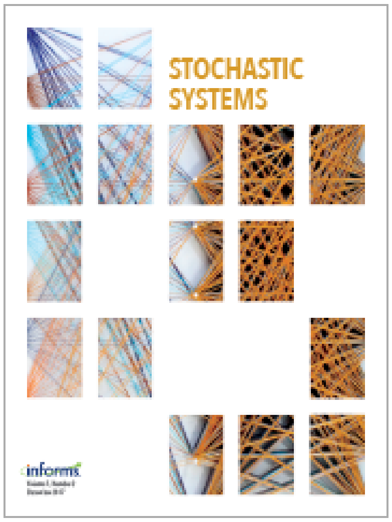

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Heavy-Traffic Analysis of Sojourn Time Under the Foreground–Background Scheduling Policy

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Abstract. We consider the steady-state distribution of the sojourn time of a job entering an M/GI/1 queue with the foreground–background scheduling policy in heavy traffic. The growth rate of its mean as well as the limiting distribution are derived under broad conditions. Assumptions commonly used in extreme value theory play a key role in both the analysis and the results.

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Keywords: M/GI/1 queue • sojourn time • heavy traffic • foreground–background • extreme value theory

1. Introduction

One of the main insights from queueing theory is that the M/GI/1 queue length and sojourn time grow at the order of $1/(1-\rho)$ as the traffic intensity of the system ρ approaches 100% utilization. This insight dates back to Kingman (1961) and Prokhorov (1963) and, appropriately reformulated, remains valid for queueing networks and multiple server queues (Whitt 2002, Gamarnik and Zeevi 2006, Braverman et al. 2017). However, the growth factor can be very different when the scheduling policy is no longer first in first out (FIFO). This observation specifically applies to the foreground–background (FB) algorithm, which we investigate in this paper.

Bansal (2005) was the first to point out that the expected sojourn time (a.k.a. response time, flow time) of a user is of $o(1/(1-\rho))$ in the M/M/1 queue when the scheduling policy is shortest remaining processing time (SRPT). In particular, he showed that the growth factor of the expected sojourn time under SRPT is $\log(1/(1-\rho))$ smaller than the growth factor under FIFO. However, because SRPT requires information on service times in advance, the question was raised if the same growth rate in heavy traffic can be reached with a blind scheduling policy.

Bansal et al. (2018) answered this question negatively for the general GI/GI/1 queueing model. Specifically, the authors showed that, for every blind scheduling policy, there exists a service-time distribution under which the growth rate in heavy traffic of the expected sojourn time is at least a factor $\log(1/(1-\rho))$ larger than the growth rate of SRPT. Bansal et al. (2018) also constructed a scheduling policy that achieves this growth rate, but this policy is rather complicated as it involves randomization.

One might wonder whether the SRPT growth rate can be achieved by a deterministic blind algorithm for specific service-time distributions. To the best of the authors' knowledge, no comprehensive answer to this question has been issued for the GI/GI/1 queue. However, researchers have derived the growth rate of the expected sojourn time in specific queueing models, thereby giving more insight into their behavior and allowing for a comparison with SRPT. On this account, there have been several contributions: for certain M/GI/1 models, there are expected sojourn-time results for the FB (Wierman et al. 2005, Bansal and Gamarnik 2006, Nuyens and Wierman 2008), preemptive shortest job first (Bansal and Gamarnik 2006), and SRPT (Bansal 2005, Lin et al. 2011) scheduling policies. All of these results utilize an explicit expression, focusing on a narrow class of job-size distributions. Furthermore, these results only concern the mean sojourn time, and it is of interest to obtain information about the distribution of the sojourn time as well.

Motivated by these developments, we consider the sojourn-time distribution in the M/GI/1 queue with the FB scheduling policy. Like in previous works, we exploit explicit expression for this distribution but do so for a comprehensive class of job-size distributions, aiming to provide as much insight as possible in how the job-size distribution affects the behavior in heavy traffic. The FB policy operates as follows: priority is given to the customer with the least-attained service, and when multiple customers satisfy this property, they are served at an equal rate. The only heavy-traffic results for FB we are aware of are of "big-O" type and are known in case of deterministic, exponential, Pareto, and specific finite-support service times (Bansal and Gamarnik 2006, Nuyens and Wierman 2008). For deterministic service times, it is straightforward to see that all customers under FB depart in one batch at the end of every busy period, and as a result, the growth rate in heavy traffic in this case $O((1-\rho)^{-2})$ is very poor. The behavior of FB is much better for service-time distributions with a decreasing failure rate as FB then optimizes the expected sojourn time among all blind policies (Righter and Shanthikumar 1989). For more background on the FB policy we refer to the survey by Nuyens and Wierman (2008).

The main results of this paper are of three types:

1. We characterize the exact growth rate (up to a constant independent of ρ) of the sojourn time in heavy traffic under very general assumptions on the service-time distribution. As in Bansal and Gamarnik (2006) and Lin et al. (2011), we find a dichotomy: when the service-time distribution has finite variance, the expected sojourn time $\mathbb{E}[T^\rho_{\mathrm{FB}}] = \Theta(\overline{F}(G^{\leftarrow}(\rho))/(1-\rho)^2)$. Here $\overline{F}(x) = 1 - F(x)$ is the tail of the service-time distribution, and $G^{\leftarrow}$ is the right-inverse of the distribution function of a residual service time; a detailed overview of notation can be found in Section 2. In the infinite variance case, we find that $\mathbb{E}[T^\rho_{\mathrm{FB}}] = \Theta(\log\frac{1}{1-\rho})$. This result is formally stated in Theorem 6. The precise conditions for these results to hold involve Matuszewska indices, a concept that is reviewed in Section 2. The behavior of $\overline{F}(G^{\leftarrow}(\rho))$ is quite rich as is illustrated by several examples.

2. Contrary to the results in Bansal and Gamarnik (2006) and Lin et al. (2011), we have been able to obtain a more precise estimate of the growth rate of $\mathbb{E}[T^\rho_{\mathrm{FB}}]$. It turns out that extreme value theory plays an essential role in our analysis, and the limiting constant factor in front of the growth rate $\overline{F}(G^{\leftarrow}(\rho))/(1-\rho)^2$ crucially depends on in which domain of attraction the service-time distribution is. This result is summarized in Theorem 7 and appended in Theorem 8. When the service-time distribution tail is regularly varying, it is shown that the growth rate of the sojourn time under FB is equal to that of SRPT up to a multiplicative constant. A comparison of the sojourn times under FB and SRPT is given in Corollary 3.

3. When analyzing the distribution, we first show that $T^\rho_{\mathrm{FB}}/\mathbb{E}[T^\rho_{\mathrm{FB}}]$ converges to zero in probability as $\rho \uparrow 1$. To still get a heavy-traffic approximation for $\mathbb{P}(T^\rho_{\mathrm{FB}} > y)$, we state a sample path representation for the sojourn-time distribution for a job that requires a known amount of service. We then use fluctuation theory for spectrally negative Lévy processes to rewrite this representation into an expression that is amenable to analysis; in particular, we obtain a representation for the Laplace transform of the *residual* sojourn-time distribution from which a heavy-traffic limit theorem follows. Finally, this Laplace transform provides an estimate for the tail distribution of T_{FB}.

More specifically, our results show that $\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}} > y)/\overline{F}(G^{\leftarrow}(\rho))$ converges to a nontrivial function $g^*(y)$, for which we give an integral expression in terms of error functions. Along the way, we derive a heavy-traffic limit for the total workload in an M/GI/1 queue with truncated service times that also seems to be of independent interest (see Proposition 1). As in the analysis for the expected sojourn time, ideas from extreme value theory play an important role in the analysis, and the limit function g^* depends on which domain of attraction the service-time distribution falls into. A precise description of this result can be found in Theorem 10.

The function $\overline{F}(G^{\leftarrow}(\rho))$ that shows up in many of our results corresponds to the probability that a customer requires at least $G^{\leftarrow}(\rho)$ units of service. Our analyses indicate that customers who require at least $G^{\leftarrow}(\rho)$ units of service determine the generic sojourn time characteristics, whereas the contribution of smaller customers is negligible. Although not mentioned explicitly, a similar phenomenon (with a different function G) can be observed in the analysis of the mean sojourn time under SRPT by Lin et al. (2011).

Even though our analysis relies on an explicit representation of the sojourn-time distribution, we hope that the insights given by our results (apart from how to separate small and large jobs, also the determination of the right scaling, which we think is not affected by the interarrival time distribution) help to design proofs that do not require explicit expressions. We hope such proofs can also deal with non-Poisson arrival streams and process limit theorems. An example of such a proof for the queue-length process for SRPT with light-tailed job sizes can be found in Puha (2015). A similar comment applies to the extension of our results from FB to a broader class of scheduling disciplines, such as the class of SMART scheduling policies considered in Wierman et al. (2005) and Nuyens et al. (2008). Developing a more probabilistic proof of our result potentially would also clarify the precise role of extreme value theory, which we feel is not entirely clear from the analysis in this paper. Finally, we want to point out that the methodology in our paper does seem to be applicable to the class of size-based scheduling disciplines that is introduced and analyzed in Scully et al. (2018).

The rest of the paper is organized as follows. Section 2 formally introduces the model that is considered. Section 3 presents all our main results on the asymptotic behavior of the expectation and the tail of the sojourn-time distribution under FB. The results concerning the expectation are then proven in Sections 4 and 5, whereas the results on the tail distribution are supported in Sections 6 and 7.

2. Preliminaries

Consider a sequence of M/GI/1 queues, indexed by n, where the ith job requires B_i units of service for all n. For convenience, we say a job that requires x units of service is a *job of size x*. All B_i are independent and identically distributed (i.i.d.) random variables with cumulative distribution function (c.d.f.) $F(x) = \mathbb{P}(B_i \le x)$ and finite mean $\mathbb{E}[B_1]$. We assume that $F(0) = 0$ and denote $x_R := \sup\{x \ge 0 : F(x) < 1\} \le \infty$. Jobs in the nth queue arrive with rate $\lambda^{(n)}$, where $\lambda^{(n)} < 1/\mathbb{E}[B_1]$ to ensure that the nth system experiences traffic intensity $\rho^{(n)} := \lambda^{(n)}\mathbb{E}[B_1] < 1$. For notational convenience, we let B denote a random variable with c.d.f. F.

Let $\overline{F}(x) := 1 - F(x)$ and $F^{\leftarrow}(y) := \inf\{x \ge 0 : F(x) \ge y\}$ denote the complementary c.d.f. (c.c.d.f.) and the right-inverse of F, respectively. The random variable B^* is defined by its c.d.f. $G(x) := \mathbb{P}(B^* \le x) = \int_0^x \overline{F}(t)/\mathbb{E}[B]\,\mathrm{d}t$ and has $(k-1)$th moment $\mathbb{E}[(B^*)^{k-1}] = \mathbb{E}[B^k]/(k\mathbb{E}[B])$. Because $G^{\leftarrow}(y)$ is continuous and strictly increasing, its (right-) inverse $G^{\leftarrow}(y)$ satisfies $G^{\leftarrow}(G(x)) = x$. Also, we recognize $h^*(x) := \frac{\overline{F}(x)}{\mathbb{E}[B]\overline{G}(x)}$ as the failure rate of B^*. One may deduce that $h^*(x)$ equals the reciprocal of the expected residual time; $h^*(x) = 1/\mathbb{E}[B - x \mid B > x]$.

2.1. Foreground–Background Scheduling Policy

Jobs are served according to the FB policy, meaning that, at any moment in time, the server equally shares its capacity over all available jobs that have received the least amount of service thus far. First, we are interested in characteristics of the sojourn time $T_{\mathrm{FB}}^{(n)}$, defined as the duration of time that a generic job spends in the system. In order to analyze this, we consider an expression for the expected sojourn time of a generic job of size x, $\mathbb{E}[T_{\mathrm{FB}}^{(n)}(x)]$, for which Schrage (1967, relation (18)) states that

$$\mathbb{E}\Big[T_{\mathrm{FB}}^{(n)}(x)\Big] = \frac{x}{1-\rho_x^{(n)}} + \frac{\mathbb{E}\big[W^{(n)}(x)\big]}{1-\rho_x^{(n)}} = \frac{x}{1-\rho_x^{(n)}} + \frac{\lambda^{(n)} m_2(x)}{2\Big(1-\rho_x^{(n)}\Big)^2}, \tag{1}$$

where $\rho_x^{(n)} := \lambda^{(n)}\mathbb{E}[B \wedge x] = \rho\mathbb{P}(B^* \le x)$ and $m_2(x) := \mathbb{E}[(B \wedge x)^2] = 2\int_0^x t\overline{F}(t)\,\mathrm{d}t$ are functions of the first and second moments of $B \wedge x := \min\{B, x\}$ and $W^{(n)}(x)$ is the steady-state waiting time in a M/GI/1/FIFO queue with arrival rate $\lambda^{(n)}$ and jobs of size $B_i \wedge x$. As a consequence of (1), the expected sojourn time $\mathbb{E}[T_{\mathrm{FB}}^{(n)}]$ of a generic job is given by

$$\mathbb{E}\Big[T_{\mathrm{FB}}^{(n)}\Big] = \int_0^\infty \frac{x}{1-\rho_x^{(n)}}\,\mathrm{d}F(x) + \int_0^\infty \frac{\lambda^{(n)} m_2(x)}{2\Big(1-\rho_x^{(n)}\Big)^2}\,\mathrm{d}F(x). \tag{2}$$

The intuition behind relation (1) is that a job J_1 of size x experiences a system in which all job sizes are truncated. Indeed, if another job J_2 of size $x + y, y > 0$ has received at least x service, then FB never dedicates its resources to job J_2 while job J_1 is incomplete. The expected sojourn time of job J_1 can now be salvaged from its own service requirement x, the truncated work already in the system upon arrival $W^{(n)}(x)$, and the rate $1-\rho_x^{(n)}$ at which it is expected to be served.

Second, we focus attention on the tail behavior of $T_{\mathrm{FB}}^{(n)}$. Write $X \overset{d}{=} Y$ if the relation $\mathbb{P}(X \le x) = \mathbb{P}(Y \le x)$ is satisfied for all $x \in \mathbb{R}$ and let $\mathcal{L}_x(y)$ denote the time required by the server to empty the system if all job sizes are truncated to $B_i \wedge x$ and the current amount of work is y. The analysis of the tail behavior is then facilitated by relation (4.28) in Kleinrock (1976), stating

$$T_{\mathrm{FB}}^{(n)}(x) \overset{d}{=} \mathcal{L}_x\Big(W_x^{(n)} + x\Big). \tag{3}$$

For both the expectation and tail behavior of $T_{\mathrm{FB}}^{(n)}$, we take specific interest in systems that experience *heavy traffic*, that is, systems in which $\rho^{(n)} \uparrow 1$ as $n \to \infty$. In the current setting, this is equivalent to sequences $\lambda^{(n)}$ that converge to $1/\mathbb{E}[B]$. Most results in this paper make no assumptions on sequence $\lambda^{(n)}$, in which case we drop the superscript n for notational convenience and just state $\rho \uparrow 1$.

The remainder of this section introduces some notation related to Matuszewska indices and extreme value theory.

2.2. Matuszewska Indices

We now introduce the notion of the upper and lower Matuszewska index.

Definition 1. Suppose that $f(\cdot)$ is positive.

• The upper Matuszewska index $\alpha(f)$ is the infimum of those α for which there exists a constant $C = C(\alpha)$ such that, for each $\mu^* > 1$,

$$\lim_{x\to\infty} f(\mu x)/f(x) \le C\mu^{\alpha} \tag{4}$$

uniformly in $\mu \in [1, \mu^*]$ as $x \to \infty$.

• The lower Matuszewska index $\beta(f)$ is the supremum of those β for which there exists a constant $D = D(\beta) > 0$ such that, for each $\mu^* > 1$,

$$\lim_{x\to\infty} f(\mu x)/f(x) \ge D\mu^{\beta} \tag{5}$$

uniformly in $\mu \in [1, \mu^*]$ as $x \to \infty$.

One may note from these definitions that $\beta(f) = -\alpha(1/f)$ holds for any positive f. Intuitively, a function f with upper and lower Matuszewska indices $\alpha(f)$ and $\beta(f)$ is bounded between functions $Dx^{\beta(f)}$ and $Cx^{\alpha(f)}$ for appropriate constants $C, D > 0$. More accurately, however, C and D could be unbounded or vanishing functions of x. Of special interest is the class of functions that satisfy $\beta(f) = \alpha(f)$.

Definition 2. A measurable function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is regularly varying (at infinity) with index $\alpha \in \mathbb{R}$ (denoted by $f \in \mathrm{RV}_{\alpha}$) if, for all $\mu > 0$,

$$\lim_{x\to\infty} f(\mu x)/f(x) = \mu^{\alpha}. \tag{6}$$

If (6) holds with $\alpha = 0$, then f is called slowly varying. If (6) holds with $\alpha = -\infty$, then f is called rapidly varying.

The following result elegantly characterizes functions of regular variation.

Theorem 1 (Bingham et al. 1989, theorem 1.4.1). *A measurable function $f(x)$ is regularly varying with index $\alpha \in \mathbb{R}$ if and only if there exists a slowly varying function $l(x)$ such that $f(x) = l(x)x^{\alpha}$.*

2.3. Extreme Value Theory

The next paragraphs introduce some notions and results from extreme value theory. The field of extreme value theory generally aims to assess the probability of an extreme event; however, for our purposes, we restrict attention to the limiting distribution of $\max\{X_1, \ldots, X_m\}$. A key result on this functional is the Fisher–Tippett theorem:

Theorem 2 (Resnick 1987, proposition 0.3). *Let $(X_m)_{m\in\mathbb{N}}$ be a sequence of i.i.d. random variables and define $M_m := \max\{X_1, \ldots, X_m\}$. If there exist norming sequences $c_m > 0$, $d_m \in \mathbb{R}$, and some nondegenerate H such that*

$$\mathbb{P}\big(c_m^{-1}(M_m - d_m) \le x\big) = F^m(c_m x + d_m) \to H(x) \tag{7}$$

weakly as $m \to \infty$, then H belongs to the type of one of the following three c.d.f.s:

$$\text{Fréchet:}\quad \Phi_{\alpha}(x) = \begin{cases} 0, & x \le 0, \\ \exp\{-x^{-\alpha}\}, & x > 0, \end{cases} \qquad \alpha > 0,$$

$$\text{Weibull:}\quad \Psi_{\alpha}(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x \le 0, \\ 1, & x > 0, \end{cases} \qquad \alpha > 0, \text{ and}$$

$$\text{Gumbel:}\quad \Lambda(x) = \exp\{-e^{-x}\}, \qquad x \in \mathbb{R}.$$

The three distributions are referred to as the extreme value distributions.

A c.d.f. F is said to be in the *maximum domain of attraction of* H if there exist norming sequences c_m and d_m such that (7) holds. In this case, we write $F \in \mathrm{MDA}(H)$. A large body of literature has identified conditions on F such that $F \in \mathrm{MDA}(H)$ and excellent collections of such and related results can be found in Embrechts et al. (1997) and Resnick (1987).

The following theorems show a particularly elegant characterization of the classes $\mathrm{MDA}(\Phi_\alpha)$ and $\mathrm{MDA}(\Psi_\alpha)$ as classes of regularly varying distributions.

Theorem 3 (Embrechts et al. 1997, theorem 3.3.7). *The c.d.f. F belongs to the maximum domain of attraction of $\Phi_\alpha, \alpha > 0$ if and only if $x_R = \infty$ and $\overline{F}$ is regularly varying with index $-\alpha$. If $F \in \mathrm{MDA}(\Phi_\alpha)$, then the norming constants can be chosen as $c_n = F^{\leftarrow}(1 - n^{-1})$ and $d_n = 0$.*

Theorem 4 (Embrechts et al. 1997, theorem 3.3.12). *The c.d.f. F belongs to the maximum domain of attraction of $\Psi_\alpha, \alpha > 0$ if and only if $x_R < \infty$ and $\overline{F}(x_R - (\cdot)^{-1})$ is regularly varying with index $-\alpha$. If $F \in \mathrm{MDA}(\Psi_\alpha)$, then the norming constants can be chosen as $c_n = x_R - F^{\leftarrow}(1 - n^{-1})$ and $d_n = x_R$.*

The class $\mathrm{MDA}(\Lambda)$ is not quite as closely related to regularly varying distributions and can be characterized as follows:

Theorem 5 (Embrechts et al. 1997, theorem 3.3.26). *The c.d.f. F with right end point $x_R \le \infty$ belongs to the maximum domain of attraction of Λ if and only if there exists some $z < x_R$ such that F has representation*

$$\overline{F}(x) = c(x)\exp\left\{-\int_z^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right\}, \quad z < x < x_R, \tag{8}$$

where c and g are measurable functions satisfying $c(x) \to c > 0, g(x) \to 1$ as $x \uparrow x_R$ and $f(\cdot)$ is a positive, absolutely continuous function (with respect to the Lebesgue measure) with density $f'(x)$ having $\lim_{x\uparrow x_R} f'(x) = 0$.

If $F \in \mathrm{MDA}(\Lambda)$, then the norming constants can be chosen as $c_m = f(d_m)$ and $d_m = F^{\leftarrow}(1 - m^{-1})$. A possible choice for the function $f(\cdot)$ is $f(\cdot) = 1/h^*(\cdot)$.

