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On Orbifolds and Free Fermion Constructions

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Abstract

This work develops the correspondence between orbifolds and free fermion models. A complete classification is obtained for orbifolds X/G with X the product of three elliptic curves and G an abelian extension of a group $(\mathbb{Z}_2)^2$ of twists acting on X. Each such quotient X/G is shown to give a geometric interpretation to an appropriate free fermion model, including the geometric NAHE+ model. However, the semi-realistic NAHE free fermion model is proved to be non-geometric: its Hodge numbers are not reproduced by any orbifold X/G. In particular cases it is shown that X/G can agree with some Borcea-Voisin threefolds, an orbifold limit of the Schoen threefold, and several further orbifolds thereof. This yields free fermion models with geometric interpretations on such special threefolds.

Introduction

This work explores a class of heterotic string theories, more precisely of heterotic conformal field theories, and their geometric interpretations. We consider quantum field theories that arise by means of so-called free fermion constructions, and we study the geometric counterparts of the resulting models. Free fermion models are interesting in this context, because mathematically, they are comparatively simple. They all yield rational conformal field theories, which makes them mathematically well behaved. On the other hand, there are free fermion models which can be interpreted as nonlinear sigma models on tori. In other words, there are special points in the moduli space of conformal field theories on tori, where the corresponding conformal field theories allow a free fermion construction. Hence for some particular models, there are geometric interpretations at hand, and the notion of "geometric interpretation" can indeed be made mathematically precise. Finally, more general free fermion models can be included into the discussion by implementing orbifold techniques.

This raises the natural question whether one can find free fermion models which on the one hand yield semi-realistic string theories, in that they produce exactly the spectrum of the minimal supersymmetric standard model in the observable massless sector, and which on the other hand allow a geometric interpretation on a geometric orbifold of a torus. In other words, do any free fermion models exist

which connect both to the real world, via the standard model of particle physics, and to geometry, via a mathematically tractable geometric interpretation?

Addressing the first part of the task, to our knowledge, [MW86] contains the first hint that free fermion models could be used to construct semi-realistic models by orbifold-like procedures. These ideas have been further developed by many authors, and models with semi-realistic gauge groups are given e.g. in [FNY90, INQ87], see also [CFN99]. Further references on free fermion models are [KLT87, GO85, ABKW86, ABK87, AB88], and the reader interested in the related topic of covariant lattice approaches could consult [FMS86, CFQS86, LLS86, LL87, BFVH87, LLS87, LNS87, LT88, LTZ88].

An example of a model of interest to us in this context is the so–called NAHE model [FGKP87, AEHN89, FN93]. It is an example of a semi-realistic heterotic string theory, and it can be obtained from a toroidal model by a chain of orbifoldings of type $\mathbb{Z}_2$. In fact, a closer study reveals that a geometric interpretation on an orbifold of a torus, if it exists, must have the form X/G with X the product of three elliptic curves and G a semidirect product of a group G_S of shifts on X and a subgroup $G_T \subset G$ which is isomorphic to $(\mathbb{Z}_2)^2$, see [Fa93, FFT06].

One is hence naturally led to a classification problem: To determine all topologically inequivalent Calabi-Yau threefolds that arise by resolving the quotient singularities in X/G with X the product of three elliptic curves and G a group of the type described above. This problem is solved in the present paper. A partial classification was already given in [DF04], under additional restrictions on the "group of twists" G_T. A classification of Calabi-Yau threefolds X/G for which G_T is isomorphic to $(\mathbb{Z}_n)^2$ with $n \neq 2$ was given by Jimmy Dillies in [Dil07]. From these classifications one finds a negative answer to the question posed above: No purely geometric interpretation of the semi-realistic NAHE free fermion model exists, since for none of the groups G described above, the Hodge numbers $h^{1,1}$, $h^{2,1}$ of the resolution of X/G yield three generations $h^{1,1} - h^{2,1} = 3$. In other words, the NAHE and other semi-realistic free fermion models must involve some non-geometric orbifolds.

The goal of this paper is to solve the geometric classification problem, to embed it into the context of free fermion models, and to point out some interesting geometric and model-building features arising from the classification.

We start in section 1 with the classification of quotients X/G with G as described above. We give a complete list, including the Hodge numbers of the resulting resolved Calabi-Yau threefolds, as well as their fundamental groups. We also include an (incomplete) discussion of possible coincidences within our list.

In section 2 we show that for each of the Calabi-Yau threefolds in our list there exists a free fermion model whose underlying geometry is X/G. We start with a mathematical review of free fermion constructions. We state and explain the rules of the game, and we discuss orbifolds in the free fermion language. We rederive the well-known fact that a particular free fermion model allows a geometric interpretation on an $SO(12)$ torus. This, along with the discussion of orbifolds, allows us to show that indeed for each model in our list of orbifolds X/G, there is an associated free fermion model.

Our list includes a number of Calabi-Yau threefolds that are familiar from other contexts. The simplest of these is the Vafa-Witten threefold $X/(\mathbb{Z}_2)^2$ studied in [VW95]. The NAHE+ model, capturing the geometric part of the NAHE model, is another example. Contrary to popular lore, it is *NOT* a $\mathbb{Z}_2$ orbifold of the Vafa-Witten threefold. We show instead that it can be obtained as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the Vafa-Witten threefold. The full NAHE model is not geometric: we do not obtain any three-generation models in our classification. For other examples, we recover six different types of Borcea-Voisin threefolds [Bor97, Voi93] within our list. We also find orbifold limits of Schoen's threefold [Sch88] and some of its orbifolds within our list of quotients X/G. This may be of considerable interest because precisely these threefolds have been successfully used in the construction of semi-realistic heterotic string theories in [DOPW02, BD06, BCD06, BD07]. If an appropriate degenerate limit of the relevant gauge bundles can be found, then our result will lead to a dramatic simplification of these heterotic constructions: Free fermion models, after all, are mathematically well understood and technically easy to handle.

Discrete torsion may be included in our orbifolds without leaving the realm of free fermion constructions. We note that turning on discrete torsion has a rather mild effect on the Hodge numbers of our threefolds: we get many of the Hodge numbers of models without torsion, and the only new Hodge pairs are mirrors of existing pairs. Similar observations in more specialized situations have been made before, e.g. in [DW00, PRRV07]. According to Vafa and Witten [VW95], full mirror symmetry (as opposed to just the Hodge theoretic matching) is indeed sometimes realized through discrete torsion. This situation may be specific to $(\mathbb{Z}_2)^2$ orbifolds though, as suggested in [KS95]. The conclusion of [KS94], suggesting that asymmetric orbifolds should be related to discrete torsion, applies in a different setting, where the emphasis lies on simple current constructions but not on geometric interpretations. The NAHE model is not obtainable as a geometric orbifold, with or without discrete torsion. Among the six Borcea-Voisin threefolds we obtain, three are their own mirrors, while the other three are exceptional in the sense that they do not have mirrors within the Borcea-Voisin construction. Our result that for these threefolds, there exist associated free fermion models, could therefore well be useful to shed some light on aspects of mirror symmetry and discrete torsion for these threefolds.

Our basic classification is accomplished with the help of some simple reduction principles, which reduce the combinatorial complexity and allow us to do everything by hand. Without these reductions, the amount of calculations required is massive. Indeed, several computer searches have been carried out recently on regions in the string landscape that overlap ours to various degrees. Nooij [CFN03, Noo06] studied $\mathbb{Z}_2$-type free fermion models based on the $SO(12)$ torus. He includes nongeometric orbifolds, and finds a handful of three generation models. A partial list of orbifolds and Hodge numbers is obtained in [PRRV07]. In work in progress, these authors are studying orbifolds with generalized discrete torsion. This apparently leads them to recover precisely the complete list of Hodge numbers obtained here. The coincidence is quite intriguing; it would be interesting to know whether

the objects themselves coincide or whether the Hodge numbers simply fail to capture the relevant data. Note that, for example, the fundamental groups of their models have not been computed. In another work in progress, Kiritisis, Lennek and Schellekens [KLS08] are searching certain free fermion models whose partition functions are left-right symmetric. Due to a language barrier, it is difficult to compare their models directly to ours. The list of Hodge numbers they get apparently agrees with ours, except that they get one additional model, with Hodge numbers (25,1). The latter is clearly not geometric in our sense: By our assumptions on the orbifolding group G, the G-invariant part of $H^*(X, \mathbb{R})$ contains three dimensional subspaces of $H^{1,1}(X, \mathbb{R})$ and of $H^{1,2}(X, \mathbb{R})$, respectively. Hence the Hodge numbers of all our geometric orbifolds arise by adding contributions of various twisted sectors to the basic (3,3) contribution of the bulk sector, so our Hodge numbers must be at least 3.

1 A classification of relevant orbifolds

In this section, we discuss a classification of orbifoldings and orbifolds. Restricting to groups whose so-called twist group G_T is isomorphic to $(\mathbb{Z}_2)^2$, we introduce a notion of equivalence among such groups, via a reduction principle. Orbifolding the product of three elliptic curves by one group yields a quotient which is isomorphic to what is obtained from the product of three different (but isogenous) elliptic curves by an equivalent group. We give a classification of all such groups up to equivalence. We also calculate some topological data of the resulting orbifolds, namely their Hodge numbers and their fundamental groups. This gives further information about possible isomorphies among the respective quotients. The main results are the tabulation of orbifolds in Section 1.6 and the somewhat incomplete analysis of coincidences in Section 1.7.

1.1 On a classification of toroidal orbifolds

We work with a 6 (real) dimensional torus $X \cong T^6$ with the complex structure of a product $E_1 \times E_2 \times E_3$ of three elliptic curves. Let $T_0 \cong (\mathbb{Z}_2)^2 \subset (\mathbb{Z}_2)^3$ be the

Klein group of twists acting on $(z_1, z_2, z_3) \in X$ by an even number of sign changes:

$$
\begin{aligned}
t_1 : & \quad (z_1, z_2, z_3) \rightarrow (z_1, -z_2, -z_3), \\
t_2 : & \quad (z_1, z_2, z_3) \rightarrow (-z_1, z_2, -z_3), \\
t_3 : & \quad (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3).
\end{aligned}
$$

An arbitrary automorphism g of X can be factored uniquely: $g = s \circ g_t$, where the twist part g_t is an automorphism sending the origin $0 \in X$ to itself, while the shift part s is translation by $g(0) \in X$. Any group G of automorphisms fits in an exact sequence

$$
0 \longrightarrow G_S \longrightarrow G \overset{\pi}{\longrightarrow} G_T^0 \longrightarrow 0,
$$

where G_S is the subgroup of shifts contained in G, and G_T^0 is the group of twist parts of all elements of G, so $G_T^0 := \{g_t | \exists$ a shift s such that $g = s \circ g_t \in G\}$. In general, G_T^0 is not a subgroup of G. However, it follows from Lemma 1.1.2 below that we can always reduce to a situation where we can choose a subgroup $G_T \subset G$ which maps isomorphically onto G_T^0 under π, and such that $G = G_S \times G_T$.

Our goal in this section is to study toroidal orbifolds, i.e. quotients X/G, for all finite groups G whose twist part is T_0. We will see that these come in a finite number of irreducible families.

Definition 1.1.1 *We say that a group G of automorphisms of X is* redundant *if it contains a translation by a non zero $x \in E_i$ for some $i \in 1, 2, 3$, and is* essential *otherwise.*

Our first observation (cf.[DF04]) is that there is a simple reduction principle: every toroidal orbifold X/G with $X = E_1 \times E_2 \times E_3$ and a given twist part G_T^0 is also of the form X'/G' for some $X' = E_1' \times E_2' \times E_3'$ and some essential group of automorphisms G' with the same twist part G_T^0. Indeed, if the redundant G contains a translation s_x by a non zero element $x \in E_i$, then x must be a torsion element, the quotient $E_i' := E_i/x$ is an elliptic curve, the quotient $X' := X/x$ is still a product of three elliptic curves with one E_i replaced by E_i', and

$$
X/G \cong X'/G',
$$

where $G' := G/\langle s_x \rangle$ fits into an exact sequence

$$
0 \rightarrow G_S' \rightarrow G' \rightarrow G_T' \rightarrow 0,
$$

with $G_S' = G_S/\langle s_x \rangle$ and $G_T' = G_T^0$ as claimed.

