1(t) - u_2(t)| \le e^{(t-\vartheta)(\omega-\alpha^{-1})}|(u_1(\vartheta) - u_2(\vartheta))| + \frac{1}{\alpha}\int_\vartheta^t e^{(t-\sigma)(\omega-\alpha^{-1})}|u_1(\sigma) - u_2(\sigma)|\,\mathrm{d}\sigma + \|z_1 - z_2\|_{C([0,T];H)}.$$

Let us denote $\varphi(t) = e^{-(t-\vartheta)(\omega-\alpha^{-1})}|u_1(t) - u_2(t)|$, then we have

$$\varphi(t) \le \varphi(\vartheta) + \frac{1}{\alpha}\int_\vartheta^t \varphi(\sigma)\,\mathrm{d}\sigma + e^{-(t-\vartheta)(\omega-\alpha^{-1})}\|z_1 - z_2\|_{C([0,T];H)}.$$

Thus, from Gronwall Lemma, we have

$$\varphi(t) \le e^{\frac{t-\vartheta}{\alpha}}\varphi(\vartheta) + \|z_1 - z_2\|_{C([0,T];H)}\int_\vartheta^t e^{-(\sigma-\vartheta)(\omega-\alpha^{-1})}\,\mathrm{d}\sigma.$$

that gives

$$|u_1(t)-u_2(t)| \le e^{(t-\vartheta)\omega}|u_1(\vartheta)-u_2(\vartheta)| + \|z_1-z_2\|_{C([0,T];H)}\int_\vartheta^t e^{(t-\sigma)(\omega-\alpha^{-1})}\,\mathrm{d}\sigma,$$

and we obtain (3.2.14). □

We can now prove the following

Theorem 3.20. *Let us assume that Hypotheses 2.1, 3.17 and 3.14 or 3.15 hold. Then the equation* (3.2.12) *has a unique generalized solution in the space* $C([0,T];H)$ *for all* $z \in C([0,T];H)$.

Proof. We shall prove the theorem in two steps. We first suppose that $x \in \mathrm{D}(A) \subset E$ and $z \in C^1([0,T];H)$, then we shall solve the general case.

Step I. Let G_α be the Yosida approximator of G, and let us consider the solution $u_\alpha \in C([0,T];\mathrm{D}(A))$ of (3.2.13).

We want to show that the solutions u_α tend to u in the space $C([0,T];H)$.

Let α, β be positive real numbers, let u_α and u_β be the solutions of (3.2.13) for α and β respectively. Then, we have:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u_\alpha(t) - u_\beta(t)) = A(u_\alpha(t) - u_\beta(t)) + (G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t))) \\ u_\alpha(0) - u_\beta(0) = 0 \end{cases}$$

Taking the scalar product with $u_\alpha(t) - u_\beta(t)$, we have:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_\alpha(t) - u_\beta(t)|^2 \leq \omega|u_\alpha(t) - u_\beta(t)|^2 + \\ + \langle G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\rangle .\end{aligned} \tag{3.2.15}$$

From the definition of $u_\alpha(t)$ and $u_\beta(t)$, recalling the properties of resolvent J_α, we have

$$\begin{aligned}u_\alpha(t) - u_\beta(t) &= (u_\alpha(t) - J_\alpha u_\alpha(t)) + (J_\alpha u_\alpha(t) - J_\beta u_\beta(t)) + (J_\beta u_\beta(t) - u_\beta(t)) \\ &= -\alpha G_\alpha(u_\alpha(t)) + (J_\alpha u_\alpha(t) - J_\beta u_\beta(t)) + \beta G_\beta(u_\beta(t)).\end{aligned}$$

From $G_\alpha(u_\alpha(t)) = G(J_\alpha(u_\alpha(t)))$ we have:

$$\begin{aligned}&\langle G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\rangle = \\ &\quad = -\langle G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t)), \alpha G_\alpha(u_\alpha(t)) - \beta G_\beta(u_\beta(t))\rangle + \\ &\quad\quad + \langle G(J_\alpha(u_\alpha(t))) - G(J_\beta(u_\beta(t))), J_\alpha u_\alpha(t) - J_\beta u_\beta(t)\rangle .\end{aligned}$$

Then, since G is dissipative,

$$\begin{aligned}&\langle G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\rangle \\ &\quad \leq -\alpha|G_\alpha(u_\alpha(t))|^2 - \beta|G_\beta(u_\beta(t))|^2 + \\ &\quad\quad + (\alpha+\beta)|G_\alpha(u_\alpha(t))||G_\beta(u_\beta(t))| \\ &\quad \leq \frac{\alpha}{4}|G_\alpha(u_\alpha(t))|^2 + \frac{\beta}{4}|G_\beta(u_\beta(t))|^2\end{aligned}$$

If we give a uniform bound C_1 to $|G_\alpha(u_\alpha(t))|$, as in Lemma 3.22, we have

$$\langle G_\alpha(u_\alpha(t)) - G_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\rangle \leq \frac{C_1}{4}(\alpha+\beta).$$

Then (3.2.15) becomes:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_\alpha(t) - u_\beta(t)|^2 \leq \omega|u_\alpha(t) - u_\beta(t)|^2 + C_1(\alpha+\beta). \tag{3.2.16}$$

Hence the sequence u_α converges uniformly to a certain u in $C([0,T];H)$. It follows that u is a solution to (3.2.12). The uniqueness of u is ensured by a standard dissipative argument.

Step II. In the general case with $x \in H$ and $z \in C([0,T];H)$, let us consider a sequence $x_n \in \mathrm{D}(A)$ and a sequence $z_n \in C^1([0,T];H)$. Denote with u_n and $u_{\alpha,n}$ respectively the corresponding solutions to equation (3.2.12) and to equation (3.2.13). Fixed $\varepsilon > 0$, from Lemma 3.19, we have that there exists $\overline{n} \in \mathbb{N}$ such that

$$\|u_{\alpha,n} - u_{\alpha,m}\|_{C([0,T];H)} \\ \leq e^{\omega T}|x_n - x_m| + \omega^{-1}(e^{\omega T} - 1)\|z_n - z_m\|_{C([0,T];H)}) \leq \frac{\varepsilon}{3},$$

for all $n, m > \overline{n}$. So, since the sequence $u_{\alpha,n}$ converges to u_n in $C([0,T];H)$, we have that there exists $\overline{\alpha} > 0$ such that

$$\|u_{\alpha,n} - u_n\|_{C([0,T];H)} \leq \frac{\varepsilon}{3}, \quad \|u_{\alpha,m} - u_m\|_{C([0,T];H)} \leq \frac{\varepsilon}{3},$$

for all $0 < \alpha \leq \overline{\alpha}$. Then the sequence of u_n is a Cauchy sequence that converges to a certain $u \in C([0,T];H)$, the generalized solution of (3.2.12). □

Corollary 3.21. *The functional* $\Gamma : C([0,T];H) \to C([0,T];H)$ *defined by* $\Gamma(z) = u$*, where* u *is the solution to* (3.2.12)*, is continuous.*

Proof. As in the Theorem 3.20, using the Lemma 3.19, we have for $z_1, z_2 \in C([0,T];H)$

$$\|\Gamma(z_1) - \Gamma(z_2)\|_{C([0,T];H)} = \|u_1 - u_2\|_{C([0,T];H)} \\ \leq \omega^{-1}(e^{\omega T} - 1)\|z_1 - z_2\|_{C([0,T];H)}$$

that ends the proof. □

Thus, it remains to prove the following Lemma.