The function $f(\cdot)$ in the preceding definition is unique up to asymptotic equivalence. We refer to f as the *auxiliary function* of $\overline{F}$. Also, we note the following property of $f(\cdot)$:

Lemma 1 (Resnick 1987, lemma 1.2). *Suppose that $f(\cdot)$ is an absolutely continuous auxiliary function with $f'(x) \to 0$ as $x \uparrow x_R$.*

i. *If $x_R = \infty$, then $\lim_{x\to\infty} \frac{f(x)}{x} = 0$.*

ii. *If $x_R < \infty$, then $\lim_{x\uparrow x_R} \frac{f(x)}{x_R - x} = 0$.*

Although $\mathrm{MDA}(\Lambda)$ does not coincide with a class of regularly varying distributions, the following lemma shows that it is related to the class of rapidly varying distributions.

Corollary 1 (Embrechts et al. 1997, corollary 3.3.32). *Assume that $F \in \mathrm{MDA}(\Lambda)$. If $x_R = \infty$, then $\overline{F} \in \mathrm{RV}_{-\infty}$. If $x_R < \infty$, then $\overline{F}(x_R - (\cdot)^{-1}) \in \mathrm{RV}_{-\infty}$.*

This section's final lemma presents a useful property for c.d.f.s in $\mathrm{MDA}(\Lambda)$:

Lemma 2. *Suppose that the c.d.f. F is in $\mathrm{MDA}(\Lambda)$ and let $G(x) = \int_0^x \overline{F}(t)/\mathbb{E}[B]\,\mathrm{d}t$. Then $G \in \mathrm{MDA}(\Lambda)$ and any auxiliary function for F is also an auxiliary function for G.*

Proof. According to theorem 3.3.27 in Embrechts et al. (1997), $G \in \mathrm{MDA}(\Lambda)$ with auxiliary function $f(\cdot)$ if and only if $\lim_{x\uparrow x_R} \overline{G}(x + tf(x))/\overline{G}(x) = e^{-t}$ for all $t \in \mathbb{R}$. It is straightforward to check that the preceding relation holds for any auxiliary function $f(\cdot)$ of F by using l'Hôpital's rule and $\lim_{x\uparrow x_R} f'(x) = 0$. □

2.4. Asymptotic Relations

Let $f(\cdot)$ and $g(\cdot)$ denote two positive functions and X and Y two random variables. We write $f \sim g$ if $\lim_{z\uparrow z^*} f(z)/g(z) = 1$, where the appropriate limit $z \uparrow z^*$ should be clear from the context; it usually equals $x \uparrow x_R$ or $\rho \uparrow 1$. Similarly, we adopt the conventions $f = o(g)$ if $\limsup_{z\uparrow z^*} f(z)/g(z) = 0$, $f = O(g)$ if $\limsup_{z\uparrow z^*} f(z)/g(z) < \infty$, and $f = \Theta(g)$ if $0 < \liminf_{z\uparrow z^*} f(z)/g(z) \le \limsup_{z\uparrow z^*} f(z)/g(z) < \infty$. We write $X \le_{st} Y$ if the relation $P(X > x) \le \mathbb{P}(Y > x)$ is satisfied for all $x \in \mathbb{R}$.

Finally, the complementary error function is defined as $\mathrm{Erfc}(x) := 2\pi^{-1/2}\int_x^\infty e^{-u^2}\,\mathrm{d}u$.

3. Main Results and Discussion

This section presents and discusses our main results. Theorems 6 and 7 consider the asymptotic behavior of the expected sojourn time $\mathbb{E}[T_{\mathrm{FB}}]$ for various classes of service-time distributions. Theorem 8 connects the asymptotic behavior of $\overline{F}(G^{\leftarrow}(\rho))$ to the literature on extreme value theory. As a consequence, the expressions obtained in Theorem 7 can be specified for many distributions in $\mathrm{MDA}(\Lambda)$. Theorem 9 shifts focus to the distribution of T_{FB} and states that the scaled sojourn time $T_{\mathrm{FB}}/\mathbb{E}[T_{\mathrm{FB}}]$ tends to zero in probability. Instead, Theorem 10 shows that a certain fraction of jobs experiences a sojourn time of order $(1-\rho)^{-2}$. This result is achieved through the Laplace transform of the remaining sojourn time T^*_{FB}, for which we give an integral representation. The proofs of the theorems are postponed to later sections.

Recall that $\overline{F}(G^{\leftarrow}(\rho)) = \mathbb{E}[B](1-\rho)h^*(G^{\leftarrow}(\rho))$. Our first theorem presents the growth rate of $\mathbb{E}[T_{\mathrm{FB}}]$.

Theorem 6. *Assume that either $x_R = \infty$ and $-\infty < \beta(\overline{F}) \le \alpha(\overline{F}) < -2$, or $x_R < \infty$ and $-\infty < \beta(\overline{F}(x_R - (\cdot)^{-1})) \le \alpha(\overline{F}(x_R - (\cdot)^{-1})) < 0$. Then the relations*

$$\mathbb{E}[T_{\mathrm{FB}}] = \Theta\left(\frac{\overline{F}(G^{\leftarrow}(\rho))}{(1-\rho)^2}\right) = \Theta\left(\frac{h^*(G^{\leftarrow}(\rho))}{1-\rho}\right) \tag{9}$$

hold as $\rho \uparrow 1$, where $\lim_{\rho\uparrow 1} h^*(G^{\leftarrow}(\rho)) = 0$ if $x_R = \infty$ and $\lim_{\rho\uparrow 1} h^*(G^{\leftarrow}(\rho)) = \infty$ if $x_R < \infty$. Alternatively, assume $x_R = \infty$ and $\beta(\overline{F}(x)) > -2$. Then the relation

$$\mathbb{E}[T_{\mathrm{FB}}] = \Theta\left(\log \frac{1}{1-\rho}\right) \tag{10}$$

holds as $\rho \uparrow 1$.

Theorem 6 shows that the behavior of $\mathbb{E}[T_{\mathrm{FB}}]$ is fundamentally different for $\alpha(\overline{F}) < -2$ and $\beta(\overline{F}(x)) > -2$. In the first case, the variance of B_1 is bounded, and therefore, the expected remaining busy period duration is of order $\Theta((1-\rho)^{-2})$. Our analysis roughly shows that all jobs of size $G^{\leftarrow}(\rho)$ and larger remain in the system until the end of the busy period and, hence, experience a sojourn time of order $\Theta((1-\rho)^{-2})$. The threshold $G^{\leftarrow}(\rho)$ itself originates as the solution of $1-\rho_x = 1-\rho^2$, which indicates that—as the traffic intensity increases to unity—jobs of size at least $G^{\leftarrow}(\rho)$ experience a truncated system that is almost as heavily congested as the nontruncated system. The theorem indicates that these jobs determine the asymptotic growth of the overall expected sojourn time.

This argumentation does not apply in case $\beta(\overline{F}(x)) > -2$ because then the expected remaining busy period duration is infinite. It turns out that, in this case, the expected sojourn time of a large job of size x is of the same order as the time that the job is in service, which has expectation $x/(1-\rho_x)$. The result follows after integrating over the service-time distribution.

Additionally, it can be shown that the statements in Theorem 6 also hold if $F \in \mathrm{MDA}(\Lambda)$, which is a special case of either $\alpha(\overline{F}) = \beta(\overline{F}) = -\infty$ or $\alpha(\overline{F}(x_R - (\cdot)^{-1})) = \beta(\overline{F}(x_R - (\cdot)^{-1}) = -\infty$ (cf. Corollary 1). In this case as well as in the case when $\overline{F}(\cdot)$ or $\overline{F}(x_R - (\cdot)^{-1})$ is regularly varying one can show that $(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]/\overline{F}(G^{\leftarrow}(\rho))$ converges. Theorem 7 specifies Theorem 6 for the aforementioned cases as well as for distributions with an atom in their end point.

Theorem 7. *The following relations hold as $\rho \uparrow 1$:*

i. *If $F \in \mathrm{MDA}(\Phi_\alpha), \alpha \in (1,2)$, then $\mathbb{E}[T_{\mathrm{FB}}] \sim \frac{\alpha}{2-\alpha}\mathbb{E}[B]\log\frac{1}{1-\rho}$.*

ii. *If $F \in \mathrm{MDA}(H)$, then $\mathbb{E}[T_{\mathrm{FB}}] \sim \frac{r(H)\mathbb{E}[B^*]\overline{F}(G^{\leftarrow}(\rho))}{(1-\rho)^2} = \frac{r(H)\mathbb{E}[B^2]h^*(G^{\leftarrow}(\rho))}{2(1-\rho)}$, where*

$$r(H) = \begin{cases} \frac{\pi/(\alpha-1)}{\sin(\pi/(\alpha-1))}\frac{\alpha}{\alpha-1} & \text{if } H = \Phi_\alpha, \alpha > 2, \\ 1 & \text{if } H = \Lambda, \text{ and} \\ \frac{\pi/(\alpha+1)}{\sin(\pi/(\alpha+1))}\frac{\alpha}{\alpha+1} & \text{if } H = \Psi_\alpha, \alpha > 0. \end{cases} \tag{11}$$

Additionally, if $H = \Phi_\alpha, \alpha > 2$, then $\lim_{\rho\uparrow 1} h^*(G^{\leftarrow}(\rho)) = 0$, whereas if either $H = \Lambda$ and $x_R < \infty$ or if $H = \Psi_\alpha, \alpha > 0$, then $\lim_{\rho\uparrow 1} h^*(G^{\leftarrow}(\rho)) = \infty$.

iii. *If F has an atom in $x_R < \infty$, say $\lim_{\delta\downarrow 0} \overline{F}(x_R - \delta) = p > 0$, then $\mathbb{E}[T_{\mathrm{FB}}] \sim \frac{p\mathbb{E}[B^*]}{(1-\rho)^2}$.*

The expressions in Theorems 6 and 7 give insight into the asymptotic behavior of $\mathbb{E}[T_{\mathrm{FB}}]$. The following corollary shows that these asymptotic expressions may be specified further if the service times are Pareto distributed. This extends the result by Bansal and Gamarnik (2006), who derived the growth factor of $\mathbb{E}[T_{\mathrm{FB}}]$ but not the exact asymptotics.

Corollary 2. *Assume $\overline{F}(x) = (x/x_L)^{-\alpha}, x \ge x_L$. Then the relations*

$$\mathbb{E}[T_{\mathrm{FB}}] \sim \begin{cases} \frac{\alpha}{2-\alpha}\mathbb{E}[B]\log\frac{1}{1-\rho} & \text{if } \alpha \in (1,2), \\ \frac{\pi/(\alpha-1)}{2\sin(\pi/(\alpha-1))}\frac{\mathbb{E}[B^2]\alpha^{\frac{\alpha}{\alpha-1}}}{x_L(1-\rho)^{\frac{\alpha-2}{\alpha-1}}} & \text{if } \alpha \in (2,\infty), \end{cases} \tag{12}$$

hold as $\rho \uparrow 1$.

Proof. One may derive that $\overline{G}(x) = \frac{1}{\alpha}\left(\frac{x}{x_L}\right)^{1-\alpha}$ for $x \geq x_L$. Consequentially, one deduces that $h^*(x) = \frac{\alpha-1}{x}$ for $x \geq x_L$ and $G^{\leftarrow}(\rho) = x_L(\alpha(1-\rho))^{\frac{-1}{\alpha-1}}$ for $\rho \geq 1 - 1/\alpha$. The result then follows from Theorem 7. □

Corollary 2 exemplifies that the asymptotic growth of $\mathbb{E}[T_{\mathrm{FB}}]$ can be specified in some cases. However, it is often nontrivial to analyze the behavior of $\overline{F}(G^{\leftarrow}(\rho))$ or equivalently $h^*(G^{\leftarrow}(\rho))$. Theorem 8 aims to overcome this problem if $F \in \mathrm{MDA}(\Lambda)$ by presenting a relation between $h^*(G^{\leftarrow}(\rho))$ and norming constants c_n of F, which can often be found in the large body of literature on extreme value theory.

Theorem 8. *Assume $F \in \mathrm{MDA}(\Lambda)$ and $x_R = \infty$ and let c_m and d_m be such that $F^m(c_m x + d_m) \to \Lambda(x)$ weakly as $m \to \infty$. Define $\lambda^{(n)} = (1-n^{-1})/\mathbb{E}[B]$ so that $\rho^{(n)} = 1 - n^{-1}$.*

i. *If there exists $\alpha > 0$ and a slowly varying function $l(x)$ such that $-\log \overline{F}(x) \sim l(x)x^{\alpha}$ as $x \to \infty$, then $h^*(x) \sim \alpha l(x) x^{\alpha-1}$ if and only if*

$$\inf_{\lambda \downarrow 1} \liminf_{x\to\infty} \inf_{t\in[1,\lambda]} \{\log h^*(tx) - \log h^*(x)\} \geq 0. \tag{13}$$

If (13) holds, then $\mathbb{E}[T_{\mathrm{FB}}^{(n)}] \sim \frac{\mathbb{E}[B^2]}{2(1-\rho^{(n)})c_n}$ as $n \to \infty$.

ii. *If there exists a function $l(x) : [0,\infty) \to \mathbb{R}$, $\liminf_{x\to\infty} l(x) > 1$ such that, for all $\lambda > 0$,*

$$\lim_{x\to\infty} \frac{-\log \overline{F}(\lambda x) + \log \overline{F}(x)}{l(x)} = \log(\lambda) \tag{14}$$

and $L = \lim_{x\to\infty} \frac{\log(x)}{l(x)}$ exists in $[0,\infty]$, then $\lim_{n\to\infty} \frac{2(1-\rho^{(n)})c_n}{\mathbb{E}[B^2]} \mathbb{E}[T_{\mathrm{FB}}^{(n)}] = e^{-L}$.

The same results hold if $x_R < \infty$ provided that the $\overline{F}(\cdot)$ and $h^*(\cdot)$ in (i) *and* (ii) *are replaced by $\overline{F}(x_R - (\cdot)^{-1})$ and $(\cdot)^{-2}h^*(x_R - (\cdot)^{-1})$, respectively.*

Remark 1. Condition (13) in part (i) of Theorem 8 is a *Tauberian condition* and originates from theorem 1.7.5 in Bingham et al. (1989). A Tauberian theorem makes assumptions on a transformed function (here h^*) and uses these assumptions to deduce the asymptotic behavior of that transform. The interested reader is referred to section 1.7 in Bingham et al. (1989) or section XIII.5 in Feller (1971).

Theorem 5 implies that c_n is asymptotically equivalent to $1/h^*(G^{\leftarrow}(1-n^{-1}))$ for many distributions in $\mathrm{MDA}(\Lambda)$. As c_n may be chosen as $1/h^*(F^{\leftarrow}(1-n^{-1}))$, Theorem 8 implicitly states conditions under which $\lim_{n\to\infty} h^*(G^{\leftarrow}(1-n^{-1}))/h^*(F^{\leftarrow}(1-n^{-1})) = \lim_{y\uparrow 1}(1-y)^{-2}\overline{F}(G^{\leftarrow}(y))\overline{G}(F^{\leftarrow}(y))$ exists and exploits this limit to write $\mathbb{E}[T_{\mathrm{FB}}^{(n)}]$ as a function of c_n rather than of the generally unknown function $h^*(G^{\leftarrow}(1-n^{-1}))$. To illustrate the implications of Theorem 8, the exact asymptotic behavior of several well-known distributions is presented in Table 1.

We take a brief moment to compare the asymptotic expected sojourn time under FB to that under SRPT in M/GI/1 models. Clearly, FB can perform no better than SRPT because of SRPT's optimality (Schrage 1968).

Table 1. Asymptotic Expressions for the Expected Sojourn Time for Several Well-Known Distributions in MDA(Λ), Characterized by Either Their Tail Distribution or Their Probability Density Function (p.d.f.)

Distribution	c.c.d.f. $\overline{F}$ or p.d.f. F'		L	$\mathbb{E}[T_{\mathrm{FB}}] \sim$
Exponential-like	$\overline{F}(x) \sim Ke^{-\mu x}$	$K, \mu > 0$	–	$\frac{\mathbb{E}[B^2]\mu}{2(1-\rho)}$
Weibull-like	$\overline{F}(x) \sim Kx^{\alpha}e^{-\mu x^{\beta}}$	$K, \mu, \beta > 0, \alpha \in \mathbb{R}$	–	$\frac{\beta\mu^{1/\beta}\mathbb{E}[B^2]}{2(1-\rho)\log\left(\frac{1}{1-\rho}\right)^{1/\beta-1}}$
Gamma	$F'(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$	$\alpha, \beta > 0$	–	$\frac{\mathbb{E}[B^2]\beta}{2(1-\rho)}$
Normal	$F'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$		–	$\frac{\mathbb{E}[B^2]\log\left(\frac{1}{1-\rho}\right)^{1/2}}{\sqrt{2}(1-\rho)}$
Lognormal	$F'(x) = \frac{1}{\sqrt{2\pi}\sigma x}e^{-(\log(x)-\mu)^2/(2\sigma^2)}$	$\sigma > 0, \mu \in \mathbb{R}$	σ^2	$\frac{\mathbb{E}[B^2]\log\left(\frac{1}{1-\rho}\right)^{1/2}}{\sigma\sqrt{2}(1-\rho)\exp\left[\mu+\sigma\left(\sigma+\sqrt{2\log\left(\frac{1}{1-\rho}\right)}-\frac{\log(4\pi)+\log\log(1/(1-\rho))}{2\sqrt{2\log(1/(1-\rho))}}\right)\right]}$
Finite exponential	$\overline{F}(x) = Ke^{-\frac{\mu}{x_R-x}}$	$K, \mu > 0, x < x_R$	–	$\frac{\mathbb{E}[B^2]\log\left(\frac{1}{1-\rho}\right)^2}{2\mu(1-\rho)}$
Benktander-I	$\overline{F}(x) = \left(1 + 2\frac{\beta}{\alpha}\log(x)\right) \times e^{-(\beta\log(x)^2+(\alpha+1)\log(x))}$	$\alpha, \beta > 0, x > 1$	$\frac{1}{2\beta}$	$\frac{\mathbb{E}[B^2]\sqrt{\beta\log\left(\frac{1}{1-\rho}\right)}}{(1-\rho)\exp\left[-\frac{\alpha+1/2}{\beta}+\sqrt{\frac{\log\left(\frac{1}{1-\rho}\right)}{\beta}}\right]}$
Benktander-II	$\overline{F}(x) = x^{-(1-\beta)}e^{-\frac{\alpha}{\beta}(x^{\beta}-1)}$	$\alpha > 0, 0 < \beta < 1, x > 1$	–	$\frac{\alpha^{1/\beta}\mathbb{E}[B^2]}{2\beta^{1/\beta-1}(1-\rho)\log\left(\frac{1}{1-\rho}\right)^{1/\beta-1}}$

Note. These expressions follow from table 3.4.4 in Embrechts et al. (1997) through Theorem 8, where it is assumed that relation (13) holds.

The ratio of their respective expected sojourn time is shown to be unbounded if the service times are exponentially distributed or if the service-time distribution has finite support (Kleinrock 1976, Bansal 2005, Nuyens and Wierman 2008, Lin et al. 2011) but bounded if the service times are Pareto distributed (Bansal and Gamarnik 2006, Lin et al. 2011). To the best of the authors' knowledge, no results of this nature are known if service times are Weibull distributed.

The following corollary specifies the asymptotic advantage of SRPT over FB if the service times are Pareto distributed and presents the first such results for Weibull-distributed service times. Its statements follow directly from corollaries 1 and 2 in Lin et al. (2011) and the results earlier in this section. Further results may be obtained by analyzing their function $G^{-1}(\rho)$ for other service-time distributions.

Corollary 3. *The following relations hold as* $\rho \uparrow 1$:

i. *If* $\overline{F}(x) = (x/x_L)^{-\alpha}, x \geq x_L > 0$ *and* $\alpha \in (1,2)$, *then* $\mathbb{E}[T_{\mathrm{FB}}]/\mathbb{E}[T_{\mathrm{SRPT}}] \sim \alpha^2$.

ii. *If* $\overline{F}(x) = (x/x_L)^{-\alpha}, x \geq x_L > 0$ *and* $\alpha > 2$, *then* $\mathbb{E}[T_{\mathrm{FB}}]/\mathbb{E}[T_{\mathrm{SRPT}}] \sim \alpha^{\frac{\alpha}{\alpha-1}}$.

iii. *If* $\overline{F}(x) = e^{-\mu x^{\beta}}, x \geq 0$ *and* $\mu, \beta > 0$, *then* $\mathbb{E}[T_{\mathrm{FB}}]/\mathbb{E}[T_{\mathrm{SRPT}}] \sim \beta \log(\frac{1}{1-\rho})$.

