Scuola Normale Superiore di Pisa &
Université d'Évry Val d'Essonne
Ecole doctorale: Sciences et Ingénierie

Ph.D. Thesis
in Mathematics Applied to Finance
(Perfezionamento)

Presented the 20$^{\text{th}}$ October by
Giorgia CALLEGARO

Credit risk models under partial information

Advisors:

Prof. Monique JEANBLANC
Université d'Évry Val d'Essonne

Prof. Wolfgang J. RUNGGALDIER
Università degli Studi di Padova

Academic Year 2010 - 2011

A mia mamma e a mio zio Franco

"Considerate la vostra semenza:
fatti non foste a viver come bruti,
ma per seguir virtute e canoscenza"
Dante Alighieri, Divina Commedia
Inf., XXVI, 118-120, 1307-1321.

"Dubium sapientiae initium", René Descartes
Meditationes De Prima Philosophia, 1641.

Acknowledgements

First and foremost, I'd like to thank my thesis advisors, Monique Jeanblanc and Wolfgang J. Runggaldier, for their patient and enthusiastic guidance during the last three years. The confidence they have in me gave me more and more strength day by day.
I am grateful to Ralf Korn and Marie-Claire Quenez for having refereed my thesis and to Stéphane Crépey, Fausto Gozzi, Stefano Marmi and Maurizio Pratelli for having accepted of being part of the defense committee.

I really have to thank all the members of the Mathematics Department in Evry, of the CREST and of the ENSIIE, who could stand me when things (and not only math computations!) did not go as I expected.
I cannot forget all the people I met at Scuola Normale in Pisa during my first year of Ph.D., especially Giuseppe Da Prato, who convinced me that infinite-dimensional stochastic calculus is an understandable subject. I am also grateful to Gianni Di Masi, who really trusts in me since 2004.

I also have to thank Valérie, Elisabetta, Raffaella and Ambra, who helped me solving all the administrative and bureaucratic problems I encountered.

I would also like to express gratitude to my co-authors Behnaz Zargari and Abass Sagna, who spent a lot of time working with me and answering to my questions.
I will never forget all the people I met during these last three years, coming from all over the world, who broadened my mind with their life experiences and who are now very good friends of mine. For this, I have to thank once more Monique Jeanblanc, who gave me the opportunity to travel all over the world with her: I will always remember the fantastic days spent this year in Tunisia, teaching Mathematics and visiting archaeological sites and the wonderful Tunis.

A special "Grazie!" goes to my father Paolo, who is my greatest reference point and who taught me to never give up and to trust in me. I will also always be grateful to my sister Camilla, who, despite the physical distance, was always with me, especially during bad periods. I would like also to thank Francesco, who waited for me four long years and who was able to make me smile every day. A final special thank goes to Giovanna, "the italian pianist", who enriched with her energy my parisian life far from Math.

This work was supported by Fondazione Aldo Gini, by the AMaMeF program of the European Science Foundation, by the CREST and by the "Chaire risque de crédit" of the French Banking Federation.

Résumé

Cette thèse se compose de cinq parties (Introduction comprise) indépendantes dédiées à la modélisation et à l'étude des problèmes liés au risque du défaut, en information partielle.

La première partie constitue l'Introduction.

La deuxième partie est dédiée au calcul de la probabilité de survie d'une firme, conditionnellement à l'information à disposition de l'investisseur, dans un modèle structurel et en information partielle. On utilise une technique numérique hybride basée sur une application de la méthode Monte Carlo et de la quantification optimale. Comme application, la courbe des spreads du crédit pour zero coupon bonds pour différentes maturités est tracée, en montrant que (comme en réalité sur le marché) les spreads au voisinage de la maturité ne sont pas nuls, i.e., en information partielle il y a du risque résiduel sur le marché, même si on est proche de la maturité. La calibration aux données réelles conclut cette deuxième partie.

Dans la troisième partie on traite, avec l'approche Programmation Dynamique, un problème en temps discret de maximisation de l'utilité de la richesse terminale, dans un marché où des titres soumis au risque du défaut sont négociés. Le risque de contagion entre les défauts est modélisé, ainsi que la possible incertitude du modèle, en travaillant en information partielle. Dans la partie numérique la robustesse de la solution trouvée en information partielle est étudiée.

Dans la quatrième partie on s'intéresse au problème de l'incertitude liée à l'horizon temporel d'investissement. En particulier, dans un marché complet soumis au risque du défaut, on résout, soit avec une approche directe du type martingale, soit avec la Programmation Dynamique, trois différents problèmes de maximisation de la consommation. Plus spécifiquement, en notant τ l'instant de défaut, où τ est une variable aléatoire positive et exogène, on considère trois problèmes de maximisation de l'utilité de la consommation: quand l'horizon temporel est fixe et égal à T, quand il est fini, mais possiblement incertain, égal à $T \wedge t$, et quand l'horizon est infini. Dans un premier temps on considère le cas général avec coefficients stochastiques, puis, afin d'obtenir une solution explicite pour les cas utilité logarithmique et puissance, on passe au cas coefficients constants.

Enfin, dans la cinquième partie on traite un problème totalement différent, dans le sens où le sujet considéré est purement théorique. Dans le contexte du grossissement de filtrations, notre but est de redémontrer, dans un cadre spécifique, les résultats déjà connus sur la caractérisation des martingales, la décomposition des martingales par rapport à la filtration de référence comme semi-martingales dans les deux filtrations progressivement et initialement grossies et le Théorème de Représentation Prévisible. Certain de ces résultats ont été utilisés dans la quatrième partie de cette thèse. L'intérêt de cette étude est pédagogique: dans notre contexte spécifique la plupart des résultats sont retrouvés d'une façon plus simple, avec des outils de "base", comme le Théorème de Girsanov et le calcul d'espérances conditionnelles.

MOTS-CLÉS: risque du défaut, information partielle, maximisation d'utilité, contrôle stochastique, méthode Monte Carlo, quantification optimale, méthode martingale, Programmation Dynamique, grossissement de filtrations.

Abstract

This Ph.D. thesis consists of five independent parts (Introduction included) devoted to the modeling and to studying problems related to default risk, under partial information.

The first part constitutes the Introduction.

The second part is devoted to the computation of survival probabilities of a firm, conditionally to the information available to the investor, in a structural model, under partial information. We exploit a numerical hybrid technique based on the application of the Monte Carlo method and of optimal quantization. As an application, we trace the credit spreads curve for zero coupon bonds for different maturities, showing that (as in practice on the market) the spreads in the neighborhood of the maturity are not null, i.e., under partial information there is some residual risk on the market, even if we are close to maturity. Calibration to real data completes this second part.

In the third part we deal, by means of the Dynamic Programming, with a discrete time maximization of the expected utility from terminal wealth problem, in a market where defaultable assets are traded. Contagion risk between the default times is modeled, as well as model uncertainty, by working under partial information. In the part devoted to numerics we study the robustness of the solution found under partial information.

In the fourth part we are interested in studying the problem linked to the uncertainty of the investment horizon. In particular, in a complete market model subject to default risk, we solve, both with a direct martingale approach and with the Dynamic Programming, three different consumption maximization problems. More specifically, denoting by τ the default time, where τ is an exogenous positive random variable, we consider three problems of maximization of expected utility from consumption: when the investment horizon is fixed and equal to T, when it is finite, but possibly uncertain, equal to $T \wedge \tau$, and when it is infinite. First we consider the general stochastic coefficients case, then, in order to obtain explicit results in the logarithmic and power utility cases, we pass to the constant coefficients case.

Finally, in the fifth part we deal with a totally different problem, given that it is purely theoretical. In the context of enlargement of filtrations our aim is to retrieve, in a specific setting, the already known results on martingales' characterization, on the decomposition of martingales with respect to the reference filtration as semi-martingales in the progressively and in the initially enlarged filtrations and the Predictable Representation Theorem. Some of these results were used in the fourth part of this thesis. The interest in this study is pedagogical: in our specific context most of the results are found more easily, by exploiting "basic" tools, such as Girsanov's Theorem and by computing conditional expectations.

KEYWORDS: default risk, partial information, utility maximization, stochastic control, Monte Carlo method, optimal quantization, martingale method, Dynamic Programming, enlargement of filtrations.

Table of contents

Part I

Introduction

Introduction

After the seminal contribution of Bachelier in 1900, with his thesis "Théorie de la spéculation", after the new developments, more than half a century later, by Samuelson and the papers from Black, Scholes and Merton in the early seventies, mathematical finance and financial engineering have been rapidly expanding domains of science.

Nevertheless, most of the literature of the eighties and nineties dealt with market models in which the assets' prices evolution was driven by a continuous stochastic process, that, in most of the cases, was Gaussian. It was impossible, then, to take into account the unpredictable jumps that are, indeed, a major characteristic of market fluctuations. For this reason, during the last fifteen years, discontinuous stochastic processes (i.e., stochastic processes whose trajectories can jump, see, e.g., Cont and Tankov [7]) have become increasingly popular in financial modeling, both for risk management and for option pricing purposes.

Focusing on risk management, we were interested in understanding, modeling and dealing with one of the fundamental sources of financial risk: *credit risk* (for an exhaustive introduction to credit risk we refer, e.g., to Bielecki and Rutkowski [3]). Credit risk embedded in a financial transaction is the possibility of loss associated with any kind of credit-linked event, such as: changes in the credit quality (e.g., credit rating), credit spreads' variations and, finally, default. Default risk is the possibility that a counter-party in a financial contract will not fulfill his/her financial obligation stated in the contract. Because of this definition, the main (challenging) tool in credit risk modeling is the definition and the analysis of the properties of the random time of default, that we denote by τ.

This thesis is divided into five parts. This introduction is the first part. The second part focuses on these modeling aspects and on some related practical problems. In the third part, we solve a discrete time portfolio optimization problem, in a partially observable market model, where defaultable assets are traded. The fourth part aims to study the impact of an uncertainty about the investment time horizon, due to the presence of an exogenously given random time, on the optimal investment-consumption strategy of an investor acting on a defaultable market. In the fifth part, we study some theoretical aspects concerning the enlargement of a reference filtration by means of a random time.

The leitmotif of this thesis is, then, the presence of an inaccessible random time and our aim is to investigate its role and, most of all, the implications of the knowledge we have about it and in particular, its unpredictability. The pivotal point in this work is, then, the analysis of the role played by the information; in particular, we focus on *partial information* settings.

▷ Overview of part II

Credit risk models come in two main varieties: the structural and the reduced form. The structural approach, introduced by Merton in 1974, consists in modeling bankruptcy as the first hitting time of a barrier by the firm value process, while in reduced form (or "intensity based") models, originally developed by Jarrow and Turnbull in 1992, the default intensity is directly modeled and it is given by a function of latent state variables or predictors of default.

The first approach, in which we are interested, is realistic from the economic point of view, but it presents some drawbacks: the firm's value process is not observable in reality, at least in continuous time and, in the case of a continuous firm's value process, default becomes predictable, leading to null credit spreads for short maturities (for surviving firms), a fact that is not observed in practice on the market. On the contrary, in reduced form models the default time is inaccessible.

Despite the apparent difference between the two models (see, e.g., the comparative survey Jarrow and Protter [11]), some recent results, starting from the seminal paper by Duffie and Lando [9], have unified the two approaches, by means of information reduction. Indeed, structural models can be "transformed" into reduced form models, by restricting the information set, from that observable by the firm's management to that observed by market participants.

We consider, then, a structural model under partial information, in which investors can not observe the firm value process, but they have access to another process whose value is related to the firm value process, such as, for example, the price of an asset issued by the firm.

We are interested in computing the conditional survival probability of the firm with respect to the restricted information set, in order to obtain yield spreads for zero coupon bonds (for surviving firms) that are strictly positive at zero maturity. This has to be the case, since investors, in practice, are uncertain about the nearness of the current firm value to the trigger level at which the firm would declare default, and this represents a source of risk.

We show that the computation of these conditional survival probabilities under partial information leads to a nonlinear filtering problem (for an overview on stochastic filtering we refer to Bain and Crişan [1]) involving the conditional survival probabilities under full information. These latter quantities are approximated (when no closed formula is available) by a Monte Carlo procedure, while the filter distribution in discrete time is approximated by exploiting optimal quantization techniques.

In the studied model, then, the shape of the term structure of credit spreads may be useful, in practice, to estimate the degree of transparency and of riskiness of a firm, from the point of view of an investor having a reduced set of information on the market.

▷ **Overview of part III**

We consider the classical portfolio optimization problem of maximization of expected utility from terminal wealth, with the innovative aspect that the assets in which one invests are subject to default-risk. We focus on a context where the assets' dynamics are affected by exogenous factor processes, some of which may have an economic interpretation, some may not but, most importantly, NOT all of them may be directly observable.

In credit risk models, factors are often used to describe contagion: "physical" and "information induced". Information induced contagion arises due to the fact that the successive updating of the distribution of the latent (not observable) factors, in reaction to incoming default observations, leads to jumps in the default intensity of the surviving firms (this is sometimes referred to as "frailty approach", see, e.g., Schönbucher [17]). It was shown in Duffie *et al.* [8] that unobservable factor processes are needed in order to explain clustering of defaults in historical credit risk data.

Notice that, in general, the formulation of a model under incomplete information on the factors allows for greater model flexibility, avoids a possible inadequate specification of the model itself and the successive updating of the distribution of the unobserved factors (for constant factors one considers them from the Bayesian point of view as random variables) allows the model to "track the market", thus avoiding classical model calibration.

We consider only one non-observable factor process, modeled as a finite state Markov chain and we explicitly take into account the possibility of default for the individual assets, as well as contagion (direct and information induced) among them. Considering a multinomial model at discrete time (with respect to continuous time models, this cin be justified since trading actually takes place in discrete time), we provide an explicit numerical solution to the optimization problem. We discuss the solution within our defaultable and partial information setup and, in particular we study its robustness. Numerical results are derived in the case of a log-utility function and they can be analogously obtained for a power utility function.

▷ **Overview of part IV**

The starting point of this work is the acknowledgement of the fact that, in most of the cases, an investment horizon is hardly known with certainty at the date when the initial investment decisions are taken. It is, then, both of practical and theoretical interest to study the influence of this uncertainty on the investor's decisions.

We consider an exogenously given nonnegative random variable τ, that is a totally inaccessible stopping time with respect to the investor's filtration, and we study three different scenarios: the first one when the investment horizon is fixed and equal to T (problem **A**), the second one when it is finite, but possibly uncertain, given by $T \wedge \tau$ (problem **B**), and the third one when it is infinite (problem **C**). Our aim is to investigate the role of the source of randomness τ in the investor's decisions, when his objective is to maximize the expected utility from consumption, in a *complete* market model in which τ affects the assets's dynamics (consider for example, a defaultable zero-coupon bond, in the case when τ is a default time, or a mortality linked security, when τ is the death time of a pensioner).

The present work can, then, be seen within the theory of optimal stochastic control problems with uncertain time horizon. Some recent works on this topic are, e.g., Karatzas and Wang [13], in which they solve an optimal dynamic investment problem in a complete market case, when the uncertain time horizon is a stopping time in the asset price's filtration; Blanchet-Scaillet *et al.* [4], where they consider a maximization of expected utility from consumption problem, in a continuous market model, in the case when the time horizon is uncertain and the source of randomness is not a stopping time in the investor's filtration and Bouchard and Pham [5], where they study, as opposite to the classical fixed time horizon setting, a wealth path-dependent utility maximization problem in an incomplete semimartingale model. In a more general setting, Zitković [20] formulates and solves a class of utility-maximization problems of the "stochastic clock" type (see the more precise Definition 2.3 in [20]) in general incomplete semimartingale markets. Finally, Menoncin [15] studies an optimal consumption-investment problem where the investment horizon is the death time of the investor and longevity bonds are traded in the market.

We solve the three problems **A**, **B** and **C** of maximization of expected utility from consumption in the case when on the market there is a risk-free asset, a defaultable risky asset and a "standard risky" asset. The investor's filtration, denoted by $\mathbb{G}$ (here "G" stands for "global"), is such that τ is $\mathbb{G}$-stopping time. We provide, in a very general stochastic coefficients case, comparison results between the optimal consumption rates of these three problems, showing that (as it should be) when the horizon is finite, but possibly uncertain (problem **B**), the investor consumes at a higher rate with respect to the case when the horizon is fixed (problem **A**). On the other hand, his consumption rate is higher in the case of problem **A** (finite horizon) than in the case of problem **C** (infinite horizon).

Furthermore, we show that, depending on whether the model coefficients are stochastic processes or deterministic functions of time, the investor's optimal investment strategy substantially changes. In the deterministic coefficients case, indeed, for an investor facing problems **A** and **C**, the optimal investment strategy consists in not investing at all in the defaultable risky asset. On the other hand, in the case of problem **B**, when the investment horizon depends on τ, the investor has to deal with this additional source of risk and it is optimal to invest in the defaultable asset.

On the contrary, in the stochastic coefficients case, the market model coefficients are adapted with respect to the investor's filtration $\mathbb{G}$, so that, on the set $\{t > \tau\}$, they depend on τ. The investor has, then, inevitably always to deal with τ (and not, as before in the deterministic coefficients' case, only in the case when τ appears in the investment horizon in problem **B**) and, as a consequence, the optimal proportion of wealth he invests in the defaultable risky asset is never equal to zero.

Part IV is divided into five chapters: in the first one we consider a stochastic coefficients market model and we solve the three problems by means of the martingale approach. We provide explicit optimal investment-consumption strategies in the log-utility case.

The second chapter is the analog to the first one in the case when model coefficients are deterministic. Explicit optimal investment-consumption strategies are found in both the logarithmic and in the exponential utility cases.

In the third chapter, still focusing on the deterministic coefficients case, we solve the problems by using the Dynamic Programming approach, as an alternative technique and, at the end, we consider the case of an investor with a reduced set of information, who does not observe τ.

In the fourth chapter we study, as a separate example, the exponential utility case in a market model with deterministic coefficients.

In the final chapter, we focus on an even more general market model and, by means of the tools developed in the subsequent Part V of this thesis, relative to *enlargement of filtrations*, we provide explicit solutions to the three problems in the log-utility case.

▷ An overview of part V

Let us consider a pair of filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}_t})_{t\geq 0}$ on the same probability space, such that $\mathcal{F}_t \subset \widetilde{\mathcal{F}_t}$, for any $t \geq 0$. In filtering theory, this structure is suitable to describe the evolution of a stochastic system that is partially observable (as in the previous Parts II and III of this thesis). In enlargement of filtration theory, the point of view is the opposite one (see, e.g., the summary in Jeulin [12]): $\mathbb{F}$ is considered to be a reference filtration, to which we add some information, thus leading us to the larger filtration $\widetilde{\mathbb{F}}$.

Here we only consider the case where the enlargement of filtration $\mathbb{F}$ is done by means of a random variable τ. Nevertheless, there are, of course, many other ways to do that, such as, for example, setting $\widetilde{\mathcal{F}_t} = \mathcal{F}_t \vee \bar{\mathcal{F}}$, $t \geq 0$, where $\bar{\mathcal{F}}$ is a σ-algebra, or defining $\widetilde{\mathcal{F}_t} = \mathcal{F}_t \vee \bar{\mathcal{F}}_t$, $t \geq 0$, where $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$ is another filtration.
There are two ways to add information to $\mathbb{F}$ by means of a random variable τ: either all of a sudden at time 0 (*initial enlargement*), or progressively, by considering the smallest σ-algebra containing $\mathbb{F}$ that makes τ a stopping time (*progressive enlargement*).

The "pioneers" who started exploring this research field were Barlow (in [2]), Jacod, Jeulin and Yor (see the references that follow in the text), at the end of the seventies. The main question that raised was the following: "Does any $\mathbb{F}$-martingale X remain an $\widetilde{\mathbb{F}}$ semi-martingale?". And, in this case: "What is the semi-martingale decomposition in $\widetilde{\mathbb{F}}$ of the $\mathbb{F}$-martingale X?"

Notice that a general (but not so practice) necessary and sufficient condition in order for an $\mathbb{F}$-local martingale to remain a $\widetilde{\mathbb{F}}$ semi-martingale is given in Jeulin [12]. Moreover, very technical existence and regularity results, which are fundamental in enlargement of filtrations theory, were proved at the very beginning, in the late seventies.

A recent detailed introduction to this subject can be found, e.g., in Chesney, Jeanblanc and Yor [6], in Mansuy and Yor [14] and in Protter [16].

The main contribution of this part is to show how, in a very specific setting, all the well-known fundamental results can be proved in an alternative (and, in some cases, simpler) way. Nevertheless, it is important to make precise that our goal is neither to present the results in the most general case, nor to study carefully regularity or existence properties.

We consider three nested filtrations

$$\mathbb{F} \subset \mathbb{G} \subset \mathbb{G}^{\tau},$$

where $\mathbb{G}$ and $\mathbb{G}^{\tau}$ stand, respectively, for the *progressive* and the *initial* enlargement of the filtration $\mathbb{F}$ with a finite positive random time τ and we address the following problems:

- Characterization of $\mathbb{G}$-martingales and $\mathbb{G}^{\tau}$-martingales in terms of $\mathbb{F}$-martingales;
- Canonical decomposition of an $\mathbb{F}$-martingale, as a semimartingale, in $\mathbb{G}$ and $\mathbb{G}^{\tau}$;
- Predictable Representation Theorem in $\mathbb{G}$ and $\mathbb{G}^{\tau}$.

The main idea is the following: assuming that the $\mathbb{F}$-conditional law of τ is equivalent to the law of τ, after an *ad hoc* change of probability measure, the problem is reduced to the case where τ and $\mathbb{F}$ are independent. It is, then, "easier" to work under this newly introduced probability measure, in the initially enlarged filtration. Then, under the original probability measure, for the initially enlarged filtration, the results are achieved by means of Girsanov's theorem. Finally, by projection, the desired results in the progressively enlarged filtration are obtained.

The "change of probability measure viewpoint" for treating the problems on enlargement of filtrations was remarked in the early eighties and it was developed by Song [18] (see also Jacod [10]). For what concerns the idea of recovering the results in the progressively enlarged filtration starting from the ones in the initially enlarged, we have to cite Yor [19].

Bibliography

[1] A. Bain and D. Crişan. *Fundamentals of stochastic filtering.* Springer, 2009.

[2] M.T. Barlow. Study of filtration expanded to include an honest time. *Z. Wahr. Verw. Gebiete*, 44:307–323, 1978.

[3] T.R. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging.* Springer Finance. Springer, 2004.

[4] C. Blanchet-Scalliet, N. El Karoui, M. Jeanblanc, and L. Martellini. Optimal investment decisions when time-horizon is uncertain. *Journal of Mathematical Economics*, 44(11):1100–1113, 2008.

[5] B. Bouchard and H. Pham. Wealth-Path Dependent Utility Maximization in Incomplete Markets. *Finance and Stochastics*, 8(4):579–603, 2004.

[6] M. Chesney, M. Jeanblanc, and M Yor. *Mathematical Methods for Financial Markets.* Springer, 2009.

[7] R. Cont and P. Tankov. *Financial Modeling with Jump Processes.* Chapman & Hall/CRC, 2004.

[8] D. Duffie, A. Eckner, G. Horel, and L. Saita. Frailty correlated defaults. *Journal of Finance*, LXIV(5):2089–2123, 2009.

[9] D. Duffie and D. Lando. Term Structures of Credit Spreads with Incomplete Accounting Information. *Econometrica*, 69(3):633–664, 2001.

[10] J. Jacod. Grossissement initial, hypothèse (H') et théorème de Girsanov. In *Lecture Notes in Mathematics*, volume 1118. Springer-Verlag, 1985.

[11] R.A. Jarrow and P. Protter. Structural versus reduced form models: a new information based perspective. *Journal of investment management*, 2(2):1–10, 2004.

[12] T. Jeulin. Semi-martingales et grossissement d'une filtration. In *Lecture Notes in Mathematics*, volume 833. Springer-Verlag, 1980.

[13] I. Karatzas and H. Wang. Utility maximization with discretionary stopping. *SIAM Journal on Control & Optimisation*, 39:306–329, 2001.

[14] R. Mansuy and M. Yor. *Random Times and (Enlargement of) Filtrations in a Brownian Setting*, volume 1873 of *Lectures Notes in Mathematics.* Springer, 2006.

[15] F. Menoncin. The role of longevity bonds in optimal portfolios. *Insurance: Mathematics and Economics*, 42:343–358, 2008.

[16] P.E. Protter. *Stochastic Integration and Differential Equations, Second edition*, volume Stochastic modelling and applied probability of *Application of Mathematics*. Springer-Verlag, Heidelberg, 2005.

[17] P.J. Schonbucher. Information-driven default contagion. Working Paper, Department of Mathematics, ETH Zurich, 2003.

[18] Sh. Song. *Grossissement de filtration et problèmes connexes.* PhD thesis, Université Paris VI, 1987.

[19] M. Yor. Grossissement de filtrations et absolue continuité de noyaux. volume 1118 of *Lecture Notes in Mathematics.* 1985.

[20] G. Zitković. Utility maximization with a stochastic clock and an unbounded random endowment. *The Annals of Applied Probability*, 15(1B):748–777, 2005.

Part II

An application to credit risk of a hybrid Monte Carlo-optimal quantization method

Chapter 1

An application to credit risk of a hybrid Monte Carlo-optimal quantization method

This is a joint work with Abass Sagna.

Abstract: in this part we use a hybrid Monte Carlo-Optimal quantization method to approximate the conditional survival probabilities of a firm, given a structural model for its credit default, under partial information.

We consider the case when the firm's value is a non-observable stochastic process $(V_t)_{t\geq 0}$ and investors in the market have access to a process $(S_t)_{t\geq 0}$, whose value at each time t is related to $(V_s, 0 \leq s \leq t)$. We are interested in the computation of the conditional survival probabilities of the firm given the "investor's information".

As an application, we analyze the shape of the credit spread term structure for zero coupon bonds in two examples in which we find (numerically) that yield spreads for surviving firms are strictly positive at zero maturity (as it is the case in practice). Calibration to available market data is also part of our study.

Keywords: credit risk, structural approach, survival probability, partial information, filtering, optimal quantization, Monte Carlo method.

1.1 Introduction

In this chapter we compute the conditional survival probabilities of a firm, in a market that is not transparent to bond investors, by using both Monte Carlo and optimal quantization techniques. This allows us to analyze the credit spread curve under partial information in some examples, in order to investigate the degree of transparency and riskiness of a firm, as viewed by bond-market participants.

To introduce the problem, recall that most of the bonds traded in the market are corporate bonds and treasury bonds, that are consequently subject to many kinds of risks, such as market risk (due for example to changes in the interest rate), counterparty risk and liquidity risk. One of the main challenges in credit risk modeling is, then, to quantify the risk associated to these financial instruments.

The methodology for modeling a credit event can be split into two main approaches: the structural approach, introduced by Merton in 1974 and the reduced form approach (or "intensity based"), originally developed by Jarrow and Turnbull in 1992.

The structural approach consists in modeling the credit event as the first hitting time of a barrier by the firm value process.

In reduced form models the default intensity is directly modeled and it is given by a function of latent state variables, or predictors of default.

The first approach, in which we are interested, is intuitive by the economic point of view, but it presents some drawbacks: the firm's value process can not be easily observed in practice, since it is not a tradeable security, and a continuous firm's value process implies a predictable credit event, leading to unnatural and undesirable features, such as null spreads for surviving firms for short maturities.

Despite the apparent difference between the two models (see, e.g., Jarrow and Protter [18]), some recent results, starting from the seminal paper Duffie and Lando [9], have unified the two approaches by means of information reduction. We also cite Cetin, Jarrow, Protter and Yildirim [5], where they consider an alternative method with respect to Duffie and Lando [9], namely, a reduction of the manager's information set, to pass from structural to reduced form models; Giesecke [12], where the role of the investor's information in a first passage model is investigated and Giesecke and Goldberg [13], where a structural model with unobservable barrier is studied. An interesting survey on different ways of restricting the information in a credit risk setting, with applications to the pricing of zero-coupon bonds, can be found in Cudennec [8].

Here we consider a structural model under partial information, in which investors can not observe the firm value process, but they have access to another process whose value is related to the firm value process. We show, in two examples, that yield spreads for surviving firms are strictly positive at zero maturity, since investors are uncertain about the nearness of the current firm's value to the trigger level at which the firm would declare default. The shape of the term structure of credit spreads may be useful, then, in practice to estimate the degree of transparency and of riskiness of a firm, from the investors' point of view.

We show that the computation of the conditional survival probabilities under partial information leads to a nonlinear filtering problem involving the conditional survival probabilities under full information. These former quantities are approximated (when no closed formula is available) by a Monte Carlo procedure. As concerns the (non)linear filtering problem, in continuous and discrete time, several computational techniques are known. An overview of some existing methods can be found in Bain and Crişan [1]. These techniques include, e.g., particle filtering, the extended Kalman filter, etc. Optimal quantization is an alternative method in discrete time. One of the advantages of this method, with respect to the others existing, is that once an optimal quantization of the signal process has been obtained, it can be kept off-line and used instantaneously to estimate the filter. This is the main reason why we use optimal quantization to estimate the discrete time filter distribution. For a comparison between particle filtering and optimal quantization see, e.g., Sellami [32].

This chapter is organized as follows. In Section 1.2, we present the market model and we decompose our problem into two problems (**P1**) and (**P2**), that are, respectively, the computation of conditional survival probability in a full information setting and the approximation of the filter distribution. Section 1.3 and Section 1.4 are devoted to the solution of the previous two problems. We provide error estimates in Section 1.5 and, finally, in Section 1.6 we present two numerical examples concerning the application to credit risk and we calibrate the given model to real data available in the market.

1.2 Market model and problem definition

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, representing all the randomness of our economic context. For the moment we concentrate our attention on a single firm model, in which the company is subject to default risk and we use a structural approach to characterize the default time.

The process representing the value of the firm, given for example by its value of financial statement, is denoted by $(V_t)_{t\geq 0}$ and we suppose that it can be modeled as the solution to the following stochastic differential equation

$$\begin{cases} dV_t &= b(t, V_t)dt + \sigma(t, V_t)dW_t, \\ V_0 &= v_0, \end{cases} \tag{1.2.1}$$

where the functions $b : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are Lipschitz in x uniformly in t and W is a standard one-dimensional Brownian motion. We suppose that $\sigma(t, x) > 0$ for every $(t, x) \in [0, +\infty) \times \mathbb{R}$.

In our setting the process V is non observable (it is also known as *state* or *signal*), but investors have access to the values of another stochastic process S, providing noisy information about the value of the firm, that can be thought, for example, as the price of an asset issued by the firm.

This observation process follows a diffusion of the type

$$\begin{cases} dS_t &= S_t \left[\psi(V_t)dt + \nu(t)dW_t + \delta(t)d\bar{W}_t\right], \\ S_0 &= s_0, \end{cases} \tag{1.2.2}$$

where ψ is locally bounded and Lipschitz, ν and δ are bounded deterministic continuous functions and $\bar{W}$ is a one-dimensional Brownian motion independent of W. Note that in this model the return on S is a (nonlinear) function of V affected by a noise. A key observation here concerns the volatility of S, that cannot be a function of V: otherwise we would be able, under suitable regularity properties of this function, to obtain estimations of the firm's value from the market observations of the quadratic variation of S. Finally, following a structural approach, we define the default of the company as

$$\tau := \inf \{t \geq 0 : V_t \leq a\}, \tag{1.2.3}$$

where, as usual, $\inf \emptyset = +\infty$ and for a given constant parameter $a \in \mathbb{R}, 0 < a < v_0$.

In numerical examples we will consider models where $V_t \in (0, +\infty)$ (eventually by stopping the process V at the default time τ by considering the process $(V_{t\wedge\tau})_{t\geq 0}$).

We will deal with two different filtrations, representing different levels of information available to agents in the market and we suppose that they satisfy the usual hypotheses: a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfies the usual hypotheses if $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets and if the filtration is right-continuous.

The first and basic information set is the "default-free" filtration, the one generated by the observation process S, that we will denote, for each $t \geq 0$,

$$\mathcal{F}_t^S := \sigma(S_s, 0 \leq s \leq t)$$

and the second one is the full information filtration $(\mathcal{G}_t)_{t\geq 0}$, i.e., the information available for example to a small number of stock holders of the company, who have access to S and V at each time t. In our case, the full information filtration is the one generated by the stochastic pair process $(W, \bar{W})$. In conclusion we have

$$\mathcal{F}_t^S \subsetneq \mathcal{G}_t, \quad \forall\, t \geq 0,$$

and we observe that the following *immersion* property holds (see Coculescu, Geman and Jeanblanc [7], Proposition 3.1, for an analogous analysis):

Lemma 1.2.1. *Any $(\mathcal{F}_t^S)_{t\geq 0}$-local martingale is a $(\mathcal{G}_t)_{t\geq 0}$-local martingale. We will say that filtration $(\mathcal{F}_t^S)_{t\geq 0}$ is* immersed *in the full information filtration $(\mathcal{G}_t)_{t\geq 0}$.*

Suppose now that a finite time horizon T is fixed. For a given s, $0 \leq s < T$, we observe the process S from 0 to s. At time s, if the firm has already defaulted we do nothing. Otherwise (i.e., on the set $\{\tau > s\}$), we invest in derivatives issued by the firm and we are, then, interested in computing the following quantity, for a given t, $s < t < T$,

$$\mathbb{1}_{\{\tau > s\}} \mathbb{P}\left(\inf_{s\leq u\leq t} V_u > a \middle| \mathcal{F}_s^S \right). \tag{1.2.4}$$

This is the conditional survival probability of the firm up to time t, given the collected information on S up to time s, on the event $\{\tau > s\}$. We will see in Section 1.6 how this quantity plays a fundamental role (if computed under a pricing measure) in the computation of credit spreads for zero coupon bonds.

1.2.1 Reduction to a nonlinear filtering problem

Using the law of iterated conditional expectations, the Markov property of V and the independence between W and $\bar{W}$, we find, for each $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+, s \leq t$,

$$\begin{aligned} \mathbb{P}\left(\inf_{s\leq u\leq t} V_u > a \middle| \mathcal{F}_s^S \right) &= \mathbb{E}\left[\mathbb{P}\left(\inf_{s\leq u\leq t} V_u > a \middle| \mathcal{G}_s \right) \middle| \mathcal{F}_s^S \right] \\ &= \mathbb{E}\left[\mathbb{P}\left(\inf_{s\leq u\leq t} V_u > a \middle| V_s \right) \middle| \mathcal{F}_s^S \right] \\ &= \mathbb{E}\left[F(s,t,V_s) | \mathcal{F}_s^S \right], \qquad \mathbb{P}-\text{a.s.} \end{aligned} \tag{1.2.5}$$

where, for every $x \in \mathbb{R}$,

$$F(s,t,x) := \mathbb{P}\left(\inf_{s \le u \le t} V_u > a \middle| V_s = x\right). \tag{1.2.6}$$

Finally, if we solve the following two problems:

(**P1**) compute $F(s,t,x)$ for every $x \in \mathbb{R}$, which is now a conditional survival probability given the full information filtration, and

(**P2**) obtain the *filter distribution* at time s, $\Pi_{V_s|\mathcal{F}_s^S}$, i.e., the conditional distribution of V_s given $\mathcal{F}_s^S$,

then we are done, since it suffices to compute the integral

$$\begin{aligned}\mathbb{E}\left[F(s,t,V_s)|\mathcal{F}_s^S\right] &= \int_{-\infty}^{\infty} F(s,t,x)\Pi_{V_s|\mathcal{F}_s^S}(\mathrm{d}x) \\ &= \int_a^{\infty} F(s,t,x)\Pi_{V_s|\mathcal{F}_s^S}(\mathrm{d}x).\end{aligned}$$

It remains to solve the two "intermediate problems" (**P1**) and (**P2**). Let us consider first problem (**P2**).

1.3 Approximation of the filter by optimal quantization

We recall in the following subsection some facts about optimal vector quantization.

1.3.1 A brief overview on optimal quantization

Consider an $\mathbb{R}^d$-valued random variable X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with finite r-th moment and probability distribution $\mathbb{P}_X$. Quantizing X on a given grid $\Gamma = \{x^1, \cdots, x^N\}$ consists in projecting X on the grid Γ following the closest neighbor rule. The induced mean L^r-error ($r > 0$)

$$\|X - \mathrm{Proj}_\Gamma(X)\|_r = \|\min_{1 \le i \le N} |X - x^i|\|_r,$$

where $||X||_r := [\mathbb{E}(|X|^r)]^{1/r}$ is called the *L^r-mean quantization error* and the projection of X on Γ, $\mathrm{Proj}_\Gamma(X)$, is called *the quantization of* X. As a function of the grid Γ the L^r-mean quantization error is continuous and reaches a minimum over all the grids with size at most N. A grid $\Gamma^\star$ minimizing the L^r-mean quantization error over all the grids with size at most N is called an *L^r-optimal quantizer* (of size N).

Moreover, the L^r-mean quantization error goes to 0 as the grid size $N \to +\infty$ and the convergence rate is ruled by the Zador theorem:

$$\min_{\Gamma,\ |\Gamma|=N} \|X - \mathrm{Proj}_\Gamma(X)\|_r = Q_r(\mathbb{P}_X)N^{-1/d} + o\big(N^{-1/d}\big),$$

where $Q_r(\mathbb{P}_X)$ is a nonnegative constant. We shall say no more about the basic results on optimal vector quantization. For a complete background on this field we refer to Graf and Luschgy [17].

The first application of optimal quantization methods to numerical probability appears in Pagès [24]. It consists in estimating $\mathbb{E}f(X)$ (it may also be a conditional expectation) by

$$\mathbb{E}f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big) = \sum_{i=1}^{N} f(x^{\star,i})\ p_i \tag{1.3.1}$$

where $\Gamma^\star = \{x^{\star,1}, \cdots, x^{\star,N}\}$ is an L^r-optimal grid for X and $p_i = \mathbb{P}\big(\mathrm{Proj}_{\Gamma^\star}(X) = x^{\star,i}\big)$. The induced quantization error estimate depends on the regularity of the function f.

• If $f : \mathbb{R}^d \mapsto \mathbb{R}$ is Lipschitz continuous and $r \geq 2$, introducing $[f]_{\mathrm{Lip}} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$, then

$$\begin{aligned} |\mathbb{E}f(X) - \mathbb{E}f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big)| &\leq \mathbb{E}|f(X) - f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big)| \\ &\leq [f]_{\mathrm{Lip}} \|X - \mathrm{Proj}_{\Gamma^\star}(X)\|_1 \\ &\leq [f]_{\mathrm{Lip}} \|X - \mathrm{Proj}_{\Gamma^\star}(X)\|_2. \end{aligned}$$

• If the derivative Df of f is Lipschitz and $r \geq 2$, then, for any optimal grid $\Gamma^\star$, we have

$$|\mathbb{E}f(X) - \mathbb{E}f\big(\mathrm{Proj}_{\Gamma^\star}(X)\big)| \leq [Df]_{\mathrm{Lip}} \|X - \mathrm{Proj}_{\Gamma^\star}(X)\|_2^2.$$

How to numerically compute the quadratic optimal quantizers or L^r-optimal (or stationary) quantizers in general, the associated weights and L^r-mean quantization errors is an important issue from the numerical point of view. Several algorithms are used in practice. In the one dimensional framework, the L^r-optimal quantizers are unique up to the grid size as soon as the density of X is strictly log-concave. In this case the Newton algorithm is a commonly used algorithm to carry out the L^r-optimal quantizers when closed or semi-closed formulas are available for the gradient (and the hessian matrix).

When the dimension d is greater than 2 the L^r-optimal grids are not uniquely determined and all L^r-optimal quantizers search algorithms are based on zero search recursive procedures like Lloyd's I algorithms (or generalized Lloyd's I algorithms which are the natural extension of the quadratic case), the Competitive Learning Vector Quantization (CLVQ) algorithm (see Gersho and Gray [11]), stochastic algorithms (see Pagès [25] and Pagès and Printems [27]), etc. From now on we consider quadratic optimal quantizers.

1.3.2 General results on discrete time nonlinear filtering

For an overview on nonlinear filtering problems in interest rate and credit risk models we refer to Frey and Runggaldier [10] and references therein and, focusing on filtering theory in credit risk, we also have to mention the seminal papers Kusuoka [20] and Nakagawa [23].

We consider a general *discrete time* setting, in which we recall the relevant formulas and the desired approximation of the filter (see, e.g., Pagès and Pham [26] and Pham, Runggaldier and Sellami [29] for a detailed background). We introduce a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (notice that $\mathbb{P}$ is not the same measure we considered in Section 1.2, but for simplicity we will use the same notation) and we suppose that:

- *the signal process* $(X_k)_{k\in\mathbb{N}}$ is a finite-state Markov chain with state space E, with known probability transition, from time $k-1$ to time k, $P_k(x_{k-1}, \mathrm{d}x_k), k \geq 1$, and with given initial law μ;

- *the observation process* is an $\mathbb{R}^q$-valued process $(Y_k)_{k\in\mathbb{N}}$ such that $Y_0 = y_0$ and the pair $(X_k, Y_k)_{k\in\mathbb{N}}$ is a Markov chain.

Furthermore, we suppose that for all $k \geq 1$

(**H**) the law of Y_k conditional on (X_{k-1}, Y_{k-1}, X_k) admits a density

$$y_k \mapsto g_k(X_{k-1}, Y_{k-1}, X_k, y_k),$$

so that the probability transition of the Markov chain $(X_k, Y_k)_{k\in\mathbb{N}}$ is given by $P_k(x_{k-1}, \mathrm{d}x_k)g_k(x_{k-1}, y_{k-1}, x_k, y_k)\mathrm{d}y_k$, with initial law $\mu(\mathrm{d}x_0)\delta_0(\mathrm{d}y_0)$.

In this discrete time setting we are interested in computing conditional expectations of the form

$$\Pi_{Y,n} f := \mathbb{E}\left[f(X_n)|Y_1, \dots, Y_n\right],$$

for suitable functions f defined on E, i.e., we are interested in computing at some time n the law $\Pi_{Y,n}$ of X_n given the past observation $Y = (Y_1, \dots, Y_n)$. Having fixed the observation $Y = (Y_1, \dots, Y_n) = (y_1, \dots, y_n) =: y$ we will write $\Pi_{y,n}$ instead of $\Pi_{Y,n}$.

It is evident that, in the case when the state space of the signal consists of a finite number of points, the filter is characterized by a finite-dimensional vector: if for example each X_k takes values in a set $\{x_k^1, \dots, x_k^{N_k}\}$ (as in the case where we quantize a process X at discrete times $t_k, k = 0, \cdots, n$ with grids of size N_k), then the discrete time filter distribution at time t_k will be fully determined by the N_k-vector with components

$$\Pi^i_{Y,k} = \mathbb{P}\left(X_k = x_k^i | Y_1, \dots, Y_k\right), \quad i = 1, \dots, N_k.$$

It is for this reason that, following Pagès and Pham [26], we apply optimal quantization results in order to obtain a spatial discretization, on a grid $\Gamma_k = \{x_k^1, \dots, x_k^{N_k}\}$, of the state $X_k, k = 0, \dots, n$, and we characterize the filter distribution by means of the finite number of points $\{x_0, x_1^1, \dots, x_1^{N_1}, x_2^1, \dots, x_2^{N_2}, \dots, x_n^1, \dots, x_n^{N_n}\}$ making up the grids $(\Gamma_k)_k$.

In what follows we recall the basic recursive filtering equation, that we will use in our numerics to approximate the filter. By applying the Markov property of X and (X, Y) and Bayes' formula, we find:

$$\Pi_{y,n} f = \frac{\pi_{y,n} f}{\pi_{y,n} \mathbb{1}}, \tag{1.3.2}$$

where $\pi_{y,n}$ is the *un-normalized filter*, defined by

$$\pi_{y,n} f = \int \cdots \int f(x_n)\mu(\mathrm{d}x_0) \prod_{k=1}^{n} g_k(x_{k-1}, y_{k-1}, x_k, y_k) P_k(x_{k-1}, \mathrm{d}x_k). \tag{1.3.3}$$

Equivalently, we recall the following recursive formula, that can be directly obtained as well by applying Bayes' formula and the Markov property:

$$\Pi_{y,k}(\mathrm{d}x_k) \propto \int g_k(x_{k-1}, y_{k-1}, x_k, y_k) P_k(x_{k-1}, \mathrm{d}x_k) \Pi_{y,k-1}(\mathrm{d}x_{k-1}),$$

where now y in $\Pi_{y,k-1}$ represents the realization of the vector $(Y_1, \dots, Y_{k-1})$ and we do not have equality because we need to re-normalize.

Now for any $k \in \{1, \cdots, n\}$ note that

$$\pi_{y,k} f = \mathbb{E}\Big(f(X_k) \prod_{i=1}^{k} g_i(X_{i-1}, y_{i-1}, X_i, y_i) \Big).$$

Therefore, introducing the natural filtration of X, $(\mathcal{F}_k^X)_{k\in\mathbb{N}}$, we have

$$\begin{aligned}
\pi_{y,k} f &= \mathbb{E}\left(\mathbb{E}\Big(f(X_k) \prod_{i=1}^{k} g_i(X_{i-1}, y_{i-1}, X_i, y_i) | \mathcal{F}_{k-1}^X \Big) \right) \\
&= \mathbb{E}\left(\mathbb{E}\left(f(X_k) g_k(X_{k-1}, y_{k-1}, X_k, y_k) | \mathcal{F}_{k-1}^X \right) \prod_{i=1}^{k-1} g_i(X_{i-1}, y_{i-1}, X_i, y_i) \right) \\
&= \mathbb{E}\left(H_{y,k}(f(X_{k-1})) \prod_{i=1}^{k-1} g_i(X_{i-1}, y_{i-1}, X_i, y_i) \right), \qquad (1.3.4)
\end{aligned}$$

where $H_{y,k}, k = 1, \ldots, n$, is a family of bounded transition kernels defined on bounded measurable functions $f : E \to \mathbb{R}$ by:

$$\begin{aligned}
H_{y,k} f(x_{k-1}) &:= \mathbb{E}\left[f(X_k) g_k(x_{k-1}, y_{k-1}, X_k, y_k) | X_{k-1} = x_{k-1} \right] \\
&= \int f(x_k) g_k(x_{k-1}, y_{k-1}, x_k, y_k) P_k(x_{k-1}, \mathrm{d}x_k), \qquad (1.3.5)
\end{aligned}$$

with $x_{k-1} \in E$. Furthermore, for every $x \in E$, we have

$$H_{y,0} f(x) := \pi_{y,0} f = \mathbb{E}\left[f(X_0) \right] = \int f(x_0) \mu(\mathrm{d}x_0).$$

It follows, then, from (1.3.4) that

$$\pi_{y,k} f = \pi_{y,k-1} H_{y,k} f, \quad k = 1, \ldots, n, \qquad (1.3.6)$$

so that we finally obtain the recursive expression

$$\pi_{y,n} = H_{y,0} \circ H_{y,1} \circ \cdots \circ H_{y,n}.$$

1.3.3 Estimation of the filter and related error

The estimation of the filter by optimal quantization is already studied in Pagès and Pham [26] and in Sellami [31]. It consists first in quantizing for every time step k the random variable X_k by considering

$$\widehat{X}_k = \mathrm{Proj}_{\Gamma_k}(X_k), \quad k = 0, \cdots, n, \qquad (1.3.7)$$

where Γ_k is a grid of N_k points x_k^i, $i = 1, \cdots, N_k$ to be optimally chosen and where Proj_{Γ_k} denotes the closest neighbor projection on the grid Γ_k.
Owing to Equation (1.3.6) our aim is to estimate the filter using an approximation of the transition probability $P_k(x_{k-1}, dx_k)$ of X_k given X_{k-1}. These probabilities are approximated by the transition probability matrix $\widehat{p}_k := (\widehat{p}_k^{ij})$ of $\widehat{X}_k$ given $\widehat{X}_{k-1}$:

$$\widehat{p}_k^{ij} = \mathbb{P}(\widehat{X}_k = x_k^j | \widehat{X}_{k-1} = x_{k-1}^i), \; i = 1, \cdots, N_{k-1}, \; j = 1, \cdots, N_k. \qquad (1.3.8)$$

Then, following Equation (1.3.5), being the observation $y := (y_0, \cdots, y_k)$ fixed, the transition kernel matrix $H_{y,k}$ is estimated by the quantized transition kernel $\widehat{H}_{y,k}$

$$\widehat{H}_{y,k} = \sum_{j=1}^{N_k} \widehat{H}_{y,k}^{ij} \delta_{x_{k-1}^i}, \quad i = 1, \cdots, N_{k-1}, \quad k = 1, \cdots, n,$$

where

$$\widehat{H}_{y,k}^{ij} = g_k(x_{k-1}^i, y_{k-1}, x_k^j, y_k)\widehat{p}_k^{ij}, \quad i = 1, \cdots, N_{k-1}, \ j = 1, \cdots, N_k$$

and where the x_k^j's, $j = 1, \cdots, N_k$ are the (quadratic) optimal quantizers of X_k. The initial kernel matrix $H_{y,0}$ is estimated by

$$\widehat{H}_{y,0} = \sum_{i=1}^{N_0} \mathbb{P}(\widehat{X}_0 = x_0^i) \ \delta_{x_0^i}.$$

This leads to the following forward induction to approximate $\pi_{y,n}$:

$$\widehat{\pi}_{y,0} = \widehat{H}_{y,0}, \qquad \widehat{\pi}_{y,k} = \widehat{\pi}_{y,k-1}\widehat{H}_{y,k}, \qquad k = 1, \cdots, n, \tag{1.3.9}$$

or, equivalently,

$$\begin{cases} \widehat{\pi}_{y,0} = \widehat{H}_{y,0} \\ \widehat{\pi}_{y,k} = \Big(\sum_{i=1}^{N_{k-1}} \widehat{H}_{y,k}^{ij} \widehat{\pi}_{y,k-1}^i\Big)_{j=1,\cdots,N_k}, \end{cases} \quad k = 1, \cdots, n.$$

Finally, the filter approximation at time t_n is

$$\widehat{\Pi}_{y,n} f = \frac{\widehat{\pi}_{y,n} f}{\widehat{\pi}_{y,n} \mathbb{1}}. \tag{1.3.10}$$

In order to have an upper bound of the quantization error estimate of $\Pi_{y,n} f$ by $\widehat{\Pi}_{y,n} f$ let us make the following assumptions.

(A1) The transition operators $P_k(x, dy)$ of X_k given X_{k-1}, $k = 1, \cdots, n$ are Lipschitz.

Recall that a probability transition P on E is C-Lipschitz (with $\mathrm{C} > 0$) if for any Lipschitz function f on E with ratio $[f]_{Lip}$, Pf is Lipschitz with ratio $[Pf]_{Lip} \leq \mathrm{C}[f]_{Lip}$. Then, one may define the Lipschitz ratio $[P]_{Lip}$ by

$$[P]_{Lip} = \sup\left\{ \frac{[Pf]_{Lip}}{[f]_{Lip}}, f \text{ a nonzero Lipschitz function } \right\} < +\infty.$$

If the transition operators $P_k(x, dy)$, $k = 1, \cdots, n$ are Lipschitz, it follows that

$$[P]_{Lip} := \max_{k=1,\cdots,n} [P_k]_{Lip} < +\infty.$$

(A2) (i) For every $k = 1, \cdots, n$, the functions g_k (recall hypothesis (**H**)) are bounded on $E \times \mathbb{R}^q \times E \times \mathbb{R}^q$ and we set

$$\mathrm{K}_g^n := \max_{k=1,\cdots,n} \|g_k\|_\infty.$$

(ii) For every $k = 1, \cdots, n$, there exist two positive functions $[g_k^1]_{Lip}$ and $[g_k^2]_{Lip}$ defined on $\mathbb{R}^q \times \mathbb{R}^q$ so that for every $x, x', \widehat{x}, \widehat{x}' \in E$ and $y, y' \in \mathbb{R}^q$,

$$|g_k(x, y, x', y') - g_k(\widehat{x}, y, \widehat{x}', y')| \le [g_k^1]_{Lip}(y, y')\ |x - \widehat{x}| + [g_k^2]_{Lip}(y, y')\ |x' - \widehat{x}'|.$$

The following result gives the error bound of the estimation of the filter (see Pagès and Pham [26] Theorem 3.1, for details of the proof).

Theorem 1.3.1. *Suppose that Assumptions **(A1)** and **(A2)** hold true. For every bounded Lipschitz function f on E and for every n-tuple of observations $y = (y_1, \cdots, y_n)$, we have for every $p \ge 1$,*

$$|\Pi_{y,n} f - \widehat{\Pi}_{y,n} f| \le \frac{\mathrm{K}_g^n}{\phi_n(y) \vee \widehat{\phi}_n(y)} \sum_{k=0}^{n} \mathrm{B}_k^n(f, y, p)\ \|X_k - \widehat{X}_k\|_p \tag{1.3.11}$$

with

$$\phi_n(y) := \pi_{y,n}\mathbf{1}, \quad \widehat{\phi}_n(y) := \widehat{\pi}_{y,n}\mathbf{1}$$

and

$$\begin{aligned} \mathrm{B}_k^n(f, y, p) \ &:= \ (2 - \delta_{2,p})[P]_{Lip}^{n-k}[f]_{Lip} + 2\bigg(\frac{\|f\|_\infty}{K_g}\big([g_{k+1}^1]_{Lip}(y_k, y_{k+1}) + [g_k^2]_{Lip}(y_{k-1}, y_k)\big) \\ &+ \ (2 - \delta_{2,p})\frac{\|f\|_\infty}{K_g} \sum_{j=k+1}^{n} [P]_{Lip}^{j-k-1}\big([g_j^1]_{Lip}(y_{j-1}, y_j) + [P]_{Lip}[g_j^2]_{Lip}(y_{j-1}, y_j)\big)\bigg). \end{aligned}$$

(Convention: $g_0 = g_{n+1} \equiv 0$ and $\delta_{n,p}$ is the usual Kronecker symbol).

Remark 1.3.1. Concerning the above L^p-error bounds, notice that in the quadratic case ($p = 2$) the coefficients B_k^n are smaller than in the L^1 case, even if the L^1 quantization error is smaller than the quadratic quantization error.

1.3.4 Application to the estimation of $\Pi_{V_s|\mathcal{F}_s^S}$

We focus now on solving problem (**P2**) and, in order to obtain the discrete time approximation of the desired filter $\Pi_{V_s|\mathcal{F}_s^S}$ at time s, we fix a time discretization grid $t_0 = 0 < \cdots < t_n = s$ in the interval $[0, s]$ and we apply the results in the previous subsections by working with the corresponding quantized process $\widehat{V}$ (we identify X with V and Y with S). From now on $(V_k)_{k=0,\cdots,n}$, will denote either the continuous time process V taken at discrete times $t_k, k = 0, \cdots, n$, or the discrete time Euler scheme relative to V.

First of all, let us make the following remark concerning the conditional law of S_t given $((V_u)_{u\in[s,t]}, S_s)$. This will ensure that in our case Hypothesis (**H**) is verified.

Remark 1.3.2. Let $s \le t$. Using the form of the solution to the SDE (1.2.2)

$$S_t = S_s \exp\left(\int_s^t \Big(\psi(V_u) - \frac{1}{2}(\nu^2(u) + \delta^2(u))\Big)du + \int_s^t \nu(u)dW_u + \int_s^t \delta(u)d\bar{W}_u \right),$$

we notice that

$$\mathcal{L}\big(S_t|(V_u)_{s\le u\le t}, S_s\big) = \mathrm{LN}(m_{s,t}; \sigma_{s,t}^2), \tag{1.3.12}$$

where

$$m_{s,t} = \log(S_s) + \int_s^t \Big(\psi(V_u) - \frac{1}{2}(\nu^2(u) + \delta^2(u)) - \nu(u)\frac{b(u,V_u)}{\sigma(u,V_u)}\Big)du + \int_s^t \frac{\nu(u)}{\sigma(u,V_u)}dV_u$$

and

$$\sigma^2_{s,t} = \int_s^t \delta^2(u)du.$$

$\mathrm{LN}(m;\sigma^2)$ stands for the lognormal distribution with mean m and variance σ^2.

Now, suppose that we temporarily have a time discretization grid from 0 to t: $u_0 = 0 < u_1 < \cdots < u_m = t$. For m large enough we can estimate the mean and the variance appearing in Equation (1.3.12) by using an Euler scheme. When the estimations of the mean $m_{s,t}$ and variance $\sigma^2_{s,t}$ between two discretization steps are respectively denoted by m_k and σ^2_k and we have:

$$\mathcal{L}\big(S_k|V_{k-1}, S_{k-1}, V_k\big) = \mathrm{LN}(m_k;\sigma^2_k) \tag{1.3.13}$$

with

$$\begin{aligned} m_k &= \log S_{k-1} + \left(\psi(V_{k-1}) - \frac{1}{2}\left(\nu^2(u_{k-1}) + \delta^2(u_{k-1})\right) - \nu(u_{k-1})\frac{b(u_{k-1},V_{k-1})}{\sigma(u_{k-1},V_{k-1})}\right)\Delta_k \\ &\quad + \frac{\nu(u_{k-1})}{\sigma(u_{k-1},V_{k-1})}\Delta V_k \end{aligned}$$

and

$$\sigma^2_k = \delta^2(u_{k-1})\Delta_k,$$

where $S_k := S_{u_k},\ V_k := V_{u_k},\ \Delta V_k = V_k - V_{k-1},\ \Delta_k = u_k - u_{k-1}.$ So, the law of S_k conditional on (V_{k-1}, S_{k-1}, V_k) admits the density (i.e., Hypothesis **(H)** is fulfilled)

$$g_k(V_{k-1}, S_{k-1}, V_k, x) = \frac{1}{\sigma_k x\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2_k}(\log x - m_k)^2\right),\ x \in (0,+\infty). \tag{1.3.14}$$

Remark 1.3.3. (a) In the case where

$$\begin{cases} dV_t = \mu V_t dt + \sigma V_t dW_t, & V_0 = v_0, \\ dS_t = rS_t dt + \sigma S_t dW_t + \delta S_t d\bar{W}_t, & S_0 = s_0, \end{cases}$$

we directly deduce from Remark 1.3.2 that, for every $s \le t$,

$$\mathcal{L}\big(S_t|(V_u)_{s\le u\le t}, S_s\big) = \mathrm{LN}\left(\log\Big(\frac{S_sV_t}{V_s}\Big) + \big(r - \mu - \frac{1}{2}(\sigma^2+\delta^2)\big)(t-s);\delta^2(t-s)\right).$$

(b) (About the transition probabilities in Equation (1.3.8)) In a general setting the transition probabilities

$$\widehat{p}^{ij}_k = \mathbb{P}(\widehat{V}_k = v^j_k|\widehat{V}_{k-1} = v^i_{k-1}),\ i = 1,\cdots,N_{k-1},\ j = 1,\cdots,N_k,$$

where $\{v^q_p, p = 0,\cdots,n; q = 1,\cdots,N_p\}$ are the quadratic optimal quantizers of the process V, can be estimated by Monte Carlo. However, in some specific cases the continuous time transition densities $p(s,t,x,dy) := \mathbb{P}(V_t \in dy | V_s = x), 0 \le s < t$, are explicitly obtained as solutions to the Kolmogorov equations. For example in the case of item (a) of the remark,

$$p(s,t,x,dy) = \frac{1}{\sigma y \sqrt{2\pi(t-s)}} \; e^{-\frac{1}{2\sigma^2(t-s)}\left[\log\left(\frac{y}{x}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t-s)\right]^2} dy. \tag{1.3.15}$$

This density can also be derived from the explicit form of V. In such situations, the $\widehat{p}_k^{ij}$'s are estimated from the $p(t_{k-1}, t_k, v_{k-1}^i, dy)$.

Once problem **(P2)** solved, owing to Equation (1.2.5) we use optimal quantization to estimate the $\mathbb{P}\big(\inf_{s \le u \le t} V_u > a | \mathcal{F}_s^S\big)$ on the set $\{\tau > s\}$ by

$$\sum_{i=1}^{N_n} F(s,t,v_n^i)\; \widehat{\Pi}_{y,n}^i, \tag{1.3.16}$$

where $v_n^i, i = 1, \cdots, N_n$ is the quadratic optimal grid of the process V at time $t_n = s$, $\widehat{\Pi}_{y,n}^i$ is the i-th coordinate of the optimal filter $\widehat{\Pi}_{y,n}$ given in (1.3.10) and, for every i, $F(s,t,v_n^i)$ is defined as in (1.2.6). Note that this last function has in general no explicit expression. In such case, we will estimate it by Monte Carlo as specified in the next section.

1.4 Approximation by Monte Carlo of survival probabilities under full information

The aim of this section is to solve problem (**P1**), i.e., to compute, for each pair of positive values $(s,t), s \le t \le T$,

$$\mathbb{P}\left(\inf_{s \le u \le t} V_u > a \middle| V_s\right) = \mathbb{E}\left(\mathbb{1}_{\{\inf_{s \le u \le t} V_u > a\}} | V_s\right), \tag{1.4.1}$$

where in our general setting the firm value V follows *a priori* a diffusion of the type (1.2.1). Notice that in the specific case where V is a geometric Brownian motion there exists a closed-formula, that we recall below.
If

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \quad V_0 = v_0,$$

then

$$\mathbb{P}\left(\inf_{s \le u \le t} V_u > a | V_s\right) = \Phi(h_1(V_s, t-s)) - \left(\frac{a}{V_s}\right)^{\sigma^{-2}(\mu - \sigma^2/2)} \Phi(h_2(V_s, t-s)) \tag{1.4.2}$$

where

$$\begin{aligned} h_1(x,u) &= \frac{1}{\sigma\sqrt{u}}\left(\log\left(\frac{x}{a}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)u\right), \\ h_2(x,u) &= \frac{1}{\sigma\sqrt{u}}\left(\log\left(\frac{a}{x}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)u\right) \end{aligned}$$

and where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-u^2/2} \mathrm{d}u$ is the cumulative distribution function of the standard Gaussian law. For an overview on the computation of boundary crossing probabilities see, e.g., Chesney, Jeanblanc and Yor, [6], Borodin and Salminen [4] or Revuz and Yor [30].

Since in general we cannot use directly the result in Equation (1.4.2), we have to resort to an approximation method. Several techniques can be used to estimate these probabilities, such as in Kahalé [19], where the crossing probabilities are calculated via Schwartz distributions in the specific case of drifted Brownian motion and in Linetsky [21] and Linetsky [22], where the survival probabilities and hitting densities relative to the CIR (Cox-Ingersoll-Ross), the CEV(Constant Elasticity of Variance) and to the OU (Ornstein-Uhlenbeck) diffusions are expressed as infinite series of exponential densities:

$$\mathbb{P}_{v_0}(\tau > t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}, \quad t > 0, \tag{1.4.3}$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ as $n \to \infty$ and $(c_n)_n$ are explicitly given in terms of the solution of the Sturm-Liouville equation and the eigenvalues of the Sturm-Liouville problem. When the basic solutions to the Sturm-Liouville equation are known, this approach provides efficient estimates of the survival probabilities.

Here, we will adopt the "regular Brownian bridge method", originally introduced in Baldi [2]. From the numerical viewpoint, if the exact c_n and λ_n in Equation (1.4.3) can be exactly computed, Linetsky's procedure may be more efficient than the "regular Brownian bridge method" (except in the Black-Scholes setting, see Section 1.6.1). Nevertheless, it will be more time consuming than the last one since obtaining, e.g., the first one hundred exact c_n's and λ_n's takes "several minutes" (see Linetsky [21]).

In order to find an approximated solution to problem (**P1**) by means of the regular Brownian bridge method, we consider the interval $[s,t]$ and we discretize it by means of $u_0 = s < u_1 < \cdots < t = u_N$. We denote by $\bar{V}$ the continuous Euler scheme relative to V. This process is defined by

$$\bar{V}_u = \bar{V}_{\underline{u}} + b(\underline{u}, \bar{V}_{\underline{u}})(u - \underline{u}) + \sigma(\underline{u}, \bar{V}_{\underline{u}})(W_u - W_{\underline{u}}), \quad \bar{V}_s = v_s,$$

with $\underline{u} = u_k$ if $u \in [u_k, u_{k+1})$, for the given time discretization grid $u_k := s + \frac{k(t-s)}{N}, k = 0, \cdots, N$, on the set $[s,t]$.

The regular Brownian bridge method is connected to the knowledge of the distribution of the minimum (or the maximum) of the continuous Euler scheme $\bar{V}$ relative to the process V over the time interval $[s,t]$, given its values at the discrete time observation points $s = u_0 < u_1 < \cdots < u_N = t$. This distribution is given in the Lemma below (see, e.g., Glasserman [14]).

Lemma 1.4.1.

$$\mathcal{L}\Big(\min_{u\in[s,t]} \bar{V}_u | \bar{V}_{u_k} = v_k, k = 0, \cdots, N\Big) = \mathcal{L}\Big(\min_{k=0,\cdots,N-1} G^{-1}_{v_k, v_{k+1}}(U_k)\Big) \tag{1.4.4}$$

where $(U_k)_{k=0,\cdots,N-1}$ are i.i.d random variables uniformly distributed over the unit interval and $G^{-1}_{x,y}$ is the inverse function of the conditional survival function $G_{x,y}$, defined by

$$G_{x,y}(u) = \exp\Big(-\frac{2N}{(t-s)\sigma^2(x)}(u-x)(u-y)\Big)\mathbb{1}_{\{\min(x,y)\geq u\}}.$$

Notice that we have omitted the dependence on time in σ.

We deduce from the previous lemma the following result.

Proposition 1.4.1.

$$\mathbb{P}\left(\min_{s\le u\le t}\bar V_u > a\big|\bar V_s\right) = \mathbb{E}\left(\prod_{k=0}^{N-1} G_{\bar V_{u_k},\bar V_{u_{k+1}}}(a)\big|\bar V_s\right).$$

Proof. We have (recall that $\bar V_s = \bar V_{u_0}$)

$$\begin{aligned}\mathbb{P}\left(\min_{s\le u\le t}\bar V_u > a\big|\bar V_s\right) &= \mathbb{E}\left(\mathbb{P}\left(\min_{s\le u\le t}\bar V_u > a\big|\bar V_{u_k}, k=0,\dots,N\right)\big|\bar V_s\right)\\ &= \mathbb{E}\left(\mathbb{P}\left(\min_{k=0,\cdots,N-1} G^{-1}_{\bar V_{u_k},\bar V_{u_{k+1}}}(U_k) > a\right)\big|\bar V_s\right).\end{aligned}$$

Since the U_k's are i.i.d uniformly distributed random variables, we have

$$\begin{aligned}\mathbb{P}\left(\min_{s\le u\le t}\bar V_u > a\big|\bar V_s\right) &= \mathbb{E}\left(\prod_{k=0}^{N-1}\mathbb{P}\left(U_k > G_{\bar V_{u_k},\bar V_{u_{k+1}}}(a)\right)\big|\bar V_s\right)\\ &= \mathbb{E}\left(\prod_{k=0}^{N-1} G_{\bar V_{u_k},\bar V_{u_{k+1}}}(a)\big|\bar V_s\right),\end{aligned}$$

which gives the announced result. □

By using Proposition 1.4.1, we estimate the survival probability under full information, i.e.,

$$\mathbb{P}\left(\inf_{s\le u\le t}\bar V_u > a\big|\bar V_s = v\right),$$

by the following Monte Carlo procedure:

- *Time grid specification.* Fix $u_0 = s < u_1 < u_2 < \cdots < t = u_N$, the set of $N+1$ points for the (discrete time) Euler scheme in the interval $[s,t]$;
- *Trajectories simulation.* Starting from v and having fixed M (number of Monte Carlo simulations), for $j = 1,\dots,M$, simulate the discrete path $(\bar V^j_{u_k})_{k=0,\dots,N}$;
- *Computation of the survival probability.* For $j=1,\dots,M$, compute (recall that, for every j, $\bar V^j_{u_0} = v$)

$$p^j_{s,t}(v;a) := \prod_{k=0}^{N-1} G_{\bar V^j_{u_k},\bar V^j_{u_{k+1}}}(a). \tag{1.4.5}$$

- *Monte Carlo procedure.* Finally, apply the Monte Carlo paradigm and obtain the following approximating value

$$\mathbb{P}\left(\inf_{s\le u\le t}\bar V_u > a\Big|\bar V_s = v\right) \approx \frac{\sum_{j=1}^M p^j_{s,t}(v;a)}{M}. \tag{1.4.6}$$

As a consequence, combining formulas (1.3.16) and (1.4.6) leads to the following hybrid Monte Carlo - Optimal quantization formula on the set $\{\tau > s\}$

$$\mathbb{P}\left(\inf_{s\le u\le t} V_u > a\Big|\mathcal{F}^S_s\right) \approx \frac{1}{M}\sum_{j=1}^M\sum_{i=1}^{N_n} p^j_{s,t}(v^i_n;a)\ \widehat\Pi^i_{y,n} \tag{1.4.7}$$

where $p^j_{s,t}(\cdot\,;a)$ was introduced in (1.4.5).