We therefore may as well restrict attention to essential groups G.

Lemma 1.1.2 *Any essential group G with twist part $G_T^0 = T_0$ is commutative and isomorphic to the direct product $G_S \times G_T^0$ of its shift and twist parts. All elements of G are of order 2, and up to conjugation G is contained in G^{max} which is the extension*

$$
0 \rightarrow X[2] \rightarrow G^{max} \rightarrow T_0 \rightarrow 0,
$$

where $X[2] \cong (\mathbb{Z}_2)^6$ is the group of all points of order 2 in X.

Proof: First we show that any $g \in G$ has order 2. Let $g = s \circ g_t \in G$ with $s \in G_S$ a shift by $x \in X$ and $g_t \in G_T^0 = T_0$. If $g_t \neq 1 \in G_T^0$ then g^2 is a shift along $x + g_t(x)$, i.e. along one of the three elliptic curves, so essence implies $g^2 = 1$. We still need to consider $g \in G_S$. The subgroup G_S of G is isomorphic to

$$\overline{G}_S := \{x \in X | s_x \in G_S\}.$$

The latter is invariant under the action of T_0. If it contains $x = (x_1, x_2, x_3)$ it must also contain $t_i(x)$ hence $x + t_i(x)$, which is in E_i. Essence therefore implies that $2x_i = 0$ for all i.

It follows that G is commutative and contains a subgroup G_T that maps isomorphically onto the twist group G_T^0. Further, it follows that G is isomorphic to the direct product $G_S \times G_T$ of its shift subgroup G_S with any such G_T. Now G_S is a group of translations by points of order 2, so it is contained in G^{max}. The twist group G_T need not be contained in G^{max}. Its generators can be written in the form:

$$(z_1, z_2, z_3) \rightarrow (x_1 + z_1, x_2 - z_2, x_3 - z_3),$$
$$(z_1, z_2, z_3) \rightarrow (y_1 - z_1, y_2 + z_2, y_3 - z_3)$$

The order-2 condition requires that x_1, y_2 and $y_3 - x_3$ be points of order 2, while the three remaining variables are unconstrained in the three elliptic curves E_i. Nevertheless, one checks immediately that conjugation by an appropriate translation of X (which also has three complex degrees of freedom, one in each E_i) can be chosen to set $x_2 = x_3 = y_1 = 0$. Such a conjugation takes G_T into G^{max} and leaves G_S unchanged, completing the proof.

$\square$

In view of the lemma, our essential group G contains a "subgroup of twists" G_T which under π maps isomorphically to G_T^0, and G is isomorphic to $G_S \times G_T$. In the next section we will see that up to conjugation there are four possible actions of the twist group G_T on X.

1.2 Classification of essential automorphism groups

Definition 1.2.1 *The* rank *of an essential automorphism group G is the rank of G_S as a module over $\mathbb{Z}_2$.*

We will study the possible automorphism groups according to their increasing rank. We will usually describe an automorphism group in terms of a minimal set of generators, listing each generator in the form of a triple $(\epsilon_1 \delta_1, \epsilon_2 \delta_2, \epsilon_3 \delta_3)$, where $\epsilon_i \in E_i$ is a point of order 2, and $\delta_i \in \{\pm\}$ indicates the pure twist part. We take the period lattice of the elliptic curve E_i to be generated by 2 and 2τ, so the ϵ_i can be one of $0, 1, \tau, 1 + \tau$. The three operations that produce equivalent groups are change of basis, permutation of the three coordinates z_i of the torus, and a shift of one or more of the z_i. We start with rank 0, where instead of listing two generators we often list all three non zero group elements.

Lemma 1.2.2 *There are 4 inequivalent groups $G = G_T$ of rank 0, given as follows:*

$$
\begin{aligned}
(0-1): \quad & (0+,0-,0-), (0-,0+,0-), (0-,0-,0+), \\
(0-2): \quad & (0+,0-,0-), (0-,0+,1-), (0-,0-,1+), \\
(0-3): \quad & (0+,0-,0-), (0-,1+,1-), (0-,1-,1+), \\
(0-4): \quad & (1+,0-,0-), (0-,1+,1-), (1-,1-,1+).
\end{aligned}
$$

Remark: In [DF04], only the first of these possibilities, as well as its further quotients, were considered, leading to the considerably shorter list there.

Proof: Any rank 0 group is generated by two elements of the form $(\epsilon_1+, \epsilon_2-, \epsilon_3-)$ and $(\epsilon_4-, \epsilon_5+, \epsilon_6-)$. By shifting the three coordinates z_i we can clearly arrange that $\epsilon_2 = \epsilon_3 = \epsilon_4 = 0$, and by changing the labeling of a homology basis for the E_i we can take each of the remaining ϵ_i to be 0 or 1. This leaves us with 8 possibilities, including the four above and

$$
\begin{aligned}
& (0+,0-,0-), (0-,1+,0-), (0-,1-,0+); \\
& (1+,0-,0-), (0-,0+,0-), (1-,0-,0+); \\
& (1+,0-,0-), (0-,0+,1-), (1-,0-,1+); \\
& (1+,0-,0-), (0-,1+,0-), (1-,1-,0+).
\end{aligned}
$$

Of these, the first two are equivalent to $(0-2)$ under a permutation of the three coordinates. The third is transformed by a shift of z_3 to $(1+,0-,1-)$, $(0-,0+,0-)$, $(1-,0-,1+)$ which is equivalent to $(0-3)$ under a permutation of z_1, z_2. Similarly, the fourth group is transformed by a shift of z_1 to $(1+,0-,0-)$, $(1-,1+,0-)$, $(0-,1-,0+)$ which under a permutation of z_2, z_3 is equivalent to the third group, hence to $(0-3)$.

One can use similar elementary means to check that the four groups in the statement of the lemma are inequivalent. In Section 1.6 we will find the stronger result that the corresponding quotients X/G are topologically inequivalent. $\qquad\square$

For a group G of higher rank, we list first two generators of G which map onto a minimal generating set for the twist group G_T in the previous $(\epsilon_1\delta_1, \epsilon_2\delta_2, \epsilon_3\delta_3)$ notation; the remaining generators are chosen to be in the shift subgroup $G_S \subset G$. Since in this case all the δ_i are 0, we can omit them, using instead the abbreviated notation $(\epsilon_1, \epsilon_2, \epsilon_3)$.

Proposition 1.2.3 *There are 11 equivalence classes of essential groups in rank 1, 14 in rank 2, 6 in rank 3, one in rank 4, and none in higher ranks. They are listed in the first two columns of Table 1, see Section 1.6.*

The proof is elementary and somewhat tedious, using the tools introduced in the proof of Lemma 1.2.2. We leave the details to the reader.

1.3 Orbifold cohomology

Though the techniques are well known, let us briefly summarize for the reader's convenience the procedure by which one calculates the Hodge numbers of a minimal

resolution of the orbifold X/G. We will assume the situation which is of interest below, that is $X = E_1 \times E_2 \times E_3$ equipped with the complex structure of the product of three elliptic curves E_i, and that G is an essential group of the type described in Section 1.1. In particular, by Lemma 1.1.2, $G = G_S \times G_T$ with $G_T \cong T_0$ under the projection π to the twist parts.

First observe that the cohomology of X is obtained by taking the wedge product between the total cohomologies of each elliptic curve E_i. With respect to a local complex coordinate z_i on E_i, the cohomology of the latter is generated by 1, dz_i, $d\overline{z}_i$, $dz_i \wedge d\overline{z}_i$. If $g \in G$ splits as $g = s \circ t_i$ with $t_i \in T_0$ into its shift and its twist part, then g acts on dz_1, dz_2, dz_3 by $dz_i \mapsto dz_i$ and $dz_j \mapsto -dz_j$ for $j \neq i$, and similarly for the $d\overline{z}_k$. Hence for the G-invariant part of the cohomology of X we find dimensions $h_{inv}^{p,q}$, $p, q \in \{0, \ldots, 3\}$, with

$$
(h_{inv}^{p,q})_{p,q} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
$$

For example, we have representatives $dz_i \wedge d\overline{z}_i$ in $H^{1,1}(X)$ and $dz_1 \wedge dz_2 \wedge d\overline{z}_3$ in $H^{2,1}(X)$.

Additional contributions to the cohomology of the minimal resolution of X/G come from the blow-ups of curves of singularities. Assume that $g \in G$, $g \neq 1$, has fixed points on X. Since by assumption $g = s \circ t_i$ for some $t_i \in T_0$ and s a shift, this implies that s is a shift by some point $x = (x_1, x_2, x_3) \in X$ of order 2 with $x_i = 0$. The fixed locus of g thus consists of 16 copies of E_i. In X/G, the image yields a curve of singularities of type A_1. Its contributions to the cohomology of a resolution of $X/\langle g \rangle$ have dimensions $h_g^{p,q}$, $p, q \in \{0, \ldots, 3\}$, with

$$
(h_g^{p,q})_{p,q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 16 & 16 & 0 \\ 0 & 16 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The contributions to the cohomology of the resolved quotient X/G are given by the G-invariant part of these vector spaces. If G has rank r, i.e. $G \cong (\mathbb{Z}_2)^{r+2}$, then the total contribution from blowing up the fixed locus of $g = s \circ t_i$ is

$$
(h_{g,inv,A}^{p,q})_{p,q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2^{3-r} & 2^{3-r} & 0 \\ 0 & 2^{3-r} & 2^{3-r} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad (h_{g,inv,B}^{p,q})_{p,q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2^{4-r} & 0 & 0 \\ 0 & 0 & 2^{4-r} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

The case $h_{g,inv,B}^{p,q}$ applies if and only if the subgroup of G which maps an irreducible component of the fixed locus of g in X onto itself is strictly larger than $\langle g \rangle$. Indeed, then G contains elements h which map each copy of E_i in the fixed locus of g onto itself, but which act by multiplication by -1 on dz_i and $d\overline{z}_i$, thus leaving none of the cohomology classes counted by $h_g^{2,1}$ and $h_g^{1,2}$ invariant, whereas all contributions to $h_g^{1,1}$ and $h_g^{2,2}$ are invariant.

1.4 Discrete torsion

In his seminal paper [Vaf86], Cumrun Vafa pointed out that in conformal field theory, there is an additional degree of freedom $\varepsilon \in H^2(G, U(1))$ when orbifolding by a group G, which is now commonly known as "discrete torsion". Roughly speaking, one introduces a twisted action of G on the contribution to the cohomology which comes from the blow-up of the singular locus in X/G. In the examples that are of interest for us, $G \cong (\mathbb{Z}_2)^{r+2}$. One checks that discrete torsion is compatible with the reduction principle of Section 1.1, and $H^2(G, U(1)) = (\mathbb{Z}_2)^m$ with $m = \binom{r+2}{2}$. Consider elements $g, h \in G - \{1\}$ such that $h \neq g$ and h maps each component of the fixed locus of g onto itself. Then the effect of non-trivial discrete torsion $\varepsilon(g, h)$ amounts to replacing the contributions $h^{p,q}_{g,inv,B}$ listed above by

$$
\left(h^{p,q}_{g,inv,\widetilde{B}} \right)_{p,q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{4-r} & 0 \\ 0 & 2^{4-r} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

1.5 Fundamental groups

There is a simple procedure for calculating the fundamental group of an orbifold, which goes back to [DHVW85] in the physics literature. A mathematical version can be found in [BH02].