On the other hand, we may also compare FB to the classic FIFO policy (Conway et al. 1967). Because $\mathbb{E}[T_{\mathrm{FIFO}}] = \mathbb{E}[B] + \rho\mathbb{E}[B^*]/(1-\rho)$, Theorems 6 and 7 indicate that FB performs better than FIFO if the service-time distribution has a heavy tail and also that FB performs worse than FIFO if the service-time distribution has finite support. If the service-time distribution has infinite support but no heavy tail, then Table 1 shows that their relationship depends on the tail of the service-time distribution. This is exemplified by Weibull-distributed service times, $\overline{F}(x) = e^{-\mu x^{\beta}}, \beta > 0$, in which case $\mathbb{E}[T_{\mathrm{FB}}]/\mathbb{E}[T_{\mathrm{FIFO}}] \sim \beta\Gamma(1+1/\beta)(\mu^{-1}\log 1/(1-\rho))^{1-1/\beta}$. In fact, Table 1 seems to suggest that FB outperforms FIFO if $-\log \overline{F}(x)/x \to 0$ as $x \to \infty$ and vice versa if $-\log \overline{F}(x)/x \to 0$ as $x \to \infty$. However, investigating this observation is beyond the scope of this paper.

Now that the asymptotic behavior of the expected sojourn time under FB has been quantified, it is natural to investigate more complex characteristics. One such characteristic is the behavior of the tail of the sojourn-time distribution, in which one usually starts by analyzing the distribution of the sojourn time normalized by its mean, $T_{\mathrm{FB}}/\mathbb{E}[T_{\mathrm{FB}}]$. The following theorem indicates that this random variable converges to zero in probability, meaning that almost every job experiences a sojourn time that is significantly shorter than the expected sojourn time as $\rho \uparrow 1$.

Theorem 9. *If either*

i. $x_R = \infty$ *and either* $\beta(\overline{F}) > -2$ *or* $-\infty < \beta(\overline{F}) \leq \alpha(\overline{F}) < -2$ *or*

ii. $x_R < \infty$ *and* $-\infty < \beta(\overline{F}(x_R - (\cdot)^{-1})) \leq \alpha(\overline{F}(x_R - (\cdot)^{-1})) < 0$ *or*

iii. $F \in \mathrm{MDA}(\Lambda)$,

then $\frac{T_{\mathrm{FB}}}{\mathbb{E}[T_{\mathrm{FB}}]} \xrightarrow{p} 0$ *as* $\rho \uparrow 1$.

Theorem 9 indicates that a decreasing fraction of jobs experiences a sojourn time of at least duration $\mathbb{E}[T_{\mathrm{FB}}]$. Our final main result aims to specify both the size of this fraction and the growth factor of the associated jobs' sojourn time.

The intuition behind Theorem 6 suggests that T_{FB} scales as $(1-\rho)^{-2}$ but only for jobs of size at least $G^{\leftarrow}(\rho)$. This makes it conceivable that the scaled probability $\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}} > y)/\overline{F}(G^{\leftarrow}(\rho))$ may be of $\Theta(1)$ as $\rho \uparrow 1$. Theorem 10 confirms this hypothesis and additionally shows that the residual sojourn time T^*_{FB} with density $\mathbb{P}(T_{\mathrm{FB}} > x)/\mathbb{E}[T_{\mathrm{FB}}]$ scales as $(1-\rho)^{-2}$.

Theorem 10. *Assume* $F \in \mathrm{MDA}(H)$, *where* H *is an extreme value distribution with a finite* $(2+\varepsilon)$*th moment for some* $\varepsilon > 0$. *Let* $r(H)$ *be as in relation* (11). *Then* $(1-\rho)^2 T^*_{\mathrm{FB}}$ *converges to a nondegenerate random variable with monotone density* g^* *as* $\rho \uparrow 1$, *and*

$$\lim_{\rho \uparrow 1} \frac{\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}} > y)}{r(H)\mathbb{E}[B^*]\overline{F}(G^{\leftarrow}(\rho))} = g^*(y) \tag{15}$$

almost everywhere. Here,

$$g^*(t) = \int_0^1 8r(H)^{-1}\nu\left(\frac{1-\nu}{\nu}\right)^{p(H)} g(t,\nu)\,\mathrm{d}\nu, \tag{16}$$

$$g(t,\nu) = \frac{e^{-\frac{t}{4\mathbb{E}[B^*]\nu^2}}}{4\mathbb{E}[B^*]\nu^2}\left(\frac{\sqrt{t}}{\nu\sqrt{\pi\mathbb{E}[B^*]}} - \frac{t}{2\mathbb{E}[B^*]\nu^2} e^{\frac{t}{4\mathbb{E}[B^*]\nu^2}}\mathrm{Erfc}\left(\frac{1}{2\nu}\sqrt{\frac{t}{\mathbb{E}[B^*]}}\right)\right), \tag{17}$$

and $p(H) = \frac{\alpha}{\alpha-1}$ *if* $H = \Phi_\alpha$, $\alpha > 2$; $p(H) = 1$ *if* $H = \Lambda$ *and* $p(H) = \frac{\alpha}{\alpha+1}$ *if* $H = \Psi_\alpha$, $\alpha > 0$.

All theorems presented in this section are now proven in order. First, Theorems 6 and 7 are proven in Section 4. Then, Theorem 8 is justified in Section 5. Finally, Sections 6 and 7 respectively validate Theorems 9 and 10.

4. Asymptotic Behavior of the Expected Sojourn Time

In this section, we prove Theorems 6 and 7 in order. The intuition behind the theorems is that jobs of size x can only be completed once the server has finished processing all jobs of size at most x. Additionally, jobs of size x experience a system with job sizes $B_i \wedge x$ because no job receives more than x units of processing as long as there are size x jobs in the system. One, thus, expects all jobs of size x to stay in the system for the duration of a remaining busy period in the truncated system, which is expected to last for $\Theta(\mathbb{E}[(B \wedge x)^2]/(1-\rho_x)^2)$ time.

Now, if $\mathbb{E}[B^2] < \infty$ and x_ρ^ν is such that $(1-\rho)/(1-\rho_{x_\rho^\nu}) = \nu \in (1-\rho, 1)$, then one can see from (1) that

$$(1-\rho)^2 \mathbb{E}\left[T_{\mathrm{FB}}\left(x_\rho^\nu\right)\right] = \nu(1-\rho)x_\rho^\nu + \nu^2 \frac{\lambda m_2\left(x_\rho^\nu\right)}{2}. \tag{18}$$

It turns out that the asymptotic behavior of $(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]$ is now determined by the fraction of jobs for which ν takes values away from zero.

If, instead, $\mathbb{E}[B^2] = \infty$, it is shown that the growth rate of the second term in (1) is bounded by the growth rate of $x\overline{G}(x)$. It then turns out that the sojourn time is of the same order as the time that a job receives service, which is of order $\Theta(x/(1-\rho_x))$.

Both theorems follow after integrating $\mathbb{E}[T_{\mathrm{FB}}(x)]$ over all possible values of x, as shown in (2). By integrating by parts, we find that the first integral in (2) can be rewritten as

$$\begin{aligned}\int_0^\infty \frac{x}{1-\rho_x}\,\mathrm{d}F(x) &= \int_0^\infty \frac{\overline{F}(x)}{1-\rho_x}\,\mathrm{d}x + \lambda\int_0^\infty \frac{x\overline{F}(x)^2}{(1-\rho_x)^2}\,\mathrm{d}x \\ &= \frac{1}{\lambda}\log\frac{1}{1-\rho} + \lambda\int_0^\infty \frac{x\overline{F}(x)^2}{(1-\rho_x)^2}\,\mathrm{d}x.\end{aligned}$$

Similarly, the second integral can be rewritten as

$$\int_0^\infty \frac{\lambda m_2(x)}{2(1-\rho_x)^2}\,\mathrm{d}F(x) = \lambda\int_0^\infty \frac{x\overline{F}(x)^2}{(1-\rho_x)^2}\,\mathrm{d}x + \lambda^2\int_0^\infty \frac{m_2(x)\overline{F}(x)^2}{(1-\rho_x)^3}\,\mathrm{d}x,$$

and, therefore,

$$\begin{aligned}\mathbb{E}[T_{\mathrm{FB}}] &= \frac{1}{\lambda}\log\frac{1}{1-\rho} + 2\lambda\int_0^\infty \frac{x\overline{F}(x)^2}{(1-\rho_x)^2}\,\mathrm{d}x + \lambda^2\int_0^\infty \frac{m_2(x)\overline{F}(x)^2}{(1-\rho_x)^3}\,\mathrm{d}x \\ &= \frac{\mathbb{E}[B]}{\rho}\log\frac{1}{1-\rho} + 2\rho\int_0^\infty \frac{x\overline{F}(x)}{(1-\rho_x)^2}\,\mathrm{d}G(x) + \frac{\rho^2}{\mathbb{E}[B]}\int_0^\infty \frac{m_2(x)\overline{F}(x)}{(1-\rho_x)^3}\,\mathrm{d}G(x).\end{aligned} \tag{19}$$

We will now derive Theorems 6 and 7 from this relation.

4.1. General Matuszewska Indices

This section proves Theorem 6. Relation (19) is analyzed separately for the cases $-\infty < \beta(\overline{F}) \le \alpha(\overline{F}) < -2$ and $-2 < \beta(\overline{F}) \le \alpha(\overline{F}) < 1$, which are referred to as the finite and the infinite variance case, respectively. The finite variance case also considers $-\infty < \beta\left(\overline{F}(x_R - (\cdot)^{-1}\right)$. Note that we always have $\beta(\overline{F}(x_R - (\cdot)^{-1})) \le \alpha(\overline{F}(x_R - (\cdot)^{-1})) \le 0$ because $\overline{F}(x_R - (\cdot)^{-1})$ is nonincreasing. Prior to further analysis, however, we introduce several results that facilitate the analysis.

Lemma 3. *Let $f_1(\cdot)$, $f_2(\cdot)$ be positive.*

i. *If $\alpha(f_1), \alpha(f_2) < \infty$, then $\alpha(f_1 \cdot f_2) \le \alpha(f_1) + \alpha(f_2)$ and, assuming that f_1 is nondecreasing, $\alpha(f_1 \circ f_2) \le \alpha(f_1)\cdot\alpha(f_2)$.*

ii. *If $\beta(f_1), \beta(f_2) > -\infty$, then $\beta(f_1 \cdot f_2) \ge \beta(f_1) + \beta(f_2)$, and assuming that f_1 is nonincreasing, $\beta(f_1 \circ f_2) \ge \beta(f_1)\cdot\beta(f_2)$.*

Lemma 4. *Let f be positive. If $\alpha(f) < 0$, then $\lim_{x\to\infty} f(x) = 0$.*

Lemma 5 (Bingham et al. 1989, theorem 2.6.1). *Let f be positive and locally integrable on $[X, \infty)$. Let $g(x) := \int_X^x f(t)/t\,\mathrm{d}t$. If $\beta(f) > 0$, then $\liminf_{x\to\infty} f(x)/g(x) > 0$.*

Lemma 6 (Bingham et al. 1989, theorem 2.6.3). *Let f be positive and measurable. Let $g(x) := \int_x^\infty f(t)/t\,\mathrm{d}t$.*

i. *If $\alpha(f) < 0$, then $g(x) < \infty$ for all large x.*

ii. *If $\beta(f) > -\infty$, then $\limsup_{x\to\infty} f(x)/g(x) < \infty$.*

Lemma 7. *If* $x_R = \infty$, *then* $\alpha(\overline{G}) \le \alpha(\overline{F}) + 1$ *and* $\beta(\overline{G}) \ge \beta(\overline{F}) + 1$. *If* $x_R < \infty$, *then* $\alpha(\overline{G}(x_R - (\cdot)^{-1})) \le \alpha(\overline{F}(x_R - (\cdot)^{-1})) - 1$ *and* $\beta(\overline{G}(x_R - (\cdot)^{-1})) \ge \beta(\overline{F}(x_R - (\cdot)^{-1})) - 1$.

Lemma 8. *If* $x_R = \infty$ *and* $\beta(\overline{F}) > -\infty$, *then* $\alpha(G^{\leftarrow}(1 - (\cdot)^{-1})) \le -\frac{1}{\alpha(\overline{F})+1}$ *and* $\beta(G^{\leftarrow}(1 - (\cdot)^{-1})) \ge -\frac{1}{\beta(\overline{F})+1}$. *Alternatively, if* $x_R < \infty$ *and* $\beta(\overline{F}(x_R - (\cdot)^{-1})) > -\infty$, *then* $\alpha(G^{\leftarrow}(1 - (\cdot)^{-1})) \le -\frac{1}{\alpha(\overline{F}(x_R-(\cdot)^{-1}))-1}$ *and* $\beta(G^{\leftarrow}(1 - (\cdot)^{-1})) \ge -\frac{1}{\beta(\overline{F}(x_R-(\cdot)^{-1}))-1}$.

Corollary 4. *If* $x_R = \infty$ *and* $\beta(\overline{F}) > -\infty$, *then* $\alpha(\overline{F}(G^{\leftarrow}(1 - (\cdot)^{-1}))) \le \frac{-\alpha(\overline{F})}{\alpha(\overline{F})+1}$ *and* $\beta(\overline{F}(G^{\leftarrow}(1 - (\cdot)^{-1}))) \ge \frac{-\beta(\overline{F})}{\beta(\overline{F})+1}$. *Alternatively, if* $x_R < \infty$ *and* $\beta(\overline{F}(x_R - (\cdot)^{-1})) > -\infty$, *then* $\alpha(\overline{F}(G^{\leftarrow}(1 - (\cdot)^{-1}))) \le \frac{-\alpha(\overline{F}(x_R-(\cdot)^{-1}))}{\alpha(\overline{F}(x_R-(\cdot)^{-1}))-1}$ *and* $\beta(\overline{F}(G^{\leftarrow}(1 - (\cdot)^{-1}))) \ge \frac{-\beta(\overline{F}(x_R-(\cdot)^{-1}))}{\beta(\overline{F}(x_R-(\cdot)^{-1}))-1}$.

Lemma 3 states some closure properties of Matuszewska indices. Lemma 4 gives a sufficient condition for f to vanish. Lemmas 5 and 6 state helpful results on the asymptotic behavior of the ratio between a function and certain integrals over this function, depending on its Matuszewska indices. Lemmas 7 and 8 and Corollary 4 specify the earlier lemmas by giving bounds on the Matuszewska indices of $\overline{G}$, $G^{\leftarrow}$, and the composition of $\overline{F}$ and $G^{\leftarrow}$. The proofs of Lemmas 3, 4, 7, and 8, along with several additional results are postponed to the appendix. Corollary 4 follows immediately from Lemmas 3 and 8.

4.1.1. Finite Variance. In this section, we assume either $x_R = \infty$ and $-\infty < \beta(\overline{F}) \le \alpha(\overline{F}) < -2$ or $x_R < \infty$ and $\beta(\overline{F}(x_R - (\cdot)^{-1})) > -\infty$. If $x_R = \infty$, then $\alpha((\cdot)^2\overline{F}(\cdot)) < 0$ and, thus, $\mathbb{E}[B^2] = 2\int_0^\infty t\overline{F}(t)\,\mathrm{d}t < \infty$ by Lemma 6(i); if $x_R < \infty$, then clearly $\mathbb{E}[B^2] < \infty$.

Noting that $G^{\leftarrow}$ is a continuous, strictly increasing function, it follows that the function $x_\rho^\nu := G^{\leftarrow}(1 - \frac{1-\rho}{\rho}\frac{1-\nu}{\nu})$ is well defined for all $\nu \in (1-\rho, 1)$. For this choice of x_ρ^ν, we have $\frac{1-\rho}{1-\rho_{x_\rho^\nu}} = \nu$ and $\frac{\mathrm{d}G(x_\rho^\nu)}{\mathrm{d}\nu} = \frac{1-\rho}{\rho}\frac{1}{\nu^2}$, and therefore, relation (19) becomes

$$
\begin{aligned}
(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}] &= \frac{\mathbb{E}[B](1-\rho)^2}{\rho}\log\frac{1}{1-\rho} + 2\rho\int_0^\infty \left(\frac{1-\rho}{1-\rho_x}\right)^2 x\overline{F}(x)\,\mathrm{d}G(x) \\
&\quad + \frac{\rho^2}{\mathbb{E}[B]}\int_0^\infty \left(\frac{1-\rho}{1-\rho_x}\right)^3 \frac{m_2(x)\overline{F}(x)}{1-\rho}\,\mathrm{d}G(x) \\
&= \frac{\mathbb{E}[B](1-\rho)^2}{\rho}\log\frac{1}{1-\rho} \\
&\quad + 2(1-\rho)\int_{1-\rho}^1 G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)\mathrm{d}\nu \\
&\quad + \frac{\rho}{\mathbb{E}[B]}\int_{1-\rho}^1 \nu\, m_2\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)\mathrm{d}\nu.
\end{aligned}
$$

Dividing both sides by $\overline{F}(G^{\leftarrow}(\rho))$ yields

$$
\begin{aligned}
\frac{(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]}{\overline{F}(G^{\leftarrow}(\rho))} &= \frac{\mathbb{E}[B](1-\rho)^2}{\rho\overline{F}(G^{\leftarrow}(\rho))}\log\frac{1}{1-\rho} \\
&\quad + \frac{2(1-\rho)}{\overline{F}(G^{\leftarrow}(\rho))}\int_{1-\rho}^1 G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)\mathrm{d}\nu \\
&\quad + \frac{\rho}{\mathbb{E}[B]}\int_{1-\rho}^1 \nu\, m_2\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)\frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\mathrm{d}\nu \\
&= \mathrm{I}(\rho) + \mathrm{II}(\rho) + \mathrm{III}(\rho).
\end{aligned}
\tag{20}
$$

We show that $\mathrm{I}(\rho) + \mathrm{II}(\rho) = o(1)$ and $\mathrm{III}(\rho) = \Theta(1)$. Assume $x_R = \infty$. Then, by Lemma 3 and Corollary 4, we find that

$$
\begin{aligned}
\alpha\big(\mathrm{I}(1-(\cdot)^{-1}\big) &\le \alpha((\cdot)^{-2}) + \alpha(1/\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))) + \alpha(\log(\cdot)) \\
&= -2 - \beta(\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))) + 0 \le -2 + \frac{\beta(\overline{F})}{\beta(\overline{F})+1} < 0,
\end{aligned}
\tag{21}
$$

and consequently, $\mathrm{I}(\rho) = o(1)$ as $\rho \uparrow 1$ by Lemma 4.

Next, fix $0 \le \varepsilon < 2 - \frac{\beta(\overline{F})}{\beta(\overline{F})+1}$. Substitution of $w = \frac{\rho}{1-\rho}\frac{\nu}{1-\nu}$ in $\mathrm{II}(\rho)$ yields

$$\mathrm{II}(\rho) = \frac{2(1-\rho)}{\overline{F}(G^{\leftarrow}(\rho))}\int_1^\infty \frac{\rho}{1-\rho}\left(\frac{\rho}{1-\rho}+w\right)^{-2} G^{\leftarrow}(1-w^{-1})\overline{F}(G^{\leftarrow}(1-w^{-1}))\,\mathrm{d}w$$
$$\le \frac{2(1-\rho)^{2-\varepsilon}}{\rho^{1-\varepsilon}\overline{F}(G^{\leftarrow}(\rho))}\int_1^\infty w^{-\varepsilon}G^{\leftarrow}(1-w^{-1})\overline{F}(G^{\leftarrow}(1-w^{-1}))\,\mathrm{d}w.$$

Let $q(w)$ denote the integrand in the last line. A similar analysis to (21) indicates that the term in front of the integral vanishes as $\rho \uparrow 1$, so we only need to show that the integral is bounded. This is implied by Lemma 6(i) after noting that

$$\alpha(q) \le -\varepsilon + \alpha(G^{\leftarrow}(1-(\cdot)^{-1})) + \alpha(\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))) \le -1-\varepsilon < 0,$$

where the inequalities follow from Lemmas 3 and 8 and Corollary 4.

Finally, we wish to show that $\mathrm{III}(\rho) = \Theta(1)$. Observe that

$$\mathrm{III}(\rho) \le \lambda\mathbb{E}[B^2]\int_{1-\rho}^{\frac{1}{1+\rho}} \nu\,\frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}\nu + \lambda\mathbb{E}[B^2]\int_{\frac{1}{1+\rho}}^{1}\frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}\nu$$
$$\le 2\rho\mathbb{E}[B^*]\int_1^{\frac{1}{1-\rho}}\frac{\rho w}{1-\rho}\left(\frac{\rho}{1-\rho}+w\right)^{-3}\frac{\overline{F}(G^{\leftarrow}(1-w^{-1}))}{\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}w + \mathbb{E}[B^*]$$
$$\le \frac{2\mathbb{E}[B^*]}{\rho}\int_1^{\frac{1}{1-\rho}}\frac{w\overline{F}(G^{\leftarrow}(1-w^{-1}))}{\frac{1}{(1-\rho)^2}\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}w + \mathbb{E}[B^*] = \frac{2\mathbb{E}[B^*]}{\rho}\int_1^{\frac{1}{1-\rho}}\frac{f(w)/w}{f(1/(1-\rho))}\,\mathrm{d}w + \mathbb{E}[B^*],$$

where $f(w) = w^2\overline{F}(G^{\leftarrow}(1-w^{-1}))$. Lemma 3 and Corollary 4 then state that $\beta(f) \ge 2 - \frac{\beta(\overline{F})}{\beta(\overline{F})+1} > 0$, and therefore, Lemma 5 implies

$$\limsup_{\rho\uparrow 1}\int_1^{\frac{1}{1-\rho}}\frac{f(w)/w}{f(1/(1-\rho))}\,\mathrm{d}v = \left[\liminf_{y\to\infty}\frac{f(y)}{\int_1^y f(w)/w\,\mathrm{d}w}\right]^{-1} < \infty.$$

As such, $\limsup_{\rho\uparrow 1}\mathrm{III}(\rho) < \infty$.