Lemma 3.22. *In the settings of Theorem 3.20, in the particular case of* $x \in \mathrm{D}(A)$ *and* $z \in C^1([0,T];H)$*, there exists a constant* C_1 *such that*

$$|G_\alpha(u_\alpha(t))| \leq C_1 \qquad \forall \alpha > 0, \forall t \in [0,T]. \tag{3.2.17}$$

Proof. First of all we shall proof that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(t)\right| \leq C_2 \quad t \in [0,T], \forall \alpha > 0.$$

From the Lemma 3.19 we have

$$|u_\alpha(t+h) - u_\alpha(t)| \leq e^{t\omega}|u_\alpha(h) - u_\alpha(0)| \\ + \omega^{-1}(e^{t\omega} - 1)\|z(\cdot + h) - z(\cdot)\|_{C([0,T];H)}. \tag{3.2.18}$$

By dividing by h, and letting $h \to 0$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(t)\right| \le e^{t\omega}\left|\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(0)\right| + \omega^{-1}(e^{t\omega}-1)\|z'\|_{C([0,T];H)}.$$

Then, since

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(0) = Ax + G_\alpha(x) + z'(0),$$

we have, recalling that $x \in \mathrm{D}(A)$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(0)\right| \le |Ax| + |G(x)| + |z'(0)| = C_3.$$

So,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(t)\right| \le C_3 + \omega^{-1}(e^{\omega T}-1)\|z'\|_{C([0,T];H)} = C_2.$$

From this we have also $|u_\alpha(t)| \le C_2 T$; in-fact,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_\alpha(t)| \le \left|\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(t)\right| \le C_2,$$

so $u_\alpha(t)$ is uniformly bounded.

We can show now that $Au_\alpha(t)$ is bounded for all $t \in [0,T], \alpha > 0$. Since u_α is a strong solution of (3.2.13), we have $u_\alpha \in \mathrm{D}(A)$. From this and from Hypotheses 3.17 follows that

$$\begin{aligned}|Au_\alpha(t)| &\le \left|\frac{\mathrm{d}}{\mathrm{d}t}u_\alpha(t)\right| + |G_\alpha(u_\alpha(t))| + |z'| \\ &\le C_2 + \|z'\|_{C([0,T];H)} + |G(u_\alpha(t))| \\ &\le C_2 + \|z'\|_{C([0,T];H)} + k|Au_\alpha(t)| + \psi(|u_\alpha(t)|).\end{aligned}$$

So, recalling that $\psi(|u_\alpha(t)|)$ is bounded by a constant C_4 since $|u_\alpha| \le TC_2$, and that $\|z'\|_{C([0,T];H)}$ is finite, we have

$$|Au_\alpha(t)| \le \frac{C_2 + \|z'\|_{C([0,T];H)} + C_4}{1-k} = C_5.$$

From this we have

$$|G_\alpha(u_\alpha(t))| \le |G_\alpha(u_\alpha(t))| \le kC_5 + C_4 = C_1,$$

that ends the proof. □

Large deviations

Let us consider the following functional $\Psi : C([0,T];H) \to C([0,T];H)$, defined as

$$\Psi(z) = u, \quad z \in C([0,T];H),$$

where u is the solution of (3.2.12). From Theorem 3.20, we have that Ψ is a bijection from $C([0,T];H)$ into itself, and, as stated in the Corollary 3.21, is continuous.

We can now prove a large deviation principle for the solutions of the semilinear stochastic equation (3.2.3).

We denote with ν_ε the law of $u_\varepsilon(\cdot)$ on the space $C([0,T];H)$.

Theorem 3.23. *Let us assume that Hypotheses 2.1, 3.17 and 3.14 or 3.15 hold. Then, the family of measure $\{\nu_\varepsilon\}_{\varepsilon>0}$ fulfills the large deviation principle with respect to the following functional J:*

$$J(u) = \begin{cases} \frac{1}{2}\int_0^T |B^{-1}(u'(\vartheta) - Au(\vartheta) - G(u(\vartheta)))|_U^2 \, \mathrm{d}\vartheta, & u \in \widetilde{R}, \\ +\infty & \textit{otherwise}, \end{cases}$$

where $\widetilde{R}$ is the the subspace of $C([0,T];H)$ given by

$$\widetilde{R} = \Bigg\{ u \in C([0,T];H)) \Bigg| \exists z \in L^2(0,T;U) : \\ u(t) = S(t)x + \int_0^t S(t-\vartheta)G(u(\vartheta))\,\mathrm{d}\vartheta + \int_0^t S(t-\vartheta)Bz(\vartheta)\,\mathrm{d}\vartheta \Bigg\}.$$

Proof. From the definition of Ψ, we have that $\nu_\varepsilon = \Psi \circ \mu_\varepsilon$, the image measure of the measure μ_ε through Ψ, and we have that $J(u) = I(\Psi^{-1}(u))$.

From Theorem 2.10, since Ψ is continuous, the theorem follows. □

Chapter 4

Large deviation for the Volterra semilinear problem

In this chapter we shall concern about the generalization to the semilinear problems of the previous chapter, to Volterra equations.

Here and in the remaining of the section we denote with $S(\cdot)$ the resolvent associated to a, A, as in (2.5.11), instead of the semigroup generated by A.

We shall need two different assumptions on the kernel a. In the first case we assume the Hypothesis 2.22 to hold, in the second case we shall assume

Hypothesis 4.1. *The kernel* $a :]0,T] \to]0,+\infty[$ *is completely monotone,* $a \in L^1_{\text{loc}}(0,T)$*, and there exists a Bernstein function* $k(t) = k_0 + \int_0^t k_1(\vartheta)\,\mathrm{d}\vartheta$ *associated to* $a(t)$*, the relation between* $a(t)$ *and* $k(t)$ *being given by*

$$k_0 a(t) + \int_0^t k_1(t-\vartheta)a(\vartheta)\,\mathrm{d}\vartheta = 1, \qquad t \in [0,T].$$

Remark 4.2. We point out that the two hypotheses are overlapping only for constant kernels, that is non Volterra case, since for instance the first allows $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ with $\alpha \in [1,2)$, while the other imposes α to be in $(0,1]$.

4.1 The case with Lipschitz non-linearity

We consider here the Volterra equation:

$$u(t) = x + \int_0^t a(t-\vartheta)[Au(\vartheta) + G(u(\vartheta))]\,\mathrm{d}\vartheta + B\,W(t), \tag{4.1.1}$$

with $x \in H$, $t \in [0,T]$.

Definition 4.3. The mild solution to (4.1.1) in H is the solution to the stochastic integral equation

$$u(t) = S(t)x + a(0)(S * G(u))(t) + (S * a' * G(u))(t) + W_S(t). \qquad (4.1.2)$$

We can state now the following theorem of existence and uniqueness for the following problem (see the author & Bonaccorsi [8]).

Theorem 4.4. *Let us assume Hypotheses 2.1, 2.18, 2.22, 2.19 and 3.1. Then equation* (4.1.1) *has a unique mild solution in* $C_{\mathcal{F}}([0,T];H)$*.*

Proof. Define $v(t) = u(t) - z(t)$, $t \geq 0$, where $z(\cdot) = W_S(\cdot)$, and note that (4.1.2) can be written as

$$v(t) = S(t)x + a(0)(S * G(v+z))(t) + (S * a' * G(v+z))(t). \qquad (4.1.3)$$

We are going to solve the previous equation pathwise, since $W_S(\cdot)$ has H-continuous trajectories.

Set $Z_T = C([0,T];H)$ and moreover

$$\Gamma(v)(t) = S(t)x + a(0)(S * G(v+z))(t) + (S * a' * G(v+z))(t).$$

Since G is continuous on H, a' is continuous in time and S is strongly continuous, $\Gamma(\cdot)$ maps Z_T in Z_T.

Using a fixed point argument, (compare the proof of Theorem 7.10 in Da Prato & Zabczyk [26]) it is possible to show that, for T_0 small enough, there exists a unique mild solution on $[0,T_0]$. To obtain global existence, it is sufficient to prove that $|v(\cdot)|$ remains bounded.