Let a group $\widetilde{G}$ act discretely on a simply connected $\widetilde{X}$. Let F be the subgroup of $\widetilde{G}$ generated by all elements which have a fixed point in $\widetilde{X}$. Then the fundamental group of the quotient space $\widetilde{X}/\widetilde{G}$ is $\widetilde{G}/F$.

In our applications, we are interested in the fundamental group of quotients of the product X of three elliptic curves, that is $X = \mathbb{C}^3/\Lambda$. We take $\widetilde{G}$ to be the extension of the orbifolding group G by the lattice Λ:

$$
0 \to \Lambda \to \widetilde{G} \to G \to 0,
$$

so the orbifold is $X/G = \mathbb{C}^3/\widetilde{G}$. The calculation of the fundamental group of each of our orbifolds is then a straightforward exercise.

1.6 Tabulation of results

Table 1: The list of automorphism groups

We list the automorphism groups by rank. For each group G we list its twist group G_T, its shift part G_S (if non-empty), the Hodge numbers $h^{1,1}, h^{2,1}$ of a small resolution of X/G, the fundamental group $\pi_1(X/G)$, and the list of contributing sectors and their contribution. For the fundamental groups we use the abbreviations:

$A:$ the extension of $\mathbb{Z}_2$ by $\mathbb{Z}^2$ (so $H_1(X) = (\mathbb{Z}_2)^3$)

$B:$ any extension of $(\mathbb{Z}_2)^2$ by $\mathbb{Z}^6$ (with various possible $H_1(X)$)

$C:$ $\mathbb{Z}_2$

$D:$ $(\mathbb{Z}_2)^2$

A shift element is denoted by a triple $(\epsilon_1, \epsilon_2, \epsilon_3)$, where $\epsilon_i \in E_i$ is a point of order 2, abbreviated as one of $0, 1, \tau, \tau 1 := 1 + \tau$. A twist element is denoted by a triple $(\epsilon_1\delta_1, \epsilon_2\delta_2, \epsilon_3\delta_3)$, where $\epsilon_i \in E_i$ is as above and $\delta_i \in \{\pm\}$ indicates the pure twist part. A two-entry contribution (a, b) adds a units to $h^{1,1}$ and b units to $h^{2,1}$. When $b = 0$ we abbreviate (a, b) to the single entry contribution a.

	G_T	G_S		$(h^{1,1}, h^{2,1})$	π_1
		sectors	contribution		
Rank 0:					
$(0-1)$	$(0+, 0-, 0-), (0-, 0+, 0-)$			$(51, 3)$	0
		$0+, 0-, 0-$	16		
		$0-, 0+, 0-$	16		
		$0-, 0-, 0+$	16		
$(0-2)$	$(0+, 0-, 0-), (0-, 0+, 1-)$			$(19, 19)$	0
		$0+, 0-, 0-$	$8, 8$		
		$0-, 0+, 1-$	$8, 8$		
$(0-3)$	$(0+, 0-, 0-), (0-, 1+, 1-)$			$(11, 11)$	A
		$0+, 0-, 0-$	$8, 8$		
$(0-4)$	$(1+, 0-, 0-), (0-, 1+, 1-)$			$(3, 3)$	B
Rank 1:					
$(1-1)$	$(0+, 0-, 0-), (0-, 0+, 0-)$	(τ, τ, τ)		$(27, 3)$	C
		$0+, 0-, 0-$	8		
		$0-, 0+, 0-$	8		
		$0-, 0-, 0+$	8		
$(1-2)$	$(0+, 0-, 0-), (0-, 0+, \tau-)$	(τ, τ, τ)		$(15, 15)$	0
		$0+, 0-, 0-$	$4, 4$		
		$0-, 0+, \tau-$	$4, 4$		
		$\tau-, \tau-, 0+$	$4, 4$		
$(1-3)$	$(0+, 0-, 0-), (0-, 0+, 1-)$	(τ, τ, τ)		$(11, 11)$	C
		$0+, 0-, 0-$	$4, 4$		
		$0-, 0+, 1-$	$4, 4$		
$(1-4)$	$(0+, 0-, 0-), (0-, 1+, 1-)$	(τ, τ, τ)		$(7, 7)$	A
		$0+, 0-, 0-$	$4, 4$		
$(1-5)$	$(1+, 0-, 0-), (0-, 1+, 1-)$	(τ, τ, τ)		$(3, 3)$	B
$(1-6)$	$(0+, 0-, 0-), (0-, 0+, 0-)$	$(\tau, \tau, 0)$		$(31, 7)$	0
		$0+, 0-, 0-$	8		
		$0-, 0+, 0-$	8		
		$0-, 0-, 0+$	8		
		$\tau-, \tau-, 0+$	$4, 4$		
$(1-7)$	$(0+, 0-, 0-), (0-, 0+, 1-)$	$(\tau, \tau, 0)$		$(11, 11)$	C
		$0+, 0-, 0-$	$4, 4$		
		$0-, 0+, 1-$	$4, 4$		

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	G_T	G_S		$(h^{1,1}, h^{2,1})$	π_1
		sectors	contribution		
$(1-8)$	$(0+,0-,0-),(0-,1+,0-)$	$(\tau,\tau,0)$		$(15,15)$	0
		$0+,0-,0-$	$4,4$		
		$0-,1-,0+$	$4,4$		
		$\tau-,\tau1-,0+$	$4,4$		
$(1-9)$	$(0+,0-,0-),(0-,1+,1-)$	$(\tau,\tau,0)$		$(7,7)$	A
		$0+,0-,0-$	$4,4$		
$(1-10)$	$(1+,0-,0-),(0-,1+,0-)$	$(\tau,\tau,0)$		$(11,11)$	A
		$1-,1-,0+$	$4,4$		
		$\tau1-,\tau1-,0+$	$4,4$		
$(1-11)$	$(1+,0-,0-),(0-,1+,1-)$	$(\tau,\tau,0)$		$(3,3)$	B
Rank 2:					
$(2-1)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,\tau,\tau)$		$(15,3)$	D
		$0+,0-,0-$	4		
		$0-,0+,0-$	4		
		$0-,0-,0+$	4		
$(2-2)$	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,1),(\tau,\tau,\tau)$		$(9,9)$	C
		$0+,0-,0-$	$2,2$		
		$0-,0+,1-$	$2,2$		
		$1-,1-,0+$	$2,2$		
$(2-3)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,\tau,0)$		$(17,5)$	C
		$0+,0-,0-$	4		
		$0-,0+,0-$	4		
		$0-,0-,0+$	4		
		$\tau-,\tau-,0+$	$2,2$		
$(2-4)$	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,1),(\tau,\tau,0)$		$(11,11)$	0
		$0+,0-,0-$	$2,2$		
		$0-,0+,1-$	$2,2$		
		$1-,1-,0+$	$2,2$		
		$\tau1-,\tau1-,0+$	$2,2$		
$(2-5)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,1),(\tau,\tau,0)$		$(7,7)$	D
		$0+,0-,0-$	$2,2$		
		$0-,0+,\tau-$	$2,2$		
$(2-6)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,1),(\tau,1,0)$		$(19,7)$	0
		$0+,0-,0-$	4		
		$0-,0+,0-$	4		
		$0-,0-,0+$	4		
		$\tau1-,0+,1-$	$2,2$		
		$\tau-,1-,0+$	$2,2$		
$(2-7)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,1),(\tau,1,0)$		$(9,9)$	C
		$0+,0-,0-$	$2,2$		
		$0-,0+,\tau-$	$2,2$		
		$\tau1-,0+,\tau1-$	$2,2$		

	G_T	G_S		$(h^{1,1}, h^{2,1})$	π_1
		sectors	contribution		
$(2-8)$	$(0+,0-,0-),(0-,\tau+,\tau-)$	$(1,1,1),(\tau,1,0)$		$(5,5)$	A
		$0+,0-,0-$	$2,2$		
$(2-9)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,1,1),(1,0,1)$		$(27,3)$	0
		$0+,0-,0-$	4		
		$0-,0+,0-$	4		
		$0-,0-,0+$	4		
		$0+,1-,1-$	4		
		$1-,0+,1-$	4		
		$1-,1-,0+$	4		
$(2-10)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(0,1,1),(1,0,1)$		$(11,11)$	0
		$0+,0-,0-$	$2,2$		
		$0+,1-,1-$	$2,2$		
		$0-,0+,\tau-$	$2,2$		
		$1-,0+,\tau1-$	$2,2$		
$(2-11)$	$(0+,0-,0-),(0-,\tau+,\tau-)$	$(0,1,1),(1,0,1)$		$(7,7)$	A
		$0+,0-,0-$	$2,2$		
		$0+,1-,1-$	$2,2$		
$(2-12)$	$(\tau+,0-,0-),(0-,\tau+,\tau-)$	$(0,1,1),(1,0,1)$		$(3,3)$	B
$(2-13)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,0),(\tau,\tau,0)$		$(21,9)$	0
		$0+,0-,0-$	4		
		$0-,0+,0-$	4		
		$0-,0-,0+$	4		
		$1-,1-,0+$	$2,2$		
		$\tau-,\tau-,0+$	$2,2$		
		$\tau1-,\tau1-,0+$	$2,2$		
$(2-14)$	$(0+,0-,0-),(0-,0+,1-)$	$(1,1,0),(\tau,\tau,0)$		$(7,7)$	D
		$0+,0-,0-$	$2,2$		
		$0-,0+,1-$	$2,2$		
Rank 3:					
$(3-1)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,\tau,1),$		$(12,6)$	0
		$(\tau,1,0),(1,0,\tau)$			
		$0+,0-,0-$	2		
		$0-,0+,0-$	2		
		$0-,0-,0+$	2		
		$0+,\tau-,1-$	$1,1$		
		$1-,0+,\tau-$	$1,1$		
		$\tau-,1-,0+$	$1,1$		

	G_T	G_S		$(h^{1,1}, h^{2,1})$	π_1
		sectors	contribution		
$(3-2)$	$(0+,0-,0-),(0-,0+,1-)$	$(0,\tau,1),$		$(12,6)$	0
		$(\tau,1,0),(1,0,\tau)$			
		$0+,0-,0-$	$1,1$		
		$0-,0+,1-$	2		
		$0-,\tau-,0+$	2		
		$0+,\tau-,1-$	2		
		$1-,0+,\tau1-$	$1,1$		
		$\tau-,\tau1-,0+$	$1,1$		
$(3-3)$	$(0+,0-,0-),(0-,0+,0-)$	$(1,1,0),$		$(17,5)$	0
		$(\tau,\tau,0),(1,\tau,1)$			
		$0+,0-,0-$	2		
		$0-,0+,0-$	2		
		$0-,0-,0+$	2		
		$0+,\tau1-,1-$	2		
		$\tau1-,0+,1-$	2		
		$1-,1-,0+$	$1,1$		
		$\tau-,\tau-,0+$	$1,1$		
		$\tau1-,\tau1-,0+$	2		
$(3-4)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(1,1,0),$		$(7,7)$	C
		$(\tau,\tau,0),(1,\tau,1)$			
		$0+,0-,0-$	$1,1$		
		$0-,0+,\tau-$	$1,1$		
		$0+,\tau1-,1-$	$1,1$		
		$\tau1-,0+,\tau1-$	$1,1$		
$(3-5)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,1,1),$		$(15,3)$	C
		$(1,0,1),(\tau,\tau,\tau)$			
		$0+,0-,0-$	2		
		$0-,0+,0-$	2		
		$0-,0-,0+$	2		
		$0+,1-,1-$	2		
		$1-,0+,1-$	2		
		$1-,1-,0+$	2		
$(3-6)$	$(0+,0-,0-),(0-,0+,\tau-)$	$(0,1,1),$		$(9,9)$	0
		$(1,0,1),(\tau,\tau,\tau)$			
		$0+,0-,0-$	$1,1$		
		$0-,0+,\tau-$	$1,1$		
		$\tau-,\tau-,0+$	$1,1$		
		$0+,1-,1-$	$1,1$		
		$1-,0+,\tau1-$	$1,1$		
		$\tau1-,\tau1-,0+$	$1,1$		