In order to show $\liminf_{\rho\uparrow 1}\mathrm{III}(\rho) > 0$, fix $c \in (0,1)$ and let $\delta_\rho := (1-\rho)/(c\rho+1-\rho)$. One may then readily verify that $\mathrm{III}(\rho) \ge \lambda m_2(G^{\leftarrow}(1-c))\int_{\delta_\rho}^{\frac{1}{1+\rho}}\nu\,\mathrm{d}\nu \to \frac{m_2(G^{\leftarrow}(1-c))}{8\mathbb{E}[B]} > 0$.

The $x_R = \infty$ case is concluded once we prove $\lim_{\rho\uparrow 1} h^*(G^{\leftarrow}(\rho)) = 0$. To this end, write $h^*(G^{\leftarrow}(\rho))$ as $x\overline{F}(G^{\leftarrow}(1-x^{-1}))/\mathbb{E}[B]$, where $x = (1-\rho)^{-1}$. The claim then follows from Lemma 4 after noting that

$$\alpha(h^*(G^{\leftarrow}(1-(\cdot)^{-1}))) \le \alpha(\cdot) + \alpha(\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))) \le 1 - \frac{\alpha(\overline{F})}{\alpha(\overline{F})+1} = \frac{1}{\alpha(\overline{F})+1} < 0,$$

where the inequalities follow from Lemma 3 and Corollary 4.

The $x_R < \infty$ case can be proven similarly. One then fixes $1 < \varepsilon < 2 - \frac{\beta(\overline{F}(x_R-(\cdot)^{-1}))}{\beta(\overline{F}(x_R-(\cdot)^{-1}))-1}$ and obtains

$$\alpha\big(\mathrm{I}(1-(\cdot)^{-1}) \le -2 + \frac{\beta(\overline{F}(x_R-(\cdot)^{-1}))}{\beta(\overline{F}(x_R-(\cdot)^{-1}))-1} < 0,$$
$$\alpha(q) \le -\varepsilon - \frac{\alpha(\overline{F}(x_R-(\cdot)^{-1}))+1}{\alpha(\overline{F}(x_R-(\cdot)^{-1}))-1} \le 1-\varepsilon < 0,$$

and

$$\beta(f) \ge 2 - \frac{\beta(\overline{F}(x_R-(\cdot)^{-1}))}{\beta(\overline{F}(x_R-(\cdot)^{-1}))-1} > 0.$$

The claim $h^*(G^{\leftarrow}(\rho)) \to \infty$ follows from Lemma 1.

4.1.2. Infinite Variance. Assume $\beta(\overline{F}) > -2$ and recall that $m_2(x) = 2\mathbb{E}[B]\int_0^x t\,\mathrm{d}G(t) = 2\mathbb{E}[B](\int_0^x \overline{G}(t)\,\mathrm{d}t - x\overline{G}(x))$. By Lemmas 3 and 7, one sees that $\beta((\cdot)\overline{G}(\cdot)) > 0$, and therefore, it follows from Lemma 5 that

$$\limsup_{x\to\infty} \frac{m_2(x)}{2\mathbb{E}[B]x\overline{G}(x)} = \limsup_{x\to\infty} \frac{\int_0^x \overline{G}(t)\,\mathrm{d}t}{x\overline{G}(x)} - 1 < \infty. \tag{22}$$

Also, because $\beta((\cdot)\overline{F}(\cdot)) > -\infty$, Lemma 6(ii) indicates that

$$\limsup_{x\to\infty} \frac{x\overline{F}(x)}{\overline{G}(x)} = \limsup_{x\to\infty} \frac{\mathbb{E}[B]x\overline{F}(x)}{\int_x^\infty \overline{F}(t)\,\mathrm{d}t} < \infty.$$

Consequently, it follows from relation (19) that, for some $C, D > 0$ and all ρ sufficiently close to one, we have

$$\begin{aligned}\mathbb{E}[T_{\mathrm{FB}}] &\le \frac{\mathbb{E}[B]}{\rho}\log\frac{1}{1-\rho} + 2\int_0^\infty \frac{x\overline{F}(x)}{(1-\rho G(x))^2}\,\mathrm{d}G(x) + \frac{1}{\mathbb{E}[B]}\int_0^\infty \frac{m_2(x)}{x\overline{G}(x)}\frac{x\overline{F}(x)}{(1-\rho G(x))^2}\,\mathrm{d}G(x)\\ &\le \frac{\mathbb{E}[B]}{\rho}\log\frac{1}{1-\rho} + C\int_0^\infty \frac{x\overline{F}(x)}{\overline{G}(x)}\frac{1}{1-\rho G(x)}\,\mathrm{d}G(x) \le D\log\frac{1}{1-\rho},\end{aligned}$$

and therefore, $\mathbb{E}[T_{\mathrm{FB}}] = \Theta(\log\frac{1}{1-\rho})$.

4.2. Special Cases

This section proves Theorem 7. The maximum domains of attraction of each of the extreme value distributions are considered in order, followed by a distribution with an atom in its right end point. The Fréchet and Weibull cases follow readily from Theorem 6 and the dominated convergence theorem. The same approach works for the Gumbel case, although Theorem 6 is not directly applicable. Finally, the atom case follows readily by analyzing the sojourn time of maximum-sized jobs.

4.2.1. Fréchet (α) and Weibull (α). Theorems 1 and 3 together state that $F \in \mathrm{MDA}(\Phi_\alpha)$ if and only if $\overline{F}(x) = L(x)x^{-\alpha}$. Karamata's theorem (Bingham et al. 1989, theorem 1.5.11) then states that $\mathbb{E}[B]\overline{G}(x) \sim x\overline{F}(x)/(\alpha-1)$ is regularly varying with index $-(\alpha-1)$. Consequently, theorem 1.5.12 in Bingham et al. (1989) states that $G^{\leftarrow}(1-1/x)$ is regularly varying with index $1/(\alpha-1)$, and therefore, $\overline{F}(G^{\leftarrow}(1-1/x))$ is regularly varying with index $-\alpha/(\alpha-1)$ (Bingham et al. 1989, proposition 1.5.7).

First, assume $\alpha > 2$. We saw in Section 4.1.1 that the asymptotic behavior of $\mathbb{E}[T_{\mathrm{FB}}]$ is identical to the asymptotic behavior of term $\mathrm{III}(\rho)$ (cf. relation (20)). Now, the uniform convergence theorem (Bingham et al. 1989, theorem 1.5.2) states that $\frac{\overline{F}(G^{\leftarrow}(1-1/x))}{\overline{F}(G^{\leftarrow}(1-1/y))} \to \left(\frac{y}{x}\right)^{\alpha/(\alpha-1)}$ uniformly for all $0 < c < x, y < \infty$. Therefore, we substitute $w = \frac{\nu-(1-\rho)}{\rho}$ and exploit the dominated convergence theorem to obtain

$$\begin{aligned}\lim_{\rho\uparrow 1}\mathrm{III}(\rho) &= \lim_{\rho\uparrow 1}\frac{\rho^2}{\mathbb{E}[B]}\int_0^1 (\rho w + 1 - \rho)\, m_2\left(G^{\leftarrow}\left(1 - \frac{(1-\rho)(1-w)}{1-\rho+\rho w}\right)\right)\frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{(1-\rho)(1-w)}{1-\rho+\rho w}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}w\\ &= \frac{\mathbb{E}[B^2]}{\mathbb{E}[B]}\int_0^1 w\left(\frac{1-w}{w}\right)^{\alpha/(\alpha-1)}\mathrm{d}w = \mathbb{E}[B^*]\frac{\pi/(\alpha-1)}{\sin(\pi/(\alpha-1))}\frac{\alpha}{\alpha-1}.\end{aligned}$$

Similarly, Theorems 1 and 4 together state that $F \in \mathrm{MDA}(\Psi_\alpha), \alpha > 0$, if and only if $x_R < \infty$ and $\overline{F}(x_R - x^{-1}) = L(x)x^{-\alpha}$. The corresponding result then follows after noting that $\mathbb{E}[B]\overline{G}(x_R - x^{-1}) \sim L(x)x^{-\alpha-1}/(\alpha+1)$ is regularly varying with index $-(\alpha+1)$ and $\frac{\overline{F}(G^{\leftarrow}(1-1/x))}{\overline{F}(G^{\leftarrow}(1-1/y))} \to \left(\frac{y}{x}\right)^{\alpha/(\alpha+1)}$ uniformly for all $0 < c < x, y < \infty$.

Finally, assume that $F \in \mathrm{MDA}(\Phi_\alpha)$, $\alpha \in (1,2)$. Then, Karamata's theorem implies $m_2(x) = 2\int_0^x y\overline{F}(y)\,\mathrm{d}y \sim 2x^2\overline{F}(x)/(2-\alpha)$ as $x \to \infty$. We analyze relation (19) and again exploit the dominated convergence theorem to find

$$\begin{aligned}\mathbb{E}[T_{\mathrm{FB}}] &= \frac{\mathbb{E}[B]}{\rho}\log\frac{1}{1-\rho} + 2\rho\int_0^\infty \frac{x\overline{F}(x)}{\overline{G}(x)}\frac{1-G(x)}{(1-\rho G(x))^2}\,\mathrm{d}G(x)\\ &\quad + \frac{\rho^2}{\mathbb{E}[B]}\int_0^\infty \frac{m_2(x)\overline{F}(x)}{\overline{G}(x)^2}\frac{(1-G(x))^2}{(1-\rho G(x))^3}\,\mathrm{d}G(x)\\ &\sim \mathbb{E}[B]\log\frac{1}{1-\rho} + 2(\alpha-1)\mathbb{E}[B]\int_0^1 \frac{1-y}{(1-\rho y)^2}\,\mathrm{d}y + \frac{2}{\mathbb{E}[B]}\frac{(\alpha-1)^2\mathbb{E}[B]^2}{2-\alpha}\int_0^1\frac{(1-y)^2}{(1-\rho y)^3}\,\mathrm{d}y\\ &\sim \mathbb{E}[B]\log\frac{1}{1-\rho} + 2(\alpha-1)\mathbb{E}[B]\log\frac{1}{1-\rho} + \frac{2(\alpha-1)^2\mathbb{E}[B]}{2-\alpha}\log\frac{1}{1-\rho} = \frac{\alpha}{2+\alpha}\mathbb{E}[B]\log\frac{1}{1-\rho}\end{aligned}$$

as $\rho \uparrow 1$.

4.2.2. Gumbel. If $F \in \mathrm{MDA}(\Lambda)$, then so is G by Lemma 2, and we may choose h^* as the auxiliary function of G. Propositions 0.9(a), 0.10, and 0.12 in Resnick (1987) together state that

$$a_G(x) := \frac{1}{h^*(G^{\leftarrow}(1-1/x))} = \frac{\mathbb{E}[B]}{x\overline{F}(G^{\leftarrow}(1-1/x))}$$

is 0 varying,[1] implying that $\overline{F}(G^{\leftarrow}(1-1/x))$ is (-1) varying.

Following the analysis in Section 4.1.1, we obtain $\alpha(\mathrm{I}) = -1 < 0$ as before. Consider term $\mathrm{II}(\rho)$. By Markov's inequality, we have $\overline{G}(x) \le \mathbb{E}[B^*]/x$. Substituting $x = G^{\leftarrow}(1-w^{-1})$ then yields $G^{\leftarrow}(1-w^{-1}) \le \mathbb{E}[B^*]w$, and hence,

$$\begin{aligned}\mathrm{II}(\rho) &= \frac{2(1-\rho)}{\overline{F}(G^{\leftarrow}(\rho))}\int_1^\infty \frac{\rho}{1-\rho}\left(\frac{\rho}{1-\rho}+w\right)^{-2} G^{\leftarrow}(1-w^{-1})\overline{F}(G^{\leftarrow}(1-w^{-1}))\,\mathrm{d}w\\ &\le \frac{2\mathbb{E}[B^*](1-\rho)^{3/2}}{\rho^{1/2}\overline{F}(G^{\leftarrow}(\rho))}\int_1^\infty w^{1/2}\overline{F}(G^{\leftarrow}(1-w^{-1}))\,\mathrm{d}w.\end{aligned}$$

The term in front of the integral and the integrand both have upper Matuszewska index $-1/2$, and therefore, $\mathrm{II}(\rho) \to 0$.

Finally, consider term $\mathrm{III}(\rho)$. The relation $\limsup_{\rho\uparrow 1}\mathrm{III}(\rho) < \infty$ follows analogously to the analysis in Section 4.1.1. Then, along the lines of Section 4.2.1, one may apply the uniform convergence theorem and the dominated convergence theorem to derive the theorem statement.

4.2.3. Atom in Right End Point. First, we show that $\mathrm{I}(\rho) + \mathrm{II}(\rho) = o(1)$. Lemma 1 states that $\lim_{x\uparrow x_R} h^*(x) = \infty$, and therefore, $\lim_{\rho\uparrow 1}\mathrm{I}(\rho) = \lim_{\rho\uparrow 1}\frac{(1-\rho)\log\frac{1}{1-\rho}}{\rho h^*(G^{\leftarrow}(\rho))} = 0$. Also, $G^{\leftarrow}$ is bounded from above by x_R, and consequently, $\lim_{\rho\uparrow 1}\cdot \mathrm{II}(\rho) \le \lim_{\rho\uparrow 1}\frac{2(1-\rho)}{\overline{F}(G^{\leftarrow}(\rho))}\cdot x_R = \lim_{\rho\uparrow 1}\frac{2x_R}{\mathbb{E}[B]h^*(G^{\leftarrow}(\rho))} = 0$.

It remains to show that $\mathrm{III}(\rho) \to \mathbb{E}[B^*]$ and $\overline{F}(G^{\leftarrow}(\rho)) \to p$ as $\rho \uparrow 1$. The following lemma facilitates the analysis of this term. The proof of the lemma is postponed until the end of this section.

Lemma 9. *Let $f : D \to \mathbb{R}$ be any function that maps $D \subseteq \mathbb{R}$ onto $\mathbb{R}$ and assume that $\lim_{y\uparrow x} f(y) = p$ for some x in the closure $\overline{D}$ of D. Then there exist $z > 0$ and $q > 0$ such that*

$$f(x-y) \le p + qy \tag{23}$$

for all $y \in (0,z]$ that satisfy $x - y \in D$.

Let $q > 0$ and $\delta^* > 0$ be such that $\overline{F}(x_R - \delta) \le p + q\delta$ for all $\delta \in (0,\delta^*]$. It follows that $\mathbb{E}[B]\overline{G}(x) = \int_x^{x_R}\overline{F}(y)\,\mathrm{d}y \sim p(x_R - x)$ as $x \uparrow x_R$, and hence, $x_R - G^{\leftarrow}(u) \sim \mathbb{E}[B](1-u)/p$ as $u \uparrow 1$. Fix $\varepsilon > 0$ and let $u^* \in (0,1)$ be such that $x_R - G^{\leftarrow}(u) \le (1+\varepsilon)\mathbb{E}[B](1-u)/p$ for all $u \in (u^*,1)$. Now, for all $u > \rho_0 := \max\{u^*, 1 - p\delta^*/((1+\varepsilon)\mathbb{E}[B])\}$, we have

$$p \le \overline{F}(G^{\leftarrow}(u)) \le p + \frac{q}{p}(1+\varepsilon)\mathbb{E}[B](1-u) =: p + p\tilde{q}(1-u) \tag{24}$$

and hence, for $\widetilde{q} = q(1+\varepsilon)\mathbb{E}[B]/p^2$, the relations

$$\frac{1}{1+\widetilde{q}(1-\rho)} \le \frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))} \le 1+\widetilde{q}\,\frac{1-\rho}{\rho}\frac{1-\nu}{\nu} \le 1+\widetilde{q}\,\frac{1-\rho}{\rho}\frac{1}{\nu}$$

hold for all $\nu > \frac{1-\rho}{1-\rho\cdot\rho_0}, \rho > \rho_0$.

Consider term III(ρ). On the one hand, we find

$$\begin{aligned}
\limsup_{\rho\uparrow 1} \mathrm{III}(\rho) &\le \limsup_{\rho\uparrow 1} \frac{\mathbb{E}[B^2]}{\mathbb{E}[B]} \int_{1-\rho}^{1} \nu\, \frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}\nu \\
&\le \limsup_{\rho\uparrow 1} \frac{\mathbb{E}[B^2]}{\mathbb{E}[B]} \int_{1-\rho}^{\frac{1-\rho}{1-\rho\cdot\rho_0}} \frac{1}{p}\,\mathrm{d}\nu + \frac{\mathbb{E}[B^2]}{\mathbb{E}[B]} \int_{\frac{1-\rho}{1-\rho\cdot\rho_0}}^{1} \left\{\nu + \widetilde{q}\,\frac{1-\rho}{\rho}\right\}\mathrm{d}\nu \\
&\le \limsup_{\rho\uparrow 1} \frac{\mathbb{E}[B^2]}{p\mathbb{E}[B]}\frac{1-\rho}{1-\rho\cdot\rho_0} + \frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\mathbb{E}[B^2]}{\mathbb{E}[B]}\,\widetilde{q}\,\frac{1-\rho}{\rho} = \mathbb{E}[B^*].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\liminf_{\rho\uparrow 1} \mathrm{III}(\rho) &\ge \liminf_{\rho\uparrow 1} \frac{\rho m_2(G^{\leftarrow}(\rho_0))}{\mathbb{E}[B]} \int_{\frac{1-\rho}{1-\rho\cdot\rho_0}}^{1} \frac{\nu}{1+\widetilde{q}(1-\rho)}\,\mathrm{d}\nu \\
&= \liminf_{\rho\uparrow 1} \frac{m_2(G^{\leftarrow}(\rho_0))}{2\mathbb{E}[B]} \frac{\rho}{1+\widetilde{q}(1-\rho)} \left(1-\left(\frac{1-\rho}{1-\rho\cdot\rho_0}\right)^2\right) = \frac{m_2(G^{\leftarrow}(\rho_0))}{\mathbb{E}[B^2]}\cdot\mathbb{E}[B^*].
\end{aligned}$$

Because ρ_0 may be chosen arbitrarily close to unity, we find $\mathbb{E}[T_{\mathrm{FB}}] \sim \frac{\mathbb{E}[B^*]\overline{F}(G^{\leftarrow}(\rho))}{(1-\rho)^2} \sim \frac{p\mathbb{E}[B^*]}{(1-\rho)^2}$ as $\rho \uparrow 1$, where the last equivalence follows from (24). The section is concluded with the proof of Lemma 9.

Proof of Lemma 9. Without loss of generality, we assume that $(x-1, x) \subset D$. For the sake of finding a contradiction, assume that the lemma statement is not true, that is, for all $z > 0$ and all $q > 0$ there exists $\xi \in (0, z]$ such that

$$f(x-\xi) > p + q\xi. \tag{25}$$

Define $z_1 := 1, q_1 := 1$ and let $\xi_1 \in (0, 1]$ be such that (25) holds with $q = q_1$ and $\xi = \xi_1$. By definition of the left limit, for any $\varepsilon > 0$, there exists $\eta^* > 0$ such that $f(x-\eta) \le p+\varepsilon$ for all $\eta \in (0, \eta^*]$. In particular, by choosing $\varepsilon = q_1\xi_1$, we obtain $\eta^* =: \eta_2^* < \xi_1 \le z_1$ such that $f(x-\eta) \le p + q_1\xi_1$ for all $\eta \in (0, \eta_2^*]$.

Define $z_2 := \min\{\eta_2^*, 1/2\}$ and set $q_2 := 1/z_2$. Again, there exists $\xi_2 \in (0, z_2]$ such that (25) holds for $q = q_2$ and $\xi = \xi_2$. By repeating this procedure, we obtain three sequences $(q_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$, and $(\xi_n)_{n\in\mathbb{N}}$ such that $q_n = 1/z_n$, $0 < z_{n+1} < \xi_n < z_n \le 1/n$, and

$$f(x-\xi_n) > p + q_n\xi_n \tag{26}$$

for all $n \in \mathbb{N}$. From these properties, one may additionally deduce that $\xi_n > 1/q_{n+1}$, $\xi_n \downarrow 0$, and $q_n \to \infty$.

We obtain a contradiction by showing that $(q_n)_{n\in\mathbb{N}}$ must also converge. If $\limsup_{n\to\infty} q_n\xi_n > 0$, then, by relation (26), we must have $\limsup_{n\to\infty} f(x-\xi_n) \ge \limsup_{n\to\infty} p + q_n\xi_n > p$. However, this contradicts the lemma assumptions, and therefore, $\limsup_{n\to\infty} q_n\xi_n$ must equal zero. As such, we find $0 \le \limsup_{n\to\infty} q_n/q_{n+1} \le \limsup_{n\to\infty} q_n\xi_n = 0$ so that the sequence $(q_n)_{n\in\mathbb{N}}$ converges by the ratio test. □

Note that Lemma 9 can be applied generally to yield lower and upper bounds for $f(y)$ around any point $x \in \overline{D}$ for which either $\lim_{y\uparrow x} f(y)$ or $\lim_{y\downarrow x} f(y)$ exists.