1.5 The error analysis

We now focus on the analysis of the error induced by approximating $\mathbb{P}\left(\inf_{s\leq u\leq t} V_u > a \middle| \mathcal{F}_s^S\right)$ by

$$\frac{1}{M}\sum_{j=1}^{M}\sum_{i=0}^{N_n} p_{s,t}^j(v_n^i; a)\ \widehat{\Pi}_{y,n}^i.$$

We distinguish three types of error. The first error is induced by the approximation of the filter $\Pi_{y,n}$, appearing in Equation (1.3.2), by $\widehat{\Pi}_{y,n}$, defined in (1.3.10). This error was already discussed in Section 1.3.3 in a general setting. The second one is the error deriving from the approximation of

$$\mathbb{P}\left(\inf_{s\leq u\leq t} V_u > a \middle| V_s = v\right) \quad \text{by} \quad \mathbb{P}\left(\inf_{s\leq u\leq t} \bar{V}_u > a \middle| \bar{V}_s = v\right),$$

where $\bar{V}$ is the (continuous) Euler scheme relative to the process V (in the Black-Scholes model, there is no need to use an Euler scheme, since Equation (1.2.1) admits an explicit solution). The last one is the error arising from the approximation of the survival probability under full information by means of Monte Carlo simulations.

We now discuss the second and third errors.

$\triangleright$ **Error induced by the Euler scheme**. We here refer to Gobet [15], in which the author starts by investigating the case of a one-dimensional diffusion and to the successive related article Gobet [16] for the multidimensional case. In the two papers the considered diffusion has homogeneous coefficients b and σ. We start by recalling here some important convergence results we find therein, we will then adapt these results to our case.

Suppose that X is a diffusion taking values in $\mathbb{R}$, with $X_0 = x$, and define τ' as the first exit time from an open set $D \subset \mathbb{R}$:

$$\tau' := \inf\left\{u \geq 0 : X_u \notin D\right\}.$$

Let τ'_c denotes the exit time from the domain D of the continuous Euler process $\bar{X}$. In order to give the error bound in the approximation of $\mathbb{E}_x\left(\mathbb{1}_{\{\tau'>t\}} f(X_t)\right)$ by $\mathbb{E}_x\left(\mathbb{1}_{\{\tau'_c>t\}} f(\bar{X}_t)\right)$ the following hypotheses are needed:

(**H1**) b is a $\mathcal{C}_b^\infty(\mathbb{R},\mathbb{R})$ function and σ is in $\mathcal{C}_b^\infty(\mathbb{R},\mathbb{R})$,

(**H2**) there exists $\sigma_0 > 0$ such that $\forall x \in \mathbb{R}, \sigma(x)^2 \geq \sigma_0^2$ (*uniform ellipticity*),

(**H3**) $\mathbb{P}_x\left(\inf_{t\in[0,T]} X_t = a\right) = 0$.

The following proposition states that, under Hypothesis (**H3**), the approximation error goes to zero as the number of time discretization steps goes to infinity.

Proposition 1.5.1 (**Convergence**). *Suppose that b and σ are Lipschitz, $D = (a, +\infty)$ and that (**H3**) holds. If $f \in \mathcal{C}_b^0(\bar{D},\mathbb{R})$ then,*

$$\lim_{N\to+\infty}\left|\mathbb{E}_x[\mathbb{1}_{\{\tau'_c>T\}} f(\bar{X}_T)] - \mathbb{E}_x[\mathbb{1}_{\{\tau'>T\}} f(X_T)]\right| = 0.$$

Note that in the homogeneous case, when $D = (a, +\infty)$, a sufficient condition in order for (**H3**) to hold is (see Gobet [15], Prop. 2.3.2)

$$\sigma(X_0) \neq 0. \tag{1.5.1}$$

On the other hand, the rate of convergence is given by the following

Proposition 1.5.2 (Rate of convergence). *Under Hypotheses (**H1**) and (**H2**), if $f \in \mathcal{C}_b^1(\bar{D}, \mathbb{R})$, then there exists an increasing function $K(T)$ such that*

$$\left|\mathbb{E}_x[\mathbb{1}_{\{\tau_c' > T\}} f(\bar{X}_T)] - \mathbb{E}_x[\mathbb{1}_{\{\tau' > T\}} f(X_T)]\right| \leq \frac{1}{\sqrt{N}} K(T) ||f||_D^{(1)},$$

where $||f||_D^{(1)} = \sum_{j=0}^{1} \sup_{x \in D} |f^{(j)}(x)|$.

Remark 1.5.1. It can be noticed, by generalizing the proof of Propositions 2.3.1, 2.4.3 and 2.3.2 in Gobet [15], that the two previous propositions and condition (1.5.1) still hold when the diffusion coefficients are in-homogeneous, as in our setting, by replacing Hypotheses (**H1**), (**H2**), (**H3**) by (**I**) and (**J**) :

(**I**) b and σ are $\mathcal{C}_b^\infty$ functions with respect to both arguments t and v, with uniformly bounded partial derivatives with respect to v,

(**J**) σ is uniformly elliptic, i.e., $\exists \alpha > 0$ such that $\sigma^2(t, v) \geq \alpha, \forall (t, v) \in [0, T] \times \mathbb{R}$ and $\sigma(0, v_0) \neq 0$.

$\triangleright$ **Error induced by Monte Carlo approximation**. This error comes from the estimation of $\mathbb{P}\left(\min_{s \leq u \leq t} \bar{V}_u > a \middle| \bar{V}_s = v_s^i\right) = \mathbb{E}\left(\prod_{k=0}^{N-1} G_{\bar{V}_{u_k}, \bar{V}_{u_{k+1}}}(a) \middle| \bar{V}_s = v_s^i\right)$, for every $i = 1, \cdots, N_s$, by

$$\frac{\sum_{j=1}^{M} p_{s,t}^j(v_s^i; a)}{M},$$

where $p_{s,t}^j(\cdot\, ; a)$ was defined in (1.4.5). We have, for every $i = 1, \cdots, N_s$,

$$\left\|\mathbb{E}\Big(\prod_{k=0}^{N-1} G_{\bar{V}_{u_k}, \bar{V}_{u_{k+1}}}(a) \big| \bar{V}_s = v_s^i\Big) - \frac{\sum_{j=1}^{M} p_{s,t}^j(v_s^i; a)}{M}\right\|_2 = \mathcal{O}\Big(\frac{1}{\sqrt{M}}\Big). \tag{1.5.2}$$

By adapting the previous results to our case, namely by identifying V with X and S with Y, one deduces an error bound for the estimation of $\Pi_{y,n} F(s, t, \cdot)$ by $\widehat{\Pi}_{y,n} F_{MN}(s, t, x)$, where n is the dimension of the observation vector y (or, equivalently, $n+1$ is the number of points in the time discretization grid of the interval $[0, s]$) and where $F_{MN}(s, t, x)$ is a Monte Carlo estimation of $F(s, t, \cdot)$ of size M, based on a time discretization grid, between s and t, of size $N + 1$. We state, then, the main result of this section.

Theorem 1.5.1. *Suppose that the transition operators of V_k given V_{k-1}, $k = 1, \dots, n$, satisfy Assumption **(A1)** and that the conditional law of S_k given (V_{k-1}, S_{k-1}, V_k) admits a density satisfying **(A2)**. Suppose, furthermore, that the coefficients b and σ of V fulfill Hypotheses **(H1)-(H2)**. Then*

$$\begin{aligned} |\Pi_{y,n}F(s,t,\cdot) - \widehat{\Pi}_{y,n}F_{MN}(s,t,\cdot)| &\leq \frac{\mathrm{K}_g^n}{\phi_n(y) \vee \widehat{\phi}_n(y)} \sum_{k=0}^{n} \mathrm{B}_k^n(F(s,t,\cdot), y, p)\ \|V_k - \widehat{V}_k\|_p \\ &+ \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{M}}\right), \end{aligned}$$

where n is the dimension of the observation vector y, N stands for the size of the time discretization grid for the Euler scheme from s to t and M is the number of Monte Carlo trials. Furthermore, $K_g^n, \phi_n(y), \widehat{\phi}_n(y)$ and $B_k^n, k = 0, \dots, n$, were introduced in Theorem 1.3.1.

Remark 1.5.2. (About the hypotheses of Theorem 1.5.1) We consider the case when V is a time homogeneous diffusion.
$\triangleright$ Concerning Assumption **(A2)** (i), the conditional density functions g_k given in Equation (1.3.14) are bounded on $\mathbb{R} \times (0, +\infty) \times \mathbb{R} \times (\varepsilon, +\infty)$ for every $\varepsilon > 0$. The Lipschitz condition **(A2)** (ii) holds.
$\triangleright$ If we suppose that the coefficients b and σ of the diffusion V are Lipschitz, we show, by using the Euler scheme relative to V, that the transition operators P_k, defined by $P_k f(x) := \mathbb{E}(f(V_k)|V_{k-1} = x)$, satisfy

$$|P_k f(x) - P_k f(x')| \leq C[f]_{\mathrm{Lip}}|x - x'|$$

for every Lipschitz function f with Lipschitz constant $[f]_{\mathrm{Lip}}$. Then Hypothesis **(A1)** holds true.
$\triangleright$ As concerns the Lipschitz property of the function $F(s,t,\cdot)$, it follows from Proposition 2.2.1 in Gobet [15], in the case when the coefficients of the diffusion satisfy Hypotheses (**H1**) - (**H2**) and for $t > s$.

Proof (of Theorem 1.5.1). We have

$$\begin{aligned} |\Pi_{y,n}F(s,t,\cdot) - \widehat{\Pi}_{y,n}F_{MN}(s,t,\cdot)| &\leq |\Pi_{y,n}F(s,t,\cdot) - \widehat{\Pi}_{y,n}F(s,t,\cdot)| \\ &+ |\widehat{\Pi}_{y,n}F(s,t,\cdot) - \widehat{\Pi}_{y,n}F_{MN}(s,t,\cdot)|. \end{aligned}$$

The error bound of the first term on the right-hand side of the above inequality is given by Theorem 1.3.1. As concerns the second term, we have

$$\begin{aligned} |\widehat{\Pi}_{y,n}F(s,t,\cdot) - \widehat{\Pi}_{y,n}F_{MN}(s,t,\cdot)| &= \Big|\sum_{i=1}^{N_s} \widehat{\Pi}_{y,n}^i (F(s,t,v_s^i) - F_{MN}(s,t,v_s^i))\Big| \\ &\leq \sup_{v \in \mathbb{R}} |F(s,t,v) - F_{MN}(s,t,v)| \sum_{i=1}^{N_s} \widehat{\Pi}_{y,n}^i \\ &= \sup_{v \in \mathbb{R}} |F(s,t,v) - F_{MN}(s,t,v)|. \end{aligned}$$

On the other hand, we have for every $v \in \mathbb{R}$

$$
\begin{aligned}
|F(s,t,v) - F_{MN}(s,t,v)| \quad &\leq \quad \Big| \mathbb{P}_v(\tau > t) - \mathbb{E}_v\Big(\prod_{k=0}^{N-1} G_{\bar{V}_{u_k},\bar{V}_{u_{k+1}}}(a)\Big)\Big| \\
&+ \quad \Big\| \mathbb{E}_v\Big(\prod_{k=0}^{N-1} G_{\bar{V}_{u_k},\bar{V}_{u_{k+1}}}(a)\Big) - \frac{\sum_{j=1}^{M} p^j_{s,t}(v;a)}{M} \Big\|_2 .
\end{aligned}
$$

We then deduce from Proposition 1.5.2 and from Equation (1.5.2) that

$$
|F(s,t,v) - F_{MN}(s,t,v)| \leq \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{M}}\right),
$$

which completes the proof since the error bounds do not depend on v. □

1.6 Numerical results

In the numerical experiments we deal with the estimation of the credit spread for zero coupon bonds. For simplicity we suppose that investors are risk neutral, so that here we work under a risk neutral probability $\mathbb{Q}$. We also suppose that the market is complete (remark that V is not a traded asset, then it will be necessary to complete the market), i.e., that $\mathbb{Q}$ is unique.

In this section S represents the stock price of an asset issued by the firm. We fix s and, given the observations of S from 0 to s, we estimate the spread curve for different maturities t ($t > s$). The credit spread for zero coupon bonds is the difference in yield between a corporate bond and a risk-less bond (Treasury bond) with the same characteristics. It can be seen as a measure of the riskiness relative to a corporate bond, with respect to a risk-free bond. If we suppose for simplicity that the face value is equal to 1 and the recovery rate is zero, the credit spread under partial information from time s to maturity t, $\mathbb{S}(s,t)$, equals (see, e.g., Bielecki and Rutkowski [3] and Coculescu, Geman and Jeanblanc [7])

$$
\mathbb{S}(s,t) = -\frac{\log\left(\mathbb{Q}(\inf_{s<u\leq t} V_u > a | \mathcal{F}^S_s)\right)}{t-s}.
$$

This section is divided into two parts. We first focus on simulations: having arbitrarily fixed the model parameters, we simulate different trajectories of S and we compute, in two examples, the credit spreads for zero coupon bonds. The second part is devoted to calibration.

1.6.1 Simulation

We consider two models for the dynamics of the firm value V: the Black-Scholes one and the CEV (Constant Elasticity of Variance) model. In both cases we fix $s = 1$ and, given the simulated trajectory of S from 0 to 1, we estimate the spreads $\mathbb{S}(1,t)$ for different maturities t varying 0.1 by 0.1 from 1.1 to 11 (the time unit is expressed in years).

▷ *The Black-Scholes model.* We consider the following model for the firm's value and the observed process' dynamics:

$$
\begin{cases}
dV_t &= V_t(\mu dt + \sigma dW_t), \qquad V_0 = v_0, \\
dS_t &= S_t(r dt + \sigma dW_t + \delta d\bar{W}_t), \quad S_0 = v_0,
\end{cases}
\tag{1.6.1}
$$

so that

$$\frac{dS_t}{S_t} = \frac{dV_t}{V_t} + (r - \mu)dt + \delta d\bar{W}_t. \tag{1.6.2}$$

For simplicity, we set $r = \mu = 0.03$, meaning that the return on S is the return on V affected by a noise (it is important to note that since V is not traded in the market, the return on V is not necessarily equal to the interest rate r). The other parameters values are $\sigma = 0.05$, $\delta = 0.1$ and $v_0 = 86.3$. The barrier a is fixed to 76.

Notice that when V evolves following a Black-Scholes dynamics, the quantization grids of the firm value process can be derived instantaneously from optimal quadratic functional quantization grids of the Brownian motion, that can be downloaded from the website `www.quantize.math-fi.com` (for more information about functional quantization for numerics see, e.g., Pagès and Printems [28]). This drastically cuts down the computational cost and allows working with grids of higher size. Furthermore, the transition probabilities are estimated using Equation (1.3.15) and the survival probabilities $F(s, t, v_n^i), i = 1, \cdots, N_n$ (under $\mathbb{Q}$) in Equation (1.3.16) are computed via Equation (1.4.2). We then obtain a single spread estimate in one second.

We set the number n of discretization points over $[0, s]$ equal to 50 and for every $k = 1, \cdots, n$, the quantization grid size N_k is set to 966, with $N_0 = 1$. Numerical results are presented in Figures 1.1 and 1.2. Figure 1.1 is relative to the partial information case, where three simulated trajectories of the observable process S and the corresponding credit spreads are depicted. Figure 1.2 treats the full information case, where we suppose that we directly observe V. In three examples, corresponding to three different trajectories of V (left hand side of Figure 1.2), we compute the corresponding credit spreads (right hand side of Figure 1.2).

As a theoretical result, we deduce from (1.6.1) (with $\mu = r$) that

$$S_t = V_t e^{-\frac{1}{2}\delta^2 t + \delta \bar{W}_t}.$$

The correlation coefficient is, then, given for every $t \geq 0$ by

$$\rho(t) := \sqrt{\frac{e^{\sigma^2 t} - 1}{e^{(\sigma^2 + \delta^2)t} - 1}},$$

meaning that the firm value V is positively correlated to the observation process S. Notice that when $\sigma < \delta$, $\rho(t)$ is a strictly decreasing function that goes to 0 as t goes to infinity. This tells us that the *a posteriori* information on V given S decreases as the maturity t increases. This is what we observe in the spreads' curves from Figure 1.1 and Figure 1.2, since for large maturities the spreads values almost coincide for analogous trajectories (e.g., for trajectories `SU` and `VU`).

Looking at the figures, first of all, we notice that the short term spreads under partial information (Figure 1.1), being the default time totally inaccessible, do not vanish, as it is the case in the full information model. Moreover, since V_t and S_t are positively correlated, it is expected that the more the trajectory of S behaves "badly", the higher the short term spreads are, as shown in Figure 1.1.

In the full information setting (Figure 1.2), on the other hand, the short term spreads are always equal to zero, but in "bad" situations (for example in the case of trajectory `VD` on

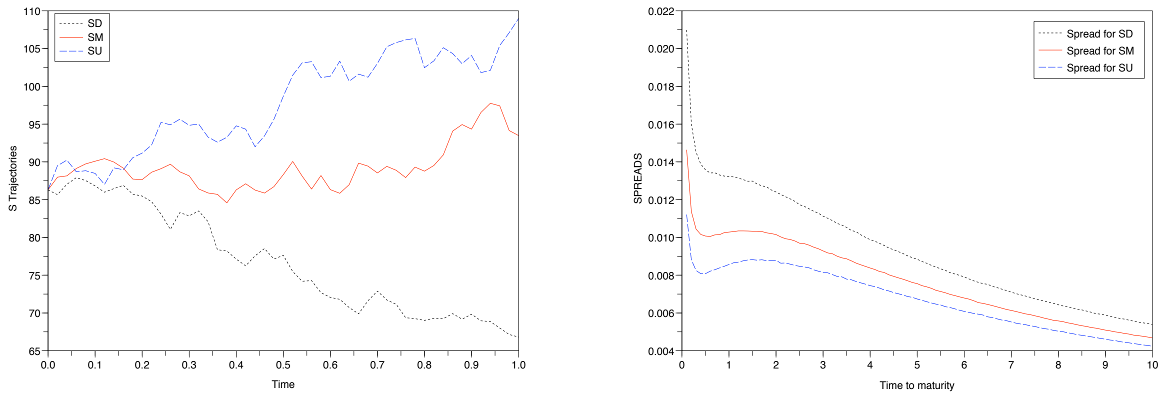


Figure 1.1: Three trajectories of the observed process S (on the left) and the corresponding spreads (on the right).

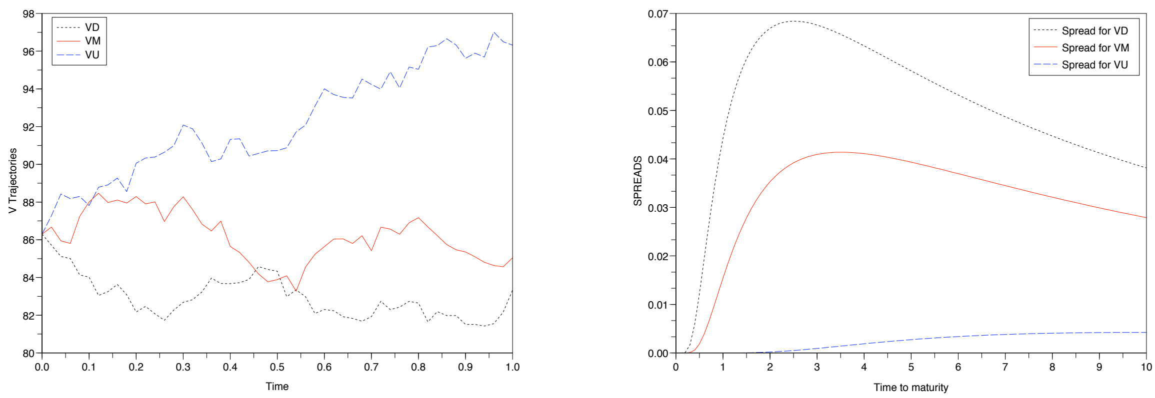


Figure 1.2: Three trajectories of the value process in the full information case (on the left) and the corresponding spreads (on the right).

the left-hand side of Figure 1.2) the medium term spreads can be higher than in the partial information model.

$\triangleright$ *The CEV model.* We suppose now that the firm's value and the observed process' dynamics are given by

$$\begin{cases} dV_t &= V_t(\mu dt + \gamma V_t^{\beta} dW_t), \qquad V_0 = v_0, \\ dS_t &= S_t(rdt + \sigma dW_t + \delta d\bar{W}_t), \quad S_0 = v_0, \end{cases} \tag{1.6.3}$$

where $\mu = r = 0.03$, $\gamma = 744.7$ (it is chosen so that the initial volatility equals 0.10), $\beta = -2$ (notice that in this case one of the characteristics of the model is the *leverage* effect: a firm's value process increase implies a decrease in the variance of the price process' return), $\sigma = 0.05, \delta = 0.1, v_0 = 86.3$. The barrier a is set here to be equal to 79.

For numerics, the number n of discretization points over $[0, s]$ equals 50 and, for every $k = 1, \cdots, n$, the quantization grid size N_k is set to 60, with $N_0 = 1$. Here, since we cannot obtain the quantization grids from the ones of the Brownian motion, we obtain the optimal grids by carrying out 80 Lloyd's I procedures. The number of Euler time discretization steps N equals 50 for t varying 0.1 by 0.1 from 1.1 to 3.0 and N is set to $N = 100$ for t varying 0.1 by 0.1 from 3.1 to 11.0. The number of Monte Carlo trials M is set to 100000.

Numerical results are presented in Figure 1.3, where three simulated trajectories of the observable process S and the corresponding spreads are depicted. We first notice that the spreads in this example are higher than the ones in the previous example. This is due to the fact that in this case the observed process S is more volatile, as it can be seen from Figure 1.3, compared to Figure 1.1.

Secondly, we remark, as in the previous example, that the more the trajectory of S behaves "badly", the higher the short term spreads are, as shown in Figure 1.3 on the right.

Moreover, notice that the spread curves corresponding to the two worst S trajectories seem to cross, however a zoom in the graph shows that it is not the case and that the spreads curve for `SD CEV` is always above the one for `SM CEV`. This can be explained by noticing that the model we use keeps the memory of all the observed path and that the trajectory `SD CEV` is globally worse than the trajectory `SM CEV`.

Remark 1.6.1. (a) The most important fact from the numerical point of view is that, as soon as the process V is quantized over $[0, s]$, the survival probability $\mathbb{Q}(\inf_{s \le u \le t} V_u > a | \mathcal{F}_s^S)$ is estimated for every maturity $t > s$ without modifying the optimal quantization grid of V.
(b) As expected, in both the Black-Scholes and the CEV models, numerical tests confirm that the spread increases as the barrier a ($a < v_0$) tends to v_0.

1.6.2 Calibration issues

For calibration to real data, we consider the Black-Scholes model

$$\begin{cases} dV_t &= V_t(\mu dt + \sigma dW_t), \qquad\qquad V_0 = v_0, \\ dS_t &= S_t(rdt + \sigma dW_t + \delta d\bar{W}_t), \quad S_0 = s_0, \end{cases} \tag{1.6.4}$$

even if the methodology presented below may be applied to other models. The calibration has been done in two steps. The first step, related to the "learning phase", consists

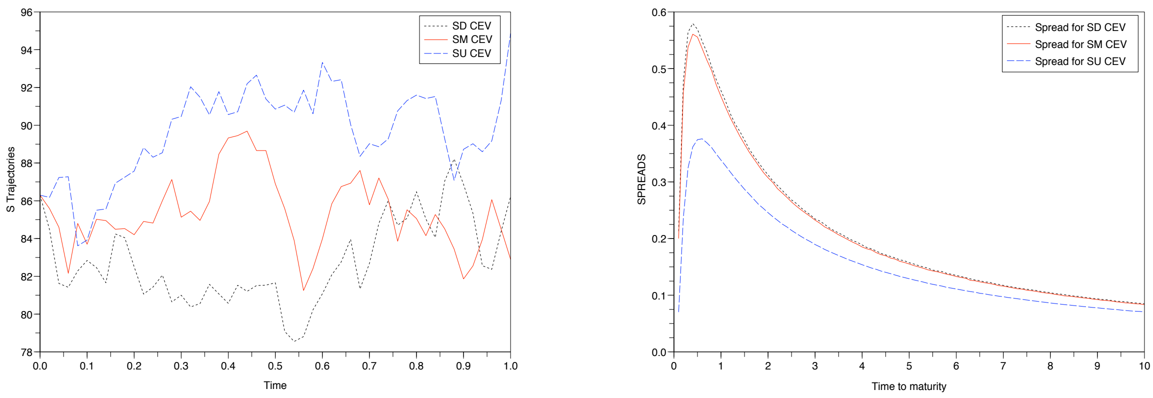


Figure 1.3: Three trajectories of the observed process S in the CEV model (on the left) and the corresponding spreads (on the right).

in calibrating the parameters of the stock price S in the observation interval $[0, s]$. The remaining parameters are, then, calibrated from the market data for credit spreads. Recall that the quantization grids of the firm value process can be derived from the optimal quadratic functional quantization grids of the Brownian motion.

$\triangleright$ *Calibration of S's parameters.* We work on JP Morgan weekly stock prices data (available on the website `www.finance.yahoo.com/`) for the period 03/22/2009 - 03/22/2010, corresponding in our setting to the observation time interval $[0, s]$ with $s = 1$. The data set is of size 53 (see Figure 1.5 on the left) and each considered stock price $S_i, i = 0, \cdots, 52$, is computed as the average between the bid and ask prices. The considered interest rate $r = 0.51\%$ is obtained as the average of the three-months U.S. Libor rates in the period March 2009 - March 2010. Given the above model for S, one can estimate the parameter $\theta := \sqrt{\sigma^2 + \delta^2}$ using elementary statistical theory. The obtained estimation $\widehat{\theta}$ from real data is $\widehat{\theta} = 0.2496$.

Before dealing with the second step of the calibration we study the impact of the noise parameter $\delta \in (0, \widehat{\theta})$ on the credit spread $\big($once δ is fixed, $\sigma = \sqrt{\widehat{\theta}^2 - \delta^2}\,\big)$. For this purpose, we set $\mu = r$ to have

$$\frac{dS_t}{S_t} = \frac{dV_t}{V_t} + \delta d\bar{W}_t. \tag{1.6.5}$$

We plot in Figure 1.4 the term structure of credit spread $\mathbb{S}(1, t)$ for t varying 0.1 by 0.1 from 1.1 to 6 and for $\delta = \{0.05, 0.10, 0.15, 0.20\}$. The considered values for v_0 and a are $v_0 = 2,079,188,000\$$ and $a = 1,908,994,000\$$. They represent, respectively, the total assets value and the total liabilities balance sheet value of the firm at the end of March 2009 (both available on `www.finance.yahoo.com/`). In this numerical implementations we have set the number of discretization points over $[0, 1]$ to 53 and the quantization grid size $N_k = 966$, for $k = 1, \cdots, 53$ and $N_0 = 1$. Numerical results show that the spreads increase as the noise parameter δ increases. This intuitively comes from Equation (1.6.5), since the more δ is large, the more the information on S is noisy and so the higher is the risk perception of the investor. Moreover, for small values of δ (as, for example, for $\delta = 0.05$), the term structure of credit spread has a form similar to the one we found in the complete information case (see Figure 1.2). Then, varying δ may allow us to obtain a rich set of different forms of the credit spread term structure.

We now focus on the calibration to real data.

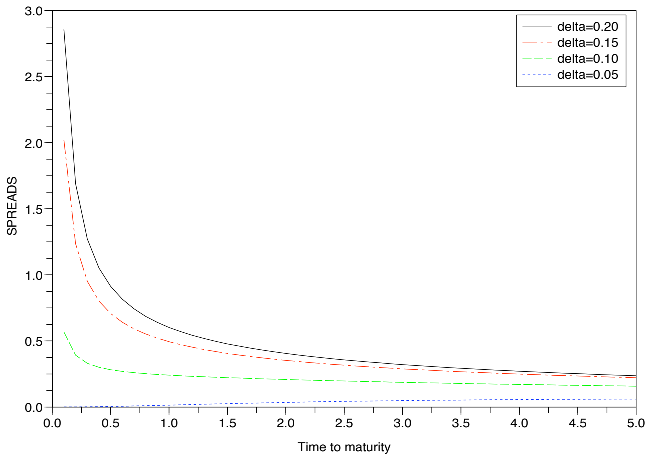


Figure 1.4: Spreads computed with different values of δ.

$\triangleright$ *Calibration.* As previously remarked, the parameters values v_0 and a are known and they correspond to the total assets value and to the total liabilities value of the firm at the end of March 2009, namely $v_0 = 2,079,188,000\$$ and $a = 1,908,994,000\$$. Furthermore, we set the initial stock price value and the interest rate to, respectively, $s_0 = 27,365$ and $r = 0.51\%$.

We calibrate μ and δ on the credit spreads (for zero coupon bonds) market data, that is, given a set of credit spreads data $\{s_{t_i}, i = 1, \cdots, 4\}$, at time $s = 1$ and for different maturities $t_1 = 7/12; t_2 = 11/12; t_3 = 1; t_4 = 13/12$, we find $(\mu^\star, \delta^\star)$ that minimize the quadratic error

$$\sum_{i=1}^{4} \left(\mathbb{S}(1, t_i) - s_{t_i}\right)^2.$$

The market data $\{s_{t_i}, i = 1, \cdots, 4\}$ are obtained as the difference between riskless Treasury bond yields and JP Morgan zero coupon bonds (Medium Term Note zero coupon SER E principal protected bond) yields. Since there is a mismatch between the maturities of corporate and Treasury bonds in the sample, we interpolate the riskless yields in order to have a continuum of maturities and we compute the spreads for all the t_i's. For the calibration we restricted our attention to the domain $[-0.1, 0.1] \times [0.01, 0.1]$. The optimal values obtained are $(\mu^\star, \delta^\star) = (0.03, 0.075)$ and the corresponding credit spread term structure over three years is depicted in Figure 1.5, right-hand side. The quadratic error equals 3.5×10^{-3}. Notice that the most challenging task in the calibration phase is the collection of real data, because zero coupon corporate bond prices at a fixed time s, issued by the same firm and with identical features, are only given for a small number of different maturities $t > s$. This is why the used set of data is of small size.

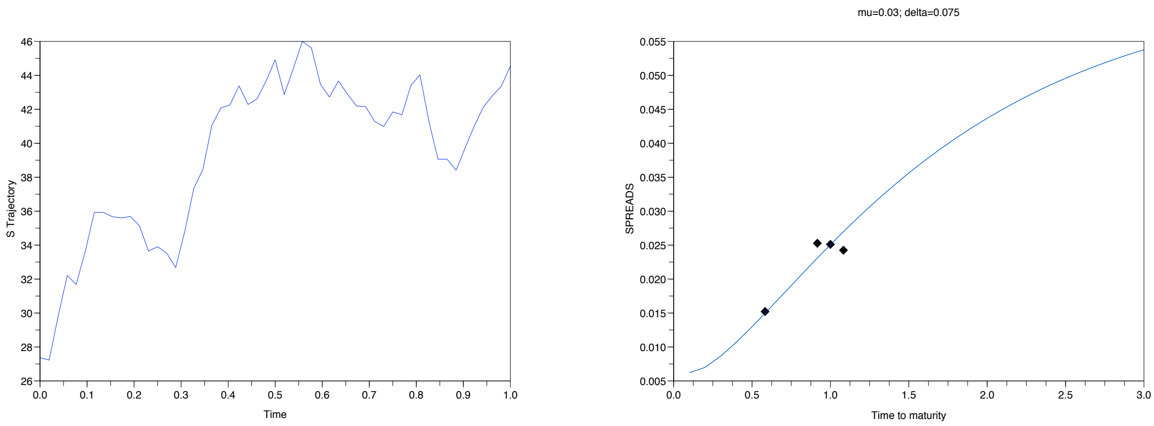


Figure 1.5: JP Morgan weekly stock prices over the period 03/22/2009 - 03/22/2010 (on the left) and corresponding credit spreads curve over three years obtained for $(\mu^\star, \delta^\star) = (0.03, 0.075)$ calibrated to market data (black square dots).

Bibliography

[1] A. Bain and D. Crişan. *Fundamentals of stochastic filtering.* Springer, 2009.

[2] P. Baldi. Exact asymptotics for the probability of exit from a domain and applications to simulations. *The Annals of Applied Probability*, 23(4):1644–1670, 1995.

[3] T.R. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging.* Springer Finance. Springer, 2004.

[4] A.N. Borodin and P. Salminen. *Handbook of Brownian motion - Facts and Formulae, 2nd edition.* Birkhauser-Verlag, 2002.

[5] U. Cetin, R. Jarrow, P. Protter, and Y. Yildirim. Modeling credit risk with partial information. *The Annals of Applied Probability*, 14(3):1167–1178, 2004.

[6] M. Chesney, M. Jeanblanc, and M Yor. *Mathematical Methods for Financial Markets.* Springer, 2009.

[7] D. Coculescu, E. Geman, and M. Jeanblanc. Valuation of Default Sensitive Claims under Imperfect Information. *Finance and Stochastics*, 12(2):195–218, 2008.

[8] C. Cudennec. *Evaluation de Dette Risquée en information imparfaite.* PhD thesis, ENSAE, 1999.

[9] D. Duffie and D. Lando. Term Structures of Credit Spreads with Incomplete Accounting Information. *Econometrica*, 69(3):633–664, 2001.

[10] R. Frey and W.J. Runggaldier. *Nonlinear filtering in models for interest rate and credit risk.* Oxford University Press, 2009.

[11] A. Gersho and R. Gray. *Vector Quantization and Signal Compression.* Boston: Kluwer Academic Press., 1992.

[12] K. Giesecke. Default and information. *Journal of Economic Dynamics and Control*, 30(11):2281–2303, 2006.

[13] K. Giesecke and L. Goldberg. Forecasting default in the face of uncertainty. *Journal of Derivatives*, 12(1):14–25, 2004.

[14] P. Glasserman. *Monte Carlo Methods in Financial Engineering.* Springer-Verlag: New York, 2003.

[15] E. Gobet. *Schémas d'Euler pour diffusion tuée. Application aux options barrière.* PhD thesis, Université Paris VII, 1998.

[16] E. Gobet. Weak approximation of killed diffusion using Euler schemes. *Stochastic Processes and their Applications*, 87:167–197, 2000.

[17] S. Graf and H. Luschgy. *Foundations of Quantization for Probability Distributions*, volume 1730 of *Lecture Notes in Mathematics.* Springer, 2000.

[18] R.A. Jarrow and P. Protter. Structural versus reduced form models: a new information based perspective. *Journal of investment management*, 2(2):1–10, 2004.

[19] N. Kahalé. Analytic crossing probabilities for certain barriers by Brownian motion. *Preprint*, 2007.

[20] S. Kusuoka. A remark on default risk models. *Advances in Mathematical Economics*, 1:69–82, 1999.

[21] V. Linestky. Computing hitting time densities for CIR and OU diffusions: applications to mean-reverting models. 2004.

[22] V. Linestky. The spectral decomposition of the option value. *International Journal of Theoretical and Applied Finance*, 7(3):337–384, 2004.

[23] H. Nakagawa. A Filtering Model on Default Risk. *J. Math. Sci. Univ. Tokyo*, 8(1):107–142, 2001.

[24] G. Pagès. A space vector quantization method for numerical integration. *Journal of Computational and Applied Mathematics*, 89:1–38, 1998.

[25] G. Pagès. *Introduction to numerical probability for finance.* 2008.

[26] G. Pagès and H. Pham. Optimal quantization methods for nonlinear filtering with discrete time observations. *Bernoulli*, 11(5):893–932, 2005.

[27] G. Pagès and J. Printems. Optimal quadratic quantization for numerics: the Gaussian case. *Monte Carlo Methods and Appl.*, 9(2):135–166, 2003.

[28] G. Pagès and J. Printems. Functional quantization for numerics with an application to option pricing. *Monte Carlo Methods and Appl.*, 11(4):407–446, 2005.

[29] H. Pham, W.J. Runggaldier, and A. Sellami. Approximation by quantization of the filter process and applications to optimal stopping problems under partial observation. *Monte Carlo Methods and Applications*, 11:57–82, 2005.

[30] D. Revuz and M. Yor. *Continuous martingales and Brownian motion, 3rd edition*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[31] A. Sellami. *Méthodes de quantification optimale pour le filtrage et applications à la finance.* PhD thesis, Paris: Université Paris Dauphine, 2005.

[32] A. Sellami. Comparative survey on nonlinear filtering methods: the quantization and the particle filtering approaches. *Journal of Statistical Computation and Simulation*, 78(2):93–113, 2008.

Part III

Portfolio optimization in a defaultable market under incomplete information

Chapter 2

Portfolio optimization in a defaultable market under incomplete information

This is a joint work with Prof. M. Jeanblanc and Prof. W.J. Runggaldier.

Abstract: we consider the problem of maximization of expected utility from terminal wealth in a market model that may be driven by a not fully observable factor process and that takes explicitly into account the possibility of default for the individual assets as well as contagion (direct and information induced) among them. It is a multinomial model in discrete time that allows for an explicit numerical solution. We discuss the solution within our defaultable and partial information setup, in particular we study its robustness. Numerical results are derived in the case of a log-utility function and they can be analogously obtained for a power utility function.

Keywords: Portfolio optimization, partial information, credit risk, dynamic programming (DP), robust solutions.

2.1 Introduction

Our study concerns the classical portfolio optimization problem of maximization of expected utility from terminal wealth when the assets, in which one invests, may default. We put ourselves in a context where the dynamics of the asset prices are affected by exogenous factor processes, some of which may have an economic interpretation, some may not but, most importantly, NOT all of them may be directly observable. In credit risk models factors are often used to describe contagion: "physical" and "information induced". Information induced contagion arises due to the fact that the successive updating of the distribution of the latent (not observable) factors in reaction to incoming default observations leads to jumps in the default intensity of the surviving firms (this is sometimes referred to as "frailty approach", see, e.g., Schönbucher [9]). As shown in Duffie *et al.* [5], unobservable factor processes are needed on top of observable covariates in order to explain clustering of defaults in historical credit risk data. In general, the formulation of a model under incomplete information on the factors allows for greater model flexibility, avoids a possible inadequate specification of the model itself, and the successive updating of the distribution of the unobserved factors (for constant factors one considers them from the Bayesian point of view as random variables) allows the model to "track the market", thus avoiding classical model calibration.

To keep the presentation at a possibly simple level, we shall consider only a single factor process that is supposed to be non directly observable and the observation history is given, in addition to the defaults, by the observed asset prices. Furthermore, we shall consider discrete time dynamics. With respect to continuous time models, this can be justified since trading actually takes place in discrete time. Moreover, a solution is easier to compute in discrete time and, while it is more difficult to obtain qualitative results than in continuous time, once an explicit numerical solution is obtained, one can evaluate its performance also with respect to alternative criteria via simulation.

The outline of the chapter is as follows. In Section 2.2 we describe our model and objective. The filter process, which allows for the transition from the partial information problem to a corresponding one under complete information, is studied in Section 2.3. Section 2.4 contains the main result on using Dynamic Programming to obtain the optimal investment strategy; we consider explicitly the log-utility case, but analogous results can be obtained for other utility functions, in particular power utility. The last Section 2.5 discusses numerical results from simulations that were performed in order to investigate the effect of shorting, as well as the robustness of the optimal strategy obtained for the partial information problem.

2.2 The model

Here we describe the model dynamics and the objective for our portfolio optimization. With a slight abuse of notation, in what follows we will use the subscript n to indicate the instant t_n. All vectors will be row vectors and $'$ will indicate transposition.

2.2.1 Model dynamics

Given a discrete time set $t_0 = 0 < t_1 < \cdots < t_N = T$, let us introduce a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ ($\mathbb{G}$ stands for "global filtration"), where $\mathbb{G} = (\mathcal{G}_n)_n$ and, in addition to a nonrisky asset with price S^0_n, $S^0_0 = 1$ (S^0_n is the price at time t_n), a set of M risky assets with prices S^m_n, $m = 1, \cdots, M$, that are subject to default, except for the first one, S^1. Both for the applications (generally one invests in a pool of assets containing at least one non defaultable asset), as well as for formal reasons (see Remark 2.3.2) it is convenient to consider investment in at least one default-free risky asset. Let τ^m be the exogenously given (i.e., independent of any other source of randomness in the market) default time of the $m-$th asset and consider the default indicator process

$$H_n := (H^1_n, \cdots, H^M_n), \quad n = 0, \cdots, N, \tag{2.2.1}$$

where

$$H^m_n := \mathbb{1}_{\{t_n \geq \tau^m\}}$$

is the default indicator for the $m-$th firm. The possible values of H_n are the $M-$tuples $h^p = (h^{p,1}, \cdots, h^{p,M})$ for $p = 1, \cdots, 2^{M-1}$ with $h^{p,m} \in \{0, 1\}$. Since S^1 is assumed to be default free, we have

$$H^1 \equiv 0.$$

Furthermore, we arrange the values h^p according to a listing

$$h^1, h^2, \cdots, h^{2^{M-1}}$$

whereby, typically, $h^1 = (0, 0, \cdots, 0)$ and $h^{2^{M-1}} = (0, 1, \cdots, 1)$.

We now let the dynamics of the asset prices be given by

$$\begin{cases} S^0_{n+1} = S^0_n(1 + r_n) \quad (\text{typically } r_n \equiv r) \\ \\ S^m_{n+1} = S^m_n \ \gamma^m(\xi_{n+1}) \ (1 - H^m_{n+1}), \quad S^m_0 = s^m_0, \quad m = 1, \cdots, M, \end{cases} \tag{2.2.2}$$

where ξ_n is a sequence of i.i.d. multinomial random variables with values in $\{\xi^1, \cdots, \xi^L\}$ and γ^m are positive measurable functions. Typically, $\gamma^m(\xi_{n+1}) \in (0, 1)$ when there is a downward movement in the dynamics of asset S^m during the period n, while $\gamma^m(\xi_{n+1}) > 1$ if the movement is an upward one. We want to point out that, while in our model the amplitude of the up- and downward movements may vary from asset to asset, in accordance with a common practice, in trinomial and multinomial price evolution models it is intended that, if $\gamma^m(\xi_{n+1}) > 1$ for one asset m, the same holds for all the other assets (analogously when $\gamma^m(\xi_{n+1}) < 1$). In vector form, we may then write

$$S_{n+1} = diag\,(S_n \gamma(\xi_{n+1}))\,(\underline{1} - H_{n+1})' =: I(S_n, \xi_{n+1}, H_{n+1}), \tag{2.2.3}$$

where $\operatorname{diag}(S_n\gamma(\xi_{n+1}))$ is an $M \times M$ diagonal matrix, with elements $S^m_n \gamma^m(\xi_{n+1})$, $m = 1, \ldots, M$. The price evolution is thus driven by (ξ_n, H_n), defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ as follows. Given a $\mathbb{G}-$adapted finite state Markov chain $(Z_n)_n$ with values $Z_n \in \{z^1, \cdots, z^J\}$, initial law μ and transition probability matrix

$$P^{ij} := \mathbb{P}(Z_n = z^j | Z_{n-1} = z^i), \quad \forall\, i, j \in \{1, \cdots, J\}, \ \forall n \ (\text{time homogeneous}), \tag{2.2.4}$$

the driving processes (ξ, H) are supposed to be independent, conditionally on Z, and their distribution is characterized by assigning

$$\begin{cases} p^\ell(z) & := & \mathbb{P}(\xi_n = \xi^\ell | Z_{n-1} = z), \ \ell = 1, \cdots, L, \\ \\ \rho^{p,q}(z) & := & \mathbb{P}(H_n = h^q | H_{n-1} = h^p, Z_{n-1} = z) \\ & & \qquad \forall p, q \in \{1, \cdots, 2^{M-1}\}, \end{cases} \tag{2.2.5}$$

where $n = 1, \cdots, N$. Notice that the dependence of $\rho^{p,q}$ on $\{Z_{n-1}\}$ allows to model contagion: only "physical" if Z_{n-1} is observed, and "information-induced" if Z_{n-1} is unobservable and its distribution is updated on the basis of the observed default state and of the defaultable asset prices.

2.2.2 Portfolios

To perform portfolio optimization, we evidently need to invest in the market and, for this purpose, we consider an investment strategy that may be defined either by specifying the number of units invested in the individual assets, namely $a_n = (a_n^0, a_n^1, \cdots, a_n^M)$ (a_n^m is the number of units of asset m held in the portfolio in period t_n), or, restricting the attention to positive portfolio values, by equivalently specifying the ratios invested in the individual assets. More precisely, we shall consider the following relationships, that differ slightly from the standard ones, for reasons that we shall explain below (see Remark 2.2.1) i.e.,

$$\phi_n^0 = \frac{a_{n+1}^0 S_n^0}{V_n^\phi}, \qquad \phi_n^m(1 - H_n^m) = \frac{a_{n+1}^m S_n^m}{V_n^\phi}, \ m = 1, \cdots, M, \tag{2.2.6}$$

where

$$V_n^\phi = V_n^a := \sum_{m=0}^M a_n^m S_n^m = \sum_{m=0}^M a_{n+1}^m S_n^m$$

is the (self-financing) portfolio value in period t_n. Notice that

$$\phi_n^0 = 1 - \sum_{m=1}^M \phi_n^m (1 - H_n^m)$$

so that, to define a self financing investment strategy $\bar{\phi}_n := (\phi_n^0, \phi_n^1, \cdots, \phi_n^M)$, it suffices to define $\phi_n := (\phi_n^1, \cdots, \phi_n^M)$.

It will be convenient to write the portfolio value at time t_{n+1} in terms of its value at time t_n and of the gain during the period n, namely

$$\begin{aligned} V_{n+1}^\phi = V_{n+1}^a &= V_n^a + a_{n+1}^0(S_{n+1}^0 - S_n^0) + \sum_{m=1}^M a_{n+1}^m (S_{n+1}^m - S_n^m) \\ &= V_n^a + a_{n+1}^0 S_n^0 r_n + \sum_{m=1}^M a_{n+1}^m S_n^m \left[\gamma^m(\xi_{n+1})(1 - H_{n+1}^m) - 1\right] \\ &= V_n^\phi + \phi_n^0 V_n^\phi r_n + \sum_{m=1}^M \phi_n^m V_n^\phi (1 - H_n^m) \left[\gamma^m(\xi_{n+1})(1 - H_{n+1}^m) - 1\right] \\ &= V_n^\phi \left\{(1 + r_n) + \sum_{m=1}^M \phi_n^m (1 - H_n^m) \left[\gamma^m(\xi_{n+1})(1 - H_{n+1}^m) - (1 + r_n)\right]\right\}. \end{aligned} \tag{2.2.7}$$

Remark 2.2.1. *With the given definitions (in particular, the presence of the factor $(1-H_n^m)$ in the definition of ϕ_n^m in Equation (2.2.6)) one has that investment in an asset automatically ceases as soon as it defaults. This implies the equivalence of the expressions for V_n^a and V_n^ϕ (namely, the next-to-last equality in Equation (2.2.7) indeed holds true).*

Assuming first that the factor process Z is observed by the investor, the definitions above also imply that we consider $(a_n)_{n\geq 0}$ to be a predictable process (a_0 is $\mathcal{G}_0$-measurable and a_n is $\mathcal{G}_{n-1}-$measurable, $n \geq 1$, meaning that investment decisions at time t_n are taken on the basis of the information available at time t_{n-1} and kept until time t_n, when new quotations are available), while $(\phi_n)_{n\geq 0}$ is adapted.

2.2.3 The partial information problem

In view of formulating our partial information problem, let the default history be given by the filtration $\mathcal{H}_n := \sigma\{H_\nu,\ \nu \leq n\}$. With this filtration, we can reexpress the global filtration as

$$\mathcal{G}_n = \mathcal{F}^\xi_n \vee \mathcal{H}_n \vee \mathcal{F}^Z_n, \quad n = 0, \cdots, N,$$

where $(\mathcal{F}^Z_n)_n$ and $(\mathcal{F}^\xi_n)_n$ denote, respectively, the natural filtration associated with Z and ξ, while, with $(\mathcal{F}^S_n)_n$ denoting the filtration given by the price observation history, the observation filtration (representing the information of an investor) is given by

$$\mathcal{F}_n = \mathcal{F}^S_n \vee \mathcal{H}_n \quad \subset \mathcal{G}_n, \quad n = 0, \cdots, N.$$

Having specified a utility function $u : \mathbb{R}^+ \to \mathbb{R}$, of class $\mathcal{C}^1$, increasing and strictly concave, that satisfies the usual Inada's conditions:

$$\lim_{x\to 0^+} u'(x) = +\infty \quad \text{and} \quad \lim_{x\to +\infty} u'(x) = 0,$$

we can now give the following

Definition 2.2.1. *A self financing investment strategy $\phi_n = (\phi^1_n, \cdots, \phi^M_n), n = 0, \cdots, N$, is called* admissible *in our partial information problem, and we write $\phi \in \mathcal{A}$, if, besides implicit technical conditions, it is $\mathcal{F}_n-$adapted and such that V^ϕ_n belongs to the domain $\mathbb{R}_+$ of $u(\cdot)$.*

Notice that, in general, the set of admissible strategies is non empty (e.g., in the log and power utility cases it contains the strategy of not investing in the risky assets) and it is a convex set that may be unbounded; by possibly bounding it (e.g., imposing that, at any time t_n, $\phi^m_n \geq -C, m = 1, \cdots, M$) it can be transformed into a set with compact closure (for details, in the log-utility case, see the proof of Theorem 2.4.1 below). We come now to define our

Problem: Given an initial wealth v_0, determine an admissible strategy ϕ^* such that

$$E\left[u\left(V^{\phi^*}_N\right)\right] \geq E\left[u\left(V^{\phi}_N\right)\right], \quad \forall\ \phi \in \mathcal{A}.$$

Our problem is a partial information problem in that the factor process Z cannot be observed; on the other hand, the investment strategy can depend only on observable quantities. The usual approach in this situation (see, e.g., Bensoussan [1], Bertsekas [2] and Van Hee [10], see also Corsi, Pham and Runggaldier [3] for a problem related to the one of the present paper) consists in transforming the partial observation problem into one under full information, by replacing the unobservable quantities Z_n by their conditional distributions, given the current observation history. These conditional distributions are the so-called *filter distributions* or just *filters* and they can be computed recursively, as we are going to show in the next section.

We conclude this section by recalling a fundamental result on the absence of arbitrage opportunities (AOA, see, e.g., Prop. 2.7.1 in Dana and Jeanblanc [4]).

Lemma 2.2.1. *If the above Problem has a solution, then there are no arbitrage opportunities. The converse also holds true, i.e., there is equivalence between the existence of an optimal solution and the AOA, in the case when the utility function u is strictly concave,* strictly *increasing and of class $\mathcal{C}^1$.*

2.3 The filter

Since the investment strategy ϕ is by definition $\mathbb{F}-$adapted, the information coming from observing (S, V, H) (namely the asset prices, the portfolio value, and the default state) is equivalent to that of observing just (S, H).

Defining $(S^n, H^n) := ((S_1, H_1), \cdots, (S_n, H_n))$, the filter distribution for Z at time t_n is the random vector $\Pi_n = (\Pi_n^1, \cdots, \Pi_n^J)$ with components

$$\Pi_n^j := \mathbb{P}\left(Z_n = z^j | \mathcal{F}_n\right) = \mathbb{P}\left(Z_n = z^j | (S^n, H^n)\right) \ , \ \ j = 1, \cdots, J \, ,$$

taking values in the J-simplex $\mathcal{K}_J \subset \mathbb{R}^J$ (here $|\cdot|_1$ denotes the l^1-norm)

$$\mathcal{K}_J = \left\{ x = (x^j) \in \mathbb{R}^J : x^j \geq 0, j = 1, \ldots, J \ \text{ and } |x|_1 = \sum_{j=1}^J x^j = 1 \right\}.$$

By applying the *recursive Bayes' formula*, one obtains, for $j = 1, \cdots, J$,

$$\begin{aligned}
\Pi_n^j &= \mathbb{P}\left(Z_n = z^j | S_n = s_n, H_n = h_n, (S^{n-1}, H^{n-1})\right) \propto \\
&\propto \textstyle\sum_{i=1}^J \mathbb{P}\left(Z_n = z^j, Z_{n-1} = z^i, S_n = s_n, H_n = h_n, (S^{n-1}, H^{n-1})\right) \propto \\
&\propto \textstyle\sum_{i=1}^J P^{ij} \ \mathbb{P}\left(S_n = s_n, H_n = h_n | Z_{n-1} = z^i, S_{n-1}, H_{n-1}\right) \ \Pi_{n-1}^i,
\end{aligned} \tag{2.3.1}$$

with the observation distribution (likelihood function) given by (recall that the processes ξ and H are conditionally independent given Z)

$$\begin{aligned}
&\mathbb{P}\left(S_n = s_n, H_n = h^q | Z_{n-1} = z^i, S_{n-1} = s_{n-1}, H_{n-1} = h^p\right) = \\
&\rho^{p,q}(z^i) \textstyle\sum_{\ell=1}^L p^\ell(z^i) \mathbb{1}_{\{s_n = I(s_{n-1}, \xi^\ell, h^q)\}} =: F(z^i; s_n, s_{n-1}, h^q, h^p)
\end{aligned} \tag{2.3.2}$$

where $I(s, \xi, h)$ was defined in (2.2.3).

Remark 2.3.1. *Since the model may not correspond exactly to reality, there may be no $\xi^\ell \in \{\xi^1, \cdots, \xi^L\}$ so that, for the actually observed values of s_{n-1} and s_n, one has $s_n = I(s_{n-1}, \xi^\ell, h^q)$. Following standard usage we shall then consider the value of ℓ for which $I(s_{n-1}, \xi^\ell, h^q)$ comes closest to the actually observed value of s_n ("nearest neighbor").*

Given the current observations (s_n, h_n) and the previous ones (s_{n-1}, h_{n-1}), setting

$$F(s_n, s_{n-1}, h_n, h_{n-1}) := diag\left(F(z; s_n, s_{n-1}, h_n, h_{n-1})\right), \tag{2.3.3}$$

which is a $J \times J$ diagonal matrix with elements $F(z^i; s_n, s_{n-1}, h_n, h_{n-1}), i = 1, \cdots, J$, the recursions (2.3.1) can be expressed in vector form as

$$\begin{cases} \Pi_0' = \mu \qquad \text{and, for } n \geq 1, \\ \\ \Pi_n' = \dfrac{P' \, F(s_n, s_{n-1}, h_n, h_{n-1}) \, \Pi_{n-1}'}{\left| P' \, F(s_n, s_{n-1}, h_n, h_{n-1}) \, \Pi_{n-1}' \right|_1} =: \bar{F}\left(\Pi_{n-1}, s_n, s_{n-1}, h_n, h_{n-1}\right). \end{cases} \tag{2.3.4}$$

Remark 2.3.2. *By having assumed that at least one asset in the market is default free, the filter is well defined at every time step. Indeed, if we had considered only defaultable assets, in the case of default of all assets by time t_n we would have found $S_n = (0, \cdots, 0)$ and we would have lost all the information on ξ_n necessary to update the filter.*

2.4 Dynamic Programming for the "equivalent full information problem"

Under full information corresponding to $\mathbb{G}$ the tuple (S, V, H, Z) is Markov. In the full information setting equivalent to the partial information problem, the process Z has to be replaced by the filter process Π. From (2.3.1) it is easily seen (for details we refer, e.g., to Pham, Runggaldier and Sellami [8]) that, in the partial information filtration $(\mathcal{F}_n)_n$, it is the tuple (S, V, H, Π) that is Markov.

Denoting by $U_n(s, v, h, \pi)$ the optimal value in period t_n for $S_n = s, V_n^\phi = v, H_n = h, \Pi_n = \pi$, i.e.,

$$U_n(s, v, h, \pi) = \sup_{\phi \in \mathcal{A}} \mathbb{E}\left\{ u(V_N^\phi) \middle| S_n = s, V_n^\phi = v, H_n = h, \Pi_n = \pi \right\}$$

(recall that $\mathcal{A}$ denotes the set of admissible strategies over the entire investment interval), an application of the Dynamic Programming Principle (see, e.g., Bertsekas [2]) leads to the backward recursions

$$\begin{cases} U_N(s, v, h, \pi) = u(v) \quad \text{and, for } n \in \{1, \cdots, N-1\}, \\ \\ U_{n-1}(s, v, h, \pi) = \\ \max_{\phi_{n-1}} \mathbb{E}\left\{ U_n(S_n, V_n^\phi, H_n, \Pi_n) \middle| (S, V^\phi, H, \Pi)_{n-1} = (s, v, h, \pi) \right\}. \end{cases} \tag{2.4.1}$$

2.4.1 Explicit solution in the log-utility case

In the log-utility case (and analogously in the power utility case), assuming for simplicity that $r_n \equiv r$, we have the following result.

Theorem 2.4.1. *For $n = 0, \cdots, N$ and supposing that $H_n = h^p$ for some $p \in \{1, \cdots, 2^{M-1}\}$ we have*

$$U_n(s, v, h^p, \pi) = \log v + K_n(s, h^p, \pi), \tag{2.4.2}$$

with

$$\begin{cases} K_N(s,h^p,\pi) &= 0 \qquad \textit{for every } s \in \mathbb{R}_+^M, p \in \{1,\cdots,2^{M-1}\}, \pi \in \mathcal{K}_J, \\ K_n(s,h^p,\pi) &= k(h^p,\pi) + \sum_{i=1}^J \pi^i \sum_{\ell=1}^L p^\ell(z^i) \sum_{q=1}^{2^{M-1}} \rho^{p,q}(z^i)\cdot \\ & \quad \cdot K_{n+1}\left(I(s,\xi^\ell,h^q), h^q, \bar{F}\left(\pi, I(s,\xi^\ell,h^q), s, h^q, h^p\right)\right), \end{cases}$$

where $\bar{F}(\cdot)$ was defined in (2.3.4) and where

$$\begin{aligned} k(h^p,\pi) = \max_{\phi=(\phi^1,\cdots,\phi^M)} \Big\{ & \sum_{i=1}^J \pi^i \sum_{\ell=1}^L p^\ell(z^i) \sum_{q=1}^{2^{M-1}} \rho^{p,q}(z^i)\cdot \\ & \cdot \log\Big[(1+r) + \sum_{m=1}^M \phi^m(1-h^{p,m})\left[\gamma^m(\xi^\ell)(1-h^{q,m}) - (1+r)\right]\Big]\Big\}. \end{aligned} \tag{2.4.3}$$

Notice that, in each period t_n, the additive term $K_n(\cdot)$ results from the sum of a current additive term $k(h^p,\pi)$ and the conditional expectation of the previously obtained $K_{n+1}(\cdot)$ given the present information. The proof can rather straightforwardly be obtained by backward induction on n, as we immediately see.

Proof. We first notice that the result holds true for $n = N$. We now suppose that Equation (2.4.2) is verified at time t_{n+1} and we show that it remains valid at time t_n. We have, given Equation (2.4.1) and recalling Equation (2.2.7), where the portfolio value at time t_{n+1} is written as a function of its value at time t_n (we omit the subscript n in the investment strategy $\phi_n = (\phi_n^1, \cdots, \phi_n^M)$)

$$\begin{aligned} U_n(s,v,h^p,\pi) &= \max_\phi \mathbb{E}\left\{U_{n+1}(S_{n+1}, V_{n+1}^\phi, H_{n+1}, \Pi_{n+1}) \big| (S,V^\phi,H,\Pi)_n = (s,v,h^p,\pi)\right\} \\ &= \max_\phi \mathbb{E}\left\{\log V_{n+1}^\phi + K_{n+1}(S_{n+1}, H_{n+1}, \Pi_{n+1}) \big| (S,V^\phi,H,\Pi)_n = (s,v,h^p,\pi)\right\} \\ &= \log v + \max_\phi \mathbb{E}\Big\{ \log\Big[(1+r) + \textstyle\sum_{m=1}^M \phi^m(1-h^{p,m})\big(\gamma^m(\xi_{n+1})(1-H_{n+1}^m) - (1+r)\big)\Big] \\ &\quad + K_{n+1}\Big(I(S_n,\xi_{n+1},H_{n+1}),\ H_{n+1},\ \bar{F}\big(\Pi_n, I(S_n,\xi_{n+1},H_{n+1}), S_n, H_{n+1}, H_n\big)\Big) \\ &\quad \big| (S,H,\Pi)_n = (s,h^p,\pi)\Big\}, \end{aligned}$$

where I and $\bar{F}$ were introduced, respectively, in Equations (2.2.3) and (2.3.4).

We now use iterated conditional expectations and we introduce a conditional expectation with respect to a larger filtration containing Z_n. This is crucial since now, due to the conditional independence of ξ_{n+1} and H_{n+1} given Z_n, we can explicitly compute this conditional expectation, that will be a function of Z_n, and we find

$$\begin{aligned} U_n(s,v,h^p,\pi) &= \log v + \max_\phi \mathbb{E}\Big\{ \sum_{\ell=1}^L p^\ell(Z_n) \sum_{q=1}^{2^{M-1}} \rho^{p,q}(Z_n) \Big[\log\Big((1+r) \\ &\quad + \sum_{m=1}^M \phi^m(1-h^{p,m})\big(\gamma^m(\xi^\ell)(1-h^{q,m}) - (1+r)\big)\Big) \\ &\quad + K_{n+1}\left(I(s,\xi^\ell,h^q), h^q, \bar{F}\big(\pi, I(s,\xi^\ell,h^q), s, h^q, h^p\big)\right)\Big] \\ &\quad \big| (S,H,\Pi)_n = (s,h^p,\pi)\Big\}. \end{aligned}$$

It suffices now to recall that the conditional distribution of Z_n given the investor's information at time t_n is, by definition, the filter at time t_n, so that we finally have

$$
\begin{aligned}
U_n(s,v,h^p,\pi) &= \log v + \max_{\phi} \Big\{ \sum_{i=1}^{J} \pi^i \sum_{\ell=1}^{L} p^\ell(z^i) \sum_{q=1}^{2^{M-1}} \rho^{p,q}(z^i) \log \Big[(1+r) \\
&\quad + \sum_{m=1}^{M} \phi^m (1-h^{p,m}) \Big(\gamma^m(\xi^\ell)(1-h^{q,m}) - (1+r) \Big) \Big] \Big\} \\
&\quad + \sum_{i=1}^{J} \pi^i \sum_{\ell=1}^{L} p^\ell(z^i) \sum_{q=1}^{2^{M-1}} \rho^{p,q}(z^i) K_{n+1}\Big(I(s,\xi^\ell,h^q), h^q, \bar{F}\big(\pi, I(s,\xi^\ell,h^q), s, h^q, h^p\big) \Big) \\
&= \log v + K_n(s,h^p,\pi).
\end{aligned}
$$

The theorem is proved once we show that $k(h^p,\pi)$ exists.
Existence of $k(h^p,\pi)$**.**
At every time step, the maximization problem is defined for $\phi = (\phi^1,\cdots,\phi^M) \in \mathcal{D}$, where $\mathcal{D}$ is such that the above logarithms are well defined. In particular, $\mathcal{D}$ is non empty (it contains at least the point $(0,\ldots,0)$) and it is delimited by the intersection of a maximum of $2^{M-1} \times L \times 2^{M-1}$ half-planes of the form

$$1 + r + \sum_{m=1}^{M} \phi^m (1-h^{p,m}) \big(\gamma^m(\xi^\ell)(1-h^{q,m}) - 1 - r \big) > 0, \tag{2.4.4}$$

where p and q vary in $\{1,\cdots,2^{M-1}\}$ and ℓ is in $\{1,\cdots,L\}$. By possibly truncating $\mathcal{D}$ from below, e.g., by imposing the condition

$$\phi^m > -C, \quad m = 1,\cdots,M, \tag{2.4.5}$$

for a "suitable" $C > 0$, we can restrict our attention to a domain $\mathcal{D}_C$ that is a subset of $\mathcal{D}$, $\mathcal{D}_C \subseteq \mathcal{D}$. The above condition (2.4.5) appears to be reasonable from an economic point of view, in that an investor should not take short positions in the risky assets for more than a proportion C of its current wealth. It is possible to show that the closure of $\mathcal{D}_C$, $\bar{\mathcal{D}}_C$, is compact. That is, denoting by $\underline{\ell}$ and $\bar{\ell} \in \{1,\ldots,L\}$, $\bar{\ell} \neq \underline{\ell}$, the indexes such that $\gamma^m(\xi^{\underline{\ell}}) \in (0,1)$ and $\gamma^m(\xi^{\bar{\ell}}) > 1$, for every $m \in \{1,\ldots,M\}$ (notice that in general, the assets's dynamics are downward as well as upward), we have to show that $\mathcal{D}_C$ is bounded from above.

For this purpose, let us set, without loss of generality, $r = 0$ and let us consider the half-plane in Equation 2.4.4 identified by $p = 1$ and $q = 2^{M-1}$ (i.e., for $h^p = (0,0,\ldots,0)$ and $h^q = (0,1,\ldots,1)$; the other cases, namely when $h^{p,m} = 1$, for some $m \in \{2,\ldots,M\}$, are even simpler to treat)

$$1 + \phi^1\big(\gamma^1(\xi^\ell) - 1\big) - \phi^2 - \cdots - \phi^M > 0. \tag{2.4.6}$$

By recalling that, by definition of $\mathcal{D}_C$, $-\phi^m < C$, for any m, focusing on ϕ^1 we find that a necessary condition for $\phi^1 \in \mathcal{D}_C$ is that

$$\phi^1\big(1 - \gamma^1(\xi^\ell)\big) < 1 - \phi^2 - \cdots - \phi^M < 1 + C(M-1), \quad \forall \ell \in \{1,\ldots,L\}.$$

By taking $\ell = \underline{\ell}$, so that $1 - \gamma^1(\xi^{\underline{\ell}}) > 0$, we find that the boundedness from below of $\mathcal{D}_C$ ensures its boundedness from above, with respect to ϕ^1. For what concerns ϕ^2 (the reasoning is the same for $\phi^3, \ldots, \phi^M$), taking $\ell = \underline{\ell}$ in Equation (2.4.6), we find that a necessary condition for $\phi^2 \in \mathcal{D}_C$ is that

$$\phi^2 < 1 - \phi^1\big(1 - \gamma^1(\xi^{\underline{\ell}})\big) - \phi^3 - \cdots - \phi^M < 1 + C\big(1 - \gamma^1(\xi^{\underline{\ell}})\big) + C(M-2)$$

and we then conclude that, given the boundedness from below, the domain $\mathcal{D}_C$ is also bounded from above in each variable and its closure is compact (see also the details in the simpler binomial example that follows in Appendix 2.6.2).

