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Preprint Nr. 09/2015 — 28. Juli 2015

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<http://www.math.uni-augsburg.de/>

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*Herausgeber:*

Institut für Mathematik

Universität Augsburg

86135 Augsburg

<http://www.math.uni-augsburg.de/de/forschung/preprints.html>

*ViSdP:*

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# THE SHARP INTERFACE LIMIT FOR THE STOCHASTIC CAHN-HILLIARD EQUATION

D.C. ANTONOPOULOU<sup>§\*</sup>, D. BLÖMKER<sup>‡</sup>, AND G.D. KARALI<sup>†\*</sup>

ABSTRACT. We study the two and three dimensional stochastic Cahn-Hilliard equation in the sharp interface limit, where the positive parameter  $\varepsilon$  tends to zero, which measures the width of transition layers generated during phase separation. We also couple the noise strength to this parameter. Using formal asymptotic expansions, we identify the limit. In the right scaling we indicate that the solutions of stochastic Cahn-Hilliard converge to a solution of a Hele-Shaw problem with stochastic forcing. In the case when the noise is sufficiently small, we rigorously prove that the limit is a deterministic Hele-Shaw problem. Finally, we discuss which estimates are necessary in order to extend the rigorous result to larger noise strength.

## 1. INTRODUCTION

In this paper we consider the sharp interface limit of the stochastic Cahn-Hilliard equation

$$(1.1) \quad \partial_t u = \Delta(-\varepsilon \Delta u + \varepsilon^{-1} f'(u) - \dot{V}(x, t; \varepsilon)) + \dot{W}(x, t; \varepsilon),$$

subject to Neumann boundary conditions on a bounded domain in  $\mathcal{D} \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$

$$(1.2) \quad \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial \mathcal{D}.$$

We always assume that  $\mathcal{D}$  has a sufficiently piece-wise smooth boundary. The typical example for the bistable nonlinearity is  $f'(u) := \partial_u f(u) = u(u^2 - 1)$ , where the primitive  $f := \frac{1}{4}(u^2 - 1)^2$  is a double well potential with equally deep wells taking its global minimum value 0 at the points  $u = \pm 1$ . The small parameter  $\varepsilon$  measures an atomistic interaction length that fixed the length-scale of transition layers between 1 and  $-1$ . Obviously the solution  $u$  depends on  $\varepsilon$ , which we usually suppress in the notation, but wherever it is needed we shall denote the solution  $u$  by  $u_\varepsilon$ .

The term  $\dot{V}$  in the chemical potential is a spatially smooth space-time noise since it is under the Laplacian, while the additive noise  $\dot{W}$  might be rougher.

The stochastic Cahn-Hilliard equation is a model for the non-equilibrium dynamics of metastable states in phase transitions, [25, 41, 44]. The deterministic Cahn-Hilliard equation with  $\dot{V} = \dot{W} = 0$  and has been extended to a stochastic version by Cook, [25] (see also in [44]), incorporating thermal fluctuations in the form of an additive noise. Such a generalized Cahn-Hilliard model, [35], is based

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*Date:* July 26, 2015.

1991 *Mathematics Subject Classification.* 35K55, 35K40, 60H30, 60H15.

*Key words and phrases.* Multi-dimensional stochastic Cahn-Hilliard equation, additive noise, stochastic sharp interface limit, Hele Shaw problem, interface motion.

on the balance law for micro-forces. In this case, the additive term  $\dot{V}$  in the chemical potential of (1.1) is given by fluctuations in an external field. See [41, 35]; cf. also [43], where the external gravity field is modeled. The free energy independent  $\dot{W}$  term stands for the Gaussian noise in Model B of [41], in accordance with the original Cahn-Hilliard-Cook model.

We can model both noise terms either as the formal derivative of a Wiener process in the sense of Da Prato & Zabczyk [28] or as a derivative of a Brownian sheet in the sense of Walsh, [49]. Dalang & Quer-Sardanyons [29] showed that both approaches are actually equivalent and we thus focus on the  $Q$ -Wiener-process the sense of [28], which may be defined by a series in the right orthonormal basis with independent coefficients given by a sequence of real valued Brownian motions.

Existence of stochastic solution for the problem (1.1) has been established under various assumptions on the noise-terms for example in [26, 27]; we also refer to the results presented in [20, 21, 9, 17]. Note that we can always combine up the two noise terms  $\dot{V}$  and  $\dot{W}$  in a single additive noise term.

**1.1. Phase transitions and noise.** Concerning the Cahn-Hilliard equation problem posed in one dimension, in [8], the authors analyzed the stochastic dynamics of the front motion of interfaces in the one dimensional equation. In the absence of noise we refer also to [14, 15] for the dynamics and construction of a finite dimensional manifold parametrized by the interface positions. This is a key tool for studying the stochastic case, but it fails in dimension two and three, as interfaces are no longer points, but interfaces of hypersurfaces.

An interesting result is this of [16], where, on the unbounded domain, a single interface moves according to a fractional Brownian motion, which is in contrast to the usual Brownian motion in most of the examples. Note that the one-dimensional case is significantly simpler, since the solutions can be fully parametrized by their finitely many zeros. Therefore, one needs only to consider the motion of solutions along a finite dimensional slow manifold, and the stability properties along such a manifold; see for example [13]. Here we try to follow similar ideas, despite the fact that the driving manifold is infinite dimensional.

A slightly simpler model, due to the absence of mass-conservation, is the stochastic Allen Cahn-equation, the so called Model A of [41]. The Interface motion of stochastic systems of Allen-Cahn type have been analyzed in [30]. In [34, 18], the authors studied the stochastic Allen-Cahn equation with initial data close to one instanton or interface and proved that, under an appropriate scaling, the solution will stay close to the instanton shape, while the random perturbation will create a dynamic motion for this single interface. This is observed now on a much faster time scale than in the deterministic case. This result has been also studied in [51] via an invariant measure approach.

If the initial data involves more than one interfaces, it is believed that these interfaces exhibit a random movement too, which is much quicker than in the deterministic case, while different interfaces should annihilate when they meet, [31]. We also refer to [48] or [36]. The limiting process should be related to a Brownian one (cf. in [32] for formal arguments). A full description of all the ideas for the analysis of the interface motion based on [24] and [8] is presented in [52]. In [47] the authors considered the stochastic Allen-Cahn equation driven by a multiplicative noise; they prove tightness of solutions for the sharp interface limit problem, and show convergence to phase-indicator functions; cf. also in [51] for the one-dimensional case with an additive space-time white noise for the proof of an exponential convergence towards a curve of minimizers of the energy.

The space-time white noise driven Allen-Cahn equation is known to be ill-posed in space dimensions greater than one, [49, 28], and a renormalization is necessary to properly define the solutions.

We refer to [37, 39] for more details. *This is in contrast with the stochastic Cahn-Hilliard equation, which for space-time white noise is still well posed in dimensions two and three.*

For a multi-dimensional stochastic Allen-Cahn equation driven by a mollified noise, in [38], it is shown that as the mollifier is removed, the solutions converge weakly to zero, independently of the initial condition. If the noise strength converges to zero at a sufficiently fast rate, then the solutions converge to those of the deterministic equation. And the behavior is well described by the Freidlin-Wentzell theory. In [50], for a smooth noise - white on the limit, the author extending the classical result of Funaki [33] to spatial dimensions more than two, derived motion by mean curvature with an additional stochastic forcing for the sharp interface limit problem. Recently, in [1], for the case of an additive ‘mild’ noise in the sense of [33, 50], the first rigorous result on the generation of interface for the stochastic Allen-Cahn equation has been derived; the authors proved layer formation for general initial data, and established that the solution’s profile near the interface remains close to that of a (squeezed) traveling wave, which means that a spatially uniform noise does not destroy this profile.

Due to phase separation, the solution  $u$  of the stochastic Cahn-Hilliard equation (1.1), which is related to the mass concentration, tends to split in regions where  $u \approx \pm 1$  and with inner layers of order  $\mathcal{O}(\varepsilon)$  between them. We shall study the motion of such layers in their sharp interface limit  $\varepsilon \rightarrow 0^+$ . *The rigorous complete description of the motion of interfaces in dimensions two and three stands for many years as a wide open question.* With this paper, we tried to contribute towards a full answer by providing asymptotics for the general noise strength case. In addition, as a first step, we derive rigorously the sharp interface limit when the noise is sufficiently small. A key problem is that spectral problems of the linearized operators are not yet understood in the generality necessary for the proof.