	G_T	G_S		$(h^{1,1}, h^{2,1})$	π_1
			sectors contribution		
Rank 4:					
$(4-1)$	$(0+,0-,0-),(0-,0+,0-)$	$(0,\tau,1),(\tau,1,0),$		$(15,3)$	0
		$(1,0,\tau),(1,1,1)$			
		$0+,0-,0-$	1		
		$0+,\tau-,1-$	1		
		$0+,1-,\tau1-$	1		
		$0+,\tau1-,\tau-$	1		
		$0-,0+,0-$	1		
		$1-,0+,\tau-$	1		
		$\tau1-,0+,1-$	1		
		$\tau-,0+,\tau1-$	1		
		$0-,0-,0+$	1		
		$\tau-,1-,0+$	1		
		$1-,\tau1-,0+$	1		
		$\tau1-,\tau-,0+$	1		

As follows from the discussion in Section 1.4, for some of the orbifolds listed above there may be a non-trivial effect on the resulting Hodge numbers, when twisted actions of G are allowed on the blow-ups of curves of singularities in X/G, that is when discrete torsion is taken into account. The most popular example of this type is our model $(0-1)$ which was extensively studied in [VW95]. In that paper, the authors discover that turning on nontrivial discrete torsion $\varepsilon \in \mathbb{Z}_2$ in this example of an orbifold by $G \cong (\mathbb{Z}_2)^2$ produces its mirror partner. From our discussion it is indeed not hard to check that the effect of $\varepsilon = -1$ instead of $\varepsilon = 1$ is a swap of the Hodge numbers $h^{1,1}$, $h^{2,1}$.

In general, for each model in our list, an interchange of $h^{1,1}$ and $h^{2,1}$ can be achieved by choosing a certain value for discrete torsion. Other types of discrete torsion exist for some of the models, typically producing Hodge number pairs that are intermediate between those of the original orbifold and its mirror. All Hodge pairs obtained this way arise also from other orbifolds without discrete torsion, so they can also be found elsewhere in our table. we list the possible Hodge numbers below:

model	possible values of $(h^{1,1}, h^{2,1})$
$(2-9)$	$(27,3),(15,15),(3,27)$
$(3-3)$	$(17,5),(11,11),(5,17)$
$(3-5)$	$(15,3),(9,9),(3,15)$
$(4-1)$	$(15,3),(12,6),(9,9),(6,12),(3,15)$

Thus we find that the main effect of discrete torsion on our list of possible Hodge numbers $(h^{1,1}, h^{2,1})$ is a symmetrization with respect to $h^{1,1} \leftrightarrow h^{2,1}$. We therefore refrain from a further study of the resulting geometries. In particular, we do not examine whether there are any Calabi-Yau threefolds obtained from allowing non-trivial discrete torsion which agree with any of the models listed in Table 1, or their mirrors.

Table 2: The orbifold family tree

Below we list all those orbifoldings which are realized as free quotients between orbifolds listed in Table 1. In the diagrams, each entry is of the form $\left(\begin{smallmatrix}(r-n)\\(h^{1,1},h^{2,1})\end{smallmatrix}\right)$, where $(r-n)$ is the label in Table 1, and $(h^{1,1}, h^{2,1})$ gives the corresponding Hodge numbers.

We first list all free quotients relating orbifolds X/G with fundamental group of type A:

$$
\begin{array}{ccccc}
 & & \left(\begin{smallmatrix}(1-4)\\(7,7)\end{smallmatrix}\right) & \longrightarrow & \left(\begin{smallmatrix}(2-8)\\(5,5)\end{smallmatrix}\right) \\
 & \nearrow & & \nearrow & \\
\left(\begin{smallmatrix}(0,3)\\(11,11)\end{smallmatrix}\right) & \longrightarrow & \left(\begin{smallmatrix}(1-9)\\(7,7)\end{smallmatrix}\right) & & \\
 & \searrow & & \searrow & \\
 & & \left(\begin{smallmatrix}(1-10)\\(11,11)\end{smallmatrix}\right) & \longrightarrow & \left(\begin{smallmatrix}(2-11)\\(7,7)\end{smallmatrix}\right)
\end{array}
$$

Next we list all free quotients relating orbifolds X/G with fundamental group of type B:

$$
\begin{array}{ccc}
\left(\begin{smallmatrix}(0,4)\\(3,3)\end{smallmatrix}\right) & \longrightarrow & \left(\begin{smallmatrix}(1-5)\\(3,3)\end{smallmatrix}\right) \\
 & \searrow & \\
 & \left(\begin{smallmatrix}(1-11)\\(3,3)\end{smallmatrix}\right) & \longrightarrow \quad \left(\begin{smallmatrix}(2-12)\\(3,3)\end{smallmatrix}\right)
\end{array}
$$

We list the remaining quotients, where the three columns give orbifolds with fundamental group 0, C, D, respectively:

$$\binom{(0-1)}{(51,3)} \longrightarrow \binom{(1-1)}{(27,3)} \longrightarrow \binom{(2-1)}{(15,3)}$$

$$\binom{(0,2)}{(19,19)} \longrightarrow \binom{(1-3)}{(11,11)} \longrightarrow \binom{(2-5)}{(7,7)}$$

$$\searrow \qquad \nearrow$$

$$\binom{(1-7)}{(11,11)} \longrightarrow \binom{(2-14)}{(7,7)}$$

$$\binom{(1-2)}{(15,15)} \longrightarrow \binom{(2-2)}{(9,9)}$$

$$\binom{(1-6)}{(31,7)} \longrightarrow \binom{(2-3)}{(17,5)}$$

$$\binom{(1-8)}{(15,15)} \longrightarrow \binom{(2-7)}{(9,9)}$$

$$\binom{(2-4)}{(11,11)}$$

$$\binom{(2-6)}{(19,7)}$$

$$\binom{(2-9)}{(27,3)} \longrightarrow \binom{(3-5)}{(15,3)}$$

$$\binom{(2-10)}{(11,11)} \longrightarrow \binom{(3-4)}{(7,7)}$$

$$\binom{(2-13)}{(21,9)}$$

$$\binom{(3-1)}{(12,6)}$$

$$\binom{(3-2)}{(12,6)}$$

$$\binom{(3-3)}{(17,5)}$$

$$\binom{(3-6)}{(9,9)}$$

$$\binom{(4-1)}{(15,3)}$$

1.7 On coincidences in the list

The Hodge numbers and fundamental group data in Table 1 do not suffice to completely distinguish the orbifolds on our list. We can obtain some additional topological information:

Lemma 1.7.1 *The four orbifolds whose fundamental group is an extension of $(\mathbb{Z}_2)^2$ by $\mathbb{Z}^6$ ("type B"), labeled $(0-4), (1-5), (1-11), (2-12)$, all with Hodge numbers $(3,3)$, are topologically inequivalent: their fundamental groups are not isomorphic.*

Proof: Each of these four orbifolds X_i is a quotient of $\mathbb{C}^3$ by a group G_i acting without fixed points, so $F_i = \{1\}, G_i = G_i/F_i = \pi_1(X_i)$.
Each G_i is generated by the two twists $(1+, 0-, 0-), (0-, 1+, 1-)$, plus a rank 6 lattice L_i of translations, with respective generators:

$$(0-4): \quad (2,0,0), (0,2,0), (0,0,2), (2\tau, 0, 0), (0, 2\tau, 0), (0, 0, 2\tau);$$
$$(1-5): \quad (2,0,0), (0,2,0), (0,0,2), (2\tau, 0, 0), (0, 2\tau, 0), (\tau, \tau, \tau);$$

$$(1-11): \quad (2,0,0),(0,2,0),(0,0,2),(2\tau,0,0),(\tau,\tau,0),(0,0,2\tau);$$
$$(2-12): \quad (2,0,0),(0,2,0),(0,0,2),(2\tau,0,0),(0,\tau,\tau),(\tau,0,\tau).$$

The commutator $[G,G]$ is generated by:

$$
\begin{aligned}
[(1+,0-,0-),(0-,1+,1-)] &= (2,-2,2); \\
[(1+,0-,0-),L_i] &= \{(0,-2b,-2c)|(a,b,c) \in L_i\}; \\
[(0-,1+,1-),L_i] &= \{(-2a,0,-2c)|(a,b,c) \in L_i\}.
\end{aligned}
$$

Consider the four quotients $H_1 = G/[G,G]$ for $G = G_i$. The images of the two twists square to the non zero elements $(2,0,0)$ and $(0,2,0)$ respectively, which in all four cases are distinct in H_1. These twists therefore generate a subgroup $(\mathbb{Z}_4)^2$ of H_1 which contains the image of the $(\mathbb{Z}_2)^2$ orbit of the $\mathbb{Z}^3$ of 1-cycles. Therefore, each of the quotients $H_1 = G/[G,G]$ is a product $H^1 \times H^\tau$ where $H^1 = (\mathbb{Z}_4)^2$ comes from the twists and the three 1-cycles, while H^τ comes from the three τ-cycles. Explicitly, the four groups H^τ are: $(\mathbb{Z}_2)^3, \mathbb{Z}_4, (\mathbb{Z}_2)^2, (\mathbb{Z}_2)^2$, so the four quotients $H_1 = G/[G,G]$ are:

$$
\begin{aligned}
(0-4): &\quad (\mathbb{Z}_4)^2 \times (\mathbb{Z}_2)^3, \\
(1-5): &\quad (\mathbb{Z}_4)^3, \\
(1-11): &\quad (\mathbb{Z}_4)^2 \times (\mathbb{Z}_2)^2, \\
(2-12): &\quad (\mathbb{Z}_4)^2 \times (\mathbb{Z}_2)^2.
\end{aligned}
$$

It remains to distinguish between the last two orbifolds. For that, note that in these cases the extension $0 \to L_i \to G_i \to (\mathbb{Z}_2)^2 \to 0$ is uniquely determined by L_i. In fact, the projection $G_i \to (\mathbb{Z}_2)^2$ is just the composition of the abelianization $G \to G/[G,G]$ with multiplication by 2 in the abelian group $G/[G,G]$. So any isomorphism of the two groups G_i induces an isomorphism of the extensions, hence of the actions of $(\mathbb{Z}_2)^2$ on L_i. This action sends $(a,b,c) \in L_i$ to $(a,-b,-c),(-a,b,-c),(-a,-b,c)$ respectively. The sum of the three sublattices fixed under these three involutions is the lattice $L_{(0-4)}$, which has index 2 in L_i for $i = (1-11)$ and index 4 for $i = (2-12)$, completing the proof that the four fundamental groups are pairwise non-isomorphic. $\square$

Unfortunately, comparable topological information is harder to get for fundamental groups of the other types. For example, one can check that the extension of $\mathbb{Z}_2$ by $\mathbb{Z}^2$, of "type A", is unique, with homology group $H_1(X) = (\mathbb{Z}_2)^3$. This leaves us with the following undistinguished cases:

$(h^{1,1}, h^{2,1})$	π_1	cases
$(15,15)$	0	$(1-2),(1-8)$
$(11,11)$	0	$(2-4),(2-10)$
$(12,6)$	0	$(3-1),(3-2)$
$(11,11)$	A	$(0-3),(1-10)$
$(7,7)$	A	$(1-4),(1-9),(2-11)$
$(11,11)$	C	$(1-3),(1-7)$
$(9,9)$	C	$(2-2),(2-7)$
$(7,7)$	D	$(2-5),(2-14)$

For some of these cases we are able to give a definite answer whether or not the corresponding threefolds agree. Specifically, the models $(0-3)$ and $(1-10)$ are of same topological type, although their complex structure is different, as we shall see in Section 3.2. Together with the orbifold family tree of Table 2 this suggests that the three models $(1-4)$, $(1-9)$, and $(2-11)$ may be topologically equivalent as well. We know that the models $(1-3)$ and $(1-7)$ are distinct as families of complex varieties, and so are $(2-5)$ and $(2-14)$, as we shall see in Section 3.3. It is not clear to us whether they are topologically equivalent.