5. Asymptotic Relation for $h^*(G^{\leftarrow}(\rho))$ in the Gumbel Case

This section is dedicated to the proof of Theorem 8. Theorem 5 states that c_n may be chosen as $1/h^*(F^{\leftarrow}(1-n^{-1}))$ so that Theorem 8 follows from Theorem 7 and an analysis of the limit $\lim_{n\to\infty} h^*(G^{\leftarrow}(1-n^{-1}))/h^*(F^{\leftarrow}(1-n^{-1})) = \lim_{y\uparrow 1}(1-y)^{-2}\overline{F}(G^{\leftarrow}(y))\overline{G}(F^{\leftarrow}(y))$. The proof heavily relies upon the work by de Haan (1974) and Resnick (1987), who both consider Γ- and Π-varying functions.

Definition 3. A function $U:(x_L,x_R)\to\mathbb{R}, \lim_{x\uparrow x_R}U(x)=\infty$ is in the class of Γ-varying functions if it is nondecreasing, and there exists a function $f:(x_L,x_R)\to\mathbb{R}_{\geq 0}$ satisfying

$$\lim_{x\uparrow x_R}\frac{U(x+tf(x))}{U(x)}=e^t \tag{27}$$

for all $t\in\mathbb{R}$. The function $f(\cdot)$ is called an auxiliary function and is unique up to asymptotic equivalence.

Definition 4. A function $V:(x_L,\infty)\to\mathbb{R}_{\geq 0}$ is in the class of Π-varying functions if it is nondecreasing and there exist functions $a(x)>0, b(x)\in\mathbb{R}$ such that

$$\lim_{x\to\infty}\frac{V(tx)-b(x)}{a(x)}=\log t \tag{28}$$

for all $t\in\mathbb{R}$. The function $a(\cdot)$ is called an auxiliary function and is unique up to asymptotic equivalence.

It turns out that Γ- and Π-varying functions are closely related to MDA(Λ). In particular, if $F\in\text{MDA}(\Lambda)$ with auxiliary function $1/h^*$, then proposition 1.9 in Resnick (1987) states that $U_F:=1/\overline{F}\in\Gamma$ with auxiliary function $f_F:=1/h^*$. Proposition 0.9(a) then states that $V_F(\cdot):=U_F^{\leftarrow}(\cdot)=\left(1/\overline{F}\right)^{\leftarrow}(\cdot)=F^{\leftarrow}\left(1-(\cdot)^{-1}\right)\in\Pi$ with auxiliary function $a_F(\cdot):=f_F(U_F^{\leftarrow}(\cdot))=1/h^*(F^{\leftarrow}(1-(\cdot)^{-1}))$. Similarly, using Lemma 2, we find that $U_G:=1/\overline{G}\in\Gamma$ and $V_G(\cdot):=U_G^{\leftarrow}(\cdot)=G^{\leftarrow}\left(1-(\cdot)^{-1}\right)\in\Pi$ with auxiliary function $a_G(\cdot):=1/h^*(G^{\leftarrow}(1-(\cdot)^{-1}))$.

Now, because Theorem 5 states that the norming constants c_n may be chosen as $1/h^*(F^{\leftarrow}(1-n^{-1}))$, we are done once we show that $\lim_{n\to\infty}c_nh^*(G^{\leftarrow}(1-n^{-1}))=\lim_{x\to\infty}\frac{a_F(x)}{a_G(x)}$ tends to the right quantity for all cases in the theorem.

Corollary 3.4 in de Haan (1974) states that[2] $\lim_{x\uparrow x_R}\frac{a_F(x)}{a_G(x)}=\xi^{-1}\in[0,\infty]$ if and only if there exist a positive function $b(x)$ with $\lim_{x\uparrow x_R}b(x)=\xi$ and constants $b_2>0$ and $b_3\in\mathbb{R}$ such that[3] $P(x)=b_3+\int_0^x b(t)\,\mathrm{d}t$ and $V_F{}^{\leftarrow}(x)\sim b_2V_G{}^{\leftarrow}(P(x))$ as $x\uparrow x_R$. As $V_\bullet{}^{\leftarrow}(x)=(U_\bullet{}^{\leftarrow})^{\leftarrow}(x)\sim U_\bullet(x)$ (Resnick 1987), this is equivalent to finding a function $P(x)$ of the given form that satisfies

$$\lim_{x\uparrow x_R}\frac{\overline{G}(P(x))}{b_2\overline{F}(x)}=\lim_{x\uparrow x_R}\frac{U_F(x)}{b_2U_G(P(x))}=\lim_{x\uparrow x_R}\frac{V_F^{\leftarrow}(x)}{b_2V_G^{\leftarrow}(P(x))}=1. \tag{29}$$

We use the following lemma, proven at the end of this section, to construct a suitable $P(x)$:

Lemma 10. *Let F be a c.d.f. Then, there exists a strictly increasing, continuous c.d.f.* $F_\uparrow(x)$ *satisfying both* $\overline{F}_\uparrow(x)\sim\overline{F}(x)$ *and* $\overline{G}(F_\uparrow(x))\sim\overline{G}(F(x))$ *as* $x\uparrow x_R$.

As $G^{\leftarrow}(F_\uparrow(x))$ is strictly increasing, there exists a positive function $b(\cdot)$ such that $\int_0^x b(t)\,\mathrm{d}t=G^{\leftarrow}(F_\uparrow(x))$. Therefore, we see that (29) is satisfied with $b_2=1$ and $b_3=0$. The result follows once we show that

$$\lim_{x\to\infty}b(x)=\lim_{x\to\infty}\frac{P(x)}{x}=\lim_{x\to\infty}\frac{G^{\leftarrow}(F(x))}{x}=\xi \tag{30}$$

if $x_R=\infty$ and once we show that

$$\lim_{x\uparrow x_R}b(x)=\lim_{x\uparrow x_R}\frac{P(x_R)-P(x)}{x_R-x}=\lim_{x\uparrow x_R}\frac{x_R-G^{\leftarrow}(F(x))}{x_R-x}=\xi \tag{31}$$

if $x_R<\infty$.

The right-hand sides of both (30) and (31) depend on the function $G^{\leftarrow}(F(x))$. The advantage of this representation is apparent from the following key relation, which connects $G^{\leftarrow}(F(x))$ to $h^*(x)$:

$$\mathbb{E}[B]h^*(x)=\exp\left[\int_{G^{\leftarrow}(F(x))}^{x}h^*(t)\,\mathrm{d}t\right]. \tag{32}$$

Relation (32) follows readily from $h^*(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\log\overline{G}(x)$. In the upcoming analysis, we first focus on (30) and then consider (31).

5.1. Infinite Support

First, assume $x_R=\infty$. The following theorem relates the assumptions on $\overline{F}(x)$ to properties of $h^*(x)$:

Theorem 11 (Beirlant et al. 1995, theorem 2.1).

i. *If there exists* $\alpha > 0$ *and a slowly varying function* $l(x)$ *such that* $-\log \overline{F}(x) \sim l(x)x^{\alpha}$ *as* $x \to \infty$, *then* $h^*(x) \sim \alpha l(x)x^{\alpha-1}$ *as* $x \to \infty$ *if and only if*

$$\lim_{\lambda \downarrow 1} \liminf_{x\to\infty} \inf_{t\in[1,\lambda]} \{\log h^*(tx) - \log h^*(x)\} \geq 0. \tag{33}$$

ii. *If there exists a function* $l(x) : [0,\infty) \to \mathbb{R}$, $\liminf_{x\to\infty} l(x) > 1$ *such that, for all* $\lambda > 0$,

$$\lim_{x\to\infty} \frac{-\log \overline{F}(\lambda x) + \log \overline{F}(x)}{l(x)} = \log(\lambda), \tag{34}$$

then $l(x)$ *is slowly varying, and* $h^*(x) \sim (l(x) - 1)/x$ *as* $x \to \infty$.

The cases in Theorem 8 correspond to the cases in Theorem 11. We consider the implications of Theorem 11 to derive the results presented in Theorem 8.

i. Assume $h^*(x) \sim \alpha l(x)x^{\alpha-1}$, $\alpha > 0$, and note that

$$\lim_{x\to\infty} \frac{-\log(\mathbb{E}[B]h^*(x))}{xh^*(x)} = \lim_{x\to\infty} \frac{-\log(\mathbb{E}[B]\alpha l(x)) - (\alpha-1)\log(x)}{\alpha l(x)x^{\alpha}} = 0.$$

We prove the asymptotic relation $\lim_{x\to\infty} G^{\leftarrow}(F(x))/x = 1$ by contradiction. Specifically, if $\limsup_{x\to\infty} G^{\leftarrow}(F(x))/x > 1$, then there exists $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \to \infty$, such that $G^{\leftarrow}(F(x_n))/x_n \geq 1+\varepsilon$ for all $n \in \mathbb{N}$. The uniform convergence theorem (Bingham et al. 1989, theorems 1.2.1 and 1.5.2) then implies

$$\begin{aligned}\frac{-\log(\mathbb{E}[B]h^*(x_n))}{x_n h^*(x_n)} &= \int_{x_n}^{G^{\leftarrow}(F(x_n))} \frac{h^*(t)}{x_n h^*(x_n)}\,\mathrm{d}t = \int_1^{G^{\leftarrow}(F(x_n))/x_n} \frac{h^*(\tau x_n)}{h^*(x_n)}\,\mathrm{d}\tau \\ &\geq \int_1^{1+\varepsilon} \frac{h^*(\tau x_n)}{h^*(x_n)}\,\mathrm{d}\tau \sim \int_1^{1+\varepsilon} \tau^{\alpha-1}\,\mathrm{d}\tau = \alpha^{-1}((1+\varepsilon)^{\alpha} - 1)\end{aligned}$$

for every $n \in \mathbb{N}$. However, this contradicts with $\lim_{x\to\infty} \frac{\log(\mathbb{E}[B]h^*(x))}{xh^*(x)} = 0$, and it follows that $\liminf_{x\to\infty} G^{\leftarrow}(F(x))/x \leq 1$. One may similarly verify $\liminf_{x\to\infty} G^{\leftarrow}(F(x))/x \geq 1$ so that $\lim_{x\to\infty} G^{\leftarrow}(F(x))/x = 1$ as claimed.

ii. Alternatively, assume $h^*(x) \sim \frac{l(x)-1}{x}$ and denote $L = \lim_{x\to\infty} \log(x)/l(x) \in [0,\infty]$. Then Lemma 1 states that $l(x) \to \infty$, and as such,

$$\lim_{x\to\infty} \frac{-\log(\mathbb{E}[B]h^*(x))}{xh^*(x)} = \lim_{x\to\infty} \frac{-\log(\mathbb{E}[B]) - \log(l(x)-1) + \log(x)}{l(x)-1} = L. \tag{35}$$

Now, if $L = 0$, then the analysis in (i) yields $\lim_{x\to\infty} G^{\leftarrow}(F(x))/x = 1$. If $L \in (0,\infty)$ then (32) and (35) imply

$$\begin{aligned}L &= \lim_{x\to\infty} \frac{-\log(\mathbb{E}[B]h^*(x))}{xh^*(x)} = \lim_{x\to\infty} \int_x^{G^{\leftarrow}(F(x))} \frac{h^*(t)}{xh^*(x)}\,\mathrm{d}t \\ &= \lim_{x\to\infty} \frac{1}{\log(x)} \int_x^{G^{\leftarrow}(F(x))} \frac{l(t)-1}{\log t} \cdot \frac{\log(x)}{l(x)-1} \cdot \frac{\log(t)}{t}\,\mathrm{d}t \\ &= \lim_{x\to\infty} \frac{1}{\log(x)} \int_x^{G^{\leftarrow}(F(x))} \frac{\log(t)}{t}\,\mathrm{d}t = \lim_{x\to\infty} \frac{\log^2(G^{\leftarrow}(F(x))) - \log^2(x)}{2\log(x)}.\end{aligned}$$

Writing $G^{\leftarrow}(F(x)) = u(x)x, u(x)x \to \infty$, now yields

$$L = \lim_{x\to\infty} \log(u(x))\left(1 + \frac{\log(u(x))}{2\log(x)}\right),$$

from which we conclude $u(x) \to e^L$, and consequently, $\lim_{x\to\infty} G^{\leftarrow}(F(x))/x = e^L$.

Finally, if $L = \infty$, then $h^*(x) \downarrow 0$, and therefore, $G^{\leftarrow}(F(x)) \geq x$ by (32). For sake of contradiction, assume $\liminf_{x\to\infty} G^{\leftarrow}(F(x))/x < \infty$. Then there exists $M_0 \geq 1$ such that, for all $M \geq M_0$, there exists a sequence $(x_n)_{n\in\mathbb{N}}, x_n \to \infty$, such that $G^{\leftarrow}(F(x_n))/x_n \leq M$ for every $n \in \mathbb{N}$. A similar analysis as in (i) then shows that this contradicts relation (35), and therefore, $\lim_{x\to\infty} G^{\leftarrow}(F(x))/x = \infty$.

5.2. Finite Support

Now assume $x_R < \infty$. Theorem 5 states that $\overline{F}(x)$ can be represented as

$$\overline{F}(x) = c(x)\exp\left\{-\int_z^x g(t)h^*(t)\,\mathrm{d}t\right\}, \quad z < x < x_R,$$

where c and g are measurable functions satisfying $c(x) \to c > 0, g(t) \to 1$ as $x \uparrow x_R$ and the auxiliary function $f_F(\cdot) = 1/h^*(\cdot)$ is positive, is absolutely continuous, and has density $f_F'(x)$ satisfying $\lim_{x\uparrow x_R} f_F'(x) = 0$. It is easily verified that the c.d.f. $\overline{F}_\infty(x) := \overline{F}(x_R - x^{-1})$, $x \geq (x_R - z)^{-1}$, is also in MDA(Λ) with auxiliary function $f_\infty(x) := x^2/h^*(x_R - x^{-1})$. From this representation, it is straightforward to obtain a finite-support equivalent of Theorem 11.

Corollary 5. *Assume $x_R < \infty$.*

i. *If there exists $\alpha > 0$ and a slowly varying function $l(x)$ such that $-\log \overline{F}(x_R - x^{-1}) \sim l(x)x^\alpha$ as $x \to \infty$, then $h^*(x_R - x^{-1}) \sim \alpha l(x)x^{\alpha+1}$ as $x \to \infty$ if and only if*

$$\lim_{\lambda\downarrow 1} \liminf_{x\to\infty} \inf_{t\in[1,\lambda]} \{\log h^*(x_R - (tx)^{-1}) - \log h^*(x_R - x^{-1}) - 2\log(t)\} \geq 0. \tag{36}$$

ii. *If there exists a function $l(x) : [0,\infty) \to \mathbb{R}$, $\liminf_{x\to\infty} l(x) > 1$, such that, for all $\lambda > 0$,*

$$\lim_{x\to\infty} \frac{-\log \overline{F}(x_R - (\lambda x)^{-1}) + \log \overline{F}(x_R - x^{-1})}{l(x)} = \log(\lambda), \tag{37}$$

then $l(x)$ is slowly varying and $h^*(x_R - x^{-1}) \sim (l(x) - 1)x$ as $x \to \infty$.

Again, the cases in Theorem 8 correspond to the cases in Corollary 5. The proof for the finite support case is similar to the infinite support case, yet we state it for completeness. Because $h^*(x) \to \infty$ as $x \uparrow x_R$ in both cases, relation (32) implies that $\frac{x_R - G^{\leftarrow}(F(x))}{x_R - x} \geq 1$ for all x sufficiently close to x_R.

i. Assume $h^*(x_R - x^{-1}) \sim \alpha l(x)x^{\alpha+1}, \alpha > 0$, and note that

$$\lim_{x\uparrow x_R} \frac{-\log(\mathbb{E}[B]h^*(x))}{(x_R - x)h^*(x)} = \lim_{y\to\infty} \frac{-\log(\mathbb{E}[B]h^*(x_R - y^{-1}))}{h^*(x_R - y^{-1})/y} = \lim_{y\to\infty} \frac{-\log(\mathbb{E}[B]\alpha l(y)) - (\alpha+1)\log(y)}{\alpha l(y)y^\alpha} = 0.$$

We show that $\lim_{x\to\infty} \frac{x_R - G^{\leftarrow}(F(x))}{x_R - x} = 1$ by contradiction. By our previous remark, we only need to show $\limsup_{x\to\infty} \frac{x_R - G^{\leftarrow}(F(x))}{x_R - x} \leq 1$. If this is false, then there exists $\varepsilon \in (0,1)$ and a sequence $(x_n)_{n\in\mathbb{N}}, x_n \uparrow x_R$ such that $\frac{x_R - x_n}{x_R - G^{\leftarrow}(F(x_n))} \leq 1 - \varepsilon$ for all $n \in \mathbb{N}$. As before, the uniform convergence theorem (Bingham et al. 1989, theorems 1.2.1 and 1.5.2) then implies

$$\begin{aligned}
-\frac{\log(\mathbb{E}[B]h^*(x_n))}{(x_R - x_n)h^*(x_n)} &= \int_{x_n}^{G^{\leftarrow}(F(x_n))} \frac{h^*(t)}{(x_R - x_n)h^*(x_n)}\,\mathrm{d}t \\
&= \int_{\frac{x_R - x_n}{x_R - G^{\leftarrow}(F(x_n))}}^1 \frac{h^*(x_R - (x_R - x_n)\tau^{-1})}{\tau^2 h^*(x_R - (x_R - x_n))}\,\mathrm{d}\tau \\
&\geq \int_{1-\varepsilon}^1 \frac{h^*(x_R - (x_R - x_n)\tau^{-1})}{\tau^2 h^*(x_R - (x_R - x_n))}\,\mathrm{d}\tau \\
&\sim \int_{1-\varepsilon}^1 \tau^{\alpha-1}\,\mathrm{d}\tau = \alpha^{-1}(1 - (1-\varepsilon)^\alpha)
\end{aligned}$$

for every $n \in \mathbb{N}$, which contradicts with $\lim_{x\uparrow x_R} \frac{\log(\mathbb{E}[B]h^*(x))}{(x_R - x)h^*(x)} = 0$.

ii. Now, assume $h^*(x_R - x^{-1}) \sim (l(x) - 1)x$ and let $L = \lim_{x\to\infty} \log(x)/l(x) \in [0,\infty]$. Lemma 1 implies $l(x) \to \infty$ so that

$$\lim_{x\uparrow x_R} \frac{-\log(\mathbb{E}[B]h^*(x))}{(x_R - x)h^*(x)} = \lim_{y\to\infty} \frac{-\log(\mathbb{E}[B](l(y) - 1)) - \log(y)}{l(y) - 1} = -L. \tag{38}$$

If $L=0$, then $\lim_{x\to\infty}\frac{x_R-G^{\leftarrow}(F(x))}{x_R-x}=1=e^0$ by the analysis in (i). Alternatively, if $L\in(0,\infty)$ then (32) and (38) imply

$$\begin{aligned}
L &= \lim_{x\uparrow x_R}\frac{\log(\mathbb{E}[B]h^*(x))}{(x_R-x)h^*(x)}=\lim_{x\uparrow x_R}\int_{G^{\leftarrow}(F(x))}^{x}\frac{h^*(t)}{(x_R-x)h^*(x)}\,\mathrm{d}t\\
&= \lim_{x\uparrow x_R}\int_{\frac{1}{x_R-G^{\leftarrow}(F(x))}}^{\frac{1}{x_R-x}}\frac{h^*(x_R-\tau^{-1})}{(x_R-x)\tau^2h^*(x_R-(x_R-x))}\,\mathrm{d}\tau\\
&= \lim_{x\uparrow x_R}\frac{1}{\log((x_R-x)^{-1})}\int_{\frac{1}{x_R-G^{\leftarrow}(F(x))}}^{\frac{1}{x_R-x}}\frac{l(\tau)-1}{\log(\tau)}\cdot\frac{\log((x_R-x)^{-1})}{l((x_R-x)^{-1})-1}\cdot\frac{\log(\tau)}{\tau}\,\mathrm{d}\tau\\
&= \lim_{x\uparrow x_R}\frac{1}{\log(x_R-x)}\int_{\frac{1}{x_R-x}}^{\frac{1}{x_R-G^{\leftarrow}(F(x))}}\frac{\log(\tau)}{\tau}\,\mathrm{d}\tau\\
&= \lim_{x\uparrow x_R}\frac{\log^2(x_R-G^{\leftarrow}(F(x)))-\log^2(x_R-x)}{2\log(x_R-x)}.
\end{aligned}$$

Write $G^{\leftarrow}(F(x))=x_R-(x_R-x)u(x)$, where $(x_R-x)u(x)\to0$ for all x sufficiently close to x_R. One then obtains

$$L=\lim_{x\uparrow x_R}\log(u(x))\left(1+\frac{\log(u(x))}{2\log(x_R-x)}\right),$$

implying $u(x)\to e^L$ and, subsequently, $\lim_{x\to\infty}\frac{x_R-G^{\leftarrow}(F(x))}{x_R-x}=e^L$.