The deterministic case (when  $\dot{V} = \dot{W} = 0$ ) for the Cahn-Hilliard equation, has been already studied in [2]. Given a solution of the deterministic Hele-Shaw problem, the authors constructed an approximation of (1.1) without noise admitting this solution, as the interface moves between  $\pm 1$ . The analysis thereof will be the foundation of some of our results. The main technical problem of this strategy, when noise is involved, is that the manifold of possible approximations is parametrized by an infinite dimensional space of closed curves. Furthermore, the spectral information provided so far for the linearized problem, necessary for a qualitative study of the approximation is insufficient. This is due mainly to the fact that most of the larger eigenvalues actually are related to the fast motion of the interface itself. Thus, this approximation can be valid only on time-scales of order  $\mathcal{O}(1)$ .

A simpler case is the motion of droplets for the two-dimensional Cahn-Hilliard or the mass conserving Allen-Cahn equation. Here, the solutions can be fully parametrized by finite dimensional data, namely the position and radius of the droplets. See [3, 12], and [7] (stochastic problem), for droplets on the boundary, and [6, 5] for droplets in the domain in the absence of noise. The approximation in these cases is valid for very long time-scales.

**1.2. Outline of the paper.** In Section 2, we present the formal derivation of the stochastic Hele-Shaw problem from (1.1) and identify the noise strength that yields to a nontrivial modification of the limiting problem.

In Section 3 we provide a rigorous definition of the setting and state the main results, which we then prove in Section 4. We concentrate on small noise strength and show in that case, that solutions of (1.1) in the sharp interface limit of  $\varepsilon \rightarrow 0^+$  are well approximated by a Hele-Shaw

problem. We will see that the main limitation towards a better approximation result is the lack of good bounds on the linearized operator.

## 2. FORMAL ASYMPTOTICS

In this section we present some formal matching asymptotics applied on (1.1) that establish a first intuition towards a rigorous proof for the stochastic sharp interface limit. We remark first that the stochastic C-H equation (1.1) can be written as a system, where we can separate the noise in the chemical potential and the noise independent of the free-energy. This representation is not unique, as we can combine both noise terms to remove one or the other from the system. If we keep both noise terms (1.1) is written as the stochastic system

$$\begin{cases} \partial_t u &= -\Delta v + \dot{W}, \\ v &= -\frac{f'(u)}{\varepsilon} + \varepsilon \Delta u + \dot{V}, \end{cases}$$

where  $v$  is the chemical potential coupled with an additional Laplacian to the first equation. This rewriting is important for a rigorous asymptotic analysis of the stochastic C-H equation in the sharp interface limit.

In order to remove one of the noise terms we define

$$\varepsilon^\sigma \dot{\mathcal{W}} := \dot{W} - \Delta \dot{V},$$

where  $\varepsilon^\sigma$  is the noise strength and  $\dot{\mathcal{W}}$  a  $\varepsilon$ -independent noise. Then we can consider the following equivalent formulation with only one noise term:

$$(2.1) \quad \begin{cases} \partial_t u &= -\Delta v + \varepsilon^\sigma \dot{\mathcal{W}}, \\ v &= -\frac{f'(u)}{\varepsilon} + \varepsilon \Delta u. \end{cases}$$

Formally in [10], and later more rigorously in [11] using a Hilbert expansions method, the asymptotic behavior for  $\varepsilon \rightarrow 0^+$  of the following deterministic system has been analyzed:

$$\begin{cases} \partial_t u &= -\Delta v + G_1, \\ v &= -\frac{f'(u)}{\varepsilon} + \varepsilon \Delta u + G_2, \end{cases}$$

where now  $G_1(x, t; \varepsilon)$  and  $G_2(x, t; \varepsilon)$  are deterministic forcing terms. The sharp interface limit problem in the multidimensional case demonstrated a local influence in phase transitions of forcing terms that stem from the chemical potential, while free energy independent terms act on the rest of the domain. In addition, the forcing may slow down the equilibrium.

Given an initial smooth closed  $n - 1$  dimensional hypersurface  $\Gamma_0$  in  $\mathcal{D}$  (this definition covers also the union of closed interfaces) then the chemical potential

$$(2.2) \quad v := \lim_{\varepsilon \rightarrow 0^+} (\varepsilon \Delta u - \varepsilon^{-1} f'(u) + G_2),$$

satisfies the following Hele-Shaw free boundary problem (assuming that the limits exist)

$$(2.3) \quad \begin{cases} \Delta v = \lim_{\varepsilon \rightarrow 0^+} G_1 & \text{in } \mathcal{D} \setminus \Gamma(t), \quad t > 0, \\ \partial_n v = 0 & \text{on } \partial \mathcal{D}, \\ v = \lambda H + \lim_{\varepsilon \rightarrow 0^+} G_2 & \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0, \end{cases}$$

where  $\Gamma(t)$  is the zero level surface of  $u(t)$ , which is for fixed time a closed  $n - 1$  dimensional hypersurface of mean curvature  $H = H(t)$  and of velocity  $V = V(t)$  that divides the domain  $\mathcal{D}$  in two open sets  $\mathcal{D}^+(t)$  and  $\mathcal{D}^-(t)$ . The constant  $\lambda$  is positive, and  $n$  is the unit outward normal vector at the inner and outer boundaries.

According to the aforementioned arguments, each noise term has a different physical meaning and appears in a different equation when C-H is presented as a system. We will use some of the ideas of the deterministic asymptotic analysis, but we will see in the following that in case the terms  $G_i$  are noise terms and small in  $\varepsilon$ , they still have an impact on the limiting behavior.

**2.1. Formal derivation of the stochastic Hele-Shaw problem.** In order to observe the limit behavior of (1.1) at larger noise strengths, we fix  $\sigma = 1$  as we expect the noise strength  $\varepsilon$  to be the critical one. In order not to calculate with formal noise terms, where the order of magnitude and the definition of products is not always obvious, we use the following change of variables

$$\hat{u}_\varepsilon := u_\varepsilon - \varepsilon \mathcal{W},$$

where we assume that the Winer process is spatially sufficiently smooth. Taking differentials, it follows that  $\hat{u}_\varepsilon$  and  $v_\varepsilon$  solve the system

$$(2.4) \quad \begin{cases} \partial_t \hat{u}_\varepsilon &= -\Delta v_\varepsilon, \\ v_\varepsilon &= -\frac{1}{\varepsilon} f'(\hat{u}_\varepsilon + \varepsilon \mathcal{W}) + \varepsilon \Delta \hat{u}_\varepsilon + \varepsilon^2 \Delta \mathcal{W}. \end{cases}$$

Observe that for spatially smooth noise on time-scales of order 1

$$\hat{u}_\varepsilon := u_\varepsilon + \mathcal{O}(\varepsilon),$$

is an approximate solution for small  $\varepsilon$ . Furthermore, the main advantage of the above system representation is based on the fact that (2.4) is now a random PDE without stochastic differentials and all terms appearing are spatially smooth and in time at least Hölder-continuous. Thus, we can treat all appearing quantities as functions and analyze the equation path-wisely, i.e., for every fixed realization of  $\mathcal{W}$ .

Therefore, we are able to follow the ideas of the formal derivation presented in [10] and derive in the limit the following stochastic Hele-Shaw problem

$$(2.5) \quad \begin{cases} \Delta v = 0 & \text{in } \mathcal{D} \setminus \Gamma(t), \quad t > 0, \\ \partial_n v = 0 & \text{on } \partial \mathcal{D}, \\ v = \lambda H + \mathcal{W} & \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0, \end{cases}$$

where again  $H$  and  $V$  are the mean curvature and velocity respectively of the zero level surface  $\Gamma(t)$ . For positive  $\varepsilon > 0$  the domain  $\mathcal{D}$  admits the following disjoint decomposition

$$\mathcal{D} = \mathcal{D}_\varepsilon^+(t) \cup \mathcal{D}_\varepsilon^-(t) \cup \mathcal{D}_\varepsilon^I(t),$$

where

$$u_\varepsilon \approx 1 \quad \text{for } x \in \mathcal{D}_\varepsilon^+(t) \quad \text{and} \quad u_\varepsilon \approx -1 \quad \text{for } x \in \mathcal{D}_\varepsilon^-(t).$$

Moreover,  $\mathcal{D}_\varepsilon^I(t)$  is a narrow interfacial region around  $\Gamma(t)$  with thickness of order  $\varepsilon$  where  $u_\varepsilon$  is neither close to  $+1$  nor  $-1$ .

In particular, we construct an inner solution close to the interface, and an outer solution away from it. Using the appropriate matching in orders of  $\varepsilon$ , we formally pass to the limit and derive the corresponding free boundary problem. To avoid additional technicalities, we also assume that the interface  $\Gamma$  does not intersect the boundary. In terms of simplicity of notation, we drop the subscript  $\varepsilon$  in all the calculations that follows.