2 Free fermion models

In this section, we briefly review free fermion models, giving the basic structure of those conformal field theories which are obtained from free fermion constructions. Moreover, we explain how the particular geometric orbifolds that we have classified in Section 1 are related to these models.

2.1 Model building with free fermions

We use free fermion models to construct heterotic string theories in $D = 4$ dimensions. As the name suggests, the basic ingredients to free fermion models are the representations of the free fermion algebra, see Appendix B. Let $\mathcal{H}_0$, $\mathcal{H}_1$ denote the irreducible Fock space representations of the free fermion algebra in the NS and the R sector, respectively, enlarged by $(-1)^F$ with F the worldsheet fermion number operator. Roughly, a free fermion model is obtained from an appropriate tensor product of Fock spaces $\mathcal{H}_0$, $\mathcal{H}_1$ by a certain projection, whose properties are partly governed by the consistency conditions of string theory.

In a heterotic theory the left handed side carries at least $N = 1$ supersymmetry, whereas the right handed side is not supersymmetric. We fermionize all internal degrees of freedom, thus allowing a description in terms of free fermions. External bosons are not fermionized, since they are free uncompactified fields, where fermionization does not apply to add degrees of freedom. The various anomaly cancellation conditions then dictate the following structure: In the left handed sector, we have four external bosons and fermions, two of which are transversal in the light cone gauge, ∂X_μ, ψ^μ, $\mu \in \{0, 1\}$. Since the superstring critical dimension is 10, where each coordinate direction corresponds to three free fermions, there are $(10 - 4) \cdot 3 = 18$ internal fermions χ^i, y^i, w^i, $i \in \{1, \ldots, 6\}$. The left handed worldsheet supercurrent, which generates the local conformal transformations, is then given in light cone gauge by: $\sum_\mu :\psi^\mu \partial X_\mu: + \sum_i :\chi^i y^i w^i:$ [ABKW86, GO85, GNO85, GKO86, DKPR85]. On the right handed side we have four external bosons, two of which are transversal, $\overline{\partial X}_\mu$, $\mu \in \{0, 1\}$. Furthermore, given the bosonic critical dimension 26 with each coordinate direction corresponding to two free fermions, there are $(26 - 4) \cdot 2 = 44$ internal fermions $\overline{\Phi}^i$, $i \in \{1, \ldots, 44\}$. All in all, including external fermions, we have $20 + 44 = 64$ fermionic degrees of freedom, and we introduce indices $j \in \{1, \ldots, 64\}$ for them in

the order ψ^0, ψ^1, $\chi^1, \ldots, \chi^6$, $y^1, \ldots, y^6$, $w^1, \ldots, w^6$, $\overline{\Phi}^1, \ldots, \overline{\Phi}^{44}$.

To construct a full theory we need to specify which combinations of spin structures for each of the 64 real free fermions contribute. Since many of our free fermions result from bosonization, the respective spin structures are coupled pairwise. This is also always true for ψ^0, ψ^1, to ensure consistency of the coupling with worldsheet gravitinos. If the y^j, w^j combine to six left handed bosons with currents $i\, {:}y^j w^j{:}$, then each pair y^j, w^j must have coupled spin structures. This is the case for free fermion models with geometric interpretation on a real six-torus, but not in general, so we will not assume such couplings in general. However, if our free fermion model arises from a heterotic compactification on a Calabi-Yau manifold with gauge group in $E_8 \times E_8$, then among the right handed $\overline{\Phi}^i$ there must be six pairs yielding the antiholomorphic partners of the y^i, w^i, which we then denote $\overline{y}^i$, $\overline{w}^i$ instead of $\overline{\Phi}^1, \ldots, \overline{\Phi}^{12}$. Then $y^i + \overline{y}^i$ and $w^i + \overline{w}^i$ are the fields corresponding to the respective fermionized coordinates, so $y^i, \overline{y}^i$ and $w^i, \overline{w}^i$ are pairs with coupled spin structures. The remaining 32 right handed $\overline{\Phi}^i$ split into two sets $\overline{\varphi}^i$, $\overline{\phi}^i$ of eight complex Dirac fermions each, to allow bosonization to 8 independent currents for each E_8 summand of the gauge Kac-Moody algebra, cf. [ABKW86, GO85, GNO85, GKO86]. Each Dirac fermion is equivalent to one boson, so the notation implies that the real and imaginary parts of $\overline{\varphi}^i$, $\overline{\phi}^i$ also have coupled spin structures. Finally, the heterotic theory on a Calabi-Yau manifold has $(N, \overline{N}) = (2, 0)$ worldsheet supersymmetry with $U(1)$ current given by, say, $\sum i\, {:}\chi^{2j-1}\chi^{2j}{:}$, which is reflected in a pairwise coupling of spin structures for the χ^i, here (χ^1, χ^2), (χ^3, χ^4), (χ^5, χ^6).

For what follows we therefore assume that the 64 fermions are arranged into pairs with coupled spin structures. In other words, we fix an injective vectorspace homomorphism $\iota : \mathbb{F}_2^{32} \longrightarrow \mathbb{F}_2^{64}$ and a fixpoint free involution σ on $\{1, \ldots, 64\}$ such that for all $\alpha \in \mathrm{im}(\iota)$ and all $i \in \{1, \ldots, 64\}$ we have $\alpha_i = \alpha_{\sigma(i)}$. For $\alpha \in \mathrm{im}(\iota)$ we then define

$$\mathcal{H}_\alpha := \mathrm{pr}\left(\bigotimes_{i=1}^{64} \mathcal{H}_{\alpha_i}^i\right),$$

where* pr acts trivially on all $\mathcal{H}_0^i$ but projects $\mathcal{H}_1^j \otimes \mathcal{H}_1^{\sigma(j)}$ onto a chosen irreducible representation of the algebra generated by the ψ_a^j, $\psi_a^{\sigma(j)}$ with $a \in \mathbb{Z}$ and the total worldsheet fermion number operator $(-1)^{F_j + F_{\sigma(j)}}$ (see Appendix B).

While a single fermion can live in one of two sectors (NS or R), states in our full theory fall into sectors $\mathcal{H}_\alpha$ characterized by $\alpha \in \mathrm{im}(\iota) \subset \mathbb{F}_2^{64}$. By the above, the relevant contributions to the partition function have the form

$$Z\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\tau, \overline{\tau}) \quad := \quad \prod_{j=1}^{20} Z\begin{bmatrix}\alpha_j\\\beta_j\end{bmatrix}(\tau) \prod_{j=21}^{64} Z\begin{bmatrix}\alpha_j\\\beta_j\end{bmatrix}(\overline{\tau})$$

$$\overset{(B.2)}{=} \quad \mathrm{tr}_{\mathcal{H}_\alpha}\left[\prod_j (-1)^{\beta_j F_j} q^{L_0 - c/24}\, \overline{q}^{\overline{L}_0 - \overline{c}/24}\right], \tag{2.1}$$

where $L_0 = \sum_{j=1}^{20} L_0^j$, $\overline{L}_0 = \sum_{j=21}^{64} \overline{L}_0^j$ are the total left and right handed Virasoro zero modes and $c = 10$, $\overline{c} = 22$ the total central charges. In the following we also

---

*Here and in the following, indices j serve as a reminder that we are considering the jth free fermion, $j \in \{1, \ldots, 64\}$.

abbreviate

$$\prod_j (-1)^{\beta_j F_j} = e^{\pi i \beta \cdot F}.$$

As mentioned above, our models include two left-handed external fermions, ψ^μ, $\mu \in \{0,1\}$, and consistency of their coupling to worldsheet gravitinos translates into the requirement that these two must always have the same spin structures. This means that we can assign a definite spin statistic $\delta_\alpha \in \{\pm 1\}$ to each sector $\mathcal{H}_\alpha$ which contributes to our theory:

$$\forall \, \alpha \in \mathbb{F}_2^{64} \text{ with } \alpha_1 = \alpha_2: \quad \delta_\alpha := (-1)^{\alpha_1},$$

where $\alpha \in \text{im}(\iota)$ implies $\alpha_1 = \alpha_2$. By definition, sectors $\mathcal{H}_\alpha$ with positive spin statistic $\delta_\alpha = 1$ contain the spacetime bosons, whereas sectors $\mathcal{H}_\alpha$ with negative spin statistic $\delta_\alpha = -1$ contain the spacetime fermions. We can thus use the spacetime fermion number operator F_S, where $(-1)^{F_S}$ acts trivially on spacetime bosons and by multiplication with (-1) on spacetime fermions. We can now make an ansatz for the Hilbert space $\mathcal{H}$ and the partition function

$$Z(\tau, \overline{\tau}) = \text{tr}_{\mathcal{H}} \left[(-1)^{F_S} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right] \tag{2.2}$$

of the total fermionized theory. Namely, we begin by requiring that Z has the form

$$Z(\tau, \overline{\tau}) = \tfrac{1}{|\mathcal{F}|} \sum_{\alpha, \beta \in \mathcal{F}} C\!\begin{bmatrix} \alpha \\ \beta \end{bmatrix} Z\!\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau, \overline{\tau}), \quad C\!\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{Z}, \tag{2.3}$$

for some $\mathcal{F} \subset \mathbb{F}_2^{64}$ such that for every $\alpha \in \mathcal{F}$ there is a $\beta \in \mathcal{F}$ with $C\!\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq 0$. Note that if Z is obtained from some pre-Hilbert space $\mathcal{H}$ by (2.2), then Z vanishes iff in $\mathcal{H}$ there is a $1\!:\!1$ correspondence between spacetime bosons and spacetime fermions which respects the $(L_0, \overline{L}_0)$ eigenvalues. In other words, $Z(\tau, \overline{\tau}) \equiv 0$ iff our theory possesses spacetime supersymmetry, see also Section 2.3.

Let us now give a description of $\mathcal{H}$ which yields (2.2) with (2.3), and let us discuss appropriate restrictions on the coefficients $C\!\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. First rewrite (2.3) with (2.1) to find

$$Z(\tau, \overline{\tau}) = \sum_{\alpha \in \mathcal{F}} \delta_\alpha Z_\alpha(\tau, \overline{\tau}), \quad Z_\alpha(\tau, \overline{\tau}) = \text{tr}_{\mathcal{H}_\alpha} \left[P^{\mathcal{F}} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right],$$

$$P^{\mathcal{F}}: \bigoplus_{\alpha \in \mathcal{F}} \mathcal{H}_\alpha \longrightarrow \bigoplus_{\alpha \in \mathcal{F}} \mathcal{H}_\alpha, \qquad P^{\mathcal{F}}_{|\mathcal{H}_\alpha} := \tfrac{1}{|\mathcal{F}|} \sum_{\beta \in \mathcal{F}} \delta_\alpha C\!\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot e^{\pi i \beta \cdot F},$$
$$\tag{2.4}$$

where by construction $(-1)^{F_S}_{|\mathcal{H}_\alpha} = \delta_\alpha$. Then (2.2) holds with $\mathcal{H} = P^{\mathcal{F}} \oplus_{\alpha \in \mathcal{F}} \mathcal{H}_\alpha$.