Notice, furthermore, that the boundary of $\mathcal{D}_C$ partly coincides with the boundary of $\mathcal{D}$. The common boundary will be called "natural boundary" of $\mathcal{D}_C$, while the boundary resulting from the truncation of $\mathcal{D}$ will be the "artificial boundary" of $\mathcal{D}_C$.

Once we have restricted our attention to a domain with a compact closure, the maximizing ϕ^* exists and it is unique. Indeed

- if $\mathcal{D}$ is bounded, we have to maximize over $\mathcal{D}$ a strictly concave and continuous function (namely the sum over p, q and ℓ of logarithms of the left hand side of (2.4.4)) that goes to $-\infty$ on $\partial\mathcal{D}$;
- otherwise, if the domain has been artificially bounded, then we have to maximize over $\mathcal{D}_C$ a strictly concave and continuous function that goes to $-\infty$ on the "natural boundary" of $\mathcal{D}_C$ and that it is well defined on the "artificial boundary" of $\mathcal{D}_C$.

The maximum point, then, exists (it is automatically admissible) and it is unique. Notice that it can be on the "artificial boundary". We only state here that ϕ^* can be numerically obtained (this will be clarified in Section 2.5, which is devoted to numerical examples). $\square$

We now consider three particular cases, namely the full information case, the case when $Z_n \equiv Z$ with Z unobservable and when it is observable. As previously done, we suppose, for simplicity, that $r_n \equiv r$.

2.4.2 Particular case: full information about Z_n

In this case the Markovian tuple is (S, V, H, Z), so that we replace Π by Z and the optimal wealth at time t_n is

$$U_n(s, v, h, z) = \sup_{\phi \in \mathcal{A}} \mathbb{E}\left\{u(V_N^\phi)\middle| S_n = s, V_n^\phi = v, H_n = h, Z_n = z\right\}.$$

In the log-utility case we find the following corollary of Theorem 2.4.1. Having fixed $Z_n = z^i$, we just substitute π by z^i in $K(\cdot)$ and in $k(\cdot)$ and we drop the $\sum_{i=1}^J \pi^i$ everywhere. Moreover, since $K_n(s, h^p, z^i)$ is the conditional expectation of $K_{n+1}(S_{n+1}, H_{n+1}, Z_{n+1})$ given the investor's information, in the definition of $K_n(s, h^p, z^i)$ we will find the sum $\sum_{j=1}^J P^{ij} K_{n+1}(\cdot, \cdot, z^j)$. We obtain

Corollary 2.4.1. *For $n = 0, \cdots, N$, supposing that $H_n = h^p$ for some $p \in \{1, \cdots, 2^{M-1}\}$ and that $Z_n = z^i$, for some $i \in \{1, \ldots, J\}$, we have*

$$U_n(s, v, h^p, z^i) = \log v + K_n(s, h^p, z^i), \tag{2.4.7}$$

with $K_N(s,h^p,z^i)=0$ *for every* $s\in\mathbb{R}^{+M}, p\in\{1,\cdots,2^{M-1}\}, i\in\{1,\ldots,J\}$ *and*

$$K_n(s,h^p,z^i)=k(h^p,z^i)+\sum_{\ell=1}^{L}p^\ell(z^i)\sum_{q=1}^{2^{M-1}}\rho^{p,q}(z^i)\sum_{j=1}^{J}P^{ij}K_{n+1}\left(I(s,\xi^\ell,h^q),h^q,z^j\right),$$

where

$$k(h^p,z^i)=\max_\phi\sum_{\ell=1}^{L}p^\ell(z^i)\sum_{q=1}^{2^{M-1}}\rho^{p,q}(z^i)\log\left[1+r+\sum_{m=1}^{M}\phi^m(1-h^{p,m})\left[\gamma^m(\xi^\ell)(1-h^{q,m})-1-r\right]\right].$$

2.4.3 Particular case: $Z_n\equiv Z$ unobserved

In the case when $Z_n\equiv Z$, the factor process reduces to an unobserved parameter that, in accordance with the Bayesian point of view, is considered as a random variable Z, with given *a priori* law μ. Even if Z is modeled as not time varying, the successive updating of its conditional distribution, i.e.,

$$\Pi_n^j:=\mathbb{P}(Z=z^j\mid(S^n,H^n)),\quad j=1,\cdots,J,\ \ n\le N,$$

makes the context dynamic. The solution is obtained as in the general case and here it simplifies considerably. In fact, the recursive Bayes' formula (2.3.1) reduces to the ordinary one, that here becomes

$$\begin{aligned}\Pi_n^j&=\mathbb{P}\left(Z=z^j|S_n=s_n,H_n=h_n,(S^{n-1},H^{n-1})\right)\\&\propto\mathbb{P}\left(S_n=s_n,H_n=h_n|Z=z^j,S_{n-1},H_{n-1}\right)\cdot\Pi_{n-1}^j.\end{aligned}$$

Having fixed the previous observations (s_{n-1},h_{n-1}) and recalling the definition (2.3.3) of the diagonal matrix F, Equation (2.3.4) then becomes

$$\begin{cases}\Pi'_0=\mu\qquad\text{and, for } n\ge 1,\\[2mm]\Pi'_n=\dfrac{F(s_n,s_{n-1},h_n,h_{n-1})\,\Pi'_{n-1}}{\left|F(s_n,s_{n-1},h_n,h_{n-1})\,\Pi'_{n-1}\right|_1}:=\bar F\left(\Pi_{n-1},s_n,s_{n-1},h_n,h_{n-1}\right).\end{cases}\tag{2.4.8}$$

With these changes the statement of Theorem 2.4.1 remains valid in the same form also for the present case.

2.4.4 Particular case: $Z_n\equiv Z$ fully observed

In this case the factor Z has no relevance anymore, the model is fully defined. Defining, in perfect analogy with Equation (2.2.5),

$$p^\ell:=\mathbb{P}(\xi_n=\xi^\ell)\quad\text{and}\quad\rho^{p,q}:=\mathbb{P}(H_n=h^q|H_{n-1}=h^p),$$

for $\ell=1,\cdots,L$ and for $p,q\in\{1,\cdots,2^{M-1}\}$, one immediately finds

Corollary 2.4.2. *For* $n=0,\cdots,N$, *supposing that* $H_n=h^p$ *for some* $p\in\{1,\cdots,2^{M-1}\}$, *we have*

$$U_n(s,v,h^p)=\log v+K_n(s,h^p),\tag{2.4.9}$$

with $K_N(s,h^p)=0$ *for every* $s\in\mathbb{R}^{+M}, p\in\{1,\cdots,2^{M-1}\}$ *and*

$$K_n(s,h^p)=k(h^p)+\sum_{\ell=1}^{L}p^\ell\sum_{q=1}^{2^{M-1}}\rho^{p,q}\,K_{n+1}\left(I(s,\xi^\ell,h^q),h^q\right),$$

where

$$k(h^p)=\max_\phi\sum_{\ell=1}^{L}p^\ell\sum_{q=1}^{2^{M-1}}\rho^{p,q}\ \log\left[1+r+\sum_{m=1}^{M}\phi^m(1-h^{p,m})\left[\gamma^m(\xi^\ell)(1-h^{q,m})-1-r\right]\right].\tag{2.4.10}$$

Remark 2.4.1. *Due to the (assumed) time homogeneity of* p *and* ρ*, i.e., of the processes* ξ *and* H*, the maximizing investment strategy* ϕ^* *does not depend on time. It does not depend on the current values* s *and* v *of the prices and the wealth either, it depends however on the current default state* h*.*

2.5 Numerical results and the issue of robustness

Numerical results from simulations are presented in the case when

- $M=3$, i.e., there are one non-defaultable and two defaultable risky assets on the market (it is the smallest value of M allowing for contagion);
- $L=2$, i.e., $\xi_n\in\{\xi^1,\xi^2\}$ (binomial model). Here ξ^1 corresponds to an "up" movement in asset prices and ξ^2 to a "down" movement;
- $J=2$, i.e., $Z_n\in\{0,1\},\ \forall\ n$, with the following economic interpretation

$$\begin{cases} Z_n=0: & \text{good state} \quad \text{(bull market)},\\ Z_n=1: & \text{bad state} \quad \text{(bear market)};\end{cases}$$

- $r_n\equiv r=0$;
- $u(x)=\log(x), x>0$.

The initial law μ of the Markov chain Z is fixed by assigning

$$\mathbb{P}(Z_0=0)=0.5,\quad \mathbb{P}(Z_0=1)=0.5$$

and its transition probability matrix is supposed to be

$$P=\begin{pmatrix}P^{11} & P^{12}\\ P^{21} & P^{22}\end{pmatrix}=\begin{pmatrix}0.6 & 0.4\\ 0.4 & 0.6\end{pmatrix}.$$

The conditional distribution of ξ given Z is also assigned as

$$p(0):=p^1(0)=\mathbb{P}(\xi_n=\xi^1|Z_{n-1}=0)=0.6,\quad p(1):=p^1(1)=0.4,$$

meaning that, when the economy is in good state, the probability of having an "up" movement in asset prices is equal to 0.6, while when the economic situation is bad, this probability decreases to 0.4. It is also useful to introduce the following notation

$$\gamma^m(\xi^1) = u^m \quad \text{and} \quad \gamma^m(\xi^2) = d^m, \quad m = 1, 2, 3,$$

where $u^{\cdot}$ stands for "up" and $d^{\cdot}$ for "down" and, typically, $0 < d^m < 1 < u^m, m = 1, 2, 3$. We fix the following listing of the possible default states $h^p, p = 1, \dots, 4$:

$$h^1 = (0,0,0), \quad h^2 = (0,1,0), \quad h^3 = (0,0,1), \quad h^4 = (0,1,1)$$

and we assign, in the next two matrices, the values for $\rho^{p,q}(z)$, $p, q \in \{1, \cdots, 4\}$, according to the value of z,

$$\{z = 0\} : \begin{pmatrix} 0.91 & 0.03 & 0.03 & 0.03 \\ 0 & 0.80 & 0 & 0.20 \\ 0 & 0 & 0.80 & 0.20 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \{z = 1\} : \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0.50 & 0 & 0.50 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the simulations we consider three cases:

- "GOOD": full information, where the true model is known and corresponds to the case $\{Z_n \equiv Z = 0\}$ (see Section 2.4.4);
- "BAD": full information, where the true model is known and corresponds to the case $\{Z_n \equiv Z = 1\}$;
- "PARTIAL": partial information, where there is uncertainty about the true model (Z_n is unobserved and evolves according to the Markov chain specified by the initial law μ and the transition probability matrix P).

We have two goals in mind:

i) investigating, for each one of the three cases, the effect of allowing for shorting in the risky assets;

ii) investigating the "robustness" of the optimal solution obtained in the partial information case (case "PARTIAL").

2.5.1 Shorting vs. no shorting

We analyzed and compared two possible situations: the first one corresponds to the case when no shorting is possible and the investment strategy is constrained from above, namely (recall that ϕ_n^m is the proportion of wealth invested in S^m at time n)

$$\phi_n^m \in [0, 2], \quad m = 1, 2, 3, \ \forall\ n,$$

while in the second one shorting is allowed and the strategy is constrained from above and below, i.e.,

$$\phi_n^m \in [-2, 2], \quad m = 1, 2, 3, \ \forall\ n.$$

It can furthermore be easily seen that, in order for V_n^ϕ to be in the domain of $u(x) = \log(x)$, in the case when no-shorting is allowed, we even have

$$\phi_n^2, \phi_n^3 < 1, \quad \forall\, n$$

(it suffices to look at the function to be maximized in Equation (2.4.3) and to consider the case when $r = 0, H_n \neq h^4, H_{n+1} = h^4$, as done in the proof of Theorem 2.4.1, to show that the domain $\mathcal{D}_C$ is bounded).

Remark 2.5.1. *In the context just described we thus consider investment strategies in a truncated domain, $\phi \in \mathcal{D}_C$ (this notation was introduced in the proof of Theorem 2.4.1), with $C = 0$ in the case of no shorting and $C = 2$ when shorting is possible.*

For the case when shorting is not allowed, in the following Table 2.1 we show, for a certain set of parameters $u^m, d^m, m = 1, 2, 3$, the optimal investment solutions $\phi^{1,*}, \phi^{2,*}, \phi^{3,*}$, in the "GOOD" and "BAD" states, varying with the default state. As pointed out in Remark 2.4.1, under full information ϕ^* does not depend on time, so that there is no need, in this table, to specify the time t_n. In the successive Table 2.2, relative to the "PARTIAL" case, we choose $N = 3$ and we have to specify the time interval in which we are working. In this case, the optimal investment solution is a function of both the default state and of the asset prices' evolution, that are the needed information in order to update the filter.

It is a key point here to specify how the optimal strategy ϕ^* was obtained, since in the proof of Theorem 2.4.1 we only stated that it exists and is unique. Here the optimizing investment strategy is obtained by means of a "search procedure", performed by means of a numerical code written in C on a grid of points constructed on the admissibility domain (an example of grid is given in Figure 2.7 in the appendix). The precision of the grid is fixed to 0.01.

When no shorting is possible, in the "BAD" state it is clear, from Table 2.1, that the optimal solution consists in not investing at all in risky assets and in placing all the money in the bank account. On the contrary, in the "GOOD" state, it is optimal to invest as much as one can in the default-free risky asset, regardless of the default state.
In the "PARTIAL" case it is never optimal to invest in the defaultable assets and $\phi_n^{1,*}$ varies with respect to n (indeed, this case can be considered as a mixture of the two previous full information cases).

Four more tables concerning the case of possible shorting follow. The difference between them is in the defaultable assets' returns, depending on the parameters $u^m, d^m, m = 2, 3$ (in particular, the case of Table 2.5, where $u^2 = u^3 = 2$, represents a very extreme case). Notice that in the case when shorting is allowed and asset returns are "reasonable" (Table 2.3), both in the "GOOD" and "BAD" states it is optimal to invest all the wealth in S^1, but if the defaultable assets have a very high yield (Table 2.5), then it becomes interesting to invest also in them. In the partial information case, too, the main difference between Tables 2.4 and 2.6 is that in the second case, when the defaultable assets have a very interesting yield, $\phi^{1,*}$ is no more equal to 2 and, in some cases, $\phi^{2,*}$ and $\phi^{3,*}$ are positive.

To conclude this analysis we show in three graphics in Figure 2.1 (corresponding to the no-shorting case, to shorting with reasonable assets' returns and to shorting with high defaultable assets' returns) the optimal expected terminal wealth, in the log-utility case, when

$$v_0 = 1, \quad H_0 = h^1 \quad \text{and} \quad N = 1, 2, \ldots, 5.$$

SHORTING not possible

Parameters	Z = {0} GOOD	Z = {1} BAD
u^1	1,01	1,01
d^1	0,99	0,99
u^2	1,3	1,3
d^2	0,9	0,9
u^3	1,35	1,35
d^3	0,8	0,8
$p(Z) = p^1 (Z)$	0,6	**0,4**
$\rho^{11}(Z)$	0,91	**0,25**
$\rho^{12} (Z)$	0,03	**0,25**
$\rho^{13} (Z)$	0,03	**0,25**
$\rho^{14} (Z)$	0,03	**0,25**
$\rho^{22} (Z)$	0,8	**0,5**
$\rho^{33} (Z)$	0,8	**0,5**
$H = h^1$ (no asset defaulted)	**Precision search: 0.01**	**Precision search: 0.01**
Dtest: [0;2]x[0;2]x[0;1]		
Φ^{1*}	**2**	**0**
Φ^{2*}	**0,37**	**0**
Φ^{3*}	**0,11**	**0**
$H = h^2$ (asset #2 defaulted)		
Dtest: [0;2]x[0;1]		
Φ^{1*}	**2**	**0**
Φ^{3*}	0	**0**
$H = h^3$ (asset #3 defaulted)		
Dtest: [0;2]x[0;1]		
Φ^{1*}	**2**	**0**
Φ^{2*}	0	**0**
$H = h^4$ (defaulted #2 e #3)		
Dtest: [0;1]		
Φ^{1*}	**2**	**0**
Computational time	1 s	1 s

Table 2.1: Optimal investment solutions under full information, "GOOD" and "BAD", shorting not possible.

SHORTING not possible

PARTIAL INFORMATION, N = 3

In the table: $(\Phi^{1*}, \Phi^{2*}, \Phi^{3*})$

t_0 - t_1	t_1 - t_2	t_2 - t_3			
(0 , 0, 0)	**u ; h1** (2, 0, 0)	**u u ; h1 h1** (2, 0, 0)	**u u; h1 h2** (0, -, 0)	**u u; h1 h3** (0, 0, -)	**u u; h1 h4** (0, -, -)
	u ; h2 (0, -, 0)	**u d ; h1 h1** (2, 0, 0)	**u d; h1 h2** (0, -, 0)	**u d; h1 h3** (0, 0, -)	**u d; h1 h4** (0, -, -)
	u ; h3 (0, 0, -)	**d u ; h1 h1** (2, 0, 0)	**d u; h1 h2** (0, -, 0)	**d u; h1 h3** (0, 0, -)	**d u; h1 h4** (0, -, -)
	u ; h4 (0, -, -)	**d d ; h1 h1** (1,93, 0, 0)	**d d; h1 h2** (0, -, 0)	**d d; h1 h3** (0, 0, -)	**d d; h1 h4** (0, -, -)
	d ; h1 (1,67, 0, 0)	**u u ; h2 h2** (1,16 , -, 0)	**u u; h2 h4** (0, - , -)		
	d ; h2 (0, -, 0)	**u d ; h2 h2** (0, - , 0)	**u d; h2 h4** (0, - , -)		
	d ; h3 (0, 0, -)	**d u ; h2 h2** (1,04, - , 0)	**d u; h2 h4** (0, - , -)		
	d ; h4 (0, -, -)	**d d ; h2 h2** (0, - , 0)	**d d; h2 h4** (0, - , -)		
		u u ; h3 h3 (1,16 , 0, -)	**u u; h3 h4** (0 , -, -)		
		u d ; h3 h3 (0 , 0, -)	**u d; h3 h4** (0 , -, -)		
		d u ; h3 h3 (1,04, 0 , -)	**d u; h3 h4** (0 , - , -)		
		d d ; h3 h3 (0 , 0 , -)	**d d; h3 h4** (0 , - , -)		
		u u ; h4 h4 (0 , - , -)			
		u d ; h4 h4 (0 , - , -)			
		d u ; h4 h4 (0 , - , -)			
		d d ; h4 h4 (0 , - , -)			

Table 2.2: Optimal investment solutions under partial information, shorting not possible.

SHORTING possible

Parameters	Z = {0} GOOD	Z = {1} BAD
u^1	1,01	1,01
d^1	0,99	0,99
u^2	1,3	1,3
d^2	0,9	0,9
u^3	1,35	1,35
d^3	0,8	0,8
$p(Z) = p^1(Z)$	0,6	**0,4**
$\rho^{11}(Z)$	0,91	**0,25**
$\rho^{12}(Z)$	0,03	**0,25**
$\rho^{13}(Z)$	0,03	**0,25**
$\rho^{14}(Z)$	0,03	**0,25**
$\rho^{22}(Z)$	0,8	**0,5**
$\rho^{33}(Z)$	0,8	**0,5**
$H = h^1$ (no asset defaulted)	**Precision search: 0.01**	**Precision search: 0.01**
Dtest: [-2;2]x[-2;2]x[-2;2]		
Φ^{1*}	**2**	**2**
Φ^{2*}	**0,37**	**-1,43**
Φ^{3*}	**0,11**	**-1,2**
$H = h^2$ (asset #2 defaulted)		
Dtest: [-2;2]x[-2;2]		
Φ^{1*}	**2**	**2**
Φ^{3*}	**-0,46**	**-1,97**
$H = h^3$ (asset #3 defaulted)		
Dtest: [-2;2]x[-2;2]		
Φ^{1*}	**2**	**2**
Φ^{2*}	**-0,49**	**-2**
$H = h^4$ (defaulted #2 e #3)		
Dtest: [-2;2]		
Φ^{1*}	**2**	**2**
Computational time	42 s	42 s

Table 2.3: Optimal investment solutions under full information, "GOOD" and "BAD", shorting possible.

SHORTING possible

PARTIAL INFORMATION, N = 3

In the table: $(\Phi^{1*}, \Phi^{2*}, \Phi^{3*})$

t_0 - t_1	t_1 - t_2	t_2 - t_3			
	u ; h1 (2, -0,54, -0,55)	**uu ; h1 h1** (2, -0,53, -0,54)	**u u; h1 h2** (2, - , -1,35)	**u u; h1 h3** (2, -1,53, -)	**u u; h1 h4** (-2 , - ,-)
	u ; h2 (2 , - , -1,36)	**ud ; h1 h1** (2, -0,57, -0,57)	**u d; h1 h2** (2, - , -1,38)	**u d; h1 h3** (2, -1,57, -)	**u d; h1 h4** (-2 , - , -)
(2, -0,67, -0,63)	**u ; h3** (2, -1,55 , -)	**du ; h1 h1** (2, -0,54, -0,54)	**d u; h1 h2** (2, - , -1,36)	**d u; h1 h3** (2, -1,54, -)	**d u; h1 h4** (-2 , - , -)
	u ; h4 (-2 , - , -)	**d d ; h1 h1** (2, -0,58 , -0,57)	**d d; h1 h2** (2, - , -1,38)	**d d; h1 h3** (2, -1,57, -)	**d d; h1 h4** (-2 , - ,-)
	d ; h1 (2, -0,59 , -0,58)	**u u ; h2 h2** (2 , - , -1,22)	**u u; h2 h4** (-1,5 , - , -)		
	d ; h2 (2 , - , -1,38)	**u d ; h2 h2** (2 , - , -1,27)	**u d; h2 h4** (-2 , - , -)		
	d ; h3 (2, -1,57, -)	**d u ; h2 h2** (2 , - , -1,22)	**d u; h2 h4** (-1,61, - , -)		
	d ; h4 (-2 , - , -)	**d d ; h2 h2** (2 , - , -1,28)	**d d; h2 h4** (-2 , - , -)		
		u u ; h3 h3 (2, -1,37, -)	**u u; h3 h4** (-1,5, - , -)		
		u d ; h3 h3 (2, -1,44, -)	**u d; h3 h4** (-2 , - , -)		
		d u ; h3 h3 (2, -1,38, -)	**d u; h3 h4** (-1,61, - , -)		
		d d ; h3 h3 (2, -1,45, -)	**d d; h3 h4** (-2 , - , -)		
		u u ; h4 h4 (0,25 , - , -)			
		u d ; h4 h4 (-1,32 , - , -)			
		d u ; h4 h4 (0,12 , - , -)			
		d d ; h4 h4 (-1,43 , - , -)			

Table 2.4: Optimal investment solutions under partial information, shorting possible.

SHORTING possible

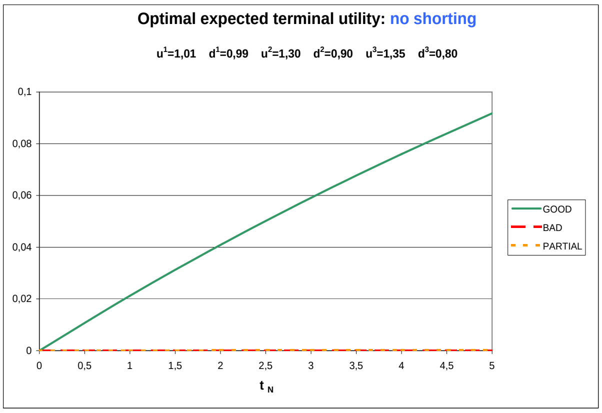
Parameters	CASE Z = {0} GOOD	CASE Z = {1} BAD
u^1	1,01	1,01
d^1	0,98	0,98
u^2	**2**	**2**
d^2	0,95	0,95
u^3	**2**	**2**
d^3	0,99	0,99
$p(Z) = p^1(Z)$	0,8	**0,1**
$\rho^{11}(Z)$	0,91	**0,25**
$\rho^{12}(Z)$	0,03	**0,25**
$\rho^{13}(Z)$	0,03	**0,25**
$\rho^{14}(Z)$	0,03	**0,25**
$\rho^{22}(Z)$	0,5	**0,5**
$\rho^{33}(Z)$	0,5	**0,5**
H = h^1 (no asset defaulted)	**Precision search: 0.01**	**Precision search: 0.01**
Dtest: [-2;2]x[-2;2]x[-2;2]		
Φ^{1*}	**1,51**	**-2**
Φ^{2*}	**0,43**	**-0,49**
Φ^{3*}	**0,49**	**-0,4**
H = h^2 (asset #2 defaulted)		
Dtest: [-2;2]x[-2;2]		
Φ^{1*}	**2**	**-2**
Φ^{3*}	**-0,12**	**-0,8**
H = h^3 (asset #3 defaulted)		
Dtest: [-2;2]x[-2;2]		
Φ^{1*}	**2**	**-2**
Φ^{2*}	**-0,12**	**-0,81**
H = h^4 (defaulted #2 e #3)		
Dtest: [-2;2]		
Φ^{1*}	**2**	**-2**
Computational time	13 s	13 s

Table 2.5: Optimal investment solutions under full information, "GOOD" and "BAD", shorting possible, high defaultable assets' returns.

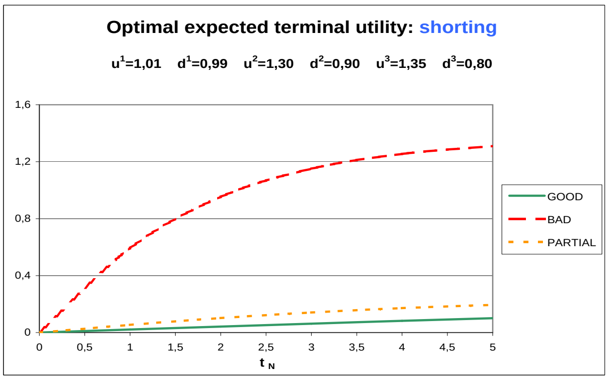
SHORTING possible

PARTIAL INFORMATION, N = 3					
In the table:	**(Φ^{1*},Φ^{2*},Φ^{3*})**				
t_0 - t_1	**t_1 - t_2**	**t_2 - t_3**			
	u ; h1	**uu ; h1 h1**	**uu; h1 h2**	**uu; h1 h3**	**uu; h1 h4**
	(-2, 0,18, 0,23)	(-2, 0,18, 0,23)	(-2, - , -0,35)	(-2, -0,37, -)	(-2 , - ,-)
	u ; h2	**ud ; h1 h1**	**ud; h1 h2**	**ud; h1 h3**	**ud; h1 h4**
	(-2, - , -0,37)	(-2, 0,1, 0,15)	(-2, - , -0,43)	(-2, -0,44, -)	(-2 , - ,-)
	u ; h3	**du ; h1 h1**	**du; h1 h2**	**du; h1 h3**	**du; h1 h4**
(-2, 0,09, 014)	(-2, -0,38, -)	(-2, 0,18, 0,23)	(-2, - , -0,37)	(-2, -0,38, -)	(-2 , - ,-)
	u ; h4	**dd ; h1 h1**	**dd; h1 h2**	**dd; h1 h3**	**dd; h1 h4**
	(-2 , - ,-)	(-2, 0,08, 0,13)	(-2, - , -0,43)	(-2, -0,44, -)	(-2 , - ,-)
	d ; h1	**uu ; h2 h2**	**uu; h2 h4**		
	(-2, 0,08, 0,13)	(-2, - , -0,32)	(-2 , - , -)		
	d ; h2	**ud ; h2 h2**	**ud; h2 h4**		
	(-2, - , -0,43)	(-2, - , -0,41)	(-2 , - , -)		
	d ; h3	**du ; h2 h2**	**du; h2 h4**		
	(-2, -0,44 , -)	(-2, - , -0,32)	(-2 , - , -)		
	d ; h4	**dd ; h2 h2**	**dd; h2 h4**		
	(-2 , - ,-)	(-2, - , -0,41)	(-2 , - , -)		
		uu ; h3 h3	**uu; h3 h4**		
		(-2, -0,33, -)	(-2 , - , -)		
		ud ; h3 h3	**ud; h3 h4**		
		(-2, -0,42, -)	(-2 , - , -)		
		du ; h3 h3	**du; h3 h4**		
		(-2, -0,33 , -)	(-2 , - , -)		
		dd ; h3 h3	**dd; h3 h4**		
		(-2, -0,43 , -)	(-2 , - , -)		
		uu ; h4 h4			
		(-2 , - ,-)			
		ud ; h4 h4			
		(-2 , - ,-)			
		du ; h4 h4			
		(-2 , - ,-)			
		dd ; h4 h4			
		(-2 , - ,-)			

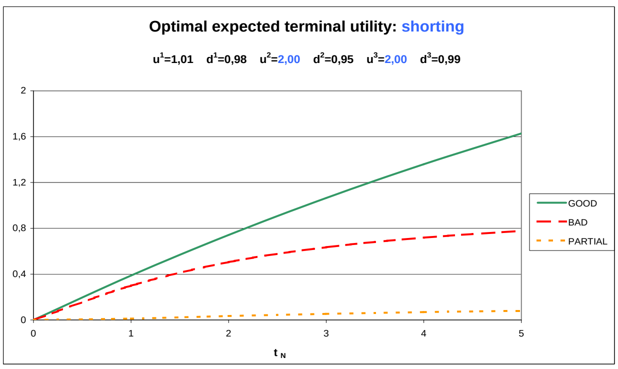
Table 2.6: Optimal investment solutions under partial information, shorting possible, high defaultable assets' returns.



(a) Shorting not allowed.



(b) Shorting allowed, reasonable assets' returns.



(c) Shorting allowed, high defaultable assets' returns.

Figure 2.1: Optimal expected utility from terminal wealth, when $V_0 = v_0, h_0 = h^1$.

Due to the fact that in the case of no shorting the optimal strategy in the "BAD" state consists in not investing in the risky assets, the corresponding optimal portfolio value (in red in Figure 2.1 (a)) remains constant over time and it is always lower than in the analogous "GOOD" state. For what concerns the optimal wealth in the partial information case of Figure 2.1 (a), it is indeed greater than the one in the "BAD" state, despite the fact that this is not clear from the figure.

When shorting is allowed, up to a certain level of "return" on the risky assets (Figure 2.1 (b)), the optimal value in the "BAD" state is superior to that in the "GOOD" state, which is due to the fact that the returns on the defaultable assets as well as the fact that they are subject to default risk make it convenient to go short in them. Beyond that level, when it becomes convenient to invest in S^2 and S^3 (Figure 2.1 (c)), the optimal value in the good state is superior than in the bad state, as one would expect.

2.5.2 Robustness

For what concerns point ii), *robustness* is here in the sense of obtaining a solution that works well for a variety of possible models. This is an important issue because the "exact model" is practically never known and, on the other hand, the solution may be rather sensitive to the model.

From the numerical calculations it turns out rather clearly, as we shall immediately see, that the solution obtained for the model under incomplete information possesses this property of robustness, in the sense that (as can be seen from Figure 2.2 and Figure 2.3)

- while it underperforms the solution under a hypothetical full information about the model,
- it performs much better with respect to using the wrong solution for the wrong model.

In this subsection we focus on the second issue.

Remark 2.5.2. *From Theorem 2.4.1 (recall Equations (2.4.3) and (2.4.10)) it follows that the optimal strategy in the case "PARTIAL" depends, in addition to the current default configuration, also on the current filter values, while under full information (cases "GOOD" and "BAD") it depends only on the default configuration. Consequently, the strategy in the case "PARTIAL" is more refined and as such it can be applied also in the cases "GOOD" and "BAD". Nevertheless, one has to take into account the fact that the optimal ϕ^* obtained under partial information is not time independent (as it is the case for ϕ^* in the full information case).*

In the four diagrams in Figures 2.2 and 2.3 we show the "robustness" of the partial information optimal strategy with respect to using the wrong optimal strategy in the wrong state and in the case where no shorting in the risky assets is allowed. In Figure 2.2 the true state is "GOOD", in Figure 2.3 it is "BAD" (the diagram on the right-hand side of each figure is the zoom of the one on the left-hand side) and we plot the optimal expected utility from terminal wealth as a function of t_n, when

$$v_0 = 1, \quad H_0 = h^1 \quad \text{and} \quad N = 1, 2, 3.$$

In particular, we plot the optimal wealth in 4 cases:

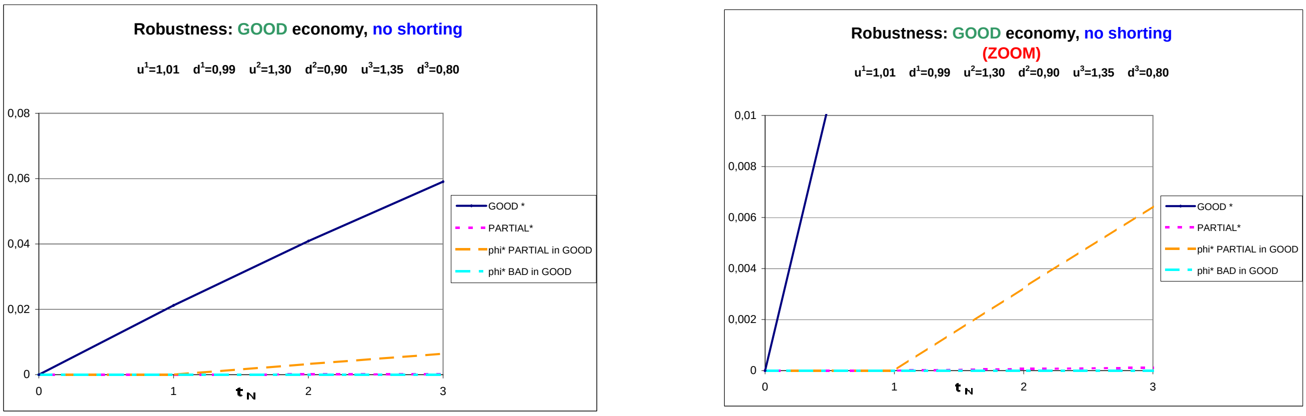


Figure 2.2: Robustness: shorting not allowed, GOOD state.

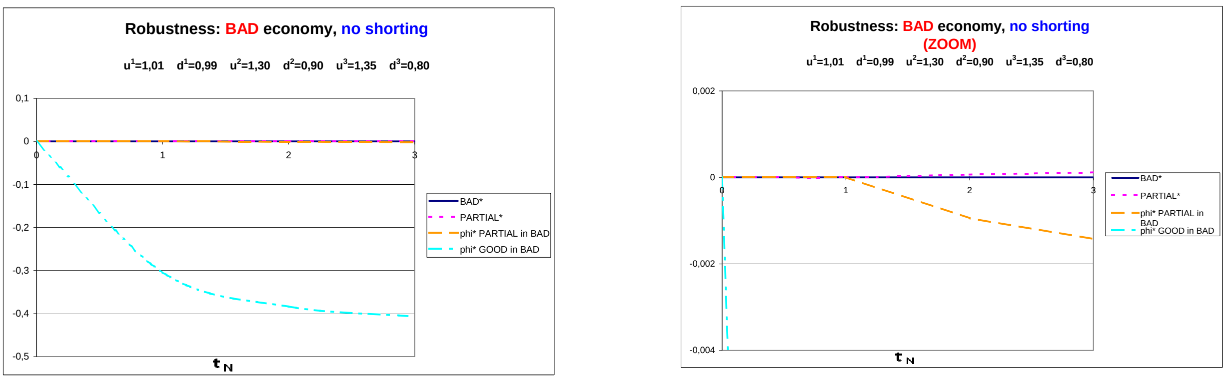


Figure 2.3: Robustness: shorting not allowed, BAD state.

- using the optimal solution for the case "GOOD" when it is indeed the true state, dark blue line (upper benchmark case);
- using the optimal solution for case "PARTIAL" in the "PARTIAL" case, fuchsia line;
- using the optimal solution for case "PARTIAL" in the case "GOOD", orange line;
- using the optimal solution for case "BAD" in the case "GOOD"(lower benchmark case), light blue line;

From the figures it is evident that the optimal investment solution obtained in the partial information case is robust, in the sense that both the fuchsia and the orange lines are above the light blue one, meaning that applying the partial information optimal strategy both when we have full knowledge about the model and when the model is uncertain is better than applying the wrong solution to the wrong model.

The last two diagrams in Figure 2.4 correspond to the case presented in Table 2.5, i.e., when shorting is allowed and the defaultable assets have a considerably high return. In this case the conclusion is once more, as claimed, the robustness of the optimal partial information strategy.

2.6 Appendix

2.6.1 Some "final" remarks: are there alternatives to DP?

In this section we focus, for simplicity, on the case when $Z_n \equiv Z$ is fully observable, $M = 2$ (S^1 is default-free, $H^1 \equiv 0$, and S^2 is defaultable) and $L = 2$ (binomial model). We list the two possible default states by setting

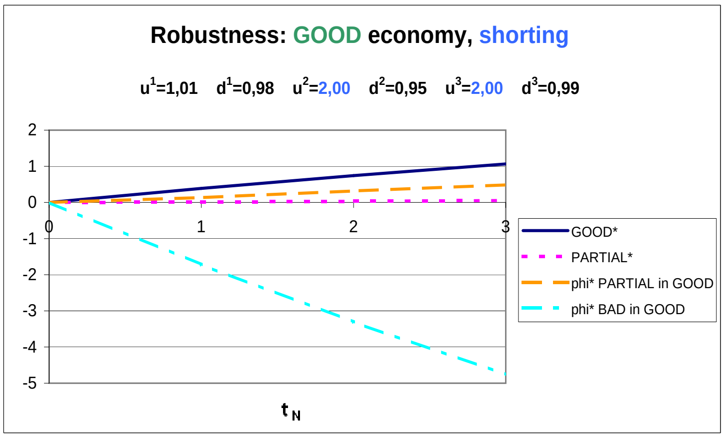
$$h^1 = (0, 0) \quad \text{and} \quad h^2 = (0, 1)$$

and we investigate the possibility of solving the problem without using DP, more precisely by means of the *"martingale method"* or of the *"convex duality method"*. Notice that an application of these methods requires, respectively, the characterization of all possible equivalent martingale measures (EMMs) and of all the Radon-Nikodým density processes. This is not straightforward when working in discrete time.

Indeed, it suffices to notice that, in general, under a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, there is no reason for the random variables ξ_n to be i.i.d. and for ξ_n to be independent of H_n, given $\mathcal{G}_n$.

Nevertheless, since the characterization of all Radon-Nikodým derivatives seems easier than finding the conditions on each time interval to have a martingale measure, we make an attempt to solve the problem by means of a suitably modified version of the duality procedure:

- Given the set $\mathcal{M}$ of all EMMs $\mathbb{Q}$ relative to $\mathbb{P}$, consider the subset $\mathcal{M}_{\mathcal{I}} \subseteq \mathcal{M}$ such that, under $\mathbb{Q}$, the random variables ξ_n are i.i.d. and ξ_n is conditionally independent of H_n given $\mathcal{G}_n$, for every n;
- Characterize the set of Radon-Nikodým derivatives corresponding to the EMMs $\mathbb{Q}$ in $\mathcal{M}_{\mathcal{I}}$ (this is smaller than the analogous set relative to $\mathcal{M}$ and easier to describe) and formulate the dual problem;



(a) Robustness: shorting allowed, high assets' returns, GOOD state.

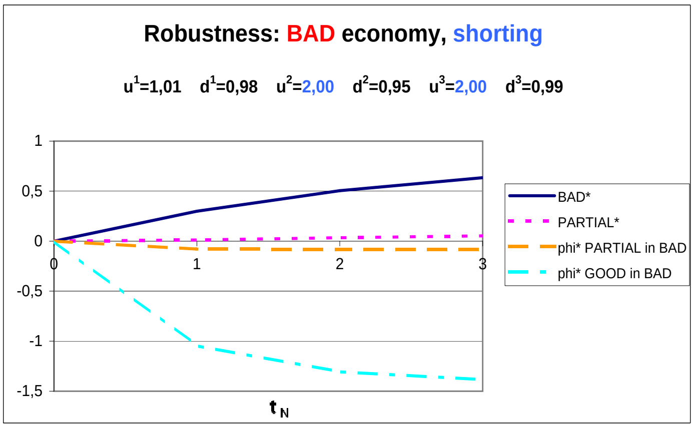


Figure 2.4: Robustness.

(b) Robustness: shorting allowed, high assets' returns, BAD state.

- Solve the dual problem. IF the optimal EMM we find is optimal also for the dual problem corresponding to the whole set of EMMs $\mathcal{M}$ (for this, verify that the "duality gap" between primal and dual optimal solutions for the original problem is equal to zero), then we are done.

Let us now apply the above "algorithm", until we find an obstacle.

Introduce $q := \mathbb{Q}(\xi_n = \xi^1)$ and $\delta_n^{p,q} = \mathbb{Q}(H_{n+1} = h^q | H_n = h^p), p, q \in \{1, 2\}$ and suppose that the default time τ^2 (recall that only S^2 is defaultable here) is an exponential random variable with parameter $\lambda^{\mathbb{P}}$ (resp., $\lambda^{\mathbb{Q}}$) under $\mathbb{P}$ (resp., under $\mathbb{Q}$), i.e.,

$$\mathbb{P}\left(\tau^2 > t_N|\mathcal{G}_n\right) = \mathbb{1}_{\{\tau^2>t_n\}} e^{-\lambda^{\mathbb{P}}(t_N - t_n)}, \quad \mathbb{Q}\left(\tau^2 > t_N|\mathcal{G}_n\right) = \mathbb{1}_{\{\tau^2>t_n\}} e^{-\lambda^{\mathbb{Q}}(t_N - t_n)},$$

so that, setting $\Delta t := t_{n+1} - t_n, \forall\, n$, we have (notice that we loose the dependence of δ from n)

$$\begin{aligned} \rho^{1,2} &= \mathbb{P}(H_{n+1} = h^2 | H_n = h^1) = 1 - \mathbb{P}(\tau^2 > t_{n+1}|\mathcal{G}_n) = 1 - e^{-\lambda^{\mathbb{P}}\Delta t} \\ \delta^{1,2} &= 1 - e^{-\lambda^{\mathbb{Q}}\Delta t}. \end{aligned} \tag{2.6.1}$$

Any measure $\mathbb{Q}$ is, then, characterized by $q \in (0, 1)$ and $\lambda^{\mathbb{Q}} > 0$ and it will be denoted $\mathbb{Q}^{q,\lambda^{\mathbb{Q}}}$. We parameterize the events ω in Ω by means of two indexes:

$$\omega = \omega^{i,k}$$

where $i \in \{0, \ldots, N\}$ is the number of "up" movements in the whole interval $[0, t_N]$ and k is the instant just before the default, i.e., $k = n \in \{0, \ldots, N-1\}$ if $H_{n+1} - H_n = (0, 1)$ and $k = N$ if default does not occur at all in $[0, t_N]$. Notice that, under $\mathbb{Q}$ (and analogously under $\mathbb{P}$)

$$\mathbb{Q}\left(\tau^2 \in (t_n, t_{n+1}]\right) = \mathbb{Q}\left(\tau^2 \in (t_n, t_{n+1}] \middle| \tau^2 > t_n\right) \mathbb{Q}\left(\tau^2 > t_n|\mathcal{G}_0\right),$$

so that, if we fix i and we consider $k \in \{0, \ldots, N-1\}$, we have

$$\mathbb{Q}\left(\{\omega^{i,k}\}\right) = q^i (1-q)^{N-i}\ e^{-\lambda^{\mathbb{Q}} k\Delta t}(1 - e^{-\lambda^{\mathbb{Q}}\Delta t}).$$

On the other hand, if $k = N$, we find

$$\mathbb{Q}\left(\{\omega^{i,N}\}\right) = q^i (1-q)^{N-i}\ e^{-\lambda^{\mathbb{Q}} N\Delta t}.$$

We finally obtain, in our specific setting, the following representation of all the Radon-Nikodým derivatives: for $k \in \{0, \ldots, N-1\}$

$$Z_N^{q,\lambda^{\mathbb{Q}}}(\omega^{i,k}) = \frac{\mathrm{d}\mathbb{Q}^{q,\lambda^{\mathbb{Q}}}}{\mathrm{d}\mathbb{P}}(\omega^{i,k}) = \frac{q^i(1-q)^{N-i} e^{-\lambda^{\mathbb{Q}} k\Delta t}(1 - e^{-\lambda^{\mathbb{Q}}\Delta t})}{p^i(1-p)^{N-i} e^{-\lambda^{\mathbb{P}} k\Delta t}(1 - e^{-\lambda^{\mathbb{P}}\Delta t})}, \tag{2.6.2}$$

and, in the case when $k = N$,

$$Z_N^{q,\lambda^{\mathbb{Q}}}(\omega^{i,N}) = \frac{\mathrm{d}\mathbb{Q}^{q,\lambda^{\mathbb{Q}}}}{\mathrm{d}\mathbb{P}}(\omega^{i,N}) = \frac{q^i(1-q)^{N-i} e^{-\lambda^{\mathbb{Q}} N\Delta t}}{p^i(1-p)^{N-i} e^{-\lambda^{\mathbb{P}} N\Delta t}}. \tag{2.6.3}$$

Remark 2.6.1. *It can be proved, by using the conditional independence of ξ and H, that we have indeed defined a Radon-Nikodým derivative.*

We then introduce the dual functional L (see, e.g., Luenberger [7] for details), that in the case when $r_n \equiv 0$, $u(y) = \log(y)$ and the initial wealth is $x = v_0$ is given by

$$\begin{aligned} L(q, \lambda^{\mathbb{Q}}, \nu) &:= \nu x + \mathbb{E}\left[\widetilde{u}\left(\nu Z_N^{q,\lambda^{\mathbb{Q}}}\right)\right] = \nu x + \mathbb{E}\left[\log\left(\frac{1}{\nu Z_N^{q,\lambda^{\mathbb{Q}}}}\right) - 1\right] \\ &= \nu x - 1 + \sum_{\omega\in\Omega} \log\left(\frac{1}{\nu Z_N^{q,\lambda^{\mathbb{Q}}}(\omega)}\right)\mathbb{P}(\omega), \end{aligned} \tag{2.6.4}$$

where $\nu > 0$ is the Lagrange multiplier and

$$\widetilde{u}(b) := \sup_{y\geq 0}\{u(y) - yb\}, \quad b > 0,$$

is the Legendre transform of u. Looking at the dual functional and at Equation (2.6.2), one sees that, in practice, the computations to minimize the dual functional involve the sum over ω (i.e., the sum over all possible i and k) and so they are not easy to treat.

In the complete information setting, then, it seems convenient to solve the portfolio optimization problem by applying the DP principle. Furthermore, we have to observe here that, to our knowledge, the DP procedure is also the best method to solve the problem under partial information, since under incomplete information the definition and identification of all the EMMs and all the Radon-Nikodým derivatives is not easy. Once again, nevertheless, also in a partial information setting, the characterization of all the Radon-Nikodým derivatives is presumably easier than the identification of all the EMMs for a time horizon t_N.

We end this section by recalling the following interesting fact (see Section 3.2 in Korn [6], see also the proof of Proposition 2.7.2 in Dana and Jeanblanc [4]): the initial problem

$$\begin{cases} \max_{a\in\mathcal{A}} \mathbb{E}\left[u(V_N^a)\right] \\ V_0^a = V_0 = v_0 > 0 \end{cases}$$

can be reformulated as an optimization problem without constraints in the following way (using a_n as strategy, we adapt equation (2.2.7) to the case $r \equiv 0$):

$$\max_{a\in\mathcal{A}} \mathbb{E}\left[u\left(v_0 + \sum_{n=0}^{N-1}\sum_{m=1}^{M} a_{n+1}^m S_n^m(\gamma^m(\xi_{n+1} - 1)\right)\right].$$

The convenience therein is that we do not consider any change of measure, but it is evident that this formulation is useful when N and M are sufficiently small.

For example, in the log-binomial model with $M = 2$ and $N = 1$, we have four possible states of the world, starting from $H_0 = (0, 0)$, namely

$$\begin{array}{lll} \omega_1 = \{\xi_1 = \xi^1, H_1 = h^1\} & \text{with probability} & p_1 := p^1(1 - \rho^{1,2}), \\ \omega_2 = \{\xi_1 = \xi^1, H_1 = h^2\} & \text{with probability} & p_2 := p^1\rho^{1,2}, \\ \omega_3 = \{\xi_1 = \xi^2, H_1 = h^1\} & \text{with probability} & p_3 := (1 - p^1)(1 - \rho^{1,2}), \\ \omega_4 = \{\xi_1 = \xi^2, H_1 = h^2\} & \text{with probability} & p_4 := (1 - p^1)\rho^{1,2}, \end{array} \tag{2.6.5}$$

and the optimal investment strategy is the solution to the following problem

$$\begin{cases} \max_{a_1=a=(a^1,a^2)} \sum_{j=1}^{4} p_j \log(v_j), \\ \\ v_1 = v_0 + a^1 S_0^1(\gamma^1(\xi^1) - 1) + a^2 S_0^2(\gamma^2(\xi^1) - 1) \\ v_2 = v_0 + a^1 S_0^1(\gamma^1(\xi^1) - 1) - a^2 S_0^2 \\ v_3 = v_0 + a^1 S_0^1(\gamma^1(\xi^2) - 1) + a^2 S_0^2(\gamma^2(\xi^2) - 1) \\ v_4 = v_0 + a^1 S_0^1(\gamma^1(\xi^2) - 1) - a^2 S_0^2. \end{cases}$$

The introduction of the Lagrangian function and of the first order necessary and sufficient conditions leads to a system of 10 equations in 10 unknown parameters, which is not straightforward to solve.

2.6.2 One clarifying (simple) example

The aim of this section is to provide a simple example in which the way we obtain the optimal investment strategy can be clarified and it is possible to show (see the following Lemma 2.6.1), as announced in the proof of Theorem 2.4.1, that we can consider sets of admissible strategies that have compact closure.

We consider here the following full information setting (in the context of Section 2.4.4)

- $M = 2$;
- $L = 2$ (binomial model) and, as in Section 2.5, we define $p = \mathbb{P}(\xi_n = \xi^1)$, $\gamma^m(\xi^1) = u^m$ and $\gamma^m(\xi^2) = d^m$, $0 < d^m < 1 < u^m, m = 1, 2$;
- $r_n \equiv r = 0$;
- $u(x) = \log(x), x > 0$.

Since in this case there are only two possible default states, that we list as follows

$$h^1 = (0, 0), \quad h^2 = (0, 1),$$

we denote by $\rho := \mathbb{P}(H_{n+1} = h^2 | H_n = h^1)$, so that $\mathbb{P}(H_{n+1} = h^1 | H_n = h^1) = 1 - \rho$.
We are interested in computing, as explicitly as possible, $\phi^{1,*}$ and $\phi^{2,*}$ (recall that under full information, this optimal strategy does not depend on time). Considering first the case when $H = h^2$, default τ^2 has already occurred and investment in S^2 has already ceased. The optimal $\phi^{1,*}$ is then the (unique) solution to

$$\max_{\phi^1} \left\{ p \log \left[1 + \phi^1(u^1 - 1) \right] + (1 - p) \log \left[1 + \phi^1(d^1 - 1) \right] \right\},$$

namely

$$\phi^{1,*} = p \frac{1}{1 - d^1} + (1 - p) \frac{-1}{u^1 - 1},$$

which is in the admissibility domain $\left(-\frac{1}{u^1 - 1}, \frac{1}{1 - d^1} \right)$.
If $H = h^1$ (default τ^2 has not yet occurred), then we have to solve

$$\max_{\phi^1, \phi^2} \left\{ p\rho \log \left[1 + \phi^1(u^1 - 1) - \phi^2 \right] + p(1 - \rho) \log \left[1 + \phi^1(u^1 - 1) + \phi^2(u^2 - 1) \right] \right.$$
$$\left. + (1 - p)\rho \log \left[1 + \phi^1(d^1 - 1) - \phi^2 \right] + (1 - p)(1 - \rho) \log \left[1 + \phi^1(d^1 - 1) + \phi^2(d^2 - 1) \right] \right\},$$

where ϕ^1 and ϕ^2, in order to be admissible, have to satisfy the following conditions (we set $x := \phi^1$ and $y := \phi^2$)

$$\begin{cases} y & < & 1 + x(u^1 - 1), \\ y & > & \dfrac{-1}{u^2 - 1} - x \dfrac{u^1 - 1}{u^2 - 1}, \\ y & < & 1 - x(1 - d^1), \\ y & < & \dfrac{1}{1 - d^2} - x \dfrac{1 - d^1}{1 - d^2}. \end{cases}$$

We denote by $\mathcal{D}_0 \subset \mathbb{R}^2$ the area delimited by the intersection of the four semi-planes above, that is the set of admissible strategies.

As we said in the proof of Theorem 2.4.1, it is possible to show that we can always restrict our attention to a domain $\mathcal{D}_0$ whose closure, $\bar{\mathcal{D}}_0$, is compact, eventually by artificially bounding the domain (as we did in Section 2.5 in the numerical examples).

Lemma 2.6.1. *If*

$$(1-d^1)(u^2-1) > (u^1-1)(1-d^2),$$

then the closure of the admissibility domain, $\bar{\mathcal{D}}_0$, i.e., the subset of $\mathbb{R}^2$ defined by

$$\begin{cases} y & \leq & 1 + x(u^1-1), \\ y & \geq & \dfrac{-1}{u^2-1} - x\dfrac{u^1-1}{u^2-1}, \\ y & \leq & 1 - x(1-d^1), \\ y & \leq & \dfrac{1}{1-d^2} - x\dfrac{1-d^1}{1-d^2}, \end{cases} \tag{2.6.6}$$

is bounded and thus compact. In all the other cases, $\bar{\mathcal{D}}_0$ is closed, but unbounded.

Proof. We start by representing on a plane the four lines

$$\begin{aligned} r_1 &: \quad y = 1 + x(u^1-1); \\ r_2 &: \quad y = \frac{-1}{u^2-1} - x\frac{u^1-1}{u^2-1}; \\ r_3 &: \quad y = 1 - x(1-d^1); \\ r_4 &: \quad y = \frac{1}{1-d^2} - x\frac{1-d^1}{1-d^2}; \end{aligned}$$

and by computing their intersection points, recalling that $0 < d^m < 1 < u^m, m = 1, 2,$

	r_2	r_3	r_4
r_1	$\left(-\frac{1}{u^1-1}, 0\right)$	$(0,1)$	$\left(\frac{d^2}{(u^1-1)(1-d^2)+1-d^1}, \frac{u^1-d^1}{(u^1-1)(1-d^2)+1-d^1}\right)$
r_2		$\left(\frac{u^2}{(1-d^1)(u^2-1)+1-u^1}, \frac{d^1-u^1}{(1-d^1)(u^2-1)+1-u^1}\right)$	$\left(\frac{u^2-d^2}{(1-d^1)(u^2-1)-(1-d^2)(u^1-1)}, \frac{d^1-u^1}{(1-d^1)(u^2-1)-(1-d^2)(u^1-1)}\right)$
r_3			$\left(\frac{1}{1-d^1}, 0\right)$

Looking at the intersection points we immediately notice that since the (negative) slope of r_3 is always greater than the (negative) slope of r_4, namely:

$$-(1-d^1) > -\frac{1-d^1}{1-d^2},$$

(so that the intersection point between r_1 and r_4 always lies in the first quadrant, while the one between r_1 and r_3 is on the vertical axis), our domain will have three of its vertexes in "$r_1 \cap r_2$", "$r_1 \cap r_3$" and "$r_3 \cap r_4$".

On the contrary, we are not able, in general, to state whether the intersection points "$r_2 \cap r_3$"

and "$r_2 \cap r_4$" are in the second or fourth quadrant and it is, then, the localization of these points in the plane that determines the compactness or not of our domain. The conclusion follows from the following observations:

- if the (negative) slope of r_2 is greater than the (negative) slope of r_4, then "$r_2 \cap r_4$" is in the fourth quadrant and the domain is compact;
- if the (negative) slope of r_2 is smaller than the (negative) slope of r_4, then "$r_2 \cap r_4$" is in the second quadrant and the domain is unbounded.

□

As examples, we consider the two situations below:

Figure 2.5: $u^1 = 1.2,\ u^2 = 1.6,\ d^1 = 0.9,\ d^2 = 0.2$ (unbounded domain);

Figure 2.6: $u^1 = 1.2,\ u^2 = 1.3,\ d^1 = 0.5,\ d^2 = 0.4$ (domain with compact closure).

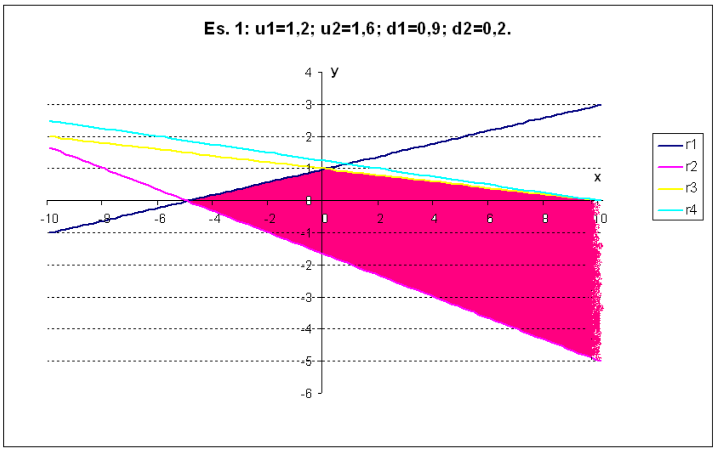


Figure 2.5: An example of unbounded domain.

The numerical approximation of the optimal maximizing values can be performed by means of a "search procedure", as we said in Section 2.5, over a finite number of points on a grid constructed on $\bar{\mathcal{D}}_0$. We show in Figure 2.7 one possible grid constructed on the domain of Figure 2.6.

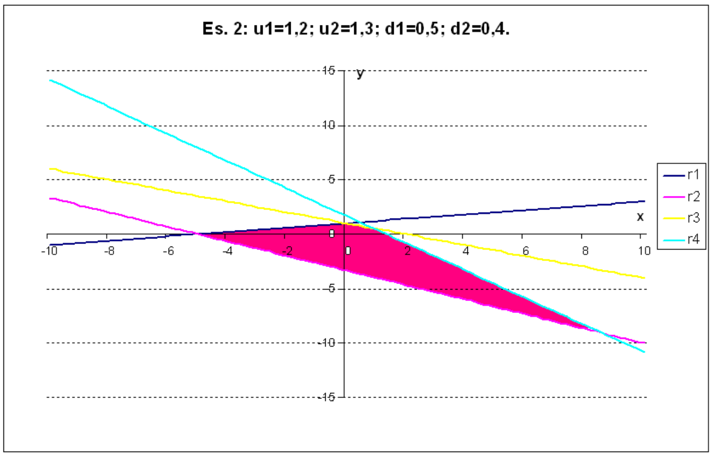


Figure 2.6: An example of domain with compact closure.

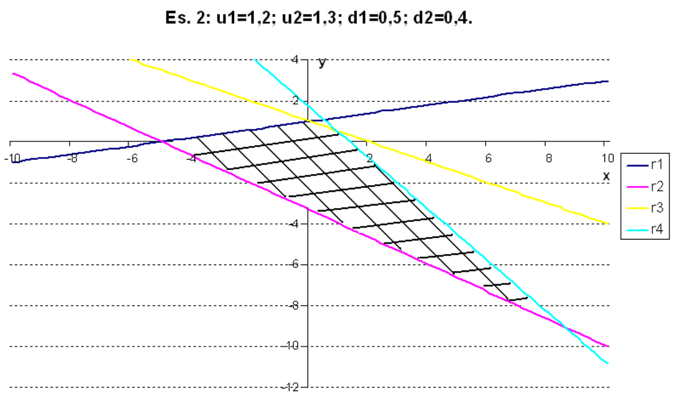


Figure 2.7: Example of grid constructed on a compact domain.

Bibliography

[1] A. Bensoussan. *Stochastic control of partially observable systems.* Cambridge University Press, 1992.

[2] D. Bertsekas. *Dynamic Programming and Stochastic Control.* Academic Press, 1976.

[3] M. Corsi, H. Pham, and W.J. Runggaldier. *Numerical Approximation by Quantization of Control Problems in Finance under Partial Observations*, volume XV of *Handbook of Numerical Analysis*, chapter Mathematical modelling and numerical methods in finance, pages 325–360. North Holland, 2008.

[4] R.-A. Dana and M. Jeanblanc. *Marchés Financières en Temps Continu, Valorisation et Équilibre, 2ème edition.* Recherche en gestion. Economica, 1998.

[5] D. Duffie, A. Eckner, G. Horel, and L. Saita. Frailty correlated defaults. *Journal of Finance*, LXIV(5):2089–2123, 2009.

[6] R. Korn. *Optimal portfolios. Stochastic Models for Optimal Investment and Risk Management in Continuous Time.* World Scientific Publishing Co. Pte. Ltd. Singapore, 1997.

[7] D.G. Luenberger. *Optimization by vector space methods.* Wiley New York, 1969.

[8] H. Pham, W.J. Runggaldier, and A. Sellami. Approximation by quantization of the filter process and applications to optimal stopping problems under partial observation. *Monte Carlo Methods and Applications*, 11:57–82, 2005.

[9] P.J. Schonbucher. Information-driven default contagion. Working Paper, Department of Mathematics, ETH Zurich, 2003.

[10] K. M. Van Hee. Bayesian Control of Markov Chains. *Mathematical Centre Tracts. Amsterdam*, 95, 1978.

Part IV

Optimal consumption problems in discontinuous markets

This is a joint work with Prof. M. Jeanblanc and Prof. W.J. Runggaldier.

Abstract: we study an extension of Merton's classical portfolio optimization problem (1969-1970) to a particular case of discontinuous market, with a single jump. The market consists of a non-risky asset, a "standard risky" asset and a risky asset with discontinuous price dynamics (e.g., a defaultable bond or a mortality linked security). We consider three different problems of maximization of the expected utility from consumption, in the cases when the investment horizon is fixed, when it is finite, but possibly uncertain, and when it is infinite.

We solve the problems by means of the martingale approach in a general stochastic coefficients model, in which, in the logarithmic utility case, we characterize the optimal investment-consumption strategy. Furthermore, we compare the optimal consumption rates for the three different problems, finding quite intuitive results.

In the constant and deterministic coefficients' cases explicit solutions are also obtained in the power utility case by applying, as an alternative technique, the Dynamic Programming approach (solving the related Hamilton-Jacobi-Bellman equation).

Explicit investment-consumption strategies are also provided in the exponential utility case, when market model coefficients are deterministic functions of time.

Keywords: Single-jump process, optimal consumption, discontinuous martingale, Dynamic Programming Principle, enlargement of filtrations.

Introduction

The starting point of this work is the acknowledgement of the fact that most investors, once entered the market, they never know with certainty when they are going to exit it. Factors that influence the decision of leaving the market are, for example, the market behavior itself, changes in the investor's endowment and exogenous shocks affecting the investor's consumption process (such as, the investor's death or the default of a firm whose assets are in his portfolio).

It is, then, both of practical and theoretical interest to study the influence of this uncertainty on the investor's decisions.

In particular, we consider an exogenously given nonnegative random variable τ, that is a totally inaccessible stopping time with respect to the investor's filtration, and we study three different scenarios: the one when the investment horizon is fixed and equal to T (problem **A**), the one when it is finite, but possibly uncertain, given by $T \wedge \tau$ (problem **B**), and when it is infinite (problem **C**). Our aim is to investigate the role of the source of randomness τ in the investor's decisions, when his objective is to maximize the expected utility from consumption, in a *complete* market model in which τ affects the assets's dynamics (think, for example, of a defaultable zero-coupon bond, in the case when τ is a default time, or of a mortality linked security, when τ is the death time of a pensioner).

The present work can, then, be seen within the theory of optimal stochastic control problems with uncertain time horizon. Some recent works in this direction are, e.g., Karatzas and Wang [15], who solve an optimal dynamic investment problem in a complete market case, when the uncertain time horizon is a stopping time in the asset price's filtration; Blanchet-Scaillet *et al.* [2], where they consider a maximization of expected utility from consumption problem, in a continuous market model, in the case when the time horizon is uncertain and the source of randomness is not a stopping-time in the investor's filtration and Bouchard and Pham [4], who consider, as opposite to the classical fixed time horizon setting, a wealth path-dependent utility maximization problem in an incomplete semimartingale model. In a more general setting, Zitković [25] formulates and solves a class of utility-maximization problems of the "stochastic clock" type (a stochastic clock is a mathematical tool to model the agent's notion of passage of time, see the more precise Definition 2.3 in [25]) in general incomplete semimartingale markets. Finally, in Menoncin [20], the author studies an optimal consumption-investment problem where the investment horizon is the death time of the investor and longevity bonds are traded in the market.

Here we solve three problems of maximization of expected utility from consumption in the case when on the market there is a risk-free asset (whose price process is denoted) S^0, a defaultable risky asset S^1, whose dynamics is driven by a Brownian motion W and a purely discontinuous martingale M and a "standard risky" asset S^2, whose dynamics only depends on W. The investor's filtration $\mathbb{G}$ (here "G" stands for "global") is the smallest filtration which contains the natural filtration of W, $\mathbb{F}$, and that makes τ a stopping time. We provide, in a very general stochastic coefficients case, comparison results between the optimal consumption rates of these three problems, showing that (as it should be) when the horizon is finite, but possibly uncertain (problem **B**), the investor consumes at a higher rate with respect to the case when the horizon is fixed (problem **A**). On the other hand, his consumption rate is higher in the case of problem **A** (finite horizon) than in the case of problem **C** (infinite horizon).

Furthermore, we show that, depending on whether the model coefficients are stochastic processes or deterministic functions of time, the investor's optimal investment strategy substantially changes. Namely, in the deterministic coefficients case, for an investor facing problems **A** and **C** the optimal investment strategy consists in not investing in the defaultable risky asset S^1: he acts in the market as if only the asset S^2 was traded. On the other hand, in the case of problem **B** (finite uncertain horizon $T \wedge \tau$), when the investment horizon depends on τ, he has to deal with this additional source of risk and it is, then, optimal to invest in the defaultable asset S^1 in order to have an optimal wealth that instantaneously jumps to zero at $T \wedge \tau$.

On the contrary, in the stochastic coefficients case, the market model coefficients are adapted with respect to the investor's filtration $\mathbb{G}$, so that, on the set $\{t > \tau\}$, they depend on τ. In this case, then, the investor has inevitably always to deal with τ (and not, as before in the deterministic coefficients case, only in the case when it appears in the investment horizon in problem **B**) and, as a consequence, the optimal proportion of wealth he invests in S^1 is never equal to zero.

This part is divided in five chapters and it is organized as follows. In the first chapter we consider a stochastic coefficients market model in which we suppose that immersion property holds between $\mathbb{F}$ and $\mathbb{G}$ (i.e, the Brownian motion W remains a martingale in the enlarged filtration $\mathbb{G}$), we study the market completeness and we solve the three problems by means of the martingale approach. We provide explicit optimal investment-consumption strategies in the log-utility case.

The second chapter is the analog to the first one in the case when model coefficients are deterministic. In this case, explicit optimal investment-consumption strategies are found in both the logarithmic and in the exponential utility cases.

In the third chapter, still focusing on the deterministic coefficients case, we solve the problems by using the Dynamic Programming approach, as an alternative technique. At the end of this chapter we study, with a mixed "martingale method - Dynamic Programming" solving method, a problem (denoted **B1**) in which the investor's strategy is $\mathbb{F}$-predictable (and no more $\mathbb{G}$-predictable). It is the case of an investor with a reduced set of information, who does not observe τ. We show that, in this case, the investor acts on the market with a modified utility function, that incorporates the conditional law (with respect to filtration $\mathbb{F}$) of the non-observable random variable τ.