**2.2. Outer expansion.** We consider that the inner interface is known and seek the outer expansion far from it, i.e., an expansion in the form

$$\begin{aligned} \hat{u} &= \hat{u}_0 + \varepsilon \hat{u}_1 + \dots, \\ v &= v_0 + \varepsilon v_1 + \dots, \end{aligned}$$

where ‘...’ denote higher order terms and  $u_0, \dots, v_0, \dots$  are smooth functions. We insert the outer expansion into the second equation of the stochastic system (2.4)<sub>2</sub> and obtain

$$(2.6) \quad \begin{aligned} v_0 + \varepsilon v_1 + \mathcal{O}(\varepsilon^2) &= -\frac{1}{\varepsilon}(f'(\hat{u}_0) + \varepsilon f''(\hat{u}_0)(\hat{u}_1 + \mathcal{W}) + \mathcal{O}(\varepsilon^2)) \\ &\quad + \varepsilon \Delta(\hat{u}_0 + \varepsilon \hat{u}_1 + \mathcal{O}(\varepsilon^2)) + \varepsilon^2 \Delta \mathcal{W} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

First collecting the  $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$  terms in the previous equation in (2.6), we arrive at

$$f'(\hat{u}_0) = 0,$$

while since we get as in Remark 4.1, (1) of [2]

$$\hat{u}_0 = \pm 1.$$

In the next step, we collect the  $\mathcal{O}(1)$  terms in (2.6) and get

$$v_0 = -f''(\hat{u}_0)(\hat{u}_1 + \mathcal{W}) + \mathcal{O}(\varepsilon^2).$$

We plug now the outer expansion into the first equation of (2.4)<sub>1</sub> and obtain

$$\partial_t(\hat{u}_0 + \varepsilon \hat{u}_1 + \mathcal{O}(\varepsilon^2)) = -\Delta(v_0 + \varepsilon v_1 + \mathcal{O}(\varepsilon^2)).$$



As  $\hat{u}_0$  is a constant we have  $\partial_t \hat{u}_0 = 0$ , and thus collecting the  $\mathcal{O}(1)$  terms yields

$$-\Delta v_0 = 0 .$$

Collecting finally the  $\mathcal{O}(\varepsilon)$  terms we arrive at

$$\partial_t \hat{u}_1 = -\Delta v_1 .$$

**2.3. Inner expansion.** Let  $x$  be a point in  $\mathcal{D}$  that at time  $t$  is near the interface  $\Gamma(t)$ . Let us introduce the stretched normal distance to the interface,  $z := \frac{d}{\varepsilon}$ , where  $d(x, t)$  is the signed distance from the point  $x$  in  $\mathcal{D}$  to the interface  $\Gamma(t)$ , such that  $d(x, t) > 0$  in  $\mathcal{D}_\varepsilon^+$  and  $d(x, t) < 0$  in  $\mathcal{D}_\varepsilon^-$ . Obviously  $\Gamma$  has the representation

$$\Gamma(t) = \{x \in \mathcal{D} : d(x, t) = 0\} .$$

If  $\Gamma$  is smooth, then  $d$  is smooth near  $\Gamma$ , and  $|\nabla d| = 1$  in a neighborhood of  $\Gamma$ . Following [2] and [46], we seek for an inner expansion valid for  $x$  near  $\Gamma$  of the form

$$\begin{aligned} \hat{u} &= q\left(\frac{d(x, t)}{\varepsilon}, x, t\right) + \varepsilon Q\left(\frac{d(x, t)}{\varepsilon}, x, t\right) + \dots , \\ v &= \tilde{q}\left(\frac{d(x, t)}{\varepsilon}, x, t\right) + \varepsilon \tilde{Q}\left(\frac{d(x, t)}{\varepsilon}, x, t\right) + \dots , \end{aligned}$$

where ‘ $\dots$ ’ denote higher order terms and  $q, Q, \dots, \tilde{q}, \tilde{Q}, \dots$  are smooth. It will be convenient to require that the quantities depending on  $z, x, t$  are defined for  $x$  in a full neighborhood of  $\Gamma$  but do not change when  $x$  varies normal to  $\Gamma$  with  $z$  held fixed, [46]. We insert the inner expansion into (2.4)<sub>2</sub>, utilize that  $|\nabla d|^2 = 1$  and obtain the following expression

$$(2.7) \quad \tilde{q} + \varepsilon \tilde{Q} + \mathcal{O}(\varepsilon^2) = -\frac{1}{\varepsilon}(f'(q) + \varepsilon f''(q)(Q + \mathcal{W})) + \varepsilon \left( \frac{\partial_z q}{\varepsilon} \Delta d + \frac{\partial_{zz} q}{\varepsilon^2} + \partial_z Q \Delta d + \frac{\partial_{zz} Q}{\varepsilon} \right) + \varepsilon^2 \Delta W .$$

We collect the order  $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$  and derive

$$\partial_{zz} q - f'(q) = 0 .$$

By matching now the terms of order  $\mathcal{O}(1)$  in (2.7), we obtain

$$\tilde{q} + \mathcal{O}(\varepsilon) = -f''(q)Q + \partial_{zz} Q - f''(q)\mathcal{W} + \partial_z q \Delta d ,$$

or equivalently

$$(2.8) \quad \tilde{q} - \partial_z q \Delta d = \partial_{zz} Q - f''(q)Q - f''(q)\mathcal{W} + \mathcal{O}(\varepsilon) .$$

We define the linearized Allen-Cahn operator

$$\mathcal{L}Q = \partial_{zz} Q - f''(q)Q .$$

Then (2.8) is written as

$$(2.9) \quad \tilde{q} - \partial_z q \Delta d = \mathcal{L}Q - f''(q)\mathcal{W} + \mathcal{O}(\varepsilon) .$$

This equation is solvable if for any  $\chi \in \text{Ker}(\mathcal{L}^*)$  it holds that  $\chi \perp (\tilde{q} - \partial_z q \Delta d + f''(q)\mathcal{W})$ , or equivalently if

$$(2.10) \quad \int_{-\infty}^{\infty} \chi \cdot (\tilde{q} - \partial_z q \Delta d + f''(q)\mathcal{W}) dz = 0 .$$

Obviously, for any  $x$  on  $\Gamma$  it holds that  $d(x, t) = 0$  and  $\Delta d(x, t) = H(x, t)$ . Replacing in (2.10) we obtain the following sufficient condition on the interface  $\Gamma$ :

$$(2.11) \quad \tilde{q} = \lambda H - f''(q)\mathcal{W} + \mathcal{O}(\varepsilon).$$

Plugging the inner expansion into (2.4)<sub>1</sub> we obtain

$$(2.12) \quad \frac{\partial_z q}{\varepsilon} d_t + \partial_z Q d_t + \mathcal{O}(\varepsilon) = - \left( \frac{\partial_z \tilde{q}}{\varepsilon} \Delta d + \frac{\partial_{zz} \tilde{q}}{\varepsilon^2} + \partial_z \tilde{Q} \Delta d + \frac{\partial_{zz} \tilde{Q}}{\varepsilon} \right).$$

We collect the terms of order  $\mathcal{O}(\varepsilon^{-2})$  and arrive at

$$\partial_{zz} \tilde{q} = 0,$$

which implies that for some functions  $a$  and  $b$

$$\tilde{q} = a(x, t)z + b(x, t).$$

To proceed further, the matching condition for the inner and outer expansions must be developed. In general, these are obtained by the following procedure ([19]). Fixing  $x \in \Gamma$ , we seek to match the expansions by requiring formally for  $z \rightarrow \infty$

$$\tilde{q} + \varepsilon \tilde{Q} + \mathcal{O}(\varepsilon^2) = v_0 + \varepsilon v_1 + \mathcal{O}(\varepsilon^2),$$

and thus in order  $\mathcal{O}(1)$

$$v_0 = \lim_{z \rightarrow \infty} \tilde{q} = \lim_{z \rightarrow \infty} (a(x, t)z + b(x, t)).$$

We obtain  $a = 0$  and thus,  $\tilde{q} = b$ . Hence, utilizing (2.11) we have that on the interface

$$v_0 = \lambda H - f''(q)\mathcal{W} + \mathcal{O}(\varepsilon) = \lambda H + \mathcal{W} + \mathcal{O}(\varepsilon),$$

where we used that  $q$  solves the Euler-Lagrange equation

$$\begin{aligned} -q''(z) + f'(q(z)) &= 0, \quad z \in \mathbb{R}, \\ \lim_{z \rightarrow \pm\infty} q(z) &= \pm 1, \quad q(0) = 0, \end{aligned}$$

while on the inner interface with  $z = d/\varepsilon = 0$  we have  $f''(q) = 3q^2 - 1 = -1$  since  $q(0) = 0$ .