We wish to interpret $P^{\mathcal{F}}$ as a projection operator. To ensure that $P^{\mathcal{F}} \circ P^{\mathcal{F}} = P^{\mathcal{F}}$ one checks that it suffices to assume that $\mathcal{F} \subset \mathbb{F}_2^{64}$ is a vector space and that

$$\forall \, \alpha, \beta, \gamma \in \mathcal{F}: \qquad C\!\begin{bmatrix} \alpha \\ \beta + \gamma \end{bmatrix} = \delta_\alpha \, C\!\begin{bmatrix} \alpha \\ \beta \end{bmatrix} C\!\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}.$$

In other words,

$$\forall \alpha \in \mathcal{F}: \qquad \chi_\alpha : \mathcal{F} \longrightarrow \{\pm 1\}, \quad \chi_\alpha(\beta) := \delta_\alpha \delta_\beta C \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

is a character. For later convenience we introduce the notation

$$\chi : \mathcal{F} \times \mathcal{F} \longrightarrow \{\pm 1\}, \quad \chi \begin{bmatrix} \alpha \\ \beta \end{bmatrix} := \delta_\alpha \delta_\beta C \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

and indeed assume for the following that $\mathcal{F}$ is a vector space and that

$$\forall \alpha, \beta, \gamma \in \mathcal{F}: \qquad \chi \begin{bmatrix} \alpha \\ \beta + \gamma \end{bmatrix} = \chi \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \chi \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \in \{\pm 1\}. \tag{2.5}$$

Then the remaining restrictions on the possible choices of $\chi \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ will come from the fact that the partition function Z must be modular invariant and that all fields in $\mathcal{H} = P^{\mathcal{F}} \bigoplus_{\alpha \in \mathcal{F}} \mathcal{H}_\alpha$ must be pairwise semi-local. We introduce

$$\forall \alpha, \beta \in \mathrm{im}(\iota): \quad \alpha \cdot \beta := \frac{1}{2} \sum_{j=1}^{20} \alpha_j \beta_j - \frac{1}{2} \sum_{j=21}^{64} \alpha_j \beta_j \quad = \quad \alpha_L \cdot \beta_L - \alpha_R \cdot \beta_R,$$
$$\alpha^2 \quad := \quad \alpha \cdot \alpha \tag{2.6}$$

with $\alpha_L, \beta_L \in \mathbb{F}_2^{20}$, $\alpha_R, \beta_R \in \mathbb{F}_2^{44}$. This induces a scalar product on $\mathbb{F}_2^{32} \cong \mathrm{im}(\iota)$ with signature (n_+, n_-, n_0). The form (2.6) encodes the conformal dimensions of the ground states in $\mathcal{H}_\alpha$ for $\alpha \in \mathcal{F}$ if we lift $\mathcal{F} \subset \mathbb{F}_2^{64}$ to $\mathbb{Z}^{64}$ with entries in $\{0, 1\}$. Namely, since NS ground states in our Fock space representation of the free fermion algebra have vanishing conformal dimension, whereas the R ground states of a single free fermion have dimension $\frac{1}{16}$, we see that the ground states of $\mathcal{H}_\alpha$ have conformal dimensions

$$(h, \overline{h}) = \left(\frac{\alpha_L \cdot \alpha_L}{8}, \frac{\alpha_R \cdot \alpha_R}{8} \right). \tag{2.7}$$

To get a well-defined fermionic theory, namely to ensure semi-locality, all conformal spins have to be half integer, $h - \overline{h} \in \frac{1}{2}\mathbb{Z}$. Since on $\mathcal{H}_\alpha$, the condition $h - \overline{h} \in \frac{1}{2}\mathbb{Z}$ depends solely on the conformal spin of the ground state as obtained from (2.7), we thus need

$$\forall \alpha \in \mathcal{F}: \quad \alpha^2 \equiv 0(4). \tag{2.8}$$

Vice versa, if this constraint holds, then one checks that on $\mathcal{H}$ a so-called OPE can be introduced consistently, as is necessary to construct a CFT.

Every theory must contain a vacuum sector $\mathcal{H}_0$, as follows from $0 \in \mathcal{F}$, and uniqueness of the vacuum dictates $C \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$. By the transformation properties listed in Appendix A, the modular transformation $\tau \mapsto -\frac{1}{\tau+1}$ maps $Z \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $Z \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since Z

must be modular invariant, this implies that $0 \in \mathcal{F}$ entails $1 \in \mathcal{F}$, the vector with all 64 entries given by 1. Hence by (2.8) we need $n_+ - n_- \equiv 0(4)$. In fact, assuming (2.5), modular invariance of the partition function holds if we also impose

$$C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = C\begin{bmatrix} \beta \\ \alpha \end{bmatrix}, \qquad C\begin{bmatrix} \alpha \\ \alpha + 1 \end{bmatrix} = -\delta_\alpha\, e^{-\pi i \alpha^2/4}. \tag{2.9}$$

The latter condition is only consistent if $n_+ - n_- \equiv 4(8)$. These rules now allow us to restrict attention to $C\begin{bmatrix} b_i \\ b_j \end{bmatrix}$ with $b_i,\, b_j \in B$ a basis of $\mathcal{F}$. For example,

$$
\begin{aligned}
B_0 &= \{1\}, \\
B_{SUSY} &= \{1, s\} && \text{with} && s_i = \begin{cases} 1 & \text{if } i \leq 8, \\ 0 & \text{otherwise}, \end{cases} \\
B_{tor} &= \{1, s, \xi_1, \xi_2\} && \text{with} && (\xi_1)_i = \begin{cases} 1 & \text{if } i > 48, \\ 0 & \text{otherwise}, \end{cases} \\
&&&&& (\xi_2)_i = \begin{cases} 1 & \text{if } 32 < i \leq 48, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
\tag{2.10}
$$

Consistent choices for the values of C are obtained, e.g., as follows:

β $\backslash$ α	0	1	s	ξ_1	ξ_2
0	1	-1	-1	1	1
1	-1	-1	-1	$(-1)^{\varepsilon_1}$	$(-1)^{\varepsilon_2}$
s	-1	-1	-1	$(-1)^{\varepsilon_1'}$	$(-1)^{\varepsilon_2'}$
ξ_1	1	$(-1)^{\varepsilon_1}$	$(-1)^{\varepsilon_1'}$	$(-1)^{\varepsilon_1}$	$(-1)^{\kappa}$
ξ_2	1	$(-1)^{\varepsilon_2}$	$(-1)^{\varepsilon_2'}$	$(-1)^{\kappa}$	$(-1)^{\varepsilon_2}$

$\varepsilon_i,\, \varepsilon_i',\, \kappa \in \{0, 1\}.$

2.2 GSO and orbifold projections

In general, from the partition function (2.2) of a CFT we can read the net number of (spacetime) bosonic minus fermionic states in $\mathcal{H}$ which are eigenvectors of the Virasoro zero modes $L_0, \overline{L}_0$ with any pair of eigenvalues $h, \overline{h} \in \mathbb{R}$. The special structure of the partition function (2.4) allows us to determine the numbers of bosonic and fermionic contributions separately by hand. Namely, for each $\beta \in \mathcal{F}$, (2.9) implies

$$
\begin{aligned}
\forall \alpha, \gamma \in \mathcal{F}: \quad C\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} Z\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \;+\; &C\begin{bmatrix} \alpha \\ \gamma + \beta \end{bmatrix} Z\begin{bmatrix} \alpha \\ \gamma + \beta \end{bmatrix} \\
&= C\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}\left(Z\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} + \delta_\alpha\, C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} Z\begin{bmatrix} \alpha \\ \gamma + \beta \end{bmatrix}\right).
\end{aligned}
\tag{2.11}
$$

Since $Z\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$ and $Z\begin{bmatrix} \alpha \\ \gamma + \beta \end{bmatrix}$ differ by an insertion of $(-1)^{F_j}$ in the trace $\mathrm{tr}_{\mathcal{H}_\alpha}$ in each component j where $\beta_j = 1$, this amounts to projecting onto states $|\sigma\rangle_\alpha \in \mathcal{H}_\alpha$ which obey

$$\delta_\alpha \, C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{\pi i \beta \cdot F} |\sigma\rangle_\alpha = |\sigma\rangle_\alpha. \tag{2.12}$$

The condition (2.12) is often called GSO-projection. Since (2.9) also implies that

$$\forall\, \alpha \in \mathcal{F}: \quad C\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \delta_\alpha,$$

so that (2.12) trivially holds for $\beta = 0$, and since $\mathcal{F}$ is a vector space over $\mathbb{F}_2$, we see that $|\sigma\rangle_\alpha$ remains in the spectrum of our theory iff (2.12) holds for all basis elements $\beta = b_j \in B$.

The interpretation of $P^{\mathcal{F}}$ as projection operator allows us to relate different choices of bases B, B' by orbifolding. Namely, if $B \subset B'$ with $r = |B|$ and $r' = |B'|$, then the corresponding theories are related by orbifolding with respect to a group G of type $(\mathbb{Z}_2)^{r'-r}$, as long as the coefficients $C\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ for $\alpha, \beta \in \mathrm{span}_{\mathbb{F}_2} B$ agree. To see this, let

$$B = \{b_1, \ldots, b_r\} \text{ and } B' = \{b_1, \ldots, b_r, b_{r+1}, \ldots, b_{r'}\} = B \cup B^\perp.$$

Define

$$\mathcal{F} := \mathrm{span}_{\mathbb{F}_2} B, \quad \mathcal{F}' := \mathrm{span}_{\mathbb{F}_2} B', \quad \mathcal{F}^\perp := \mathrm{span}_{\mathbb{F}_2} B^\perp.$$

Recall that the CFTs $\mathcal{C}, \mathcal{C}'$ corresponding to the bases B and B', respectively, have underlying pre-Hilbert spaces

$$\mathcal{H} = P^{\mathcal{F}} \bigoplus_{\alpha \in \mathcal{F}} \mathcal{H}_\alpha, \quad \mathcal{H}' = P^{\mathcal{F}'} \bigoplus_{\alpha' \in \mathcal{F}'} \mathcal{H}_{\alpha'}.$$

We now wish to reinterpret $\mathcal{H}'$ as arising from $\mathcal{H}$ by orbifolding, i.e. by rewriting

$$\mathcal{H}' = \mathcal{H}^G \oplus \left(\mathcal{H}^{twist} \right)^G,$$

where a superscript G denotes the G-invariant subspace of a given vectorspace, and $\mathcal{H}^{twist} := \bigoplus_{\gamma \neq 0} \mathcal{H}_\gamma$ is the sum of the twisted sectors. We will see that this amounts to a simple reordering of the summands of $\mathcal{H}'$. Indeed, the sectors $\mathcal{H}_{\alpha'}$, $\alpha' \in \mathcal{F}'$, which contribute to the theory $\mathcal{C}'$ associated to B' can be listed as follows:

$$\forall\, \gamma \in \mathcal{F}^\perp: \quad \widetilde{\mathcal{H}}_\gamma^{orb} := \bigoplus_{\widetilde{\alpha} \in \mathcal{F}} \mathcal{H}_{\widetilde{\alpha} + \gamma}, \quad \text{so } \mathcal{H}' = P^{\mathcal{F}'} \bigoplus_{\gamma \in \mathcal{F}^\perp} \widetilde{\mathcal{H}}_\gamma^{orb}.$$

Using $P^{\mathcal{F}'} = P^{\mathcal{F}^\perp} \circ P^{\mathcal{F}}$ and for all $\gamma \in \mathcal{F}^\perp$: $\mathcal{H}_\gamma^{orb} := P^{\mathcal{F}} \widetilde{\mathcal{H}}_\gamma^{orb}$, we observe $\mathcal{H}' = P^{\mathcal{F}^\perp} \bigoplus_{\gamma \in \mathcal{F}^\perp} \mathcal{H}_\gamma^{orb}$. It thus remains to argue that $P^{\mathcal{F}^\perp} \mathcal{H}_\gamma^{orb} = \mathcal{H}_\gamma^G$ for $\gamma \in \mathcal{F}^\perp$ with $\mathcal{F}^\perp$ acting as orbifolding group $G \cong (\mathbb{Z}_2)^{r'-r}$ and $\mathcal{H}_\gamma$ the γ-twisted sector. First,

one immediately checks $\mathcal{H}_0^{orb} = \mathcal{H}$, the original pre-Hilbert space of the theory $\mathcal{C}$ associated to B, as necessary. On each $\mathcal{H}_\gamma^{orb}$ we have an action of a group $G \cong (\mathbb{Z}_2)^{r'-r}$ which is generated by g_{b_j} with $b_j \in B^\perp$, where

$$\text{for } |\sigma\rangle_\alpha \in \mathcal{H}_\alpha \subset \mathcal{H}_\gamma^{orb}: \qquad g_{b_j}|\sigma\rangle_\alpha := \delta_\alpha\, C\begin{bmatrix} \alpha \\ b_j \end{bmatrix} e^{\pi i b_j \cdot F}|\sigma\rangle_\alpha. \qquad (2.13)$$

Invariance under G then is equivalent to $|\sigma\rangle$ obeying (2.12) for all $\beta \in \mathcal{F}^\perp$. Hence $P^{\mathcal{F}^\perp}$ is the projection onto G-invariant states, as claimed. One also checks that $\mathcal{H}_\gamma^{orb}$ is indeed a γ-twisted representation of the OPE of $\mathcal{H}$.