In the fourth chapter we study, as a separate case, the exponential utility case, in the case of deterministic coefficients. We provide optimal solutions to the three problems that are explicit, but not so "talkative".

In the final chapter, we focus on an even more general market model, in which we do not suppose that immersion property holds between $\mathbb{F}$ and $\mathbb{G}$. By means of the tools developed in the fifth part of this thesis, relative to *enlargement of filtrations*, we provide explicit solutions to every considered problem in the log-utility case.

Chapter 3

A stochastic model

3.1 Market model and problem definition

On a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, equipped with a Brownian motion $(W_t)_{t\geq 0}$, we consider an (exogenously given) non-negative random variable τ, satisfying $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$, for any $t \in \mathbb{R}^+$. The law of τ is denoted by $\upsilon, \upsilon(d\theta) = \mathbb{P}(\tau \in d\theta)$. We assume that υ is absolutely continuous with respect to the Lebesgue measure and that (with a slight abuse of notation) $\upsilon(d\theta) = \upsilon(\theta)d\theta$. We can think, for example, of τ as the time of default of a firm issuing assets in the market, as the death time of a pensioner, or as a generic time occurrence of a shock in the market. As common, all the considered filtrations will be assumed to satisfy the "usual hypotheses" of right-continuity and completeness. We denote by $\mathbb{F} := \mathbb{F}^W = (\mathcal{F}_t^W)_{t\geq 0}$ the filtration generated by W, representing the information at disposal to investors before τ. When the shock τ occurs, this information becomes immediately accessible to investors, that add this knowledge to the reference filtration $\mathbb{F}$. Introducing the single-jump process $(H_t)_{t\geq 0}$,

$$H_t := \mathbb{1}_{\{t\geq\tau\}}, \quad \forall\, t \geq 0,$$

and denoting by $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ the natural filtration of H, this increase of information is modeled by saying that the investors' filtration $\mathbb{G}$ is, indeed, the (so-called) *progressively enlarged* filtration of $\mathbb{F}$ with τ (under $\mathbb{P}$). Namely, we define $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ by

$$\mathcal{G}_t := \bigcap_{\epsilon>0} \left\{\mathcal{F}_{t+\epsilon}^W \vee \mathcal{H}_{t+\epsilon}\right\},$$

and we have $\mathcal{F}_t^W \subsetneq \mathcal{G}_t$ for every $t \geq 0$. Being $\mathbb{H}$ the smallest filtration that makes τ a stopping time, τ is a $\mathbb{G}$-stopping time, too.

The financial market consists of a non-risky asset S^0, whose strictly positive price process has the dynamics

$$\mathrm{d}S_t^0 = r_t S_t^0 \mathrm{d}t, \quad S_0^0 = 1, \tag{3.1.1}$$

where the interest rate r is assumed to be a nonnegative uniformly bounded $\mathbb{G}$-adapted process, and of two risky assets (think for example of a defaultable zero-coupon bond, or a mortality-linked security, and of a "standard risky" asset, respectively), whose prices S^1 and S^2 evolve following the linear stochastic differential equations

$$\begin{cases} \mathrm{d}S_t^1 &= S_{t-}^1 \left(\mu_t^1 \mathrm{d}t + \sigma_t^1 \mathrm{d}W_t + \phi_t^1 \mathrm{d}M_t\right), \quad S_0^1 = s_0^1, \\ \mathrm{d}S_t^2 &= S_t^2 \left(\mu_t^2 \mathrm{d}t + \sigma_t^2 \mathrm{d}W_t\right), \quad\quad\quad\quad\; S_0^2 = s_0^2. \end{cases} \tag{3.1.2}$$

The coefficients μ^1, σ^1 and ϕ^1 are by hypothesis $\mathbb{G}$-predictable and uniformly bounded processes, with $\sigma_t^1 > 0, t \geq 0$, a.s., $\phi_t^1 > -1, t \geq 0$, a.s. (to guarantee that the price process S^1 always remains strictly positive) and $\phi_t^1 \neq 0, t \geq 0$, a.s. (if $\phi^1 = 0$ the market is a Black-Scholes market, that can present arbitrage opportunities). In the case when $\phi^1 = -1$ the asset's price S^1 jumps to zero and it remains at this level after the shock τ and results are different.

On the other hand, the processes μ^2 and σ^2, $\sigma^2 \neq \sigma^1$, are taken to be $\mathbb{F}$-predictable (in particular, they are also $\mathbb{G}$-predictable) and uniformly bounded, with $\sigma_t^2 > 0, t \geq 0$, a.s.

In the assets' dynamics (3.1.2) M represents the compensated (purely discontinuous) martingale associated with H, that we suppose being equal to

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda_s \mathrm{d}s = H_t - \int_0^t \mathbb{1}_{\{s<\tau\}} \lambda_s \mathrm{d}s = H_t - \int_0^t \bar{\lambda}_s \mathrm{d}s, \quad t \geq 0. \tag{3.1.3}$$

The process λ in the above Equation (3.1.3) denotes the non-negative and $\mathbb{F}-$adapted intensity rate of τ and we have introduced the $\mathbb{G}$-adapted process $\bar{\lambda}$, $\bar{\lambda}_t := \mathbb{1}_{\{t<\tau\}}\lambda_t$, $t \geq 0$. Notice that, by introducing the above representation of M, we have assumed that the compensator of H is absolutely continuous with respect to the Lebesgue measure, so that, in particular, τ is a $\mathbb{G}-$totally inaccessible stopping time (see Dellacherie and Meyer [8, Ch. IV, 107]).

Before defining the investment strategies, let us recall that any $\mathbb{G}$-predictable process Y (all the details concerning the characterization of $\mathbb{G}$-predictable processes can be found in the following Part V of this thesis) can be written in the form

$$Y_t(\omega) = \widetilde{y}_t(\omega)\mathbb{1}_{\{t\leq\tau(\omega)\}} + y_t(\omega, \tau(\omega))\mathbb{1}_{\{t>\tau(\omega)\}}, \quad t \geq 0,$$

where $\widetilde{y}$ is $\mathbb{F}$-predictable and where the function $(t, \omega, u) \to y_t(\omega, u)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable. Here $\mathcal{P}(\mathbb{F})$ denotes the predictable σ-algebra corresponding to $\mathbb{F}$ on $\mathbb{R}^+ \times \Omega$.

The economic interpretation of our assumptions on the coefficients is that when the shock perturbs the market, the interest rate r and the coefficients μ^1 and σ^1 of the stock price S^1 (and not the ones of S^2) switch from given processes $\widetilde{r}, \widetilde{\mu}^1, \widetilde{\sigma}^1$ to processes $r(\tau), \mu^1(\tau), \sigma^1(\tau)$. Their values after the perturbation on the market can obviously depend on τ. Since the martingale M is constant after τ, ϕ_t^1 plays no more role in the assets' dynamics when $t > \tau$ and we will simply use the notation ϕ_t^1 instead of $\widetilde{\phi}_t^1$, for $t \leq \tau$. For clarity we represent the notation used for the coefficients of the model in the following table:

	r	μ^1	σ^1	ϕ^1	μ^2	σ^2
$\{t \leq \tau\}$	$\widetilde{r}_t$	$\widetilde{\mu}_t^1$	$\widetilde{\sigma}_t^1$	ϕ_t^1	μ_t^2	σ_t^2
$\{t > \tau\}$	$r_t(\tau)$	$\mu_t^1(\tau)$	$\sigma_t^1(\tau)$	$\times$	μ_t^2	σ_t^2

Furthermore, we make the following assumption, that will be necessary, when working with deterministic coefficients, to have a well-defined model, to obtain explicit results and in order to avoid arbitrage opportunities, as we will see later on in Section 3.1.1.

Assumption 3.1.1. *a) (Immersion property) The Brownian motion W, that is an $\mathbb{F}$-martingale, is also a $\mathbb{G}$-martingale, i.e., the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$. This will be also referred to as the $(\mathcal{H})$-Hypothesis;*

b) The model coefficients satisfy

$$\begin{cases} \dfrac{\sigma_t^2(\widetilde{r}_t - \widetilde{\mu}_t^1) - \widetilde{\sigma}_t^1(\widetilde{r}_t - \mu_t^2)}{\phi_t^1 \sigma_t^2 \lambda_t} > -1 \quad a.s., \quad t \le \tau; \\ \\ \dfrac{r_t(\tau) - \mu_t^1(\tau)}{\sigma_t^1(\tau)} = \dfrac{r_t(\tau) - \mu_t^2}{\sigma_t^2} \qquad a.s., \quad t > \tau. \end{cases} \tag{3.1.4}$$

The above Assumption 3.1.1 b) implies that after the shock τ the market is "redundant". We would expect, then, to have the possibility to arbitrarily invest in S^1 (resp., S^2), since our portfolio can be re-balanced also by means of S^2 (resp., S^1), that, on the set $\{t > \tau\}$, has a dynamics proportional to the one of S^1 (resp., S^2).

In order to state our optimization problems, we now consider an investor, having an initial wealth $x_0 \ge 0$, who trades continuously in the financial market according to the *self-financing* investment strategy $\alpha_t = (\alpha_t^0, \alpha_t^1, \alpha_t^2)$, where α_t^i denotes the number of assets of S^i in his portfolio at time t. If, in addition, we suppose that he consumes at a consumption rate $c_t \ge 0$, then his wealth process X is driven by the stochastic differential equation

$$\mathrm{d}X_t = r_t X_t \mathrm{d}t + \alpha_t^1(\mathrm{d}S_t^1 - r_t S_t^1 \mathrm{d}t) + \alpha_t^2(\mathrm{d}S_t^2 - r_t S_t^2 \mathrm{d}t) - c_t \mathrm{d}t, \quad X_0 = x_0. \tag{3.1.5}$$

Equivalently, denoting by $\pi_t = (\pi_t^0, \pi_t^1, \pi_t^2)$ the proportion of wealth invested in the three assets (it is necessary here to suppose that, before the maturity, the wealth remains almost surely positive at any time), the investor's wealth dynamics is

$$\begin{aligned} \mathrm{d}X_t &= \left[r_t X_t + \pi_t^1 X_t(\mu_t^1 - r_t - \phi_t^1 \bar{\lambda}_t) + \pi_t^2 X_t(\mu_t^2 - r_t) - c_t\right] \mathrm{d}t + \\ &\quad \pi_t^1 \phi_t^1 X_{t-} \mathrm{d}H_t + \left[\pi_t^1 \sigma_t^1 X_t + \pi_t^2 \sigma_t^2 X_t\right] \mathrm{d}W_t, \quad X_0 = x_0. \end{aligned} \tag{3.1.6}$$

We now introduce the notion of admissible pair (π, c) that is general enough to be suitable for the three different problems we want to solve, namely in the case of a finite horizon optimization problem, in the case of a (possibly) uncertain finite horizon and for an infinite horizon problem.

Definition 3.1.1. *A pair (π, c) of portfolio and consumption processes satisfying*

- $(\pi_t)_{t\ge0}$ *$\mathbb{G}$-predictable and, for every $t \ge 0$ and for $i = 1, 2$,*

$$\int_0^t |\pi_s^i|^2 ds < +\infty \quad a.s. \quad and \quad \pi_\tau^1 \phi_\tau^1 \ge -1 \quad a.s.,$$

- $(c_t)_{t\ge0}$ *$\mathbb{G}$-adapted, non negative and such that, for every $t \ge 0$, $\int_0^t c_s ds < +\infty$, a.s.,*

is said to be admissible *for the initial endowment $x_0 \ge 0$, and we write $(\pi, c) \in Adm(x_0)$, if the corresponding wealth process X satisfies*

$$X_t \ge 0 \quad for\ every \quad 0 < t < \infty, \quad a.s.$$

Remark 3.1.1. *No sign restriction is made for what concerns the investment strategy, meaning that the agent may borrow or sell assets short.*
The condition $\pi_\tau^1 \phi_\tau^1 \ge -1$ a.s. ensures that the wealth X remains nonnegative after the shock τ and it also tells that, before τ,

- *If $\phi^1 > 0$ a.s., then it is not possible to take "too short" positions in S^1;*
- *If $\phi^1 \in (-1, 0)$ a.s., then it is not possible to take "too long" positions in S^1.*

For a given fixed time horizon $T \in (0, +\infty)$, we are interested in solving the following three types of maximization of the expected utility from consumption problems under the historical measure $\mathbb{P}$ (notice that the path-wise constraint of positivity of the wealth process X is "hidden" in the admissibility set $\mathrm{Adm}(x_0)$):

$$\underbrace{\sup_{(\pi,c)\in\mathcal{A}(x_0)} \mathbb{E}\int_0^T u(c_s)\mathrm{d}s,}_{\mathbf{A}} \quad \underbrace{\sup_{(\pi,c)\in\mathcal{A}_\tau(x_0)} \mathbb{E}\int_0^{T\wedge\tau} u(c_s)\mathrm{d}s,}_{\mathbf{B}} \quad \underbrace{\sup_{(\pi,c)\in\mathcal{A}_\infty(x_0)} \mathbb{E}\int_0^{+\infty} e^{-\rho s}u(c_s)\mathrm{d}s}_{\mathbf{C}} \tag{3.1.7}$$

where

$$\mathcal{A}(x_0) := \left\{(\pi, c) \in Adm(x_0) : \mathbb{E}\int_0^T \min\{0, u(c_s)\}\mathrm{d}s > -\infty\right\}, \tag{3.1.8}$$

$$\mathcal{A}_\tau(x_0) := \left\{(\pi, c) \in Adm(x_0) : \mathbb{E}\int_0^{T\wedge\tau} \min\{0, u(c_s)\}\mathrm{d}s > -\infty\right\}, \tag{3.1.9}$$

$$\mathcal{A}_\infty(x_0) := \left\{(\pi, c) \in Adm(x_0) : \mathbb{E}\int_0^{+\infty} e^{-\rho s}\min\{0, u(c_s)\}\mathrm{d}s > -\infty\right\} \tag{3.1.10}$$

(so that the three problems are well-defined) and where $\rho > 0$ is a discounting factor. Furthermore, for what concerns $u(c)$, the utility of consuming at a rate c, we assume that (the exponential utility case will be treated as a separate example in Chapter 6) $u : (0, \infty) \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable and satisfies

$$u'(0^+) = \lim_{c\downarrow 0} u'(c) = +\infty \quad \text{and} \quad u'(\infty) = \lim_{c\to+\infty} u'(c) = 0. \tag{3.1.11}$$

We allow $u(0) = \lim_{c\downarrow 0} u(c)$ to be equal to $-\infty$. Notice that under the above assumptions the marginal utility function u' is invertible.

Definition 3.1.2. *We denote by I the continuous and strictly decreasing inverse of u', defined on $]u'(\infty), u'(0^+)[$, namely $I : (0, \infty) \to (0, \infty)$, that satisfies $I(0^+) = \infty$ and $I(\infty) = 0$.*

3.1.1 The unique EMM $\mathbb{Q}^*$

From the predictable representation property in the case of filtration $\mathbb{G}$, under the $(\mathcal{H})$-Hypothesis (see, e.g., Chesney, Jeanblanc and Yor [5, Th. 7.5.5.1]), if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures, we know that there exist two $\mathbb{G}-$predictable processes ψ and γ, with $\gamma > -1$ a.s., such that the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ can be written as

$$Z_t := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}_{|\mathcal{G}_t} = 1 + \int_{]0,t]} Z_{u-}\left(\psi_u \mathrm{d}W_u + \gamma_u \mathrm{d}M_u\right), \quad t \geq 0.$$

Notice that the above result does not require the independence of W and M, but we will need their orthogonality (that is, indeed, obvious).

In our case we show that, under Assumption 3.1.1 b), the market is complete, i.e., there exists a unique equivalent martingale measure $\mathbb{Q}^*$. In fact, by imposing the (local) martingale property to the discounted value processes of S^1 and S^2, under the measure $\mathbb{Q}^*$, we find that the processes ψ^* and γ^* in the Radon-Nikodým density Z^* (provided that this process is a true martingale, as in our case, given that the model coefficients are uniformly bounded) have to satisfy the following two conditions, in order to have the existence of at least one EMM

$$\begin{cases} \mu_t^1 - r_t + \sigma_t^1 \psi_t^* + \phi_t^1 \gamma_t^* \bar{\lambda}_t = 0 \\ \\ \mu_t^2 - r_t + \sigma_t^2 \psi_t^* = 0. \end{cases} \tag{3.1.12}$$

By distinguishing between values before and after the shock, we find that there exists at least one EMM $\mathbb{Q}^*$ if $\widetilde{\psi}^*, \psi(\tau)$ and γ^* satisfy

$$\begin{aligned} \psi_t^* &= \begin{cases} \widetilde{\psi}_t^* = \dfrac{\widetilde{r}_t - \mu_t^2}{\sigma_t^2} \quad \text{a.s.,} & \text{if} \quad t \le \tau; \\ \psi_t^*(\tau) = \dfrac{r_t(\tau) - \mu_t^1(\tau)}{\sigma_t^1(\tau)} = \dfrac{r_t(\tau) - \mu_t^2}{\sigma_t^2} \quad \text{a.s.,} & \text{if} \quad t > \tau; \end{cases} \\ \gamma_t^* &= \begin{cases} \dfrac{\sigma_t^2(\widetilde{r}_t - \widetilde{\mu}_t^1) - \widetilde{\sigma}_t^1(\widetilde{r}_t - \mu^2)}{\sigma_t^2 \phi_t^1 \lambda_t} > -1 \quad \text{a.s.,} & \text{if} \quad t \le \tau; \\ \\ \text{any predictable } \gamma_t^* > -1 \quad \text{a.s.,} & \text{if} \quad t > \tau. \end{cases} \end{aligned} \tag{3.1.13}$$

Given Assumption 3.1.1 b), such an EMM exists (notice that ψ^* and γ^* are, indeed, $\mathbb{G}$-predictable) and the market is arbitrage free. Furthermore, the processes ψ^* and γ^* are uniquely determined, so that (from the Second Fundamental Theorem of Asset Pricing) the market is complete.

Even if γ^* is not uniquely defined after τ, the process Z^* is uniquely assigned, since γ^* does not affect the dynamics of Z^* after the shock. The Radon-Nikodým density is then given, for every $t \ge 0$, by

$$Z_t^* = e^{\int_0^t \psi_s^* dW_s - \frac{1}{2}\int_0^t (\psi_s^*)^2 ds} \, e^{-\int_0^t \gamma_s^* \bar{\lambda}_s \mathrm{d}s} \, (1 + \gamma_\tau^*)^{H_t} \quad \text{a.s.}$$

or, more precisely, by

$$Z_t^* = \begin{cases} e^{\int_0^t \widetilde{\psi}_s^* dW_s - \frac{1}{2}\int_0^t (\widetilde{\psi}_s^*)^2 ds} e^{-\int_0^t \gamma_s^* \lambda_s \mathrm{d}s} & \text{if} \quad t < \tau; \\ Z_{\tau-}^* (1 + \gamma_\tau^*) & \text{if} \quad t = \tau; \\ e^{\int_0^\tau \widetilde{\psi}_s^* dW_s + \int_\tau^t \psi_s^*(\tau) dW_s - \frac{1}{2}\int_0^\tau (\widetilde{\psi}_s^*)^2 ds - \frac{1}{2}\int_\tau^t (\psi_s^*(\tau))^2 ds} e^{-\int_0^\tau \gamma_s^* \lambda_s \mathrm{d}s} \, (1 + \gamma_\tau^*) & \text{if} \quad t > \tau; \end{cases} \tag{3.1.14}$$

We can summarize the results of this section in the following Lemma.

Lemma 3.1.1. *The conditions*

$$\frac{\sigma_t^2(\widetilde{r}_t - \widetilde{\mu}_t^1) - \widetilde{\sigma}_t^1(\widetilde{r}_t - \mu_t^2)}{\phi_t^1 \sigma_t^2 \lambda_t} > -1 \quad a.s., \quad t \leq \tau;$$

$$\frac{r_t(\tau) - \mu_t^1(\tau)}{\sigma_t^1(\tau)} = \frac{r_t(\tau) - \mu_t^2}{\sigma_t^2} \quad a.s., \qquad t > \tau.$$

are necessary and sufficient to ensure the absence of arbitrage in our market.

3.1.2 From the admissibility conditions to the budget constraints

In this subsection, we show how the infinite dimensional constraint $X_t \geq 0$ for every $t \geq 0$, a.s., that is required in the Definition 3.1.1 of admissible investment-consumption strategies, can be rewritten in a form that is more practical to use.

For the moment we concentrate on the case when the investment horizon is bounded, i.e., $t \in [0, T]$, namely we focus on problems **A** and **B**. For the infinite horizon case, we refer to Section 3.2.5.

First of all notice that the condition $X_t \geq 0, t \in [0, T]$, $\mathbb{P}-$a.s., remains valid under $\mathbb{Q}$, if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures. In our setting, furthermore, looking at Equation (3.1.5), it is clear that if the consumption-investment strategy is admissible, the process

$$\left(e^{-\int_0^t r_s ds} X_t + \int_0^t e^{-\int_0^s r_u du} \; c_s \mathrm{d}s\right)_{t \geq 0} \tag{3.1.15}$$

is a positive $\mathbb{G}$-local martingale (hence a super-martingale, by Fatou's Lemma) under the unique equivalent martingale measure $\mathbb{Q}^*$.

Given the definition of an admissible investment-consumption strategy, it is then clear that the following (so-called) budget constraint is a necessary condition for admissibility

$$\mathbb{E}^{\mathbb{Q}^*}\left(e^{-\int_0^T r_s ds} X_T + \int_0^T e^{-\int_0^s r_u du} \; c_s \mathrm{d}s\right) \leq x_0 \tag{3.1.16}$$

and that the following inequality holds true

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{-\int_0^s r_u du} \; c_s \mathrm{d}s\right) \leq x_0. \tag{3.1.17}$$

Our aim being to maximize the expected utility from consumption, it is clear that, at the optimum, we would like to saturate the above inequality in order to solve our problems, so that in the case of problems **A** and **B**, the necessary conditions for the optimality of the consumption strategy c^*, that are, respectively, $X_T^* = 0$ and $X_{T\wedge\tau}^* = 0$, are equivalent to

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{-\int_0^s r_u du} \; c_s^* \mathrm{d}s\right) = x_0 \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{T\wedge\tau} e^{-\int_0^s r_u du} \; c_s^* \mathrm{d}s\right) = x_0. \tag{3.1.18}$$

By considering their analog at time t, we find expressions for the optimal wealth at time $t \leq T$, in the case of problems **A** and **B**:

$$\begin{aligned} X_t^* &= e^{\int_0^t r_s ds} \, \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-\int_0^s r_u du} \; c_s^* \mathrm{d}s | \mathcal{G}_t\right) && \text{a.s., for } t \leq T, \\ X_t^* &= e^{\int_0^t r_s ds} \, \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^{T\wedge\tau} e^{-\int_0^s r_u du} \; c_s^* \mathrm{d}s | \mathcal{G}_t\right) && \text{a.s., for } t \leq (T \wedge \tau). \end{aligned}$$

From the previous equations, the optimal wealth is clearly positive (indeed, any wealth associated with an admissible nonnegative consumption, having nonnegative final value, is positive).

Furthermore, inspired by Proposition 2.1 in Jeanblanc and Pontier [13], we can show that the budget constraint (3.1.16) (resp., its analog where $T \wedge \tau$ replaces T) is also a sufficient condition for the *admissibility* in problem **A** (resp., **B**), as stated in the following Proposition.

Proposition 3.1.1. *Let $x_0 \geq 0$ be given, let c be a consumption process, satisfying a suitable integrability condition as in Definition 3.1.1, and let ξ be a nonnegative and square integrable $\mathcal{G}_T$-measurable random variable, such that*

$$\mathbb{E}^{\mathbb{Q}^*}\left(e^{-rT}\xi + \int_0^T e^{-rs} c_s ds\right) = x_0.$$

Then, there exists a portfolio process α, such that the pair (α, c) is admissible for the initial endowment x_0 and the associated terminal wealth X_T is equal to ξ.

Proof. The proof is based on the predictable representation property and it can be straightforwardly obtained from the one in Proposition 2.1 in Jeanblanc and Pontier [13], by recalling that in the case when $\mathbb{F}$ is the Brownian filtration and the $(\mathcal{H})$-Hypothesis holds between $\mathbb{F}$ and $\mathbb{G}$, a $\mathbb{G}$-martingale representation theorem was proved by Kusuoka in [19, Th. 2.3] (see also Section 7.5.5 in Chesney, Jeanblanc and Yor [5]). □

The sufficiency of the admissibility condition in the case of problem **B** is shown in the same way, using the key fact that τ is a $\mathbb{G}$-stopping time.

It is interesting to notice that, as observed in Karatzas and Shreve [14, Remark 3.4], from the $(\mathbb{Q}^*, \mathbb{G})$-martingale property of the process defined in Equation (3.1.15): "Bankruptcy is an absorbing state for the wealth process X when (π, c) is an admissible control".

3.2 The solution: martingale approach

3.2.1 Problem A: optimal consumption

In this case, being the investment horizon fixed, in order to maximize his consumption, the investor's aim is necessarily to end up at time T with an optimal wealth satisfying $X_T^* = 0$. In the following proposition, we provide a general result concerning the optimal consumption process. Notice that it can be found, as clarified in Remark 3.2.1 b), by adapting a martingale approach (recall that here the equivalent martingale measure $\mathbb{Q}^*$ is unique).

Proposition 3.2.1. *Given the market structure (3.1.2), the optimal consumption rate solving problem **A** in (3.1.7) is given by*

$$c_s^{*,A} = I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) \quad a.s., \tag{3.2.1}$$

where I denotes the inverse function of u', $\nu > 0$ is a real parameter satisfying

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{-\int_0^s r_u du} I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) ds\right) = \mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du} Z_s^* I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) ds\right) = x_0$$

and we recall that Z^* is the Radon-Nikodým density process introduced in Equation (3.1.14).

Proof. Given the concavity property of u and the definition of $c^{*,A}$ in (3.2.1), we have:

$$\begin{aligned}\mathbb{E}\left(\int_0^T [u(c_s)-u(c_s^{*,A})]\mathrm{d}s\right) &\le \mathbb{E}\left(\int_0^T (c_s-c_s^{*,A})u'(c_s^{*,A})\mathrm{d}s\right)\\ &= \mathbb{E}\left(\int_0^T (c_s-c_s^{*,A})\nu e^{-\int_0^s r_u du}Z_s^*\mathrm{d}s\right)\\ &\le \nu(x_0-x_0)=0,\end{aligned}$$

where in the last inequality we have used the fact that (see Section 3.1.2), under the measure $\mathbb{P}$, any admissible consumption strategy c and so the optimal one $c^{*,A}$ as well satisfy, respectively,

$$\mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du}c_s Z_s^*\mathrm{d}s\right)\le x_0,\quad \mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du}c_s^{*,A} Z_s^*\mathrm{d}s\right)= x_0.$$

The optimality of $c^{*,A}$ is proved. □

What about its existence? Problem **A** admits a solution under the following assumption (it corresponds to Assumption 7.1 in Karatzas and Shreve [14], see also a similar analysis in Korn and Korn [17] and in Dana and Jeanblanc [6]).

Assumption 3.2.1. *The function*

$$\Psi^A(\nu):=\mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du}Z_s^* I\left(\nu e^{-\int_0^s r_u du}Z_s^*\right)ds\right)$$

is finite for every $0<\nu<\infty$.

Indeed, in this case it can be proved, as already done in the literature (see, e.g., Lemma 4.5.2 in Dana and Jeanblanc [6]), that Ψ^A is non-increasing and continuous on $(0,\infty)$, with $\Psi^A(0+)=\infty$. Furthermore, it is strictly decreasing on $(0,\bar{\nu})$, where $\bar{\nu}:=\inf\{\nu|\Psi(\nu)=0\}$ and, when restricted to this interval, it admits then a continuous strictly decreasing inverse. This implies that ν satisfying the budget constraint, i.e., $\nu=(\Psi^A)^{-1}(x_0)$ exists and so does $c^{*,A}$ (notice that, given the above Assumption 3.2.1, such a $c^{*,A}$ is admissible: the optimal consumption strategy is continuous, except at time $t=\tau$). A detailed analysis on the regularity conditions to be required on the model coefficients for Assumption 3.2.1 to hold true can be found, e.g., in Remark 3.6.8 and Remark 3.6.9 in Karatzas and Shreve [14]. Let us only remark here that in the deterministic coefficients case, for the logarithmic and power utility cases, explicit solutions are available, as we will see later on.

Remark 3.2.1. *a) As expected, the optimal consumption strategy is made of two parts: one before the shock and the other one after the shock.*
*b) Equation (3.2.1) can be "directly" obtained by considering the Lagrangian function associated with problem **A**, with the admissibility constraint*
$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{-\int_0^s r_u du}\ c_s ds\right)\le x_0$*, namely*

$$\begin{aligned}\mathcal{L}(c,\nu;x_0) &:= \mathbb{E}\left(\int_0^T u(c_s)ds-\nu Z_T^*\int_0^T e^{-\int_0^s r_u du}c_s ds\right)+\nu x_0\\ &= \mathbb{E}\left(\int_0^T u(c_s)ds-\nu\int_0^T e^{-\int_0^s r_u du}\ Z_s^*\ c_s ds\right)+\nu x_0\end{aligned}$$

and by formally maximizing the expectation by looking for the supremum in the integrand function "ω per ω" (see also some comments about this method in Korn and Korn [17], page 208).

▷ **The log-utility case.**
In the case when u is the logarithmic function, the inverse of its derivative is $I(y) = 1/y$, so that for every $0 \le s \le T$ we find

$$c_s^{*,A} = \frac{1}{\nu e^{-\int_0^s r_u du} Z_s^*} = \frac{x_0}{T e^{-\int_0^s r_u du} Z_s^*} \quad \text{a.s.,} \tag{3.2.2}$$

since it is straightforward to prove that ν in this case is equal to T/x_0.
▷ **The power-utility case.**
As a second example, we consider the power utility case, i.e., we choose $u(x) = \frac{x^\gamma}{\gamma}, x \ge 0, \gamma < 1, \gamma \neq 0$, so that $I(y) = (y)^{\frac{1}{\gamma-1}}$. In this case, then, we have

$$c_s^{*,A} = \left(\nu e^{-\int_0^s r_u du} Z_s^*\right)^{\frac{1}{\gamma-1}} = x_0 \frac{\left(e^{-\int_0^s r_u du} Z_s^*\right)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^T \left(e^{-\int_0^u r_v dv} Z_u^*\right)^{\frac{\gamma}{\gamma-1}} \mathrm{d}u} \quad \text{a.s.,} \tag{3.2.3}$$

since it can be checked that ν is given by

$$\nu = \left[\frac{\mathbb{E}\int_0^T \left(e^{-\int_0^u r_v dv} Z_u^*\right)^{\frac{\gamma}{\gamma-1}} \mathrm{d}u}{x_0}\right]^{1-\gamma}.$$

We will provide a more explicit result in the case of deterministic coefficients.

3.2.2 Problem A: optimal investment strategy

In order to completely solve our problem, we need to obtain the optimal investment strategy π^*. This can be done, knowing c^*, through the knowledge of X^*, as we presently explain. We first obtain the optimal wealth by computing the following conditional expectation

$$X_t^* = e^{\int_0^t r_s ds}\, \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-\int_0^s r_u du} c_s^* \,\mathrm{d}s \mid \mathcal{G}_t\right) \qquad \text{a.s.}$$

Then, we compute the stochastic differential of X^* and we finally identify, term by term, the dynamics dX_t^* and the theoretical dynamics dX_t (under $\mathbb{Q}^*$), thus leading us to $\pi_t^{1,*}$ and $\pi_t^{2,*}$ (and, as a consequence, to $\pi_t^{0,*}$).

We now explicitly characterize the optimal investment strategies in the case of a logarithmic utility function. In the power utility case the computations do not lead to an explicit solution, so that we refer the reader to the deterministic coefficient case that follows.
▷ **The log-utility case.**
From Equation (3.2.2), we know that (under $\mathbb{P}$)

$$c_s^* = \frac{x_0\, e^{\int_0^s r_u du}}{T Z_s^*} \quad \text{a.s.,} \quad s \le T.$$

Notice that this expression, as any other expression depending on Z^*, is not a real investment rule, since Z^* is not observable in the market. We refer to Equation (3.2.7) below for a more useful form of c^*.

A direct computation, applying the conditional version of Fubini-Tonelli's theorem and recalling that $(Z^*)^{-1}$ is a $(\mathbb{Q}^*, \mathbb{G})-$martingale, gives us

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-\int_0^s r_u du} c_s^* \,\mathrm{d}s \mid \mathcal{G}_t\right) = \frac{x_0}{T}\int_t^T \mathbb{E}^{\mathbb{Q}^*}\left[(Z_s^*)^{-1}|\mathcal{G}_t\right]\mathrm{d}s = \frac{x_0(T-t)}{T\ Z_t^*} \quad \text{a.s.},$$

so that the optimal wealth is (under $\mathbb{Q}^*$)

$$X_t^* = e^{\int_0^t r_s ds}\,\frac{x_0(T-t)}{T\ Z_t^*}, \quad \text{a.s.}, \quad t \le T.$$

Notice that $X_T^* = 0$. In order to obtain the stochastic differential of X^*, we introduce the $(\mathbb{Q}^*, \mathbb{G})$-martingales

$$W_t^* := W_t - \int_0^t \psi_s^* ds \quad \text{and} \quad M_t^* := M_t - \int_0^t \gamma_s^* \bar{\lambda}_s ds, \quad t \ge 0,$$

and we first compute (recall that ψ^* and γ^* are $\mathbb{G}$-predictable)

$$d\left(\frac{1}{Z_t^*}\right) = \frac{1}{Z_{t-}^*}\left[-\psi_t^* dW_t^* - \frac{\gamma_t^*}{1+\gamma_t^*} dM_t^*\right]. \tag{3.2.4}$$

We then easily find

$$\mathrm{d}X_t^* = X_{t-}^*\left[\left(r_t - \frac{1}{T-t}\right)\mathrm{d}t - \psi_t^*\mathrm{d}W_t^* - \frac{\gamma_t^*}{1+\gamma_t^*} dM_t^*\right], \quad X_0^* = x_0. \tag{3.2.5}$$

To determine $\pi^{1,*}$ and $\pi^{2,*}$ it suffices, as it is standard in continuous time, to identify term by term the above equation and Equation (3.1.6) written under the measure $\mathbb{Q}^*$ (notice that in order to obtain the dynamics in the equations that follow we need the explicit characterization of ψ^* and γ^* in Equation (3.1.13) and, *a priori*, we have to distinguish between the cases when $t \le \tau$ and when $t > \tau$), namely

$$\mathrm{d}X_t = (r_t X_t - c_t)\,\mathrm{d}t + \pi_t^1 \phi_t^1 X_{t-}\mathrm{d}M_t^* + X_t\left(\pi_t^1\sigma_t^1 + \pi_t^2\sigma_t^2\right)\mathrm{d}W_t^*, \quad X_0 = x_0. \tag{3.2.6}$$

We finally find that

$$c_t^* = \frac{X_t^*}{T-t}, \quad 0 \le t \le T \tag{3.2.7}$$

and

$$\begin{cases} \pi_t^{1,*} = -\dfrac{\gamma_t^*}{\phi_t^1(1+\gamma_t^*)}, \quad \pi_t^{2,*} = -\dfrac{\widetilde{\psi}_t^*}{\sigma_t^2} + \dfrac{\gamma_t^*\widetilde{\sigma}_t^1}{\sigma_t^2\phi_t^1(1+\gamma_t^*)} \quad \text{a.s.}, \quad t \le \tau; \\ \pi_t^{1,*}\sigma_t^1(\tau) + \pi_t^{2,*}\sigma_t^2 = -\psi_t^*(\tau) \quad \text{a.s.}, \qquad t > \tau. \end{cases} \tag{3.2.8}$$

In our complete market, after τ, the investment strategy is not unique, as expressed in the previous equation. This is due to the fact that, being the market arbitrage free, we have $\frac{r_t(\tau)-\mu_t^1(\tau)}{\sigma_t^1(\tau)} = \frac{r_t(\tau)-\mu_t^2}{\sigma_t^2}$ a.s., $t > \tau$, and so after the shock the two assets S^1 and S^2 have proportional dynamics, i.e., the market is redundant.

3.2.3 Problem B: optimal consumption

As previously remarked, in order to maximize his consumption, the investor's aim in this case is to end up with an optimal wealth satisfying

$$X^*_{T\wedge\tau} = 0 \quad \text{a.s.} \tag{3.2.9}$$

(this condition immediately implies that c^* is equal to zero after τ). It is important to underline that the constraint $X^*_{T\wedge\tau} = 0$ makes sense since the investor's filtration is $\mathbb{G}$ (τ is a $\mathbb{G}$-stopping time), while it would not be the case if working with the filtration $\mathbb{F}$. We will go more into details on this aspect later on, in Section 5.4. We have already observed (recall Equation (3.1.18)) that the condition $X^*_{T\wedge\tau} = 0$ can be equivalently written as

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{T\wedge\tau} e^{-\int_0^s r_u du} c_s^{*,B} \mathrm{d}s\right) = \mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T (1-H_s) e^{-\int_0^s r_u du} c_s^{*,B} \mathrm{d}s\right) = x_0,$$

so that problem **B** consists in maximizing, over all the admissible investment-consumption strategies, the expected utility from consumption

$$\mathbb{E}\int_0^{T\wedge\tau} u(c_s)\mathrm{d}s = \mathbb{E}\int_0^T (1-H_s)u(c_s)\mathrm{d}s,$$

with the above constraint. An application of the martingale approach in this case provides (with no additional technical difficulties linked to the presence of the factor $(1-H_s)$ in the integrand) the optimal consumption rate process before the shock and we state the analog to Proposition 3.2.1 (for this reason, we omit its proof).

Proposition 3.2.2. *Given the market structure (3.1.2), the optimal consumption rate solving problem **B**, with the terminal condition $X^*_{T\wedge\tau} = 0$, is given by*

$$c_s^{*,B} = I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) \quad \textit{a.s.}, \quad s \le (T\wedge\tau), \tag{3.2.10}$$

where I denotes the inverse function of u' and $\nu > 0$ is a real parameter satisfying

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{T\wedge\tau} e^{-\int_0^s r_u du} I\left(e^{-\int_0^s r_u du}\nu Z_s^*\right) ds\right) = \mathbb{E}\left(\int_0^{T\wedge\tau} e^{-\int_0^s r_u du} Z_s^* I\left(e^{-\int_0^s r_u du}\nu Z_s^*\right) ds\right) = x_0.$$

As previously remarked in the case of problem **A** here, too, the existence of the optimal $c^{*,B}$ is not immediate and it derives (the reasoning is the same as in Section 3.2.1) from the following

Assumption 3.2.2. *The function*

$$\Psi^B(\nu) := \mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du}(1-H_s) Z_s^* I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) ds\right)$$

is finite for every $0 < \nu < \infty$.

Concerning a comparison between the two optimal investment strategies $c^{*,A}$ and $c^{*,B}$, we now show that, before the shock τ, an investor facing problem **B** consumes at a higher rate than an investor facing problem **A**, as one might expect.

Proposition 3.2.3. *Before the shock τ, i.e., for $s \leq (T \wedge \tau)$, under Assumption 3.2.1 and Assumption 3.2.2,*

$$c_s^{*,B} \geq c_s^{*,A} \quad a.s.$$

Proof. We have to compare, for $s \leq \tau$,

$$c_s^{*,B} = I\left(\nu^B e^{-\int_0^s r_u du} Z_s^*\right) \quad \text{and} \quad c_s^{*,A} = I\left(\nu^A e^{-\int_0^s r_u du} Z_s^*\right),$$

where ν^B and ν^A are positive (their existence is ensured by Assumption 3.2.1 and Assumption 3.2.2) and they satisfy, respectively,

$$\mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du} Z_s^* (1 - H_s) I\left(e^{-\int_0^s r_u du} \nu^B Z_s^*\right) \mathrm{d}s\right) = x_0$$

and

$$\mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du} Z_s^* I\left(e^{-\int_0^s r_u du} \nu^A Z_s^*\right) \mathrm{d}s\right) = x_0.$$

Recall that I denotes the strictly decreasing inverse function of u'. Equivalently, $\nu^A = (\Psi^A)^{-1}(x_0)$ and $\nu^B = (\Psi^B)^{-1}(x_0)$. Furthermore, since $(1 - H_u) \leq 1, \forall u \in [0, T]$, $\Psi^B \leq \Psi^A$ and so $(\Psi^B)^{-1} \leq (\Psi^A)^{-1}$. We then have $\nu^B \leq \nu^A$ and we are done, since I is decreasing. □

The explicit solution to problem **B** and an explicit comparison with the optimal strategy in case **A** are given below in the two usual examples.

▷ **The log-utility case.**

Here ν is found to be the solution of the following equation

$$\mathbb{E}\int_0^{T\wedge\tau} \frac{1}{\nu} ds = \frac{1}{\nu}\, \mathbb{E}(T \wedge \tau) = x_0,$$

namely $\nu = \mathbb{E}(T \wedge \tau)/x_0$. We then find

$$c_s^{*,B} = \frac{x_0}{\mathbb{E}(T \wedge \tau) Z_s^* e^{-\int_0^s r_u du}} \quad \text{a.s.,} \quad s \leq (T \wedge \tau). \tag{3.2.11}$$

Being $T\wedge\tau$ always smaller than T, it is evident that the investor facing problem **B** consumes (before the shock τ) at a higher rate than the investor facing problem **A** (recall Equation (3.2.2)), as we expected from Proposition 3.2.3.

▷ **The power-utility case.**

Here we have

$$c_s^{*,B} = x_0 \frac{\left(e^{-\int_0^s r_u du} Z_s^*\right)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^T (1 - H_u)\left(e^{-\int_0^u r_v dv} Z_u^*\right)^{\frac{\gamma}{\gamma-1}} \mathrm{d}u} \quad \text{a.s.,} \quad s \leq (T \wedge \tau), \tag{3.2.12}$$

since ν is given by

$$\nu = \left[\frac{\mathbb{E}\int_0^T (1 - H_u)(e^{-\int_0^u r_v dv} Z_u^*)^{\frac{\gamma}{\gamma-1}} \mathrm{d}u}{x_0}\right]^{1-\gamma}.$$

In this power utility case, too, as expected, being $1 - H_u \leq 1$ a.s. for every u, the optimal consumption rate is greater (before τ) than the one we found for problem **A**, that is the one in Equation (3.2.3). An explicit result will be provided in the case of deterministic coefficients.

3.2.4 Problem B: optimal investment strategy

As previously done for problem **A**, we now focus on the computation of the optimal investment strategy. The optimal wealth here is given by (recall Section 3.1.2)

$$X_t^* = e^{\int_0^t r_s ds}\, \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T \mathbb{1}_{\{s<\tau\}} e^{-\int_0^s r_u du} c_s^* \,\mathrm{d}s \mid \mathcal{G}_t\right) \quad \text{a.s.}$$

Before focusing on the logarithmic utility case, that is the only one in which the computation can be performed up to the end, in the following lemma we state an interesting and quite general result.

Lemma 3.2.1. *Let Y be an $\mathbb{F}-$predictable process. If $Y_\tau \geq 0$ then*

$$Y_t \geq 0 \quad \textit{a.s.} \quad \textit{for every } t \leq \tau.$$

Proof. Given $h>0$, we have $Y_\tau \mathbb{1}_{t<\tau\leq t+h} \geq 0$ a.s and $\mathbb{E}\left(Y_\tau \mathbb{1}_{t<\tau\leq t+h} | \mathcal{F}_t^W\right) \geq 0$ a.s. From the definition of *predictable compensator* (see, e.g., Definition 5.2.1.5 in Chesney, Jeanblanc and Yor [5]) associated with τ, we also have

$$\mathbb{E}\left(Y_\tau \mathbb{1}_{t<\tau\leq t+h}\ |\mathcal{F}_t^W\right) = \mathbb{E}\left(\int_t^{t+h} Y_u dA_u^\tau\ |\mathcal{F}_t^W\right) \geq 0 \quad \text{a.s.},$$

where A^τ is the $\mathbb{F}$-dual predictable projection of the process H. Writing the (general) Doob-Meyer decomposition of the super-martingale G as $G_t = \mu_t^\tau - A_t^\tau$ (see, e.g., Proposition 5.9.4.3 in Chesney, Jeanblanc and Yor [5]), we then find

$$\mathbb{E}\left(Y_\tau \mathbb{1}_{t<\tau\leq t+h}\ |\mathcal{F}_t^W\right) = -\mathbb{E}\left(\int_t^{t+h} Y_u dG_u\ |\mathcal{F}_t^W\right) = \mathbb{E}\left(\int_t^{t+h} Y_u G_u \lambda_u du\ |\mathcal{F}_t^W\right) \geq 0,$$

where we used the fact that, under the immersion property, we have $G_t = e^{-\Lambda_t}, t \geq 0$, where the process Λ here corresponds to $\Lambda = \int_0^\cdot \lambda_s ds$ and λ is the intensity rate introduced in Equation (3.1.3). Finally, dividing by h and passing to the limit for $h \to 0$, we have $\mathbb{E}\left(Y_t G_t \lambda_t | \mathcal{F}_t^W\right) \geq 0$ a.s., that immediately gives

$$Y_t G_t \lambda_t \geq 0 \quad \text{a.s.},$$

which concludes the proof. □

As a consequence, we find the following interesting result concerning the optimal investment strategy $\pi_t^{1,*}$, for $t \leq \tau$.

Remark 3.2.2. *The terminal condition $X_{T\wedge\tau}^* = 0$ becomes, on the set $\{\tau < T\}$, $X_\tau^* = 0$. Since $X_\tau = X_{\tau^-} + (X_\tau - X_{\tau^-}) = X_{\tau^-} + \Delta X_\tau$ and since, recalling Equation (3.1.6),*

$$\Delta X_t := X_{t^-} \pi_t^1 \phi_t^1 \Delta H_t \quad \textit{a.s.},$$

we have

$$X_\tau = X_{\tau^-}(1 + \pi_\tau^1 \phi_\tau^1) \quad \textit{a.s.}$$

and so we are obliged to set (recall that $\phi^1 \neq 0$)

$$\pi_\tau^{1,*} = -\frac{1}{\phi_\tau^1} \quad \textit{a.s.},$$

so that at time $t = \tau$ the optimal wealth jumps to zero. We then conclude, by means of Lemma 3.2.1, that, more generally,

$$\pi_t^{1,*} = -\frac{1}{\phi_t^1} \quad a.s., \quad 0 \le t \le \tau. \tag{3.2.13}$$

For an economic interpretation of the above optimal $\pi^{1,*}$ in the constant coefficients case, we refer to the following Remark 4.2.3.

▷ **The log-utility case.**

From Equation (3.2.11), we know that (under the measure $\mathbb{P}$)

$$c_s^* = \frac{x_0}{\mathbb{E}(T \wedge \tau) Z_s^* e^{-\int_0^s r_u du}} \quad \text{a.s.,} \quad s \le (T \wedge \tau).$$

By passing under the measure $\mathbb{P}$ by means of $Z_t^* = \frac{d\mathbb{Q}^*}{d\mathbb{P}}_{|\mathcal{G}_t}$ and by applying Fubini-Tonelli's theorem we find

$$\begin{aligned} X_t^* &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T \wedge \tau)} \frac{1}{Z_t^*} \mathbb{E}\left(Z_t^* \int_t^T \mathbb{1}_{\{s<\tau\}} \frac{1}{Z_s^*} \, ds \mid \mathcal{G}_t \right) \\ &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T \wedge \tau)} \frac{1}{Z_t^*} \int_t^T \mathbb{E}\left(\mathbb{1}_{\{s<\tau\}} | \mathcal{G}_t \right) ds \qquad \text{a.s.} \end{aligned}$$

In order to obtain a more explicit result, we now exploit the following "key-lemma" (see, e.g., Lemma 7.3.4.1 in Chesney, Jeanblanc and Yor [5]), originally stated (in a completely different context) by Dellacherie (see [7], page 65). In what follows $(G_t)_{t \ge 0}$ denotes the $\mathbb{F}$ super-martingale defined as $G_t := \mathbb{P}(\tau > t | \mathcal{F}_t^W), t \ge 0$ (notice that, under the immersion property, G is decreasing).

Lemma 3.2.2. *Let $Y \in \mathcal{F}_T$ be an integrable random variable. Then*

$$\mathbb{E}(Y \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{1}{G_t} \mathbb{E}\left(Y G_T | \mathcal{F}_t^W \right).$$

The computations that follow are done in a "general" setting, without using the fact that in this chapter the immersion property hold between $\mathbb{F}$ and $\mathbb{G}$, under $\mathbb{P}$.

Applying the above Lemma, we then have, under $\mathbb{P}$,

$$\begin{aligned} X_t^* &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T \wedge \tau) Z_t^*} \int_t^T \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}\left(G_s | \mathcal{F}_t^W \right)}{G_t} ds \\ &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T \wedge \tau)} \frac{(1 - H_t)}{Z_t^* \, G_t} \int_t^T \mathbb{P}\left(\tau > s | \mathcal{F}_t^W \right) ds \qquad \text{a.s.} \end{aligned}$$

In particular, $X_{T\wedge\tau}^* = 0$ a.s. At this point in order to go ahead we need to make an assumption concerning the conditional law of τ with respect to $\mathbb{F}$ (it is exactly the same assumption as in Part V, for all the details we refer to that part).

Assumption 3.2.3. ($\mathcal{E}$)-***Hypothesis.*** *The $\mathbb{F}$-(regular) conditional law of τ is equivalent to the law of τ, i.e.,*

$$\mathbb{P}\left(\tau \in d\theta \; | \mathcal{F}_t^W \right) \sim \upsilon(\theta) d\theta, \quad \textit{for every } t \ge 0, \quad \mathbb{P} - a.s. \tag{3.2.14}$$

One of the consequences of the above assumption (for all the details we refer to the following Part V) is the following: there exists a "regular" family of strictly positive martingales $(p_t(\theta))_{t\geq 0}$, $\theta \geq 0$, such that for $s \geq 0$

$$\mathbb{P}\left(\tau > s | \mathcal{F}_t^W\right) = \int_s^\infty p_t(\theta) \upsilon(\theta) d\theta \quad \text{for every } t \geq 0, \quad \mathbb{P} - \text{a.s.} \tag{3.2.15}$$

By means of Equation (3.2.15), we can then write the optimal wealth as follows

$$X_t^* = e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T \wedge \tau)} \frac{(1 - H_t)}{Z_t^* \; G_t} \int_t^T \mathrm{d}s \int_s^\infty p_t(\theta) \upsilon(\theta) d\theta \qquad \text{a.s.}$$

and this stochastic expression has to be differentiated in order to obtain the optimal investment strategy π^*. Because of the presence of t as a subscript in the random variable $p_t(\theta)$, we need the following differentiating rule, also known as the Itô-Kunita-Ventzell formula (see, e.g., Kunita [18]).

Theorem 3.2.1. *Let $F_t(x), t \geq 0, x \in \mathbb{R}^d$, be a family of stochastic processes, continuous in (t, x) a.s., satisfying*

(i) For each $t > 0$, $F_t(\cdot)$ is a $\mathcal{C}^2$-map from $\mathbb{R}^d$ into $\mathbb{R}$;

(ii) For each x, $(F_t(x))_{t\geq 0}$ is a continuous semi-martingale, represented as

$$F_t(x) = F_0(x) + \sum_{j=1}^m \int_0^t f_s^j(x) dN_s^j,$$

where $N^1, \ldots, N^m$ are continuous semi-martingales and $f^j(x), x \in \mathbb{R}^d$, are stochastic processes continuous in (t, x), such that

(a) For each $s > 0$, $f_s^j(\cdot)$ are $\mathcal{C}^1$-maps from $\mathbb{R}^d$ into $\mathbb{R}$;

(b) For each x, $f^j(x)$ are adapted processes.

Let $X = (X^1, \ldots, X^d)$ be a continuous semi-martingale. Then

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \sum_{j=1}^m \int_0^t f_s^j(X_s) dN_s^j + \sum_{i=1}^d \int_0^t \frac{\partial F_s}{\partial x_i}(X_s) dX_s^i \\ &+ \sum_{i=1}^d \sum_{j=1}^m \int_0^t \frac{\partial f_s^j}{\partial x_i}(X_s) \; d< N^j, X^i >_s + \frac{1}{2} \sum_{i,k=1}^d \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_k}(X_s) \; d< X^i, X^k >_s. \end{aligned} \tag{3.2.16}$$

In order to obtain the stochastic differential of X^*, we first use the Itô-Kunita-Ventzell formula to differentiate $\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta) \upsilon(\theta) d\theta$, by setting

$$F_t(x) := \int_x^T \mathrm{d}s \int_s^\infty p_t(\theta) \upsilon(\theta) d\theta.$$

Notice that in this case $d = m = 1$, $N_t^1 = W_t, t \geq 0$ and $X_t^1 = t, t \geq 0$. Now we use the predictable representation theorem in $(\mathbb{P}, \mathbb{F})$ to write the strictly positive martingale $p(\theta)$ as

$$p_t(\theta) = 1 + \int_0^t p_u(\theta) q_u(\theta) dW_u, \quad p_0(\theta) = 1, \quad t \geq 0,$$

for some family of $\mathbb{F}$-predictable integrable processes $q(\theta)$, $\theta \geq 0$. In particular, we find

$$f_u(x) = \int_x^T ds \int_s^\infty p_u(\theta) q_u(\theta) \upsilon(\theta) d\theta,$$

so that the hypotheses of the above Theorem 3.2.1 are satisfied and we have

$$\begin{aligned} dF_t(t) &= d\left(\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)\upsilon(\theta)d\theta\right) \\ &= \left(\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\right) dW_t - \left(\int_t^\infty p_t(\theta)\upsilon(\theta)d\theta\right) dt \\ &= \left(\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\right) dW_t - G_t dt. \end{aligned}$$

Analogously, we find

$$dG_t = \left(\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\right) dW_t - p_t(t)\upsilon(t)dt, \quad G_0 = 1.$$

We can finally compute the differential of X^*, that we re-write below for the reader's ease (recall Equation (3.1.14) and notice that we emphasize the jump factor),

$$X_t^* = \frac{x_0}{\mathbb{E}(T \wedge \tau)} e^{\int_0^t (r_s + \gamma_s^* \bar{\lambda}_s + \frac{1}{2}(\psi_s^*)^2) ds - \int_0^t \psi_s^* dW_s} \frac{(1 - H_t)}{(1 + \gamma_\tau^*)^{H_t}} \frac{1}{G_t} \int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)\upsilon(\theta)d\theta \quad \text{a.s.}$$

namely (notice that the jump factor $(1 + \gamma_\tau^*)^{H_t}$ equals one on the set $\{t < \tau\}$, where $H_t = 0$, so that in practice it does not affect the optimal wealth value)

$$\begin{aligned} dX_t^* &= X_{t-}^* \left[(r_t + \gamma_t^* \bar{\lambda}_t + (\psi_t^*)^2) dt - \psi_t^* dW_t - dH_t\right] \\ &+ \frac{X_t^*}{G_t} \left[-\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta \cdot dW_t + p_t(t)\upsilon(t)dt + \frac{1}{G_t}\left(\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\right)^2 dt\right] \\ &+ \frac{X_t^*}{F_t(t)} \left[\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta \cdot dW_t - G_t dt\right], \end{aligned} \tag{3.2.17}$$

where $F_t(t) = \int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)\upsilon(\theta)d\theta$. By identification with Equation (3.1.6)

$$\mathrm{d}X_t = [\ldots]dt + \pi_t^1 \phi_t^1 X_{t-} \mathrm{d}H_t + \left[\pi_t^1 \sigma_t^1 X_t + \pi_t^2 \sigma_t^2 X_t\right] \mathrm{d}W_t, \quad X_0 = x_0,$$

we find the optimal investment strategies $\pi_t^{1,*}$ and $\pi_t^{2,*}$ as the solution to the following system of equations (notice that the first one is valid over $\{t \leq \tau\}$, while the second is valid everywhere)

$$
\begin{cases}
\pi_t^{1,*} & = & -\frac{1}{\phi_t^1} & \text{a.s.,} \quad t \leq \tau; \\
\pi_t^{1,*}\sigma_t^1 + \pi_t^{2,*}\sigma_t^2 & = & -\psi_t^* - \dfrac{1}{G_t}\displaystyle\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta & \\
& + & \dfrac{1}{F_t(t)}\displaystyle\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta & \text{a.s.,} \quad 0 \leq t \leq T.
\end{cases}
\tag{3.2.18}
$$

$\triangleright$ **Immersion property.**

In this chapter, the immersion property holds between $\mathbb{F}$ and $\mathbb{G}$, under $\mathbb{P}$ (recall Assumption 3.1.1 a)). It is well known (see, e.g., Lemma 5.9.4.2 in Chesney, Jeanblanc and Yor [5]) that $\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if, for every $s \leq t$,

$$\mathbb{P}(\tau \leq s|\mathcal{F}_\infty) = \mathbb{P}(\tau \leq s|\mathcal{F}_t),$$

or, equivalently, if and only if, for every $t \geq 0$,

$$\mathbb{P}(\tau \leq t|\mathcal{F}_\infty) = \mathbb{P}(\tau \leq t|\mathcal{F}_t).$$

Taking $s \leq t$, we then have $\mathbb{P}(\tau \leq s|\mathcal{F}_t) = \mathbb{P}(\tau \leq s|\mathcal{F}_s)$, so that, under the $(\mathcal{E})$-Hypothesis, from Equation (3.2.15),

$$p_t(\theta) = p_\theta(\theta), \quad \text{a.s., for } t \geq \theta.$$

In particular, $q_t(\theta) = 0$ a.s., for $t \geq \theta$.

Furthermore, under the immersion property, we have $G_t = e^{-\Lambda_t}, t \geq 0$, where the process Λ here corresponds to $\Lambda = \int_0^\cdot \lambda_s ds$ and λ is the intensity rate introduced in Equation (3.1.3), so that in the stochastic differential of G the diffusion coefficient is equal to zero.

Remark 3.2.3. *When working with a deterministic intensity λ, as we will see in the following Lemma 4.1.1, the process G becomes a deterministic function of time, so that $\mathbb{P}(\tau > t|\mathcal{F}_t^W) = \mathbb{P}(\tau > t) = \int_t^\infty \upsilon(\theta)d\theta$ and $p(\theta) \equiv 1$ for any $\theta \geq 0$. As a consequence, in the integral representation of $p_t(\theta)$ we have $q(\theta) = 0$, for any θ, and the above computations considerably simplify, as we will see in Section 4.2.4.*

3.2.5 Problem C: optimal consumption

In order to obtain the optimal c^* by means of the martingale approach, we first need the analog in this case to the budget constraint given in Equation (3.1.17), namely

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{-\int_0^s r_u du} c_s \mathrm{d}s\right) \leq x_0.$$

Since $c_t \geq 0$ by definition, by means of Fatou's lemma we can pass to the limit as T tends to infinity and we obtain a generalized version of the budget constraint in the case of an infinite horizon, i.e.,

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{+\infty} e^{-\int_0^s r_u du} c_s \mathrm{d}s\right) \leq x_0. \tag{3.2.19}$$

As previously, it is evident that, in order to solve problem **C**, we would like to saturate the above inequality. We, then, obtain the analog to Propositions 3.2.1 and 3.2.2 (whose proof is directly obtained from the one relative to Proposition 3.2.1 and it is omitted).

Proposition 3.2.4. *Given the market structure (3.1.2), the optimal consumption rate solving problem $\boldsymbol{C}$ is given by*

$$c_s^{*,C} = I\left(\nu e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) \quad a.s., \tag{3.2.20}$$

where I denotes the inverse function of u' and $\nu > 0$ is a real parameter satisfying

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{+\infty} e^{-\int_0^s r_u du} I\left(\nu e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) ds\right) = \mathbb{E}\left(\int_0^{+\infty} e^{-\int_0^s r_u du} Z_s^* I\left(\nu e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) ds\right) = x_0.$$

As previously, the existence of $c^{*,C}$ is not immediate and it is a consequence of the following assumption (the equivalent of Assumption 3.9.9 in Karatzas and Shreve [14] and the analog to Assumptions 3.2.1 and 3.2.2).

Assumption 3.2.4. *The function*

$$\Psi^C(\nu) := \mathbb{E}\left(\int_0^{+\infty} e^{-\int_0^s r_u du} Z_s^* I\left(\nu e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) ds\right)$$

is finite for every $0 < \nu < \infty$.

A sufficient condition, in the case of constant coefficients, for this to hold is given in Theorem 3.9.14 in Karatzas and Shreve [14]. Notice that here, in the stochastic coefficients case, the above assumption is automatically verified in the logarithmic utility case (as we are going to see), while in the power utility case no explicit solution is available. In the latter case, we will then pass to the deterministic coefficient case in order to provide an explicit solution to problem **C** and, in this case, in Section 4.2.5, we will give a sufficient condition in order for $c^{*,C}$ to exist.

Before passing to the explicit computation of $c^{*,C}$ in the case of logarithmic utility, let us compare $c^{*,A}$ and $c^{*,C}$, as we did for $c^{*,A}$ and $c^{*,B}$ in Proposition 3.2.3. To do that, as it is intuitive, we need to consider a slightly different version of problem **A**, namely (for simplicity we omit the admissibility set, that should be re-defined)

$$\underbrace{\sup_{(\pi,c)} \ \mathbb{E}\int_0^T e^{-\rho s} u(c_s) \mathrm{d}s}_{\mathbf{A1}}.$$

Its solution is immediately found to be

$$c_s^{*,A1} = I\left(\nu^{A1} e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) \quad \text{a.s.}, \quad s \le T, \tag{3.2.21}$$

where ν^{A1} satisfies

$$\mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du} Z_s^* \ I\left(\nu^{A1} e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) ds\right) = x_0,$$

under the assumption, that is the analog to Assumption 3.2.1,

Assumption 3.2.5. *The function*

$$\Psi^{A1}(\nu) := \mathbb{E}\left(\int_0^T e^{-\int_0^s r_u du} Z_s^* I\left(\nu e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) ds\right)$$

is finite for every $0 < \nu < \infty$.

We now prove that an investor facing problem **A1** consumes at a higher rate than an investor facing problem **C**, as one might expect.

Proposition 3.2.5. *Under Assumption 3.2.5 and Assumption 3.2.4,*

$$c_s^{*,A1} \geq c_s^{*,C} \quad a.s., \quad s \leq T.$$

Proof. We have to compare, for $s \leq T$,

$$c_s^{*,A1} = I\left(\nu^{A1} e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right) \quad \text{and} \quad c_s^{*,C} = I\left(\nu e^{\rho s} e^{-\int_0^s r_u du} Z_s^*\right),$$

where $\nu^{A1} > 0$ and $\nu^C > 0$ are given by $\nu^{A1} = \left(\Psi^{A1}\right)^{-1}(x_0)$ and $\nu^C = \left(\Psi^C\right)^{-1}(x_0)$. It is clear that $\Psi^{A1} \leq \Psi^C$ and so $\left(\Psi^{A1}\right)^{-1} \leq \left(\Psi^C\right)^{-1}$. We then have $\nu^{A1} \leq \nu^C$ and we are done, since I is decreasing. □

$\triangleright$ **The log-utility case.**
Here

$$c_s^{*,C} = \frac{1}{\nu Z_s^* e^{\rho s} e^{-\int_0^s r_u du}} \quad \text{a.s.}$$

and ν satisfies

$$\mathbb{E}\left(\int_0^{+\infty} \frac{e^{-\rho s}}{\nu} \mathrm{d}s\right) = x_0.$$

So, in this case, as we said, Assumption 3.2.4 is verified, given that ρ is, by definition, positive. We obtain $\nu = \frac{1}{\rho x_0}$ and finally

$$c_s^{*,C} = \frac{\rho x_0}{Z_s^* e^{\rho s} e^{-\int_0^s r_u du}} \quad \text{a.s.} \tag{3.2.22}$$

$\triangleright$ **The power-utility case.**
In this case we find,

$$c_s^{*,C} = \left(\nu e^{\rho s} Z_s^* e^{-\int_0^s r_u du}\right)^{-\frac{1}{1-\gamma}} = x_0 \frac{\left(e^{\rho s} Z_s^* e^{-\int_0^s r_u du}\right)^{-\frac{1}{1-\gamma}}}{\mathbb{E}\int_0^{+\infty} \left(e^{-\int_0^s r_u du} Z_s^*\right)^{\frac{\gamma}{\gamma-1}} \left(e^{-\rho s}\right)^{\frac{1}{1-\gamma}} \mathrm{d}s}, \quad \text{a.s.,} \tag{3.2.23}$$

since ν is given (if it exists) by

$$\nu = \left[\frac{\mathbb{E}\int_0^{+\infty} \left(e^{-\int_0^s r_u du} Z_s^*\right)^{\frac{\gamma}{\gamma-1}} \left(e^{-\rho s}\right)^{\frac{1}{1-\gamma}} \mathrm{d}s}{x_0}\right]^{1-\gamma}.$$

A sufficient condition for the existence of ν and an explicit optimal consumption strategy are given in the deterministic coefficients case, in Section 4.2.5.

3.2.6 Problem C: optimal investment strategy

As previously done for problem **A**, we now derive the optimal investment strategy, in the logarithmic utility case, by means of direct computations, based on the fact that the optimal wealth can be now expressed as

$$X_t^* = e^{\int_0^t r_s ds} \, \mathbb{E}^{\mathbb{Q}^*} \left(\int_t^{+\infty} e^{-\int_0^s r_u du} c_s^* \, \mathrm{d}s \mid \mathcal{G}_t \right), \quad \text{a.s.,} \quad t < +\infty.$$

▷ **The log-utility case.**
From Equation (3.2.22), we know that

$$c_s^* = \frac{\rho x_0}{Z_s^* e^{\rho s} e^{-\int_0^s r_u du}} \quad \text{a.s.,} \quad s < +\infty.$$

A direct computation, applying the conditional version of Fubini-Tonelli's theorem and recalling that $(Z^*)^{-1}$ is a $(\mathbb{G}, \mathbb{Q}^*)-$martingale, gives us

$$\begin{aligned} X_t^* &= e^{\int_0^t r_s ds} \, \mathbb{E}^{\mathbb{Q}^*} \left(\int_t^{+\infty} e^{-\int_0^s r_u du} c_s^* \, \mathrm{d}s \mid \mathcal{G}_t \right) \\ &= e^{\int_0^t r_s ds} \rho x_0 \int_t^{+\infty} e^{-\rho s} \, \mathbb{E}^{\mathbb{Q}^*} \left(\frac{1}{Z_s^*} | \mathcal{G}_t \right) \mathrm{d}s \\ &= x_0 \, e^{\int_0^t r_s ds} \, \frac{e^{-\rho t}}{Z_t^*} \qquad \text{a.s.} \end{aligned}$$

Equivalently, in differential form (under $\mathbb{Q}^*$), recalling Equation (3.2.4),

$$\mathrm{d}X_t^* = X_{t-}^* \left[(r_t - \rho) \, \mathrm{d}t - \psi_t^* \mathrm{d}W_t^* - \frac{\gamma_t^*}{1 + \gamma_t^*} dM_t^* \right], \quad X_0^* = x_0. \tag{3.2.24}$$

To determine $\pi^{1,*}$ and $\pi^{2,*}$ it suffices to identify, term by term, the above equation and Equation (3.2.6), namely

$$\mathrm{d}X_t = (r_t X_t - c_t) \, \mathrm{d}t + \pi_t^1 \phi_t^1 X_{t-} \mathrm{d}M_t^* + X_t \left(\pi_t^1 \sigma_t^1 + \pi_t^2 \sigma_t^2 \right) \mathrm{d}W_t^*, \quad X_0 = x_0.$$

We finally find that $c_t^* = \rho X_t^*$ and

$$\begin{cases} \pi_t^{1,*} = -\dfrac{\gamma_t^*}{\phi_t^1 (1 + \gamma_t^*)}, \quad \pi_t^{2,*} = -\dfrac{\widetilde{\psi}_t^*}{\sigma_t^2} + \dfrac{\gamma_t^* \widetilde{\sigma}_t^1}{\sigma_t^2 \phi_t^1 (1 + \gamma_t^*)} & \text{a.s.,} \quad t \leq \tau; \\ \pi_t^{1,*} \sigma_t^1(\tau) + \pi_t^{2,*} \sigma_t^2 = -\psi_t^*(\tau) \quad \text{a.s.,} & \qquad t > \tau. \end{cases} \tag{3.2.25}$$

Once more, after τ the investment strategy is not unique, due to the redundancy of the market.