What is still missing is the evolution law, which should come from the inner expansion. From (2.12), we collect the terms of order  $\mathcal{O}(1/\varepsilon)$  and obtain

$$\partial_z q d_t = -\partial_z \tilde{q} \Delta d - \partial_{zz} \tilde{Q} + \mathcal{O}(\varepsilon^2).$$

Recall that  $-d_t = V$ , while  $\Delta d = H$ , [2], and integrate over  $z$  from  $-\infty$  to  $\infty$  to get

$$- \int_{-\infty}^{\infty} \partial_z q V dz = - \int_{-\infty}^{\infty} \partial_{zz} \tilde{Q} dz + \mathcal{O}(\varepsilon^2).$$

From the matching conditions we get

$$q(+\infty) = 1 + \mathcal{O}(\varepsilon^3) \quad \text{and} \quad q(-\infty) = -1 + \mathcal{O}(\varepsilon^3).$$

Hence,

$$V = \frac{1}{2} [\partial_z \tilde{Q}(+\infty) - \partial_z \tilde{Q}(-\infty)] + \mathcal{O}(\varepsilon^2).$$

Thus the stochastic Hele-Shaw problem (2.5) is established formally as the sharp interface limit, in the case  $\sigma = 1$ .

**Remark 2.1.** *In the case  $\sigma > 1$ , we follow the same construction of inner and outer solutions as above and obtain in the limit, the deterministic Hele-Shaw problem, ([2])*

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{D} \setminus \Gamma(t), \quad t > 0, \\ \partial_n v = 0 & \text{on } \partial \mathcal{D}, \\ v = \lambda H & \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0, \end{cases}$$

where  $H$  and  $V$  are the mean curvature and velocity respectively of the zero level surface  $\Gamma(t)$  contained in the interfacial region  $\mathcal{D}_\varepsilon^I(t)$ .

**Remark 2.2.** *Note that the change of variables*

$$\hat{u}_\varepsilon := u_\varepsilon - \varepsilon^\sigma \mathcal{W},$$

implies that

$$\hat{u}_\varepsilon := u_\varepsilon + \mathcal{O}(\varepsilon^\sigma).$$

Thus, only if  $\sigma \geq 1$ ,  $\hat{u}$  is permitted to be expanded as in the presented inner expansion using  $q$ . The key difference to the deterministic analysis is that in the nonlinearity we have  $\frac{1}{\varepsilon} f(\hat{u}_\varepsilon + \varepsilon^\sigma \mathcal{W})$ . In case  $\sigma > 1$  there is no contribution of  $\mathcal{W}$  in an asymptotic expansion in terms of order  $\mathcal{O}(1)$  and  $\mathcal{O}(1/\varepsilon)$ , while for  $\sigma = 1$  there is an impact of  $\mathcal{W}$  on terms of order  $\mathcal{O}(1)$ .

**Remark 2.3.** *When  $0 \leq \sigma < 1$  the strategy presented in this section fails. For this case, we might think of avoiding the change of variables and apply the formal asymptotics of [10] directly instead. The sharp interface limit coincides to (2.3), but for  $\lim_{\varepsilon \rightarrow 0^+} G_1$ ,  $\lim_{\varepsilon \rightarrow 0^+} G_2$  replaced by*

$$\lim_{\varepsilon \rightarrow 0^+} \dot{W}(\cdot, \varepsilon), \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \dot{V}(\cdot, \varepsilon),$$

respectively. When  $0 < \sigma < 1$  we would obtain that both these limits are 0 and thus the limiting problem is a deterministic Hele-Shaw problem without the contribution of the noise. But this is a very dangerous reasoning, as noise terms, even if they are  $\varepsilon$ -independent are not of order  $\mathcal{O}(1)$ , and we would still expect an impact of the noise terms on the limiting problem.

### 3. THE SHARP INTERFACE LIMIT

The main result of this paper is that (1.1), as  $\varepsilon$  tends to zero, may have a deterministic or a stochastic profile depending on the strength of the additive noise in terms of  $\varepsilon$ . Only large noise perturbations with  $\sigma = 1$  generates a stochastic limit problem. Here we discuss the limit for smaller noise strength.

Let us first precisely state our problem. We combine both sources of noise in one term, and assume that noise is induced by the formal derivative of a  $Q$ -Wiener process  $\mathcal{W}$  in a Fourier series representation (see [28]); for simplicity, the only  $\varepsilon$ -dependence will appear in the noise strength, and thus, for the rest of this paper we shall use the notation  $\varepsilon^\sigma d\mathcal{W}(x, t)$  for the additive noise, where  $\sigma \in \mathbb{R}$ .

**Assumption 3.1.** *Let  $\mathcal{W}$  be a  $Q$ -Wiener process such that*

$$\mathcal{W}(t) = \sum_{k \in \mathbb{N}} \alpha_k \beta_k(t) e_k,$$

for an orthonormal basis  $(e_k)_{k \in \mathbb{N}}$ , independent real-valued Brownian motions  $(\beta_k)_{k \in \mathbb{N}}$ , and coefficients  $\alpha_k$  such that  $Qe_k = \alpha_k^2 e_k$ . Furthermore, we assume that the noise is sufficiently smooth in space, i.e.,  $Q$  satisfies

$$(3.1) \quad \text{trace}(\Delta^{-1}Q) < \infty.$$

To deal with a mass-conserving stochastic problem, we impose the condition

$$\int_{\mathcal{D}} \mathcal{W}(t) dx = 0.$$

Note that (3.1) implies that the Wiener-process  $\mathcal{W}(t)$  is  $H^{-1}$ -valued. This is the minimal requirement for the approximation theorem presented in the sequel; we might need more regularity, in order to have the stochastic Hele-Shaw limit problem well defined, or while performing the formal asymptotics.

The solution is still mass conservative and satisfies

$$(3.2) \quad du_\varepsilon = \Delta(-\varepsilon \Delta u_\varepsilon + \varepsilon^{-1} f'(u_\varepsilon)) dt + \varepsilon^\sigma d\mathcal{W}(x, t),$$

associated to Neumann conditions on the boundary.

The following theorem is well known. See for example [26].

**Theorem 3.2.** *Let  $\mathcal{D}$  be a rectangle in dimensions 1, 2, 3. If  $Q = I$  or  $\text{trace}(\Delta^{-1+\delta}Q) < \infty$  for  $\delta > 0$ , then the following holds true:*

- (1) *if  $u_0$  is in  $H^{-1}(\mathcal{D})$ , there exists a unique solution for the problem (1.1) in  $C([0, T]; H^{-1}(\mathcal{D}))$ ,*
- (2) *if  $u_0$  is in  $L^2(\mathcal{D})$ , then the solution for the problem (1.1) is in  $L^\infty(0, T; L^2(\mathcal{D}))$ .*

Note that the previous theorems could be extended for general Lipschitz domains in dimensions 2 and 3 under some additional assumptions of minimum eigenfunctions growth, cf. the arguments in [9]. For the analysis underlying our results, we will for the remainder of the paper always assume:

**Assumption 3.3.** *There exists a unique solution for the problem (1.1) in  $C([0, T]; H^{-1}(\mathcal{D}))$ .*

The solution is more regular. Actually it is as regular as the stochastic convolution. Moreover, we have additional regularity and can apply Itô-formula to the  $H^{-1}$ -norm.

Introducing the chemical potential  $v_\varepsilon$ , the equation is as in the formal derivation rewritten as a stochastic system. Indeed, for  $T > 0$ , let  $\mathcal{D}_T := \mathcal{D} \times (0, T)$ , then (3.2) is written as

$$(3.3) \quad \begin{aligned} du_\varepsilon &= -\Delta v_\varepsilon dt + \varepsilon^\sigma d\mathcal{W} & \text{in } \mathcal{D}_T, \\ v_\varepsilon &= -\frac{1}{\varepsilon} f'(u_\varepsilon) + \varepsilon \Delta u_\varepsilon & \text{in } \mathcal{D}_T, \end{aligned}$$

subject to Neumann boundary conditions

$$\frac{\partial u_\varepsilon}{\partial n} = \frac{\partial \Delta u_\varepsilon}{\partial n} = 0.$$

Our main analytic theorem considers *a sufficiently small noise resulting to a deterministic sharp interface* limiting behavior. In particular, we analyze the case

$$\sigma \gg \sigma_0 = 1,$$

where  $\sigma_0$  is the borderline case, where according to our formal calculation the noise has an impact on the limiting model. Under some assumptions on the initial condition  $u_\varepsilon(0)$ , the limit of  $u_\varepsilon$  and  $v_\varepsilon$  as  $\varepsilon \rightarrow 0^+$  solves, on a given time interval  $[0, T]$ , the deterministic Hele-Shaw problem. We

will state the precise formulation of this argument in Theorem 3.10, and then present the rigorous proof.