To obtain a well-defined G orbifold CFT of $\mathcal{C}$ we cannot allow arbitrary G actions as in (2.13). Namely, G must obey the so-called level matching conditions [Vaf86]. These conditions have been translated into the language of free fermion constructions in [MW86]. Note that for even n, in [MW86, (7)] the first condition gives a necessary and sufficient condition for level matching [MW86, (5)] to hold. Since there, external fermions ψ^μ, $\mu \in \{0, 1\}$, are never twisted, in our notation these conditions read $\alpha^2 \equiv 0(4)$ for all $\alpha \in \mathcal{F}^\perp$ and hence are equivalent to our condition (2.8) on $\mathcal{F}^\perp$. The remaining conditions [MW86, (7)] are necessary to ensure that G in fact acts as $(\mathbb{Z}_2)^{r'-r}$ on $\mathcal{H}_0^{orb}$. In our language they guarantee that also $2\alpha \cdot \beta \equiv 0(4)$ for all $\alpha \in \mathcal{F}$, $\beta \in \mathcal{F}^\perp$, i.e. that all $\gamma \in \mathcal{F} \oplus \mathcal{F}^\perp$ obey $\gamma^2 \equiv 0(4)$. It should be kept in mind that the orbifoldings for free fermion models $\mathcal{C}$ described above in any given interpretation of $\mathcal{C}$ as a nonlinear sigma model on some Calabi-Yau variety do not necessarily translate into geometric orbifoldings of that variety.

2.3 Supersymmetry

Consider a free fermion model specified by a choice of $\mathcal{F}$ and of coefficients C. Assume that for every $\beta \in \mathcal{F}$ we have $\beta_1 = \cdots = \beta_8$. We claim that the theory is automatically spacetime supersymmetric, i.e. $Z(\tau, \overline{\tau}) \equiv 0$, if $s \in \mathcal{F}$, and if

$$\chi\begin{bmatrix} s \\ \beta \end{bmatrix} = \delta_\beta, \quad \text{i.e.} \quad C\begin{bmatrix} s \\ \beta \end{bmatrix} = -1 \quad \text{for all} \quad \beta \in \mathcal{F},$$

amounting to $\varepsilon_i' = 1$ in the table below (2.10). In terms of the partition function this can be seen as follows: By assumption, $\mathcal{F} = \mathcal{F}^0 \cup \mathcal{F}^1$ where $\delta_\beta = (-1)^b$ for $\beta \in \mathcal{F}^b$, and $\beta \in \mathcal{F}^0 \Leftrightarrow \beta + s \in \mathcal{F}^1$. Moreover,

$$\forall \alpha \in \mathcal{F}^a,\, \beta \in \mathcal{F}^b: \quad Z\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau, \overline{\tau}) = Z\begin{bmatrix} a \\ b \end{bmatrix}^8(\tau) \cdot \prod_{j=9}^{20} Z\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}(\tau) \prod_{j=21}^{64} Z\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}(\overline{\tau}).$$

Hence by (2.9)

$$
\begin{aligned}
Z(\tau,\overline{\tau}) \;&=\; \tfrac{1}{|\mathcal{F}|}\sum_{\alpha,\beta\in\mathcal{F}^0}\left\{ C\!\begin{bmatrix}\alpha\\\beta\end{bmatrix} Z\!\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\tau,\overline{\tau}) + C\!\begin{bmatrix}\alpha\\\beta+s\end{bmatrix} Z\!\begin{bmatrix}\alpha\\\beta+s\end{bmatrix}(\tau,\overline{\tau})\right.\\
&\qquad\qquad \left. + C\!\begin{bmatrix}\alpha+s\\\beta\end{bmatrix} Z\!\begin{bmatrix}\alpha+s\\\beta\end{bmatrix}(\tau,\overline{\tau}) + C\!\begin{bmatrix}\alpha+s\\\beta+s\end{bmatrix} Z\!\begin{bmatrix}\alpha+s\\\beta+s\end{bmatrix}(\tau,\overline{\tau})\right\}\\[4pt]
&=\; \tfrac{1}{|\mathcal{F}|}\sum_{\alpha,\beta\in\mathcal{F}^0} C\!\begin{bmatrix}\alpha\\\beta\end{bmatrix}\left\{ Z\!\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\tau,\overline{\tau}) + \delta_\alpha C\!\begin{bmatrix}\alpha\\s\end{bmatrix} Z\!\begin{bmatrix}\alpha\\\beta+s\end{bmatrix}(\tau,\overline{\tau})\right.\\
&\qquad\qquad \left. + \delta_\beta C\!\begin{bmatrix}s\\\beta\end{bmatrix} Z\!\begin{bmatrix}\alpha+s\\\beta\end{bmatrix}(\tau,\overline{\tau}) + \delta_\beta C\!\begin{bmatrix}s\\\beta\end{bmatrix} Z\!\begin{bmatrix}\alpha+s\\\beta+s\end{bmatrix}(\tau,\overline{\tau})\right\}\\[4pt]
&=\; \tfrac{1}{|\mathcal{F}|}\sum_{\alpha,\beta\in\mathcal{F}^0} C\!\begin{bmatrix}\alpha\\\beta\end{bmatrix}\left\{ Z\!\begin{bmatrix}0\\0\end{bmatrix}^8\!\!(\tau) - Z\!\begin{bmatrix}0\\1\end{bmatrix}^8\!\!(\tau) - Z\!\begin{bmatrix}1\\0\end{bmatrix}^8\!\!(\tau) - Z\!\begin{bmatrix}1\\1\end{bmatrix}^8\!\!(\tau)\right\}\\
&\qquad\qquad \cdot \prod_{j=9}^{20} Z\!\begin{bmatrix}\alpha_j\\\beta_j\end{bmatrix}(\tau) \prod_{j=21}^{64} Z\!\begin{bmatrix}\alpha_j\\\beta_j\end{bmatrix}(\overline{\tau})\\[4pt]
&\overset{(A.1)}{=}\; 0.
\end{aligned}
$$

$$(2.14)$$

2.4   Gauge algebras

The gauge algebra of a free fermion model is the Lie algebra generated by the zero modes of its gauge bosons. The gauge bosons are those massless spacetime bosons Φ which generate deformations $\delta_\Phi S = \Phi(z,\overline{z})dz\wedge d\overline{z}$ of the action of our free fermion CFTs and which transform appropriately under the action of the "external" space-time Lorentz group. In particular this means that we are looking for fields Φ with conformal weights $h = \overline{h} = 1$ which transform in the vector representation of the Lorentz group. In the literature, fields with $h = \overline{h} = 1$ are often called massless, since $\delta_\Phi S$ is invariant under infinitesimal conformal transformations of $\mathbb{C}$, i.e. $\delta_\Phi S$ defines a "massless deformation" of the theory. To preserve the left handed worldsheet supersymmetry of our heterotic CFT, the field Φ must also be the top entry of an $(N,\overline{N}) = (1,0)$ supermultiplet. So equivalently to listing the appropriate massless fields we can count states with $(h,\overline{h}) = (\tfrac{1}{2},1)$ in our CFT, the lowest components of multiplets containing a massless field, if we keep in mind that we need to apply a left handed worldsheet supersymmetry to obtain the actual massless state.

Again, the particular form of our models allows us to count these states by hand. Namely, by the discussion of Section 2.1, each state in our theory belongs to a sector $\mathcal{H}_\alpha$ with $\alpha \in \mathrm{im}(\iota) \subset \mathbb{F}_2^{64}$, and $\alpha_j = 1$ iff the jth free fermion belongs to the R-sector. Using (2.7) to determine the conformal dimensions of the ground states in $\mathcal{H}_\alpha$ we see: To list all gauge bosons we need to list all states in our theory which are obtained from the ground state of $\mathcal{H}_\alpha$ by the action of creation operators, which obey the GSO projection (2.12), and such that

$$\frac{\alpha_L \cdot \alpha_L}{8} + N_L = \frac{1}{2}, \qquad \frac{\alpha_R \cdot \alpha_R}{8} + N_R = 1. \tag{2.15}$$

Here N_L, $N_R \in \tfrac{1}{2}\mathbb{N}$ count the energy coming from creation operators with integer

(half integer) contributions from the jth component iff $\alpha_j = 1$ ($\alpha_j = 0$), i.e. iff the jth free fermion is in the R (NS) sector. N_R can also get contributions from the two right moving bosons $\overline{\partial X}_\mu$, $\mu \in \{0, 1\}$, whose creation operators are integer moded. Finally, we have to inspect the contributions coming from the external fields to find all massless fields that transform in the vector representation of the external Lorentz group.

Let us count the gauge bosons in two of the examples listed in (2.10) with the choices of C given there:

For $B_0 = \{1\}$, we have $\mathcal{F} = \{0, 1\}$, and $\mathcal{H}_1$ does not contain massless states since the ground state already has conformal dimensions $(h, \overline{h}) = (\frac{10}{8}, \frac{22}{8})$, i.e. $h > \frac{1}{2}$. In fact, $\mathcal{H}_1$ is spacetime fermionic and as such cannot contain gauge bosons anyway. For $\mathcal{H}_0$, the GSO condition (2.12) enforces

$$-e^{\pi i 1 \cdot F} |\sigma\rangle_0 = |\sigma\rangle_0,$$

i.e. the total number of fermionic creation operators must be odd. Together with $N_L = \frac{1}{2}$, $N_R = 1$ from (2.15), since $\alpha = 0$ leaves all free fermions in the NS sector, we must have one creation operator on the left and two fermionic or one bosonic one on the right handed side. This yields the following fields:

$$\psi^\mu \overline{\partial X}_\nu, \ \chi^i \overline{\partial X}_\mu, y^i \overline{\partial X}_\mu, w^i \overline{\partial X}_\mu \ \ (i \in \{1, \ldots, 6\}),$$
$$\psi^\mu \overline{\Phi}^i \overline{\Phi}^j, \ \chi^i \overline{\Phi}^j \overline{\Phi}^k, y^i \overline{\Phi}^j \overline{\Phi}^k, w^i \overline{\Phi}^j \overline{\Phi}^k \ \ (i \in \{1, \ldots, 6\}, \ j, k \in \{1, \ldots, 44\}).$$

Counting only those fields which transform in the vector representation of the external Lorentz group, we find the following gauge bosons:

$$\chi^i \overline{\partial X}_\mu, y^i \overline{\partial X}_\mu, w^i \overline{\partial X}_\mu \quad \text{giving} \quad \mathfrak{so}(3)^6 = \mathfrak{su}(2)^6,$$
$$\psi^\mu \overline{\Phi}^i \overline{\Phi}^j \quad \text{giving} \quad \mathfrak{so}(44).$$

Moreover, $\psi^\mu \overline{\partial X}_\nu$ also gives a massless field (the graviton), and so do $\chi^i \overline{\Phi}^j \overline{\Phi}^k$, $y^i \overline{\Phi}^j \overline{\Phi}^k$, $w^i \overline{\Phi}^j \overline{\Phi}^k$ (Lorentz scalars in the $(3)_i \times \text{ad}_{\mathfrak{so}(44)}$).