Finally, consider $L=\infty$ and assume $\limsup_{x\to\infty}\frac{x_R-G^{\leftarrow}(F(x))}{x_R-x}<\infty$ for the sake of contradiction. Then there exists $M_0\ge1$ such that, for all $M\ge M_0$, there exists a sequence $(x_n)_{n\in\mathbb{N}}, x_n\uparrow x_R$, such that $\frac{x_R-G^{\leftarrow}(F(x_n))}{x_R-x_n}\le M$ for every $n\in\mathbb{N}$. A similar analysis as in (i) then shows that this contradicts relation (38), and therefore, $\lim_{x\to\infty}\frac{x_R-G^{\leftarrow}(F(x))}{x_R-x}=\infty$.

5.3. Proof of Lemma 10

For any positive, nonincreasing $\phi:[0,1)\to(0,1)$ that vanishes as the argument tends to unity, we may define

$$F_\phi(x):=\begin{cases}F(x) & \text{if } x<s_1,\text{ and}\\ F(x)+\frac{x-s_n}{s_{n+1}-s_n}(F(s_{n+1})-F(x)) & \text{if } s_n\le x<s_{n+1}, n\ge1,\end{cases}\tag{39}$$

where $s_1:=0$ and $s_{n+1}:=\inf\left\{x\ge0:F(x)\ge\frac{F(s_n)+\phi(F(s_n))}{1+\phi(F(s_n))}\right\}$ forms a strictly increasing sequence. Now, if $s_n\uparrow s^*<x_R$, then $F(s_n)\uparrow p$ for some $p\in(0,1)$ and therefore, for any $\varepsilon\in(0,1)$ and all n sufficiently large, we have $(1-\varepsilon)p\le F(s_n)\le p$. Consequently, s_{n+1} must satisfy $p\ge F(s_{n+1})\ge\frac{(1-\varepsilon)p+\phi(p)}{1+\phi(p)}$, which yields a contradiction if $\varepsilon<\phi(p)\frac{1-p}{p}$. We conclude that F_ϕ is a strictly increasing, continuous c.d.f. that satisfies $\overline{F}_\phi(x)\le\overline{F}(x)$ for all x.

Define $n(x):=\sup\{n\in\mathbb{N}:s_{n-1}\le x\}$. Then

$$\begin{aligned}
\frac{\overline{F}_\phi(x)}{\overline{F}(x)} &= 1-\frac{x-s_{n(x)}}{s_{n(x)+1}-s_{n(x)}}\frac{F(s_{n(x)+1})-F(x)}{\overline{F}(x)}\ge1-\frac{F(s_{n(x)+1})-F(s_{n(x)})}{1-F(s_{n(x)+1})}\\
&\ge 1-\frac{\frac{F(s_{n(x)})+\phi(F(s_{n(x)}))}{1+\phi(F(s_{n(x)}))}-F(s_{n(x)})}{1-\frac{F(s_{n(x)})+\phi(F(s_{n(x)}))}{1+\phi(F(s_{n(x)}))}}=1-\phi(F(s_{n(x)}))\to1
\end{aligned}\tag{40}$$

as $x\uparrow x_R$ so that $\overline{F}_\uparrow(x)\sim\overline{F}(x)$ by our earlier remark.

Let $(s_n)_{n\in\mathbb{N}}$ and $(\tilde{s}_n)_{n\in\mathbb{N}}$ be the sequences associated with F_ϕ and $F_{\tilde{\phi}}$ and assume $\tilde{\phi}(y)\le\phi(y)$ for all $y\in[0,1)$. We prove $\tilde{s}_n\le s_n$ for all $n\in\mathbb{N}$ by induction. The inequality $\tilde{s}_1\le s_1$ is immediate from the definition. Now, assume that $\tilde{s}_n\le s_n$ and observe that $(F(s)+q)/(1+q)$ is nondecreasing in s for every $q\ge0$ and in q for every $s\in\mathbb{R}$. Thus, any x that satisfies $F(x)\ge(F(s_n)+\phi(F(s_n)))/(1+\phi(F(s_n)))$ evidently satisfies $F(x)\ge(F(\tilde{s}_n)+\tilde{\phi}(F(\tilde{s}_n)))/(1+\tilde{\phi}(F(\tilde{s}_n)))$, and hence, $\tilde{s}_{n+1}\le s_{n+1}$.

As $F_\phi(x)\ge F(x)$ implies $G^{\leftarrow}(F_\phi(x))\ge G^{\leftarrow}(F(x))$, the proof is complete once we show that there is a version of ϕ such that $\limsup_{x\uparrow x_R}\frac{G^{\leftarrow}(F(x))}{G^{\leftarrow}(F_\phi(x))}\ge1$. To this end, we construct a suitable ϕ inductively.

Fix $\phi_1:=1/2$. Then, for $n=1,2,\ldots$, let $r_{n+1}:=\inf\left\{x\ge0:F(x)\ge\frac{F(s_n)+\phi_n}{1+\phi_n}\right\}$, denote $\phi_{n+1}:=\min\{\phi_n,\overline{F}(G^{\leftarrow}(r_{n+1}))^2/(4\mathbb{E}[B]^2)\}$, and define $\phi(y):=\phi_{n+1}$ for $y\in[F(s_n),F(s_{n+1}))$.

Because $\phi(F(s_n)) \le \phi_n$, it must be that $s_n \le r_n$ for all $n \in \mathbb{N}$. As a consequence, $\phi(F(s_n)) \le 2^{-2}\mathbb{E}[B]^{-2}\overline{F}(G^{\leftarrow}(s_{n+1}))^2$. Writing $\eta(x) := \phi(F(s_{n(x)}))$ for notational convenience, one may now use (40) to deduce

$$\begin{aligned}
G^{\leftarrow}(F(x)) &= \inf\{z \in \mathbb{R} : G(z) \ge F(x)\} = \inf\{z \in \mathbb{R} : \overline{G}(z) \le \overline{F}(x)\}\\
&\ge \inf\left\{z \in \mathbb{R} : \overline{G}(z) \le \frac{\overline{F}_\phi(x)}{1-\eta(x)}\right\}\\
&= \inf\left\{z - \sqrt{\eta(x)} \in \mathbb{R} : \overline{G}(z) + \mathbb{E}[B]^{-1}\int_{z-\sqrt{\eta(x)}}^{z} \overline{F}(t)\,\mathrm{d}t \le \overline{F}_\phi(x) + \frac{\eta(x)}{1-\eta(x)}\overline{F}_\phi(x)\right\}\\
&\ge \inf\left\{z \in \mathbb{R} : \overline{G}(z) + \mathbb{E}[B]^{-1}\sqrt{\eta(x)}\overline{F}(z) \le \overline{F}_\phi(x) + \frac{\eta(x)}{1-\eta(x)}\right\} - \sqrt{\eta(x)}\\
&\ge G^{\leftarrow}(F_\phi(x)) - \sqrt{\eta(x)},
\end{aligned}$$

where the last inequality follows from the relation

$$\begin{aligned}
\frac{\eta(x)}{1-\eta(x)} - \mathbb{E}[B]^{-1}\sqrt{\eta(x)}\overline{F}(z) &\le \frac{\phi(F(s_{n(x)}))}{1-\phi(F(s_{n(x)}))} - \mathbb{E}[B]^{-1}\sqrt{\phi(F(s_{n(x)}))}\overline{F}(G^{\leftarrow}(F(s_{n(x)+1})))\\
&\le \sqrt{\phi(F(s_{n(x)}))}\left[2\sqrt{\phi(F(s_{n(x)}))} - \mathbb{E}[B]^{-1}\overline{F}(G^{\leftarrow}(F(s_{n(x)+1})))\right] \le 0
\end{aligned}$$

for all $z \le G^{\leftarrow}(F_\phi(x)) \le G^{\leftarrow}(F(s_{n(x)+1}))$. We conclude that $\overline{G}(F_\uparrow(x)) \sim \overline{G}(F(x))$ as $x \uparrow x_R$.

6. Scaled Sojourn Time Tends to Zero in Probability

The current section is dedicated to the proof of Theorem 9. The intuition behind the proof is that the sojourn times of all jobs of size at most $\widetilde{x}_\rho$ grow slower than $\mathbb{E}[T_{\mathrm{FB}}]$, where $\widetilde{x}_\rho$ is a function that depends on F. Alternatively, the fraction of jobs of size at least $\widetilde{x}_\rho$ tends to zero because $\widetilde{x}_\rho \to x_R$ as $\rho \uparrow 1$. Section 7 discusses the sojourn time of these jobs in more detail.

For any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\frac{T_{\mathrm{FB}}}{\mathbb{E}[T_{\mathrm{FB}}]} > \varepsilon\right) = \int_0^\infty \mathbb{P}(T_{\mathrm{FB}}(x) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}])\,\mathrm{d}F(x) \le \mathbb{P}(T_{\mathrm{FB}}(\widetilde{x}_\rho) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}]) + \overline{F}(\widetilde{x}_\rho), \tag{41}$$

where the final term vanishes as $\rho \uparrow 1$ by choice of $\widetilde{x}_\rho$. The proof is completed if the first probability at the right-hand side also vanishes as $\rho \uparrow 1$.

In preparation for the analysis of $\mathbb{P}(T_{\mathrm{FB}}(\widetilde{x}_\rho) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}])$, reconsider the busy period representation $T_{\mathrm{FB}}(x) \stackrel{d}{=} \mathcal{L}_x(W_x + x)$. This relation states that the sojourn time of a job of size x is equal in distribution to a busy period with job sizes $B_i \wedge x$, initiated by the job of size x itself and the time W_x required to serve all jobs already in the system up to level x. Here, the random variable W_x is equal in distribution to the steady-state waiting time in an M/GI/1/FIFO queue with job sizes $B_i \wedge x$.

Let $N_x(t)$ denote a Poisson process with rate $\rho_x/\mathbb{E}[B \wedge x]$. Then, it follows from the busy period representation of T_{FB} that

$$\begin{aligned}
\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(x) > y) &= \mathbb{P}(\mathcal{L}_x(W_x + x) > (1-\rho)^{-2}y)\\
&= \mathbb{P}\left(\inf\left\{t \ge 0 : \sum_{i=1}^{N(t)}(B_i \wedge x) - t \le -(W_x + x)\right\} > (1-\rho)^{-2}y\right)\\
&= \mathbb{P}\left(\inf_{t\in[0,(1-\rho)^{-2}y]}\left\{\sum_{i=1}^{N(t)}(B_i \wedge x) - t\right\} \ge -(W_x + x)\right)\\
&= \mathbb{P}\left(\sup_{t\in[0,y]}\left\{\frac{t}{(1-\rho)^2} - \sum_{i=1}^{N((1-\rho)^{-2}t)}(B_i \wedge x)\right\} \le W_x + x\right).
\end{aligned} \tag{42}$$

Additionally, application of Chebychev's inequality to the preceding relation yields

$$
\begin{aligned}
\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(x) > y) &\le \mathbb{P}\left(\frac{y}{(1-\rho)^2} - \sum_{i=1}^{N((1-\rho)^{-2}y)} (B_i \wedge x) \le W_x + x\right) \\
&\le \mathbb{P}\Bigg(\left| W_x + \sum_{i=1}^{N((1-\rho)^{-2}y)} (B_i \wedge x) - \frac{\rho_x}{1-\rho_x}\mathbb{E}[(B\wedge x)^*] - \frac{\rho_x}{(1-\rho)^2} y\right| \\
&\quad \ge \frac{1-\rho_x}{(1-\rho)^2} y - x - \frac{\rho_x}{1-\rho_x}\mathbb{E}[(B\wedge x)^*]\Bigg) \\
&\le \frac{\mathbb{V}\mathrm{ar}[W_x] + \mathbb{V}\mathrm{ar}\left[\sum_{i=1}^{N((1-\rho)^{-2}y)} (B_i \wedge x)\right]}{\left(\frac{1-\rho_x}{(1-\rho)^2} y - x - \frac{\rho_x}{1-\rho_x}\mathbb{E}[(B\wedge x)^*]\right)^2} \\
&= \frac{\frac{\rho_x^2}{(1-\rho_x)^2}\mathbb{E}[(B\wedge x)^*]^2 + \frac{\rho_x}{1-\rho_x}\mathbb{E}[((B\wedge x)^*)^2] + \frac{2\rho_x \mathbb{E}[(B\wedge x)^*]}{(1-\rho)^2} y}{\left(\frac{1-\rho_x}{(1-\rho)^2} y - x - \frac{\rho_x}{1-\rho_x}\mathbb{E}[(B\wedge x)^*]\right)^2}.
\end{aligned} \tag{43}
$$

At this point, similar to the approach in Section 4, we distinguish between the finite and infinite variance cases.

6.1. Finite Variance

This section considers all functions F that satisfy one of the conditions in the theorem statement and have finite variance. Specifically, this excludes the case $x_R = \infty$ for $\beta(\overline{F}) > -2$. Fix

$$
\widetilde{p}(F) := \begin{cases} \frac{\beta(\overline{F})}{\beta(\overline{F})+1} & \text{if } F \notin \mathrm{MDA}(\Lambda) \text{ and } x_R = \infty, \\ \frac{\beta(\overline{F}(x_R - (\cdot)^{-1})}{\beta(\overline{F}(x_R-(\cdot)^{-1}))-1} & \text{if } F \notin \mathrm{MDA}(\Lambda) \text{ and } x_R < \infty, \text{ and} \\ 1 & \text{if } F \in \mathrm{MDA}(\Lambda), \end{cases} \tag{44}
$$

and $\widetilde{\gamma} \in (\widetilde{p}(F)/2, 1)$ and define $\nu(\rho) := (1-\rho)^{\widetilde{\gamma}}$ and $\widetilde{x}_\rho := x_\rho^{\nu(\rho)} = G^{\leftarrow}(1 - \frac{1-\rho}{\rho}\frac{1-\nu(\rho)}{\nu(\rho)})$. Indeed $\widetilde{x}_\rho \to x_R$, and we proceed with the analysis in (43). Noting that $\mathbb{E}[((B\wedge x)^*)^2] = \frac{\mathbb{E}[(B\wedge x)^3]}{3\mathbb{E}[B]} \le \frac{x\mathbb{E}[B^2]}{3\mathbb{E}[B]} = \frac{2}{3}\mathbb{E}[B^*]x$ and substituting $x = \widetilde{x}_\rho$ gives

$$
\begin{aligned}
\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(\widetilde{x}_\rho) > y) &\le \frac{\left(\frac{1-\rho}{1-\rho_{\widetilde{x}_\rho}}\right)^2 \mathbb{E}[B^*]^2 + \frac{1-\rho}{1-\rho_{\widetilde{x}_\rho}}\frac{2}{3}\mathbb{E}[B^*](1-\rho)\widetilde{x}_\rho + 2\mathbb{E}[B^*]y}{\left(\frac{1-\rho_{\widetilde{x}_\rho}}{1-\rho} y - (1-\rho)\widetilde{x}_\rho - \frac{1-\rho}{1-\rho_{\widetilde{x}_\rho}}\rho_{\widetilde{x}_\rho}\mathbb{E}[B^*]\right)^2} \\
&= \frac{\mathbb{E}[B^*]^2 \nu(\rho)^2 + \frac{2}{3}\mathbb{E}[B^*]\nu(\rho)(1-\rho)x_\rho^{\nu(\rho)} + 2\mathbb{E}[B^*]y}{\left(\nu(\rho)^{-1}y - (1-\rho)x_\rho^{\nu(\rho)} - \rho_{x_\rho^{\nu(\rho)}}\mathbb{E}[B^*]\nu\right)^2}.
\end{aligned}
$$

We now return to the probability $\mathbb{P}(T_{\mathrm{FB}}(\widetilde{x}_\rho) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}])$ in relation (41). By Theorems 6 and 7, there exists $C > 0$ such that the inequality $(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}] \ge C\overline{F}(G^{\leftarrow}(\rho))$ holds true for all ρ sufficiently close to one. Denoting $\widetilde{\varepsilon} := \varepsilon C$, this gives

$$
\begin{aligned}
\mathbb{P}(T_{\mathrm{FB}}(\widetilde{x}_\rho) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}]) &\le \mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(\widetilde{x}_\rho) > \widetilde{\varepsilon}\overline{F}(G^{\leftarrow}(\rho))) \\
&\le \frac{\mathbb{E}[B^*]^2\nu(\rho)^2 + \frac{2}{3}\mathbb{E}[B^*]\nu(\rho)(1-\rho)x_\rho^{\nu(\rho)} + 2\widetilde{\varepsilon}\mathbb{E}[B^*]\overline{F}(G^{\leftarrow}(\rho))}{\left(\widetilde{\varepsilon}\nu(\rho)^{-1}\overline{F}(G^{\leftarrow}(\rho)) - (1-\rho)x_\rho^{\nu(\rho)} - \rho_{x_\rho^{\nu(\rho)}}\mathbb{E}[B^*]\nu(\rho)\right)^2} \\
&= \frac{\mathbb{E}[B^*]^2 \frac{\nu(\rho)^4}{\overline{F}(G^{\leftarrow}(\rho))^2} + \frac{2\mathbb{E}[B^*]}{3}\frac{\nu(\rho)^3(1-\rho)x_\rho^{\nu(\rho)}}{\overline{F}(G^{\leftarrow}(\rho))^2} + 2\widetilde{\varepsilon}\mathbb{E}[B^*]\frac{\nu(\rho)^2}{\overline{F}(G^{\leftarrow}(\rho))}}{\left(\widetilde{\varepsilon} - \frac{\nu(\rho)(1-\rho)x_\rho^{\nu(\rho)}}{\overline{F}(G^{\leftarrow}(\rho))} - \rho_{x_\rho^{\nu(\rho)}}\mathbb{E}[B^*]\frac{\nu(\rho)^2}{\overline{F}(G^{\leftarrow}(\rho))}\right)^2}.
\end{aligned}
$$

Subsequently, we observe for any $\nu \in (0,1)$ that

$$\lim_{\rho\uparrow 1}(1-\rho)x_\rho^\nu = \lim_{\rho\uparrow 1}(1-\rho)G^{\leftarrow}\left(1-\frac{1-\nu}{\nu}\frac{1-\rho}{\rho}\right) = \lim_{z\to x_R}\frac{\frac{\nu}{1-\nu}\overline{G}(z)\cdot z}{1+\frac{\nu}{1-\nu}\overline{G}(z)} \le \lim_{z\to x_R}\frac{\nu\cdot z\overline{G}(z)}{1-\nu}, \tag{45}$$

where $z\overline{G}(z) \to 0$ as $z \to x_R$ because $\mathbb{E}[B^2] < \infty$ (cf. Section 4.1.1). It, therefore, follows that $(1-\rho)x_\rho^{\nu(\rho)} = o(\nu(\rho))$ as $\rho \uparrow 1$, and consequently, $\lim_{\rho\uparrow 1}\mathbb{P}(T_{\mathrm{FB}} > \varepsilon\mathbb{E}[T_{\mathrm{FB}}]) = 0$ provided that $\lim_{\rho\uparrow 1}\frac{\nu(\rho)^2}{\overline{F}(G^{\leftarrow}(\rho))} = 0$.

Write $x = (1-\rho)^{-1}$. By Lemma 4, it suffices to show $\alpha\big((\cdot)^{-2\widetilde{\gamma}}\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))\big) < 0$. This relation follows from Lemma 3, Corollary 4, and our choice of $\widetilde{\gamma}$:

$$\alpha\left(\frac{(\cdot)^{-2\widetilde{\gamma}}}{\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))}\right) \le -2\widetilde{\gamma} - \beta\big(\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))\big) \le -2\widetilde{\gamma} + \widetilde{p}(F) < 0.$$

6.2. Infinite Variance

This section regards all functions F that satisfy $x_R = \infty, \beta(\overline{F}) > -2$. In this case, $\widetilde{x}_\rho$ can be any function that satisfies both $\lim_{\rho\uparrow 1}\widetilde{x}_\rho = \infty$ and $\lim_{\rho\uparrow 1}\frac{\widetilde{x}_\rho}{\overline{G}(\widetilde{x}_\rho)\log\left(\frac{1}{1-\rho}\right)} = 0$.