Let us take the norm in (4.1.3):

$$|v(t)| \leq |S(t)x| + a(0)\left|(S * G(v+z))(t)\right| + \left|(S * a' * G(v+z))(t)\right|.$$

From the assumption on $S(\cdot)$ the following estimate holds:

$$|S(t)x| \leq M|x|, \qquad \forall t \in [0,T];$$

the convolution integral is bounded by:

$$
\begin{aligned}
|(S * a' * G(v+z))(t)| &\leq \int_0^t \left| S(t-r) \int_0^r a'(r-s) G(v(s)+z(s))\,\mathrm{d}s \right| \mathrm{d}r \\
&\leq M \int_0^t \int_s^t a'(r-s)\,\mathrm{d}r |G(v(s)+z(s))|\,\mathrm{d}s \\
&\leq M \int_0^t (a(t-s)-a(0)) |G(v(s)+z(s))|\,\mathrm{d}s \\
&\leq 2M\|a\|_\infty \int_0^t |G(v(s)+z(s))|\,\mathrm{d}s \\
&\leq 2M\|a\|_\infty \int_0^t C(|z(s)|)[1+|v(s)|]\,\mathrm{d}s \\
&\leq 2MC(\|z\|_{Z_T})\|a\|_\infty \int_0^t [1+|v(s)|]\,\mathrm{d}s;
\end{aligned}
$$

finally the middle term is bounded by

$$
\begin{aligned}
a(0)|(S * G(v+z))(t)| &\leq a(0) \int_0^t |S(t-r)||G(v(r)+z(r))|\,\mathrm{d}r \\
&\leq a(0)M \int_0^t |G(v(r)+z(r))|\,\mathrm{d}r \\
&\leq a(0)M \int_0^t C(|z(r)|)[1+|v(r)|]\,\mathrm{d}r \\
&\leq a(0)MC(\|z\|_{Z_T}) \int_0^t [1+|v(r)|]\,\mathrm{d}r.
\end{aligned}
$$

Summing up the above computations we get

$$|v(t)| \le M|x| + tMC(\|z\|_{Z_T})(2\|a\|_\infty + a(0)) + MC(\|z\|_{Z_T})(2\|a\|_\infty + a(0)) \int_0^t |v(s)| \,\mathrm{d}s.$$

Therefore, from Gronwall Lemma, $|v(t)|$ is bounded on $[0,T]$ by a constant that depends on x, T, M, a and that depends on z only through its norm in Z_T. □

4.1.1 Large deviation in the Lipschitz case

In order to examine the large deviation associated to the solution of the problem (4.1.1) we replace, here, the operator B with $\sqrt{\varepsilon}B$, leading to the equation:

$$u_\varepsilon(t) = x + \int_0^t a(t-\vartheta)[Au_\varepsilon(\vartheta) + G(u_\varepsilon(\vartheta))] \,\mathrm{d}\vartheta + \sqrt{\varepsilon}B\,W(t),$$

for $x \in H$, $t \in [0,T]$.

We define $\Psi : Z_T \to Z_T$ as the solution functional that maps each given trajectory $z(\cdot)$ of $W_S(\cdot)$ to the corresponding solution $u(\cdot)$, as follows

$$u(t) = S(t)x + a(0)(S * G(u(\cdot)))(t) + (S * a' * G(u(\cdot)))(t) + z(t). \tag{4.1.4}$$

We see that Ψ maps solutions of the linear problem (2.5.9) into solutions of the non-linear problem (4.1.1).

We can now state the large deviation property for the family of laws of $u_\varepsilon(\cdot)$.

Theorem 4.5. *Suppose that he assumptions of Theorem 4.4 hold. Then the family of laws of $\mu_\varepsilon = \mathfrak{L}(u_\varepsilon(\cdot))$ satisfies a large deviation principle with respect to the rate functional*

$$J(u) = \begin{cases} \frac{1}{2}\int_0^T \big|B^{-1}[u'(\vartheta) - a(0)(Au(\vartheta) + G(u(\vartheta))) - a' * (Au + G(u))(\vartheta)]\big|^2 \,\mathrm{d}\vartheta & \text{for } u \in R \\ +\infty & \text{otherwise,} \end{cases}$$

where R is the the subspace of $C([0,T];H)$ given by

$$R = \Big\{ u \in C([0,T];H)) \Big| \exists z \in L^2(0,T;U) : u(t) = S(t)x$$
$$+ a(0)(S * G(u))(t) + (S * a' * G(u))(t) + (S * Bz)(t) \Big\}.$$

As before we have
$$J(\cdot) = I(\Psi^{-1}(\cdot)).$$

Proof. We have that $\mu_\varepsilon = \Psi \circ \mathfrak{L}\left(\sqrt{\varepsilon} z(\cdot)\right)$, so if we show that the bijection Ψ is continuous, the result follows from Theorem 2.10 and Remark 1.5.

Consider $z_1(\cdot), z_2(\cdot)$ in Z_T and the corresponding solutions $u_1(\cdot), u_2(\cdot)$ of (4.1.4). Suppose $\|z_i\|_{Z_T} \le K_0$, then from Theorem 4.4 we have $\|u_i\|_{Z_T} \le K$, so without loss of generality we can consider G globally Lipschitz.

Let us define $v_i(\cdot) = u_i(\cdot) - z_i(\cdot)$. Then we have

$$|v_1(t) - v_2(t)| \le |a(0)[(S * G(v_1 + z_1))(t) - (S * G(v_2 + z_2))(t)]|$$
$$+ |(S * a' * G(v_1 + z_1))(t) - (S * a' * G(v_2 + z_2))(t)|.$$

As in Theorem 4.4, from Lipschitzianity of G, we have

$$\begin{aligned} |v_1(t) - v_2(t)| \le{}& |a(0)| M L \int_0^t \big(|v_1(s) - v_2(s)| + |z_1(s) - z_2(s)|\big)\,\mathrm{d}s \\ &+ 2M\|a\|_\infty L \int_0^t \big(|v_1(s) - v_2(s)| + |z_1(s) - z_2(s)|\big)\,\mathrm{d}s \\ \le{}& ML(|a(0)| + 2\|a\|_\infty) \int_0^t |v_1(s) - v_2(s)|\,\mathrm{d}s \\ &+ ML(|a(0)| + 2\|a\|_\infty)\|z_1 - z_2\|_{Z_T} \end{aligned}$$

So from Gronwall's lemma we have

$$\|v_1 - v_2\|_{Z_T} \le \tilde{K} \|z_1 - z_2\|_{Z_T};$$

since $u_i(\cdot) = v_i(\cdot) + z_i(\cdot)$, this ends the proof. □

Remark 4.6. As in the linear case the rate functional is related to the control system given by (4.1.4), so it is possible to give the following definition for the rate functional in terms of z:

$$J(u) = \inf \left\{ \frac{1}{2} \int_0^T |z(\vartheta)|^2 \, \mathrm{d}\vartheta \; : z \text{ such that (4.1.4) has solution } u \right\}$$

and this formulation brings us again to interpret $J(u)$ as the minimal "energy", given by the forcing term z, to allow the system to remain in u.

Opposite to the linear case considered in Remark 2.25, in the nonlinear case, even for a ordinary differential equation, the rate functional J defined above does not coincide with the Onsager-Machlup functional, compare for instance Bardina *et al.* [4].

4.2 The case with dissipative non-linearity

As before we are concerned with a Volterra generalization of previous problem as follows:

$$u(t) = x + \int_0^t a(t-\vartheta)[Au(\vartheta) + G(u(\vartheta))] \, \mathrm{d}\vartheta + B\, W(t). \tag{4.2.5}$$

Here we assume that Hypotheses 2.18, 2.19, 2.26, 2.27, 4.1 and 3.4 hold.

To solve the problem (4.2.5) as usual we define $v(t) = u(t) - W_S(t)$ and note that for the E-valued process $v(t)$, (4.2.5) may be written in the form

$$v(t) = x - \int_0^t a(t-\vartheta) Av(\vartheta) \, \mathrm{d}\vartheta + \int_0^t a(t-\vartheta) G(v(\vartheta) + z(\vartheta)) \, \mathrm{d}\vartheta, \tag{4.2.6}$$

where $z(t) = W_S(t) \in C([0,T]; E)$ is a trajectory of the stochastic convolution process.