Chapter 4

The deterministic coefficients case

4.1 Introduction: peculiarities of the setting

In this chapter we focus on the case when market model coefficients are constant (or deterministic functions of time), in order to obtain the explicit optimal investment-consumption strategies for the three problems. As we will see, in this specific setting the obtained results are quite surprising. We start by giving the assets's dynamics and studying the consequences of our model assumptions.

In the constant coefficients case the assets' dynamics given in Equations (3.1.1) and (3.1.2) become, respectively,

$$\begin{cases} \mathrm{d}S_t^0 &= rS_t^0 \mathrm{d}t, & S_0^0 = 1, \\ \mathrm{d}S_t^1 &= S_{t-}^1 \left(\mu^1 \mathrm{d}t + \sigma^1 \mathrm{d}W_t + \phi^1 \mathrm{d}M_t\right), & S_0^1 = s_0^1, \\ \mathrm{d}S_t^2 &= S_t^2 \left(\mu^2 \mathrm{d}t + \sigma^2 \mathrm{d}W_t\right), & S_0^2 = s_0^2. \end{cases} \tag{4.1.1}$$

Notice that here the total information filtration $\mathbb{G}$ is the filtration generated by the price processes S^1 and S^2, namely, for $t \geq 0$,

$$\mathcal{G}_t := \bigcap_{\epsilon > 0} \left\{ \mathcal{F}_{t+\epsilon}^1 \vee \mathcal{F}_{t+\epsilon}^2 \right\},$$

with $\mathcal{F}_t^i := \sigma(S_s^i, s \leq t)$, $i = 1, 2$. Furthermore, Assumption 3.1.1 b) (that is equivalent to the absence of arbitrage opportunities, as we will verify in the next Section 4.1.1) here becomes:

Assumption 4.1.1. *The following proportionality relation holds true*

$$\frac{r - \mu^1}{\sigma^1} = \frac{r - \mu^2}{\sigma^2}. \tag{4.1.2}$$

In this chapter, furthermore, we make the standing assumption:

Assumption 4.1.2. *The $\mathbb{F}$-intensity rate of τ is a deterministic function of time $\lambda(t)$.*

The lemma below provides an interesting necessary and sufficient condition in order for λ to be a deterministic function of time.

Lemma 4.1.1. *Under the immersion property, a necessary and sufficient condition for the intensity rate of τ to be a deterministic function of time is to have τ independent of $\mathbb{F}$.*

Proof. ($\Rightarrow$) Given that λ is deterministic, by using the definition of independence between a random variable and a σ-algebra, we have to prove that, for any measurable and bounded function f and for any $t \geq 0$, we have

$$\mathbb{E}\left(f(\tau)|\mathcal{F}_t^W\right) = \mathbb{E}\left(f(\tau)\right), \quad \text{a.s.}$$

We will show that this is true for $t = +\infty$.

Because of the immersion property between $\mathbb{F}$ and $\mathbb{G}$ (that is the progressive enlargement of $\mathbb{F}$ with τ) we have $\mathbb{E}(\mathbb{1}_{\{\tau \leq t\}}|\mathcal{F}_\infty^W) = \mathbb{E}(\mathbb{1}_{\{\tau \leq t\}}|\mathcal{F}_t^W)$, so that the process $F_{\infty,t} := \mathbb{P}\left(\tau \leq t|\mathcal{F}_\infty^W\right) = \mathbb{P}\left(\tau \leq t|\mathcal{F}_t^W\right), t \geq 0$, is here a deterministic function of time, namely $F_{\infty,t} = F_\infty(t) = 1 - e^{-\int_0^t \lambda(s)\mathrm{d}s}$, where the last equality follows from the fact that if F is continuous, then working under the $(\mathcal{H})$-Hypothesis is equivalent to a Cox modeling (see, e.g., Lemma 2 in Blanchet-Scalliet *et al.* [3]). We, then, have

$$\begin{aligned}
\mathbb{E}\left(f(\tau)|\mathcal{F}_\infty^W\right) &= \mathbb{E}\int_0^\infty f(t)\mathrm{d}F_\infty(t) = \mathbb{E}\int_0^\infty f(t)\lambda(t)e^{-\int_0^t \lambda(s)\mathrm{d}s}\mathrm{d}t \\
&= \int_0^\infty f(t)\lambda(t)e^{-\int_0^t \lambda(s)\mathrm{d}s}\mathrm{d}t
\end{aligned}$$

and, being λ deterministic, the conclusion follows.

($\Leftarrow$) Let us now suppose that τ is independent of $\mathbb{F}$. The independence of τ from $\mathcal{F}_\infty^W$ implies that, for any $t \geq 0$,

$$\mathbb{E}\left(\mathbb{1}_{\{\tau \leq t\}}|\mathcal{F}_\infty^W\right) = \mathbb{P}(\tau \leq t) \tag{4.1.3}$$

(and hence the immersion property holds true). To conclude, it suffices to remark that, from (4.1.3), the $\mathbb{F}-$hazard process of τ, Γ, defined for every t as

$$\Gamma_t = -\ln \mathbb{P}(\tau > t),$$

is deterministic and this implies that the jump intensity (i.e., the derivative of Γ) is deterministic, too. $\square$

For what concerns the admissibility condition (recall Definition 3.1.1) $\pi_\tau^1 \phi^1 \geq -1$ a.s., by applying Lemma 3.2.1 to $Y_t := \pi_t^1 \phi^1 - 1, t \leq \tau$, we find that the above condition is equivalent to

$$\pi_t^1 \phi^1 \geq -1 \quad \text{a.s. for every } t \leq \tau. \tag{4.1.4}$$

We now characterize the unique Radon-Nikodým density process Z^* in this specific setting.

4.1.1 The unique EMM $\mathbb{Q}^*$

Equation (3.1.12), that established a relation between the coefficients ψ^*, γ^* and the model parameters, here reads

$$\begin{cases} \mu^1 - r + \sigma^1 \psi_t^* + \phi^1 \gamma_t^* \bar{\lambda}_t = 0 \quad \text{a.s.} \\ \mu^2 - r + \sigma^2 \psi_t^* = 0 \quad \text{a.s.}, \end{cases} \tag{4.1.5}$$

so that we find

$$\begin{cases} \psi_t^* \equiv \psi^* = \frac{r-\mu^2}{\sigma^2} \quad \text{a.s.} \\ \mu^1 - r + \sigma^1 \psi_t^* + \phi^1 \gamma_t^* \bar{\lambda}_t = 0 \quad \text{a.s.} \end{cases} \tag{4.1.6}$$

Notice, furthermore, that, on the set $\{t \geq \tau\}$, ψ^* has to satisfy

$$\psi^* = \frac{r-\mu^2}{\sigma^2} = \frac{r-\mu^1}{\sigma^1} \quad \text{a.s.}, \tag{4.1.7}$$

that corresponds to our Assumption 4.1.1 (it represents the usual non-arbitrage condition in a Black-Scholes non defaultable market) and γ can be arbitrarily chosen, since it does not affect the dynamics of Z^* after the shock τ. Moreover, being the model parameters constant, it is clear that (4.1.7) has to be valid everywhere, so that, before the shock, on the set $\{t < \tau\}$, we find

$$\gamma_t^* = \frac{1}{\phi^1 \lambda(t)} \left[r - \mu^1 - \sigma^1 \frac{r-\mu^2}{\sigma^2} \right] = 0 \quad \text{a.s.}$$

In conclusion, recalling that γ^* is $\mathbb{G}-$predictable, so that there exists a *unique* $\widetilde{\gamma}$, $\mathbb{F}-$predictable, such that $\gamma_t^* 1\!\!1_{\{t \leq \tau\}} = \widetilde{\gamma}_t 1\!\!1_{\{t \leq \tau\}}$, we find that at the jump time, too, $\gamma_\tau^* = 0$ and finally

$$\begin{cases} \psi_t^* = \psi^* = \frac{r-\mu^2}{\sigma^2} = \frac{r-\mu^1}{\sigma^1} \quad \text{a.s.}; \\ \\ \gamma_t^* = \begin{cases} 0 \quad \text{a.s.} & \text{if } t \leq \tau; \\ \text{any predictable } \gamma_t^* > -1 & \text{if } t > \tau. \end{cases} \end{cases} \tag{4.1.8}$$

The processes ψ^* and γ^* define a *unique* martingale measure $\mathbb{Q}^*$ (the process Z^* is, indeed, a true martingale). The Radon-Nikodým density process Z^* is given, for every t, by

$$Z_t^* = e^{\psi^* W_t - \frac{1}{2}(\psi^*)^2 t} \, e^{-\int_0^t \gamma_u^* \bar{\lambda}_u \mathrm{d}u} \, (1+\gamma_\tau^*)^{H_t} = e^{\psi^* W_t - \frac{1}{2}(\psi^*)^2 t} \quad \text{a.s.} \tag{4.1.9}$$

Remark 4.1.1. *a) The unique change of probability does not affect the martingale M (on the set $\{t < \tau\}$, $\gamma_t^* = 0$), so that, in particular, the historical jump intensity is equal to the risk-neutral one.*
b) Being $\psi^* = \frac{r-\mu^2}{\sigma^2}$ a.s. and $\gamma^* = 0$ a.s., the Radon-Nikodým density process is $\mathbb{F}$-adapted and the immersion property is preserved when passing under the measure $\mathbb{Q}^*$ (for a detailed analysis on the stability of the $(\mathcal{H})$-Hypothesis we refer, e.g., to Section 3 in Blanchet-Scalliet and Jeanblanc [3], for the links between the $(\mathcal{H})$-Hypothesis and market completeness, and to Propositions 5.9.1.2 and 5.9.1.3 in Chesney, Jeanblanc and Yor [5] and Theorem 5.11 in El Karoui, Jeanblanc and Jiao [9]).
c) Another consequence of the $\mathbb{F}$-adaptability of Z^* is the following

$$\mathbb{Q}^*(\tau > t | \mathcal{F}_t^W) = \frac{1}{Z_t^*} \, \mathbb{E}(Z_t^* 1\!\!1_{\{\tau > t\}} | \mathcal{F}_t^W) = \mathbb{P}(\tau > t | \mathcal{F}_t^W).$$

The above Remark will be crucial to understand the forthcoming results. We end this section checking that if Assumption 4.1.1 is not verified, then the market is no more arbitrage free. More precisely, we provide an example of arbitrage opportunity.

Lemma 4.1.2. *If Equation (4.1.7) does not hold true, then there are arbitrage opportunities.*

Proof. We consider the set $\{t \geq \tau\}$ and we suppose, for simplicity, that $r = 0$, so that the prices' dynamics are given by

$$\begin{aligned} \mathrm{d}S_t^1 &= S_t^1 \left(\mu^1 \mathrm{d}t + \sigma^1 \mathrm{d}W_t \right), \quad t \geq \tau, \\ \mathrm{d}S_t^2 &= S_t^2 \left(\mu^2 \mathrm{d}t + \sigma^2 \mathrm{d}W_t \right), \quad t \geq \tau. \end{aligned}$$

We show that if Equation (4.1.7) is not satisfied, namely if

$$\mu^1\sigma^2 \neq \mu^2\sigma^1, \tag{4.1.10}$$

we find a self-financing strategy that produces an arbitrage, i.e., such that, starting with an initial wealth equal to zero, the corresponding terminal wealth is a.s. nonnegative and strictly positive with strictly positive probability.

It suffices to choose

$$\alpha_t^1 = \frac{\sigma^2}{S_t^1}, \quad \alpha_t^2 = -\frac{\sigma^1}{S_t^2}, \quad \text{for } t \geq \tau$$

and $\alpha_t^1 = \alpha_t^2 = 0$, for $t < \tau$ and, by means of α_t^0, to re-balance the portfolio to make it self-financing. We then have

$$\mathrm{d}V_t = \alpha_t^1 \mathrm{d}S_t^1 + \alpha_t^2 \mathrm{d}S_t^2 = (\mu^1\sigma^2 - \mu^2\sigma^1)1\!\!1_{\{t\geq\tau\}}\mathrm{d}t$$

and it is clear that if Equation (4.1.10) holds true then we have an arbitrage opportunity, since, in particular, if $\mu^1\sigma^2 > \mu^2\sigma^1$ we can obtain a strictly positive terminal wealth with an initial investment equal to zero. Indeed, it suffices, starting with $x_0 = 0$, to borrow one euro from the bank and to invest it in V at time $t = \tau$. Notice that $\mathbb{P}(t \geq \tau) = 1 - e^{-\int_0^t \lambda(s)\mathrm{d}s} > 0$, since, for every $t > 0$, $\int_0^t \lambda(s)\mathrm{d}s > 0$ (otherwise the jump intensity is identically equal to zero), meaning that our terminal wealth will be strictly positive with strictly positive probability. □

We can summarize the results in this section in the following Lemma.

Lemma 4.1.3. *The condition*

$$\frac{r-\mu^1}{\sigma^1} = \frac{r-\mu^2}{\sigma^2}$$

is necessary and sufficient to ensure the absence of arbitrage in our market.

To conclude, notice that if the market is arbitrage free, then it is also complete, since in this case there exists a unique EMM. On the other hand, if $\frac{r-\mu^2}{\sigma^2} \neq \frac{r-\mu^1}{\sigma^1}$ the market is complete, with arbitrage opportunities.

4.2 The solution: martingale approach

4.2.1 Problem A: optimal consumption

We adapt the result obtained in Section 3.2.1 in the general stochastic coefficients case to our particular setting. In the log-utility case c^* is exactly the same we found in Section 3.2.1, in Equation (3.2.2) (where the stochastic interest rate $(r_t)_{t\geq0}$ has to be replaced by the constant r), but in the power utility case we obtain an explicit expression for c^*, in contrast with what we had in the previous chapter.

▷ **The power-utility case.**

In this case we have (recall Equation (3.2.3))

$$c_s^{*,A} = x_0 \frac{(e^{-rs}Z_s^*)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^T (e^{-ru}Z_u^*)^{\frac{\gamma}{\gamma-1}}\mathrm{d}u} \quad \text{a.s.} \tag{4.2.1}$$

Given that $0 \le u \le T$, $Z_u^* = e^{\psi^* W_u - \frac{1}{2}(\psi^*)^2 u}$, we can explicitly compute

$$\mathbb{E}\left[(Z_u^*)^{\frac{\gamma}{\gamma-1}}\right] = e^{\frac{1}{2}(\psi^*)^2 \frac{\gamma^2}{(\gamma-1)^2}u - \frac{1}{2}(\psi^*)^2 \frac{\gamma}{\gamma-1}u} = e^{\frac{1}{2}(\psi^*)^2 \frac{\gamma}{(\gamma-1)^2}u}$$

and Fubini-Tonelli's theorem gives us the denominator in (4.2.1), namely

$$\mathbb{E}\int_0^T (e^{-ru}Z_u^*)^{\frac{\gamma}{\gamma-1}}\mathrm{d}u = \frac{1-\gamma}{\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma} + r\gamma}\left(e^{\frac{T}{1-\gamma}\left(\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma}+r\gamma\right)} - 1\right).$$

In conclusion

$$c_s^{*,A} = \frac{x_0}{(e^{-rs}Z_s^*)^{\frac{1}{1-\gamma}}} \frac{\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma} + r\gamma}{(1-\gamma)\left(e^{\frac{T}{1-\gamma}\left(\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma}+r\gamma\right)} - 1\right)} \quad \text{a.s.} \tag{4.2.2}$$

We end this subsection with a curious remark.

Remark 4.2.1. *In the logarithmic utility case, the* discounted *optimal consumption rate process, given by*

$$\widetilde{c}_t^* := e^{-rt}c_t^* = \frac{x_0}{TZ_t^*}, \quad t \le T,$$

is a $(\mathbb{G}, \mathbb{Q}^*)-$*martingale.*
Inspired, then, by the ideas developed in the well known benchmark approach of Platen, more precisely by the notion of "growth optimal portfolio", that coincides with the one of "numeraire portfolio" when maximizing $\mathbb{E}[\ln(X_T)]$ *(see, e.g., Section 2.3 in Korn [16]), we then ask ourselves the following question, supposing for simplicity that* $r = 0$*: "If we are given a* $(\mathbb{G}, \mathbb{Q}^*)-$*martingale* $\bar{c}$*, defined as* $\bar{c}_t = I(\nu Z_t^*)$*, for some positive constant* ν *and where* I *is the inverse of some marginal utility* u'*, is necessarily* u *a logarithmic utility function?"*
The answer is: "Yes!". In fact, $\bar{c}Z^*$ *is a* $(\mathbb{G}, \mathbb{P})-$*martingale and, since (under* $\mathbb{P}$*)* $dZ_t^* = Z_t^*\psi^* dW_t$ *(recall Equation (4.1.9)), we have*

$$d\bar{c}_t = I'(\nu Z_t^*)\nu Z_t^* \psi^* dW_t + \frac{1}{2}I''(\nu Z_t^*)\nu^2 d\langle Z^*\rangle_t$$

and

$$\begin{aligned} d(\bar{c}_t Z_t^*) &= Z_t^* d\bar{c}_t + \bar{c}_t dZ_t^* + d\langle \bar{c}, Z^*\rangle_t = Z_t^* I'(\nu Z_t^*)\nu Z_t^* \psi^* dW_t + \frac{1}{2}Z_t^* I''(\nu Z_t^*)\nu^2 Z_t^{*2}\psi^{*2} dt \\ &\quad + \bar{c}_t Z_t^* \psi^* dW_t + I'(\nu Z_t^*)\nu Z_t^{*2}\psi^{*2} dt. \end{aligned}$$

The martingale representation theorem tells us that, in order to have a $(\mathbb{G}, \mathbb{P})-$*martingale, the following relation has to be satisfied*

$$\frac{1}{2}Z_t^* I''(\nu Z_t^*)\nu^2 Z_t^{*2}\psi^{*2} + I'(\nu Z_t^*)\nu Z_t^{*2}\psi^{*2} = 0,$$

or, equivalently, by defining $y := \nu Z_t^*$, $y \in \mathbb{R}_*^+$,

$$\frac{1}{2}yI''(y) + I'(y) = 0. \tag{4.2.3}$$

$\bar{c}Z^*$ is, then, a $(\mathbb{G},\mathbb{P})-$martingale if and only if the function I solves the above second order ordinary differential equation (4.2.3), namely if

$$I(y) = \frac{c_1}{y} + c_2,$$

with c_1, c_2 strictly positive constants, so that $I : (0,\infty) \to (0,\infty)$ is strictly decreasing. In order to have $I(0^+) = \infty$ and $I(\infty) = 0$ (recall Definition 3.1.2) we find $c_2 = 0$, so that the inverse of u' is

$$I(y) = \frac{c_1}{y} \tag{4.2.4}$$

and this implies that u is a logarithmic utility function.

4.2.2 Problem A: optimal investment strategy

First of all we notice the following interesting fact, that arises from the peculiarity of our model.

Proposition 4.2.1. *In our setting, the optimal proportion of wealth to be invested in S^1 to solve problem **A** is, before the shock τ,*

$$\pi_t^{1,*} = 0 \quad a.s. \tag{4.2.5}$$

In other words, the optimal investment strategy is the one of an investor that acts in the market as if only the asset S^2 was traded.

Proof. The key remark is that the optimal consumption strategy c^*, given in Equation (3.2.1), is HERE $\mathbb{F}$-adapted (this derives from the $\mathbb{F}-$adaptability of Z^*). This implies that

$$\begin{aligned} X_t^* &= e^{rt}\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs}c_s^*\,\mathrm{d}s \mid \mathcal{G}_t\right) \quad \text{a.s.} \\ &= e^{rt}\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs}c_s^*\,\mathrm{d}s \mid \mathcal{F}_t^W\right) \quad \text{a.s.} \end{aligned}$$

since the immersion property holds under $\mathbb{Q}^*$ (recall Remark 4.1.1), so that X^* is $\mathbb{F}-$adapted, for any admissible π. In particular, X^* cannot have a jump at time τ, meaning that any investment strategy π is such that $\pi_\tau^1 = 0$ a.s. Because of the predictability of π, from Lemma 3.2.1 this has to be true everywhere before τ and it remains valid for the optimal $\pi^{1,*}$, namely $\pi_t^{1,*} = 0, t \leq \tau,$ a.s. □

By means of direct computations, we now explicitly characterize the optimal investment strategies in the examples.

▷ **The log-utility case.**

Exactly as in Section 3.2.2 (with the only difference that here the interest rate is constant), we find, recalling that $W_t^* := W_t - \psi^* t,\ t \geq 0$, is a $(\mathbb{G},\mathbb{Q}^*)$-Brownian motion,

$$X_t^* = e^{rt}\frac{x_0(T-t)}{T}e^{-\psi^* W_t^*}e^{-\frac{1}{2}(\psi^*)^2 t}, \quad t \leq T, \text{ a.s.}$$

or, equivalently,

$$\mathrm{d}X_t^* = X_t^* \left[\left(r - \frac{1}{T-t} \right) \mathrm{d}t - \psi^* \mathrm{d}W_t^* \right], \quad X_0^* = x_0. \tag{4.2.6}$$

As usual, an identification term by term with

$$\mathrm{d}X_t = (rX_t - c_t)\,\mathrm{d}t + \pi_t^1 \phi^1 X_{t-} \mathrm{d}M_t + X_t \left(\pi_t^1 \sigma^1 + \pi_t^2 \sigma^2 \right) \mathrm{d}W_t^*, \quad X_0 = x_0,$$

gives us $\pi^{1,*}$ and $\pi^{2,*}$, that are different from the ones we found in Equation (3.2.8) (see, in particular, the case before the shock),

$$\begin{cases} \pi_t^{1,*} = 0, \quad \pi_t^{2,*} = -\dfrac{\psi^*}{\sigma^2} & \text{a.s.,} \quad t \le \tau; \\ \pi_t^{1,*}\sigma^1 + \pi_t^{2,*}\sigma^2 = -\psi^* & \text{a.s.,} \quad t > \tau. \end{cases} \tag{4.2.7}$$

The investment strategy after τ remains not unique, due to the redundancy of the market.
$\triangleright$ **The power-utility case.**
The reasoning is the same as in the previous example. From Equation (4.2.2), passing under the measure $\mathbb{Q}^*$, we have, for $s \le T$,

$$c_s^* = \frac{x_0}{\left(e^{-rs} e^{\psi^* W_s^* + \frac{1}{2}(\psi^*)^2 s} \right)^{\frac{1}{1-\gamma}}} \frac{K}{(1-\gamma)\left(e^{\frac{T}{1-\gamma}K} - 1 \right)} \quad \text{a.s.,}$$

where $K = \frac{1}{2}(\psi^*)^2 \frac{\gamma}{1-\gamma} + r\gamma$, and so we explicitly compute

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs} c_s^* \, \mathrm{d}s \mid \mathcal{G}_t \right) &= \frac{x_0 K}{(1-\gamma)\left(e^{\frac{T}{1-\gamma}K} - 1 \right)} \int_t^T e^{\frac{r\gamma}{1-\gamma}s} e^{-\frac{1}{2}\frac{(\psi^*)^2}{1-\gamma}s} \, \mathbb{E}^{\mathbb{Q}^*}\left[e^{-\frac{\psi^*}{1-\gamma}W_s^*} | \mathcal{G}_t \right] \mathrm{d}s \\ &= \cdots = \frac{x_0}{\left(e^{\frac{T}{1-\gamma}K} - 1 \right)} e^{-\frac{\psi^*}{1-\gamma}W_t^*} e^{-\frac{1}{2}\frac{(\psi^*)^2}{(1-\gamma)^2}t} \left(e^{\frac{TK}{1-\gamma}} - e^{\frac{tK}{1-\gamma}} \right) \text{ a.s.} \end{aligned}$$

This gives us the optimal wealth for $t \le T$ (notice that, indeed, $X_T^* = 0$ a.s.)

$$X_t^* = e^{rt} \frac{x_0}{\left(e^{\frac{T}{1-\gamma}K} - 1 \right)} e^{-\frac{\psi^*}{1-\gamma}W_t^*} e^{-\frac{1}{2}\frac{(\psi^*)^2}{(1-\gamma)^2}t} \left(e^{\frac{TK}{1-\gamma}} - e^{\frac{tK}{1-\gamma}} \right), \text{ a.s.,}$$

or, in differential form,

$$\mathrm{d}X_t^* = X_t^* \left(r - \frac{K}{1-\gamma} \frac{1}{\left(e^{\frac{K}{1-\gamma}(T-t)} - 1 \right)} \right) \mathrm{d}t - X_t^* \frac{\psi^*}{1-\gamma} \mathrm{d}W_t^*, \quad X_0^* = x_0$$

and by identification with the coefficients in the dynamics of X under $\mathbb{Q}^*$ as before, we find

$$\begin{cases} \pi_t^{1,*} = 0, \quad \pi_t^{2,*} = -\dfrac{\psi^*}{\sigma^2(1-\gamma)} & \text{a.s.,} \quad t \le \tau; \\ \pi_t^{1,*}\sigma^1 + \pi_t^{2,*}\sigma^2 = -\dfrac{\psi^*}{1-\gamma} & \text{a.s.,} \quad t > \tau. \end{cases} \tag{4.2.8}$$

4.2.3 Problem B: optimal consumption

▷ **The log-utility case.**
As in Section 3.2.3, we find

$$c_s^{*,B} = \frac{x_0}{\mathbb{E}(T \wedge \tau) Z_s^* e^{-rs}} \quad \text{a.s.,} \quad s \leq T \wedge \tau. \tag{4.2.9}$$

Given Assumption 4.1.2, and supposing, furthermore, that the intensity of τ is constant, namely $\lambda(t) \equiv \lambda > 0$, the conditional survival probability $G(t) := \mathbb{P}(\tau > t)$ (that here coincides with $G_t := \mathbb{P}(\tau > t|\mathcal{F}_t^W), t \geq 0$) is equal to $G(t) = e^{-\lambda t}$ and so, by Fubini-Tonelli's theorem,

$$\mathbb{E}(T \wedge \tau) = \mathbb{E}\int_0^T \mathbb{1}_{\{s<\tau\}} \mathrm{d}s = \int_0^T \mathbb{P}(\tau > s)\mathrm{d}s = \int_0^T e^{-\lambda s}\mathrm{d}s = \frac{1-e^{-\lambda T}}{\lambda}.$$

The optimal consumption rate is, then, in the specific constant intensity case,

$$c_s^{*,B} = \frac{x_0 \lambda}{(1-e^{-\lambda T}) Z_s^* e^{-rs}} \quad \text{a.s.,} \quad s \leq (T \wedge \tau).$$

▷ **The power-utility case.**
Here we have

$$c_s^{*,B} = x_0 \frac{(e^{-rs} Z_s^*)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^T (1-H_u)(e^{-ru} Z_u^*)^{\frac{\gamma}{\gamma-1}} \mathrm{d}u} \quad \text{a.s.,} \quad s \leq (T \wedge \tau), \tag{4.2.10}$$

since ν is given by

$$\nu = \left[\frac{\mathbb{E}\int_0^T (1-H_u)(e^{-ru} Z_u^*)^{\frac{\gamma}{\gamma-1}} \mathrm{d}u}{x_0}\right]^{1-\gamma}.$$

If $\lambda(t) \equiv \lambda$, given the independence of τ of $\mathbb{F}$ and recalling that

$$\mathbb{E}\left[(Z_u^*)^{\frac{\gamma}{\gamma-1}}\right] = e^{\frac{1}{2}(\psi^*)^2 \frac{\gamma}{(\gamma-1)^2} u},$$

we explicitly obtain ν, by computing (with an application of Fubini-Tonelli's theorem)

$$\begin{aligned}
\mathbb{E}\int_0^T (1-H_u)(e^{-ru} Z_u^*)^{\frac{\gamma}{\gamma-1}} \mathrm{d}u &= \int_0^T e^{-\lambda u} e^{\frac{r\gamma}{1-\gamma} u} e^{\frac{1}{2}(\psi^*)^2 \frac{\gamma}{(\gamma-1)^2} u} \mathrm{d}u \\
&= \frac{e^{T\left(\frac{r\gamma}{1-\gamma} + \frac{1}{2}(\psi^*)^2 \frac{\gamma}{(\gamma-1)^2} - \lambda\right)} - 1}{\frac{r\gamma}{1-\gamma} + \frac{1}{2}(\psi^*)^2 \frac{\gamma}{(\gamma-1)^2} - \lambda}.
\end{aligned}$$

The optimal consumption rate is, then,

$$c_s^{*,B} = \frac{x_0}{(e^{-rs} Z_s^*)^{\frac{1}{1-\gamma}}} \frac{\frac{r\gamma}{1-\gamma} + \frac{1}{2}(\psi^*)^2 \frac{\gamma}{(\gamma-1)^2} - \lambda}{e^{T\left(\frac{r\gamma}{1-\gamma} + \frac{1}{2}(\psi^*)^2 \frac{\gamma}{(\gamma-1)^2} - \lambda\right)} - 1} \quad \text{a.s.,} \quad s \leq (T \wedge \tau).$$

4.2.4 Problem B: optimal investment strategy

▷ **The log-utility case.**
From Equation (4.2.9) we know that (under the measure $\mathbb{P}$)

$$c_s^* = \frac{x_0}{\mathbb{E}(T \wedge \tau) Z_s^* e^{-rs}} \quad \text{a.s.,} \quad s \leq (T \wedge \tau).$$

By applying the conditional version of Fubini-Tonelli's theorem and passing under the measure $\mathbb{P}$ by means of Z^*, we find, for $t \leq (T \wedge \tau)$,

$$\begin{aligned} X_t^* &= e^{rt}\, \mathbb{E}^{\mathbb{Q}^*} \left(\int_t^T \mathbb{1}_{\{s \leq \tau\}} e^{-rs} c_s^* \, \mathrm{d}s \mid \mathcal{G}_t \right) = \frac{x_0}{\mathbb{E}(T \wedge \tau)} \int_t^T \mathbb{E}^{\mathbb{Q}^*} \left(\mathbb{1}_{\{s \leq \tau\}} \frac{1}{Z_s^*} | \mathcal{G}_t \right) \mathrm{d}s \\ &= \frac{x_0}{\mathbb{E}(T \wedge \tau)} \int_t^T \frac{\mathbb{E}\left(\mathbb{1}_{\{s \leq \tau\}} | \mathcal{G}_t\right)}{Z_t^*} \mathrm{d}s \quad \text{a.s.,} \end{aligned}$$

while, if $t \geq (T \wedge \tau)$, then $X_t^* = 0$. In order to explicitly obtain X^*, we can exploit the independence of τ of $\mathbb{F}$, using the "key-Lemma" 3.2.2 to have a conditional expectation with respect to the smaller filtration $\mathbb{F}$. Then, under $\mathbb{P}$, recalling that $G_t = \mathbb{P}(\tau > t | \mathcal{F}_t^W) = G(t) = \mathbb{P}(\tau > t)$, we obtain

$$X_t^* = e^{rt} \frac{x_0 (Z_t^*)^{-1}}{\mathbb{E}(T \wedge \tau)} \int_t^T \frac{\mathbb{E}\left(\mathbb{1}_{\{s \leq \tau\}} | \mathcal{F}_t^W\right)}{\mathbb{E}\left(\mathbb{1}_{\{t < \tau\}} | \mathcal{F}_t^W\right)} \mathrm{d}s = \frac{x_0 (Z_t^*)^{-1}}{\mathbb{E}(T \wedge \tau) G(t)} \int_t^T G(s) \mathrm{d}s \quad \text{a.s.}$$

namely

$$X_t^* = \frac{x_0 e^{rt}}{\mathbb{E}(T \wedge \tau)} \frac{e^{-\psi^* W_t + \frac{1}{2}(\psi^*)^2 t}}{G(t)} \int_t^T G(s) \mathrm{d}s \quad \text{a.s.}$$

Equivalently, in differential form, being $G(t) = e^{-\int_0^t \lambda(s) \mathrm{d}s}$ and noticing that at time $t = \tau$ the optimal wealth jumps to zero (the absorbing state) we have, for $t < T$,

$$\mathrm{d}X_t^* = X_{t^-}^* \left[\left(r + (\psi^*)^2 + \lambda(t) - \frac{G(t)}{\int_t^T G(s) \mathrm{d}s} \right) \mathrm{d}t - \mathrm{d}H_t - \psi^* \mathrm{d}W_t \right], \quad X_0^* = x_0. \quad (4.2.11)$$

To determine $\pi^{1,*}$ and $\pi^{2,*}$ it suffices to identify, term by term, the above equation and Equation (3.1.6) (we are now under $\mathbb{P}$) and we finally find

$$\pi_t^{1,*} = -\frac{1}{\phi^1}, \quad \pi_t^{2,*} = -\frac{\psi^*}{\sigma^2} + \frac{\sigma^1}{\sigma^2 \phi^1} \quad \text{a.s.,} \quad t \leq (T \wedge \tau). \quad (4.2.12)$$

Remark 4.2.2. *The above stochastic differential Equation (4.2.11) is, indeed, the analog to Equation (3.2.17) in the case of deterministic model coefficients and deterministic intensity λ, as explained in Remark 3.2.3. It suffices to understand the link between the super-martingale G and the deterministic function $G(\cdot)$, that here coincide. We have (recall that the law of τ is by definition υ and it is assumed to be absolutely continuous with respect to the Lebesgue measure) $G_t = \mathbb{P}(\tau > t | \mathcal{F}_t^W) = \mathbb{P}(\tau > t) = \int_t^\infty \upsilon(\theta) d\theta = G(t) = e^{-\int_0^t \lambda(\theta) d\theta}$.*

▷ **The power-utility case.**
The reasoning is the same as in the previous example, but now we work under the measure $\mathbb{Q}^*$, instead of $\mathbb{P}$, in order to show an alternative way to obtain the result. From Equation

(4.2.10), denoting for simplicity by $A(T)$ the denominator in c_s^* (it does not depend on s) we have

$$c_s^* = x_0 \frac{(e^{-rs}Z_s^*)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^T (1-H_u)(e^{-ru}Z_u^*)^{\frac{\gamma}{\gamma-1}}\mathrm{d}u} =: \frac{x_0}{A(T)}(e^{-rs}Z_s^*)^{\frac{1}{\gamma-1}} \quad \text{a.s.,} \quad s \leq (T\wedge\tau),$$

and so the computation that has to be done is

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T \mathbb{1}_{\{s\leq\tau\}}e^{-rs}c_s^*\,\mathrm{d}s \mid \mathcal{G}_t\right) = \frac{x_0}{A(T)}\int_t^T e^{\frac{r\gamma}{1-\gamma}s}\,\mathbb{E}^{\mathbb{Q}^*}\left(\mathbb{1}_{\{s\leq\tau\}}(Z_s^*)^{\frac{1}{\gamma-1}}|\mathcal{G}_t\right)\mathrm{d}s \text{ a.s..}$$

If we remark that the "key-Lemma" 3.2.2 is also valid under the measure $\mathbb{Q}^*$, that here $(\mathcal{F}_t^W)_{t\geq0} = (\mathcal{F}_t^{W^*})_{t\geq0}$ and that (from Remark 4.1.1 c)) $G(t) = \mathbb{P}(\tau > t|\mathcal{F}_t^W) = \mathbb{Q}^*(\tau > t|\mathcal{F}_t^{W^*})$, recalling that $Z_s^* = e^{\psi^*W_s^*+\frac{1}{2}(\psi^*)^2 s}$ (under $\mathbb{Q}^*$), we find, a.s.,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T \mathbb{1}_{\{s\leq\tau\}}e^{-rs}c_s^*\,\mathrm{d}s \mid \mathcal{G}_t\right) &= \frac{x_0}{A(T)}\int_t^T e^{\frac{r\gamma}{1-\gamma}s}\frac{\mathbb{E}^{\mathbb{Q}^*}\left(\mathbb{1}_{\{s\leq\tau\}}(Z_s^*)^{\frac{1}{\gamma-1}}|\mathcal{F}_t^{W^*}\right)}{\mathbb{E}^{\mathbb{Q}^*}\left(\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t^{W^*}\right)}\mathrm{d}s \\
&= \frac{x_0}{A(T)}\frac{1}{G(t)}\int_t^T e^{\frac{r\gamma}{1-\gamma}s}G(s)\mathbb{E}^{\mathbb{Q}^*}\left(e^{\frac{\psi^*}{\gamma-1}W_s^*+\frac{1}{2}\frac{(\psi^*)^2}{\gamma-1}s}|\mathcal{F}_t^{W^*}\right)\mathrm{d}s \\
&= \cdots = \frac{x_0}{A(T)}\frac{1}{G(t)}e^{\frac{\psi^*}{\gamma-1}W_t^*}e^{-\frac{1}{2}\frac{(\psi^*)^2}{(\gamma-1)^2}t}\int_t^T G(s)e^{\frac{K}{1-\gamma}s}\mathrm{d}s,
\end{aligned}$$

where we recall that $K = \frac{1}{2}\frac{\gamma}{1-\gamma}(\psi^*)^2 + r\gamma$ (it was introduced in Section 4.2.2 in the power utility case). This gives us the optimal wealth for $t < (T\wedge\tau)$

$$X_t^* = e^{rt}\frac{x_0}{A(T)}\frac{1}{G(t)}e^{-\frac{\psi^*}{1-\gamma}W_t^*}e^{-\frac{1}{2}\frac{(\psi^*)^2}{(\gamma-1)^2}t}\int_t^T G(s)e^{\frac{K}{1-\gamma}s}\mathrm{d}s \quad \text{a.s.}$$

while for $t \geq (T\wedge\tau)$ we have $X_t^* = 0$ (in particular, the optimal wealth jumps to zero at time $t = \tau$, as it is also clear from the following stochastic differential equation, given that the coefficient of the jump part is equal to -1). In differential form, for $t < T$,

$$\mathrm{d}X_t^* = X_{t-}^*\left[\left(r + \lambda(t) - \frac{G(t)e^{\frac{tK}{1-\gamma}}}{\int_t^T G(s)e^{\frac{K}{1-\gamma}s}\mathrm{d}s}\right)\mathrm{d}t - \mathrm{d}H_t - \frac{\psi^*}{1-\gamma}\mathrm{d}W_t^*\right], \quad X_0^* = x_0$$

and by identifying term by term the above equation with the dynamics of the optimal wealth X^* written under the measure $\mathbb{Q}^*$ (Equation (3.2.6)), we find

$$\pi_t^{1,*} = -\frac{1}{\phi^1}, \quad \pi_t^{2,*} = -\frac{\psi^*}{\sigma^2(1-\gamma)} + \frac{\sigma^1}{\sigma^2\phi^1} \quad \text{a.s.,} \qquad t \leq (T\wedge\tau). \tag{4.2.13}$$

Remark 4.2.3. *What is the economic/financial interpretation of the optimal solution we find in both the logarithmic and power utility cases, namely* $\pi_t^{1,*} = -\frac{1}{\phi^1}, t \leq (T\wedge\tau)$*?*

- *If* $\phi^1 > 0$*, the asset's price* S^1 *has an upward jump at time* τ *and the optimal investment consists in selling* S^1*;*
- *If* $\phi^1 \in (-1, 0)$*, the asset's price* S^1 *has an downward jump at time* τ *and the optimal investment consists in buying* S^1*.*

This has to be understood in the following sense: the investor's final aim is to end up at time $T \wedge \tau$ with an optimal wealth satisfying $X^*_{T\wedge\tau} = 0$ (recall also Remark 3.2.2), i.e., he wants his optimal wealth to instantaneously jump to zero at the maturity. For this reason, if he knows a priori *that S^1 will upwardly jump at τ, he will sell this asset, since, being by definition his wealth nonnegative almost surely at any time, having this asset in his portfolio will not help him to reach his optimal final condition. The reasoning is the opposite for the case $\phi^1 \in (-1, 0)$.*

4.2.5 Problem C: optimal consumption

In the log-utility case, c^* is exactly the same as that we found in Section 3.2.5, Equation (3.2.22) (where the stochastic interest rate has to be replaced by the constant one). In the power utility case, we obtain an explicit expression for c^*, in contrast with what we had in the previous chapter.

▷ **The power-utility case.**

In this case, we find

$$c_s^{*,C} = \left(\nu Z_s^* e^{(\rho-r)s}\right)^{\frac{1}{\gamma-1}} = x_0 \frac{\left(e^{(\rho-r)s} Z_s^*\right)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^{+\infty} (e^{-rs} Z_s^*)^{\frac{\gamma}{\gamma-1}} (e^{\rho s})^{\frac{1}{\gamma-1}} \mathrm{d}s} \quad \text{a.s.,} \tag{4.2.14}$$

since ν is given by

$$\nu = \left[\frac{\mathbb{E}\int_0^{+\infty} (e^{-rs} Z_s^*)^{\frac{\gamma}{\gamma-1}} (e^{\rho s})^{\frac{1}{\gamma-1}} \mathrm{d}s}{x_0}\right]^{1-\gamma}.$$

By applying Fubini-Tonelli's theorem, we have

$$\mathbb{E}\int_0^{+\infty} (e^{-rs} Z_s^*)^{\frac{\gamma}{\gamma-1}} (e^{\rho s})^{\frac{1}{\gamma-1}} \mathrm{d}s = \int_0^{+\infty} e^{-s\frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - \frac{1}{2}\frac{(\psi^*)^2}{1-\gamma} - r\right]} \mathrm{d}s$$

and it is clear that Assumption 3.2.4 is satisfied (and so problem **C** admits an optimal solution $c^{*,C}$) if

$$k := \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - \frac{(\psi^*)^2}{2(1-\gamma)} - r\right] > 0. \tag{4.2.15}$$

In this case

$$c_s^{*,C} = x_0 k \left(e^{(\rho-r)s} Z_s^*\right)^{\frac{1}{\gamma-1}} \quad \text{a.s.} \tag{4.2.16}$$

We conclude this example by noticing that the above condition (4.2.15) exactly corresponds to [21, Equation (40), page 62] found by Merton in his Ph.D. thesis.

4.2.6 Problem C: optimal investment strategy

As expected, given that here the optimal consumption rate c^* is $\mathbb{F}$-adapted, a result analogous to Proposition 4.2.1 holds here.

Proposition 4.2.2. *The optimal proportion of wealth that has to be invested in S^1 in order to optimally solve problem $\boldsymbol{C}$ is, before the shock τ,*

$$\pi_t^{1,*} = 0 \quad a.s. \tag{4.2.17}$$

Proof. It is exactly the same as the one in Proposition 4.2.1 (based on the $\mathbb{F}-$adaptability of c^*) and we omit it. $\square$

▷ **The log-utility case.**
Exactly as in Section 3.2.6 we find, under $\mathbb{Q}^*$,

$$X_t^* = x_0 e^{(r-\rho)t}\frac{1}{Z_t^*} = x_0 e^{(r-\rho)t} e^{-\psi^* W_t^*} e^{-\frac{1}{2}(\psi^*)^2 t}, \quad t \leq T \quad \text{a.s.}$$

or, equivalently,

$$\mathrm{d}X_t^* = X_t^*\left[(r-\rho)\,\mathrm{d}t - \psi^* \mathrm{d}W_t^*\right], \quad X_0^* = x_0. \tag{4.2.18}$$

To determine $\pi^{1,*}$ and $\pi^{2,*}$ it suffices to identify, term by term, the above equation and Equation (3.1.6) written under the measure $\mathbb{Q}^*$, finding

$$\begin{cases} \pi_t^{1,*} = 0, \quad \pi_t^{2,*} = -\dfrac{\psi^*}{\sigma^2} \quad \text{a.s.}, \quad t \leq \tau; \\ \pi_t^{1,*}\sigma^1 + \pi_t^{2,*}\sigma^2 = -\psi^* \quad \text{a.s.}, \quad t > \tau. \end{cases} \tag{4.2.19}$$

▷ **The power-utility case.**
From Equation (4.2.16) we know that if $k = \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - \frac{(\psi^*)^2}{2(1-\gamma)} - r\right] > 0$ the optimal consumption rate exists and it is given by

$$c_s^* = x_0 k \left(e^{(\rho-r)s} Z_s^*\right)^{\frac{1}{\gamma-1}} \quad \text{a.s.}, \quad s < +\infty$$

and in this case we can compute (recall that, under $\mathbb{Q}^*$, $Z_t^* = e^{\psi^* W_t^* + \frac{1}{2}(\psi^*)^2 t}$)

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^{+\infty} e^{-rs} c_s^* \,\mathrm{d}s \mid \mathcal{G}_t\right) &= x_0 k \int_t^{+\infty} e^{\frac{\rho}{\gamma-1}s} e^{-\frac{r\gamma}{\gamma-1}s}\, \mathbb{E}^{\mathbb{Q}^*}\left[e^{\frac{\psi^*}{\gamma-1}W_s^* + \frac{1}{2}\frac{(\psi^*)^2}{\gamma-1}s} | \mathcal{G}_t\right] \mathrm{d}s \\
&= x_0 k e^{\frac{\psi^*}{\gamma-1}W_t^*} e^{-\frac{1}{2}\frac{(\psi^*)^2}{(\gamma-1)^2}t} \int_t^{+\infty} e^{\frac{\rho}{\gamma-1}s} e^{-\frac{r\gamma}{\gamma-1}s}\, e^{\frac{1}{2}\frac{(\psi^*)^2}{\gamma-1}s} e^{\frac{1}{2}\frac{(\psi^*)^2}{(\gamma-1)^2}s} \mathrm{d}s \\
&= x_0 k e^{-\frac{\psi^*}{1-\gamma}W_t^*} e^{-\frac{1}{2}\frac{(\psi^*)^2}{(\gamma-1)^2}t} \int_t^{+\infty} e^{-ks}\mathrm{d}s \\
&= x_0 e^{-\frac{\psi^*}{1-\gamma}W_t^*} e^{-\frac{1}{2}\frac{(\psi^*)^2}{(\gamma-1)^2}t} e^{-kt} \quad \text{a.s.}
\end{aligned}$$

This gives the optimal wealth for $t < +\infty$

$$X_t^* = x_0 e^{(r-k)t} e^{-\frac{\psi^*}{1-\gamma}W_t^*} e^{-\frac{1}{2}\frac{(\psi^*)^2}{(\gamma-1)^2}t},$$

or, in differential form (under $\mathbb{Q}^*$),

$$\mathrm{d}X_t^* = X_t^*\left[(r-k)\,\mathrm{d}t - \frac{\psi^*}{1-\gamma}\mathrm{d}W_t^*\right], \quad X_0^* = x_0.$$

By identification, as before, we find

$$\begin{cases} \pi_t^{1,*} = 0, \quad \pi_t^{2,*} = -\dfrac{\psi^*}{\sigma^2(1-\gamma)} \quad \text{a.s.}, \quad t \leq \tau; \\ \pi_t^{1,*}\sigma^1 + \pi_t^{2,*}\sigma^2 = -\dfrac{\psi^*}{1-\gamma} \quad \text{a.s.}, \quad t > \tau. \end{cases} \tag{4.2.20}$$

Chapter 5

Solution via the Dynamic Programming approach

For simplicity, in this chapter we consider the same deterministic coefficients setting as in Chapter 4 (in particular, we will see that in the case of problem **C**, in order to obtain explicit solutions, we will have to consider a constant intensity rate λ). We briefly recall here the model and the working assumptions. The assets' dynamics are given by the following stochastic differential equations

$$\begin{cases} \mathrm{d}S_t^0 &= rS_t^0\mathrm{d}t, & S_0^0 = 1, \\ \mathrm{d}S_t^1 &= S_{t-}^1\left(\mu^1\mathrm{d}t + \sigma^1\mathrm{d}W_t + \phi^1\mathrm{d}M_t\right), & S_0^1 = s_0^1, \\ \mathrm{d}S_t^2 &= S_t^2\left(\mu^2\mathrm{d}t + \sigma^2\mathrm{d}W_t\right), & S_0^2 = s_0^2. \end{cases} \tag{5.0.1}$$

The total information filtration $\mathbb{G}$ is here the filtration generated by the price processes S^1 and S^2, while $\mathbb{F}$ is the filtration generated by W. Furthermore, we work under the following hypothesis.

Assumption 5.0.1. *The proportionality relation*

$$\frac{r-\mu^1}{\sigma^1} = \frac{r-\mu^2}{\sigma^2} \tag{5.0.2}$$

holds true.

In this chapter, as in the previous one, the following will be our standing assumption.

Assumption 5.0.2. *The $\mathbb{F}$-intensity rate of τ is a deterministic function of time $\lambda(t)$.*

5.1 Problem A

In this section, we determine the optimal investment strategy $\pi_t^* = (\pi_t^{1,*}, \pi_t^{2,*})$ for every $0 \le t \le T$ (so that the optimal $\pi^{0,*}$ automatically follows) by applying the Dynamic Programming Principle (DPP) and by solving the Hamilton-Jacobi-Bellman (HJB) equation. The main difference with respect to the results obtained in the previous chapters (that were obtained using the martingale approach) is that here we directly find an optimal consumption strategy $c_t^*, 0 \le t \le T$ (and, at the same time, the optimal $\pi_t^{1,*}, \pi_t^{2,*}, 0 \le t \le T$) that depends on time (except in the case of problem **C**), on the optimal wealth X_t^* and on the jump indicator process H_t, that are accessible to the investor, while previously, in a first analysis, the optimal investment-consumption strategy was a function of the initial wealth x_0, of time (except in the case of problem **C**) and of the Radon-Nikodým derivative Z_t^*, that is non observable on the market.

Remark 5.1.1. *In order to apply the DPP and to obtain the HJB equation relative to our problem, we need the Markovianity of the state process $(X_t, H_t)_{t\geq 0}$. In the case of problems **A** and **B**, this is guaranteed by the fact that the intensity of τ is deterministic (Assumption 5.0.2), in the case of problem **C** by the fact that we will consider a constant intensity λ.*

Being the investor's information set at time t given by $\mathcal{G}_t$, the agent at time t has immediately access to his wealth X_t and to the value of the jump indicator process, i.e., H_t. This enables us to introduce the objective function

$$J(t, x, h; \pi, c) := \mathbb{E}\left[\int_t^T u(c_s)\mathrm{d}s | X_t = x, H_t = h\right], \tag{5.1.1}$$

where $h \in \{0, 1\}$ and the value function

$$V(t, x, h) := \sup_{(\pi,c)\in\mathcal{A}(t,x)} J(t, x, h; \pi, c), \tag{5.1.2}$$

where now $\mathcal{A}(t, x)$ is the analog to $\mathcal{A}(x_0)$ defined in Equation (3.1.8), when the wealth X_t at time t is equal to x. A pair (π^*, c^*) is optimal for problem **A** if it is admissible and if $J(t, x, h; \pi^*, c^*) = V(t, x, h)$ for every t, x, h.

Remark 5.1.2. *In order for our problem to be meaningful, we need the finiteness of $V(0, x, h)$ for all $(x, h) \in \mathbb{R}^+ \times \{0, 1\}$. A sufficient condition for this to hold is that u is continuous and satisfies the polynomial growth condition*

$$u(c) \leq K(1 + c^p) \qquad \forall\, c \in (0, +\infty),$$

for some $0 < K < \infty$ and $p \in (0, 1)$ (it suffices to adapt to our case the hypothesis in Remark 3.6.8 in Karatzas and Shreve [14]).

It is fairly straightforward that the function $x \mapsto V(t, x, h)$ is increasing on $(0, \infty)$ and, furthermore, it is strictly concave (provided that an optimal control, indeed, exists), as the following lemma shows.

Lemma 5.1.1. *For any $(t, h) \in [0, T] \times \{0, 1\}$, the function $x \mapsto V(t, x, h)$ is strictly concave.*

Proof. For a given $\lambda \in (0, 1)$ and $x, y \in \mathbb{R}^+$ our aim is to prove that for every $t \in [0, T]$ and $h \in \{0, 1\}$

$$V(t, \lambda x + (1 - \lambda)y, h) > \lambda V(t, x, h) + (1 - \lambda)V(t, y, h).$$

By definition of V, we have

$$V(t, \lambda x + (1 - \lambda)y, h) = \mathbb{E}_{t,z,h}\left[\int_t^T u(c_s^{*,z})\mathrm{d}s\right],$$

where $c^{*,z}$ is the optimal consumption strategy over the horizon $[t,T]$, corresponding to the initial wealth $X_t = z := \lambda x + (1-\lambda)y$ and where $\mathbb{E}_{t,z,h}$ denotes the expectation given that the state at time t is $(X_t, H_t) = (z, h)$. By recalling the admissibility condition (3.1.17), that now, being the starting investment date t, reads

$$X_t \geq e^{rt}\, \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs} c_s \, \mathrm{d}s \mid \mathcal{G}_t\right) \quad \text{a.s.},$$

it is evident that the admissible consumption rate $c_s^z = c_s^{\lambda x+(1-\lambda)y}$ satisfies (we divide on both sides by e^{rt})

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs} c_s^{*,z} \, \mathrm{d}s \mid \mathcal{G}_t\right) &= z = \lambda x + (1-\lambda)y \\
&\geq \lambda \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs} c_s^{x} \, \mathrm{d}s \mid \mathcal{G}_t\right) + (1-\lambda)\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs} c_s^{y} \, \mathrm{d}s \mid \mathcal{G}_t\right) \\
&= \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs} [\lambda c_s^x + (1-\lambda) c_s^y] \, \mathrm{d}s \mid \mathcal{G}_t\right).
\end{aligned}$$

We then have

$$c_s^{*,z} \geq \lambda c_s^x + (1-\lambda) c_s^y,$$

for every admissible consumption process c_s^x and c_s^y, corresponding, respectively, to the initial wealth x and y. Recalling the u is strictly increasing and strictly concave, we find

$$\begin{aligned}
V(t, \lambda x + (1-\lambda)y, h) &= \mathbb{E}_{t,z,h}\left[\int_t^T u(c_s^{*,z}) \mathrm{d}s\right] \\
&\geq \mathbb{E}_{t,z,h}\left[\int_t^T u\left(\lambda c_s^x + (1-\lambda) c_s^y\right) \mathrm{d}s\right] \\
&> \mathbb{E}_{t,z,h}\left[\int_t^T [\lambda u(c_s^x) + (1-\lambda) u(c_s^y)] \, \mathrm{d}s\right] \\
&\geq \lambda \mathbb{E}_{t,x,h}\left[\int_t^T u\left(c_s^x\right) \mathrm{d}s\right] + (1-\lambda)\mathbb{E}_{t,y,h}\left[\int_t^T u\left(c_s^y\right) \mathrm{d}s\right],
\end{aligned}$$

where in the last inequality we have automatically transformed $\mathbb{E}_{t,z,h}$ into $\mathbb{E}_{t,x,h}$ and $\mathbb{E}_{t,y,h}$, since by splitting the problem into two parts we work with c_s^x and c_s^y, that are consumption processes corresponding to a fixed initial wealth equal to, respectively, x ad y. It suffices, to conclude, to consider the supremum over all the admissible consumption rates c_s^x and c_s^y. □

Furthermore, we need the following Assumption (see the following Remark 5.1.3 for some related comments):

Assumption 5.1.1. *For $h = 0$ and $h = 1$, $V(t,x,h)$ is $\mathcal{C}^1$ with respect to t and $\mathcal{C}^2$ with respect to x.*

Given the above Lemma 5.1.1 and given Assumption 5.1.1, the function $x \mapsto V'_x(t,x,h)$ admits an inverse, defined on $\mathbb{R}^+$, that we denote by $\chi(t,\cdot,h)$.

5.1.1 The Hamilton-Jacobi-Bellman equation

By adapting to our (constant coefficients) setting the results in Øksendal and Sulem [22, Section 3.1], that are presented in the context of Lévy processes, and, in particular, by distinguishing between the two possible cases $h = 0$ and $h = 1$, it is easily seen that the value function satisfies the following Hamilton-Jacobi-Bellman equation.

Lemma 5.1.2. *The value function $V : [0, T) \times \mathbb{R}^+ \times \{0, 1\} \to \mathbb{R}$, defined in (5.1.2), satisfies the following fully nonlinear partial differential equation*

$$V_t'(t, x, h) + \max_{(\pi,c)\in\mathcal{A}(t,x)} [\mathcal{A}(t, \pi, c, x, h) + u(c)] = 0, \tag{5.1.3}$$

where

$$\begin{aligned}\mathcal{A}(t, \pi, c, x, 0) &= V_x'(t, x, 0) \left[rx + \pi^1 x(\mu^1 - r - \phi^1\lambda(t)) + \pi^2 x(\mu^2 - r) - c\right] \\ &\quad + \frac{1}{2} V_{xx}''(t, x, 0)\left(\pi^1 x\sigma^1 + \pi^2 x\sigma^2\right)^2 \\ &\quad + \lambda(t) \left[V(t, x + x\pi^1\phi^1, 1) - V(t, x, 0)\right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}(t, \pi, c, x, 1) &= V_x'(t, x, 1) \left[rx + \pi^1 x(\mu^1 - r) + \pi^2 x(\mu^2 - r) - c\right] \\ &\quad + \frac{1}{2} V_{xx}''(t, x, 1)(\pi^1 x\sigma^1 + \pi^2 x\sigma^2)^2,\end{aligned}$$

with the boundary condition

$$V(T, x, h) = 0, \qquad \forall\ (x, h) \in \mathbb{R}^+ \times \{0, 1\}. \tag{5.1.4}$$

Remark 5.1.3. ***Important!***
Note that, in practice, the problem naturally splits into two sub-problems, that are solved recursively. In a first step, we solve the partial differential equation (PDE) satisfied by the post-default value function $V(t, x, 1)$, then, we substitute this function into the analogous PDE for $V(t, x, 0)$, and we solve it. As a consequence, there are no jumps in the above PDEs and we do not deal with integro-partial differential equations, but with classical ones. Assumption 5.1.1 is, then, a "standard" one. A similar analysis can be found in Bielecki, Jeanblanc and Rutkowski [1], in the context of hedging of defaultable derivatives.

Results on the existence and uniqueness of a solution to equations of the above form (that in our case is the value function V) in the more general case of stochastic Markovian coefficients are known in the case of a bounded domain (see for example the overview in Fleming and Soner [10, IV.4] in the case when the PDE is uniformly parabolic, and Gilbarg and Trudinger [11, Ch. 17] in the case of a uniformly elliptic equation, arising in the infinite horizon optimization setting). For the unbounded domain case, when the coefficients are deterministic, we refer to the existence and uniqueness result in Jeanblanc and Pontier [13, Proposition 4.1].

We now characterize, in the following proposition, the optimal consumption and investment strategies in terms of the value function. We will then provide explicit solutions in two classic examples, namely in the logarithmic and power utility cases.

Proposition 5.1.1. *Suppose that there exists an optimal pair process* (π^*, c^*) *and that the value function satisfies the HJB equation. Then, the optimal consumption-investment strategies, corresponding to a wealth* x *at time* t*, in the two possible cases* $\{h = 1\}$ *and* $\{h = 0\}$ *are characterized as follows:*

- *after the shock* τ

$$c^*(t, x, 1) \quad = \quad I\left[V'_x(t, x, 1)\right]$$

and the optimal investment strategy $(\pi^{1,*}(t, x, 1), \pi^{2,*}(t, x, 1))$ *solves*

$$\pi^{1,*}(t, x, 1)\; x\sigma^1 + \pi^{2,*}(t, x, 1)\; x\sigma^2 = \psi^* \frac{V'_x(t, x, 1)}{V''_{xx}(t, x, 1)}. \tag{5.1.5}$$

- *Before the shock* τ

$$\begin{aligned}
c^*(t, x, 0) \quad &= \quad I\left[V'_x(t, x, 0)\right], \\
\pi^{1,*}(t, x, 0) \quad &= \quad \frac{1}{x\phi^1}\left\{\chi(t, V'_x(t, x, 0), 1) - x\right\}, \\
\pi^{2,*}(t, x, 0) \quad &= \quad \frac{\psi^*}{x\sigma^2}\frac{V'_x(t, x, 0)}{V''_{xx}(t, x, 0)} - \frac{\sigma^1}{x\sigma^2\phi^1}\left\{\chi(t, V'_x(t, x, 0), 1) - x\right\},
\end{aligned}$$

where I *is the inverse function of* u'*,* ψ^* *was introduced in Equation (4.1.7) and we recall that* χ *is the inverse function of* $V'_x(t, \cdot, h)$*.*

As suggested in the literature, we say that such pairs are given in feedback form, since, in both cases $h = 0$ and $h = 1$, they are determined, at time t, as functions of the optimal wealth $x = X^*_t$.

Proof. Being V solution to the HJB Equation (5.1.3), together with the boundary condition (5.1.4), the optimal consumption-investment strategies in the two cases $h = 1$ and $h = 0$, maximize, for all t, respectively,

$$\begin{aligned}
V'_x(t, X^*_t, 1)&\left[rX^*_t + \pi^1 X^*_t(\mu^1 - r) + \pi^2 X^*_t(\mu^2 - r) - c\right] \\
&+\tfrac{1}{2}V''_{xx}(t, X^*_t, 1)\left(\pi^1 X^*_t\sigma^1 + \pi^2 X^*_t\sigma^2\right)^2 + u(c)
\end{aligned}$$

and

$$\begin{aligned}
V'_x(t, X^*_t, 0)&\left[rX^*_t + \pi^1 X^*_t(\mu^1 - r - \phi^1\lambda(t)) + \pi^2 X^*_t(\mu^2 - r) - c\right] \\
&+\tfrac{1}{2}V''_{xx}(t, X^*_t, 0)\left(\pi^1 X^*_t\sigma^1 + \pi^2 X^*_t\sigma^2\right)^2 \\
&+\lambda(t)\left[V(t, X^*_t + X^*_t\pi^1\phi^1, 1) - V(t, X^*_t, 0)\right] + u(c).
\end{aligned}$$

The optimal consumption rate is immediately given in feedback form, in both cases and so we focus on the optimal investment strategy.

For what concerns the case $h = 1$, the conclusion follows by considering the first order conditions, by recalling that $\psi^* = \frac{r-\mu^2}{\sigma^2} = \frac{r-\mu^1}{\sigma^1}$ and by setting $X^*_t = x$.

In the case $h = 0$ first order conditions can be considered, but a deeper analysis is required to be sure that the value $\pi^{1,*}$ found by setting the first derivative equal to zero is optimal. We have, indeed, to solve a maximization problem under the constraint (4.1.4)

$$\pi^1\phi^1 \geq -1,$$

namely, to obtain the optimal pair $(\pi^{1,*},\pi^{2,*})$, before the shock, we have to solve the following problem (in a simplified notation):

$$\begin{cases} \max\limits_{\pi^1,\pi^2} \quad f(\pi^1,\pi^2) \\ \pi^1\phi^1 \geq -1,\ \pi^2 \in \mathbb{R}, \end{cases} \tag{5.1.6}$$

where (notice that we set $X_t^* = x$)

$$\begin{aligned} f(\pi^1,\pi^2) \quad := \quad & V_x'(t,x,0)\left[\pi^1 x(\mu^1 - r - \phi^1\lambda(t)) + \pi^2 x(\mu^2 - r)\right] + \frac{x^2}{2}V_{xx}''(t,x,0) \\ & \cdot\left(\pi^1\sigma^1 + \pi^2\sigma^2\right)^2 + \lambda(t)\left[V(t,x+x\pi^1\phi^1,1) - V(t,x,0)\right]. \end{aligned}$$

Here we have, then, to pay attention to the fact that we can have $\pi^{1,*}$ on the boundary of its domain, namely $\pi^{1,*} = -\frac{1}{\phi^1}$, without having $\dfrac{\partial f}{\partial \pi^1}(\pi^{1,*},\pi^2) = 0$. More specifically, by considering first order conditions for a regular *interior* maximum

$$\begin{cases} \frac{\partial f}{\partial \pi^1}(\pi^1,\pi^2) = 0 \\ \frac{\partial f}{\partial \pi^2}(\pi^1,\pi^2) = 0 \end{cases}$$

we find the optimal desired $\pi^{2,*}$ and a local (candidate global) maximum $\pi^{1,*}_{\text{loc}}$, in the interior of the domain. In order to be sure that $\pi^{1,*}_{\text{loc}} = \pi^{1,*}$, we have to verify that (recall that $\phi^1 > -1, \phi^1 \neq 0$)

$$\begin{cases} \dfrac{\partial f}{\partial \pi^1}\left(-\dfrac{1}{\phi^1},\pi^{2,*}\right) < 0, \quad \text{if} \quad \phi^1 \in (-1,0), \\ \dfrac{\partial f}{\partial \pi^1}\left(-\dfrac{1}{\phi^1},\pi^{2,*}\right) > 0, \quad \text{if} \quad \phi^1 > 0. \end{cases}$$

We obtain

$$\frac{\partial f}{\partial \pi^1}\left(-\frac{1}{\phi^1},\pi^{2,*}\right) = \phi^1\left\{x\lambda(t)\left[V_x'(t,0,1) - V_x'(t,x,0)\right] - x\left(\frac{\sigma^1}{\phi^1}\right)^2 V_{xx}''(t,x,0)\chi(t,V_x'(t,x,0),1)\right\},$$

and we observe that, since $c^*(t,0,h) = 0$ for every $t \geq 0$ and $h \in \{0,1\}$ (if we have no wealth, we cannot consume), then

$$V_x'(t,0,1) - V_x'(t,x,0) = u'\left(c^*(t,0,1)\right) - u'\left(c^*(t,x,0)\right) = u'(0) - u'\left(c^*(t,x,0)\right) \geq 0$$

and the optimality of $\pi^{1,*}_{\text{loc}}$ follows by recalling that V is increasing and strictly concave with respect to x and that χ is positive. $\square$

Once more, as in the previous chapters, we deduce from Equation (5.1.5) that in the considered complete market, after τ, the investment strategy is not unique, due to redundancy of the market.

In this specific setting we can obtain, as a corollary and using the results of the previous chapter, the same result obtained in Proposition 4.2.1, concerning the investment strategy before τ.

Corollary 5.1.1. *The optimal consumption-investment strategy, corresponding to a wealth x at time t, on the set $\{t \leq \tau\}$, is:*

$$
\begin{aligned}
c^*(t,x,0) &= I\left(V'_x(t,x,0)\right) = c^*(t,x,1),\\
\pi^{1,*}(t,x,0) &= 0,\\
\pi^{2,*}(t,x,0) &= \frac{\psi^*}{x\sigma^2}\frac{V'_x(t,x,0)}{V''_{xx}(t,x,0)}.
\end{aligned}
$$

Proof. It follows immediately by definition of the value function. Indeed, we already know, from Proposition 3.2.1, that the optimal consumption rate is here $\mathbb{F}$-adapted (this derives from the $\mathbb{F}$-adaptability of Z^*). In particular, the trajectories of the consumption process do not have any discontinuity at time τ, namely $c^*(t,x,0) = c^*(t,x,1)$, so that $V(t,x,0) = V(t,x,1)$ and

$$\chi(t, V'_x(t,x,0),1) - x = \chi(t, V'_x(t,x,1),1) - x = \chi(t, V'_x(t,x,0),0) - x = x - x = 0.$$

□

The following result, well known as *Verification theorem* (see, e.g., Fleming and Soner [10, Th. IV.3.1] in the context of controlled Markov diffusions in $\mathbb{R}^n$, or Øksendal and Sulem [22, Th. 3.1] in the context of jump-diffusions) provides a useful tool to determine the optimal feedback controls explicitly in two examples. Nevertheless, it assumes the knowledge of a candidate value function and of the optimal wealth.