Theorem 6 implies that there exists $C > 0$ such that $\mathbb{E}[T_{\mathrm{FB}}] \ge C\log(\frac{1}{1-\rho})$ for all ρ sufficiently close to one. Again, denote $\widetilde{\varepsilon} = \varepsilon C$. The analysis resumes with relation (43), where we substitute y by $\widetilde{\varepsilon}(1-\rho)^2\log(\frac{1}{1-\rho})$ to obtain

$$\mathbb{P}(T_{\mathrm{FB}}(x) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}]) \le \mathbb{P}\left((1-\rho)^2T_{\mathrm{FB}}(x) > \widetilde{\varepsilon}(1-\rho)^2\log\left(\frac{1}{1-\rho}\right)\right)$$
$$\le \frac{\frac{1}{(1-\rho_x)^2}\mathbb{E}[(B\wedge x)^*]^2 + \frac{1}{1-\rho_x}\mathbb{E}[((B\wedge x)^*)^2] + 2\widetilde{\varepsilon}\mathbb{E}[(B\wedge x)^*]\log\left(\frac{1}{1-\rho}\right)}{\left(\widetilde{\varepsilon}(1-\rho_x)\log\left(\frac{1}{1-\rho}\right) - x - \frac{\rho_x}{1-\rho_x}\mathbb{E}[(B\wedge x)^*]\right)^2}.$$

By relation (22), there exists a function $b(x)$ that is bounded for all x sufficiently large and satisfies $m_2(x) = \mathbb{E}[B]b(x)x\overline{G}(x)$. As such, $\mathbb{E}[((B\wedge x)^*)^2] = \frac{\mathbb{E}[(B\wedge x)^3]}{3\mathbb{E}[B]} \le \frac{xm_2(x)}{3\mathbb{E}[B]} = b(x)x^2\overline{G}(x)/3$ and similarly $\mathbb{E}[(B\wedge x)^*] = \frac{m_2(x)}{2\mathbb{E}[B]} = b(x)x\overline{G}(x)/2$. Substituting this into the preceding relation yields

$$\mathbb{P}(T_{\mathrm{FB}}(x) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}]) \le \frac{\frac{b(x)^2}{4}\frac{x^2\overline{G}(x)^2}{(1-\rho_x)^2} + \frac{b(x)}{3}\frac{x^2\overline{G}(x)}{1-\rho_x} + \widetilde{\varepsilon}b(x)x\overline{G}(x)\log\left(\frac{1}{1-\rho}\right)}{\left(\widetilde{\varepsilon}(1-\rho_x)\log\left(\frac{1}{1-\rho}\right) - x - \frac{\rho_x b(x)}{2}\frac{x\overline{G}(x)}{1-\rho_x}\right)^2},$$

so that

$$\mathbb{P}(T_{\mathrm{FB}}(x) > \varepsilon\mathbb{E}[T_{\mathrm{FB}}])$$
$$\le \frac{\frac{b(x)^2}{4}\frac{\overline{G}(x)^2}{(1-\rho_x)^2}\frac{x^2}{(1-\rho_x)^2\log^2\left(\frac{1}{1-\rho}\right)} + \frac{b(x)}{3}\frac{\overline{G}(x)}{1-\rho_x}\frac{x^2}{(1-\rho_x)^2\log^2\left(\frac{1}{1-\rho}\right)} + \widetilde{\varepsilon}b(x)\frac{\overline{G}(x)}{1-\rho_x}\frac{x}{(1-\rho_x)\log\left(\frac{1}{1-\rho}\right)}}{\left(\widetilde{\varepsilon} - \frac{x}{(1-\rho_x)\log\left(\frac{1}{1-\rho}\right)} - \frac{\rho_x b(x)}{2}\frac{\overline{G}(x)}{1-\rho_x}\frac{x}{(1-\rho_x)\log\left(\frac{1}{1-\rho}\right)}\right)^2}.$$

The result follows after noting that $1-\rho_x = 1-\rho G(x) \ge \overline{G}(x)$ and substituting $\widetilde{x}_\rho$ for x.

7. Asymptotic Behavior of the Sojourn Time Tail

In this section, we prove Theorem 10 after presenting two facilitating propositions. The proofs of the propositions are postponed to Sections 7.1 and 7.2. Throughout this section, $\mathbf{e}(q)$ denotes an exponentially distributed random variable with rate $q > 0$. We abuse notation by writing $\mathbf{e}(0) = +\infty$.

Reconsider the relation $T_{\mathrm{FB}}(x) \stackrel{d}{=} \mathscr{L}_x(W_x + x)$ to gain some intuition. A rough approximation of the duration of a busy period given $W_x + x$ units of work at time $t = 0$ is $(W_x + x)/(1-\rho_x)$. The scaled sojourn time $(1-\rho)^2T_{\mathrm{FB}}(x)$ is then approximated by $\frac{1-\rho}{1-\rho_x}(1-\rho)(W_x + x)$. As in Section 4, define $x_\rho^\nu = G^{\leftarrow}(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu})$, $\nu \in (1-\rho, 1)$, so that $\frac{1-\rho}{1-\rho_x} = \nu$. Then, for all $\nu \in (0,1)$, we have $(1-\rho)^2T_{\mathrm{FB}}(x_\rho^\nu) \stackrel{d}{\approx} \nu(1-\rho)(W_{x_\rho^\nu} + x_\rho^\nu)$. We show that

$(1-\rho)x_\rho^\nu \to 0$ for all fixed $\nu \in (0,1)$. Instead, the following proposition shows that $(1-\rho)W_{x_\rho^\nu}$ behaves as an exponentially distributed random variable as $\rho \uparrow 1$.

Proposition 1. *Let* $x_\rho^\nu = G^{\leftarrow}(1 - \frac{1-\rho}{\rho}\frac{1-\nu}{\nu})$, $\nu \in (1-\rho, 1)$, *and let* W_x^ρ *denote the steady-state waiting time in an* M/GI/1/FIFO *queue with job sizes* $B_i \wedge x$ *and arrival rate* $\rho_x/\mathbb{E}[B \wedge x]$. *Then, for any fixed* $\nu \in (0,1)$, $(1-\rho)W_{x_\rho^\nu} \xrightarrow{d} \mathrm{Exp}((\nu\mathbb{E}[B^*])^{-1})$ *as* $\rho \uparrow 1$.

Kingman (1961) proved that, if $W^\rho = W_\infty^\rho$ denotes the steady-state waiting time in the nontruncated system, then $(1-\rho)W^\rho \xrightarrow{d} \mathrm{Exp}(\mathbb{E}[B^*]^{-1})$. Proposition 1 shows how jobs can be truncated such that the exponential behavior is preserved and quantifies how the truncation affects the parameter of the exponential distribution.

Substituting the result in Proposition 1 into our approximation yields $(1-\rho)^2 T_{\mathrm{FB}}(x_\rho^\nu) \overset{d}{\approx} \mathrm{Exp}((\nu^2\mathbb{E}[B^*])^{-1})$ for every fixed $\nu \in (0,1)$. We show that the fraction of jobs for which ν is in $(\varepsilon, 1-\varepsilon)$ scales as $\overline{F}(G^{\leftarrow}(\rho))$ and that the contribution of other jobs to the tail of $(1-\rho)^2 T_{\mathrm{FB}}$ is negligible. The result is presented in Proposition 2, in which we focus on the probability $\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}} > \mathbf{e}(q))$ for its connection to the Laplace transform of T_{FB}^*.

Proposition 2. *Assume* $F \in \mathrm{MDA}(H)$, *where* H *is an extreme value distribution. Let* $p(H) = \frac{\alpha}{\alpha-1}$ *if* $H = \Phi_\alpha$, $\alpha > 2$; $p(H) = 1$ *if* $H = \Lambda$, *and* $p(H) = \frac{\alpha}{\alpha+1}$ *if* $H = \Psi_\alpha$, $\alpha > 0$. *Then*

$$\lim_{\rho\uparrow 1} \frac{\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}} > \mathbf{e}(q))}{\overline{F}(G^{\leftarrow}(\rho))} = \int_0^1 \frac{8\mathbb{E}[B^*]q\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)} \mathrm{d}\nu \tag{46}$$

for all $q \geq 0$. *Here, the integral is finite for all* $q \geq 0$.

We are now ready to prove Theorem 10. Using the relation $\mathbb{E}[e^{-qY}] = \mathbb{P}(\mathbf{e}(q) > Y)$, one sees that $\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}^\rho > \mathbf{e}(q)) = 1 - \mathbb{E}[e^{-q(1-\rho)^2 T_{\mathrm{FB}}^\rho}]$, and consequently,

$$\begin{aligned}\frac{\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}} > \mathbf{e}(q))}{\overline{F}(G^{\leftarrow}(\rho))} &= \frac{(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]}{\overline{F}(G^{\leftarrow}(\rho))} \cdot \frac{1-\mathbb{E}\left[e^{-q(1-\rho)^2 T_{\mathrm{FB}}}\right]}{(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]} \\ &= \frac{(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]}{\overline{F}(G^{\leftarrow}(\rho))} \cdot q \cdot \mathbb{E}\left[e^{-q(1-\rho)^2 T_{\mathrm{FB}}^*}\right],\end{aligned}$$

where T_{FB}^* is the residual sojourn time and has density $\mathbb{P}(T_{\mathrm{FB}} > t)/\mathbb{E}[T_{\mathrm{FB}}]$. Consequently,

$$\begin{aligned}\lim_{\rho\uparrow 1} \mathbb{E}\left[e^{-q(1-\rho)^2 T_{\mathrm{FB}}^*}\right] &= \lim_{\rho\uparrow 1} \frac{\overline{F}(G^{\leftarrow}(\rho))}{(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]} \int_0^1 \frac{8\mathbb{E}[B^*]\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)} \mathrm{d}\nu \\ &= r(H)^{-1} \int_0^1 \frac{8\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)} \mathrm{d}\nu\end{aligned} \tag{47}$$

for all $q \geq 0$, where $r(H)$ was introduced in Theorem 7. It follows from Sections 4.2.1 and 4.2.2 that $\lim_{q\downarrow 0}\lim_{\rho\uparrow 1}\mathbb{E}[e^{-q(1-\rho)^2 T_{\mathrm{FB}}^*}] = 1$. Additionally, the right-hand side is continuous in q so that $(1-\rho)^2 T_{\mathrm{FB}}^*$ converges to some nondegenerate random variable by the continuity theorem (Feller 1971, section XIII.1, theorem 2a).

The Laplace transform inversion equation (12) in Bateman (1954) states that $f(t) = \frac{2\sqrt{t}}{\sqrt{\pi}} - 2te^t\mathrm{Erfc}(\sqrt{t})$ is the Laplace inverse of $s^{-1/2}(s^{1/2}+1)^{-2}$, that is, $\int_0^\infty e^{-qt}f(t)\,\mathrm{d}t = \frac{1}{\sqrt{q}(\sqrt{q}+1)^2}$. Consequently, we have

$$\int_0^\infty e^{-qt}g(t,\nu)\,\mathrm{d}t = \frac{1}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1\right)^2} \tag{48}$$

for $g(t,\nu) = \frac{e^{-\frac{t}{4\mathbb{E}[B^*]\nu^2}}}{4\mathbb{E}[B^*]\nu^2} f\left(\frac{t}{4\mathbb{E}[B^*]\nu^2}\right)$, and hence relation (47) may be rewritten as

$$\begin{aligned}\lim_{\rho\uparrow 1} \mathbb{E}\left[e^{-q(1-\rho)^2 T_{\mathrm{FB}}^*}\right] &= \int_0^\infty e^{-qt}\left[\int_0^1 8r(H)^{-1}\nu\left(\frac{1-\nu}{\nu}\right)^{p(H)} g(t,\nu)\,\mathrm{d}\nu\right]\mathrm{d}t \\ &=: \int_0^\infty e^{-qt}g^*(t)\,\mathrm{d}t.\end{aligned}$$

We conclude that the limiting random variable $\lim_{\rho\uparrow 1}(1-\rho)^2 T^*_{\mathrm{FB}}$ has density g^*. Furthermore, as

$$\lim_{\rho\uparrow 1}\mathbb{E}\Big[e^{-q(1-\rho)^2 T^*_{\mathrm{FB}}}\Big] = \lim_{\rho\uparrow 1}\int_0^\infty e^{-q\tau}\frac{\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}>\tau)}{(1-\rho)^2\mathbb{E}[T_{\mathrm{FB}}]}\,\mathrm{d}\tau = \lim_{\rho\uparrow 1}\int_0^\infty e^{-q\tau}\frac{\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}>\tau)}{r(H)\mathbb{E}[B^*]\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}\tau,$$

for all $q\geq 0$, we also see that $\lim_{\rho\uparrow 1}\frac{\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}>y)}{r(H)\mathbb{E}[B^*]\overline{F}(G^{\leftarrow}(\rho))} = g^*(y)$ almost everywhere.

To see that g^* is monotone, it suffices to show that $f(t)$ is monotone. To this end, we exploit the continued fraction representation (13.2.20a in Cuyt et al. (2008)) and find

$$\mathrm{Erfc}(x) = \frac{x}{\sqrt{\pi}}e^{-x^2}\cfrac{1}{x^2+\cfrac{1/2}{1+\cfrac{1}{x^2+\cfrac{3/2}{1+\dots}}}} \geq \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1-\frac{x^2+3/2}{2x^4+6x^2+3/2}\right). \tag{49}$$

As a consequence, one sees that

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = \frac{1+2t}{\sqrt{\pi}\sqrt{t}} - 2(1+t)e^t\mathrm{Erfc}(\sqrt{t}) \leq \frac{1+2t-2(1+t)\left(1-\frac{t+3/2}{2t^2+6t+3/2}\right)}{\sqrt{\pi}\sqrt{t}} = \frac{-1+\frac{2t^2+5t+3}{2t^2+6t+3/2}}{\sqrt{\pi}\sqrt{t}},$$

which is negative for all $t\geq 0$. We conclude the section with the postponed proofs of Propositions 1 and 2.

7.1. Proof of Proposition 1

The Pollaczek–Khintchine formula states that $\mathbb{E}[e^{-s(1-\rho)W_x}] = \frac{1-\rho_x}{1-\rho_x\mathbb{E}[e^{-s(1-\rho)(B\wedge x)^*}]}$. In this representation, we expand the Laplace–Stieltjes transform $\mathbb{E}[e^{-s(1-\rho)(B\wedge x)^*}]$ around $\rho = 1$ to find

$$\mathbb{E}[e^{-s(1-\rho)W_x}] = \frac{1-\rho_x}{1-\rho_x\big(1-\mathbb{E}[(B\wedge x)^*](1-\rho)s+o(1-\rho)\big)}$$

and, hence,

$$\mathbb{E}[e^{-s(1-\rho)W_{x^\nu_\rho}}] = \frac{1}{1+\frac{1-\rho}{1-\rho_{x^\nu_\rho}}\rho_{x^\nu_\rho}\mathbb{E}[(B\wedge x^\nu_\rho)^*]s+o\left(\frac{1-\rho}{1-\rho_{x^\nu_\rho}}\right)} = \frac{1}{1+\nu\rho_{x^\nu_\rho}\mathbb{E}[(B\wedge x^\nu_\rho)^*]s+o(1)},$$

where $o(1)$ vanishes as $\rho\uparrow 1$. By definition of x^ν_ρ, $x^\nu_\rho\to\infty$ and $\rho_{x^\nu_\rho}\uparrow 1$ as $\rho\uparrow 1$ for any fixed $\nu\in(0,1)$. In particular, $\lim_{\rho\uparrow 1}\mathbb{E}[e^{-s(1-\rho)W_{x^\nu_\rho}}] = \frac{1}{1+\nu\mathbb{E}[B^*]s}$. The proof is completed by applying the continuity theorem (Feller 1971, section XIII.1, theorem 2a).

7.2. Proof of Proposition 2

We require functions $\nu_l(\rho)\downarrow 0$ and $\nu_u(\rho)\uparrow 1$ that distinguish the jobs that significantly contribute to the tail of $(1-\rho)^2 T_{\mathrm{FB}}$ and those that do not. For the former function, fix $\gamma\in(p(H)/2,1)$ and let $\nu_l(\rho) = (1-\rho)^\gamma$ as in Section 6.1. This is possible as $p(H)<2$ for all H to which the theorem applies. For the latter function, we refer to relation (45) to verify that there exists a function $\nu(\rho)\uparrow 1$ such that $(1-\rho)x_\rho^{\nu(\rho)}\to 0$. Let $\nu_u(\rho)$ be a function with this property and write

$$\begin{aligned}\frac{\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}>\mathbf{e}(q))}{\overline{F}(G^{\leftarrow}(\rho))} &= \int_{\nu=0}^{\nu_l(\rho)}\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(x^\nu_\rho)>\mathbf{e}(q))\frac{\mathrm{d}F(x^\nu_\rho)}{\overline{F}(G^{\leftarrow}(\rho))}\\ &\quad+\int_{\nu=\nu_l(\rho)}^{\nu_u(\rho)}\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(x^\nu_\rho)>\mathbf{e}(q))\frac{\mathrm{d}F(x^\nu_\rho)}{\overline{F}(G^{\leftarrow}(\rho))}\\ &\quad+\int_{\nu=\nu_u(\rho)}^{1}\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(x^\nu_\rho)>\mathbf{e}(q))\frac{\mathrm{d}F(x^\nu_\rho)}{\overline{F}(G^{\leftarrow}(\rho))}\\ &=: \widehat{\mathrm{I}}(\rho)+\widehat{\mathrm{II}}(\rho)+\widehat{\mathrm{III}}(\rho).\end{aligned}$$

The next paragraphs study the behavior of $\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(x) > \mathbf{e}(q))$, which then facilitates the analysis of the preceding three regions. Specifically, we derive the asymptotic behavior of $\widehat{\mathrm{II}}(\rho)$ in terms of q and show that $\widehat{\mathrm{I}}(\rho) + \widehat{\mathrm{III}}(\rho) = o(1)$ for any $q \geq 0$.

Define $X_x^\rho(t) := \frac{t}{1-\rho} - \sum_{i=1}^{N((1-\rho)^{-2}t)} (1-\rho)(B_i \wedge x)$. Then $X_x^\rho(t)$ is a spectrally negative Lévy process, and we obtain

$$\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}(x) > \mathbf{e}(q)) = \mathbb{P}\left(\sup_{t\in[0,\mathbf{e}(q)]} X_x^\rho(t) \leq (1-\rho)W_x + (1-\rho)x\right) \tag{50}$$

from relation (42). The Laplace exponent of $X_x^\rho(t)$ is given by $\psi(s) := t^{-1}\log \mathbb{E}[e^{sX_x^\rho(t)}]$ and has right-inverse $\varphi(x,\rho,q) := \sup\{s \geq 0 : \psi(x,\rho,s) = q\}$. With these notions, relation (8.4) in Kyprianou (2014) states that

$$\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}^\rho(x) > \mathbf{e}(q)) = \mathbb{P}\big(\mathbf{e}(\varphi(x,\rho,q)) \leq (1-\rho)W_x + (1-\rho)x\big). \tag{51}$$

Because

$$\begin{aligned}\psi(x,\rho,s) &= t^{-1}\log\mathbb{E}\left[e^{\frac{st}{1-\rho} - \sum_{i=1}^{N((1-\rho)^{-2}t)}(1-\rho)s(B_i\wedge x)}\right]\\ &= \frac{s}{1-\rho} + t^{-1}\log\mathbb{E}\left[e^{-\sum_{i=1}^{N((1-\rho)^{-2}t)}(1-\rho)s(B_i\wedge x)}\right]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[e^{-\sum_{i=1}^{N((1-\rho)^{-2}t)}(1-\rho)s(B_i\wedge x)}] &= \sum_{n=0}^{\infty}\mathbb{E}[e^{-(1-\rho)s(B\wedge x)}]^n \frac{\left(\frac{\lambda t}{(1-\rho)^2}\right)^n}{n!} e^{-\frac{\lambda t}{(1-\rho)^2}}\\ &= e^{-\frac{\lambda t}{(1-\rho)^2}\left(1-\mathbb{E}[e^{-(1-\rho)s(B\wedge x)}]\right)},\end{aligned}$$

we obtain $\psi(x,\rho,s) = \frac{s}{1-\rho} - \frac{\lambda}{(1-\rho)^2}\left(1-\mathbb{E}[e^{-(1-\rho)s(B\wedge x)}]\right)$. A Taylor expansion around $\rho = 1$ now yields

$$\begin{aligned}&\psi(x,\rho,s)\\ &= \frac{s}{1-\rho} - \frac{\lambda}{(1-\rho)^2}\left(1-\left(1-(1-\rho)s\mathbb{E}[B\wedge x] + \frac{(1-\rho)^2 s^2}{2}\mathbb{E}[(B\wedge x)^2] + o((1-\rho)^2 s^2)\right)\right)\\ &= \frac{s}{1-\rho} - \left(\frac{\rho_x s}{1-\rho} - \frac{\lambda\mathbb{E}[(B\wedge x)^2]}{2}s^2 + o(s^2)\right) = \frac{1-\rho_x}{1-\rho}s + \frac{\rho\mathbb{E}[(B\wedge x)^2]}{2\mathbb{E}[B]}s^2 + o(s^2),\end{aligned}$$

so that $\lim_{\rho\uparrow 1}\psi(x_\rho^\nu,\rho,s) = \nu^{-1}s + \mathbb{E}[B^*]s^2$ for all $\nu > 0$, and consequently,

$$\lim_{\rho\uparrow 1}\varphi(x_\rho^\nu,\rho,q) = \frac{\sqrt{\nu^{-2}+4\mathbb{E}[B^*]q}-\nu^{-1}}{2\mathbb{E}[B^*]} = \frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{2\mathbb{E}[B^*]\nu} =: \varphi(\nu,q). \tag{52}$$

Similarly, one deduces that $\lim_{\rho\uparrow 1}\nu_l(\rho)\psi(x_\rho^{\nu_l(\rho)},\rho,s) = s$ and

$$\lim_{\rho\uparrow 1}\nu_l(\rho)^{-1}\varphi(x_\rho^{\nu_l(\rho)},\rho,q) = q. \tag{53}$$

We have now gathered sufficient tools to analyze the asymptotic behavior of $\widehat{\mathrm{II}}(\rho)$.