Since $a(t)$ is a completely monotone kernel, if we introduce the linear Volterra operator

$$Lu(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left[k_0 u(t) + \int_0^t k_1(t-\vartheta) u(\vartheta) \, \mathrm{d}\vartheta \right], \qquad t \in [0,T],$$

it is possible to rewrite (4.2.6) as an equivalent integro-differential problem:

$$\begin{cases} L[v-x](t) + Av(t) = G(v(t)+z(t)), \\ k_0 v(0+) + (k_1 * v)(0+) = k_0 x. \end{cases} \tag{4.2.7}$$

We arrive, therefore, to a (family of) deterministic equation (depending on ω) in a real separable Banach space E, that we shall study via the techniques of integro-differential Volterra equations.

Theorem 4.7. *Assume that E is a real Banach space and let the coefficients in* (4.2.6) *satisfy Hypotheses 2.18, 2.19, 2.26, 2.27, 3.4 and 4.1. Then, for any $x \in \mathrm{D}(A)$, there exists a unique generalized solution $v(t)$ to the abstract non-linear Volterra equation* (4.2.7)*.*

Then, we shall say that $u(t) = v(t) + W_S(t)$ is a generalized mild solution to (4.2.5) *in $C([0,T];E)$.*

The complete proof of this theorem is beyond the scope of this thesis, so we briefly recall the ideas in the author & Bonaccorsi [7] that lead to the proof.

The problem under consideration is the following Volterra integral equation

$$L[v-x](t) + Av(t) = F(t, v(t)); \tag{4.2.8}$$

in our case $F(t, v(t)) = G(v(t) + z(t))$ The problem was introduced since the early 1970s in the case where $F(t,v) = f(t)$; this case, that we shall call the "inhomogeneous problem", is an important step also in our construction.

The next step in the literature is to consider functional perturbations of such problem, see for instance Crandall & Nohel [24] or Gripenberg [38]. In the author & Bonaccorsi [7], we consider perturbation operators acting on E, but we can allow such operators to be non-autonomous. The study of (4.2.8) with the operator $F(t,v)$ is based on the results for the inhomogeneous problem $F = f(t)$ and a fixed point argument; this should justify the appellative of "perturbation term" given to $F(t,v)$.

Volterra operators

We first discuss some properties of the linear Volterra operator

$$Lv(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[k_0 v(t) + \int_0^t k_1(t-\vartheta) v(\vartheta)\,\mathrm{d}\vartheta \right], \qquad t > 0, \tag{4.2.9}$$

with domain

$$\mathrm{D}(L) = \{f \in L^p((0,+\infty);E) \mid k_0 f + (k_1 * f) \in W^{1,p}((0,+\infty);E)\}$$

The operator L is m-accretive in $L^p((0,+\infty);E)$, for any $1 \le p < \infty$, and densely defined, see Clément [18], Proposition 3.2. There is a natural representation of its inverse operator L^{-1} in terms of the kernel $a(t)$.

$$L^{-1}v(t) = \int_0^t a(t-\vartheta)v(\vartheta)\,\mathrm{d}\vartheta. \tag{4.2.10}$$

Now, we focus, for a moment, on the explicit form of the Yosida approximation $L_\mu = L(I+\mu^{-1}L)^{-1}$. The following result is proved in the author & Bonaccorsi [7].

Lemma 4.8. *The operator $L_\mu = L(I+\mu^{-1}L)^{-1}$ is given by*

$$L_\mu v(t) = \mu \frac{\mathrm{d}}{\mathrm{d}t}(v * s_\mu)(t). \tag{4.2.11}$$

Inhomogeneous problem

We consider the inhomogeneous problem

$$L[v-x](t) + Av(t) = f(t). \tag{4.2.12}$$

In order to define a generalized solution to (4.2.12), we shall consider an approximate equation, where the operator L is replaced by its Yosida approximation L_μ, $\mu > 0$. Let v_μ be the solution of the following equation

$$L_\mu[v_\mu(\cdot) - x](t) + Av_\mu(t) = f(t). \tag{4.2.13}$$

In the next theorem, we establish the existence of a generalized solution of (4.2.12).

Theorem 4.9. *Assume that the coefficients in* (4.2.12) *satisfy, 2.26 and 4.1, and let $x \in \overline{\mathrm{D}(A)} = E$ and $f \in C([0,+\infty);E)$. Then, for every $\mu > 0$* (4.2.13) *has a unique solution $v_\mu(\cdot) \in C([0,+\infty);E)$.*

As $\mu \to \infty$, there exists a function $v = U(x,f)$ with $v \in L^1_{\mathrm{loc}}((0,+\infty);E)$ such that $v_\mu \to v$ in $L^1_{\mathrm{loc}}((0,+\infty);E)$.

If $x \in \mathrm{D}(A)$ then the convergence takes place also in $L^\infty_{\mathrm{loc}}((0,+\infty);E)$ and the limit function v belongs to $C([0,+\infty);E)$.

The function $v = U(x,f)$, that exists according to Theorem 4.9, is called the generalized solution for problem (4.2.12).

Non-autonomous perturbations

Now we return to (4.2.8). Before we discuss the case of dissipative non-linearities, which is the object of Theorem 4.7, we shall consider the case of a Lipschitz non-linearity. We shall say that $v(t)$ is a generalized solution of (4.2.8) if $v = U(x, F(\cdot, v))$.

Theorem 4.10. *Let the assumptions of Theorem 4.9 be fulfilled and assume that the nonlinear term $F : [0,T] \times E \to E$ is a continuous function, and that there exists a function $\eta(t) \in L^\infty_{\rm loc}(0,+\infty)$ such that, for any $t \in (0,+\infty)$,*

$$\|F(t, v_1) - F(t, v_2)\| \le \eta(t)\|v_1 - v_2\|. \qquad (4.2.14)$$

Then there exists a unique generalized solution to (4.2.8)

$$\begin{cases} L[v(\cdot) - x](t) + Av(t) = F(t, v(t)), \\ t \in (0,+\infty), \qquad v(0+) = x. \end{cases}$$

Let us give an idea of the proof of Theorem 4.7. Preliminarily, we notice that, in our setting, we consider as non-autonomous perturbation $F(t, v(t))$ the function $F(v(t) + z(t))$; next, we introduce, for any $\alpha > 0$, the approximating equation

$$L(v_\alpha - x)(t) + Av_\alpha(t) = F_\alpha(v_\alpha(t) + z(t)), \qquad (4.2.15)$$

where $F_\alpha(\cdot)$ are the Yosida approximations of $F(\cdot)$. Since F_α is Lipschitz continuous and bounded in norm by F, we obtain an a priori estimate for the approximating solution $v_\alpha(\cdot)$ as follows:

$$\|v_\alpha(t)\| \le s_{-\omega}(t)\,\|x\| - \frac{1}{\omega}\big(r_{-\omega} * \|F(z(\cdot))\|\big)(t).$$

This assures that the sequence $\{v_\alpha(\cdot)\}$ is bounded uniformly in α.

To show the convergence of the sequence, we set, for any $\alpha, \beta > 0$,

$$g^{\alpha,\beta}(t) = v_\alpha(t) - v_\beta(t),$$

and we estimate the relevant norm with techniques analogous to those used in the proof of 3.7, based on the a priori estimate of $\|v_\alpha(\cdot)\|$ and dissipativity of F. Thanks to a convolutional Gronwall lemma (see for instance the author & Bonaccorsi [7]), we obtain

$$\|g^{\alpha,\beta}(t)\| \le [\rho_F(\tfrac{2}{\alpha}R) + \rho_F(\tfrac{2}{\beta}R)]\big(a * s_{-\omega}\big)(t),$$

where ρ_F is the modulus of continuity of $F(\cdot)$ on the bounded set $B(0, 2R)$.

This yields the convergence of the sequence $v_\alpha(t)$ in $C([0,T];E)$ to a function v, which is easily seen to be the unique generalized solution for problem (4.2.7). □

We conclude this section with a last remark about equation (4.2.8).