Theorem 5.1.1. *Let $v(t,x,h)$ be a real valued function defined over $[0,T) \times \mathbb{R}^+ \times \{0,1\}$, of class $\mathcal{C}^{1,2}$ with respect to, respectively, t and x, solution to the HJB Equation (5.1.3), together with the boundary condition (5.1.4) and let (π^*, c^*) the pair defined in Proposition 5.1.1. If the pair is admissible, then v is the value function of our problem and (π^*, c^*) is optimal.*

We now apply the above theorem and the corollary in two examples, namely in the case of the logarithmic and power utility function cases and we obtain explicitly the optimal consumption rate and the investment strategy.

▷ The logarithmic and power utility cases

Proposition 5.1.2. *Let us suppose that $u(c) = ln(c)$. Then the explicit optimal solution to our consumption maximization problem $\boldsymbol{A}$, distinguishing between the cases $h = 1$ and $h = 0$ is:*

- *after the shock τ,*

$$
\begin{aligned}
c^*(t,x,1) &= \frac{x}{T-t},\\
\pi^{1,*}(t,x,1)\sigma^1 + \pi^{2,*}(t,x,1)\sigma^2 &= -\psi^*.
\end{aligned}
$$

The value function is

$$V(t,x,1) = \ln(x)(T-t) + q_1(t),$$

where

$$q_1(t) = \left(r + \frac{(\psi^*)^2}{2}\right)\frac{1}{2}(T-t)^2 - (T-t)\ln(T-t).$$

- *Before the shock τ,*

$$\begin{aligned} c^*(t,x,0) &= \frac{x}{T-t}, \\ \pi^{1,*}(t,x,0) &= 0, \\ \pi^{2,*}(t,x,0) &= -\frac{\psi^*}{\sigma^2}. \end{aligned}$$

The value function is

$$V(t,x,0) = V(t,x,1).$$

Proof. It suffices (notice that, thanks to Corollary 5.1.1, we only consider the case $h=1$) to make the following *ansatz*: $V(t,x,1) = \ln(x)p_1(t) + q_1(t)$ and to choose the two functions p_1 and q_1 that solve the HJB equation, together with the boundary condition. In particular, we find that $p_1(t) = T-t$ and that $q_1(t)$ has to satisfy

$$q_1'(t) = 1 + \ln(T-t) - (T-t)\left(r + \frac{1}{2}(\psi^*)^2\right), \quad q_1(T) = 0,$$

whose solution is given in the statement. Notice that the optimal consumption strategy $c^*(t,x,1) = c^*(t,x,0)$ is nonnegative.

To conclude, it remains to show that the wealth associated with the pair (π^*, c^*) is a.s. positive, for every $0 \le t \le T$. We find the dynamics

$$\mathrm{d}X_t^* = X_t^*\left[\left(r + (\psi^*)^2 - \frac{1}{T-t}\right)\mathrm{d}t - \psi^*\mathrm{d}W_t\right], \quad X_0^* = x_0, \tag{5.1.7}$$

that corresponds to

$$X_t^* = x_0\, e^{\left(r+\frac{1}{2}(\psi^*)^2\right)t}\, e^{-\psi^* W_t}\, \frac{T-t}{T} = \frac{x_0 e^{rt}(T-t)}{Z_t^* T} \quad \text{a.s.} \tag{5.1.8}$$

and the optimal wealth remains therefore always nonnegative over the time interval $[0,T]$, since $X_0 = x_0 \ge 0$, and it satisfies $X_T^* = 0$ a.s. $\square$

By performing analogous computations, we find the optimal solution in the power utility case.

Proposition 5.1.3. *Let us suppose that $u(c) = \frac{c^\gamma}{\gamma}, c \ge 0, \gamma < 1, \gamma \ne 0$. Then the explicit optimal solution to our consumption maximization problem $\boldsymbol{A}$, distinguishing between the cases $h=1$ and $h=0$ is:*

- *after the shock τ,*

$$\begin{aligned} c^*(t,x,1) &= x(\beta_1(t))^{\frac{1}{\gamma-1}}, \\ \pi^{1,*}(t,x,1)\sigma^1 + \pi^{2,*}(t,x,1)\sigma^2 &= -\psi^*\frac{1}{1-\gamma} \end{aligned}$$

and the value function is

$$V(t,x,1) = \frac{x^\gamma}{\gamma}\beta_1(t),$$

where $\beta_1(t) = \left[\frac{1-\gamma}{K}\left(e^{\frac{K}{1-\gamma}(T-t)} - 1\right)\right]^{1-\gamma}$ and $K = \frac{1}{2}\frac{\gamma}{1-\gamma}(\psi^*)^2 + r\gamma$.

- *Before the shock* τ,

$$\begin{aligned} c^*(t,x,0) &= x(\beta_1(t))^{\frac{1}{\gamma-1}}, \\ \pi^{1,*}(t,x,0) &= 0 \\ \pi^{2,*}(t,x,0) &= -\frac{\psi^*}{\sigma^2}\frac{1}{1-\gamma}. \end{aligned}$$

The value function is

$$V(t,x,0) = V(t,x,1).$$

Proof. The proof is analogous to the one in the previous Proposition, but in this case, the *ansatz* we make is: $V(t,x,1) = \frac{x^\gamma}{\gamma}\beta_1(t)$.
In both cases $\gamma \in (0,1)$ and $\gamma < 0$, it can be shown that $\beta_1(t)$ is positive and this implies that the optimal consumption $c^*(t,x,1) = c^*(t,x,0)$ is positive, too. The admissibility of the optimal solution to our problem follows immediately, as before in the logarithmic case, noticing that the wealth dynamics is given by

$$\mathrm{d}X_t^* = X_t^*\left[\left(r + \frac{(\psi^*)^2}{1-\gamma} - (\beta_1(t))^{\frac{1}{\gamma-1}}\right)\mathrm{d}t - \frac{\psi^*}{1-\gamma}\mathrm{d}W_t\right], \quad X_0^* = x_0 \geq 0, \tag{5.1.9}$$

or, equivalently, by

$$X_t^* = x_0\, e^{\left(r-\frac{(\psi^*)^2\gamma}{(1-\gamma)^2}\right)t}\, e^{-\frac{\psi^*}{1-\gamma}W_t} e^{-\int_0^t (\beta_1(s))^{\frac{1}{\gamma-1}}\mathrm{d}s} \quad \text{a.s.}$$

An explicit computation of $e^{-\int_0^t (\beta_1(s))^{\frac{1}{\gamma-1}}\mathrm{d}s}$ finally gives

$$X_t^* = x_0\, e^{\left(r-\frac{(\psi^*)^2\gamma}{(1-\gamma)^2}\right)t}\, \frac{e^{\frac{K}{1-\gamma}T} - e^{\frac{K}{1-\gamma}t}}{e^{\frac{K}{1-\gamma}T} - 1}\, e^{-\frac{\psi^*}{1-\gamma}W_t} \quad \text{a.s.}, \tag{5.1.10}$$

so that, in particular, $X_T^* = 0$ a.s. □

We conclude this section with the following remark.

Remark 5.1.4. *The optimal solutions, relative to the logarithmic and power utility cases, found in this subsection coincide with the solutions found in Section 4.2.1, namely, respectively,*

$$c_s^{*,A} = \frac{x_0}{TZ_s^* e^{-rs}} = c^*(s, X_s^*, 0) = c^*(s, X_s^*, 1) = \frac{X_s^*}{T-s}$$

and, in the power utility case,

$$\begin{aligned} c_s^{*,A} &= \frac{x_0}{(e^{-rs}Z_s^*)^{\frac{1}{1-\gamma}}} \frac{\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma} + r\gamma}{(1-\gamma)\left(e^{\frac{T}{1-\gamma}\left(\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma}+r\gamma\right)} - 1\right)} = c^*(s, X_s^*, 0) = c^*(s, X_s^*, 1) \\ &= X_s^*\left[\frac{1-\gamma}{K}\left(e^{\frac{K}{1-\gamma}(T-s)} - 1\right)\right]^{-1} = X_s^* \frac{\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma} + r\gamma}{(1-\gamma)\left(e^{\frac{(T-s)}{1-\gamma}\left(\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma}+r\gamma\right)} - 1\right)}, \end{aligned}$$

where in the last equality we have simply substituted the value of $K = \frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma} + r\gamma$.
This can be proved by exploiting Equations (5.1.8) and (5.1.10), since x *in* $c^*(s,x,\cdot)$ *represents the wealth that we have at time* s*, that is* X_s^* *(when we are at time* s*, we are supposed to have optimally invested in the market up to that time).*

5.2 Problem B

We introduce the objective function (notice that the investment-consumption strategy is "hidden" in the wealth process X)

$$J_{(\tau)}(t, x, h; \pi, c) := \mathbb{E}\left[\int_t^T (1 - H_s)u(c_s)\mathrm{d}s | X_t = x, H_t = h\right], \tag{5.2.1}$$

where $x \in \mathbb{R}^+$ and $h \in \{0, 1\}$, and the value function

$$V_{(\tau)}(t, x, h) := \sup_{(\pi,c)\in\mathcal{A}_\tau(t,x)} J_{(\tau)}(t, x, h; \pi, c), \tag{5.2.2}$$

where $\mathcal{A}_\tau(t, x)$ is the analog to $\mathcal{A}_\tau(x_0)$ defined in Equation (3.1.9), when the wealth X_t, at time t, is equal to x.

Remark 5.2.1. *By definition of $V_{(\tau)}$ we have*

$$V_{(\tau)}(t, x, 1) = 0, \quad \forall\ (t, x) \in [0, T] \times \mathbb{R}^+. \tag{5.2.3}$$

A pair (π^*, c^*) is optimal for problem **B** if it is admissible and if $J_{(\tau)}(t, x, h; \pi^*, c^*) = V_{(\tau)}(t, x, h)$ for every t, x, h.

Since we work in the pre-shock time interval, π^1_t is admissible if it satisfies suitable integrability conditions and if, for every $t \leq \tau$, almost surely,

$$\pi^1_t \phi^1 \geq -1. \tag{5.2.4}$$

Furthermore, the results in Remark 5.1.2 and Lemma 5.1.1 can be easily adapted here and also in this case the value function is increasing and strictly concave in x. The following assumption will be necessary to derive the HJB equation.

Assumption 5.2.1. *$V_{(\tau)}(t, x, 0)$ is $\mathcal{C}^1$ with respect to t and $\mathcal{C}^2$ with respect to x.*

Lemma 5.2.1. *The value function $V_{(\tau)}(\cdot, \cdot, 0) : [0, T) \times \mathbb{R}^+ \to \mathbb{R}$, defined in (5.2.2), satisfies the following fully nonlinear partial differential equation*

$$V'_{(\tau),t}(t, x, 0) + \max_{(\pi,c)\in\mathcal{A}_\tau(t,x)} \left[\mathcal{A}_{(\tau)}(t, \pi, c, x, 0) + u(c)\right] = 0, \tag{5.2.5}$$

where

$$\begin{aligned}\mathcal{A}_{(\tau)}(t, \pi, c, x, 0) &= V'_{(\tau),x}(t, x, 0)\left[rx + \pi^1 x(\mu^1 - r - \phi^1\lambda(t)) + \pi^2 x(\mu^2 - r) - c\right] \\ &\quad + \frac{1}{2}V''_{(\tau),xx}(t, x, 0)\left(\pi^1 x\sigma^1 + \pi^2 x\sigma^2\right)^2 - \lambda(t)V_{(\tau)}(t, x, 0),\end{aligned}$$

with the boundary condition

$$V_{(\tau)}(T, x, 0) = 0, \qquad \forall\ x \in \mathbb{R}^+. \tag{5.2.6}$$

Proof. The HJB equation can be obtained from the Dynamic Programming Principle, noticing that the Bellman principle can be written, for every pair $(\pi, c) \in \mathcal{A}_{(\tau)}(t,x)$ and for $\epsilon \geq 0, \epsilon \leq T-t$, as

$$V_{(\tau)}(t,x,0) \geq \mathbb{E}\left[\int_t^{t+\epsilon} (1-H_s)u(c_s)\mathrm{d}s + V_{(\tau)}(t+\epsilon, X_{t+\epsilon}, H_{t+\epsilon})\big| X_t = x, H_t = 0\right].$$

We now apply Itô's lemma to $V_{(\tau)}(t+\epsilon, X_{t+\epsilon}, H_{t+\epsilon})$ (with starting time t) and, recalling that (see Equation (3.1.3)) $\mathrm{d}H_t = \mathrm{d}M_t + \bar{\lambda}_t \mathrm{d}t = \mathrm{d}M_t + \lambda_t \mathbb{1}_{\{t<\tau\}}\mathrm{d}t$ and assuming that the local martingale we find (that is given by a stochastic integral in $\mathrm{d}W$ plus a stochastic integral in $\mathrm{d}M$) is a martingale, a formal computation gives us

$$\begin{aligned}
0 \quad \geq \quad & \mathbb{E}\Big\{\int_t^{t+\epsilon} \Big[(1-H_s)u(c_s) + \frac{\partial V_{(\tau)}}{\partial t}(s, X_s, H_s) + \frac{\partial V_{(\tau)}}{\partial x}(s, X_s, H_s) \\
& \cdot\Big(rX_s + \pi_s^1 X_s(\mu^1 - r - \phi^1\lambda(s)\mathbb{1}_{\{s<\tau\}}) + \pi_s^2 X_s(\mu^2 - r) - c_s\Big) \\
& + \frac{1}{2}\frac{\partial^2 V_{(\tau)}}{\partial x^2}(s, X_s, H_s)\big(\pi_s^1\sigma^1 X_s + \pi_s^2\sigma^2 X_s\big)^2 + \lambda(s)\mathbb{1}_{\{s<\tau\}} \\
& \cdot[V_{(\tau)}(s, X_{s-} + X_{s-}\pi_s^1\phi^1, 1) - V_{(\tau)}(s, X_{s-}, 0)]\Big]\mathrm{d}s \mid X_t = x, H_t = 0\Big\}.
\end{aligned}$$

It is crucial, now, to notice that the above integral is a continuous function of time, despite of the fact that the integrand has a discontinuity at $s = \tau$. We conclude, then, by standard arguments (we have $V_{(\tau)}(s, X_{s-} + X_{s-}\pi_s^1\phi^1, 1) = 0$), by dividing the right-hand side by ϵ and taking the limit as ϵ goes to zero and noticing that equality holds for the optimal pair (π^*, c^*). ☐

In the following proposition, we provide the optimal consumption rate and the optimal investment strategy in feedback form, we will give, then, explicit formulas in two examples.

Proposition 5.2.1. *Suppose that there exists an optimal pair process (π^*, c^*) and that the value function satisfies the HJB equation. Then, the optimal consumption-investment strategy, corresponding to a wealth x at time t, is characterized as follows:*

$$\begin{aligned}
c^*(t,x,0) &= I\left[V'_{(\tau),x}(t,x,0)\right], \\
\pi^{1,*}(t,x,0) &= -\frac{1}{\phi^1}, \\
\pi^{2,*}(t,x,0) &= \frac{\psi^*}{x\sigma^2}\frac{V'_{(\tau),x}(t,x,0)}{V''_{(\tau),xx}(t,x,0)} + \frac{\sigma^1}{\sigma^2\phi^1}.
\end{aligned}$$

Proof. Being $V_{(\tau)}$ solution to the HJB Equation (5.2.5), together with the boundary condition (5.2.6), the optimal consumption rate and the investment strategy have to maximize, for all t,

$$\begin{aligned}
& V'_{(\tau),x}(t, X_t^*, 0)\left[rX_t^* + \pi^1 X_t^*(\mu^1 - r - \phi^1\lambda(t)) + \pi^2 X_t^*(\mu^2 - r) - c\right] \\
& + \tfrac{1}{2}V''_{(\tau),xx}(t, X_t^*, 0)\big(\pi^1 X_t^*\sigma^1 + \pi^2 X_t^*\sigma^2\big)^2 - \lambda(t)V_{(\tau)}(t, X_t^*, 0) + u(c).
\end{aligned}$$

c^* is immediately obtained by considering the usual first order condition and so we focus on $(\pi^{1,*}, \pi^{2,*})$. We have, indeed, to solve a maximization problem under the constraint

$$\pi^1\phi^1 \geq -1,$$

namely, with a simplified notation, in order to obtain the optimal investment strategy we have to solve:

$$\begin{cases} \max\limits_{\pi^1,\pi^2} \quad f_{(\tau)}(\pi^1,\pi^2) \\ \pi^1\phi^1 \geq -1,\ \pi^2 \in \mathbb{R}, \end{cases} \tag{5.2.7}$$

where (notice that we set $X_t^* = x$)

$$\begin{aligned} f_{(\tau)}(\pi^1,\pi^2) \quad &:= \quad V'_{(\tau),x}(t,x,0)\left[\pi^1 x(\mu^1 - r - \phi^1\lambda(t)) + \pi^2 x(\mu^2 - r)\right] \\ &\quad +\frac{1}{2}V''_{(\tau),xx}(t,x,0)\big(\pi^1 x\sigma^1 + \pi^2 x\sigma^2\big)^2. \end{aligned}$$

It is convenient to make the following change of variable, introducing

$$z := \pi^1 x\sigma^1 + \pi^2 x\sigma^2 \quad \in \mathbb{R},$$

so that, recalling that $\psi^* = \frac{r-\mu^1}{\sigma^1} = \frac{r-\mu^2}{\sigma^2}$, we can equivalently maximize

$$f_{(\tau)}(\pi^1,z) = -\psi^* z V'_{(\tau),x}(t,x,0) - \pi^1 x\phi^1\lambda(t)V'_{(\tau),x}(t,x,0) + \frac{1}{2}z^2 V''_{(\tau),xx}(t,x,0)$$

over the domain $\{(\pi^1,z) \in \mathbb{R}^2 : \pi^1\phi^1 \geq -1\}$. First order conditions, for a regular *interior* maximum are given below:

$$\begin{cases} \frac{\partial f_{(\tau)}}{\partial \pi^1}(\pi^1,z) = -x\phi^1\lambda(t)V'_{(\tau),x}(t,x,0) = 0 \\ \frac{\partial f_{(\tau)}}{\partial z}(\pi^1,z) = -\psi^* V'_{(\tau),x}(t,x,0) + zV''_{(\tau),xx}(t,x,0) = 0 \\ \pi^1\phi^1 \geq -1,\ z \in \mathbb{R} \end{cases}$$

and so z^* is

$$z^* = \psi^* \frac{V'_{(\tau),x}(t,x,0)}{V''_{(\tau),xx}(t,x,0)},$$

but we do not find $\pi^{1,*}$. The reason for this is evident *a posteriori*, since the optimal π^1 is on the boundary of its domain, i.e., $\pi^{1,*} = -\frac{1}{\phi^1}$. In fact, we have (recall that we have to distinguish between the two possible cases $\phi^1 \in (-1,0)$ and $\phi^1 > 0$):

- if $\phi^1 \in (-1,0)$, the domain with respect to π^1 is $\pi^1 \leq -\frac{1}{\phi^1}$ and $f_{(\tau)}$ is increasing as a function of $\pi^1 \Longrightarrow$ the maximum is attained at $\pi^{1,*} = -\frac{1}{\phi^1}$;

- if $\phi^1 > 0$, the domain with respect to π^1 is $\pi^1 \geq -\frac{1}{\phi^1}$ and $f_{(\tau)}$ is decreasing as a function of $\pi^1 \Longrightarrow$ the maximum is attained at $\pi^{1,*} = -\frac{1}{\phi^1}$.

The optimal $\pi^{2,*}$ follows by simply recalling that $z = \pi^1 x\sigma^1 + \pi^2 x\sigma^2$. □

We apply now an analog to the Verification Theorem 5.1.1 to obtain the optimal feedback controls explicitly in the two usual examples.

▷ **The logarithmic and power utility cases**

Proposition 5.2.2. *Let us consider the case when $u(c) = ln(c)$. The explicit optimal solution to problem **B** is:*

$$\begin{aligned} c^*(t,x,0) &= \frac{x}{a(t)} \\ \pi^{1,*}(t,x,0) &= -\frac{1}{\phi^1}, \\ \pi^{2,*}(t,x,0) &= -\frac{\psi^*}{\sigma^2} + \frac{\sigma^1}{\sigma^2\phi^1}, \end{aligned}$$

where

$$a(t) = e^{\int_0^t \lambda(u)du} \int_t^T e^{-\int_0^s \lambda(u)du} ds = \frac{\int_t^T G(s)ds}{G(t)}.$$

The value function is

$$V_{(\tau)}(t,x,0) = \ln(x)a(t) + b(t),$$

where

$$b(t) = -e^{\int_0^t \lambda(u)du} \int_t^T e^{-\int_0^s \lambda(u)du}(1 + f(s))ds,$$

with $f(s) = \ln a(s) - a(s)\left(r + \lambda(s) + \frac{1}{2}(\psi^*)^2\right)$.

Proof. It suffices to make the following *ansatz*: $V_{(\tau)}(t,x,0) = \ln(x)a(t) + b(t)$ and find a and b that solve the HJB equation, together with the boundary condition. In particular, we find that $a(t)$ has to satisfy

$$a'(t) = \lambda(t)a(t) - 1, \quad a(T) = 0$$

and that $b(t)$ is a solution to the following differential equation

$$b'(t) = \lambda(t)b(t) + 1 + \ln a(t) - a(t)\left(r + \lambda(t) + \frac{1}{2}(\psi^*)^2\right), \quad b(T) = 0.$$

Notice that the optimal consumption strategy is nonnegative. To conclude, it remains to show that the wealth associated with the pair (π^*, c^*) is a.s. nonnegative, for every $0 \le t \le T$. We find, by substituting the optimal strategy into Equation (3.1.6), the dynamics

$$\begin{aligned} \mathrm{d}X_t^* &= X_{t^-}^*\left[\left(r + \lambda(t) + (\psi^*)^2 - \frac{1}{a(t)}\right)\mathrm{d}t - \mathrm{d}H_t - \psi^*\mathrm{d}W_t\right] \\ &= X_{t^-}^*\left[\left(r + (\psi^*)^2 - \frac{1}{a(t)}\right)\mathrm{d}t - \mathrm{d}M_t - \psi^*\mathrm{d}W_t\right], \quad X_0^* = x_0 \ge 0, \end{aligned} \tag{5.2.8}$$

that corresponds to

$$\begin{cases} X_t^* = x_0\, e^{\left(r+\frac{1}{2}(\psi^*)^2\right)t}\, e^{\int_0^t \lambda(s)\mathrm{d}s}\, e^{-\psi^* W_t}\, e^{-\int_0^t \frac{1}{a(s)}\mathrm{d}s} \quad \text{a.s.}, & t < \tau, \\ X_t^* = 0 \quad \text{a.s.}, & t \ge \tau. \end{cases} \tag{5.2.9}$$

More precisely, by computing explicitly $e^{-\int_0^t \frac{1}{a(s)}\mathrm{d}s}$, we find

$$\begin{cases} X_t^* = x_0\, e^{\left(r+\frac{1}{2}(\psi^*)^2\right)t}\, e^{\int_0^t \lambda(s)\mathrm{d}s}\, \frac{\int_t^T G(s)ds}{\int_0^T G(s)ds}\, e^{-\psi^* W_t} \quad \text{a.s.}, & t<\tau, \\ X_t^* = 0 \quad \text{a.s.}, & t \geq \tau \end{cases} \tag{5.2.10}$$

and the optimal wealth remains therefore always nonnegative over the time interval $[0,T]$ and we always have $X^*_{T\wedge\tau} = 0$ a.s. Furthermore, when $\{\tau > T\}$, the optimal consumption rate satisfies

$$\lim_{t\to T} c^*(t,x,0) = +\infty,$$

meaning that, when T approaches, the optimal consumption rule consists in consuming at a maximum rate, in order to get $X_T^* = 0$ a.s. $\square$

By performing analogous computations, we find the optimal solution in the power utility case.

Proposition 5.2.3. *Let us suppose that $u(c) = \frac{c^\gamma}{\gamma}, c \geq 0, \gamma < 1, \gamma \neq 0$. Then the optimal solution to problem $\boldsymbol{B}$ is:*

$$\begin{aligned} c^*(t,x,0) &= x(\alpha(t))^{\frac{1}{\gamma-1}}, \\ \pi^{1,*}(t,x,0) &= -\frac{1}{\phi^1}, \\ \pi^{2,*}(t,x,0) &= -\frac{\psi^*}{\sigma^2}\frac{1}{1-\gamma} + \frac{\sigma^1}{\sigma^2\phi^1} \end{aligned}$$

and the value function is

$$V_{(\tau)}(t,x,0) = \frac{x^\gamma}{\gamma}\alpha(t),$$

where

$$\alpha(t) = \left(e^{-\frac{K}{1-\gamma}t} e^{\int_0^t \lambda(s)\,ds} \int_t^T e^{\frac{K}{1-\gamma}s} e^{-\int_0^s \lambda(u)\,du} ds\right)^{1-\gamma} = \left(\frac{e^{-\frac{K}{1-\gamma}t}}{G(t)} \int_t^T e^{\frac{K}{1-\gamma}s} G(s)\,ds\right)^{1-\gamma},$$

and where (recall Proposition 5.1.3) $K = \frac{1}{2}\frac{\gamma}{1-\gamma}(\psi^*)^2 + r\gamma$.

Proof. We make the *ansatz*: $V_{(\tau)}(t,x,0) = \frac{x^\gamma}{\gamma}\alpha(t)$ and we find that $V_{(\tau)}$ solves the HJB Equation (5.2.5), together with the boundary condition (5.2.6) if α satisfies the following Bernoulli-type differential equation

$$\alpha'(t) = \left[-K + \lambda(t)(1-\gamma)\right]\alpha(t) - (1-\gamma)(\alpha(t))^{\frac{\gamma}{\gamma-1}}, \quad \alpha(T) = 0,$$

where $K = \frac{1}{2}\frac{\gamma}{1-\gamma}(\psi^*)^2 + r\gamma$, whose solution is given in the statement of the proposition.

The optimal consumption $c^*(t,x,1) = c^*(t,x,0)$ is positive and the admissibility of the optimal solution follows immediately, noticing that the wealth dynamics is given by

$$\mathrm{d}X_t^* = X_{t-}^* \left\{ \left[r + \lambda(t) + \frac{(\psi^*)^2}{1-\gamma} - (\alpha(t))^{\frac{1}{\gamma-1}} \right] \mathrm{d}t - \mathrm{d}H_t - \frac{\psi^*}{1-\gamma}\mathrm{d}W_t \right\}, X_0^* = x_0, \tag{5.2.11}$$

that corresponds to

$$\begin{cases} X_t^* = x_0\, e^{\left(r+\frac{(\psi^*)^2}{1-\gamma}-\frac{1}{2}\frac{(\psi^*)^2}{(1-\gamma)^2}\right)t}\, e^{\int_0^t \lambda(s)\mathrm{d}s}\, e^{-\frac{\psi^*}{1-\gamma}W_t}\, e^{-\int_0^t \alpha(s)^{\frac{1}{\gamma-1}}\mathrm{d}s} \quad \text{a.s.}, & t<\tau, \\ X_t^* = 0 \quad \text{a.s.}, & t \geq \tau. \end{cases} \tag{5.2.12}$$

By explicitly computing the integral $\int_0^t \alpha(s)^{\frac{1}{\gamma-1}}\mathrm{d}s$ we finally have, a.s.,

$$\begin{cases} X_t^* = x_0\, e^{\left(r+\frac{(\psi^*)^2}{1-\gamma}-\frac{1}{2}\frac{(\psi^*)^2}{(1-\gamma)^2}\right)t}\, e^{\int_0^t \lambda(s)\mathrm{d}s}\, \dfrac{\int_t^T e^{\frac{K}{1-\gamma}s}G(s)ds}{\int_0^T e^{\frac{K}{1-\gamma}s}G(s)ds}\, e^{-\frac{\psi^*}{1-\gamma}W_t} \quad \text{a.s.}, & t<\tau, \\ X_t^* = 0 \quad \text{a.s.}, & t \geq \tau. \end{cases} \tag{5.2.13}$$

In particular, $X^*_{T\wedge\tau} = 0$ a.s. and it is interesting to notice, as previously done in the logarithmic case, that, if $t < (T \wedge \tau)$,

$$\lim_{t\to T} c^*(t,x,0) = +\infty,$$

meaning that the optimal strategy, when approaching T, consists in consuming at a maximum rate, in order to get $X_T^* = 0$ a.s. □

Remark 5.2.2. *Easy computations show immediately that, in the two examples, the optimal consumption rate $c^{*,B}$ found by means of the "direct approach" in Section 4.2.3 coincides with the solution c^* to the HJB equation (5.2.5) (recall the analog Remark 5.1.4).*

5.3 Problem C

We introduce the objective function (for notational simplicity in what follows we will use x instead of x_0 to denote the initial wealth)

$$J_\infty(x,h;\pi,c) = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho s}u(c_s)\mathrm{d}s | X_0 = x, H_0 = h\right]$$

and we define the value function

$$V_\infty(x,h) := \sup_{(\pi,c)\in\mathcal{A}_\infty(x)} J_\infty(x,h;\pi,c). \tag{5.3.1}$$

Our purpose is to find a pair (π^*, c^*) that is admissible and that satisfies $J_\infty(x,h;\pi^*,c^*) = V_\infty(x,h)$. Notice that the value function does not depend on time and it is crucial to point out that x in V_∞ denotes the initial investor's wealth.

For what concerns the properties of the value function, given the specific form of the Radon-Nikodým derivative Z^*, that in the case of deterministic coefficients is given in Equation (4.1.9) (despite the presence of a jump in the market, Z^* is a diffusion process), we can use Theorem 3.9.18 in Karatzas and Shreve [14] to state that, under Assumption

3.2.4, V_∞ is finite and continuously differentiable for every $h \in \{0,1\}$. Furthermore, by its definition, V_∞ is increasing as a function of x and we can extend to this case the proof of Lemma 5.1.1 to show that, for any $h \in \{0,1\}$, $x \mapsto V_\infty(x,h)$ is strictly concave. It is, then, evident that the function $x \mapsto V_x'(x,h)$ admits an inverse defined on $\mathbb{R}^+$, that we denote $\chi_\infty(\cdot,h)$. Nevertheless, we have to suppose that the value function is two times differentiable with respect to the space variable (some comments on this assumption are given below in Remark 5.3.1).

Assumption 5.3.1. *For $h=0$ and $h=1$, $V_\infty(x,h)$ is $\mathcal{C}^2$ with respect to x.*

5.3.1 The Hamilton-Jacobi-Bellman equation

As before in the case of the finite horizon T, it is possible to show that V_∞ solves a fully nonlinear partial differential equation. Notice that in this case, in order to write the HJB equation, we need the intensity rate of τ, $\lambda(t), t \geq 0$, to be constant. We set $\lambda(t) \equiv \lambda$.

Lemma 5.3.1. *The value function $V_\infty : \mathbb{R}^+ \times \{0,1\} \to \mathbb{R}$ satisfies the Hamilton-Jacobi-Bellman equation of dynamic programming*

$$-\rho V_\infty(x,h) + \max_{(\pi,c)\in\mathcal{A}_\infty(x)} \left[\mathcal{A}_\infty(\pi,c,x,h) + u(c)\right] = 0, \tag{5.3.2}$$

where

$$\begin{aligned}\mathcal{A}_\infty(\pi,c,x,0) &= V'_{\infty,x}(x,0)\left[rx + \pi^1 x(\mu^1 - r - \phi^1\lambda(t)) + \pi^2 x(\mu^2 - r) - c\right] \\ &\quad + \frac{1}{2}V''_{\infty,xx}(x,0)\big(\pi^1 x\sigma^1 + \pi^2 x\sigma^2\big)^2 \\ &\quad + \lambda\left[V_\infty(x + x\pi^1\phi^1, 1) - V_\infty(x,0)\right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_\infty(\pi,c,x,1) &= V'_{\infty,x}(x,1)\left[rx + \pi^1 x(\mu^1 - r) + \pi^2 x(\mu^2 - r) - c\right] \\ &\quad + \frac{1}{2}V''_{\infty,xx}(x,1)(\pi^1 x\sigma^1 + \pi^2 x\sigma^2)^2.\end{aligned}$$

Remark 5.3.1. ***Important!***
*As noticed in Remark 5.1.3 relative to problem **A**, also here the problem naturally splits into two sub-problems, that are solved recursively. First, we solve the PDE satisfied by $V_\infty(x,1)$ and then we substitute this function into the analogous PDE for $V_\infty(x,0)$. We then deal with two classical PDEs and Assumption 5.3.1 is "standard" in this context.*

For an overview on existence and uniqueness results concerning the solution to the HJB equation in the infinite horizon case we refer to Fleming and Soner [10, Section IV.5, pag. 165-166]. We now provide the optimal consumption-investment strategy in feedback form, i.e., in terms of the value function V_∞.

Proposition 5.3.1. *Suppose that there exists an optimal pair process (π^*, c^*) and that the value function satisfies the HJB Equation (5.3.2). Then, the optimal consumption-investment strategies, corresponding to an initial wealth x, in the two possible cases $\{h=1\}$ and $\{h=0\}$, are characterized as follows:*

- *after the shock τ the optimal consumption rate is*

$$c^*(x,1) \;=\; I\left[V'_{\infty,x}(x,1)\right]$$

and the optimal investment strategy $(\pi^{1,*}(x,1), \pi^{2,*}(x,1))$ solves

$$\pi^{1,*}(x,1)\; x\sigma^1 + \pi^{2,*}(x,1)\; x\sigma^2 = \psi^* \frac{V'_{\infty,x}(x,1)}{V''_{\infty,xx}(x,1)}. \tag{5.3.3}$$

- *Before the shock τ*

$$\begin{aligned}
c^*(x,0) &= I\left[V'_{\infty,x}(x,0)\right] \\
\pi^{1,*}(x,0) &= \frac{1}{x\phi^1}\left\{\chi_\infty\left(V'_{\infty,x}(x,0),1\right) - x\right\} \\
\pi^{2,*}(x,0) &= \frac{\psi^*}{x\sigma^2}\frac{V'_{\infty,x}(x,0)}{V''_{\infty,xx}(x,0)} - \frac{\sigma^1}{x\sigma^2\phi^1}\left\{\chi_\infty\left(V'_{\infty,x}(x,0),1\right) - x\right\},
\end{aligned}$$

where I is the inverse function of u', ψ^* was introduced in Equation (4.1.7) and we recall that χ_∞ is the inverse function of $V'_{\infty,x}$.

Proof. It is exactly the same as the proof of Proposition 5.1.1 and we omit it. We only remark *a posteriori* that the optimal solutions are actually admissible. □

We now state the analog to Corollary 5.1.1 and we provide an interesting result based on the fact that, from previous results, we know that the value function is the same before and after the shock.

Corollary 5.3.1. *The optimal consumption-investment strategy, corresponding to an initial wealth x, before the shock τ, is:*

$$\begin{aligned}
c^*(x,0) &= I\left[V'_{\infty,x}(x,0)\right] = c^*(x,1), \\
\pi^{1,*}(x,0) &= 0, \\
\pi^{2,*}(x,0) &= \frac{\psi^*}{x\sigma^2}\frac{V'_{\infty,x}(x,0)}{V''_{\infty,xx}(x,0)}.
\end{aligned}$$

Proof. It is exactly the same as the one of Corollary 5.1.1, namely it is based on the fact that, for every x, $V_\infty(x,1) = V_\infty(x,0)$ and we omit it. □

The Verification Theorem below will be the key tool in order to obtain explicit solutions in the two following examples.

Theorem 5.3.1. *Let $v(x,h)$ be a real valued function defined over $\mathbb{R}^+ \times \{0,1\}$, of class $\mathcal{C}^2$ with respect to x, solution to the HJB Equation (5.3.2) and let (π^*, c^*) the pair defined in Proposition 5.3.1. If the pair is admissible, then v is the value function of our problem and this pair is optimal.*

▷ The logarithmic and power utility cases

Proposition 5.3.2. *Let us suppose that $u(c) = ln(c)$. The explicit optimal solution to problem **C** is:*

- *after the shock τ,*

$$
\begin{aligned}
c^*(x,1) &= \rho x,\\
\pi^{1,*}(x,1)\sigma^1 + \pi^{2,*}(x,1)\sigma^2 &= -\psi^*.
\end{aligned}
$$

The value function is

$$V_\infty(x,1) = \ln(x)A_1 + B_1 = \frac{1}{\rho}\left[\ln(x) + \frac{r}{\rho} + \frac{(\psi^*)^2}{2\rho} - 1 + \ln\rho\right],$$

where $A_1 = \frac{1}{\rho}$ and

$$B_1 = \frac{1}{\rho}\left[\frac{r}{\rho} + \frac{(\psi^*)^2}{2\rho} - 1 + \ln\rho\right].$$

- *Before the shock τ,*

$$
\begin{aligned}
c^*(x,0) &= \rho x,\\
\pi^{1,*}(x,0) &= 0,\\
\pi^{2,*}(x,0) &= -\frac{\psi^*}{\sigma^2}.
\end{aligned}
$$

The value function is

$$V_\infty(x,0) = V_\infty(x,1).$$

Proof. We start by making an *ansatz* concerning the value function (notice that, given Corollary 5.3.1, we only consider the case $h = 1$), namely we suppose that $V_\infty(x,1) = \ln(x)A_1 + B_1$ and then we solve the HJB equation, finding A_1 and B_1. To conclude, we have to check that the optimal solution leads to a positive wealth process and here it is the case (if $x > 0$), since we have, for every t,

$$\mathrm{d}X_t^* = X_t^*\left[\left(r + (\psi^*)^2 - \rho\right)\mathrm{d}t - \psi^*\mathrm{d}W_t\right], \quad X_0 = x \geq 0,$$

that corresponds to

$$X_t^* = xe^{(r-\rho)t}e^{\frac{1}{2}(\psi^*)^2 t}e^{-\psi^* W_t} \quad \text{a.s.}$$

$\square$

Analogously, in the power utility case we obtain the following result.

Proposition 5.3.3. *Let us suppose that $u(c) = \frac{c^\gamma}{\gamma}, c \geq 0, \gamma < 1, \gamma \neq 0$. Then the explicit optimal solution to our consumption maximization problem $\boldsymbol{C}$ is:*

- *after the shock τ,*

$$
\begin{aligned}
c^*(x,1) &= xA^{\frac{1}{\gamma-1}},\\
\pi^{1,*}(x,1)\sigma^1 + \pi^{2,*}(x,1)\sigma^2 &= -\frac{\psi^*}{1-\gamma}
\end{aligned}
$$

and the value function is

$$V_\infty(x,1) = \frac{x^\gamma}{\gamma}A,$$

where $A = \left[\frac{\gamma}{1-\gamma}\left(\frac{\rho}{\gamma} - \frac{1}{2}\frac{(\psi^*)^2}{1-\gamma} - r\right)\right]^{\gamma-1}$.

- *Before the shock τ,*

$$\begin{aligned} c^*(x,0) &= xA^{\frac{1}{\gamma-1}} = c^*(x,1), \\ \pi^{1,*}(x,0) &= 0 \\ \pi^{2,*}(x,0) &= -\frac{\psi^*}{\sigma^2(1-\gamma)}. \end{aligned}$$

The value function is

$$V_\infty(x,0) = V_\infty(x,1).$$

Proof. The result can be shown, without additional difficulties, as in the previous cases. Here it is only interesting to notice that the admissibility of the optimal consumption rate derives from the positivity of A, that was required in Equation (4.2.15). Furthermore, the optimal wealth always remains positive (provided that $x > 0$), since we have, for every t,

$$\mathrm{d}X_t^* = X_t^* \left[\left(r + \frac{(\psi^*)^2}{1-\gamma} - (A)^{\frac{1}{\gamma-1}} \right) \mathrm{d}t - \frac{\psi^*}{1-\gamma} \mathrm{d}W_t \right], \quad X_0 = x \geq 0,$$

that corresponds to

$$X_t^* = xe^{rt} e^{\frac{(\psi^*)^2}{1-\gamma}t} e^{-\frac{1}{2}\frac{(\psi^*)^2}{(1-\gamma)^2}t} e^{-(A)^{\frac{1}{\gamma-1}}t} e^{-\frac{\psi^*}{1-\gamma}W_t} \quad \text{a.s.}$$

□

Remark 5.3.2. *a) The admissibility of the optimal consumption rate process, i.e., the positivity of A, derives from (4.2.15), that was introduced in the power utility case, when applying a direct method, in order to ensure the existence of an optimal solution c^*.*
b) As done in Remark 5.1.4 and Remark 5.2.2, it is possible to show that, in both examples,

$$c_s^{*,C} = c^*(X_s^*, 0) = c^*(X_s^*, 1).$$

Notice that the optimal solution $c^*(x,\cdot)$ does not depend on time: x in $c^*(x,\cdot)$ represents the investor's wealth at the beginning of the investment period.

5.4 A reduced information setting: problem B1

We are now interested in studying problem **B** from a different point of view, namely we consider consumption-investment strategies that are $\mathbb{F}-$predictable (and no more $\mathbb{G}-$predictable, recall Definition 3.1.1). It is the problem of an investor with a reduced set of information, who has not access to the full information filtration $\mathbb{G}$. In particular, he does not observe τ (think for example of τ as a power supply interruption of a power plant). We will see that, in order to maximize his consumption up to time T, he will consider an alternative problem, denoted **B1**, in the filtration $\mathbb{F}$, by modifying his utility function u.

Indeed, since the filtration $\mathbb{F}$ does not include the observation of the event τ, we re-write the objective function as follows

$$\begin{aligned}
\mathbb{E}\left(\int_0^{T\wedge\tau} u(c_s)\mathrm{d}s\right) &= \mathbb{E}\left(\mathbb{1}_{\{\tau>T\}}\int_0^T u(c_s)\mathrm{d}s + \mathbb{1}_{\{\tau\leq T\}}\int_0^\tau u(c_s)\mathrm{d}s\right)\\
&= \mathbb{E}\left[\mathbb{E}\left(\mathbb{1}_{\{\tau>T\}}\int_0^T u(c_s)\mathrm{d}s|\mathcal{F}_T^W\right)\right] + \mathbb{E}\left(\mathbb{1}_{\{\tau\leq T\}}\int_0^\tau u(c_s)\mathrm{d}s\right)\\
&= \mathbb{E}\left(G(T)\int_0^T u(c_s)\mathrm{d}s\right) + \mathbb{E}\left(\int_0^T \upsilon(\theta)d\theta\int_0^\theta u(c_s)\mathrm{d}s\right),
\end{aligned}$$

where we have used the fact that the consumption rate c is $\mathbb{F}$-adapted, that $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t^W) = \mathbb{P}(\tau > t) = G(t)$ (it is a consequence of the independence of τ of $\mathbb{F}$, recall Lemma 4.1.1) and we have integrated with respect to the law υ of τ. We finally have, by applying Fubini-Tonelli's theorem,

$$\begin{aligned}
\mathbb{E}\left(\int_0^{T\wedge\tau} u(c_s)\mathrm{d}s\right) &= \mathbb{E}\left(G(T)\int_0^T u(c_s)\mathrm{d}s\right) + \mathbb{E}\left(\int_0^T u(c_s)\left[G(s) - G(T)\right]\mathrm{d}s\right)\\
&= \mathbb{E}\left(\int_0^T G(s)u(c_s)\mathrm{d}s\right) = \mathbb{E}\left(\int_0^T \widetilde{u}(s,c_s)\mathrm{d}s\right),
\end{aligned}$$

where $\widetilde{u} : [0,T]\times\mathbb{R}^+ \to [-\infty,+\infty)$ is

$$\widetilde{u}(s,c_s) := G(s)u(c_s) = e^{-\int_0^s \lambda(u)\mathrm{d}u}u(c_s). \tag{5.4.1}$$

Notice that, for each $s\in[0,T]$, $\widetilde{u}(s,\cdot)$ is a utility function, so that the investor with a reduced set of information acts on the market with a modified utility function, that incorporates the conditional law of the *non-observable* random variable τ. With respect to the objective function of problem **B**, here the investment horizon is larger, but the utility of consuming is lower, since in the integrand $u(c_s)$ is multiplied by $G(s)$ satisfying $0 < G(s) < 1$.

We can now state problem **B1**, that is analogous to problem **B**, where $\mathcal{A}_{\mathbb{F}}(x_0)$ corresponds to $\mathcal{A}_\tau(x_0)$ (we use the subscript "$\mathbb{F}$" to indicate that consumption-investment strategies are $\mathbb{F}-$ predictable):

$$\mathbf{B1}\quad \sup_{(\pi,c)\in\mathcal{A}_{\mathbb{F}}(x_0)} \mathbb{E}\int_0^T \widetilde{u}(s,c_s)\mathrm{d}s. \tag{5.4.2}$$

We solve **B1** by means of a mixed use of both the techniques exploited in the previous sections: we characterize the optimal consumption strategy $c^{*,B1}$ using the martingale approach and we obtain the optimal investment strategies by solving the HJB equation relative to problem **B1**.

Proposition 5.4.1. *Given the market structure (3.1.2), the optimal consumption rate solving problem **B1** in (5.4.2), with the terminal condition $X_T^* = 0$, is given by*

$$c_s^{*,B1} = I\left(\frac{\nu e^{-rs}Z_s^*}{G(s)}\right) \quad a.s., \tag{5.4.3}$$

where I denotes the inverse function of u' and $\nu > 0$ is a real parameter satisfying

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T e^{-ru} I\left(\frac{\nu e^{-ru}Z_u^*}{G(u)}\right)du\right] = \mathbb{E}\left[\int_0^T e^{-ru}Z_u^* I\left(\frac{\nu e^{-ru}Z_u^*}{G(u)}\right)du\right] = x_0.$$

Proof. It is analogous to the one of Proposition 3.2.1: given the concavity property of u and the definition of $c^{*,B1}$ in (5.4.3), we have:

$$\begin{aligned}\mathbb{E}\left(\int_0^T G(s)[u(c_s)-u(c_s^{*,B1})]\mathrm{d}s\right) &\leq \mathbb{E}\left(\int_0^T G(s)(c_s-c_s^{*,B1})u'(c_s^{*,B1})\mathrm{d}s\right)\\ &= \mathbb{E}\left(\int_0^T (c_s-c_s^{*,B1})\nu e^{-rs}Z_s^*\mathrm{d}s\right)\\ &\leq \nu(x_0-x_0)=0,\end{aligned}$$

where in the last inequality we have used the fact that c and $c^{*,B1}$ are admissible (recall Section 3.1.2). The optimality of $c^{*,B1}$ is proved. □

In order to compare the optimal consumption strategies of the two investors with different levels of information, $c_s^{*,B1}$ and $c_s^{*,B}$, we explicitly characterize them in the usual two cases.

▷ **The log-utility case.**

Straightforward computations show that the $\mathbb{F}$−adapted optimal consumption rate is given by

$$c_s^{*,B1}=\frac{x_0G(s)}{Z_s^*e^{-rs}\mathbb{E}\int_0^T G(u)\mathrm{d}u}=\frac{x_0G(s)}{Z_s^*e^{-rs}\int_0^T G(u)\mathrm{d}u} \quad \text{a.s.,} \tag{5.4.4}$$

meaning that an investor not having information concerning τ consumes, at time s, at a rate that depends on $\mathbb{P}(\tau>s|\mathcal{F}_s)=\mathbb{P}(\tau>s)$, i.e., on the law of the random time τ in his filtration. The comparison with the solution in Equation (4.2.9) can be summarized as follows (notice that we have distinguished between the case "before" and "after" the shock, even if $c_s^{*,B1}$ in practice does not depend on τ)

$$\begin{cases} c_s^{*,B1} \leq c_s^{*,B} \quad \text{a.s.,} & s\leq\tau,\\ c_s^{*,B1} > c_s^{*,B}=0 \quad \text{a.s.,} & s>\tau.\end{cases} \tag{5.4.5}$$

▷ **The power-utility case.**

We find

$$c_s^{*,B1}=\frac{x_0G(s)^{\frac{1}{1-\gamma}}(e^{-rs}Z_s^*)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^T G(u)^{\frac{1}{1-\gamma}}(e^{-ru}Z_u^*)^{\frac{\gamma}{\gamma-1}}\mathrm{d}u} \quad \text{a.s.} \tag{5.4.6}$$

Explicit computations in the case when the intensity of τ is constant and equal to λ, recalling that $G(u)=e^{-\lambda u}$, give us

$$c_s^{*,B1}=\frac{x_0G(s)^{\frac{1}{1-\gamma}}}{(e^{-rs}Z_s^*)^{\frac{1}{1-\gamma}}}\frac{\frac{r\gamma}{1-\gamma}+\frac{1}{2}(\psi^*)^2\frac{\gamma}{(\gamma-1)^2}-\frac{\lambda}{1-\gamma}}{e^{T\left(\frac{r\gamma}{1-\gamma}+\frac{1}{2}(\psi^*)^2\frac{\gamma}{(\gamma-1)^2}-\frac{\lambda}{1-\gamma}\right)}-1} \quad \text{a.s.}$$

First of all, in order to make a comparison with $c_s^{*,B}$, that on the set $\{s\leq\tau\}$, is equal to

$$c_s^{*,B}=x_0\frac{(e^{-rs}Z_s^*)^{\frac{1}{\gamma-1}}}{\mathbb{E}\int_0^T(1-H_u)(e^{-ru}Z_u^*)^{\frac{\gamma}{\gamma-1}}\mathrm{d}u} \quad \text{a.s.}$$

(on the set $\{s>\tau\}, c_s^{*,B}=0$ a.s.), we observe that, being $\frac{1}{1-\gamma}>0$ and $0<G(s)<1$,

$$G(s)^{\frac{1}{1-\gamma}}<1.$$

We then have to compare the denominators in $c_s^{*,B1}$ and in $c_s^{*,B}$, namely, if we apply Fubini-Tonelli's theorem and we use the independence of τ of $\mathbb{F}$,

$$\int_0^T G(u)^{\frac{1}{1-\gamma}}\mathbb{E}(e^{-ru}Z_u^*)^{\frac{\gamma}{\gamma-1}}\mathrm{d}u \quad \text{and} \quad \int_0^T G(u)\mathbb{E}(e^{-ru}Z_u^*)^{\frac{\gamma}{\gamma-1}}\mathrm{d}u.$$

We have, for every $u \in [0,T]$,

$$\begin{cases} G(u)^{\frac{1}{1-\gamma}} & < & G(u), \quad 0<\gamma<1, \\ G(u)^{\frac{1}{1-\gamma}} & > & G(u), \quad \gamma<0, \end{cases}$$

and so the comparison is possible only in the case $\gamma < 0$, in which, for $s \leq \tau$,

$$c_s^{*,B1} < c_s^{*,B} \quad \text{a.s.}$$

We now obtain the $\mathbb{F}-$predictable optimal investment strategies $\pi_t^{1,*}, \pi_t^{2,*}$, for every $0 \leq t \leq T$, by solving the corresponding Hamilton-Jacobi-Bellman equation. For this purpose, we introduce the objective function

$$J_{\mathbb{F}}(t,x;\pi,c) := \mathbb{E}\left[\int_t^T G(s)u(c_s)\mathrm{d}s|\mathcal{F}_t^W\right] \tag{5.4.7}$$

and the value function

$$V_{\mathbb{F}}(t,x) := \sup_{(\pi,c)\in\mathcal{A}_{\mathbb{F}}(t,x)} J_{\mathbb{F}}(t,x;\pi,c), \tag{5.4.8}$$

where $\mathcal{A}_{\mathbb{F}}(t,x)$ is the equivalent of $\mathcal{A}_{\mathbb{F}}(x_0)$ for a wealth equal to x at time t. Notice that here H is no more a state variable, since the investor's filtration is $\mathbb{F}$, but we suppose, nevertheless, that at time t the investor observes his wealth process.

Due to the similarity with problem **A** and given that $0 < G(s) < 1$, for every $s \geq 0$, the results in Remark 5.1.2 and Lemma 5.1.1 can be easily adapted here. Before deriving the HJB equation we make the following assumption.

Assumption 5.4.1. *$V_{\mathbb{F}}(t,x)$ is $\mathcal{C}^1$ with respect to t and $\mathcal{C}^2$ with respect to x.*

Lemma 5.4.1. *The value function $V_{\mathbb{F}} : [0,T) \times \mathbb{R}^+ \to \mathbb{R}$, defined in (5.4.8), satisfies the following fully nonlinear partial differential equation*

$$V'_{\mathbb{F},t}(t,x) + \max_{(\pi,c)\in\mathcal{A}_{\mathbb{F}}(t,x)} \left[\mathcal{A}_{\mathbb{F}}(t,\pi,c,x) + \widetilde{u}(t,c)\right] = 0, \tag{5.4.9}$$

where

$$\begin{aligned} \mathcal{A}_{\mathbb{F}}(t,\pi,c,x) &= V'_{\mathbb{F},x}(t,x)\left[rx + \pi^1 x(\mu^1 - r - \phi^1\lambda(t)G(t)) + \pi^2 x(\mu^2 - r) - c\right] \\ &\quad + \frac{1}{2}V''_{\mathbb{F},xx}(t,x)\left(\pi^1 x\sigma^1 + \pi^2 x\sigma^2\right)^2 \\ &\quad + \lambda(t)G(t)\left[V_{\mathbb{F}}(t,x+x\pi^1\phi^1) - V_{\mathbb{F}}(t,x)\right], \end{aligned}$$

with the boundary condition

$$V_{\mathbb{F}}(T,x) = 0, \qquad \forall\, x \in \mathbb{R}^+. \tag{5.4.10}$$

Proof. The HJB equation above can be obtained, as usual, starting from the Dynamic Programming Principle, namely by noticing that the Bellman principle can be written, for every pair $(\pi, c) \in \mathcal{A}_{\mathbb{F}}(t,x)$ and for $h \geq 0, h \leq T-t$, as

$$V_{\mathbb{F}}(t,x) \geq \mathbb{E}\left[\int_t^{t+h} G(s)u(c_s)\mathrm{d}s + V_{\mathbb{F}}(t+h, X_{t+h})|\mathcal{F}_t^W\right].$$

We then apply Itô's lemma to $V_{\mathbb{F}}(t+h, X_{t+h})$ and, assuming that the local martingale we find is a martingale, a formal computation gives us

$$\begin{aligned} 0 \quad \geq \quad & \mathbb{E}\Big\{\int_t^{t+h}\Big[G(s)u(c_s) + \frac{\partial V_{\mathbb{F}}}{\partial t}(s,X_s) + \frac{\partial V_{\mathbb{F}}}{\partial x}(s,X_s)\Big(rX_s + \pi_s^1 X_s(\mu^1 - r - \phi^1\lambda(s)\mathbb{1}_{\{s<\tau\}}) \\ & +\pi_s^2 X_s(\mu^2 - r) - c_s\Big) + \frac{1}{2}\frac{\partial^2 V_{\mathbb{F}}}{\partial x^2}(s,X_s)\big(\pi_s^1\sigma^1 X_s + \pi_s^2\sigma^2 X_s\big)^2 \\ & +\lambda(s)\mathbb{1}_{\{s<\tau\}}[V_{\mathbb{F}}(s, X_{s-} + X_{s-}\pi_s^1\phi^1) - V_{\mathbb{F}}(s,X_{s-})]\Big]\mathrm{d}s \mid \mathcal{F}_t^W\Big\}. \end{aligned}$$

At this point we have to pay attention to the fact that here the reference filtration is $\mathbb{F}$ and, even if the consumption-investment strategy is, by definition here, $\mathbb{F}$-predictable and the constant and deterministic coefficients, too, $(\mathbb{1}_{\{t<\tau\}})_{t\geq 0}$ is not. That is why in the HJB equation we find the conditional survival probability $G(t)$, that we introduce in the integrand by using the "tower property" of the conditional expectation, namely by considering a second conditional expectation made with respect to $\mathcal{F}_s^W \supseteq \mathcal{F}_t^W$. We conclude by standard arguments by dividing the right-hand side by h and taking the limit as h goes to zero and noticing that equality holds for the optimal pair (π^*, c^*). □

Remark 5.4.1. *The above HJB Equation (5.4.9) is similar to Equation (5.1.3) found for problem* ***A****, but here we are working under the filtration $\mathbb{F}$, so that, as we have already pointed out, we do not have access to τ and so the process H cannot be included in the state variable's set. As a consequence, in the HJB equation, the intensity $\lambda(t)$ is multiplied by $G(t) = \mathbb{P}(\tau > t|\mathcal{F}_t^W) = \mathbb{P}(\tau > t)$: the information we have about τ is its distribution.*

Explicit solutions are given in feedback form.

Proposition 5.4.2. *Suppose that there exists an optimal pair process (π^*, c^*) and that the value function satisfies the HJB equation. Then, the optimal consumption-investment strategy is characterized, at time t, as follows*

$$c^*(t,x) = I\left(\frac{V'_{\mathbb{F},x}(t,x)}{G(t)}\right), \tag{5.4.11}$$

$$\pi^{1,*}(t,x) = 0, \tag{5.4.12}$$

$$\pi^{2,*}(t,x) = \frac{\psi^*}{x\sigma^2}\frac{V'_{\mathbb{F},x}(t,x)}{V''_{\mathbb{F},xx}(t,x)}. \tag{5.4.13}$$

Remark 5.4.2. *The above result shows that the investor with reduced information, who cannot observe the random time τ, does not invest in the risky asset S^1 at all. This strategy seems to us an intuitive one and it is, indeed, different from the optimal strategy of an investor facing problem* ***B****.*

By applying a Verification theorem analogous to Theorem 5.1.1 we obtain the explicit solutions in the two usual examples.

$\triangleright$ **The logarithmic and power utility cases.**

Proposition 5.4.3. *Let us suppose that $u(c) = ln(c)$. The explicit $\mathbb{F}-$adapted optimal solution to problem **B1** is:*

$$
\begin{aligned}
c^*(t,x) &= \frac{xG(t)}{\int_t^T G(s)\,ds},\\
\pi^{1,*}(t,x) &= 0,\\
\pi^{2,*}(t,x) &= -\frac{\psi^*}{\sigma^2}.
\end{aligned}
$$

The value function is

$$V_{\mathbb{F}}(t,x) = \ln(x)\int_t^T G(s)\,ds + q(t),$$

where

$$q(t) = \int_t^T \left[p(s)\left(r + \frac{1}{2}(\psi^*)^2\right) - G(s) + G(s)\ln\frac{G(s)}{p(s)}\right] ds$$

and $p(s) = \int_s^T G(u)\,du$.

Proof. As usual we make an *ansatz*: $V_{\mathbb{F}}(t,x) = \ln(x)p(t) + q(t)$ and we choose the two functions p and q that solve the HJB equation, together with the boundary condition.

Notice that the optimal consumption strategy is positive and that the optimal solution is $\mathbb{F}-$adapted, as required. Furthermore, the optimal wealth has the dynamics

$$\mathrm{d}X_t^* = X_t^*\left[\left(r + (\psi^*)^2 - \frac{G(t)}{p(t)}\right)\mathrm{d}t - \psi^*\mathrm{d}W_t\right],$$

that is equivalent to

$$
\begin{aligned}
X_t^* &= x_0 e^{\left(r+\frac{1}{2}(\psi^*)^2\right)t} e^{-\int_0^t \frac{G(s)}{p(s)}ds} e^{-\psi^* W_t}\\
&= x_0 e^{\left(r+\frac{1}{2}(\psi^*)^2\right)t} e^{-\psi^* W_t} \frac{\int_t^T G(s)ds}{\int_0^T G(s)ds} \quad \text{a.s.}
\end{aligned}
$$

The optimal wealth is, therefore, positive, given that $X_0 = x_0 > 0$ and the terminal condition $X_T^* = 0$ a.s. is satisfied. $\square$

Remark 5.4.3. *The comparison with the optimal consumption rate and the optimal wealth found in problem **A** (recall Proposition 5.1.2) is possible if we compare, at time t,*

$$\frac{G(t)}{\int_t^T G(s)\,ds} \quad \textit{and} \quad \frac{1}{T-t}.$$

Since $0 < G(s) < 1, s \in (0,T]$, we have

$$\frac{1}{\int_t^T G(s)\,ds} > \frac{1}{T-t}$$

*so that if $G(t)=1$, meaning that we are sure that τ will arrive after time t, then the optimal consumption rate solving problem **B1** is greater than the one found in problem **A**, otherwise no general comparison is possible.*

Analogously, in the power utility case we find the following result.

Proposition 5.4.4. *Let us suppose that $u(c)=\frac{c^\gamma}{\gamma}, c\geq 0, \gamma<1, \gamma\neq 0$. Then the explicit $\mathbb{F}-$adapted optimal solution to our consumption maximization problem **B1** is:*

$$\begin{aligned} c^*(t,x) &= x\left(\frac{\beta(t)}{G(t)}\right)^{\frac{1}{\gamma-1}}, \\ \pi^{1,*}(t,x) &= 0, \\ \pi^{2,*}(t,x) &= \frac{\psi^*}{\sigma^2(\gamma-1)} \end{aligned}$$

and the value function is

$$V_{\mathbb{F}}(t,x)=\frac{x^\gamma}{\gamma}\beta(t),$$

where

$$\beta(t)=\left[e^{\frac{L}{1-\gamma}t}\int_t^T e^{-\frac{L}{1-\gamma}s}G(s)^{\frac{1}{1-\gamma}}\,ds\right]^{1-\gamma}$$

and

$$L=-\frac{1}{2}(\psi^*)^2\frac{\gamma}{1-\gamma}-r\gamma.$$

In this case too, the optimal investment-consumption strategy is $\mathbb{F}$-adapted and c^* is positive. Furthermore, we easily find

$$\mathrm{d}X_t^*=X_t^*\left\{\left[r+\frac{(\psi^*)^2}{1-\gamma}-\left(\frac{\beta(t)}{G(t)}\right)^{\frac{1}{\gamma-1}}\right]\mathrm{d}t-\frac{\psi^*}{1-\gamma}\mathrm{d}W_t\right\},\quad X_0^*=x_0,$$

that corresponds to

$$X_t^*=x_0\, e^{\left(r+\frac{(\psi^*)^2}{1-\gamma}-\frac{1}{2}\frac{(\psi^*)^2}{(1-\gamma)^2}\right)t}\; e^{-\frac{\psi^*}{1-\gamma}W_t}\; e^{-\int_0^t\left(\frac{\beta(s)}{G(s)}\right)^{\frac{1}{\gamma-1}}\mathrm{d}s}\qquad \text{a.s.}$$

By explicitly computing the integral $\int_0^t\left(\frac{\beta(s)}{G(s)}\right)^{\frac{1}{\gamma-1}}\mathrm{d}s$, we finally have,

$$X_t^*=x_0\, e^{\left(r+\frac{(\psi^*)^2}{1-\gamma}-\frac{1}{2}\frac{(\psi^*)^2}{(1-\gamma)^2}\right)t}\; e^{-\frac{\psi^*}{1-\gamma}W_t}\;\frac{\int_t^T e^{-\frac{L}{1-\gamma}s}G(s)^{\frac{1}{1-\gamma}}ds}{\int_0^T e^{-\frac{L}{1-\gamma}s}G(s)^{\frac{1}{1-\gamma}}ds}\qquad \text{a.s.}$$

In particular, $X_T^*=0$ a.s.

Remark 5.4.4. *Also in this case, as previously in the logarithmic utility case, no general comparison with the optimal solutions to problem **A** (recall Proposition 5.1.3) is possible, since we would need to compare*

$$x\left(\frac{\beta(t)}{G(t)}\right)^{\frac{1}{\gamma-1}}\quad \textit{and}\quad x(\beta_1(t))^{\frac{1}{\gamma-1}}$$

where

$$\beta(t) = \left[e^{\frac{L}{1-\gamma}t} \int_t^T e^{-\frac{L}{1-\gamma}s} G(s)^{\frac{1}{1-\gamma}} \, ds \right]^{1-\gamma} \quad \text{and} \quad \beta_1(t) = \left[e^{\frac{-K}{1-\gamma}t} \int_t^T e^{\frac{K}{1-\gamma}s} ds \right]^{1-\gamma}$$

with $L = -K$*. The fact is that we have* $G(t)^{\frac{1}{1-\gamma}} < 1$ *and*

$$(\beta(t))^{\frac{1}{1-\gamma}} < (\beta_1(t))^{\frac{1}{1-\gamma}}$$

and we cannot conclude.

Remark 5.4.5. *As observed in Remark 5.1.4, easy computations show immediately that, in the two examples, the optimal consumption rate found by means of the martingale method* $c^{*,B1}$ *coincides with the solution* c^* *to the HJB equation.*

Chapter 6

The exponential utility case

In this section, we consider the utility function

$$u(c) = -e^{-\eta c}, \quad c \in \mathbb{R},\ \eta > 0,$$

as a separate example, since u is strictly increasing, strictly concave and continuously differentiable, but its domain is $\mathbb{R}$ (and not, respectively, $\mathbb{R}_*^+$ and $\mathbb{R}^+$, as in the previous logarithmic and power utility cases) and so it does not satisfy the first condition in Equation (3.1.11). We denote by $I : \mathbb{R}_*^+ \to \mathbb{R}$ the continuous and strictly decreasing inverse of u', that is

$$I(y) = -\frac{1}{\eta} \ln\left(\frac{y}{\eta}\right).$$

Notice that we are not interested in solving problems of maximization of the utility from terminal wealth, so that we will not generalize the definition of admissible investment-consumption strategy (Definition 3.1.1), requiring that the corresponding wealth satisfies, for all $t \in [0, +\infty)$,

$$X_t \geq -\bar{K}, \quad \text{a.s.},$$

for a sufficiently large $\bar{K} > 0$ (that is the usual requirement that can be found in the literature focusing on such problems). We will rather continue working with admissible strategies in the sense of Definition 3.1.1, namely, whose corresponding wealth remains always positive over time.

For simplicity, in order to obtain "explicit" results, we directly consider here the same deterministic coefficients case introduced in Chapter 4. All the general remarks and considerations in Section 4.1 remain valid here.

6.1 Problem A

In this case, Proposition 3.2.1 (adapted to the deterministic coefficients case) is no more valid here, since I takes now values in $\mathbb{R}$ and so c^* is no more guaranteed to be positive. On the contrary, Lemma 5.1.2 remains valid, but we did not manage to obtain an explicit solution to the HJB equation in this exponential case. We focused, then, on the solution to problem **A** by means of a direct approach.