Fix $\varepsilon \in (0,1/3)$. We have already proven the relations $(1-\rho)W_{x_\rho^\nu}^\rho \to \mathbf{e}((\nu\mathbb{E}[B^*])^{-1})$ and $(1-\rho)x_\rho^\nu \to 0$ as $\rho\uparrow 1$ for all $\nu\in(0,1)$. Because $\mathbf{e}(q_1) \leq_{st} \mathbf{e}(q_2)$ whenever $q_1 \geq q_2$, relations (51) and (52) imply

$$\begin{aligned}\mathbb{P}((1-\rho)^2 T_{\mathrm{FB}}^\rho(x_\rho^\nu) > \mathbf{e}(q)) &\leq \mathbb{P}\big(\mathbf{e}((1+\varepsilon)\varphi(\nu,q)) \leq \mathbf{e}((1-\varepsilon)(\nu\mathbb{E}[B^*])^{-1}) + \varepsilon\big)\\ &= e^{-(1+\varepsilon)\varepsilon\varphi(\nu,q)}\frac{(1+\varepsilon)\varphi(\nu,q)}{(1+\varepsilon)\varphi(\nu,q)+(1-\varepsilon)(\nu\mathbb{E}[B^*])^{-1}} + 1 - e^{-\varepsilon(1+\varepsilon)\varphi(\nu,q)}\\ &\leq \frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1-\frac{4\varepsilon}{1+\varepsilon}} + 1 - e^{-\varepsilon\cdot\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\mathbb{E}[B^*]\nu}}\end{aligned}$$

for all $\rho \geq \rho_\varepsilon$, where $\rho_\varepsilon \in (0,1)$ is fixed sufficiently close to one. Consequently, for all $\rho \geq \rho_\varepsilon$,

$$\begin{aligned}
\widehat{\text{II}}(\rho) \leq& \int_{\nu_l(\rho)}^{\nu_u(\rho)} \frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1-\frac{4\varepsilon}{1+\varepsilon}} \frac{\mathrm{d}F\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))} \\
&+ \int_{\nu_l(\rho)}^{\nu_u(\rho)} \left(1-e^{-\varepsilon\cdot\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\mathbb{E}[B^*]\nu}}\right) \frac{\mathrm{d}F\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))} \\
\leq& -\left[\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1-\frac{4\varepsilon}{1+\varepsilon}} \frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\right]_{\nu=\nu_l(\rho)}^{\nu_u(\rho)} \\
&+ \int_{\nu_l(\rho)}^{\nu_u(\rho)} \frac{8\mathbb{E}[B^*]q\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1-\frac{4\varepsilon}{1+\varepsilon}\right)^2} \frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}\nu \\
&- \left[\left(1-e^{-\varepsilon\cdot\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\mathbb{E}[B^*]\nu}}\right) \frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\right]_{\nu=\nu_l(\rho)}^{\nu_u(\rho)} \\
&+ 4q\int_{\nu_l(\rho)}^{\nu_u(\rho)} \varepsilon\cdot e^{-\varepsilon\cdot\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\mathbb{E}[B^*]\nu}} \frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu}{\nu}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\,\mathrm{d}\nu.
\end{aligned}$$

In Sections 4.2.1 and 4.2.2, we deduced that $\overline{F}(G^{\leftarrow}(1-(\cdot)^{-1}))$ is regularly varying with index $-p(H)$. The uniform convergence theorem, hence, implies

$$\begin{aligned}
\limsup_{\rho\uparrow 1}\widehat{\text{II}}(\rho) \leq& -\left[\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1-\frac{4\varepsilon}{1+\varepsilon}}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\right]_{\nu=0}^{1} \\
&+ \int_0^1 \frac{8\mathbb{E}[B^*]q\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1-\frac{4\varepsilon}{1+\varepsilon}\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\mathrm{d}\nu \\
&- \left[\left(1-e^{-\varepsilon\cdot\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\mathbb{E}[B^*]\nu}}\right)\left(\frac{1-\nu}{\nu}\right)^{p(H)}\right]_{\nu=0}^{1} \\
&+ 4q\int_0^1 \varepsilon\cdot e^{-\varepsilon\cdot\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\mathbb{E}[B^*]\nu}}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\mathrm{d}\nu \\
=& \int_0^1 \frac{8\mathbb{E}[B^*]q\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1-\frac{4\varepsilon}{1+\varepsilon}\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\mathrm{d}\nu \\
&+ 4q\int_0^1 \varepsilon\cdot e^{-\varepsilon\cdot\frac{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}-1}{\mathbb{E}[B^*]\nu}}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\mathrm{d}\nu.
\end{aligned}$$

Both these integrals are bounded for all $\varepsilon \in (0,1/3)$ and all $q \geq 0$. Additionally, both integrands are increasing in ε for all ε sufficiently small. One may, thus, take the limit $\varepsilon\downarrow 0$ and apply the dominated convergence theorem to find

$$\limsup_{\rho\uparrow 1}\widehat{\text{II}}(\rho) \leq \int_0^1 \frac{8\mathbb{E}[B^*]q\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\mathrm{d}\nu. \tag{54}$$

Similarly, one may show that

$$\liminf_{\rho\uparrow 1}\widehat{\text{II}}(\rho) \geq \int_0^1 \frac{8\mathbb{E}[B^*]q\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1+\frac{4\varepsilon}{1-\varepsilon}\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\mathrm{d}\nu,$$

and we conclude

$$\lim_{\rho\uparrow 1}\widehat{\text{II}}(\rho)=\int_0^1 \frac{8\mathbb{E}[B^*]q\nu}{\sqrt{1+4\mathbb{E}[B^*]q\nu^2}\left(\sqrt{1+4\mathbb{E}[B^*]q\nu^2}+1\right)^2}\left(\frac{1-\nu}{\nu}\right)^{p(H)}\mathrm{d}\nu. \tag{55}$$

Second, consider $\widehat{\text{I}}(\rho)$. Define $M(\rho):=(1-\rho)^{-\widehat{\gamma}}$ for some $\widehat{\gamma}\in\left(p(H)/2,\gamma\right)$ and recall that $(1-\rho)x\to 0$ and $(1-\rho)\cdot W_x^\rho \xrightarrow{d} 0$ (and, hence, in probability) for all $x\le x_\rho^{\nu_l(\rho)}$. Thus, for all $x\le x_\rho^{\nu_l}$ and all ρ sufficiently large, we have

$$\begin{aligned}\widehat{\text{I}}(\rho)&=\int_{\nu=0}^{\nu_l(\rho)}\mathbb{P}\left(\mathbf{e}(\varphi(x_\rho^\nu,\rho,q))\le(1-\rho)W_{x_\rho^\nu}+(1-\rho)x_\rho^\nu\right)\frac{\mathrm{d}F(x_\rho^\nu)}{\overline{F}(G^{\leftarrow}(\rho))}\\&\le\frac{\mathbb{P}\left(\mathbf{e}(\varphi(x_\rho^{\nu_l(\rho)},\rho,q))\le 2M(\rho)\right)}{\overline{F}(G^{\leftarrow}(\rho))}+\frac{\mathbb{P}((1-\rho)W_x^\rho\ge M(\rho))}{\overline{F}(G^{\leftarrow}(\rho))}=:\widehat{\text{Ia}}(\rho)+\widehat{\text{Ib}}(\rho).\end{aligned}$$

Fix $\delta\in\left(0,p(H)-\gamma-\widehat{\gamma}\right)$. Potter's theorem (Bingham et al. 1989, theorem 1.5.6) states that $\overline{F}(G^{\leftarrow}(\rho))\ge C(1-\rho)^{p(H)+\delta}$ for some constant $C>0$ and all ρ sufficiently close to one. Also, one may readily deduce from relation (53) that $\mathbf{e}(\varphi(x_\rho^{\nu_l(\rho)},\rho,q))\ge_{st}\mathbf{e}(2q\nu_l(\rho))$ for all $x\le x_\rho^{\nu_l(\rho)}$ and ρ sufficiently large. Consequently,

$$\begin{aligned}\limsup_{\rho\uparrow 1}\widehat{\text{Ia}}(\rho)&\le\limsup_{\rho\uparrow 1}\frac{1-e^{-4q\nu_l(\rho)M(\rho)}}{\overline{F}(G^{\leftarrow}(\rho))}\le\lim_{\rho\uparrow 1}\frac{1-e^{-4q(1-\rho)^{\gamma-\widehat{\gamma}}}}{C(1-\rho)^{p(H)+\delta}}\\&=\lim_{\rho\uparrow 1}\frac{4q(\gamma-\widehat{\gamma})(1-\rho)^{\gamma-\widehat{\gamma}-1}e^{-4q(1-\rho)^{\gamma-\widehat{\gamma}}}}{C\left(p(H)+\delta\right)(1-\rho)^{p(H)-1+\delta}}\\&=\lim_{\rho\uparrow 1}\frac{4q(\gamma-\widehat{\gamma})}{C\left(p(H)+\delta\right)}\cdot\exp\left[-4q(1-\rho)^{\gamma-\widehat{\gamma}}+\left(\gamma-\widehat{\gamma}-p(H)-\delta\right)\log(1-\rho)\right]=0.\end{aligned}$$

For term $\widehat{\text{Ib}}(\rho)$, we apply Markov's inequality and Potter's theorem to obtain

$$\begin{aligned}\limsup_{\rho\uparrow 1}\widehat{\text{Ib}}(\rho)&\le\limsup_{\rho\uparrow 1}\frac{\frac{1-\rho}{1-\rho_x}\rho_x\mathbb{E}[(B\wedge x)^*]}{M(\rho)\overline{F}(G^{\leftarrow}(\rho))}\le\lim_{\rho\uparrow 1}C_1\frac{\mathbb{E}[B^*]\nu_l(\rho)}{M(\rho)(1-\rho)^{p(H)+\delta}}\\&=\lim_{\rho\uparrow 1}C_1\mathbb{E}[B^*](1-\rho)^{\gamma+\widehat{\gamma}-p(H)-\delta}=0.\end{aligned}$$

Finally, consider term $\widehat{\text{III}}(\rho)$. For this term, the claim follows readily from the uniform convergence theorem and the property $\nu_u(\rho)\uparrow 1$:

$$\begin{aligned}\limsup_{\rho\uparrow 1}\widehat{\text{III}}(\rho)&\le\limsup_{\rho\uparrow 1}\frac{\overline{F}(x_\rho^{\nu_u})}{\overline{F}(G^{\leftarrow}(\rho))}=\limsup_{\rho\uparrow 1}\frac{\overline{F}\left(G^{\leftarrow}\left(1-\frac{1-\rho}{\rho}\frac{1-\nu_u(\rho)}{\nu_u(\rho)}\right)\right)}{\overline{F}(G^{\leftarrow}(\rho))}\\&=\limsup_{\rho\uparrow 1}\left(\frac{1-\nu_u(\rho)}{\rho\nu_u(\rho)}\right)^{p(H)}=0.\end{aligned}$$

This concludes the proof of Proposition 2. The paper is concluded with some additional Matuszewska theory and the postponed proofs of the lemmas in Section 4.1.

Appendix. Additional Matuszewska Theory

This appendix gathers some results on Matuszewska indices. Lemmas 3 and 4 are proven directly from Definition 1. Then a generalized version of Potter's theorem allows us to prove Lemmas 7 and 8.

Proof of Lemma 3. Let $\alpha_1>\alpha(f_1)$ and $\alpha_2>\alpha(f_2)$. Then, by definition of the upper Matuszewska index, there exist $C_1,C_2>0$ such that, for all $\mu\in[1,\mu^*]$, $\mu^*>1$, and all x sufficiently large, we have $f_1(\mu x)\le C_1\mu^{\alpha_1}f_1(x)$ and $f_2(\mu x)\le C_2\mu^{\alpha_2}f_2(x)$. Consequently, we have $\limsup_{x\to\infty}\frac{f_1(\mu x)f_2(\mu x)}{f_1(x)f_2(x)}\le C_1C_2\mu^{\alpha_1+\alpha_2}$, and thus, $\alpha(f_1\cdot f_2)\le\alpha(f_1)+\alpha(f_2)$.

Similarly, if f_1 is nondecreasing, we have

$$f_1(f_2(\mu x))\le f_1(C_2\mu^{\alpha_2}f_2(x))\le C_1C_2^{\alpha_1}\mu^{\alpha_1\alpha_2}f_1(f_2(x)),$$

and thus, $\alpha(f_1\circ f_2)\le\alpha(f_2)\cdot\alpha(f_2)$. The results on the lower Matuszewska indices are proven analogously. □

Proof of Lemma 4. As f is positive, it suffices to show that $\limsup_{x\to\infty} f(x) = 0$. For the sake of contradiction, assume that this is false. Then there exists a constant $m > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}, x_n \to \infty$, such that $f(x_n) \geq m$ for all $n \in \mathbb{N}$. Now, by definition of the upper Matuszewska index, there exists $C > 0$ such that, for all $\mu \in [1, \mu^*], \mu^* > 1$, we have $f(x) \geq C\mu^{-\alpha(f)/2} f(\mu x)$ for all x sufficiently large. As a consequence, for some $N \in \mathbb{N}$ we have $f(x_N) \geq C(x_n/x_N)^{-\alpha(f)/2} f(x_n) \geq Cm(x_n/x_N)^{-\alpha(f)/2}$ for any fixed $n \geq N$. This is a contradiction for any x_n that satisfies $x_n > x_N(Cm/f(x_N))^{2/\alpha(f)}$. □

The following result is a generalized version of Potter's theorem and gives bounds on the ratio $f(y)/f(x)$.

Theorem A.1 (Bingham et al. 1989, proposition 2.2.1). *Let f be positive.*

i. *If $\alpha(f) < \infty$, then, for every $\alpha > \alpha(f)$, there exist positive constants C and X such that $f(y)/f(x) \leq C(y/x)^{\alpha}$ for all $y \geq x \geq X$.*
ii. *If $\beta(f) > -\infty$, then, for every $\beta < \beta(f)$, there exist positive constants D and X such that $f(y)/f(x) \geq D(y/x)^{\beta}$ for all $y \geq x \geq X'$.*

Theorem A.1 allows us to derive a relation between the Matuszewska indices of f to those of $f^{\leftarrow}$, which is presented as Lemma A.1:

Lemma A.1. *Let f be positive and locally integrable on $[X, \infty)$. If f is strictly increasing, unbounded above, and $\alpha(f) < \infty$, then $\beta(f^{\leftarrow}) = 1/\alpha(f)$. If $\beta(f) > 0$, then $\alpha(f^{\leftarrow}) = 1/\beta(f)$.*

Proof. By definition of the upper Matuszewska index, for all $\alpha > \alpha(f)$, there exists a constant $C > 0$ such that, for each $\mu^* > 1$, $f(\mu x)/f(x) \leq C\mu^{\alpha}$ uniformly in $\mu \in [1, \mu^*]$ as $x \to \infty$. In particular, for all x sufficiently large we have $f((\mu/C)^{1/\alpha}x) \leq \mu f(x)$. As f is strictly increasing and unbounded above, one can, hence, see that

$$\lim_{x\to\infty} \frac{f^{\leftarrow}(\mu x)}{f^{\leftarrow}(x)} = \lim_{y\to\infty} \frac{f^{\leftarrow}(\mu f(y))}{f^{\leftarrow}(f(y))} \geq \lim_{y\to\infty} \frac{f^{\leftarrow}(f((\mu/C)^{1/\alpha} y))}{y} \geq (C)^{-1/\alpha}\mu^{1/\alpha} \tag{A.1}$$

uniformly for $\mu \in [1, \mu^*]$. As a consequence, $\beta(f^{\leftarrow}) \geq 1/\alpha(f)$.

On the other hand, if $\beta(f^{\leftarrow}) > 1/\alpha(f), \alpha(f) > 0$, then Theorem A.1(ii) claims that, for some $\varepsilon > 0$ sufficiently small, there exists a constant $C' > 0$ such that $f^{\leftarrow}(y)/f^{\leftarrow}(z) \geq C'(y/z)^{1/\alpha(f)+\varepsilon}$ for all $y \geq z$ sufficiently large. By substitution of $y = f(\mu x)$ and $z = f(x)$, we obtain

$$C'\left(\frac{f(\mu x)}{f(x)}\right)^{1/\alpha(f)+\varepsilon} \leq \frac{f^{\leftarrow}(f(\mu x))}{f^{\leftarrow}(f(x))} = \mu$$

and, hence, $\lim_{x\to\infty} f(\mu x)/f(x) \leq ((C')^{-1}\mu)^{\frac{\alpha(f)}{1+\varepsilon\alpha(f)}}$. This inequality, however, indicates that $\alpha(f)$ was not the infimum over all α satisfying (4), which is a contradiction.

The relation $\alpha(f^{\leftarrow}) = 1/\beta(f)$ is proven similarly. □

A more general version of this lemma has been stated in several other works (Bingham et al. 1989, Lin et al. 2011); however, these works refer to an unpublished manuscript by de Haan and Resnick for the corresponding proof.

Our final results relate the Matuszewska indices of $\overline{F}$ to those of related functions. First, Lemma 7 relates the Matuszewska indices of $\overline{F}$ to those of $\overline{G}$. Its proof is similar to the proof of lemma 6 in Lin et al. (2011).

Proof of Lemma 7. Assume $x_R = \infty$. Then, by definition of $\alpha(\overline{F})$, we have, for all $\alpha > \alpha(\overline{F})$, that $\overline{F}(\mu t)/\overline{F}(t) \leq C(1 + o(1))\mu^{\alpha}$ uniformly in $\mu \in [1, \mu^*]$, and hence,

$$\begin{aligned}\mathbb{E}[B]\overline{G}(\mu x) &= \mu\int_x^{\infty} \overline{F}(\mu\tau)\,\mathrm{d}\tau \leq C(1 + o(1))\mu^{\alpha+1}\int_x^{\infty}\overline{F}(\tau)\,\mathrm{d}\tau\\ &= C(1 + o(1))\mu^{\alpha+1}\mathbb{E}[B]\overline{G}(x)\end{aligned}$$

as $x \to \infty$. On the other hand, if $x_R < \infty$, then

$$\begin{aligned}\mathbb{E}[B]\overline{G}(x_R - (\mu x)^{-1}) &= \int_{x_R-(\mu x)^{-1}}^{x_R} \overline{F}(t)\,\mathrm{d}t = \int_x^{\infty} \mu^{-1}\tau^{-2}\overline{F}(x_R - (\mu\tau)^{-1})\,\mathrm{d}\tau\\ &\leq C(1 + o(1))\mu^{\alpha-1}\int_x^{\infty}\tau^{-2}\overline{F}(x_R - \tau^{-1})\,\mathrm{d}\tau\\ &= C(1 + o(1))\mu^{\alpha-1}\mathbb{E}[B]\overline{G}(x_R - x^{-1})\end{aligned}$$

as $x \to \infty$. The claims on the lower Matuszewska index can be proven analogously. □

Second, Lemma 8 relates the Matuszewska indices of $\overline{F}$ to those of $G^{\leftarrow}$. It does so by combining Lemmas 3, 7, and A.1.

Proof of Lemma 8. We only prove the relation between the lower Matuszewska indices as the relation between the upper Matuszewska indices can be proven similarly.

First, assume $x_R = \infty$. Because $\beta(\overline{F}) > -\infty$, it follows from Lemma 7 that $\beta(\overline{G}) > -\infty$ and, hence, by Lemma 3, that $\alpha(1/\overline{G}) = -\alpha(\overline{G}) \leq -\beta(\overline{G}) < \infty$. The result follows readily from Lemma A.1 through $\beta(G^{\leftarrow}(1 - (\cdot)^{-1})) = \beta((1/\overline{G})^{\leftarrow}) = 1/\alpha(1/\overline{G}) = -1/\beta(\overline{G})$ and subsequent application of Lemma 7.

Similarly, if $x_R < \infty$, then $\alpha(1/\overline{G}(x_R - (\cdot)^{-1})) < \infty$ and

$$\begin{aligned}\frac{1}{x_R - G^{\leftarrow}(1-x^{-1})} &= \frac{1}{x_R - \inf\{z : G(z) > 1 - x^{-1}\}} = \inf\left\{\frac{1}{x_R - z} : G(z) > 1 - x^{-1}\right\}\\ &= \inf\{y : G(x_R - y^{-1}) > 1 - x^{-1}\} = \inf\{y : 1/\overline{G}(x_R - y^{-1}) > x\}\\ &= \left(\frac{1}{\overline{G}\left(x_R - \frac{1}{\cdot}\right)}\right)^{\leftarrow}(x).\end{aligned}$$

The result follows from the equalities $\beta\left(\frac{1}{x_R - G^{\leftarrow}(1-(\cdot)^{-1})}\right) = \beta\left(\left(\frac{1}{\overline{G}(x_R-(\cdot)^{-1})}(\cdot)\right)^{\leftarrow}\right) = 1/\alpha\left(\frac{1}{\overline{G}(x_R-(\cdot)^{-1})}\right) = -1/\beta(\overline{G}(x_R - (\cdot)^{-1}))$ and application of Lemma 7. □

Endnotes

[1] The propositions regard Π- and Γ-varying functions; we consider these classes in Section 5.

[2] Here, we denote $0^{-1} = +\infty$.

[3] Their paper only considers the $x_R = \infty$ case; however, the proof also holds for finite x_R.

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