By taking a larger noise strength, we fix  $\sigma = 1$ . The formal derivation of Section 2.1 motivates the following conjecture implying a *stochastic* sharp interface limit:

**Conjecture 3.4.** *For  $\sigma = 1$  the limit of  $u_\varepsilon$  and  $v_\varepsilon$  solves the stochastic Hele-Shaw problem*

$$(3.4) \quad \begin{cases} \Delta v = 0 & \text{in } \mathcal{D} \setminus \Gamma(t), \quad t > 0, \\ \partial_n v = 0 & \text{on } \partial \mathcal{D}, \\ v = \lambda H + \mathcal{W} & \text{on } \Gamma(t), \\ V = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) & \text{on } \Gamma(t), \\ \Gamma(0) = \Gamma_0. \end{cases}$$

**Remark 3.5.** *In section 2 by formal asymptotics, we only presented an indication for the correctness of the conjecture. A rigorous proof of this conjecture remains open at the moment. We hope to attack the problem to its full generality in the near future.*

**Remark 3.6.** *Note that  $\mathcal{W}$  is a Wiener process, and the equation  $v = \lambda H + \mathcal{W}$  on  $\Gamma(t)$  appearing in (3.4), has a rigorous mathematical meaning as functions. In fact, no noise is present, while a random equation appears on  $\mathcal{D} \setminus \Gamma(t)$  in the following sense. For any given  $t$ ,  $\Gamma(t)$  is defined by its velocity  $V$  and thus is known, and the unknown function  $v$  on  $\Gamma(t)$  is a stochastic process. Thus, the problem for fixed  $t$  is posed in between the inner boundary  $\Gamma = \Gamma(t)$  and the outer boundary  $\mathcal{D}$  as follows*

$$(3.5) \quad \Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma, \quad \partial_n v = 0 \text{ on } \partial \mathcal{D}, \quad v = \lambda H + \mathcal{W} \text{ on } \Gamma,$$

*the inner boundary condition  $v|_\Gamma$  being a realization of a  $t$ -dependent stochastic process.*

**3.1. Statement of the Main Theorem.** In this section, we shall state the main analytic theorem of this paper, concerning the sharp interface limiting profile for sufficiently small noise strength. To approximate the stochastic solution we use the same approximations  $u_\varepsilon^A$  and  $v_\varepsilon^A$  as in [2] proposed in the absence of noise. For a precise definition see further below. In our proof we follow the ideas of the proof of their Theorem 2.1, and need to adapt the analysis to presence of the noise.

The main difference is the noise in the equation for the residual

$$(3.6) \quad R := u_\varepsilon - u_\varepsilon^A.$$

**Assumption 3.7.** *Let the family  $\{\Gamma(t)\}_{t \in [0, T]}$  of smooth closed hypersurfaces together with the functions  $\{v(t)\}_{t \in [0, T]}$  be a solution of the deterministic Hele-Shaw problem (i.e., equation (3.4) with  $\mathcal{W} = 0$ ) such that the interfaces do not intersect with the boundary  $\partial \mathcal{D}$ , i.e.,  $\Gamma(t) \subset \mathcal{D}$  for all  $t \in [0, T]$ .*

With  $\Gamma$  from Assumption 3.7, in [2], the authors construct a pair of approximate solutions  $(u_\varepsilon^A, v_\varepsilon^A)$ , so that  $\Gamma(t)$  is the zero level set of  $u_\varepsilon^A(t)$ , and which satisfies

$$(3.7) \quad \begin{aligned} du_\varepsilon^A &= -\Delta v_\varepsilon^A dt \quad \text{in } \mathcal{D}_T, \\ v_\varepsilon^A &= -\frac{1}{\varepsilon} f'(u_\varepsilon^A) + \varepsilon \Delta u_\varepsilon^A + r_\varepsilon^A \quad \text{in } \mathcal{D}_T, \end{aligned}$$

for boundary conditions

$$\frac{\partial u_\varepsilon^A}{\partial n} = \frac{\partial \Delta u_\varepsilon^A}{\partial n} = 0 \quad \text{on } \partial \mathcal{D}.$$

We recall that  $u_\varepsilon^A$  approximates the deterministic version of equation (1.1) (i.e., for  $\dot{V} = \dot{W} = 0$  or  $\mathcal{W} = 0$ ). The error term  $r_\varepsilon^A$  is bounded in terms of  $\varepsilon$ , and depending on the smoothness of  $\Gamma$  and the number of approximation steps, the bound on  $r_\varepsilon^A$  can be arbitrarily small. For details see relation (4.30) and Theorem 4.12 in [2].

We will summarize the results of [2] that we need for our proof in the following Theorem.

**Theorem 3.8.** *Under the Assumption 3.7, for any  $K > 0$  there exists a pair  $(u_\varepsilon(t)^A, v_\varepsilon(t)^A)$  of solutions to (3.7), such that*

$$\|r_\varepsilon^A\|_{C^0(\mathcal{D}_T)} \leq C\varepsilon^{K-2}.$$

Moreover, it holds that

$$\|v_\varepsilon^A - v\|_{C^0(\mathcal{D}_T)} \leq C\varepsilon,$$

and finally for  $x$  away from  $\Gamma(t)$  (i.e.,  $d(x, \Gamma(t)) \geq c\varepsilon$ )

$$|u_\varepsilon^A(t, x) - 1| \leq C\varepsilon \quad \text{or} \quad |u_\varepsilon^A(t, x) + 1| \leq C\varepsilon.$$

We present now the following spectral estimate, useful in our proof; we refer to [23] for dimensions larger than two, and to [4] for dimensions two. Unfortunately, this is also the key problem to extend the approximation result beyond time-scales of order 1.

**Proposition 3.9** (Proposition 3.1 of [2]). *Let  $u_\varepsilon^A$  be the approximation given in Theorem 3.8. Then for all  $w \in H^1(\mathcal{D})$  satisfying Neumann boundary conditions such that  $\int_{\mathcal{D}} w dx = 0$ , the following estimate is valid*

$$\int_{\mathcal{D}} [\varepsilon |\nabla w|^2 + \frac{1}{\varepsilon} f''(u_\varepsilon^A) w^2] dx \geq -C_0 \|\nabla w\|_{L^2}^2.$$

Our main theorem will provide bounds for the residual  $R$  and is stated as follows:

**Theorem 3.10. (Main Theorem)** *Let  $u_\varepsilon$  be the solution of the stochastic Cahn-Hilliard equation (1.1) and  $u_\varepsilon^A$  and  $v_\varepsilon^A$  as described above the approximation constructed in [2], which approximates on a given interval  $[0, T]$  the deterministic Hele-Shaw problem.*

*Let  $Ru_\varepsilon - u_\varepsilon^A$  be the error and fix  $p \in (2, 3]$ . Then for*

$$\gamma > \frac{1}{p-2} \left[ 1 + \frac{2p+d(p-2)}{2p-d(p-2)} \cdot \frac{p+2}{p} \right]$$

and if

$$\sigma > \gamma + \frac{2p+d(p-2)}{2p-d(p-2)} \cdot \frac{p+2}{p}$$

the probability is larger than  $1 - C\varepsilon^\ell$  for any  $\ell > 0$  that the following estimate holds:

$$\|R\|_{L^p([0, T] \times \mathcal{D})} \leq \varepsilon^\gamma.$$

Moreover, with the same probability for some sufficiently small  $\kappa > 0$  we have:

$$\|R\|_{L^\infty(0, T, H^{-1})}^2 \leq C[\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa}],$$

$$\|R(t)\|_{L^2(0, T, H^1)}^2 \leq C[\varepsilon^{-1-2/p+2\gamma} + \varepsilon^{-1+\sigma+\gamma-\kappa}].$$

**Remark 3.11.** *Let us remark that in dimension  $d = 2$  one can easily check that we obtain the smallest possible value both for  $\sigma$  and  $\gamma$  for  $p = 3$ . In that case  $\gamma > 6$  and  $\sigma > 23/3$ . This is in well agreement with the  $\gamma$  derived in [2], but unfortunately we can only consider very small noise strength. But it seems that using the  $H^{-1}$ -norm and spectral information available there is no improvement possible.*

For dimension  $d = 3$  again the noise strength is small, but the result is not that clear. While the smallest value for  $\gamma$  is still attained at  $p = 3$  (with  $\gamma > 6$  and  $\sigma > 11$ ) we obtain the smallest value of  $\sigma$  for some  $p < 3$ .

**Remark 3.12.** Let us state two main problems with the approach presented.

First, the spectral estimate in Theorem 3.9 yields an unstable eigenvalue of order  $\mathcal{O}(1)$ . This immediately restricts any approximation result to time scales of order  $\mathcal{O}(1)$ . But we strongly believe that this eigenvalue represents only a motion of the interfaces itself. One would need spectral information orthogonal to the space of all possible approximations  $u_\varepsilon^A$ , which are parametrized by the hypersurfaces  $\Gamma$ . But this does not seem to be available at the moment.

Moreover, later in the closure of the estimate we can only allow  $\sigma > \sigma_0$  large enough, i.e., for sufficiently small noise strength. Here, an additional problem is that the  $H^{-1}$ -norm is not strong enough to control the nonlinearity, and from the spectral theorem 3.9, we do not get any higher order norms that would help in the estimate. Nevertheless, if we start with higher order norms like  $L^2$ , for instance, then there are no spectral estimates available at all.