Proposition 6.1.1. *Given the market structure (4.1.1), the optimal consumption rate solving problem* $\boldsymbol{A}$ *in the exponential utility case is*

$$c_s^{*,A} = I(\nu e^{-rs} Z_s^*) \vee 0 = \left[-\frac{1}{\eta} \left(\ln \frac{\nu}{\eta} - rs + \psi^* W_s - \frac{1}{2} (\psi^*)^2 s \right) \right] \vee 0 \quad a.s., \tag{6.1.1}$$

where $\nu > 0$ *is a real parameter satisfying*

$$\mathbb{E}^{\mathbb{Q}^*} \left(\int_0^T e^{-ru} \left[I(\nu e^{-ru} Z_u^*) \vee 0 \right] du \right) = \mathbb{E} \left(\int_0^T e^{-ru} Z_u^* \left[I(\nu e^{-ru} Z_u^*) \vee 0 \right] du \right) = x_0 \tag{6.1.2}$$

and we recall that Z^* *is the Radon-Nikodým density process introduced in Equation (4.1.9).*

Proof. For simplicity we define $I_s := I(\nu e^{-rs} Z_s^*)$. We have, given the concavity of u,

$$\begin{aligned} u(c_s) - u(c_s^{*,A}) &\le (c_s - c_s^{*,A}) u'(c_s^{*,A}) = (c_s - c_s^{*,A}) u'(c_s^{*,A}) \left(\mathbb{1}_{\{I_s > 0\}} + \mathbb{1}_{\{I_s \le 0\}} \right) \\ &= (c_s - c_s^{*,A}) \nu e^{-rs} Z_s^* \mathbb{1}_{\{I_s > 0\}} + c_s u'(0) \mathbb{1}_{\{I_s \le 0\}} \\ &= c_s u'(0) \mathbb{1}_{\{I_s \le 0\}} + (c_s - c_s^{*,A}) \nu e^{-rs} Z_s^* - (c_s - c_s^{*,A}) \nu e^{-rs} Z_s^* \mathbb{1}_{\{I_s \le 0\}} \\ &= c_s u'(0) \mathbb{1}_{\{I_s \le 0\}} + (c_s - c_s^{*,A}) \nu e^{-rs} Z_s^* - c_s \nu e^{-rs} Z_s^* \mathbb{1}_{\{I_s \le 0\}} \\ &= c_s \mathbb{1}_{\{I_s \le 0\}} \left(u'(0) - u'(I_s) \right) + (c_s - c_s^{*,A}) \nu e^{-rs} Z_s^* \\ &\le (c_s - c_s^{*,A}) \nu e^{-rs} Z_s^*, \end{aligned}$$

where the last inequality follows from the fact that u' is decreasing and $c_s \ge 0$. We then find (we proceed as in the proof of Proposition 3.2.1)

$$\mathbb{E} \int_0^T \left[u(c_s) - u(c_s^{*,A}) \right] \mathrm{d}s \le \mathbb{E} \int_0^T (c_s - c_s^{*,A}) \nu e^{-rs} Z_s^* \mathrm{d}s \le \nu (x_0 - x_0) = 0,$$

where in the last inequality we have used the budget constraint (recall Section 3.1.2) and Equation (6.1.2), namely the fact that any admissible consumption rate c and the optimal one $c^{*,A}$ satisfy, respectively,

$$\mathbb{E} \int_0^T e^{-ru} c_u Z_u^* \mathrm{d}u \le x_0, \qquad \mathbb{E} \int_0^T e^{-ru} c_u^{*,A} Z_u^* \mathrm{d}u = x_0.$$

□

The existence of such an optimal consumption strategy is the subject of Remark 6.1.2.

As done before in Section 4.2.2, we can now directly compute the optimal wealth corresponding to $c^{*,A}$, by recalling that

$$X_t^* = e^{rt} \, \mathbb{E}^{\mathbb{Q}^*} \left(\int_t^T e^{-rs} c_s^* \mathrm{d}s | \mathcal{G}_t \right) \quad \text{a.s.,} \quad t \le T.$$

Remark 6.1.1. *By definition, given the positivity of the optimal consumption rate process, the optimal wealth is* positive *at any time.*

Proposition 6.1.2. *The optimal wealth process corresponding to the optimal consumption rate given in Proposition 6.1.1 is, for $t \le T$, a.s.*

$$X_t^* = e^{rt} \int_t^T e^{-rs} \frac{1}{\eta} \left[\frac{|\psi^*|\sqrt{s-t}}{\sqrt{2\pi}} e^{-\frac{Y_{t,s}^2}{2(\psi^*)^2(s-t)}} + Y_{t,s} \Phi\left(\frac{Y_{t,s}}{|\psi^*|\sqrt{s-t}} \right) \right] ds, \tag{6.1.3}$$

where $Y_{t,s} := \ln\left(\frac{\eta}{\nu}\right) + rs - \frac{1}{2}(\psi^*)^2 s - \psi^* W_t^*$ *and* $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$.

Proof. The optimal wealth at time $t \le T$ is given by $e^{rt}\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-rs} c_s^* \mathrm{d}s | \mathcal{G}_t \right)$, namely, if we apply Fubini-Tonelli's theorem, we have to compute $\mathbb{E}^{\mathbb{Q}^*}(c_s^*|\mathcal{G}_t)$, for s in $[t,T]$. We have, from Equation (6.1.1) and passing under the measure $\mathbb{Q}^*$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}(c_s^*|\mathcal{G}_t) &= \mathbb{E}^{\mathbb{Q}^*}\left(-\frac{1}{\eta}\left(\ln\frac{\nu}{\eta} - rs + \psi^* W_s^* + \frac{1}{2}(\psi^*)^2 s \right) \vee 0 \Big| \mathcal{G}_t \right) \\
&= \frac{1}{\eta}\mathbb{E}^{\mathbb{Q}^*}\left(\left(\ln\frac{\eta}{\nu} + rs - \psi^* W_s^* - \frac{1}{2}(\psi^*)^2 s \right) \mathbb{1}_{\{K_s>0\}} \Big| \mathcal{G}_t \right).
\end{aligned}$$

where $K_s := \ln\frac{\eta}{\nu} + rs - \psi^* W_s^* - \frac{1}{2}(\psi^*)^2 s$. If we then recall that W^* is a $(\mathbb{G}, \mathbb{Q}^*)-$Brownian motion, we find

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}(c_s^*|\mathcal{G}_t) &= \frac{1}{\eta}\mathbb{E}^{\mathbb{Q}^*}\left(\left(\ln\frac{\eta}{\nu} + rs - \psi^*(W_s^* - W_t^*) - \psi^* W_t^* - \frac{1}{2}(\psi^*)^2 s \right) \mathbb{1}_{\{K_s>0\}} \Big| \mathcal{G}_t \right) \\
&= \frac{1}{\eta}\mathbb{E}^{\mathbb{Q}^*}\left((-\psi^*(W_s^* - W_t^*) + Y_{t,s}) \mathbb{1}_{\{-\psi^*(W_s^*-W_t^*)+Y_{t,s}>0\}} \Big| \mathcal{G}_t \right),
\end{aligned}$$

where $Y_{t,s} = \ln\left(\frac{\eta}{\nu}\right) + rs - \frac{1}{2}(\psi^*)^2 s - \psi^* W_t^*$ is a $\mathcal{G}_t-$measurable random variable (that depends also on s). It now suffices to use the properties of conditional expectation to find

$$\mathbb{E}^{\mathbb{Q}^*}(c_s^*|\mathcal{G}_t) = \frac{1}{\eta}\mathbb{E}^{\mathbb{Q}^*}\left((-\psi^*(W_s^* - W_t^*) + y) \mathbb{1}_{\{-\psi^*(W_s^*-W_t^*)+y>0\}} \right)\Big|_{y=Y_{t,s}}$$

and we now make the direct computations, distinguishing between the two possible cases $\psi^* \ge 0$ and $\psi^* < 0$ and recalling that $W_s^* - W_t^*$ has same law as $\sqrt{s-t}Z_0$, with $Z_0 \backsim \mathcal{N}(0,1)$.

- If $\psi^* \ge 0$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}\left(-\psi^*(W_s^* - W_t^*)\mathbb{1}_{\{-\psi^*(W_s^*-W_t^*)+y>0\}} \right) &= \mathbb{E}\left(-\psi^* Z_0 \sqrt{s-t}\mathbb{1}_{\{Z_0 < \frac{y}{\psi^*\sqrt{s-t}}\}} \right) \\
&= \int_{-\infty}^{\frac{y}{\psi^*\sqrt{s-t}}} -\psi^* z \sqrt{s-t} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathrm{d}z \\
&= \frac{\psi^*\sqrt{s-t}}{\sqrt{2\pi}} e^{-\frac{y^2}{2(\psi^*)^2(s-t)}}
\end{aligned}$$

and

$$\mathbb{E}^{\mathbb{Q}^*}\left(y\mathbb{1}_{\{-\psi^*(W_s^*-W_t^*)+y>0\}} \right) = y\,\mathbb{P}\left(Z_0 < \frac{y}{\psi^*\sqrt{s-t}} \right) = y\,\Phi\left(\frac{y}{\psi^*\sqrt{s-t}} \right);$$

- If $\psi^* < 0$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*}\left(-\psi^*(W_s^*-W_t^*)\mathbb{1}_{\{-\psi^*(W_s^*-W_t^*)+y>0\}}\right) &= \mathbb{E}\left(-\psi^* Z_0\sqrt{s-t}\,\mathbb{1}_{\{Z_0>-\frac{y}{(-\psi^*)\sqrt{s-t}}\}}\right)\\
&= \int_{-\frac{y}{(-\psi^*)\sqrt{s-t}}}^{+\infty} -\psi^* z\sqrt{s-t}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\mathrm{d}z\\
&= \frac{(-\psi^*)\sqrt{s-t}}{\sqrt{2\pi}}e^{-\frac{y^2}{2(\psi^*)^2(s-t)}}
\end{aligned}$$

and

$$\mathbb{E}^{\mathbb{Q}^*}\left(y\mathbb{1}_{\{-\psi^*(W_s^*-W_t^*)+y>0\}}\right) = y\,\mathbb{P}\left(Z_0 > -\frac{y}{(-\psi^*)\sqrt{s-t}}\right) = y\,\Phi\left(\frac{y}{(-\psi^*)\sqrt{s-t}}\right).$$

The conclusion follows by replacing y with $Y_{t,s}$. □

Being the optimal consumption strategy $\mathbb{F}$-adapted, the optimal wealth has the same property and, as a consequence, it cannot have a jump at time τ. For this reason, in the stochastic differential dX^* there will be no term in dM and so we immediately find

$$\pi_t^{1,*} = 0 \quad \text{a.s.,} \quad t \leq \tau.$$

In order to fully obtain the optimal investment strategy we are only interested in terms in dW^*. We have, then, differentiating Equation (6.1.3) with respect to t in $Y_{t,s}$,

$$\begin{aligned}
dX_t^* &= e^{rt}\Big\{\int_t^T e^{-rs}\frac{1}{\eta}\Big[\frac{|\psi^*|\sqrt{s-t}}{\sqrt{2\pi}}e^{-\frac{Y_{t,s}^2}{2(\psi^*)^2(s-t)}}\frac{Y_{t,s}}{(\psi^*)^2(s-t)}\psi^*\\
&- \psi^*\Phi\left(\frac{Y_{t,s}}{|\psi^*|\sqrt{s-t}}\right) + Y_{t,s}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y_{t,s}^2}{2(\psi^*)^2(s-t)}}\frac{-\psi^*}{|\psi^*|\sqrt{s-t}}\Big]\mathrm{d}s\Big\}\,dW_t^*\\
&= -e^{rt}\frac{\psi^*}{\eta}\int_t^T e^{-rs}\Phi\left(\frac{Y_{t,s}}{|\psi^*|\sqrt{s-t}}\right)ds\,dW_t^*
\end{aligned}$$

and the optimal $\pi_t^{1,*}, \pi_t^{2,*}$ satisfy the following equation

$$\begin{cases} \pi_t^{1,*} = 0 & \text{a.s.,} \quad t \leq \tau;\\ \pi_t^{1,*}\sigma^1 + \pi_t^{2,*}\sigma^2 = -\frac{e^{rt}}{X_t^*}\frac{\psi^*}{\eta}\int_t^T e^{-rs}\Phi\left(\frac{Y_{t,s}}{|\psi^*|\sqrt{s-t}}\right)ds & \text{a.s.,} \quad t \leq T, \end{cases} \tag{6.1.4}$$

which is, unfortunately, not so expressive.

Remark 6.1.2. *Despite of the fact that we have already found the optimal solution, its existence is,* a priori*, based on the assumption (the analog to Assumption 3.2.1) that the function*

$$\Psi_e^A(\nu) := \mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{rs}[I(\nu e^{-rs}Z_s^*)\vee 0]ds\right) \tag{6.1.5}$$

is finite for every $0 < \nu < \infty$. The computations performed to obtain the optimal wealth, setting $t = 0$, give us

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{rs}[I(\nu e^{-rs}Z_s^*)\vee 0]ds\right) = \frac{1}{\eta}\int_0^T e^{-rs}\left[\frac{|\psi^*|\sqrt{s}}{\sqrt{2\pi}}e^{-\frac{Y_{0,s}^2}{2(\psi^*)^2 s}} + Y_{0,s}\Phi\left(\frac{Y_{0,s}}{|\psi^*|\sqrt{s}}\right)\right]ds,$$

where $Y_{0,s}$ is now deterministic, $Y_{0,s} = \ln\left(\frac{\eta}{\nu}\right) + rs - \frac{1}{2}(\psi^*)^2 s$. Then, $\Psi_e^A(\nu)$ is, indeed, finite for any $\nu \in (0,\infty)$ (notice, e.g., that Φ takes values in $[0,1]$).

6.2 Problem B

As previously in the case of problem **A**, we were not able to solve the HJB Equation (5.2.5) in the case when u is exponential. Nevertheless, Remark 3.2.2 remains valid here and we immediately find

$$\pi_t^{1,*} = -\frac{1}{\phi^1} \quad \text{a.s.} \quad 0 \leq t \leq \tau.$$

The following two results are the analog to Proposition 6.1.1 and of Proposition 6.1.2, respectively.

Proposition 6.2.1. *Given the market structure (4.1.1), the optimal consumption rate solving problem **B** in the exponential utility case is*

$$c_s^{*,B} = I(\nu e^{-rs} Z_s^*) \vee 0 = \left[-\frac{1}{\eta} \left(\ln \frac{\nu}{\eta} - rs + \psi^* W_s - \frac{1}{2}(\psi^*)^2 s \right) \right] \vee 0 \quad a.s., \tag{6.2.1}$$

where $\nu > 0$ is a real parameter satisfying

$$\mathbb{E}^{\mathbb{Q}^*} \left(\int_0^{T \wedge \tau} e^{-ru} \left[I(\nu e^{-ru} Z_u^*) \vee 0 \right] du \right) = \mathbb{E} \left(\int_0^{T \wedge \tau} e^{-ru} Z_u^* \left[I(\nu e^{-ru} Z_u^*) \vee 0 \right] du \right) = x_0.$$

Proof. It is exactly the same as the proof of Proposition 6.1.1 and we omit it. □

Remark 6.2.1. *a) The existence of the optimal c^* will be justified in Remark 6.2.2.*
*b) It is also possible to prove that Proposition 3.2.3 holds here too, meaning that before the shock τ, an investor facing problem **B** consumes at a higher rate than an investor facing problem **A**.*

We can now obtain the *positive* optimal wealth corresponding to $c^{*,B}$, by computing

$$X_t^* = e^{rt} \, \mathbb{E}^{\mathbb{Q}^*} \left(\int_t^T \mathbb{1}_{\{s \leq \tau\}} e^{-rs} c_s^* \mathrm{d}s | \mathcal{G}_t \right), \quad t \leq T.$$

Proposition 6.2.2. *The optimal wealth process corresponding to the optimal consumption rate given in Proposition 6.2.1 is, for $t \leq T$, a.s.,*

$$X_t^* = \frac{e^{rt}}{G(t)} \int_t^T e^{-rs} G(s) \left[\frac{1}{\eta} \left(\frac{|\psi^*| \sqrt{s-t}}{\sqrt{2\pi}} e^{-\frac{Y_{t,s}^2}{2(\psi^*)^2 (s-t)}} + Y_{t,s} \Phi \left(\frac{Y_{t,s}}{|\psi^*| \sqrt{s-t}} \right) \right) \right] ds, \tag{6.2.2}$$

where $Y_{t,s} := \ln\left(\frac{\eta}{\nu}\right) + rs - \frac{1}{2}(\psi^*)^2 s - \psi^* W_t^*$ and $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$.

Proof. The proof is completely analogous to the one of Proposition 6.1.2, the only extra difficulty here is that, instead of $\mathbb{E}^{\mathbb{Q}^*}(c_s^* | \mathcal{G}_t)$, we have to compute $\mathbb{E}^{\mathbb{Q}^*}\left(\mathbb{1}_{\{s \leq \tau\}} c_s^* | \mathcal{G}_t\right)$. However, we can pass to a conditional expectation made with respect to $\mathcal{F}_t^W$, thanks to the "key-Lemma" 3.2.2, in order to exploit the independence of τ of $\mathbb{F}$ and we are, then, led to the same computations as in the proof of Proposition 6.1.2, since the survival probability function $G(t), t > 0$, is a deterministic function of time (that can be taken out of the conditional expectation) and W^* is not only a $(\mathbb{G}, \mathbb{Q}^*)-$Brownian motion, but also a $(\mathbb{F}, \mathbb{Q}^*)-$Brownian motion. □

The computations that lead to the optimal investment strategies are the same as the ones in the previous section and we obtain

$$\begin{cases} \pi_t^{1,*} = -\frac{1}{\phi^1} & \text{a.s.,} \quad t \le \tau; \\ \pi_t^{1,*}\sigma^1 + \pi_t^{2,*}\sigma^2 = -\frac{e^{rt}}{X_t^* G(t)} \frac{\psi^*}{\eta} \int_t^T e^{-rs} G(s) \Phi\left(\frac{Y_{t,s}}{|\psi^*|\sqrt{s-t}}\right) ds & \text{a.s.,} \quad t \le T. \end{cases} \tag{6.2.3}$$

Remark 6.2.2. *The existence of the optimal solution was based on the assumption (the analog to Assumption 3.2.2) that the function*

$$\Psi_e^B(\nu) := \mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{T\wedge\tau} e^{-rs}\ [I(\nu e^{-rs} Z_s^*) \vee 0] ds\right)$$

is finite for every $0 < \nu < \infty$. The computations performed to obtain the optimal wealth, setting $t = 0$, give us (exactly as in Remark 6.1.2)

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{T\wedge\tau} e^{-rs}\ [I(\nu e^{-rs} Z_s^*) \vee 0] ds\right) = \frac{1}{\eta G(0)} \int_0^T e^{-rs} G(s) \left(\frac{|\psi^*|\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{Y_{0,s}^2}{2(\psi^*)^2 s}} + Y_{0,s} \Phi\left(\frac{Y_{0,s}}{|\psi^*|\sqrt{s}}\right)\right) ds,$$

where $Y_{0,s} = \ln\left(\frac{\eta}{\nu}\right) + rs - \frac{1}{2}(\psi^*)^2 s$. Then, $\Psi_e^B(\nu)$ is, indeed, finite for any $\nu \in (0,\infty)$.

6.3 Problem C

Since we were not able to solve the HJB Equation (5.3.2) in the exponential utility case, we focused on the direct approach.

Proposition 6.3.1. *Given the market structure (4.1.1), the optimal consumption rate solving problem **C** in the exponential utility case is, almost surely,*

$$c_s^{*,C} = I(\nu e^{(\rho-r)s} Z_s^*) \vee 0 = \left[-\frac{1}{\eta}\left(\ln\frac{\nu}{\eta} + (\rho - r)s + \psi^* W_s - \frac{1}{2}(\psi^*)^2 s\right)\right] \vee 0, \tag{6.3.1}$$

where $\nu > 0$ is a real parameter satisfying

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{T\wedge\tau} e^{-ru}\left[I(\nu e^{(\rho-r)u} Z_u^*) \vee 0\right] du\right) = \mathbb{E}\left(\int_0^{T\wedge\tau} e^{-ru} Z_u^* \left[I(\nu e^{(\rho-r)u} Z_u^*) \vee 0\right] du\right) = x_0.$$

Proof. It is exactly the same as the proof of Proposition 6.1.1 and we omit it. □

Remark 6.3.1. *a) The existence of c^* will be justified in Remark 6.2.2.*
b) It is also possible to prove that Proposition 3.2.5 (that is about a comparison between $c^{*,A1}$ and $c^{*,C}$) holds here, too.

We can now obtain the *positive* optimal wealth corresponding to $c^{*,C}$, by computing

$$X_t^* = e^{rt}\ \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^{+\infty} e^{-rs} c_s^* \mathrm{d}s | \mathcal{G}_t\right), \quad t < +\infty.$$

Proposition 6.3.2. *The optimal wealth process corresponding to the optimal consumption rate given in Proposition 6.3.1 is, for $t < \infty$, a.s.,*

$$X_t^* = e^{rt} \int_t^{+\infty} e^{-rs} \frac{1}{\eta} \left(\frac{|\psi^*|\sqrt{s-t}}{\sqrt{2\pi}} e^{-\frac{Z_{t,s}^2}{2(\psi^*)^2(s-t)}} + Z_{t,s}\Phi\left(\frac{Z_{t,s}}{|\psi^*|\sqrt{s-t}} \right) \right) ds, \tag{6.3.2}$$

where $Z_{t,s} := \ln\left(\frac{\eta}{\nu}\right) + (r-\rho)s - \frac{1}{2}(\psi^*)^2 s - \psi^* W_t^*$ and $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$.

Proof. The proof is completely analogous to the one of Proposition 6.1.2 and we omit it. □

The computations that lead to the optimal investment strategies are the same as the ones in the two previous sections. In particular, being the optimal consumption $\mathbb{F}$-adapted, we find here, too, that

$$\pi_t^{1,*} = 0 \ \text{ a.s.,} \quad t \le \tau.$$

By identification with the coefficients in the stochastic differential dX under $\mathbb{Q}^*$, we finally obtain,

$$\begin{cases} \pi_t^{1,*} = 0 & \text{a.s.,} \quad t \le \tau; \\ \pi_t^{1,*}\sigma^1 + \pi_t^{2,*}\sigma^2 = -\frac{e^{rt}}{X_t^*}\frac{\psi^*}{\eta}\int_t^{\infty} e^{-rs}\Phi\left(\frac{Z_{t,s}}{|\psi^*|\sqrt{s-t}}\right) ds & \text{a.s.,} \quad t \le T. \end{cases} \tag{6.3.3}$$

Remark 6.3.2. *The existence of the optimal solution was based on the assumption (the analog to Assumption 3.2.4) that the function*

$$\Psi_e^C(\nu) := \mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{\infty} e^{-rs} \ [I(\nu e^{(\rho-r)s} Z_s^*) \vee 0] ds \right)$$

is finite for every $0 < \nu < \infty$. By considering the optimal wealth X_0^* we find (exactly as in Remark 6.1.2)

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{\infty} e^{-rs} \ [I(\nu e^{(\rho-r)s} Z_s^*) \vee 0] ds \right) = \int_0^{+\infty} e^{-rs} \frac{1}{\eta} \left(\frac{|\psi^*|\sqrt{s}}{\sqrt{2\pi}} e^{-\frac{Z_{0,s}^2}{2(\psi^*)^2 s}} + Z_{0,s}\Phi\left(\frac{Z_{0,s}}{|\psi^*|\sqrt{s}} \right) \right) ds,$$

where $Z_{0,s} = \ln\left(\frac{\eta}{\nu}\right) + (r-\rho)s - \frac{1}{2}(\psi^*)^2 s$. $\Psi_e^C(\nu)$ is, indeed, finite for any $\nu \in (0, \infty)$.

Chapter 7

A more general stochastic model

7.1 Market model

In this chapter, we consider a market model similar to the one introduced in Chapter 3, but where we do not suppose (recall Assumption 3.1.1 a)) that the reference filtration $\mathbb{F}$ is immersed in the progressively enlarged filtration $\mathbb{G}$. Let us now briefly introduce the model and the working hypotheses.

On a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, equipped with a Brownian motion $(W_t)_{t\geq 0}$, we consider a non-negative random variable τ, satisfying $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$, for any $t \in \mathbb{R}^+$. The law of τ is denoted by $\upsilon, \upsilon(d\theta) = \mathbb{P}(\tau \in d\theta)$. We assume that υ is absolutely continuous with respect to Lebesgue measure and we write, with a slight abuse of notation,

$$\mathbb{P}(\tau \in d\theta) = \upsilon(d\theta) = \upsilon(\theta)d\theta.$$

We denote by $\mathbb{F} := \mathbb{F}^W = (\mathcal{F}_t^W)_{t\geq 0}$ the filtration generated by W, representing the information at disposal to investors before τ and by $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ the progressively enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ is the natural filtration of the process $H_t := \mathbb{1}_{\{t\geq\tau\}},\ t \geq 0$. The filtration $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$, that makes τ a stopping time.

In the sequel, the following will be our standing assumption (it is exactly the same as Assumption 3.2.3, introduced in Section 3.2.4 and it is similar to the main hypothesis of the following Part V).

Assumption 7.1.1. $(\mathcal{E})$-***Hypothesis***
The $\mathbb{F}$-(regular) conditional law of τ is equivalent to the law of τ, i.e.,

$$\mathbb{P}(\tau \in d\theta|\mathcal{F}_t^W) \sim \upsilon(\theta)d\theta \quad \textit{for every } t \geq 0,\ \ \mathbb{P} - a.s.$$

One of the consequences of the above assumption (for all the details we refer to Part V) is that there exists a "regular" family of strictly positive $(\mathbb{P}, \mathbb{F})$-martingales $(p_t(\theta))_{t\geq 0}$, $\theta \geq 0$, such that, for $s \geq 0$,

$$\mathbb{P}\left(\tau > s|\mathcal{F}_t^W\right) = \int_s^\infty p_t(\theta)\upsilon(\theta)d\theta \quad \text{for every } t \geq 0, \quad \mathbb{P} - \text{a.s.} \tag{7.1.1}$$

A similar hypothesis (called "*density hypothesis*") is made in Pham and Jiao [23] when studying problems of maximization of expected utility from terminal wealth. An exhaustive study of the role of this hypothesis for credit risk modeling has been recently made by El Karoui, Jeanblanc and Jiao in [9].

The market dynamics are given by

$$\begin{cases} \mathrm{d}S_t^0 &= r_t S_t^0 \mathrm{d}t, & S_0^0 = 1, \\ \mathrm{d}S_t^1 &= S_{t-}^1 \left(\mu_t^1 \mathrm{d}t + \sigma_t^1 \mathrm{d}W_t + \phi_t^1 \mathrm{d}M_t\right), & S_0^1 = s_0^1, \\ \mathrm{d}S_t^2 &= S_t^2 \left(\mu_t^2 \mathrm{d}t + \sigma_t^2 \mathrm{d}W_t\right), & S_0^2 = s_0^2, \end{cases} \tag{7.1.2}$$

where the interest rate r is assumed to be a nonnegative uniformly bounded $\mathbb{G}$-adapted process, and the coefficients μ^1, $\sigma^1, \phi^1, \mu^2, \sigma^2$ are $\mathbb{G}$-predictable and uniformly bounded processes, with $\sigma_t^1 > 0, \phi_t^1 > -1, \phi_t^1 \neq 0, \sigma_t^2 > 0, t \geq 0$, a.s. and $\sigma^2 \neq \sigma^1$ a.s.

In the assets' dynamics (7.1.2), M represents the compensated martingale associated with H. In this setting, it is known (see, e.g., Proposition 4.4 in El Karoui, Jeanblanc and Jiao [9]) that the process M, defined as

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda_s \upsilon(s) ds = H_t - \int_0^t \bar{\lambda}_s \upsilon(s) ds, \quad t \geq 0, \tag{7.1.3}$$

is a $(\mathbb{P}, \mathbb{G})$-martingale, where

$$\bar{\lambda}_t := \mathbb{1}_{\{t<\tau\}} \frac{p_t(t)}{G_t} = \mathbb{1}_{\{t<\tau\}} \lambda_t, \quad t \geq 0,$$

is the $\mathbb{G}$-adapted intensity of τ.
As previously in Chapter 3, the compensator of H is absolutely continuous with respect to the Lebesgue measure, meaning that τ is a $\mathbb{G}-$totally inaccessible stopping time.

At this stage, without the $(\mathcal{H})$-Hypothesis between $\mathbb{F}$ and $\mathbb{G}$ (recall Assumption 3.1.1 a)), we do not know, in all generality, that the $\mathbb{F}$-martingale W is a $\mathbb{G}$ semi-martingale. Nevertheless, because of the $(\mathcal{E})$-Hypothesis, Jacod's criterion ensures that the continuous $\mathbb{F}$-martingale W is a semi-martingale in the initially enlarged filtration $\mathbb{G}^\tau := \mathbb{F} \vee \sigma(\tau)$ (for all the details we refer to the following Part V and to the lecture notes [12]). Furthermore, W is $\mathbb{G}$-adapted and so, from Stricker's Theorem in [24], it is a $\mathbb{G}$ semi-martingale. The explicit canonical decomposition of W as a $\mathbb{G}$ semi-martingale is given in terms of the density p as follows (see, e.g., Part V or Section 2.5 in the lecture notes [12])

$$W_t = W_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle W, G\rangle_s}{G_{s-}} + \int_{t\wedge\tau}^t \frac{d\langle W, p_.(\theta)\rangle_s}{p_{s-}(\theta)}\Big|_{\theta=\tau} =: W_t^{\mathbb{G}} + A_t, \tag{7.1.4}$$

where $W^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-Brownian motion and A is a $\mathbb{G}$-adapted finite variation process. Moreover, it can be shown (see, e.g., Proposition 2.5.1 in [12]) that here the process A admits a representation in the form $A_t = \int_0^t a_s ds$. Indeed (recall what was done in Section 3.2.4), $p(\theta)$ is, for any $\theta \geq 0$, a $(\mathbb{P}, \mathbb{F})$-martingale, that admits the (predictable) representation

$$d_t p_t(\theta) = p_t(\theta) q_t(\theta) dW_t, \quad p_0(\theta) = 1,$$

for some family of $\mathbb{F}$-predictable integrable processes $q(\theta), \theta \geq 0$, so that the survival process $G_t = \mathbb{P}(\tau > t | \mathcal{F}_t^W) = \int_t^{+\infty} p_t(u) \upsilon(u) du$ satisfies

$$dG_t = \left(\int_t^{+\infty} p_t(\theta) q_t(\theta) \upsilon(\theta) d\theta\right) dW_t - p_t(t) \upsilon(t) dt, \quad G_0 = 1$$

and the predictable brackets in the above decomposition (7.1.4) can be explicitly computed. We find

$$d\langle W, p_\cdot(\theta)\rangle_t = p_t(\theta)q_t(\theta)dt \quad \text{and} \quad d\langle W, G\rangle_t = \left(\int_t^{+\infty} p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\right)dt,$$

so that the $\mathbb{G}$-canonical decomposition of W is (notice that G and $p(\theta), \theta \geq 0$, are continuous)

$$W_t = W_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{1}{G_s}\int_s^{\infty} p_s(\theta)q_s(\theta)\upsilon(\theta)d\theta\ ds + \int_{t\wedge\tau}^{t} q_s(\tau)ds = W_t^{\mathbb{G}} + A_t. \tag{7.1.5}$$

It is, then, clear that the $\mathbb{G}$-adapted process a in $A = \int_0^{\cdot} a_s ds$ can be explicitly written, for $t \geq 0$, as

$$a_t = 1\!\!1_{\{t<\tau\}}\frac{1}{G_t}\int_t^{\infty} p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta + 1\!\!1_{\{\tau\geq t\}}q_t(\tau) =: 1\!\!1_{\{t<\tau\}}\widetilde{a}_t + 1\!\!1_{\{\tau\geq t\}}a_t(\tau). \tag{7.1.6}$$

The market structure is, then, equivalently given by

$$\begin{cases} \mathrm{d}S_t^0 &= r_t S_t^0 \mathrm{d}t, & S_0^0 = 1, \\ \mathrm{d}S_t^1 &= S_{t-}^1\left((\mu_t^1 + a_t)\mathrm{d}t + \sigma_t^1 \mathrm{d}W_t^{\mathbb{G}} + \phi_t^1 \mathrm{d}M_t\right), & S_0^1 = s_0^1, \\ \mathrm{d}S_t^2 &= S_t^2\left((\mu_t^2 + a_t)\mathrm{d}t + \sigma_t^2 \mathrm{d}W_t^{\mathbb{G}}\right), & S_0^2 = s_0^2, \end{cases} \tag{7.1.7}$$

where $W^{\mathbb{G}}$ and M are $(\mathbb{P}, \mathbb{G})$-martingales. As previously done in Chapter 3, recalling that any $\mathbb{G}$-predictable process Y can be written in the form

$$Y_t(\omega) = \widetilde{y}_t(\omega)1\!\!1_{\{t\leq\tau(\omega)\}} + y_t(\omega, \tau(\omega))1\!\!1_{\{t>\tau(\omega)\}}, \quad t \geq 0,$$

where $\widetilde{y}$ is $\mathbb{F}$-predictable and where the function $(t, \omega, u) \to y_t(\omega, u)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, we distinguish between the values of the coefficients before and after τ, as shown in the following table (that is different from the corresponding one in Section 3.1, due to the $\mathbb{G}$-predictability, here, of the processes μ^2 and σ^2).

	r	μ^1	σ^1	ϕ^1	μ^2	σ^2
$\{t \leq \tau\}$	$\widetilde{r}_t$	$\widetilde{\mu}_t^1$	$\widetilde{\sigma}_t^1$	ϕ_t^1	$\widetilde{\mu}_t^2$	$\widetilde{\sigma}_t^2$
$\{t > \tau\}$	$r_t(\tau)$	$\mu_t^1(\tau)$	$\sigma_t^1(\tau)$	×	$\mu_t^2(\tau)$	$\sigma_t^2(\tau)$

Furthermore, we make the following assumption, that will be necessary in order to avoid arbitrage opportunities, as we will see later on in Section 7.1.1. Notice that on the set $\{t < \tau\}$ we have, by hypothesis, $\upsilon(t) > 0$.

Assumption 7.1.2. *The model coefficients satisfy*

$$\begin{cases} \dfrac{\widetilde{\sigma}_t^2(\widetilde{r}_t - \widetilde{\mu}_t^1 - \widetilde{a}_t) - \widetilde{\sigma}_t^1(\widetilde{r}_t - \widetilde{\mu}_t^2 - \widetilde{a}_t)}{\phi_t^1 \sigma_t^2 \lambda_t \upsilon(t)} > -1 & a.s., \quad t \leq \tau; \\ \\ \dfrac{r_t(\tau) - \mu_t^1(\tau) - a_t(\tau)}{\sigma_t^1(\tau)} = \dfrac{r_t(\tau) - \mu_t^2(\tau) - a_t(\tau)}{\sigma_t^2(\tau)} & a.s., \quad t > \tau. \end{cases} \tag{7.1.8}$$

Introducing the investment-consumption strategy (π, c) as in Chapter 3, the investor's wealth dynamics (starting with an initial wealth $x_0 \geq 0$) is given here by the following stochastic differential equation (the analog to Equation (3.1.6))

$$\begin{aligned} \mathrm{d}X_t &= \left[r_t X_t + \pi_t^1 X_t \left(\mu_t^1 + a_t - r_t - \phi_t^1 \bar{\lambda}_t \upsilon(t)\right) + \pi_t^2 X_t(\mu_t^2 + a_t - r_t) - c_t\right] \mathrm{d}t + \\ & \quad \pi_t^1 \phi_t^1 X_{t-} \mathrm{d}H_t + \left[\pi_t^1 \sigma_t^1 X_t + \pi_t^2 \sigma_t^2 X_t\right] \mathrm{d}W_t^{\mathbb{G}}, \quad X_0 = x_0. \end{aligned} \tag{7.1.9}$$

The Definition 3.1.1 of admissible investment-consumption strategy, the statement of problems **A**, **B** and **C** (recall Equation (3.1.7), together with Equations (3.1.8), (3.1.9) and (3.1.10)) and the utility function's properties are the same as in Chapter 3.
We now pass to the characterization of the unique EMM $\mathbb{Q}^*$.

7.1.1 The unique EMM $\mathbb{Q}^*$

From the predictable representation theorem in the case of filtration $\mathbb{G}$ (see, e.g., Part V) if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures, we know that there exist two $\mathbb{G}-$predictable processes ψ and γ, with $\gamma > -1$ a.s., such that the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ (that is a strictly positive $(\mathbb{P}, \mathbb{G})$-martingale) admits the representation

$$Z_t := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}_{|\mathcal{G}_t} = 1 + \int_{]0,t]} Z_{u-} \left(\psi_u \mathrm{d}W_u^{\mathbb{G}} + \gamma_u \mathrm{d}M_u\right), \quad t \geq 0.$$

In this case,

$$\bar{W}_t := W_t^{\mathbb{G}} - \int_0^t \psi_s ds, \quad t \geq 0,$$

is a $(\mathbb{Q}, \mathbb{G})$-Brownian motion and the process

$$\bar{M}_t := M_t - \int_{]0,t]} \gamma_s \bar{\lambda}_s \upsilon(s) ds = H_t - \int_{]0,t]} (1 + \gamma_s) \bar{\lambda}_s \upsilon(s) ds$$

is a $(\mathbb{Q}, \mathbb{G})$-martingale, orthogonal to $\bar{W}$.

Here we show that, under Assumption 7.1.2, the market is complete. In fact, by imposing the (local) martingale property to the discounted value processes of S^1 and S^2, under $\mathbb{Q}^*$, we find that ψ^* and γ^* in the Radon-Nikodým density Z^* (provided that this process is a true martingale, as in our case, given the uniform boundedness of the model coefficients) have to satisfy the following two conditions, for $t \geq 0$, in order to have the existence of at least one EMM,

$$\begin{cases} \mu_t^1 + a_t - r_t + \sigma_t^1 \psi_t^* + \phi_t^1 \gamma_t^* \bar{\lambda}_t \upsilon(t) = 0 \\ \\ \mu_t^2 + a_t - r_t + \sigma_t^2 \psi_t^* = 0. \end{cases} \tag{7.1.10}$$

By distinguishing between values before and after the shock, we find that there exists at least one EMM $\mathbb{Q}^*$ if (recall that ψ^* and γ^* are, by definition, $\mathbb{G}$-predictable)

$$\psi_t^* = \begin{cases} \widetilde{\psi}_t^* = \dfrac{\widetilde{r}_t - \widetilde{\mu}_t^2 - \widetilde{a}_t}{\widetilde{\sigma}_t^2}, & \text{a.s.}, \quad t \le \tau; \\ \psi_t^*(\tau) = \dfrac{r_t(\tau) - \mu_t^1(\tau) - a_t(\tau)}{\sigma_t^1(\tau)} = \dfrac{r_t(\tau) - \mu_t^2(\tau) - a_t(\tau)}{\sigma_t^2(\tau)}, & \text{a.s.}, \quad t > \tau; \end{cases}$$

$$\gamma_t^* = \begin{cases} \dfrac{\widetilde{\sigma}_t^2(\widetilde{r}_t - \widetilde{\mu}_t^1 - \widetilde{a}_t) - \widetilde{\sigma}_t^1(\widetilde{r}_t - \widetilde{\mu}^2 - \widetilde{a}_t)}{\widetilde{\sigma}_t^2 \phi_t^1 \lambda_t \upsilon(t)} > -1, & \text{a.s.}, \quad t \le \tau; \\ \\ \text{any predictable } \gamma_t^* > -1, & \text{a.s.}, \quad t > \tau. \end{cases}$$

Given Assumption 7.1.2, such an EMM exists and the market is arbitrage free. Furthermore, the processes ψ^* and γ^* are uniquely determined, so that the market is complete. The Radon-Nikodým density Z^* is unique and it is given, for every $t \ge 0$, by

$$Z_t^* = e^{\int_0^t \psi_s^* dW_s^{\mathbb{G}} - \frac{1}{2}\int_0^t (\psi_s^*)^2 ds} \, e^{-\int_0^t \gamma_s^* \bar{\lambda}_s \upsilon(s) \mathrm{d}s} \, (1 + \gamma_\tau^*)^{H_t} \quad \text{a.s.} \tag{7.1.11}$$

Furthermore, the wealth dynamics under $\mathbb{Q}^*$ is given by

$$\mathrm{d}X_t = (r_t X_t - c_t)\,\mathrm{d}t + \pi_t^1 \phi_t^1 X_{t-} \mathrm{d}M_t^* + X_t \left(\pi_t^1 \sigma_t^1 + \pi_t^2 \sigma_t^2\right) \mathrm{d}W_t^*, \quad X_0 = x_0, \tag{7.1.12}$$

where

$$W_t^* := W_t^{\mathbb{G}} - \int_0^t \psi_s^* ds, \qquad M_t^* := M_t - \int_{]0,t]} \gamma_s^* \bar{\lambda}_s \upsilon(s) ds.$$

7.2 The solution to problem A: the log-utility case

Since the results presented in Section 3.1.2, that concern the budget constraint, are valid here too, we focus now on the optimal solution to problem **A**. The optimal consumption process c^* is given as in Proposition 3.2.1, namely

$$c_s^{*,A} = I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) \quad \text{a.s.},$$

where I denotes the inverse function of u', $\nu > 0$ is a real parameter satisfying the budget constraint

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^T e^{-\int_0^s r_u du} I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) \mathrm{d}s\right) = x_0$$

and where Z^* is here the Radon-Nikodým density process introduced in Equation (7.1.11).

As previously in Chapter 3, in this more general setting, too, we provide an explicit optimal solution (π^*, c^*) only in the logarithmic utility case.

$\triangleright$ **The optimal consumption**

As in Chapter 3 (the difference here is in the Radon-Nikodým density process) we easily find $\nu = T/x_0$, so that, for every $0 \le s \le T$,

$$c_s^{*,A} = \frac{1}{\nu e^{-\int_0^s r_u du} Z_s^*} = \frac{x_0}{T e^{-\int_0^s r_u du} Z_s^*} \quad \text{a.s.} \tag{7.2.1}$$

▷ **The optimal investment strategy**
A direct computation, applying the conditional version of Fubini-Tonelli's theorem and recalling that $(Z^*)^{-1}$ is a $(\mathbb{Q}^*,\mathbb{G})-$martingale, gives us

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_t^T e^{-\int_0^s r_u du} c_s^* \,\mathrm{d}s \mid \mathcal{G}_t\right) \quad = \quad \frac{x_0}{T}\int_t^T \mathbb{E}^{\mathbb{Q}^*}\left[(Z_s^*)^{-1}|\mathcal{G}_t\right]\mathrm{d}s = \frac{x_0(T-t)}{T\ Z_t^*} \quad \text{a.s.},$$

so that the optimal wealth is (notice that $X_T^*=0$)

$$X_t^* = e^{\int_0^t r_s ds}\ \frac{x_0(T-t)}{T\ Z_t^*}, \quad \text{a.s.}, \quad t\le T.$$

In order to obtain the stochastic differential of X^*, we we first compute

$$d\left(\frac{1}{Z_t^*}\right) = \frac{1}{Z_{t-}^*}\left[-\psi_t^* dW_t^* - \frac{\gamma_t^*}{1+\gamma_t^*}dM_t^*\right], \tag{7.2.2}$$

so that, exactly as in Section 3.2.2,

$$\mathrm{d}X_t^* = X_{t-}^*\left[\left(r_t - \frac{1}{T-t}\right)\mathrm{d}t - \psi_t^*\mathrm{d}W_t^* - \frac{\gamma_t^*}{1+\gamma_t^*}dM_t^*\right], \quad X_0^* = x_0. \tag{7.2.3}$$

Comparing the coefficients with the ones in Equation (7.1.12) we finally find the following optimal investment strategy, that is more general than the one in Equation (3.2.8), due to the $\mathbb{G}$-predictability of σ^2,

$$\begin{cases} \pi_t^{1,*} = -\dfrac{\gamma_t^*}{\phi_t^1(1+\gamma_t^*)}, \quad \pi_t^{2,*} = -\dfrac{\widetilde{\psi}_t^*}{\widetilde{\sigma}_t^2} + \dfrac{\gamma_t^*\widetilde{\sigma}_t^1}{\widetilde{\sigma}_t^2\phi_t^1(1+\gamma_t^*)} & \text{a.s.}, \quad t\le\tau; \\ \pi_t^{1,*}\sigma_t^1(\tau) + \pi_t^{2,*}\sigma_t^2(\tau) = -\psi_t^*(\tau) \quad \text{a.s.}, & \qquad\quad t>\tau. \end{cases}$$

7.3 The solution to problem B: the log-utility case

All the general results presented in Section 3.2.3 are still valid here, so that the optimal consumption process c^* is given as in Proposition 3.2.2, namely

$$c_s^{*,B} = I\left(\nu e^{-\int_0^s r_u du} Z_s^*\right) \quad \text{a.s.}, \quad s\le (T\wedge\tau),$$

where $\nu>0$ is a real parameter satisfying the budget constraint

$$\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^{T\wedge\tau} e^{-\int_0^s r_u du} I\left(e^{-\int_0^s r_u du}\nu Z_s^*\right)\mathrm{d}s\right) = x_0.$$

As done in the previous section, we provide an explicit optimal solution (π^*, c^*) in the logarithmic utility case.
▷ **The optimal consumption**
As in Section 3.2.3, ν is found to be equal to $\nu = \mathbb{E}(T\wedge\tau)/x_0$, so that

$$c_s^{*,B} = \frac{x_0}{\mathbb{E}(T\wedge\tau) Z_s^* e^{-\int_0^s r_u du}} \quad \text{a.s.}, \quad s\le (T\wedge\tau). \tag{7.3.1}$$

▷ **The optimal investment strategy**

The optimal wealth, exactly as in Section 3.2.4, is obtained by performing explicit computations. Indeed, passing under $\mathbb{P}$, applying Fubini-Tonelli's theorem and the "key-Lemma" 3.2.2, we find (notice that, in particular, $X^*_{T\wedge\tau} = 0$ a.s.)

$$\begin{aligned} X^*_t &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T\wedge\tau)} \frac{1}{Z^*_t} \mathbb{E}\left(Z^*_t \int_t^T \mathbb{1}_{\{s<\tau\}} \frac{1}{Z^*_s} \,\mathrm{d}s \mid \mathcal{G}_t \right) \\ &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T\wedge\tau)} \frac{1}{Z^*_t} \int_t^T \mathbb{E}\left(\mathbb{1}_{\{s<\tau\}} | \mathcal{G}_t \right) \mathrm{d}s \\ &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T\wedge\tau)} \frac{(1-H_t)}{Z^*_t\, G_t} \int_t^T \mathbb{P}\left(\tau > s | \mathcal{F}^W_t \right) \mathrm{d}s \\ &= e^{\int_0^t r_s ds} \frac{x_0}{\mathbb{E}(T\wedge\tau)} \frac{(1-H_t)}{Z^*_t\, G_t} \int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)\upsilon(\theta)d\theta \qquad \text{a.s.,} \end{aligned}$$

where in the last equality we used Equation (7.1.1). In order to differentiate X^*, to obtain the optimal investment strategy π^*, we apply the Itô-Kunita-Ventzell formula, given in Theorem 3.2.1.

Noticing that the strictly positive $(\mathbb{P}, \mathbb{F})$-martingale $p(\theta)$ admits the (predictable) representation

$$p_t(\theta) = 1 + \int_0^t p_u(\theta) q_u(\theta) dW_u, \quad t \geq 0,$$

with $q(\theta)$ an integrable $\mathbb{F}$-predictable process, for every $\theta \geq 0$, we find (for all the details we refer to Section 3.2.4)

$$\begin{aligned} d\left(\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)\upsilon(\theta)d\theta \right) &= \left(\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta \right) dW_t - \left(\int_t^\infty p_t(\theta)\upsilon(\theta)d\theta \right) dt \\ &= \left(\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta \right) dW_t - G_t dt \\ &= \left(\int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta \right) \left(dW^{\mathbb{G}}_t + a_t dt \right) - G_t dt \end{aligned}$$

and, analogously,

$$dG_t = \left(\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta \right) \left(dW^{\mathbb{G}}_t + a_t dt \right) - p_t(t)\upsilon(t)dt,$$

where we have used the canonical decomposition of W in $\mathbb{G}$, i.e., Equation (7.1.4). We now finally compute the differential of X^*, that we re-write below using Equation (7.1.11),

$$X^*_t = \frac{x_0}{\mathbb{E}(T\wedge\tau)} e^{\int_0^t \left[r_s + \gamma^*_s \bar{\lambda}_s \upsilon(s) + \frac{1}{2}(\psi^*_s)^2 \right] ds - \int_0^t \psi^*_s dW^{\mathbb{G}}_s} \frac{(1-H_t)}{(1+\gamma^*_\tau)^{H_t}} \frac{1}{G_t} \int_t^T \mathrm{d}s \int_s^\infty p_t(\theta)\upsilon(\theta)d\theta,$$

namely (notice that the jump factor $(1+\gamma^*_\tau)^{H_t}$ equals one on the set $\{t < \tau\}$, where $H_t = 0$, so that in practice it does not affect the above equation)

$$
\begin{aligned}
dX_t^* &= X_{t-}^*\left[(r_t+\gamma_t^*\bar{\lambda}_t\upsilon(t)+(\psi_t^*)^2)dt-\psi_t^* dW_t^{\mathbb{G}}-dH_t\right]\\
&+ \frac{X_t^*}{G_t}\left[-\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\left(dW_t^{\mathbb{G}}+a_t dt\right)+p_t(t)\upsilon(t)dt+\frac{1}{G_t}\left(\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\right)^2 dt\right]\\
&+ \frac{X_t^*}{F_t(t)}\left[\int_t^T \mathrm{d}s\int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta\left(dW_t^{\mathbb{G}}+a_t dt\right)-G_t dt\right],
\end{aligned}
$$

where $F_t(t):=\int_t^T \mathrm{d}s\int_s^\infty p_t(\theta)\upsilon(\theta)d\theta$. By identification with Equation (7.1.9), namely,

$$
\mathrm{d}X_t=[\dots]dt+\pi_t^1\phi_t^1X_{t-}\mathrm{d}H_t+\left[\pi_t^1\sigma_t^1X_t+\pi_t^2\sigma_t^2X_t\right]\mathrm{d}W_t^{\mathbb{G}},\quad X_0=x_0,
$$

we find the optimal investment strategies $\pi_t^{1,*}$ and $\pi_t^{2,*}$ as solutions to the following system of equations

$$
\begin{cases}
\pi_t^{1,*} = -\frac{1}{\phi_t^1} & \text{a.s.},\quad t\le\tau;\\
\pi_t^{1,*}\sigma_t^1+\pi_t^{2,*}\sigma_t^2 = -\psi_t^*-\dfrac{1}{G_t}\displaystyle\int_t^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta & \\
\qquad +\dfrac{1}{F_t(t)}\displaystyle\int_t^T\mathrm{d}s\int_s^\infty p_t(\theta)q_t(\theta)\upsilon(\theta)d\theta & \text{a.s.},\quad 0\le t\le T.
\end{cases}
\tag{7.3.2}
$$

7.4 The solution to problem C: the log-utility case

All the general results presented in Section 3.2.5 are still valid here, so that the optimal consumption process c^* is given as in Proposition 3.2.4, namely

$$
c_s^{*,C}=I\left(\nu e^{\rho s}e^{-\int_0^s r_u du}Z_s^*\right)\quad \text{a.s.},\quad s\ge 0,
$$

where $\nu>0$ is a real parameter satisfying the budget constraint

$$
\mathbb{E}^{\mathbb{Q}^*}\left(\int_0^\infty e^{-\int_0^s r_u du}I\left(\nu e^{\rho s}e^{-\int_0^s r_u du}Z_s^*\right)\mathrm{d}s\right)=x_0.
$$

As previously done, we provide an explicit optimal solution in the log-utility case.
▷ **The optimal consumption**
Here, as in Section 3.2.5,

$$
c_s^{*,C}=\frac{1}{\nu Z_s^* e^{\rho s}e^{-\int_0^s r_u du}}\quad \text{a.s.},\quad s\ge 0
$$

and ν satisfies

$$
\mathbb{E}\left(\int_0^{+\infty}\frac{e^{-\rho s}}{\nu}\mathrm{d}s\right)=x_0.
$$

In this case, then, the optimal consumption is, indeed, well defined (recall Assumption 3.2.4), since the above integral is finite, given that ρ is, by definition, positive. We then find that $\nu=\frac{1}{\rho x_0}$ and finally we have

$$
c_s^{*,C}=\frac{\rho x_0}{Z_s^* e^{\rho s}e^{-\int_0^s r_u du}}\quad \text{a.s.},\quad s\ge 0.
$$

▷ **The optimal investment strategy**
A direct computation, applying the conditional version of Fubini-Tonelli's theorem and recalling that $(Z^*)^{-1}$ is a $(\mathbb{G},\mathbb{Q}^*)-$martingale, gives us

$$\begin{aligned} X_t^* &= e^{\int_0^t r_s ds}\ \mathbb{E}^{\mathbb{Q}^*}\left(\int_t^{+\infty} e^{-\int_0^s r_u du} c_s^*\ \mathrm{d}s \mid \mathcal{G}_t\right) \\ &= e^{\int_0^t r_s ds}\rho x_0 \int_t^{+\infty} e^{-\rho s}\ \mathbb{E}^{\mathbb{Q}^*}\left(\frac{1}{Z_s^*}\big|\mathcal{G}_t\right)\mathrm{d}s \\ &= x_0\ e^{-\rho t}\ e^{\int_0^t r_s ds}\ \frac{1}{Z_t^*} \qquad \text{a.s.} \end{aligned}$$

Equivalently, in differential form, recalling Equation (7.2.2), we find

$$\mathrm{d}X_t^* = X_{t-}^*\left[(r_t-\rho)\,\mathrm{d}t - \psi_t^*\mathrm{d}W_t^* - \frac{\gamma_t^*}{1+\gamma_t^*}dM_t^*\right], \quad X_0^* = x_0. \tag{7.4.1}$$

To determine $\pi^{1,*}$ and $\pi^{2,*}$ it suffices to identify, term by term, the above equation and Equation (7.1.12), that is,

$$\mathrm{d}X_t = (r_tX_t - c_t)\,\mathrm{d}t + \pi_t^1\phi_t^1X_{t-}\mathrm{d}M_t^* + X_t\left(\pi_t^1\sigma_t^1 + \pi_t^2\sigma_t^2\right)\mathrm{d}W_t^*, \quad X_0 = x_0.$$

We finally have

$$\begin{cases} \pi_t^{1,*} = -\dfrac{\gamma_t^*}{\phi_t^1(1+\gamma_t^*)}, \quad \pi_t^{2,*} = -\dfrac{\widetilde{\psi}_t^*}{\widetilde{\sigma}_t^2} + \dfrac{\gamma_t^*\widetilde{\sigma}_t^1}{\widetilde{\sigma}_t^2\phi_t^1(1+\gamma_t^*)} & \text{a.s.}, \quad t\le\tau; \\ \pi_t^{1,*}\sigma_t^1(\tau) + \pi_t^{2,*}\sigma_t^2(\tau) = -\psi_t^*(\tau) \quad \text{a.s.}, & \qquad t>\tau, \end{cases}$$

that is an optimal investment strategy more general than the one in Equation (3.2.25), because of the $\mathbb{G}$-predictability of σ^2.

Bibliography

[1] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. PDE approach to valuation and hedging of credit derivatives. *Quantitative Finance*, 5(3):257–270, 2005.

[2] C. Blanchet-Scalliet, N. El Karoui, M. Jeanblanc, and L. Martellini. Optimal investment decisions when time-horizon is uncertain. *Journal of Mathematical Economics*, 44(11):1100–1113, 2008.

[3] C. Blanchet-Scalliet and M. Jeanblanc. Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics*, (8):145–159, 2004.

[4] B. Bouchard and H. Pham. Wealth-Path Dependent Utility Maximization in Incomplete Markets. *Finance and Stochastics*, 8(4):579–603, 2004.

[5] M. Chesney, M. Jeanblanc, and M Yor. *Mathematical Methods for Financial Markets.* Springer, 2009.

[6] R.-A. Dana and M. Jeanblanc. *Financial Markets in Continuous Time (Corrected Second Printing).* Springer-Finance, 2007.

[7] C. Dellacherie. Un exemple de la théorie générale des processus. In P.A. Meyer, editor, *Séminaire de Probabilités IV*, volume 124 of *Lecture Notes in Mathematics.* Springer, 1970.

[8] C. Dellacherie and P.A. Meyer. *Probabilités et Potentiel - Chapitres I à IV.* Hermann, Paris, 1975.

[9] N. El Karoui, M. Jeanblanc, and Y. Jiao. What happens after a default: the conditional density approach. *Stochastic Processes and their Applications*, 120(7):1011–1032, 2010.

[10] W.H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions.* Number 25 in Stochastic modelling and applied probability. Springer, 2nd edition, 2006.

[11] D. Gilbarg and N. Trudinger. *Elliptic Differential Equations of Second Order.* Classics in Mathematics (originally published as Vol. 224 of the Grundlehren der Mathematischen Wissenschaften). Springer, 3rd edition, 2001.

[12] M. Jeanblanc. *Enlargement of filtrations.* Lecture notes, 2010.

[13] M. Jeanblanc and M. Pontier. Optimal portfolio for a small investor in a market model with discontinuous prices. *Applied Mathematics and Optimization*, (22):287–310, 1990.

[14] I. Karatzas and S.E. Shreve. *Methods of Mathematical Finance*. Springer-Verlag, New York, 1998.

[15] I. Karatzas and H. Wang. Utility maximization with discretionary stopping. *SIAM Journal on Control & Optimisation*, 39:306–329, 2001.

[16] R. Korn. *Optimal portfolios. Stochastic Models for Optimal Investment and Risk Management in Continuous Time.* World Scientific Publishing Co. Pte. Ltd. Singapore, 1997.

[17] R. Korn and E. Korn. Option pricing and portfolio optimization - Modern methods of financial Mathematics. In *Graduate studies in Mathematics*, volume 31. AMS, 2001.

[18] H. Kunita. *Some extensions of Itô's formula*, volume 850 of *Lecture Notes in Mathematics*, chapter Séminaire de Probabilités XV, pages 118–141. Springer-Verlag, 1981.

[19] S. Kusuoka. A remark on default risk models. *Advances in Mathematical Economics*, 1:69–82, 1999.

[20] F. Menoncin. The role of longevity bonds in optimal portfolios. *Insurance: Mathematics and Economics*, 42:343–358, 2008.

[21] R.C. Merton. *Analytical Optimal Control Theory as Applied to Stochastic and Non-Stochastic Economics.* PhD thesis, MIT, 1970.

[22] B. Øksendal and A. Sulem. *Applied stochastic control of jump diffusions.* Springer, 2005.

[23] H. Pham and Y. Jiao. Optimal investment with counterparty risk: a default-density modeling approach. *To appear in Finance and Stochastics*, 2009.

[24] C. Stricker. Quasi-martingales, martingales locales et filtrations naturelles. *Zeitschrift fur Wahr*, 39:55–63, 1977.

[25] G. Zitković. Utility maximization with a stochastic clock and an unbounded random endowment. *The Annals of Applied Probability*, 15(1B):748–777, 2005.

Part V

Enlargement of filtrations

Chapter 8

“Carthaginian” enlargement of filtrations

This is a joint work with Prof. M. Jeanblanc and B. Zargari.

Abstract: in this work we provide, having a pedagogical aim in mind, an overview of some well-known key results in the theory of initial and progressive enlargement of a reference filtration $\mathbb{F}$ with a random time τ, providing, in a very specific setting, alternative proofs to the already existing ones.

Keywords: initial and progressive enlargement of filtration, predictable projection, canonical decomposition, predictable representation theorem.

8.1 Introduction and preliminaries

Let us consider a pair of filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}_t})_{t\geq 0}$ on the same probability space, such that $\mathcal{F}_t \subset \widetilde{\mathcal{F}_t}$, for any $t \geq 0$. In filtering theory, this structure is suitable to describe the evolution of a stochastic system that is partially observable (as in the previous Parts II and III of this thesis). In enlargement of filtration theory, the point of view is the opposite one (see, e.g., the summary in Jeulin [24]): $\mathbb{F}$ is considered to be a reference filtration, to which we add some information, thus leading us to the larger filtration $\widetilde{\mathbb{F}}$.

Here we only consider the case where the enlargement of filtration $\mathbb{F}$ is done by means of a random variable τ. Nevertheless, there are, of course, many other ways to do that, such as, for example, setting $\widetilde{\mathcal{F}_t} = \mathcal{F}_t \vee \bar{\mathcal{F}}$, $t \geq 0$, where $\bar{\mathcal{F}}$ is a σ-algebra, or defining $\widetilde{\mathcal{F}_t} = \mathcal{F}_t \vee \bar{\mathcal{F}}_t$, $t \geq 0$, where $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$ is another filtration.

There are two ways to add information to $\mathbb{F}$ by means of a random variable τ: either all of a sudden at time 0 (*initial enlargement*), or progressively, by considering the smallest filtration, satisfying the usual conditions, containing $\mathbb{F}$ that makes τ a stopping time (*progressive enlargement*).

The "pioneers" who started exploring this research field were Barlow (in [4]), Jacod, Jeulin and Yor (see the references that follow in the text) at the end of the seventies. The main question that raised was the following: "Does any $\mathbb{F}$-martingale X remain an $\widetilde{\mathbb{F}}$ semi-martingale?". And, in this case: "What is the semi-martingale decomposition in $\widetilde{\mathbb{F}}$ of the $\mathbb{F}$-martingale X?"

Notice that a general (but not so practice) necessary and sufficient condition in order for any $\mathbb{F}$-local martingale to remain a $\widetilde{\mathbb{F}}$ semi-martingale is given in Jeulin [24], page 12. Moreover, very technical existence and regularity results (that we will need in the sequel), which are fundamental in enlargement of filtrations theory, were proved at the very beginning, in the late seventies.

A recent detailed introduction to this subject can be found, e.g., in Chesney, Jeanblanc and Yor [7], in Mansuy and Yor [27] and in Protter [29]. Furthermore, many authors, such as Ankirchner (see, e.g., [3]), Amendinger (e.g., in [2]), Baudoin [5], Corcuera *et al.* [9], Eyraud-Loisel [14], Gasbarra *et al.* [16], Grorud and Pontier (see, e.g., [17]), Hillairet [20], Imkeller [21], Kohatsu-Higa and Øksendal [25] and Wu [34] were recently interested in applying enlargement of filtration theory to insider trading in finance.

The main contribution of this work is to show how, in a very specific setting, some well-known fundamental results can be proved in an alternative (and, in some cases, simpler) way. Nevertheless, it is important to make precise that the goal of this work is neither to present the results in the most general case, nor to study the needed and difficult regularity and existence properties, for which we refer to existing papers.

Let us start, then, by motivating the title, by introducing some notation and by stating the preliminary results that are needed henceforth. Inspired by a visit to the Tunisian archaeological site of Carthage, where one can find remains of THREE levels of different civilizations, we decided to use the catchy adjective "Carthaginian" associated with filtration, since in this chapter there will be THREE levels of filtrations.

We consider, then, three nested filtrations

$$\mathbb{F} \subset \mathbb{G} \subset \mathbb{G}^\tau,$$

where $\mathbb{G}$ and $\mathbb{G}^\tau$ stand, respectively, for the *progressive* and the *initial* enlargement of $\mathbb{F}$

with a finite random time (i.e., a finite non-negative random variable). Under a specific assumption (see the $(\mathcal{E})$-Hypothesis below), we address the following problems:

- Characterization of $\mathbb{G}$-martingales and $\mathbb{G}^\tau$-martingales in terms of $\mathbb{F}$-martingales (in Section 8.2);
- Canonical decomposition of an $\mathbb{F}$-martingale, as a semimartingale, in $\mathbb{G}$ and $\mathbb{G}^\tau$ (in Section 8.3);
- Predictable Representation Theorem in $\mathbb{G}$ and $\mathbb{G}^\tau$ (in Section 8.4).

The exploited idea is the following: assuming that the $\mathbb{F}$-conditional law of τ is equivalent to the law of τ, after an *ad hoc* change of probability measure, the problem is reduced to the case where τ and $\mathbb{F}$ are independent. Working under this newly introduced probability measure, in the initially enlarged filtration, is, then, "easy". Then, under the original probability measure, for the initially enlarged filtration, the results are achieved by means of Girsanov's theorem. Finally, by projection, one obtains the results of interest in the progressively enlarged filtration (notice that, alternatively, they can be obtained with another application of Girsanov's theorem, starting from the newly introduced probability measure, with respect to the progressively enlarged filtration).

The "change of probability measure viewpoint" for treating the problems on enlargement of filtrations was remarked in the early 80's and developed by Song [31] (see also Jacod [22], Section 5). For what concerns the idea of recovering the results in the progressively enlarged filtration starting from the ones in the initially enlarged, we have to cite Yor [35].

Let us now become more precise about the setup and the preliminary results. We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses of right-continuity and completeness and where $\mathcal{F}_0$ is the trivial σ-field. Let τ be a finite random time with law ν, $\nu(du) = \mathbb{P}(\tau \in du)$. In what follows, we assume, moreover, that the probability measure ν has no atoms.

We denote by $\mathcal{P}(\mathbb{F})$ (resp., $\mathcal{O}(\mathbb{F})$) the predictable (resp., optional) σ-algebra corresponding to $\mathbb{F}$ on $\mathbb{R}^+ \times \Omega$ (an accurate characterization of predictable and optional $\sigma-$fields is given, e.g., in Dellacherie and Meyer [10, Ch. IV, Th. 67 and Th. 64]). In the sequel, the natural filtration associated with a process X will be denoted by $\mathbb{F}^X$.

Our standing assumption is the following:

Assumption 8.1.1. $(\mathcal{E})$-***Hypothesis***
The $\mathbb{F}$-(regular) conditional law of τ is equivalent to the law of τ. Namely,

$$\mathbb{P}(\tau \in du|\mathcal{F}_t) \sim \nu(du) \quad \textit{for every } t \geq 0, \;\; \mathbb{P} - a.s.$$

Notice that this assumption, in the case when $t \in [0, T]$, corresponds to the *equivalence assumption* in Amendinger's thesis [1, Assumption 0.2] and to Hypothesis (HJ) in the papers from Grorud and Pontier (see, e.g., [17]).

Amongst the consequences of the $(\mathcal{E})$-Hypothesis, one has the existence and regularity of the conditional density, for which we refer to Amendiger's reformulation (see Remarks, page 17 of [1]) of Jacod's result (Lemma 1.8 in [22]): there exists a strictly positive $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable function $(t, \omega, u) \to p_t(\omega, u)$, such that for every $u \in \mathbb{R}^+$, $p(u)$ is a càdlàg $(\mathbb{P}, \mathbb{F})$-martingale and

$$\mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty p_t(u)\nu(du) \quad \text{for every } t \geq 0, \;\; \mathbb{P} - a.s.$$

In particular, $p_0(u) = 1$ for every $u \in \mathbb{R}^+$. The family $p_t(\cdot)$ is called the $(\mathbb{P}, \mathbb{F})$-*conditional density* of τ with respect to ν, given $\mathcal{F}_t$, or the *density* of τ if there is no ambiguity. Furthermore, under the $(\mathcal{E})$-Hypothesis, the assumption that ν has no atoms implies that the default time τ avoids the $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \xi) = 0$ for every $\mathbb{F}$-stopping time ξ (see, e.g., Corollary 2.2 in El Karoui, Jeanblanc and Jiao [13]).

The initial enlargement of $\mathbb{F}$ with τ, denoted by $\mathbb{G}^\tau = (\mathcal{G}_t^\tau, t \geq 0)$ is defined as $\mathcal{G}_t^\tau = \mathcal{F}_t \vee \sigma(\tau)$. It was shown in Amendinger [1] (cf. Proposition 1.10 therein) that the strict positiveness of p implies the right-continuity of filtration $\mathbb{G}^\tau$.

Let $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ denote the smallest filtration with respect to which τ is a stopping time, i.e., $\mathcal{H}_t = \sigma(\mathbb{1}_{\tau \leq s}, s \leq t)$. This filtration is right-continuous (since the law of τ has no atoms). The progressive enlargement of $\mathbb{F}$ with the random time τ, denoted by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, is defined as the right-continuous regularization of $\mathbb{F} \vee \mathbb{H}$.

In the sequel, we will consider the right-continuous version of all the martingales (recall that if a filtration is right-continuous, any martingale with respect to this filtration has a càdlàg version).