Remark 4.11. Notice that we are concerned with a continuous and m-dissipative operator G; however, since this term is non-autonomous, we cannot consider the sum $(A - G)$ as a unique operator, even if we assume that $(-A + G)$ is m-dissipative.

Now we can focus on the transfer functional Ψ that associates the trajectories of the stochastic convolution process to a solution of the nonlinear problem (4.2.7).

We may be more precise on the regularity of Ψ;

Theorem 4.12. *Suppose that the assumptions of Theorem 4.7 hold. Then the functional $\Psi : C([0,T];E) \to C([0,T];E)$ that associates a trajectory of the stochastic convolution process to the solution of* (4.2.7) *is continuous.*

Proof. Our argument is divided into two steps. In the first step we suppose that the non linear term F is locally Lipschitz on E, while in the second we prove the theorem in the general case.

Step I. Let $z_1(t)$ be a continuous function on E; since we want to show that Ψ is continuous at the point z_1 of $C([0,T];E)$, we can restrict ourselves to a bounded neighborhood B around z_1. Then, since F is locally Lipschitz, we can suppose, without loss of generality, that F is totally Lipschitz on B, with Lipschitz constant equal to Λ.

Let z_2 belongs to B and denote by $v_1 = \Psi(z_1)$ and $v_2 = \Psi(z_2)$ respectively. From definition of generalized solution, we have that there exist two sequences $v_{1,\mu}, v_{2,\mu}$ such that

$$v_{i,\mu} \to v_i \in L^1_{loc}((0,+\infty);E) \cap C((0,+\infty);E), \qquad i \in \{1,2\},$$

and

$$L_\mu(v_{i,\mu} - x)(t) + Av_{i,\mu}(t) = F(v_i(t) + z_i(t)), \qquad i \in \{1,2\}.$$

Then subtracting term to term we have

$$L_\mu(v_{1,\mu} - v_{2,\mu})(t) + A(v_{1,\mu}(t) - v_{2,\mu}) = F(v_1(t) + z_1(t)) - F(v_2(t) + z_2(t)).$$

Choose an element y^* in the sub-differential $\partial\|v_{1,\mu}(t) - v_{2,\mu}(t)\|$; taking the scalar product of both members in previous equation with y^*, we have

$$\langle L_\mu(v_{1,\mu} - v_{2,\mu})(t), y^*\rangle + \langle A(v_{1,\mu}(t) - v_{2,\mu}(t)), y^*\rangle \\ = \langle F(v_1(t) + z_1(t)) - F(v_2(t) + z_2(t)), y^*\rangle\,.$$

Recalling the definition of L_μ, we get

$$\mu\Big(\|v_{1,\mu}(t) - v_{2,\mu}(t)\| - \big(\|v_{1,\mu}(\cdot) - v_{2,\mu}(t)\| * r_\mu\big)(t)\Big) \\ - \omega\|v_{1,\mu}(t) - v_{2,\mu}(t)\| \le \Lambda(\|v_1(t) - v_2(t)\| + \|z_1(t) - z_2(t)\|).$$

From this equation we obtain an estimate on the norm $\|v_{1,\mu}(t)-v_{2,\mu}(t)\|$ via a convolutional Gronwall lemma (see the author & Bonaccorsi[9, Lemma 4.1]):

$$\|v_{1,\mu}(t) - v_{2,\mu}(t)\| \le \Lambda\tfrac{\omega_\mu}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mu^{-1}\big[\|v_1 - v_2\| + \|z_1 - z_2\|\big] * s_{-\omega_\mu} \\ + a * \big[(\|v_1 - v_2\| + \|z_1 - z_2\|)\big] * s_{-\omega_\mu}\Big)(t).$$

where $\omega_\mu = \frac{\mu\omega}{\mu-\omega}$.

Since, passing to the limit $\mu \to \infty$, we have $((a + \mu^{-1}) * s_{-\omega_\mu})(t) \to (a * s_{-\omega})(t)$, we obtain:

$$\|v_1(t) - v_2(t)\| \le \Lambda\frac{\mathrm{d}}{\mathrm{d}t}\Big(a * \big[(\|v_1 - v_2\| + \|z_1 - z_2\|)\big] * s_{-\omega}\Big)(t),$$

that becomes

$$\begin{aligned}\|v_1(t) - v_2(t)\| &\le \Lambda\Big(a * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t) \\ &\quad - \Lambda\Big(a * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t) \\ &\quad + \Lambda\Big(-\tfrac{1}{\omega}r_{-\omega} * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t) \\ &= \Lambda\Big(-\tfrac{1}{\omega}r_{-\omega} * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t).\end{aligned}$$

Now since $-\frac{1}{\omega}r_{-\omega}(t)$ is a completely monotone kernel, we can apply again the convolutional Gronwall lemma and we have:

$$\|v_1(t) - v_2(t)\| \le \big(-\widetilde{r}_{-\Lambda} * \|z_1 - z_2\|\big)(t),$$

where $\widetilde{r}_{-\Lambda}(t)$ satisfies

$$\widetilde{r}_{-\Lambda}(t) + \tfrac{\Lambda}{\omega}\big(\widetilde{r}_{-\Lambda} * r_{-\omega}\big)(t) = \tfrac{\Lambda}{\omega}r_{-\omega}(t).$$

Since $-\widetilde{r}_{-\Lambda}(t) \le -r_{-(\omega+\Lambda)}(t)$ for all $t \ge 0$, we have

$$\|\Psi(z_1)(t) - \Psi(z_2)(t)\| \le \big(-r_{-(\omega+\Lambda)} * \|z_1 - z_2\|\big)(t). \tag{4.2.16}$$

Therefore Ψ is continuous in the Lipschitz case.

Step II. In the general case we can approximate F with its Yosida approximations F_α. So, denoting with Ψ_α the functional corresponding to Ψ in (4.2.7) with F_α in place of F, we have:

$$\|\Psi(z_1) - \Psi(z_2)\| \le \|\Psi(z_1) - \Psi_\alpha(z_1)\| + \|\Psi_\alpha(z_1) - \Psi_\alpha(z_2)\| + \|\Psi_\alpha(z_2) - \Psi(z_2)\|.$$

Remark 4.13. Since the a priori estimate of solution $v(t)$ to Theorem 4.7 depends on the function $z(t)$ only via its supreme norm, we have that also the coefficients involved in the proof of the theorem depends "only" on supreme norm of $z(t)$ (see author & Bonaccorsi[7, Theorem 4.1]).

Then, we can state that from Theorem 4.7 for all ε there exists a α small enough such that

$$\|\Psi(z_1) - \Psi_\alpha(z_1)\| \le \varepsilon,$$
$$\|\Psi_\alpha(z_2) - \Psi(z_2)\| \le \varepsilon,$$

for all $z_1(t), z_2(t)$ in the same bounded set of $C([0,T];E)$. Now continuity of Ψ follows from continuity of Ψ_α. □

4.2.1 Large deviations in the dissipative case

We shall consider now problem (4.2.5) with B replaced by $\sqrt{\varepsilon}B$, and the family of its solutions $u_\varepsilon(t)$. We denote by $\nu_\varepsilon = \mathfrak{L}(u_\varepsilon)$ the law of $u_\varepsilon(\cdot)$ on the space $C([0,T];E)$. From Theorem 4.12, via the contraction principle, we prove

Theorem 4.14. *Suppose that the assumptions of Theorem 4.7 hold and that E is densely and continuously embedded in H as Borel subspace. Then, the family of laws ν_ε satisfies the large deviation principle with respect to the following explicit functional $J : C([0,T];E) \to [0;+\infty]$*

$$J(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|B^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}[u(\vartheta)+(a*Au)(\vartheta) - (a*G(u))(\vartheta)]\right|^2 \mathrm{d}\vartheta & \textit{for } u \in \widetilde{R} \\ +\infty & \textit{otherwise,} \end{cases} \tag{4.2.17}$$

where $\widetilde{R}$ *is the subset of* $C([0,T];E)$ *defined as*

$$\widetilde{R} = \Big\{ u \in C([0,T];E) \;\Big|\; \exists g \in L^2(0,T;H) : u(t) = S(t)x \\ + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t S(t-\vartheta)\,(a * G(u))(\vartheta)\,\mathrm{d}\vartheta\right] + \int_0^t S(t-\vartheta) B g(\vartheta)\,\mathrm{d}\vartheta \Big\}. \quad (4.2.18)$$