#### 4. THE PROOF OF MAIN THEOREM 3.10

**4.1. Idea of Proof.** For the proof we need to define for some  $p \in (2, 3]$  and  $\sigma > \gamma > 0$  (both fixed later), the stopping time

$$(4.1) \quad T_\varepsilon := \inf \left\{ t \in [0, T] : \left( \int_0^t \|R(s)\|_{L^p}^p ds \right)^{1/p} > \varepsilon^\gamma \right\},$$

where the convention is that  $T_\varepsilon = T$  if the condition is never true.

The general strategy for the proof of the main theorem is the following:

- (1) Use Itô-formula for  $d\|R\|_{H^{-1}}^2$ .
- (2) Consider all estimates up to  $T_\varepsilon$  only.
- (3) Bound the stochastic integrals (at least on a set with high probability).
- (4) Show that  $T_\varepsilon = T$  with high probability using the bound derived for  $\int_0^t \|R\|_{L^p} dt$  up to  $T_\varepsilon$ .

**4.2. A differential equation for the error.** Let us first derive an SPDE for  $R$  from (3.6), using (3.7) and (3.3), as follows

$$(4.2) \quad \begin{aligned} dR &= du_\varepsilon - du_\varepsilon^A = \Delta v_\varepsilon^A dt - \Delta v_\varepsilon dt + \varepsilon^\sigma dW \\ &= \left[ -\frac{1}{\varepsilon} \Delta f'(u_\varepsilon^A) + \varepsilon \Delta^2 u_\varepsilon^A + \Delta r_\varepsilon^A + \frac{1}{\varepsilon} \Delta f'(u_\varepsilon) - \varepsilon \Delta^2 u_\varepsilon \right] dt + \varepsilon^\sigma dW \\ &= \frac{1}{\varepsilon} \left[ \Delta f'(u_\varepsilon^A + R) - \Delta f'(u_\varepsilon^A) \right] dt + \left[ -\varepsilon \Delta^2 R + \Delta r_\varepsilon^A \right] dt + \varepsilon^\sigma dW. \end{aligned}$$

**4.3. The  $H^{-1}$ -norm of  $R$ .** The approximate solutions  $u_\varepsilon^A$  and  $v_\varepsilon^A$  are, by their construction, functions in  $C^2(\overline{\mathcal{D}_T})$ , while  $u_\varepsilon^A$  satisfies for all  $t \in [0, T]$

$$\int_{\mathcal{D}} u_\varepsilon^A(t) dx = 0.$$

Since (3.2) is mass conservative, we can conclude mass conservation also holds for  $R$ , i.e., for all  $t \in [0, T]$

$$\int_{\mathcal{D}} R(t) dx = 0.$$

Observe that the operator  $-\Delta$  is a symmetric positive operator on the space

$$\mathcal{H}^2 := \left\{ w \in C^2(\overline{\mathcal{D}}) : \int_{\mathcal{D}} w \, dx = 0 \text{ and } \partial_n w = 0 \text{ on } \partial\mathcal{D} \right\}.$$

Therefore, by elliptic regularity, the operator  $-\Delta : \mathcal{H}^2 \rightarrow L^2$  is bijective. So, we can invert it and for any  $t \in [0, T]$  there exists a unique  $\psi(t) \in \mathcal{H}^2$  such that

$$(4.3) \quad -\Delta\psi(t) = R(t), \quad \text{or equivalently} \quad (-\Delta)^{-1}R(t) = \psi(t).$$

With the scalar product  $\langle \cdot, \cdot \rangle$  in  $L^2$  the  $H^{-1}$ -norm of  $R$  is given by

$$\|R\|_{H^{-1}}^2 = \|(-\Delta)^{-1/2}R\|_{L^2}^2 = \|(-\Delta)^{1/2}\psi\|_{L^2}^2 = \|\nabla\psi\|_{L^2}^2 = \langle \psi, R \rangle.$$

Since

$$\langle d\psi, R \rangle = \langle -\Delta dR, R \rangle = \langle dR, -\Delta R \rangle,$$

considering the Itô-differential, we obtain

$$(4.4) \quad \begin{aligned} \frac{1}{2}d\|R\|_{H^{-1}}^2 &= \langle \psi, dR \rangle + \frac{1}{2}\langle d\psi, dR \rangle = \langle \psi, dR \rangle + \frac{1}{2}\varepsilon^{2\sigma}\langle (-\Delta)^{-1}d\mathcal{W}, d\mathcal{W} \rangle \\ &= \langle \psi, dR \rangle + \frac{1}{2}\varepsilon^{2\sigma}\text{tr}(Q^{1/2}(-\Delta)^{-1}Q^{1/2}). \end{aligned}$$

Here, by assumption 3.1 the trace in the previous estimate is bounded. So, using (3.3) and (3.7), we arrive at

$$(4.5) \quad \begin{aligned} \langle \psi, dR \rangle &= \langle \psi, d(u_\varepsilon - u_\varepsilon^A) \rangle = \langle \psi, (-\Delta)(v_\varepsilon - v_\varepsilon^A)dt + \varepsilon^\sigma d\mathcal{W} \rangle \\ &= \langle R, (v_\varepsilon - v_\varepsilon^A) \rangle dt + \varepsilon^\sigma \langle \psi, d\mathcal{W} \rangle. \end{aligned}$$

Using again (3.3) and (3.7) in order to replace the  $v$ 's, this yields the following equality

$$(4.6) \quad \begin{aligned} \langle \psi, dR \rangle &= -\varepsilon^{-1}\langle R, f'(u_\varepsilon) - f'(u_\varepsilon^A) \rangle dt + \varepsilon\langle R, \Delta(u_\varepsilon - u_\varepsilon^A) \rangle dt - \langle R, r_\varepsilon^A \rangle dt + \varepsilon^\sigma \langle \psi, d\mathcal{W} \rangle. \\ &= -\varepsilon^{-1}\langle R, (f'(u_\varepsilon) - f'(u_\varepsilon^A)) \rangle dt - \varepsilon\|\nabla R\|^2 dt - \langle R, r_\varepsilon^A \rangle dt + \varepsilon^\sigma \langle \psi, d\mathcal{W} \rangle. \end{aligned}$$

For any positive integer  $p$ , we define

$$\|f\|_{p, \mathcal{D}} := \left( \int_{\mathcal{D}} |f|^p dx \right)^{1/p}, \quad \text{and} \quad \|f\|_{p, \mathcal{D}_t} := \left( \int_0^t \int_{\mathcal{D}} |f|^p dx ds \right)^{1/p}.$$

Also, by  $\|\cdot\|$  we denote the usual  $L^2(\mathcal{D})$ -norm and by  $\|\cdot\|_{L^p}$  the  $L^p(\mathcal{D})$ -norm.

Applying Taylor's formula to expand  $f'(u_\varepsilon)$  around  $u_\varepsilon^A$ , with residua  $\mathcal{N}(u_\varepsilon^A, R)$ , we have

$$f'(u_\varepsilon) - f'(u_\varepsilon^A) = f''(u_\varepsilon^A)R + \mathcal{N}(u_\varepsilon^A, R).$$

The crucial bound for the nonlinearity in the residual is the following result from Lemma 2.2 of [2]. It is based on a direct representation of the remainder  $\mathcal{N}$  in the Taylor expansion together with the fact that  $u_\varepsilon^A$  is uniformly bounded.

**Lemma 4.1.** *Let  $p \in (2, 3]$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then it holds that*

$$(4.7) \quad -\int \varepsilon^{-1}\mathcal{N}(u_\varepsilon^A, R)R \leq c\varepsilon^{-1}\|R\|_{p, \mathcal{D}}^p.$$

Thus, we obtain

$$(4.8) \quad \begin{aligned} -\frac{1}{\varepsilon}\langle R, (f'(u_\varepsilon) - f'(u_\varepsilon^A)) \rangle &= -\frac{1}{\varepsilon}\langle R, f''(u_\varepsilon^A)R \rangle - \frac{1}{\varepsilon}\langle R, \mathcal{N}(u_\varepsilon^A, R) \rangle \\ &\leq -\frac{1}{\varepsilon}\langle R, f''(u_\varepsilon^A)R \rangle + c\varepsilon^{-1}\|R\|_{p, \mathcal{D}}^p. \end{aligned}$$



Relations (4.4), (4.6) and (4.8) yield the following first key estimate

$$(4.9) \quad \begin{aligned} & \frac{1}{2}d\|\nabla\psi\|^2 + \varepsilon\|\nabla R\|^2 dt + \frac{1}{\varepsilon}\langle R, f''(u_\varepsilon^A)R \rangle dt \\ & \leq c\varepsilon^{-1}\|R\|_{p,\mathcal{D}}^p dt + \|R\|_{p,\mathcal{D}}\|r_\varepsilon^A\|_{q,\mathcal{D}} dt + \varepsilon^\sigma\langle\psi, d\mathcal{W}\rangle + C_{\text{tr}}\varepsilon^{2\sigma} dt. \end{aligned}$$

From this *a-priori* estimate, we now can derive a uniform bound for  $\|\nabla\psi\|$  and later a mean square bound on  $\|\nabla R\|$ . Both estimates still involve the  $L^p$ -norm of  $R$  on the right hand side, and we use the stopping time  $T_\varepsilon$  to control this.