Next, we consider a useful (equivalent) change of probability measure introduced, independently, by Grorud and Pontier in [18] and by Amendinger in [1]. Having verified that the process L, defined by $L_t = \frac{1}{p_t(\tau)}$, $t \geq 0$, is a $(\mathbb{P}, \mathbb{G}^\tau)$-martingale, with $\mathbb{E}(L_t) = L_0 = 1$, these authors defined a locally equivalent probability measure $\mathbb{P}^*$ by setting

$$d\mathbb{P}^*{}_{|\mathcal{G}_t^\tau} = L_t \; d\mathbb{P}_{|\mathcal{G}_t^\tau} = \frac{1}{p_t(\tau)} d\mathbb{P}_{|\mathcal{G}_t^\tau} \; .$$

They proved that, under $\mathbb{P}^*$, the random time τ is independent of $\mathcal{F}_t$ for any $t \geq 0$ and, moreover, that

$$\mathbb{P}^*{}_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t}, \quad \text{for any } t \geq 0, \quad \mathbb{P}^*{}_{|\sigma(\tau)} = \mathbb{P}_{|\sigma(\tau)}.$$

The above properties imply that $\mathbb{P}^*(\tau \in du|\mathcal{F}_t) = \mathbb{P}^*(\tau \in du)$, so that the $(\mathbb{P}^*, \mathbb{F})$-density of τ, denoted by $p^*(u), u \geq 0$, is a constant equal to one, $\mathbb{P}^* \otimes \nu$-a.s.

Remark 8.1.1. *The probability measure $\mathbb{P}^*$, being defined on $\mathcal{F}_t$ for $t \geq 0$, is (uniquely) defined on $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. Then, as τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$, it immediately follows that τ is also independent of $\mathcal{F}_\infty$, under $\mathbb{P}^*$. However, one can not claim that: "$\mathbb{P}^*$ is equivalent to $\mathbb{P}$ on $\mathcal{G}_\infty^\tau$", since we do not know* a priori *whether $\frac{1}{p(\tau)}$ is a* closed $(\mathbb{P}, \mathbb{G}^\tau)$-*martingale or not. A similar problem is studied by Föllmer and Imkeller in [15] (it is therein called "paradox") in the case when the reference (canonical) filtration is enlarged by the information about the endpoint at time $t = 1$.*

Remark 8.1.2. *Let $x = (x_t, t \geq 0)$ be a $(\mathbb{P}, \mathbb{F})$-martingale. Since $\mathbb{P}$ and $\mathbb{P}^*$ coincide on $\mathbb{F}$, x is a $(\mathbb{P}^*, \mathbb{F})$-martingale, too. Hence, using the fact that τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$, x is a $(\mathbb{P}^*, \mathbb{G})$-martingale (and also a $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingale). Because of these facts, the measure $\mathbb{P}^*$ is called by Amendinger* "martingale preserving probability measure under initial enlargement of filtrations".

Notation 8.1.1. *In this paper, as we mentioned, we deal with three different levels of information and two equivalent probability measures. In order to distinguish objects defined under $\mathbb{P}$ and under $\mathbb{P}^*$, we will use a superscript $*$ when working under $\mathbb{P}^*$. For example, $\mathbb{E}$ and $\mathbb{E}^*$ stand for the expectations under $\mathbb{P}$ and $\mathbb{P}^*$, respectively. For what concerns the filtrations, when necessary, we will use the following illustrating notation: x, X, X^τ to denote* *processes adapted to $\mathbb{F}, \mathbb{G}$ and $\mathbb{G}^\tau$, respectively (we shall not use the same notation for processes stopped at τ, so that there is no possible confusion with the notation X^τ). Furthermore, for simplicity in this chapter we drop the double brackets "{" and "}" in the functions $1\!\!1_{t<\tau}$ and $1\!\!1_{\tau\le t}$.*

We conclude this introductive section with the following general result, that will be very useful in the sequel.

Proposition 8.1.1. ***Projection***
Let x be a uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingale and $\widetilde{\mathbb{F}}$ a filtration larger than $\mathbb{F}$, i.e., $\mathbb{F} \subset \widetilde{\mathbb{F}}$. Then there exists a $(\mathbb{P}, \widetilde{\mathbb{F}})$-martingale X such that $\mathbb{E}(X_t|\mathcal{F}_t) = x_t, t \ge 0$.

Proof. First note that in view of the uniform integrability assumption on x, x_∞ exists (it is the L^1 limit of x_t for $t \to \infty$) and, furthermore, $\mathbb{E}(x_\infty|\mathcal{F}_t) = x_t$ (see, e.g., Chesney, Jeanblanc and Yor [7, pag. 22] and [30, Ch. II, Th. 2.10]). Now, the process X, defined by $X_t = \mathbb{E}(x_\infty|\widetilde{\mathcal{F}}_t)$, $t \ge 0$, is an $\widetilde{\mathbb{F}}$-martingale. Moreover, $\mathbb{E}(X_t|\mathcal{F}_t) = \mathbb{E}\big(\mathbb{E}(x_\infty|\widetilde{\mathcal{F}}_t)|\mathcal{F}_t\big) = x_t$. □

Remark 8.1.3. *The uniqueness of such a martingale X is not claimed in the above proposition and it is not true in general.*

8.1.1 Characterization of different measurability properties

Before focusing on the three topics announced from the beginning, we recall some important results on the characterization of $\mathcal{G}_t$ and $\mathcal{G}_t^\tau$-measurable random variables and $\mathbb{G}$ and $\mathbb{G}^\tau$-predictable processes, that will be useful in the sequel. The necessary part of the result below, in the case of predictable processes, is due to Jeulin [24, Lemma 3.13]. See also Yor [35].

Proposition 8.1.2. *One has*

*(i) A random variable Y_t^τ is $\mathcal{G}_t^\tau$-*measurable *if and only if it is of the form $Y_t^\tau(\omega) = y_t(\omega, \tau(\omega))$ for some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable $y_t(\cdot, u)$.*

*(ii) A process Y^τ is $\mathbb{G}^\tau$-*predictable *if and only if it is of the form $Y_t^\tau(\omega) = y_t(\omega, \tau(\omega))$, $t \ge 0$, where $(t, \omega, u) \mapsto y_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function.*

Proof. It is fundamental here that (see, e.g., Jacod [22]): $\mathcal{P}(\mathbb{F} \otimes \mathcal{B}(\mathbb{R}^+)) = \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$.

The proof of part (i) is based on the fact that $\mathcal{G}_t^\tau$-measurable random variables are generated by random variables of the form $X_t(\omega) = x_t(\omega) f\big(\tau(\omega)\big)$, with $x_t \in \mathcal{F}_t$ and f bounded Borel on $\mathbb{R}^+$.

(ii) For the necessity (see [24, Lemma 3.13 a)]), it suffices to notice that processes of the form $X_t := x_t f(\tau)$, $t \ge 0$, where x is $\mathbb{F}$-predictable and f is a bounded Borel function on $\mathbb{R}^+$, generate the $\mathbb{G}^\tau$-predictable σ-field. We then obtain the result applying a monotone class argument (see, e.g., Theorem 21, Ch. I in Dellacherie and Meyer [10]).

Conversely, if $(t, \omega, u) \mapsto y_t(\omega, u)$ is an elementary $\mathcal{P}(\mathcal{F}_t) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function, we have $y_t(\omega, u) = h_t(\omega) f(u)$, where h is $\mathbb{F}$-predictable and f bounded Borel on $\mathbb{R}^+$. It is, then, clear that by substituting u by $\tau(\omega)$, we find $y_t(\omega, \tau(\omega)) = h_t(\omega) f(\tau(\omega))$, $t \ge 0$, that is by definition a predictable process in the enlarged filtration $\mathbb{G}^\tau$. □

For the sake of simplicity in notation, we will often drop the dependence on ω and we will write $Y_t^\tau = y_t(\tau)$.

For what concerns the progressive enlargement setting, the following result is the analog to Proposition 8.1.2. The necessity of part (ii) is already proved in Jeulin [24, Lemma 4.4].

Proposition 8.1.3. *One has*

(i) A random variable Y_t is $\mathcal{G}_t$-measurable if and only if it is of the form $Y_t(\omega) = \widetilde{y}_t(\omega)\mathbb{1}_{t<\tau(\omega)} + \widehat{y}_t(\omega, \tau(\omega))\mathbb{1}_{\tau(\omega)\le t}$ for some $\mathcal{F}_t$-measurable random variable $\widetilde{y}_t$ and some family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variables $\widehat{y}_t(\cdot, u), t \geq u$.

(ii) A process Y is $\mathbb{G}$-predictable if and only if it is of the form $Y_t(\omega) = \widetilde{y}_t(\omega)\mathbb{1}_{t\le\tau(\omega)} + \widehat{y}_t(\omega, \tau(\omega))\mathbb{1}_{\tau(\omega)<t}$, $t \geq 0$, where $\widetilde{y}$ is $\mathbb{F}$-predictable and $(t, \omega, u) \mapsto \widehat{y}_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function.

Proof. For part (i) it suffices to recall that $\mathcal{G}_t$-measurable random variables are generated by random variables of the form $X_t(\omega) = x_t(\omega)f\big(t \wedge \tau(\omega)\big)$, with $x_t \in \mathcal{F}_t$ and f bounded Borel on $\mathbb{R}^+$.

(ii) As previously done in the proof of Proposition 8.1.2, it suffices to notice that $\mathbb{G}$-predictable processes are generated by processes of the form $X_t = x_t\mathbb{1}_{t\le\tau} + \widehat{x}_t f(\tau)\mathbb{1}_{\tau<t}$, $t \geq 0$, where $x, \widehat{x}$ are $\mathbb{F}$-predictable and f is a bounded Borel function, defined on $\mathbb{R}^+$. $\square$

8.1.2 Expectation and projection tools

Lemma 8.1.1. *Let $Y_t^\tau = y_t(\tau)$ be a $\mathcal{G}_t^\tau$-measurable random variable.*
(i) If $y_t(\tau)$ is $\mathbb{P}$-integrable and $y_t(\tau) = 0$ $\mathbb{P}$-a.s., then, for ν-a.e. $u \geq 0$, $y_t(u) = 0$ $\mathbb{P}$-a.s.
(ii) If $y_t(\tau)$ is $\mathbb{P}^*$ and $\mathbb{P}$-integrable and if $y_t(u)$ is $\mathbb{P}$ (or $\mathbb{P}^*$)-integrable for any $u \geq 0$, for $s \leq t$ one has, $\mathbb{P}^*$ and $\mathbb{P}$-a.s.,

$$\begin{aligned}
\mathbb{E}^*\big(y_t(\tau)|\mathcal{G}_s^\tau\big) &= \mathbb{E}^*\big(y_t(u)|\mathcal{F}_s\big)\big|_{u=\tau} = \mathbb{E}\big(y_t(u)|\mathcal{F}_s\big)\big|_{u=\tau} \\
\mathbb{E}\big(y_t(\tau)|\mathcal{G}_s^\tau\big) &= \frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\big|_{u=\tau}.
\end{aligned}$$

Proof. (i) We have, by applying Fubini-Tonelli's Theorem,

$$0 = \mathbb{E}\big(\,|y_t(\tau)|\,\big) = \mathbb{E}\Big(\mathbb{E}\big(\,|y_t(\tau)|\,\big|\mathcal{F}_t\big)\Big) = \mathbb{E}\Big(\int_0^\infty |y_t(u)|\,p_t(u)\nu(du)\big)\Big).$$

Then $\int_0^\infty |y_t(u)|\,p_t(u)\nu(du) = 0$ $\mathbb{P}$-a.s. and, given that $p_t(u)$ is strictly positive for any u and that ν is non atomic, we have that for ν-almost every u, $y_t(u) = 0$ $\mathbb{P}$-a.s.
(ii) The result under $\mathbb{P}^*$ is straightforward for elementary random variables of the form $f(\tau)x_t$, given the independence between τ and $\mathcal{F}_t$, for any $t \geq 0$. It is, then, extended via the Monotone Class Theorem (see Lemma 8.6.1 in the Appendix).
The result under $\mathbb{P}$ is an immediate consequence, since it suffices, by means of (conditional) Bayes' formula, to pass under the measure $\mathbb{P}^*$. Namely, for $s < t$, we have

$$\mathbb{E}\big(y_t(\tau)|\mathcal{G}_s^\tau\big) = \frac{\mathbb{E}^*\big(y_t(\tau)p_t(\tau)|\mathcal{G}_s^\tau\big)}{\mathbb{E}^*\big(p_t(\tau)|\mathcal{G}_s^\tau\big)} = \frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\big|_{u=\tau},$$

where in the last equality we have also used the previous result under $\mathbb{P}^*$ and the fact that $p(\tau)$ is a $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingale. Note that if $y_t(\tau)$ is $\mathbb{P}$-integrable, then $\mathbb{E}(\int_0^\infty |y_t(u)|p_t(u)\nu(du)) = \mathbb{E}(|y_t(\tau)|) < \infty$, which implies that $\mathbb{E}(|y_t(u)|p_t(u)) < \infty$. $\square$

When working with the progressively enlarged filtration $\mathbb{G}$, it is convenient to introduce the notation G (resp., $G(\cdot)$) for the Azéma super-martingale associated with τ under the probability measure $\mathbb{P}$ (resp. $\mathbb{P}^*$). More precisely,

$$G_t \;:=\; \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty p_t(u)\nu(du), \tag{8.1.1}$$

$$G(t) \;:=\; \mathbb{P}^*(\tau > t|\mathcal{F}_t) = \mathbb{P}^*(\tau > t) = \mathbb{P}(\tau > t) = \int_t^\infty \nu(du). \tag{8.1.2}$$

Note, in particular, that $(G_t)_{t\geq 0}$ is an $\mathbb{F}$ super-martingale, whereas $G(\cdot)$ is a (deterministic) continuous and decreasing function. Furthermore, it is clear that, under the $(\mathcal{E})$-Hypothesis, G and $G(\cdot)$ do not vanish.

We will also need the lemma below.

Lemma 8.1.2. *Let $Y_t^\tau = y_t(\tau)$ be a $\mathcal{G}_t^\tau$-measurable, $\mathbb{P}$-integrable random variable. Then, for $s \leq t$,*

$$\mathbb{E}(Y_t^\tau|\mathcal{G}_s) = \mathbb{E}(y_t(\tau)|\mathcal{G}_s) = \widetilde{y}_s \mathbb{1}_{s<\tau} + \widehat{y}_s(\tau)\mathbb{1}_{\tau\leq s},$$

with

$$\begin{aligned}\widetilde{y}_s &= \frac{1}{G_s}\mathbb{E}\Big(\int_s^{+\infty} y_t(u)p_t(u)\nu(du)|\mathcal{F}_s\Big) ,\\ \widehat{y}_s(u) &= \frac{1}{p_s(u)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big) .\end{aligned}$$

Proof. From the above Proposition 8.1.3 it is clear that $\mathbb{E}(y_t(\tau)|\mathcal{G}_s)$ has to be written in the form $\widetilde{y}_s\mathbb{1}_{s<\tau} + \widehat{y}_s(\tau)\mathbb{1}_{\tau\leq s}$. On the set before τ we have, applying Lemma 3.1.2 in Bielecki et al. [6] and using the $(\mathcal{E})$-Hypothesis (see also El Karoui, Jeanblanc and Yiao [13] for analogous computations),

$$\begin{aligned}\mathbb{1}_{s<\tau}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) &= \mathbb{1}_{s<\tau}\frac{\mathbb{E}\left[\mathbb{E}(y_t(\tau)\mathbb{1}_{s<\tau}|\mathcal{F}_t)|\mathcal{F}_s\right]}{G_s} = \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^{+\infty} y_t(u)p_t(u)\nu(du)|\mathcal{F}_s\right)\\ &=: \mathbb{1}_{s<\tau}\widetilde{y}_s.\end{aligned}$$

On the complementary set we have, by applying Lemma 8.1.1,

$$\mathbb{1}_{\tau\leq s}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) = \mathbb{1}_{\tau\leq s}\mathbb{E}\left[\mathbb{E}(y_t(\tau)|\mathcal{G}_s^\tau)|\mathcal{G}_s\right] = \mathbb{1}_{\tau\leq s}\frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau} =: \mathbb{1}_{\tau\leq s}\widehat{y}_s(\tau).$$

$\square$

For $s > t$, we obtain $\mathbb{E}(Y_t^\tau|\mathcal{G}_s) = \frac{1}{G_s}\int_s^\infty y_t(u)p_s(u)\nu(du)\mathbb{1}_{s<\tau} + y_t(\tau)\mathbb{1}_{\tau\leq s}$. Remark, then, that we have obtained the right-continuous version (recall that we work with right-continuous versions of all the martingales) of the $\mathbb{G}$-martingale $(\mathbb{E}(Y_t^\tau|\mathcal{G}_s), s\geq 0)$.

As an application, projecting the martingale L (defined earlier as $L_t = \frac{1}{p_t(\tau)}, t \geq 0$) on $\mathbb{G}$ yields to the corresponding Radon-Nikodým density on $\mathbb{G}$:

$$d\mathbb{P}^*{}_{|\mathcal{G}_t} = \ell_t \; d\mathbb{P}_{|\mathcal{G}_t} \; ,$$

with

$$\begin{aligned}\ell_t &= \mathbb{E}(L_t|\mathcal{G}_t) = 1\!\!1_{t<\tau}\frac{1}{G_t}\int_t^\infty \nu(du) + 1\!\!1_{\tau\le t}\frac{1}{p_t(\tau)} \\ &= 1\!\!1_{t<\tau}\frac{G(t)}{G_t} + 1\!\!1_{\tau\le t}\frac{1}{p_t(\tau)}\ .\end{aligned}$$

Obviously, one has

$$d\mathbb{P}_{|\mathcal{G}^\tau_t} = L^*_t\ d\mathbb{P}^*{}_{|\mathcal{G}^\tau_t},$$

where $L^* = 1/L$ is a $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingale and

$$d\mathbb{P}_{|\mathcal{G}_t} = \ell^*_t\ d\mathbb{P}^*{}_{|\mathcal{G}_t},$$

where $\ell^* = 1/\ell$ is a $(\mathbb{P}^*, \mathbb{G})$-martingale.

We now recall some important facts concerning the compensated martingale of H. We know, from the general theory (see for example El Karoui et al. [13]), that denoting by H the default indicator process $H_t = 1\!\!1_{\tau\le t}, t \ge 0$, the process M defined as

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda_s\ \nu(ds), \quad t \ge 0, \tag{8.1.3}$$

with $\lambda_t = \frac{p_t(t)}{G_t}$ is a $(\mathbb{P}, \mathbb{G})$-martingale and that

$$M^*_t := H_t - \int_0^{t\wedge\tau} \lambda^*(s)\ \nu(ds), \quad t \ge 0, \tag{8.1.4}$$

with $\lambda^*(t) = \frac{1}{G(t)}$, is a $(\mathbb{P}^*, \mathbb{G})$-martingale. Furthermore, being λ^* deterministic, M^* (being $\mathbb{H}$-adapted) is a $(\mathbb{P}^*, \mathbb{H})$-martingale, too.

We conclude this subsection with the following two propositions, concerning the predictable projection (see Theorem 8.6.1 in the Appendix for the definition and existence of the predictable projection), respectively on $\mathbb{F}$ and on $\mathbb{G}$, of a $\mathbb{G}^\tau$-predictable process. The first result is due to Jacod [22, Lemme 1.10].

Proposition 8.1.4. *Let $Y^\tau = y(\tau)$ be a $\mathbb{G}^\tau$-predictable, positive or bounded, process. Then, the predictable projection of Y^τ on $\mathbb{F}$ is given by*

$$^{(p)}(Y^\tau)_t = \int_0^\infty y_t(u) p_{t-}(u)\nu(du)\ .$$

Proof. It is obtained by a monotone class argument and by using the definition of density of τ, writing, for "elementary" processes, $Y^\tau_t = y_t f(\tau)$, with y a bounded $\mathbb{F}$-predictable process and f a bounded Borel function. For this, we refer to the proof of Lemma 1.10 in Jacod [22]. □

Proposition 8.1.5. *Let $Y^\tau = y(\tau)$ be a $\mathbb{G}^\tau$-predictable, positive or bounded, process. Then, the predictable projection of Y^τ on $\mathbb{G}$ is given by*

$$^{(p)}(Y^\tau)_t = 1\!\!1_{t\le\tau}\frac{1}{G_{t-}}\int_t^\infty y_t(u)p_{t-}(u)\nu(du) + 1\!\!1_{\tau<t}y_t(\tau)\ .$$

Proof. In this proof, for clarity, the left-hand side superscript "$(p_{\mathbb{G}})$" denotes the predictable projection on $\mathbb{G}$, while the left-hand side superscript "$(p_{\mathbb{F}})$" indicates the predictable projection on $\mathbb{F}$. By the definition of predictable projection, we know (from Proposition 8.1.3 (ii)) that we are looking for a (unique) process of the form

$$^{(p_{\mathbb{G}})}(Y^\tau)_t = \widetilde{y}_t \mathbb{1}_{t\le\tau} + \widehat{y}_t(\tau)\mathbb{1}_{\tau<t}, \quad t \ge 0,$$

where $\widetilde{y}$ is $\mathbb{F}$-predictable, positive or bounded, and $(t,\omega,u) \mapsto \widehat{y}_t(\omega,u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable positive or bounded function, to be identified.

- On the predictable set $\{\tau < t\}$, being Y^τ a $\mathbb{G}^\tau$-predictable, positive or bounded, process (recall Proposition 8.1.2 (ii)), we immediately find $\widehat{y}(\tau) = y(\tau)$;
- On the complementary set $\{t \le \tau\}$, introducing the $\mathbb{G}$-predictable process

  $$Y :=^{(p_{\mathbb{G}})} Y^\tau$$

  it is possible to use Remark 4.5, page 64 of Jeulin [24] (see also Dellacherie and Meyer [12], Ch. XX, page 186), to write

  $$Y\mathbb{1}_{]\!]0,\tau]\!]} = \frac{1}{G_-}\ ^{(p_{\mathbb{F}})}\big(Y\mathbb{1}_{]\!]0,\tau]\!]}\big)\mathbb{1}_{]\!]0,\tau]\!]} = \frac{1}{G_-}\ ^{(p_{\mathbb{F}})}\Big(^{(p_{\mathbb{G}})}(Y^\tau)\mathbb{1}_{]\!]0,\tau]\!]}\Big)\mathbb{1}_{]\!]0,\tau]\!]}.$$

  We then have, being $\mathbb{1}_{]\!]0,\tau]\!]}$, by definition, $\mathbb{G}$-predictable (recall that τ is a $\mathbb{G}$-stopping time),

  $$Y\mathbb{1}_{]\!]0,\tau]\!]} = \frac{1}{G_-}\ ^{(p_{\mathbb{F}})}\Big(^{(p_{\mathbb{G}})}(Y^\tau\mathbb{1}_{]\!]0,\tau]\!]})\Big)\mathbb{1}_{]\!]0,\tau]\!]} = \frac{1}{G_-}\ ^{(p_{\mathbb{F}})}\big(Y^\tau\mathbb{1}_{]\!]0,\tau]\!]}\big)\mathbb{1}_{]\!]0,\tau]\!]},$$

  where the last equality follows by the definition of predictable projection, being $\mathbb{F} \subset \mathbb{G}$. Finally, given the result in Proposition 8.1.4 we have

  $$^{(p_{\mathbb{F}})}\big(Y^\tau\mathbb{1}_{]\!]0,\tau]\!]}\big)_t = \int_t^{+\infty} y_t(u)p_{t-}(u)\nu(du)$$

  and the proposition is proved.

□

8.2 Martingales' characterization

The aim of this section is to characterize $(\mathbb{P},\mathbb{G}^\tau)$ and $(\mathbb{P},\mathbb{G})$-martingales in terms of $(\mathbb{P},\mathbb{F})$-martingales.

Proposition 8.2.1. ***Characterization of*** $(\mathbb{P},\mathbb{G}^\tau)$***-martingales in terms of*** $(\mathbb{P},\mathbb{F})$***-martingales***
A process $Y^\tau = y(\tau)$ *is a* $(\mathbb{P},\mathbb{G}^\tau)$*-martingale if and only if* $(y_t(u)p_t(u), t \ge 0)$ *is a* $(\mathbb{P},\mathbb{F})$*-martingale, for* ν*-almost every* $u \ge 0$.

Proof. The sufficiency is a direct consequence of Proposition 8.1.2 and Lemma 8.1.1 (ii). Conversely, assume that $y(\tau)$ is a $\mathbb{G}^\tau$-martingale. Then, for $s \le t$, from Lemma 8.1.1 (ii),

$$y_s(\tau) = \mathbb{E}\left(y_t(\tau)|\mathcal{G}_s^\tau\right) = \frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)_{|u=\tau}$$

and the result follows from Lemma 8.1.1 (i). □

Passing to the progressive enlargement setting, we state and prove a martingale characterization result, essentially established by El Karoui et al. in [13] (see Theorem 5.7).

Proposition 8.2.2. ***Characterization of*** $(\mathbb{P},\mathbb{G})$ ***martingales in terms of*** $(\mathbb{P},\mathbb{F})$***-martingales*** *A* $\mathbb{G}$*-adapted process* $Y_t := \widetilde{y}_t \mathbb{1}_{t<\tau} + \widehat{y}_t(\tau)\mathbb{1}_{\tau\le t}, t\ge 0$*, is a* $(\mathbb{P},\mathbb{G})$*-martingale if and only if the following two conditions are satisfied*

(i) for ν*-almost every* $u\ge 0$*,* $\big(\widehat{y}_t(u)p_t(u), t\ge u\big)$ *is a* $(\mathbb{P},\mathbb{F})$*-martingale;*

(ii) the process $m=(m_t, t\ge 0)$*, defined by*

$$m_t := \mathbb{E}(Y_t|\mathcal{F}_t) = \widetilde{y}_t G_t + \int_0^t \widehat{y}_t(u)p_t(u)\nu(du)\ , \tag{8.2.1}$$

is a $(\mathbb{P},\mathbb{F})$*-martingale.*

Proof. For the necessity, in a first step, we show that we can reduce our attention to the case where Y is u.i. (uniformly integrable). Indeed, let Y be a $(\mathbb{P},\mathbb{G})$-martingale: for any T, let $Y^{(T)} = (Y_{t\wedge T}, t\ge 0)$ be the associated stopped martingale, which is u.i. Assuming that the result is established for u.i. martingales will prove that the processes in *(i)* and *(ii)* are martingales up to time T. Since T can be chosen as large as possible, we shall have the result.

Assume that Y is a u.i. $(\mathbb{P},\mathbb{G})$-martingale. From Proposition 8.1.1, $Y_t = \mathbb{E}(Y^\tau_t|\mathcal{G}_t)$ for some $(\mathbb{P},\mathbb{G}^\tau)$-martingale Y^τ. Proposition 8.2.1, then, implies that $Y^\tau_t = y_t(\tau)$, where for ν-almost every $u\ge 0$ the process $\big(y_t(u)p_t(u), t\ge 0\big)$ is a $(\mathbb{P},\mathbb{F})$-martingale. One then has

$$\mathbb{1}_{\tau\le t}\widehat{y}_t(\tau) = \mathbb{1}_{\tau\le t}Y_t = \mathbb{1}_{\tau\le t}\mathbb{E}(Y^\tau_t|\mathcal{G}_t) = \mathbb{E}(\mathbb{1}_{\tau\le t}Y^\tau_t|\mathcal{G}_t) = \mathbb{1}_{\tau\le t}y_t(\tau)\ ,$$

which implies, in view of Lemma 8.1.1 (i), that for ν-almost every $u\le t$, the identity $y_t(u) = \widehat{y}_t(u)$ holds $\mathbb{P}$-a.s. Part *(i)* is, then, proved.

Next, Y being a $(\mathbb{P},\mathbb{G})$-martingale, its projection on the smaller filtration $\mathbb{F}$, namely the process m in (8.2.1), is a $(\mathbb{P},\mathbb{F})$-martingale.

Conversely, assuming *(i)* and *(ii)*, we verify that $\mathbb{E}(Y_t|\mathcal{G}_s) = Y_s$ for $s\le t$. We start by noting that

$$\mathbb{E}(Y_t|\mathcal{G}_s) = \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}(Y_t\mathbb{1}_{s<\tau}|\mathcal{F}_s) + \mathbb{1}_{\tau\le s}\mathbb{E}(Y_t\mathbb{1}_{\tau\le s}|\mathcal{G}_s)\ . \tag{8.2.2}$$

We then compute the two conditional expectations in (8.2.2), as specified, for $s\le t$:

$$\begin{aligned}
\mathbb{E}(Y_t\mathbb{1}_{s<\tau}|\mathcal{F}_s) &= \mathbb{E}(Y_t|\mathcal{F}_s) - \mathbb{E}(Y_t\mathbb{1}_{\tau\le s}|\mathcal{F}_s)\\
&= \mathbb{E}(m_t|\mathcal{F}_s) - \mathbb{E}\big(\mathbb{E}(\widehat{y}_t(\tau)\mathbb{1}_{\tau\le s}|\mathcal{F}_t)|\mathcal{F}_s\big)\\
&= m_s - \mathbb{E}\Big(\int_0^s \widehat{y}_t(u)p_t(u)\nu(du)|\mathcal{F}_s\Big)\\
&= \widetilde{y}_s G_s + \int_0^s \widehat{y}_s(u)p_s(u)\nu(du) - \int_0^s \widehat{y}_s(u)p_s(u)\nu(du) = \widetilde{y}_s G_s\ ,
\end{aligned}$$

where we used Fubini-Tonelli's theorem and the condition *(i)* to obtain the next-to-last identity.

Furthermore, an application of Lemma 8.1.2 yields to (recall that $s\le t$)

$$\begin{aligned}
\mathbb{E}(Y_t\mathbb{1}_{\tau\le s}|\mathcal{G}_s) &= \mathbb{E}(\widehat{y}_t(\tau)\mathbb{1}_{\tau\le s}|\mathcal{G}_s) = \mathbb{1}_{\tau\le s}\frac{1}{p_s(\tau)}\mathbb{E}\big(\widehat{y}_t(u)p_t(u)|\mathcal{F}_s\big)\big|_{u=\tau}\\
&= \mathbb{1}_{\tau\le s}\frac{1}{p_s(\tau)}\widehat{y}_s(\tau)p_s(\tau) = \mathbb{1}_{\tau\le s}\widehat{y}_s(\tau)
\end{aligned}$$

where the next-to-last identity holds in view of the condition *(ii)*. The proposition is proved. □

We end this section with a "curiosity" linking martingales in the filtrations $\mathbb{G}$ and $\mathbb{G}^\tau$: we already know, from Remark 8.1.2, that any $(\mathbb{P}^*, \mathbb{F})$-martingale remains a $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingale, but it is not true that any $(\mathbb{P}^*, \mathbb{G})$-martingale remains a $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingale. Indeed, we have the following result.

Lemma 8.2.1. *Any $(\mathbb{P}^*, \mathbb{G})$-martingale X^* is a $(\mathbb{P}^*, \mathbb{G}^\tau)$ semi-martingale. In particular, any $(\mathbb{P}, \mathbb{G})$-martingale is a $(\mathbb{P}, \mathbb{G}^\tau)$ semi-martingale.*

Proof. In all the proof, we work under $\mathbb{P}^*$. The result follows immediately from Proposition 8.2.2 (under $\mathbb{P}^*$), noticing that the $(\mathbb{P}^*, \mathbb{G})$ martingale Y^* can be written as $Y_t^* = \widetilde{y}_t^* \mathbb{1}_{t<\tau} + \widehat{y}_t^*(\tau)\mathbb{1}_{\tau \le t}$. Therefore, in the filtration $\mathbb{G}^\tau$, it is the sum of two $\mathbb{G}^\tau$ semi-martingales: the processes $\mathbb{1}_{t<\tau}$ and $\mathbb{1}_{\tau\le t}$ are $\mathbb{G}^\tau$ semi-martingales, as well as the processes $\widetilde{y}, \widehat{y}^*(\tau)$. Indeed, from Proposition 8.2.2, recalling that the $(\mathbb{P}^*, \mathbb{F})$-density of τ, p^*, is a constant equal to one, we know that, for every $u > 0$, $\big(\widehat{y}_t^*(u), t \ge u\big)$ is an $\mathbb{F}$-martingale and that the process $\big(\widetilde{y}_t^* G(t) + \int_0^t \widehat{y}_t^*(u)\nu(du), t \ge 0\big)$ is an $\mathbb{F}$-martingale, hence $\widetilde{y}^*$ is a $\mathbb{G}$ semi-martingale. □

8.3 Canonical decomposition

In this section, we show that any $\mathbb{F}$-local martingale x is a semi-martingale in both the initially enlarged filtration $\mathbb{G}^\tau$ and in the progressively enlarged filtration $\mathbb{G}$ and that any $\mathbb{G}$-martingale is a $\mathbb{G}^\tau$ semi-martingale. We also provide the canonical decomposition of the $\mathbb{F}$-local martingale as a semi-martingale in $\mathbb{G}^\tau$ and in $\mathbb{G}$. Under the assumption that the $\mathbb{F}$-conditional law of τ is absolutely continuous w.r.t. the law of τ, these questions were answered in Jacod [22], in the initial enlargement setting, and in Jeanblanc and Le Cam [23] and El Karoui *et al.* [13], in the progressive enlargement case. Our aim here is to retain their results in an alternative manner.

We will need the following technical result, concerning the existence of the predictable bracket $\langle x, p_.(u)\rangle$: from Theorem 2.5 a) in Jacod [22], it follows immediately that, under the equivalence assumption, for every $(\mathbb{P}, \mathbb{F})$-(local)martingale x, there exists a ν-negligible set B (depending on x), such that $\langle x, p_.(u)\rangle$ is well-defined for $u \notin B$. Hereafter, by $\langle x, p_.(\tau)\rangle_s$ we mean $\langle x, p_.(u)\rangle_s \big|_{u=\tau}$.

Furthermore, according to Theorem 2.5 b) in Jacod [22], under the $(\mathcal{E})$-Hypothesis, there exists an $\mathbb{F}$-predictable increasing process A and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(t, \omega, u) \to k_t(\omega, u)$ such that, for any $u \notin B$,

$$\langle x, p_.(u)\rangle_t = \int_0^t k_s(u) p_{s-}(u) dA_s \quad \text{a.s., for any } t > 0. \tag{8.3.1}$$

Notice that the two processes A and k depend on x, but, in order to keep a simple notation, we do not write $A^{(x)}$, nor $k^{(x)}$.

Furthermore, if A and k exist and they satisfy the above requirements, then

$$\int_0^t |k_s(u)| dA_s \; < \; \infty \quad \text{a.s., for any } t > 0. \tag{8.3.2}$$

The following two propositions provide, under the $(\mathcal{E})$-Hypothesis, the canonical decomposition of any $(\mathbb{P},\mathbb{F})$-local martingale x in the enlarged filtrations $\mathbb{G}^\tau$ and in $\mathbb{G}$, respectively. The first result is due to Jacod [22, Theorem 2.5 c)]. Our proof is different, but less general.

Proposition 8.3.1. ***Canonical Decomposition in*** $\mathbb{G}^\tau$
Any $(\mathbb{P},\mathbb{F})$-local martingale x is a $(\mathbb{P},\mathbb{G}^\tau)$-semimartingale with canonical decomposition

$$x_t = X_t^\tau + \int_0^t \frac{d\langle x, p_.(\tau)\rangle_s}{p_{s-}(\tau)},$$

for some $(\mathbb{P},\mathbb{G}^\tau)$-local martingale X^τ.

Proof. In view of Remark 8.1.2, if x is a $(\mathbb{P},\mathbb{F})$-martingale, it is a $(\mathbb{P}^*,\mathbb{G}^\tau)$-martingale, too. Noting that $\frac{d\mathbb{P}}{d\mathbb{P}^*} = p_t(\tau)$ on $\mathcal{G}_t^\tau$, Girsanov's Theorem tells us that the process X^τ, defined by

$$X_t^\tau = x_t - \int_0^t \frac{d\langle x, p_.(\tau)\rangle_s}{p_{s-}(\tau)}$$

is a $(\mathbb{P},\mathbb{G}^\tau)$-martingale. $\square$

Now, any $\mathbb{F}$-local martingale is a $\mathbb{G}$-adapted process and a $\mathbb{G}^\tau$ semi-martingale (from the above Proposition 8.3.1), so in view of Stricker's Theorem (see, e.g., [32]: in the case when two filtrations $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ satisfy $\mathbb{F}\subset\widetilde{\mathbb{F}}$, if X is an $\widetilde{\mathbb{F}}$ semi-martingale and it is $\mathbb{F}$-adapted, then it is also an $\mathbb{F}$ semi-martingale), it is also a $\mathbb{G}$ semi-martingale. The following proposition aims to obtain the $\mathbb{G}$-canonical decomposition of an $\mathbb{F}$-local martingale. We refer to Jeanblanc and Le Cam [23] for an alternative proof.
We need some preliminary results, as well as the following assumption:

Assumption 8.3.1. *There exists a version of the process $(p_t(t), t\geq 0)$, such that $(t,\omega)\to p_t(t)$ is $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^+)$-measurable.*

Recall that the Doob-Meyer decomposition of the Azéma super-martingale G, introduced in Equation (8.1.1), writes $G_t = \mu_t - \int_0^t p_u(u)\nu(du), t\geq 0$, where μ is the $\mathbb{F}$-martingale defined as

$$\mu_t := 1 - \int_0^t \left(p_t(u) - p_u(u)\right)\nu(du)$$

(see, e.g., Section 4.2.1 in El Karoui *et al.* [13]). The following lemma provides a formula for the predictable quadratic covariation process $\langle x, G\rangle = \langle x,\mu\rangle$ in terms of the density p.

Lemma 8.3.1. *Let x be an $\mathbb{F}$-martingale and μ the $\mathbb{F}-$martingale part in the Doob-Meyer decomposition of G. If kp_- is $dA\otimes d\nu$-integrable, then*

$$\langle x,\mu\rangle_t = \int_0^t dA_s \int_s^\infty k_s(u)p_{s-}(u)\nu(du), \tag{8.3.3}$$

where k was introduced in Equation (8.3.1).

Proof. First consider the right-hand-side of (8.3.3), that is, by definition, predictable and apply Fubini-Tonelli's Theorem (recall Equation (8.3.2))

$$\begin{aligned}
\xi_t &:= \int_0^t dA_s \int_s^\infty k_s(u)p_{s-}(u)\nu(du) \\
&= \int_0^t dA_s \int_s^t k_s(u)p_{s-}(u)\nu(du) + \int_0^t dA_s \int_t^\infty k_s(u)p_{s-}(u)\nu(du) \\
&= \int_0^t \nu(du) \int_0^u k_s(u)p_{s-}(u)dA_s + \int_t^\infty \nu(du) \int_0^t k_s(u)p_{s-}(u)dA_s \\
&= \int_0^t \langle x, p.(u)\rangle_u \ \nu(du) + \int_t^\infty \langle x, p.(u)\rangle_t \ \nu(du) \\
&= \int_0^\infty \langle x, p.(u)\rangle_t \ \nu(du) + \int_0^t \left(\langle x, p.(u)\rangle_u - \langle x, p.(u)\rangle_t\right) \nu(du) \ .
\end{aligned}$$

To verify (8.3.3), it suffices to show that the process $x\mu - \xi$ is an $\mathbb{F}$-local martingale (since ξ is a predictable, finite variation process). Note that for ν-almost every $u \in \mathbb{R}^+$, the process $(m_t(u) := x_t p_t(u) - \langle x, p.(u)\rangle_t, t \geq 0)$ is an $\mathbb{F}$-local martingale. Then, given that $1 = \int_0^\infty p_t(u)\nu(du)$ for every $t \geq 0$, a.s., we have

$$\begin{aligned}
x_t\mu_t - \xi_t &= x_t \int_0^\infty p_t(u)\nu(du) - x_t \int_0^t \left(p_t(u) - p_u(u)\right)\nu(du) \\
&- \int_0^\infty \langle x, p.(u)\rangle_t \ \nu(du) + \int_0^t \left(\langle x, p.(u)\rangle_t - \langle x, p.(u)\rangle_u\right) \nu(du) \\
&= \int_0^\infty m_t(u)\nu(du) - \int_0^t \left(m_t(u) - m_u(u)\right)\nu(du) + \int_0^t p_u(u)(x_t - x_u)\nu(du) \ .
\end{aligned}$$

The first two terms are martingale (because of the martingale property of $m(u)$ for ν almost every $u \in \mathbb{R}^+$). As for the last term, using the fact that ν has no atoms, we find

$$\begin{aligned}
d\left(x_t \int_0^t p_u(u)\nu(du) - \int_0^t p_u(u)x_u\nu(du)\right) &= dx_t \int_0^t p_u(u)\nu(du) + x_t p_t(t)\nu(dt) - p_t(t)x_t\nu(dt) \\
&= dx_t \int_0^t p_u(u)\nu(du)
\end{aligned}$$

and we have, indeed, proved that $x\mu - \xi$ is an $\mathbb{F}$-local martingale. □

Proposition 8.3.2. ***Canonical Decomposition in*** $\mathbb{G}$
Any (càdlàg) $(\mathbb{P}, \mathbb{F})$*-local martingale* x *is a* $(\mathbb{P}, \mathbb{G})$ *semi-martingale with canonical decomposition*

$$x_t = X_t + \int_0^{t\wedge\tau} \frac{d\langle x, G\rangle_s}{G_{s-}} + \int_{t\wedge\tau}^t \frac{d\langle x, p.(\tau)\rangle_s}{p_{s-}(\tau)} \tag{8.3.4}$$

where X *is a* $(\mathbb{P}, \mathbb{G})$*-local martingale.*

Proof. From Proposition 8.3.1, any $\mathbb{F}$-local martingale x can be decomposed as $x = X^\tau + C$ where X^τ is a $(\mathbb{P}, \mathbb{G}^\tau)$-local martingale and (recall Equation (8.3.1))

$$C_t = \int_0^t \frac{d\langle x, p.(\tau)\rangle_s}{p_{s-}(\tau)} = \int_0^t k_s(\tau)dA_s \ .$$

The idea is to project this decomposition on the filtration $\mathbb{G}$:

$$x_t = {}^{(o)}x_t = {}^{(o)}X_t^\tau + {}^{(o)}C_t - C_t^{(p)} + C_t^{(p)} ,$$

where the left-hand side superscript "(o)" indicates the optional projection (on $\mathbb{G}$) and where the right-hand side superscript "(p)" denotes the dual predictable projection (on $\mathbb{G}$). Now, the process $X := {}^{(o)}X^\tau + \left({}^{(o)}C - C^{(p)}\right)$ is the sum of two $\mathbb{G}$-martingales, hence it is a $\mathbb{G}$-martingale. From classical results on the predictable projection of processes (see for instance Theorem 57, Chapter VI of [11]), we also have, being A predictable,

$$C_t^{(p)} = \int_0^t {}^{(p)}(k(\tau))_s dA_s . \tag{8.3.5}$$

From Proposition 8.1.5, moreover,

$${}^{(p)}(k(\tau))_s = \mathbb{1}_{s\le\tau}\frac{1}{G_{s-}}\int_s^\infty k_s(u)p_{s-}(u)\nu(du) + \mathbb{1}_{\tau<s}k_s(\tau) . \tag{8.3.6}$$

Thus, substituting (8.3.6) in (8.3.5) and using Lemma 8.3.1, one obtains decomposition (8.3.4). □

As in Lemma 8.2.1, we deduce that any $(\mathbb{P},\mathbb{G})$-martingale is a $(\mathbb{P},\mathbb{G}^\tau)$ semi-martingale.

8.4 Predictable Representation Theorems

The aim of this section is to obtain Predictable Representation Theorems (PRT hereafter) in the enlarged filtrations $\mathbb{G}$ and $\mathbb{G}^\tau$, both under $\mathbb{P}$ and $\mathbb{P}^*$. We start by assuming that there exists a $(\mathbb{P},\mathbb{F})$-local martingale z (possibly multidimensional), such that the Predictable Representation Property (PRP hereafter) holds in $(\mathbb{P},\mathbb{F})$. Notice that z is not necessarily continuous.

Beforehand we introduce some notation: $\mathcal{M}^{\mathrm{loc}}(\mathbb{P},\mathbb{F})$ denotes the set of $(\mathbb{P},\mathbb{F})$-local martingales, while $\mathcal{M}^2(\mathbb{P},\mathbb{F})$ denotes the set of $(\mathbb{P},\mathbb{F})$-martingales x, such that

$$\mathbb{E}\left(x_t^2\right) < \infty, \quad \forall\, t \ge 0.$$

Also, for a $(\mathbb{P},\mathbb{F})$-local martingale m, we denote by $\mathcal{L}(m,\mathbb{P},\mathbb{F})$ the set of $\mathbb{F}$-predictable processes which are integrable with respect to m (in the sense of local martingale), namely (see, e.g., Definition 9.1 and Theorem 9.2. in [19])

$$\mathcal{L}(m,\mathbb{P},\mathbb{F}) = \left\{\varphi \in \mathcal{P}(\mathbb{F}) : \left(\int_0^\cdot \varphi_s^2 d[m]_s\right)^{1/2} \text{ is } \mathbb{P}-\text{locally integrable}\right\} .$$

Assumption 8.4.1. ***PRT for*** $(\mathbb{P},\mathbb{F})$
There exists a process $z \in \mathcal{M}^{\mathrm{loc}}(\mathbb{P},\mathbb{F})$ *such that every* $x \in \mathcal{M}^{\mathrm{loc}}(\mathbb{P},\mathbb{F})$ *can be represented as*

$$x_t = x_0 + \int_0^t \varphi_s dz_s$$

for some $\varphi \in \mathcal{L}(z,\mathbb{P},\mathbb{F})$.

For simplicity, we start investigating what happens under the measure $\mathbb{P}^*$, in the initially enlarged filtration $\mathbb{G}^\tau$, since here τ is independent of $\mathcal{F}_t$, for any $t \geq 0$, so that we expect things to be easier.

Notice that under immersion property, Kusuoka in [26] established a PRT in the progressively enlarged filtration, in the case when the process z in Assumption 8.4.1 is a Brownian motion. Furthermore, under the *equivalence assumption* in $[0, T]$ and assuming a martingale representation theorem in the reference filtration $\mathbb{F}$, Amendinger (see [1, Th. 2.4]) proved a martingale representation theorem in $(\mathbb{P}^*, \mathbb{G}^\tau)$. This result was extended to $(\mathbb{P}, \mathbb{G}^\tau)$, in the case when the underlying (local) martingale in the reference filtration is continuous.

Proposition 8.4.1. *PRT for* $(\mathbb{P}^*, \mathbb{G}^\tau)$
Under Assumption 8.4.1, every $X^\tau \in \mathcal{M}^{\text{loc}}(\mathbb{P}^*, \mathbb{G}^\tau)$ admits a representation

$$X_t^\tau = X_0^\tau + \int_0^t \Phi_s^\tau dz_s, \tag{8.4.1}$$

where $\Phi^\tau \in \mathcal{L}(z, \mathbb{P}^*, \mathbb{G}^\tau)$. In the case where $X^\tau \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G}^\tau)$, one has $\mathbb{E}^*\big(\int_0^t (\Phi_s^\tau)^2 d[z]_s\big) < \infty$, for all $t \geq 0$ and the representation is unique.

Proof. From Theorem 13.4 in [19], it suffices to prove that any bounded martingale admits a predictable representation in terms of z. Let $X^\tau \in \mathcal{M}^{\text{loc}}(\mathbb{P}^*, \mathbb{G}^\tau)$ be bounded by K. From Proposition 8.2.1, $X_t^\tau = x_t(\tau)$, where for ν-almost every $u \in \mathbb{R}^+$, the process $\big(x_t(u), t \geq 0\big)$ is a $(\mathbb{P}^*, \mathbb{F})$-martingale, hence a $(\mathbb{P}, \mathbb{F})$-martingale. Thus (for ν-almost every $u \in \mathbb{R}^+$), Assumption 8.4.1 implies that

$$x_t(u) = x_0(u) + \int_0^t \varphi_s(u) dz_s \ ,$$

where $(\varphi_t(u), t \geq 0)$ is an $\mathbb{F}$-predictable process.

The process X^τ being bounded by K, it follows by an application of Lemma 8.1.1 (i) that for ν-almost every $u \geq 0$, the process $(x_t(u), t \geq 0)$ is bounded by K. Then, using the Itô isometry,

$$\begin{aligned}
\mathbb{E}^* \int_0^t \varphi_s^2(u) d[z]_s &= \mathbb{E}^*\big(\int_0^t \varphi_s(u) dz_s\big)^2 \\
&= \mathbb{E}^*\big((x_t(u) - x_0(u))^2\big) \leq \mathbb{E}^*(x_t^2(u)) \leq K^2 \ .
\end{aligned}$$

Furthermore, from [33, Lemma 2], one can consider a version of the process $\int_0^\cdot \varphi_s^2(u) d[z]_s$ which is measurable with respect to u. Using this fact,

$$\mathbb{E}^*\Big[\big(\int_0^t \varphi_s^2(\tau) d[z]_s\big)^{1/2}\Big] = \int_0^\infty \nu(du)\Big(\mathbb{E}^*\big(\int_0^t \varphi_s^2(u) d[z]_s\big)\Big)^{1/2} \leq \int_0^\infty \nu(du) K = K \ .$$

Now, the process Φ^τ defined by $\Phi_t^\tau = \varphi_t(\tau)$ is $\mathbb{G}^\tau$-predictable, according to Proposition 8.1.2, it satisfies (8.4.1), with $X_0(\tau) = x_0(\tau)$ and it belongs to $\mathcal{L}(z, \mathbb{P}^*, \mathbb{G}^\tau)$.

If $X^\tau \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G}^\tau)$, from Ito's isometry,

$$\mathbb{E}^* \int_0^t (\Phi_s^\tau)^2 d[z]_s = \mathbb{E}^* \left(\int_0^t \Phi_s^\tau dz_s\right)^2 = \mathbb{E}^*(X_t^\tau - X_0^\tau)^2 < \infty \ .$$

Also, from this last equation, if $X^\tau \equiv 0$, then $\Phi^\tau \equiv 0$, from which the uniqueness of the representation follows. $\square$

Passing to the progressively enlarged filtration $\mathbb{G}$, that is given by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, intuitively one needs two martingales to establish a PRT. Apart from z, intuition tells us that a candidate for the second martingale might be the compensated martingale of H, that was introduced, respectively under $\mathbb{P}$ (it was denoted by M) and under $\mathbb{P}^*$ (denoted by M^*), in Equation (8.1.3) and in Equation (8.1.4).

Proposition 8.4.2. ***PRT for*** $(\mathbb{P}^*, \mathbb{G})$
Under Assumption 8.4.1, every $X \in \mathcal{M}^{\text{loc}}(\mathbb{P}^*, \mathbb{G})$ *admits a representation*

$$X_t = X_0 + \int_0^t \Phi_s dz_s + \int_0^t \Psi_s dM_s^*$$

for some processes $\Phi \in \mathcal{L}(z, \mathbb{P}^*, \mathbb{G})$ *and* $\Psi \in \mathcal{L}(M^*, \mathbb{P}^*, \mathbb{G})$*. Moreover, if* $X \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G})$*, one has, for any* $t \geq 0$*,*

$$\mathbb{E}\left(\int_0^t \Phi_s^2 d[z]_s\right) < \infty \quad , \quad \mathbb{E}\left(\int_0^t \Psi_s^2 \lambda_s^* \nu(ds)\right) < \infty \, ,$$

and the representation is unique.

Proof. It is known that any $(\mathbb{P}^*, \mathbb{H})$ local martingale ξ can be represented as $\xi_t = \xi_0 + \int_0^t \psi_s dM_s^*$ for some process $\psi \in \mathcal{L}(M^*, \mathbb{P}^*, \mathbb{H})$ (see, e.g., the proof in Chou and Meyer [8]). Notice that ψ has a role only before τ and, for this reason (recall that $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$, where $\mathcal{H}_t = \sigma(\mathbb{1}_{\tau \leq s}, s \leq t)$), ψ can be chosen deterministic.

Under $\mathbb{P}^*$, we then have

- the PRP holds in $\mathbb{F}$ with respect to z,
- the PRP holds in $\mathbb{H}$ with respect to M^*,
- $\mathbb{F}$ and $\mathbb{H}$ are independent.

From classical literature (see Lemma 9.5.4.1 (ii) of Chesney, Jeanblanc and Yor [7], for instance), the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ enjoys the predictable representation property under $\mathbb{P}^*$ with respect to the pair (z, M^*).

Now suppose that $X \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G})$. We find

$$\begin{aligned} \infty > \mathbb{E}(X_t - X_0)^2 &= \mathbb{E}\left(\int_0^t \Phi_s dz_s + \int_0^t \Psi_s dM_s^*\right)^2 \\ &= \mathbb{E}\int_0^t \Phi_s^2 d[z]_s + 2\mathbb{E}\left(\int_0^t \Phi_s dz_s \int_0^t \Psi_s dM_s^*\right) + \mathbb{E}\int_0^t \Psi_s^2 \lambda_s^* \nu(ds), \end{aligned}$$

where in the last equality we used the Itô isometry. The cross-product term in the last equality is zero due to the orthogonality of z and M^* (under $\mathbb{P}^*$). From this inequality, the desired integrability conditions hold and the uniqueness of the representation follows (as in the previous proposition). □

Remark 8.4.1. *In order to establish a PRT for the initially enlarged filtration* $\mathbb{G}^\tau$ *and under* $\mathbb{P}^*$*, one could have proceeded as in the proof of Proposition 8.4.2, noting that any martingale* ξ *in the "constant" filtration* $\sigma(\tau)$ *satisfies* $\xi_t = \xi_0 + 0$ *and that under* $\mathbb{P}^*$ *the two filtrations* $\mathbb{F}$ *and* $\sigma(\tau)$ *are independent.*

Proposition 8.4.3. ***PRT under*** $\mathbb{P}$

Under Assumption 8.4.1, one has:

(i) Every $X^\tau \in \mathcal{M}^{\mathrm{loc}}(\mathbb{P}, \mathbb{G}^\tau)$ *can be represented as*

$$X_t^\tau = X_0^\tau + \int_0^t \Phi_s^\tau dZ_s^\tau$$

where Z^τ *is the martingale part in the* $\mathbb{G}^\tau$*-canonical decomposition of* z *and* $\Phi \in \mathcal{L}(Z^\tau, \mathbb{P}, \mathbb{G}^\tau)$.

(ii) Every $X \in \mathcal{M}^{\mathrm{loc}}(\mathbb{P}, \mathbb{G})$ *can be represented as*

$$X_t = X_0 + \int_0^t \Phi_s dZ_s + \int_0^t \Psi_s dM_s$$

where Z *is the martingale part in the* $\mathbb{G}$*-canonical decomposition of* z*,* M *is the* $(\mathbb{P}, \mathbb{G})$*-compensated martingale associated with* H *and* $\Phi \in \mathcal{L}(Z, \mathbb{P}, \mathbb{G})$ *and* $\Psi \in \mathcal{L}(M, \mathbb{P}, \mathbb{G})$.

Proof. The assertion (i) (resp. (ii)) follows from Proposition 8.4.1 (resp. Proposition 8.4.2) and the stability of the PRP under an equivalent change of measure (see for example He, Wang and Yan [19]).

For part (ii), it is important to note that, if z is a $(\mathbb{P}, \mathbb{F})$-martingale, it is a $(\mathbb{P}^*, \mathbb{G})$-martingale, too. Hence, by applying a Girsanov type transformation, Z defined as $dZ_t := dz_t - \frac{1}{\ell^*_{t_-}} d\langle z, \ell^* \rangle_t$, $Z_0 = z_0$, is a $(\mathbb{P}, \mathbb{G})$-martingale, where we recall that $\ell^* = 1/\ell$ is a $(\mathbb{P}^*, \mathbb{G})$-martingale (in fact, $d\mathbb{P}_{|\mathcal{G}_t} = \ell_t^* \; d\mathbb{P}^*{}_{|\mathcal{G}_t}$). From the uniqueness of the canonical decomposition of the $(\mathbb{P}, \mathbb{G})$-semimartingale z (which is, indeed, special) and from Proposition 8.3.2, it follows that the $(\mathbb{P}, \mathbb{G})$-martingale Z is, in particular, given by

$$Z_t = z_t - \int_0^{t \wedge \tau} \frac{d\langle z, G \rangle_s}{G_{s_-}} - \int_{t \wedge \tau}^t \frac{d\langle z, p.(\tau) \rangle_s}{p_{s_-}(\tau)} \; .$$

□

8.5 Concluding Remarks

- In the multi-dimensional case, that is when $\tau = (\tau_1, \cdots, \tau_d)$ is a vector of finite random times, the same machinery can be applied. More precisely, under the assumption

  $$\mathbb{P}(\tau_1 \in \theta_1, \cdots, \tau_d \in \theta_d | \mathcal{F}_t) \sim \mathbb{P}(\tau_1 \in \theta_1, \cdots, \tau_d \in \theta_d)$$

  one defines the probability $\mathbb{P}^*$ on $\mathcal{G}_t^\tau = \mathcal{F}_t \vee \sigma(\tau_1) \vee \cdots \vee \sigma(\tau_d)$, with respect to $\mathbb{P}$, by

  $$\frac{d\mathbb{P}^*}{d\mathbb{P}}_{|\mathcal{G}_t^\tau} = \frac{1}{p_t(\tau_1, \cdots, \tau_d)},$$

  where $p_t(\tau_1, \cdots, \tau_d)$ is the (multidimensional) analog to $p_t(\tau)$, and the results for the initially enlarged filtration are obtained in the same way as the one-dimensional case.

  As for the progressively enlarged filtration, one has to note that, in this case, a measurable process is decomposed into 2^d terms, depending on whether $t < \tau_i$ or $\tau_i \leq t$.

- Notice that honest times (recall that a random time L is honest if it is equal to an $\mathcal{F}_t$-measurable random variable on $\{L < t\}$; in particular, an honest time is $\mathcal{F}_\infty$-measurable) cannot be included in this study. Indeed, it was shown by Nikeghbali and Yor in [28], Theorem 4.1, that, in the case when all $\mathbb{F}$-martingales are continuous and if the honest time L avoids any $\mathbb{F}$-stopping time, then there exists a continuous and nonnegative local martingale $(N_t)_{t\geq 0}$, with $N_0 = 1$ and $\lim_{t\to+\infty} N_t = 0$, such that:
$$P(L > t|\mathcal{F}_t) = \frac{N_t}{S_t},$$
where $S_t := \sup_{s\leq t} N_s$. In our case, under the $(\mathcal{E})$-Hypothesis, the above equation does not hold true, since the Azéma supermartingale G admits the multiplicative decomposition $G_t = N_t e^{-\Lambda_t}$, where the intensity process Λ is strictly increasing (so that it is not possible that $e^{\Lambda_t} = \sup_{s\leq t} N$).

- Under immersion property and under the $(\mathcal{E})$-Hypothesis, $p_t(u) = p_u(u), t \geq u$. In particular, as expected (for all the details see, e.g., Corollary 1 in Jeanblanc and Le Cam [23]), the canonical decomposition's formulae presented in Section 8.3 are trivial.

8.6 Appendix

Lemma 8.6.1. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two independent σ-fields and X is an integrable random variable independent of $\mathcal{F}_2$ then*
$$\mathbb{E}(X|\mathcal{F}_1 \vee \mathcal{F}_2) = \mathbb{E}(X|\mathcal{F}_1).$$

Proof. On may equivalently check that, for every $Y \in \mathcal{F}_1 \vee \mathcal{F}_2$,
$$\mathbb{E}(XY) = \mathbb{E}\big(\mathbb{E}(X|\mathcal{F}_1)Y\big)$$
In view of the monotone class theorem it suffices to show the above assertion for $Y = Y_1Y_2$ where $Y_1 \in \mathcal{F}_1$ and $Y_2 \in \mathcal{F}_2$. We have
$$\mathbb{E}\big(\mathbb{E}(X|\mathcal{F}_1)Y_1Y_2\big) = \mathbb{E}\big(\mathbb{E}(XY_1|\mathcal{F}_1)Y_2\big) = \mathbb{E}\big(\mathbb{E}(XY_1|\mathcal{F}_1)\big)\mathbb{E}(Y_2) = \mathbb{E}(XY_1)\mathbb{E}(Y_2) = \mathbb{E}(XY_1Y_2)$$
and the result is proved. □

In order to define the optional and predictable projections of a process X with respect to a filtration $\mathbb{F}$, we first introduce the following two notions of σ-algebra associated with a stopping time (for this we refer to Dellacherie and Meyer [10], Ch. IV, Definitions 52 and 54, page 186).

Definition 8.6.1. *Let τ be a stopping time with respect to a filtration $\mathbb{F}$ satisfying the usual hypotheses. The σ-algebra of events prior to τ, denoted $\mathcal{F}_\tau$, is defined as follows:*
$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t,\ \forall\ t\}\,.$$
The σ-algebra of events strictly *prior to τ, denoted $\mathcal{F}_{\tau-}$, is the smallest σ-algebra that contains $\mathcal{F}_0$ and all the sets of the form $A \cap \{t < \tau\}$, $t \geq 0$, for $A \in \mathcal{F}_t$.*

The following result can be found in Dellacherie and Meyer [11], Ch. VI, Th. 43.

Theorem 8.6.1. ***Optional and predictable projections***
Let X be a bounded or positive measurable process and $\mathbb{F}$ a filtration satisfying the usual hypotheses. There exists a unique *optional process Y and a* unique *predictable process Z such that*

$$\begin{aligned}
\mathbb{E}\left[X_\tau \mathbb{1}_{\tau<\infty} | \mathcal{F}_\tau\right] &= Y_\tau \mathbb{1}_{\tau<\infty}, \quad \textit{a.s., for every stopping time } \tau, \\
\mathbb{E}\left[X_\theta \mathbb{1}_{\theta<\infty} | \mathcal{F}_{\theta-}\right] &= Z_\theta \mathbb{1}_{\theta<\infty}, \quad \textit{a.s., for every } \text{predictable} \textit{ stopping time } \theta.
\end{aligned}$$

The process Y is called the ***optional projection*** *of X, while Z is called the* ***predictable projection*** *of X.*

Bibliography

[1] J. Amendinger. *Initial Enlargement of Filtrations and Additional Information in Financial Markets.* PhD thesis, Technischen Universität, Berlin, 1999.

[2] J. Amendinger. Martingale representation theorems for initially enlarged filtrations. *Stochastic Processes and their Appl.*, 89:101–116, 2000.

[3] S. Ankirchner. *Information and Semimartingales.* PhD thesis, Humboldt Universität, Berlin, 2005.

[4] M.T. Barlow. Study of filtration expanded to include an honest time. *Z. Wahr. Verw. Gebiete*, 44:307–323, 1978.

[5] F. Baudoin. *Modeling anticipations on financial markets*, volume 1814 of *Lecture Notes in Mathematics*, chapter Paris-Princeton Lecture on mathematical Finance 2002, pages 43–92. Springer-Verlag, 2003.

[6] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. *Credit Risk Modeling.* CSFI lecture note series. Osaka University Press, 2009.

[7] M. Chesney, M. Jeanblanc, and M Yor. *Mathematical Methods for Financial Markets.* Springer, 2009.

[8] C.S. Chou and P.A. Meyer. Sur la représentation des martingales comme intégrales stochastiques dans les processus ponctuels. *Séminaire de probabilités*, IX:226–236, 1975.

[9] J.M. Corcuera, P. Imkeller, A. Kohatsu-Higa, and D. Nualart. Additional utility of insiders with imperfect dynamical information. *Finance and Stochastics*, 8:437–450, 2004.

[10] C. Dellacherie and P.A. Meyer. *Probabilités et Potentiel - Chapitres I à IV.* Hermann, Paris, 1975.

[11] C. Dellacherie and P.A. Meyer. *Probabilités et Potentiel, Chapitres V à VIII, Théorie des Martingales.* Hermann, Paris, 1980.

[12] C. Dellacherie and P.A. Meyer. *Probabilités et Potentiel, Chapitres XXVII à XXIV. Processus de Markov(fin). Compléments de calcul stochastique.* Hermann, Paris, 1992.

[13] N. El Karoui, M. Jeanblanc, and Y. Jiao. What happens after a default: the conditional density approach. *Stochastic Processes and their Applications*, 120(7):1011–1032, 2010.

[14] A. Eyraud-Loisel. Bsde with enlarged filtration. Option hedging of an insider trader in a financial market with jumps. *Stochastic Processes and their Appl.*, 115:1745–1763, 2005.

[15] H. Föllmer and P. Imkeller. Anticipation cancelled by a girsanov transformation: a paradox on wiener space. *Ann. Inst. Henri Poincaré*, 29(4):569–586, 1993.

[16] D. Gasbarra, E. Valkeika, and L. Vostrikova. Enlargement of filtration and additional information in pricing models: a bayesian approach. In R. Lipster Yu. Kabanov and editors J. Stoyanov, editors, *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift*, pages 257–286. 2006.

[17] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *IJTAF*, 1(3):331–347, 1998.

[18] A. Grorud and M. Pontier. Asymmetrical information and incomplete markets. *IJTAF*, 4(3):285–302, 2001.

[19] Sh. He, J. Wang, and J. Yan. *Semimartingale theory and stochastic calculus.* CRC Press, 1992.

[20] C. Hillairet. Comparison of insiders' optimal strategies depending on the type of side-information. *Stochastic Processes and their Appl.*, 115:1603–1627, 2005.

[21] P. Imkeller. Random times at which insiders can have free lunches. *Stochastics and Stochastics Reports*, 74:465–487, 2002.

[22] J. Jacod. Grossissement initial, hypothèse (H') et théorème de Girsanov. In *Lecture Notes in Mathematics*, volume 1118. Springer-Verlag, 1985.

[23] M. Jeanblanc and Y. Le Cam. Progressive enlargement of filtrations with initial times. *Stochastic Processes and their Applications*, 119:2523–2543, 2009.

[24] T. Jeulin. Semi-martingales et grossissement d'une filtration. In *Lecture Notes in Mathematics*, volume 833. Springer-Verlag, 1980.

[25] A. Kohatsu-Higa and B. Øksendal. Enlargement of filtration and insider trading. *Preprint*, 2004.

[26] S. Kusuoka. A remark on default risk models. *Advances in Mathematical Economics*, 1:69–82, 1999.

[27] R. Mansuy and M. Yor. *Random Times and (Enlargement of) Filtrations in a Brownian Setting*, volume 1873 of *Lectures Notes in Mathematics.* Springer, 2006.

[28] A. Nikeghbali and M. Yor. Doob's maximal identity, multiplicative decompositions and enlargements of filtrations. In D. Burkholder, editor, *Joseph Doob: A Collection of Mathematical Articles in his Memory*, volume 50 of *Illinois Journal of Mathematics*, pages 791–814. 2007.

[29] P.E. Protter. *Stochastic Integration and Differential Equations, Second edition*, volume Stochastic modelling and applied probability of *Application of Mathematics.* Springer-Verlag, Heidelberg, 2005.

[30] D. Revuz and M. Yor. *Continuous martingales and Brownian motion, 3rd edition*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[31] Sh. Song. *Grossissement de filtration et problèmes connexes*. PhD thesis, Université Paris VI, 1987.

[32] C. Stricker. Quasi-martingales, martingales locales et filtrations naturelles. *Zeitschrift fur Wahr*, 39:55–63, 1977.

[33] C. Stricker and M. Yor. Calcul stochastique dépendant d'un paramètre. *Zeitschrift fur Wahr*, 45:109–133, 1978.

[34] C.-T. Wu. *Construction of Brownian motions in enlarged filtrations and their role in mathematical models of insider trading*. PhD thesis, Humbolt-Universität Berlin, 1999.

[35] M. Yor. Grossissement de filtrations et absolue continuité de noyaux. volume 1118 of *Lecture Notes in Mathematics*. 1985.