Proof. We have that $\nu_\varepsilon = \Psi \circ \mu_\varepsilon$, where by Theorem 4.12 the functional Ψ is continuous. Thus, from a slight variation of Theorem 2.28, due to the Volterra case and the different hypotheses on a, the family of laws ν_ε has the large deviation property with respect to the functional $J = \Psi^{-1}(I)$. Eventually the result follows since the definition of Ψ implies that J has the explicit formulation (4.2.17). □

Chapter 5

The exit problem

In this chapter we consider the following semilinear problem, already considered in the Section 3.2:

$$\begin{cases} du_\varepsilon(t) = Au_\varepsilon(t)dt + G(u_\varepsilon(t))dt + \sqrt{\varepsilon}BdW(t), \quad \varepsilon > 0, \\ u_\varepsilon(0) = x, \end{cases}$$

and the related control system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} f^{x,z}(t) = Af^{x,z}(t) + G(f^{x,z}(t)) + Bz(t), \\ f^{x,z}(0) = x. \end{cases} \tag{5.0.1}$$

We assume in this chapter that $G(0) = 0$ so that in the case $z \equiv 0$, 0 is an equilibrium point for the control system. Moreover, we impose that there exists an open bounded neighborhood $D \subset E$ of 0 which is uniformly attracted to 0 by (5.0.1) in the case $z \equiv 0$:

Hypothesis 5.1.

$$\forall r > 0, \ \exists T > 0, \ \textit{such that} \ \|f^{x,0}(t)\| \leq r \qquad \forall t \geq T, \ x \in D. \tag{5.0.2}$$

Note that $u_0(\cdot) = f^{x,0}(\cdot)$. So the Assumption 5.1 means that

$$\lim_{t \to +\infty} u_0(t) = 0.$$

However, for $\varepsilon > 0$, the behavior of $u_\varepsilon(t)$ will be totally different due to the influence of the additive noise: the solution $u_\varepsilon(t)$, starting from inside D will eventually reach the boundary ∂D.

To see this denote by $\tau^{x,\varepsilon}$ the exit time from D of the process $u_\varepsilon(t)$ originating in x:

$$\tau^{x,\varepsilon} = \inf\{t \geq 0 \mid u_\varepsilon(t) \in D^c, u_\varepsilon(0) = x\}. \tag{5.0.3}$$

Now we can prove the following result.

Proposition 5.2. *If the process $BW(\cdot)$ is not identically* 0*, then for arbitrary $x \in D$ and $\varepsilon > 0$,* $\mathbb{E}\left[\tau^{x,\varepsilon}\right] < +\infty$.

Proof. We can assume that $D = \{x \in E \mid \|x\| \leq R\}$ for some $R > 0$ and that G is a bounded transformation. Since the process $BW(\cdot)$ is non degenerate, there exists $\varphi \in E^*$, $\|\varphi\|_{E^*} = 1$, such that the one-dimensional Gaussian variable $\varphi(W_S(1))$ is non degenerate. Define

$$q_\varepsilon(x) = \mathbb{P}(\tau^{x,\varepsilon} > 1),$$

and note that

$$q_\varepsilon(x) \leq \mathbb{P}(\|u_\varepsilon(1)\| \leq R) \leq \mathbb{P}(|\varphi(u_\varepsilon(1))| \leq R).$$

Since G is bounded, there exists $R_1 > 0$ such that

$$|\varphi(u_\varepsilon(1))| \geq \sqrt{\varepsilon}|\varphi(W_S(1))| - R_1, \quad \mathbb{P}\text{-a.e.}.$$

It follows that

$$q_\varepsilon(x) \leq \mathbb{P}\left(|\varphi(W_S(1))| \leq \tfrac{R+R_1}{\sqrt{\varepsilon}}\right) = p_\varepsilon < 1, \qquad \forall x \in D.$$

Moreover, for arbitrary $k = 0, 1, 2, \ldots$

$$\mathbb{P}(\tau^{x,\varepsilon} > k+1) = \mathbb{P}(A_k \cap B_k),$$

where

$$A_k = \{\|u_\varepsilon(t)\| < R, \forall t \in [0,k]\} \in \mathcal{F}_k,$$

$$B_k = \{\|u_\varepsilon(k+s)\| < R, \forall s \in [0,1]\}.$$

It follows that

$$\mathbb{P}(\tau^{x,\varepsilon} > k+1) = \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}_{A_k} \cap \mathbb{I}_{B_k} \middle| \mathcal{F}_k\right]\right] = \mathbb{E}\left[1_{A_k}\mathbb{E}\left[\mathbb{I}_{B_k} \middle| \mathcal{F}_k\right]\right].$$

On the other hand

$$\mathbb{E}\left[\mathbb{I}_{B_k} \middle| \mathcal{F}_k\right] = \mathbb{E}\left[\mathbb{I}_{u_\varepsilon(k+\cdot)\in\Gamma} \middle| \mathcal{F}_k\right],$$

where

$$\Gamma = \{f \in C([0,+\infty[;E) \mid \|f(t)\| < R \quad \forall t \in [0,1]\}.$$

By the Markov property of solution (see Da Prato & Zabczyk [26, Corollary 9.13])

$$\mathbb{E}\left[\mathbb{I}_{u_\varepsilon(k+\cdot)\in\Gamma} \middle| \mathcal{F}_k\right] = q_\varepsilon(u_\varepsilon(k)) \leq p_\varepsilon.$$

Consequently, by induction, we have

$$\mathbb{P}(\tau^{x,\varepsilon} > k) \leq p_\varepsilon^k, \quad k = 0, 1, 2, \ldots.$$

Therefore it easy to show that

$$\mathbb{E}\left[\tau^{x,\varepsilon}\right] \leq 1/(1 - p_\varepsilon).$$

□

5.1 Exit rate estimates

It is intuitively clear that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\tau^{x,\varepsilon}\right]=+\infty.$$

In this section we will calculate the rate of divergence in the previous equation. To achieve this we will use the following exponential estimates obtained applying Proposition 1.6 and Proposition 1.7 to Theorem 3.11. Let us define

$$K_T^x(r)=\{f\in C([0,T];E)\mid f=f^{x,z};\frac{1}{2}\|z\|_{L^2(0,T;H)}\leq r^2\}.$$

Let $R_0>0, r_0>0$ and $T>0$ be numbers such that all sets $K_T^x(r_0)$, and $\{\|x\|\leq R_0\}$ are contained in a bounded subset of $C([0,T];E)$. Then

I. $\forall\delta>0$, $\forall\gamma>0$, $\exists\varepsilon_0>0$ such that $\forall\varepsilon\in]0,\varepsilon_0[$, $\forall x\in E$ with $\|x\|\leq R_0$:

$$\mathbb{P}(\mathrm{d}_{C([0,T];E)}\left(u_\varepsilon,K_T^x(r)\right)<\delta)\geq 1-e^{-\frac{1}{\varepsilon}(r^2-\gamma)};\tag{5.1.4}$$

II. $\forall\delta>0$, $\forall\gamma>0$, $\exists\varepsilon_0>0$ such that $\forall\varepsilon\in]0,\varepsilon_0[$, $\forall z\in L^2(0,T;U)$ satisfying $\frac{1}{2}\|z\|_{L^2(0,T;H)}\leq r_0^2$, $\forall x\in E$ with $\|x\|\leq R_0$

$$\mathbb{P}\big[\sup_{t\in[0,T]}\|u_e p(t)-f^{x,z}(t)\|_E<\delta\big]\geq e^{-\frac{1}{\varepsilon}\left(\frac{1}{2}\int_0^T|z(\vartheta)|^2\,\mathrm{d}\vartheta+\gamma\right)}.\tag{5.1.5}$$

We define

$$\overline{e}=\inf\left\{\frac{1}{2}\int_0^T|z(\vartheta)|^2\,\mathrm{d}\vartheta\,\Big|\,f^{0,z}(T)\in\left(\overline{D}\right)^c,T>0\right\}.$$

We will call $\overline{e}$ the *upper exit rate.*

For any $r>0$ let

$$e_r=\inf\left\{\frac{1}{2}\int_0^T|z(\vartheta)|^2\,\mathrm{d}\vartheta\,\Big|\,f^{x,z}(T)\in\left(\overline{D}\right)^c,T>0,\|x\|\leq r\right\}.$$

We call the number

$$\underline{e}=\lim_{r\downarrow 0}e_r$$

the *lower exit rate.* Note that we have always $\underline{e}\leq\overline{e}$.