**4.4. Technical Lemmas.** We first need the following Lemma of Burkholder-Davis-Gundy type for stochastic integrals. Recall the stopping time  $T_\varepsilon$  from (4.1).

**Lemma 4.2.** *Let  $f$  be a continuous real valued function, and  $\Delta\psi = R$  as before. Then for all  $\kappa > 0$ ,  $\ell > 1$  there exists a constant  $C = C(\ell, T, \kappa)$  such that*

$$\mathbb{P}\left(\sup_{[0, T_\varepsilon]} \left| \int_0^t f\langle\psi, d\mathcal{W}\rangle \right| \geq \varepsilon^{\gamma-\kappa}\right) \leq C\varepsilon^{\ell\kappa}\|f\|_{L^{2p/(p-2)}}^\ell.$$

*Proof.* We shall use the Chebychev's inequality. Thus, we need to bound the moments first. Applying Burkholder-Davis-Gundy inequality (using that  $\Delta^{-1}Q\Delta^{-1}$  is a bounded operator by assumption), we obtain

$$\begin{aligned} \mathbb{E} \sup_{[0, T_\varepsilon]} \left| \int_0^t f(s)\langle\psi(s), d\mathcal{W}(s)\rangle \right|^\ell & \leq C_\ell \mathbb{E} \left| \int_0^{T_\varepsilon} f^2(s)\langle\psi(s), Q\psi(s)\rangle ds \right|^{\ell/2} \\ & \leq C \mathbb{E} \left| \int_0^{T_\varepsilon} f^2(s)\|R(s)\|_{L^2}^2 ds \right|^{\ell/2} \\ & \leq C \mathbb{E} \left| \left( \int_0^{T_\varepsilon} \|R(s)\|_{L^p}^p ds \right)^{2/p} \right|^{\ell/2} \left| \int_0^{T_\varepsilon} f^{2p/(p-2)} \right|^{(p-2)\ell/(2p)} \\ & \leq C \|f\|_{L^{2p/(p-2)}}^\ell \varepsilon^{\gamma\ell}. \end{aligned}$$

Here, we applied Hölder's inequality and the definition of  $T_\varepsilon \leq T$ .

Furthermore, using Chebychev's inequality, we obtain the result as follows

$$\mathbb{P}\left(\sup_{[0, T_\varepsilon]} \left| \int_0^t f\langle\psi, d\mathcal{W}\rangle \right| \geq \varepsilon^{\gamma-\kappa}\right) \leq \varepsilon^{-\ell(\gamma-\kappa)} \mathbb{E} \sup_{[0, T_\varepsilon]} \left| \int_0^t f\langle\psi, d\mathcal{W}\rangle \right|^\ell \leq C\varepsilon^{\ell\kappa}\|f\|_{L^{2p/(p-2)}}^\ell.$$

□

Now we present the following stochastic version of Gronwall's Lemma.

**Lemma 4.3.** *Let  $X$ ,  $\mathcal{F}_i$ ,  $\lambda$  be real valued processes, and  $\mathcal{G}$  be a Hilbert-space valued one. Furthermore, assume that*

$$dX := [\lambda X + \mathcal{F}_1]dt + \langle\mathcal{G}, d\mathcal{W}\rangle,$$

and that

$$\mathcal{F}_1 \leq \mathcal{F}_2.$$

Then the following inequality holds true

$$X(t) \leq \int_0^t e^{\Lambda(t)} X(0) + \int_0^t e^{\Lambda(t)-\Lambda(s)} \mathcal{F}_1(s) ds + \int_0^t e^{\Lambda(t)-\Lambda(s)} \langle\mathcal{G}(s), d\mathcal{W}(s)\rangle,$$

for

$$\Lambda(t) := \int_0^t \lambda(s) ds.$$

*Proof.* We define

$$Y(t) := X(t)e^{-\Lambda(t)}.$$

By the definition of the process  $Y$ , we obtain easily

$$dY = e^{-\Lambda} dX - \lambda Y dt = e^{-\Lambda} \mathcal{F}_1 dt + e^{-\Lambda} \langle \mathcal{G}, d\mathcal{W} \rangle,$$

and

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t e^{-\Lambda(s)} \mathcal{F}_1(s) ds + \int_0^t e^{-\Lambda(s)} \langle \mathcal{G}(s), d\mathcal{W}(s) \rangle \\ &\leq X(0) + \int_0^t e^{-\Lambda(s)} \mathcal{F}_2(s) ds + \int_0^t e^{-\Lambda(s)} \langle \mathcal{G}(s), d\mathcal{W}(s) \rangle. \end{aligned}$$

Multiplying the above with  $e^{\Lambda(t)}$ , using the definition of  $Y$  and integrating, we derive the stated stochastic version of Gronwall's inequality.  $\square$

**4.5. Uniform Bound on  $\nabla\psi$ .** Using the spectral estimate of Proposition 3.9, we get from (4.9)

$$(4.10) \quad d\|\nabla\psi\|^2 \leq \left[ C\|\nabla\psi\|^2 + c\varepsilon^{-1}\|R\|_{p,\mathcal{D}}^p + 2\|R\|_{p,\mathcal{D}}\|r_\varepsilon^A\|_{q,\mathcal{D}} + C\varepsilon^{2\sigma} \right] dt + 2\varepsilon^\sigma \langle \psi, d\mathcal{W} \rangle.$$

Application of Lemma 4.3) and since  $\nabla\psi(0) = 0$  as  $R(0) = 0$ , yields

$$\begin{aligned} \|\nabla\psi(t)\|^2 &\leq \int_0^t e^{C(t-s)} \left[ c\varepsilon^{-1}\|R\|_{p,\mathcal{D}}^p + \|R\|_{p,\mathcal{D}}\|r_\varepsilon^A\|_{q,\mathcal{D}} + C_{\text{tr}}\varepsilon^{2\sigma} \right] ds + \int_0^t e^{C(t-s)} \varepsilon^\sigma \langle \psi, d\mathcal{W}(s) \rangle \\ &\leq e^{CT} \int_0^t \left[ c\varepsilon^{-1}\|R\|_{p,\mathcal{D}}^p + \|R\|_{p,\mathcal{D}}\|r_\varepsilon^A\|_{q,\mathcal{D}} + C_{\text{tr}}\varepsilon^{2\sigma} \right] ds + \varepsilon^\sigma e^{CT} \left| \int_0^t e^{-Cs} \langle \psi, d\mathcal{W}(s) \rangle \right|. \end{aligned}$$

Furthermore, from Lemma 4.2 we obtain on a subset with high probability

$$\sup_{t \in [0, T_\varepsilon]} \left| \int_0^t e^{-Cs} \langle \psi(s), d\mathcal{W}(s) \rangle \right| \leq C\varepsilon^{\gamma-\kappa}.$$

And thus we arrive at

$$(4.11) \quad \begin{aligned} \|\nabla\psi(t)\|^2 &\leq C\varepsilon^{-1}\|R\|_{p,\mathcal{D}_t}^p + C\|R\|_{p,\mathcal{D}_t}\|r_\varepsilon^A\|_{q,\mathcal{D}_t} + C\varepsilon^{2\sigma}t + C\varepsilon^{\sigma+\gamma-\kappa} \\ &\leq C[\varepsilon^{p\gamma-1} + \varepsilon^\gamma\|r_\varepsilon^A\|_{q,\mathcal{D}_T} + \varepsilon^{2\sigma} + \varepsilon^{\sigma+\gamma-\kappa}] \\ &\leq C[\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa}], \end{aligned}$$

where we used that  $\gamma < \sigma$  and that  $\kappa$  is sufficiently small, together with Theorem 3.8. We verified the following Lemma:

**Lemma 4.4.** *For all  $p \in [2, 3)$ ,  $\sigma > 1$ ,  $\kappa > 0$ , and  $\gamma < \sigma$  we have*

$$(4.12) \quad \|R(t)\|_{L^\infty(0, T_\varepsilon, H^{-1})}^2 \leq C[\varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa}]$$

with probability larger than  $1 - C_\ell \varepsilon^\ell$  for all  $\ell > 0$ .