The main result of this section is the following theorem. In its formulation we set

$$D^0=\{x\in D\mid f^{x,0}(t)\in D,\forall t\geq 0\}.$$

Theorem 5.3. *We assume that the Hypotheses of Theorem 3.11 and 5.1 hold. Then we have*

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{E}\left[\tau^{x,\varepsilon}\right] \leq \overline{e}, \qquad x \in D, \tag{5.1.6}$$

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{E}\left[\tau^{x,\varepsilon}\right] \geq \underline{e}, \qquad x \in D^0. \tag{5.1.7}$$

Theorem 5.3 is a generalization of finite dimensional results due to Freĭdlin & Wentzell [35]. The presentation follows Zabczyk. [61, 62].

Proof. Without loss of generality, we can assume that G is globally Lipschitz, with constant L and that $\|S(t)\| \leq M$, $\forall t > 0$.

Step I. Proof of the estimate (5.1.6). Let us fix a control $\hat{z}$ such that $f^{0,\hat{z}} \in (D)^c$. It is enough to show that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log \mathbb{E}\left[\tau^{x,\varepsilon}\right] \leq \frac{1}{2} \int_0^T |\hat{z}(\vartheta)|^2 \,\mathrm{d}\vartheta = \hat{r}.$$

It follows, from the continuous dependence on initial data of the solution of (3.1.2), that there exists two positive numbers δ_1, δ_2, such that

$$\|x\| \leq \delta_1 \implies \mathrm{d}_E\left(f^{x,\hat{z}}(T), \partial D\right) \geq \delta_2.$$

Since

$$\mathbb{P}(\tau^{x,\varepsilon} < T) \geq \mathbb{P}\left(\sup_{t\in[0,T]} \|u_e p(t) - f^{x,\hat{z}}(T)\| < \delta_2\right),$$

from (5.1.4) one obtains that, for any $\gamma > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, and $x \in E$ with $\|x\| \leq \delta_1$

$$q_\varepsilon(x) = \mathbb{P}(\tau^{x,\varepsilon} < T) \geq e^{-\frac{1}{\varepsilon}(\hat{r}+\gamma)}.$$

Taking into account Hypothesis 5.1 and the continuous dependence on initial data of solutions of (5.0.1), one can find positive numbers T_1 and p_1 such that

$$\mathbb{P}(\|u_\varepsilon(T_1)\| \leq \delta_1) \geq p_1,$$

for all $x \in D$ and all sufficiently small $\varepsilon > 0$. Consequently, by the Markov property, we obtain that

$$\begin{aligned}
\mathbb{P}(\tau^{x,\varepsilon} < T + T_1) &\geq \mathbb{P}\big[(\|u_\varepsilon(T_1)\| \leq \delta_1) \cap (\exists s \in [0,T] \mid u_\varepsilon(T_1 + s) \in \partial D)\big] \\
&\geq \mathbb{E}\left[q_\varepsilon(u_\varepsilon(T_1))\right] 1_{\{\|u_\varepsilon(T_1)\| \leq \delta_1\}} \\
&\geq e^{-\frac{1}{\varepsilon}(\hat{r}+\gamma)} \mathbb{P}(\|u_\varepsilon(T_1)\| \leq \delta_1) \geq p_1 e^{-\frac{1}{\varepsilon}(\hat{r}+\gamma)},
\end{aligned}$$

for $x \in D$, or, equivalently,

$$\mathbb{P}(\tau^{x,\varepsilon} \geq T + T_1) \leq p = 1 - p_1 e^{-\frac{1}{\varepsilon}(\hat{r}+\gamma)}, \quad x \in D.$$

By successive applications of the Markov property, we have

$$\mathbb{P}(\tau^{x,\varepsilon} \geq k(T + T_1)) \leq p^k, \quad k = 0, 1, 2, \dots .$$

Then, we have,

$$\mathbb{E}\left[\frac{\tau^{x,\varepsilon}}{T + T_1}\right] \leq \frac{1}{1-p} = {p_1}^{-1} e^{\frac{1}{\varepsilon}(\hat{r}+\gamma)}.$$

Thus,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \mathbb{E}\left[\tau^{x,\varepsilon}\right] \leq \hat{r} + \gamma;$$

since γ is arbitrary, we have (5.1.6).

Step II. Proof of the estimate (5.1.7). Fix $\gamma > 0$, choose $r > 0$ such that $e_r \geq \underline{e} - \gamma$ and fix $0 < r_0 < r$. We shall show that estimate (5.1.7) holds in the case $\|x\| = r$. Define

$$\sigma_1^{x,\varepsilon} = \inf\{t \geq 0 \mid \|u_\varepsilon(t)\| = r, \|u_\varepsilon(t_1)\| = r_0 \text{ for some } t_1 \in [0, t]\}.$$

We shall show first that there exists $\varepsilon_1 > 0$ such that, for all $\varepsilon \in]0, \varepsilon_1[$,

$$p_1^\varepsilon(x) = \mathbb{P}(\sigma_1^{x,\varepsilon} < \tau^{x,\varepsilon}) \geq 1 - e^{-\frac{1}{\varepsilon}(e_r - \gamma)}, \quad \text{for } |x| = r. \tag{5.1.8}$$

Note that

$$p_1^\varepsilon(x) = \mathbb{P}(\|u_\varepsilon(t)\| = r_0, \text{ for some } t < \tau^{x,\varepsilon}).$$

Then,

$$q_1^\varepsilon(x) = 1 - p_1^\varepsilon(x) = \mathbb{P}(\|u_\varepsilon(t)\| > r_0, \forall t < \tau^{x,\varepsilon}).$$

For arbitrary $T > 0$, we have therefore

$$q_1^\varepsilon(x) \leq \mathbb{P}(\tau^{x,\varepsilon} \leq T) + \mathbb{P}(u_\varepsilon(t) \in K, \forall t \in [0, T]) = P_1 + P_2,$$

where $K = \overline{D} \cap \{x \in E \mid \|x\| \geq r_0\}$. From Lemma 5.4 below, there exists $\varepsilon_0 > 0$, such that

$$P_2 < e^{-\frac{1}{\varepsilon}(e_r - \gamma)}, \quad \forall \varepsilon \in]0, \varepsilon_0[.$$

It remains to estimate P_1. Let us remark that, again by (1.1.7) applied to our case, and by the definition of e_r,

$$\mathbb{P}(\tau^{x,\varepsilon} \leq T) \leq \mathbb{P}(\mathrm{d}_{C([0,T];E)}(u_\varepsilon, K_T^x(e_r)) \geq \tfrac{r}{2}) \leq e^{-\frac{1}{\varepsilon}(e_r - \gamma)},$$

for all sufficiently small ε and $x \in D$ such that $\|x\| = r$. In this way the proof of (5.1.8) is complete.

We will generalize the estimate (5.1.8) and show

$$p_k^\varepsilon(x) = \mathbb{P}(\sigma_k^{x,\varepsilon} < \tau^{x,\varepsilon}) \geq (1 - e^{-\frac{1}{\varepsilon}(e_r - \gamma)})^k, \tag{5.1.9}$$

for $|x| = r$, $k = 0, 1, 2, \dots$, where $\sigma_0^{x,\varepsilon} = 0$, and

$$\sigma_{k+1}^{x,\varepsilon} = \inf \{t > \sigma_k^{x,\varepsilon} \mid \|u_\varepsilon(t)\| = r, \exists t_1 \in [\sigma_k^{x,\varepsilon}, t] : \|u_\varepsilon(t_1)\| = r_0\}.$$

However, by the strong Markov property,

$$\begin{aligned} p_{k+1}^\varepsilon(x) =& \mathbb{E}\left[p_1(u_\varepsilon(\sigma_k^{x,\varepsilon}))1_{\{\sigma_k^{x,\varepsilon} < \tau^{x,\varepsilon}\}}\right] \\ \geq& (1 - e^{-\frac{1}{\varepsilon}(e_r - \gamma)})\mathbb{P}(\sigma_k^{x,\varepsilon} < \tau^{x,\varepsilon}), \quad \|x\| = r, \end{aligned}$$

so the generalization (5.1.9) follows by induction. Let $C_k = \{\sigma_k^{x,\varepsilon} < \tau^{x,\varepsilon}\}$, then by (5.1.9) we have

$$\mathbb{P}(C_k) \geq (1 - e^{-\frac{1}{\varepsilon}(e_r - \gamma)})^k.$$

Let moreover

$$s_k = \begin{cases} \inf \{s \geq 0 \mid \|u_\varepsilon(\sigma_k^{x,\varepsilon} + s) - u_\varepsilon(\sigma_k^{x,\varepsilon})\| \geq \frac{r - r_0}{2}\} & \text{if } \sigma_k^{x,\varepsilon} < +\infty, \\ +\infty & \text{otherwise} \end{cases}$$

It is clear that on $C_k \setminus C_{k+1}$

$$\tau^{x,\varepsilon} \geq s_0 + \cdots + s_k$$

and, therefore,

$$\mathbb{E}\,[\tau^{x,\varepsilon}] \geq \sum_{k=0}^{\infty} \mathbb{E}\left[(s_0 + \cdots + s_k)1_{C_k \setminus C_{k+1}}\right] = \sum_{k=0}^{\infty} \mathbb{E}\,[s_k 1_{C_k}].$$

This finishes the proof of (5.1.7) for any $x \in D$ such that $\|x\| = r$.

If now $x \in D^0$ then there exists $p \in]0, 1[$ and $T_2 > 0$ such that

$$\mathbb{P}(u_\varepsilon(t) \in S \text{ for some } t \in [0, T_2]) \geq p,$$

where $S = \{x \in D \mid \|x\| = r\}$. Using the strong Markov property one obtains immediately

$$p_k^\varepsilon(x) \geq p(1 - e^{-\frac{1}{\varepsilon}(e_r - \gamma)})^k, \quad k = 0, 1, 2, \dots .$$

So by the above argument (5.1.7) follows in the general case. □

Lemma 5.4. *Assume that Hypothesis 5.1 holds. Then* $\forall r_0 > 0$, $\forall L > 0$, $\exists \varepsilon_0 > 0$ *such that* $\forall x \in K = \overline{D} \cap \{x \in E \mid \|x\| \geq r_0\}$, $\forall \varepsilon \in]0, \varepsilon_0[$*:*

$$\mathbb{P}(u_\varepsilon(s) \in K, \ \textit{for } s \in [0,T]) \leq e^{-\frac{L}{\varepsilon}}. \tag{5.1.10}$$

Proof. Let K_δ be a δ neighborhood of K, $\delta \leq \frac{r_0}{3}$. There exists $T_1 > 0$, $\delta > 0$ such that, if $t \geq T_1$ and $x \in K_\delta$, then $\|w^x(t)\| \leq \frac{r_0}{3}$.

Let $x \in K$, $z \in L^2(0,T;U)$ be a control such that $f^{x,z}(s) \in K_\delta$, $\forall s \in [0, T_1]$. If $M_1 = \|B\|$, then

$$\|w^x(t) - f^{x,z}(t)\| \leq ML \int_0^t \|w^x(\vartheta) - f^{x,z}(\vartheta)\| \,\mathrm{d}\vartheta + MM_1 \int_0^t |z(\vartheta)| \,\mathrm{d}\vartheta,$$

consequently

$$\frac{r_0}{3} \leq \|w^x(T_1) - f^{x,z}(T_1)\| \leq e^{MLT_1} MM_1 T_1^{\frac{1}{2}} \left(\int_0^{T_1} |z(\vartheta)|^2 \,\mathrm{d}\vartheta \right)^{\frac{1}{2}},$$

and

$$\frac{1}{2} \int_0^{T_1} |z(\vartheta)|^2 \,\mathrm{d}\vartheta \geq \left(\frac{r_0}{3MM_1} T_1^{-\frac{1}{2}} e^{MLT_1} \right)^2 = M_2.$$

By a simple induction argument, we have that, if $f^{x,z}(t) \in K_\delta$ for all $t \in [0, jT_1]$, for some $j = 1, 2, \ldots$, then

$$\frac{1}{2} \int_0^{jT_1} |z(\vartheta)|^2 \,\mathrm{d}\vartheta \geq jM_2.$$

Let us remark that, if $T = jT_1 > 2L$, then

$$\mathbb{P}(u_\varepsilon(s) \in K, \ \text{for } s \in [0,T]) \leq \mathbb{P}(\mathrm{d}_{C([0,T];E)}\,(u_\varepsilon, K_T^x(2L)) \geq \tfrac{r_0}{3}).$$

Taking into account again (1.1.7), we can find $\varepsilon_0 > 0$ such that (5.1.10) holds for $\varepsilon \in]0, \varepsilon_0[$ and all $x \in K$. This finishes the proof of this lemma. □

5.2 Exit place determination

A closed set $C \subset \partial D$ is called an *exit* set for the problem (3.2.3) and set D if for arbitrary $\delta > 0$ and all x sufficiently close to 0:

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\mathrm{d}_E\,(u_\varepsilon(\tau^{x,\varepsilon}), C) > \delta) = 0.$$

It turns out that for a large class of problems and domains D one can often find an exit set occupying only a small portion of the boundary ∂D. We will introduce a family of exit sets which have a useful control theoretic interpretation and are sufficiently small.

Define for $x \in E$, $T > 0$ and $r > 0$[†]

$$\gamma_T^x(r) = \operatorname{cl}\left\{ y \in E : y = f^{x,z}(t),\, t \in [0,T],\, \frac{1}{2}\int_0^T |z(\vartheta)|^2 \,\mathrm{d}\vartheta \le r \right\},$$

$$\gamma^x(r) = \operatorname{cl}\left\{ \bigcup_{T \ge 0} \gamma_T^x(r) \right\}, \qquad r \ge 0,\, T \ge 0,\, x \in E.$$

Set

$$C_r = \operatorname{cl}\left\{ y \in \partial D : \mathrm{d}_E\left(y, \bigcup_{\|x\| \le r} \gamma^x(\bar{e} + r) \right) \le r \right\}.$$

Note that C_0 is exactly the closure of the set of all elements of ∂D which can be reached from 0, by the system (5.0.1) in the case $z \equiv 0$, with the minimal possible energy $\bar{e}$. In several cases it is possible to show that $\bigcap_{r>0} C_r = C_0$.

We shall state, without proof, that for $r > 0$ the sets C_r are exit sets. This is an extension of a finite dimensional result due to Freĭdlin & Wentzell [35].

Theorem 5.5. *Under Hypothesis 5.1, for all $r > 0$ and $x \in D^0$,*

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(u_\varepsilon(\tau^{x,\varepsilon}) \in C_r) = 1.$$

[†] With $\operatorname{cl}\{\Gamma\}$ we denote the closure of Γ

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