**4.6. Mean Square Bound on  $\nabla R$ .** We return to relation (4.9) and shall use an estimate presented on pg. 171 of [2], based on the fact that the set where the value of  $u_\varepsilon^A$  is not close to either  $+1$  or  $-1$ , has a small measure. More precisely, the measure is controlled by

$$\text{measure}\left\{(x, t) \in \mathcal{D}_T : f''(u_\varepsilon^A) < 0\right\} \leq C\varepsilon, \quad \varepsilon \in (0, 1].$$

The aforementioned estimate is

$$-\varepsilon^{-1} \int_0^t \int_{\mathcal{D}} f'(u_\varepsilon^A) R^2 dx \leq \varepsilon^{-2/p} \|R\|_{p, \mathcal{D}_t}^2.$$

Therefore, integrating (4.9) and since  $\nabla\psi(0) = 0$  as  $R(0) = 0$ , we arrive at

$$\varepsilon \|\nabla R\|_{2, \mathcal{D}_t}^2 \leq \varepsilon^{-2/p} \|R\|_{p, \mathcal{D}_t}^2 + c\varepsilon^{-1} \|R\|_{p, \mathcal{D}_t}^p + \|R\|_{p, \mathcal{D}_t} \|r_\varepsilon^A\|_{q, \mathcal{D}_t} + \varepsilon^\sigma \int_0^t \langle \psi, d\mathcal{W} \rangle + C_{\text{tr}} \varepsilon^{2\sigma} t.$$

Revoking again Lemma 4.2, we obtain on a set of high probability

$$\varepsilon \|\nabla R\|_{2, \mathcal{D}_{T_\varepsilon}}^2 \leq \varepsilon^{-2/p} \|R\|_{p, \mathcal{D}_{T_\varepsilon}}^2 + c\varepsilon^{-1} \|R\|_{p, \mathcal{D}_{T_\varepsilon}}^p + \|R\|_{p, \mathcal{D}_{T_\varepsilon}} \|r_\varepsilon^A\|_{q, \mathcal{D}_T} + \varepsilon^{\sigma+\gamma-\kappa} + C_{\text{tr}} \varepsilon^{2\sigma} T,$$

where we used that  $T_\varepsilon \leq T$ . Moreover, the definition of  $T_\varepsilon$  implies for all  $t \in [0, T_\varepsilon]$

$$\varepsilon \|\nabla R\|_{2, \mathcal{D}_{T_\varepsilon}}^2 \leq \varepsilon^{-2/p} \varepsilon^{2\gamma} + c\varepsilon^{-1} \varepsilon^{p\gamma} + \varepsilon^\gamma \|r_\varepsilon^A\|_{q, \mathcal{D}_T} + \varepsilon^{\sigma+\gamma-\kappa} + C\varepsilon^{2\sigma}.$$

Here, the constant depends on the final time  $T$ . Using again  $\gamma < \sigma$  and  $\kappa$  sufficiently small, together with Theorem 3.8, we obtain

$$(4.13) \quad \varepsilon \|\nabla R\|_{2, \mathcal{D}_{T_\varepsilon}}^2 \leq C[\varepsilon^{-2/p} \varepsilon^{2\gamma} + \varepsilon^{-1} \varepsilon^{p\gamma} + \varepsilon^{\sigma+\gamma-\kappa}].$$

Note that as  $p > 2$ , a short calculation shows that

$$\varepsilon^{-2/p} \varepsilon^{2\gamma} > \varepsilon^{-1} \varepsilon^{p\gamma} \quad \iff \quad \frac{1}{p} < \gamma,$$

which we assume from now on, as we expect both  $\gamma$  and  $\sigma$  to be bigger than 1. We verified the following Lemma:

**Lemma 4.5.** *For all  $p \in [2, 3)$ ,  $\kappa > 0$ ,  $\sigma > 1$ , and  $\frac{1}{p} < \gamma < \sigma$  we have*

$$(4.14) \quad \|R(t)\|_{L^2(0, T_\varepsilon, H^1)}^2 \leq C[\varepsilon^{-1-2/p+2\gamma} + \varepsilon^{-1+\sigma+\gamma-\kappa}]$$

with probability larger than  $1 - C_\ell \varepsilon^\ell$  for all  $\ell > 0$ .

**4.7. Final step.** In the final part of the proof it remains to show that  $T_\varepsilon = T$  on our set of high probability. Thus, we shall use our estimates of the previous two Lemmas to show that  $\|R\|_{p, \mathcal{D}}$  is not larger than  $\varepsilon^\gamma$ .

Observe first, that the following trivial interpolation inequality holds true

$$(4.15) \quad \|R\|_{2, \mathcal{D}}^2 = - \int_{\mathcal{D}} R \Delta \psi dx = \int_{\mathcal{D}} \nabla R \nabla \psi dx \leq \|\nabla R\|_{2, \mathcal{D}} \|\nabla \psi\|_{2, \mathcal{D}}.$$

We use the Sobolev's embedding of  $H^\alpha$  into  $L^p$  with  $\alpha := d(\frac{1}{2} - \frac{1}{p}) = \frac{d(p-2)}{2p}$ , and then interpolate  $H^\alpha$  between  $L^2$  and  $H^1$ . We need  $\alpha \in [0, 1]$ , which is assured by  $2 < p \leq 3 < \frac{2d}{(d-2)}$ . This gives,

$$\|R\|_{p, \mathcal{D}} \leq C \|R\|_{H^\alpha} \leq C \|R\|_{2, \mathcal{D}}^{1-\alpha} \|\nabla R\|_{2, \mathcal{D}}^\alpha,$$

and thus, using (4.15) we obtain

$$\begin{aligned} \|R\|_{p,\mathcal{D}}^p &\leq C \|R\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{2}} \|\nabla R\|_{2,\mathcal{D}}^{\frac{d(p-2)}{2}} \\ &\leq C \|\nabla\psi\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{4}} \|\nabla R\|_{2,\mathcal{D}}^{\frac{d(p-2)}{2} + \frac{2p-d(p-2)}{4}} = C \|\nabla\psi\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{4}} \|\nabla R\|_{2,\mathcal{D}}^{\frac{d(p-2)+2p}{4}}. \end{aligned}$$

Integration yields

$$\|R\|_{p,\mathcal{D}_t}^p \leq C \sup_{[0,t]} \|\nabla\psi\|_{2,\mathcal{D}}^{\frac{2p-d(p-2)}{4}} \cdot \|\nabla R\|_{2,\mathcal{D}_t}^{\frac{d(p-2)+2p}{4}}.$$

Now we use (4.12) and (4.14) and arrive at

$$(4.16) \quad \|R\|_{p,\mathcal{D}}^p \leq C \left[ \varepsilon^{p\gamma-1} + \varepsilon^{\sigma+\gamma-\kappa} \right]^{\frac{2p-d(p-2)}{8}} \left[ \varepsilon^{-1-2/p+2\gamma} + \varepsilon^{-1+\sigma+\gamma-\kappa} \right]^{\frac{d(p-2)+2p}{8}}$$

and thus, pulling out  $\varepsilon^{2\gamma}$  from both brackets, and as  $\gamma - \sigma \geq 0$  and  $\kappa$  small, we arrive at

$$\begin{aligned} \varepsilon^{-\gamma} \|R\|_{p,\mathcal{D}} &\leq C \cdot \left[ \varepsilon^{(p-2)\gamma-1} + \varepsilon^{\sigma-\gamma-\kappa} \right]^{\frac{2p-d(p-2)}{8p}} \left[ \varepsilon^{-1-2/p} + \varepsilon^{-1+\sigma-\gamma-\kappa} \right]^{\frac{d(p-2)+2p}{8p}} \\ &\leq C \cdot \left[ \varepsilon^{(p-2)\gamma-1} + \varepsilon^{\sigma-\gamma-\kappa} \right]^{\frac{2p-d(p-2)}{8p}} \cdot \varepsilon^{-(1+2/p)\frac{d(p-2)+2p}{8p}}. \end{aligned}$$

The previous bound holds with probability larger than  $1 - C_\ell \varepsilon^\ell$ . In order to show that  $T_\varepsilon = T$  holds with high probability, we need to prove that the right hand side of the previous equation is smaller than 1.

As the second factor is larger than one, we need the first one to be smaller than 1 and to compensate the larger factor. Hence, we need

$$\gamma > \frac{1}{p-2} \left[ 1 + \frac{2p+d(p-2)}{2p-d(p-2)} \cdot \frac{p+2}{p} \right] > \frac{1}{p}$$

and provided  $\kappa$  is sufficiently small

$$\sigma > \gamma + \frac{2p+d(p-2)}{2p-d(p-2)} \cdot \frac{p+2}{p} > \gamma.$$

Hence, the proof of theorem is now complete.

**Acknowledgments.** Dimitra Antonopoulou and Georgia Karali are supported by the ‘‘ARIS-TEIA’’ Action of the ‘‘Operational Program Education and Lifelong Learning’’ co-funded by the European Social Fund (ESF) and National Resources.

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