For B_{tor} and with $\varepsilon_i' = 1$ the theory is spacetime supersymmetric, as explained in Section 2.3. To find gauge bosons, it suffices to consider those $\mathcal{H}_\alpha$ with $\delta_\alpha = 1$. Proceeding as above we see that in $\mathcal{H}_0$, the GSO projection with $s = \beta$ breaks the gauge group $\mathfrak{su}(2)^6$ into $\mathfrak{u}(1)^6$, and the GSO projections with $\beta = \xi_k$ break $\mathfrak{so}(44)$ into $\mathfrak{so}(12) \oplus \mathfrak{so}(16) \oplus \mathfrak{so}(16)$:

$$\chi^i \overline{\partial X}_\mu \quad \text{giving} \quad \mathfrak{u}(1)^6,$$
$$\psi^\mu \overline{\Phi}^i \overline{\Phi}^j \ \ (i, j \in \{1, \ldots, 12\}) \quad \text{giving} \quad \mathfrak{so}(12),$$
$$\psi^\mu \overline{\varphi}^i \overline{\varphi}^j, \ \psi^\mu \overline{\phi}^i \overline{\phi}^j \quad \text{giving} \quad \mathfrak{so}(16) \oplus \mathfrak{so}(16).$$

In each $\mathcal{H}_{\xi_k}$ the condition (2.15) yields $N_L = \frac{1}{2}$, $N_R = 0$. Now let $|\pm\rangle_i$ denote the R ground states associated to $\overline{\varphi}^i$ or $\overline{\phi}^i$, respectively. If $\kappa = 1$ in our tabular below (2.10), then no additional gauge bosons arise from $\mathcal{H}_{\xi_k}$. But if $\kappa = 0$, then we get $\psi^\mu \otimes_{i=1}^8 |\pm\rangle_i$ with an even or odd number of $|+\rangle_i$, depending on the ε_k.

In other words, we get a $2^7 = \mathbf{128}$ spinor representation of $\mathfrak{so}(16)$, built from the same 16 free fermions which give the adjoint $\mathbf{120}$ representation of $\mathfrak{so}(16)$ in $\mathcal{H}_0$. Consistency of the spacetime theory requires that the gauge bosons must transform in the adjoint of the gauge algebra; indeed, $\mathbf{128} + \mathbf{120} = \mathbf{248}$ gives $\mathrm{ad}_{\mathfrak{e}_8}$. All in all, since no other sector $\mathcal{H}_\alpha$ contains massless states, we have the gauge algebra

$$\mathfrak{u}(1)^6 \oplus \mathfrak{so}(12) \oplus \mathfrak{e}_8 \oplus \mathfrak{e}_8 \qquad \text{if } \kappa = 0,$$
$$\mathfrak{u}(1)^6 \oplus \mathfrak{so}(12) \oplus \mathfrak{so}(16) \oplus \mathfrak{so}(16) \qquad \text{if } \kappa = 1.$$

The case $\kappa = 0$ gives precisely the gauge algebra of the toroidally compactified heterotic string theory with enhanced symmetry $SO(12) \times E_8 \times E_8$.

2.5 Examples of free fermion models

It is known that one can use free fermion models to construct toroidal CFTs with enhanced symmetry. In other words, for a particular choice of $\mathcal{F}$ and the coefficients C in the partition function (2.3), a geometric interpretation will be easy to obtain. As explained in Section 2.2, adding any basis element to a basis B of $\mathcal{F}$ results in orbifolding by a group of type $\mathbb{Z}_2$. In CFT, orbifolding by $\mathbb{Z}_2$ can be reversed by orbifolding with respect to another group of type $\mathbb{Z}_2$. Hence omitting basis elements in a free fermion model also amounts to orbifolding. Thus all free fermion models can be interpreted as orbifolds of a toroidal CFT. These may or may not be geometric orbifoldings in a given geometric interpretation: For example, purely geometric group actions can never yield the desired spectra of Faraggi's semi-realistic free fermion models [FNY90, Fa92], an observation made in [DF04]. This was checked in [DF04] for the particular class of geometric orbifolds considered there, and extended to all geometric orbifolds in our Section 1.

In this subsection we will derive geometric interpretations for several important examples of free fermion models. In particular, we will see that there exists a free fermion model with geometric interpretation on the product of three elliptic curves. The corresponding B-field is nontrivial, but it is compatible with all geometric orbifoldings classified in Section 1. In other words, all the corresponding geometric orbifoldings can be lifted to the level of CFT, and for each of the resulting moduli spaces of orbifold CFTs, there are special points giving models which allow a free fermion construction. This also holds for orbifolds with discrete torsion.

2.5.1 Free fermion model on the $SO(12)$ torus

It was already noted in [MW86] that the toroidally compactified heterotic string with enhanced symmetry $SO(12) \times E_8 \times E_8$ can be described using a free fermion formulation. Let us briefly give the argument in our language:

First, we identify the partition function of the free fermion model with basis B_{tor}. By what was said in Sections 2.2 and 2.4, we must choose $\varepsilon_1' = \varepsilon_2' = 1$ and $\kappa = 0$ in the table below (2.10) in order to reproduce the correct gauge algebra. We

introduce $\xi_3 := 1 + s + \xi_1 + \xi_2$ and by (2.14) and (B.1) find

$$
\begin{aligned}
Z_{SUSY}(\tau,\overline{\tau}) \;&:=\; \tfrac{1}{2}\left\{\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_1(\tau)}{\eta(\tau)}\right)^4\right\} \equiv 0, \\
Z(\tau,\overline{\tau}) \;&=\; Z_{SUSY}(\tau,\overline{\tau}) \cdot Z_{Narain}(\tau,\overline{\tau}), \\
Z_{Narain}(\tau,\overline{\tau}) \;&=\; \tfrac{1}{8}\sum_{\alpha,\beta\in\mathrm{span}_{\mathbb{F}_2}(\xi_1,\xi_2,\xi_3)} C\begin{bmatrix}\alpha\\\beta\end{bmatrix}\prod_{j=9}^{20} Z\begin{bmatrix}\alpha_j\\\beta_j\end{bmatrix}(\tau)\prod_{j=21}^{64} Z\begin{bmatrix}\alpha_j\\\beta_j\end{bmatrix}(\overline{\tau}),
\end{aligned}
$$

where (2.9) allows us to calculate the relevant coefficients C, in particular

β α	0	ξ_1	ξ_2	ξ_3
0	1	1	1	1
ξ_1	1	$(-1)^{\varepsilon_1+1}$	1	1
ξ_2	1	1	$(-1)^{\varepsilon_2+1}$	1
ξ_3	1	1	1	$(-1)^{\varepsilon_2+\varepsilon_2}$

$\varepsilon_i \in \{0,1\}$.

Note that for all α, $\beta \in \mathrm{span}_{\mathbb{F}_2}(\xi_1,\xi_2)$:

$$
C\begin{bmatrix}\alpha\\\beta+\xi_3\end{bmatrix} \overset{(2.9)}{=} C\begin{bmatrix}\alpha\\\beta\end{bmatrix} C\begin{bmatrix}\alpha\\\xi_3\end{bmatrix} = C\begin{bmatrix}\alpha\\\beta\end{bmatrix},
$$

and similarly for all α, $\beta \in \{0,\xi_1\}$:

$$
C\begin{bmatrix}\alpha\\\beta+\xi_2\end{bmatrix} \overset{(2.9)}{=} C\begin{bmatrix}\alpha\\\beta\end{bmatrix} C\begin{bmatrix}\alpha\\\xi_2\end{bmatrix} = C\begin{bmatrix}\alpha\\\beta\end{bmatrix}.
$$

Together with $Z\begin{bmatrix}1\\1\end{bmatrix}(\overline{\tau}) \equiv 0$ this implies that a calculation similar to the one performed in (2.14) allows to decompose Z_{Narain} as follows:

$$
Z_{Narain}(\tau,\overline{\tau}) = \tfrac{1}{2}\sum_{i=1}^{4}\left|\frac{\vartheta_i(\tau)}{\eta(\tau)}\right|^{12} \cdot \left(\tfrac{1}{2}\sum_{i=1}^{4}\left(\frac{\vartheta_i(\overline{\tau})}{\eta(\overline{\tau})}\right)^8\right)^2.
$$

Since $\tfrac{1}{2}\sum_i \vartheta_i^8$ is the unique modular form of weight 8 and constant coefficient 1, it agrees with the theta function E_4 of the E_8 lattice, and

$$
\begin{aligned}
Z_{Narain}(\tau,\overline{\tau}) \;&=\; Z_{SO(12)}(\tau,\overline{\tau}) \cdot (Z_{E_8}(\overline{\tau}))^2, \\
Z_{SO(12)}(\tau,\overline{\tau}) \;&=\; \frac{1}{2}\sum_{i=1}^{4}\left|\frac{\vartheta_i(\tau)}{\eta(\tau)}\right|^{12}, \qquad Z_{E_8}(\overline{\tau}) = \frac{E_4(\overline{\tau})}{\eta^8(\overline{\tau})}.
\end{aligned} \qquad (2.16)
$$

This was to be expected, since Z_{Narain} is the partition function of the free fermion model constructed from 12 left moving fermions y^i, w^i, $i \in \{1,\ldots,6\}$ and 44 right

moving fermions $\overline{y}^i, \overline{w}^i$, $i \in \{1, \ldots, 6\}$, $\overline{\varphi}^i, \overline{\phi}^i$, $i \in \{1, \ldots, 16\}$, with spin structures coupled among the y^i, w^i, $\overline{y}^i$, $\overline{w}^i$, among the $\overline{\varphi}^i$, and among the $\overline{\phi}^i$. In fact, since we find a $\mathfrak{u}(1)_L^6 \oplus \mathfrak{u}(1)_R^{22}$ current algebra generated by $:y^i w^i:$, $:\overline{y}^i \overline{w}^i:$, $:\overline{\Phi}^{2j-1}\overline{\Phi}^{2j}:$ $(j > 6)$, this is a toroidal CFT, and the determination of its charge lattice will suffice to specify the theory. By the above we already know that we have an $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ gauge symmetry, and thus a geometric interpretation in terms of a toroidal theory with trivial $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ bundle on some torus.

It remains to be shown that $Z_{SO(12)}$ is the partition function of the toroidal CFT at $c = \overline{c} = 6$ with enhanced $SO(12)$ symmetry. To this end first note that according to the formulas given in Appendix A,

$$
\frac{1}{2}\left(\vartheta_3^6(\tau)\overline{\vartheta}_3^6(\overline{\tau}) + \vartheta_4^6(\tau)\overline{\vartheta}_4^6(\overline{\tau})\right) = \sum_{\substack{x,y \in \mathbb{Z}^6, \\ x-y \in D_6}} q^{\frac{x^2}{2}} \overline{q}^{\frac{y^2}{2}},
$$

$$
\frac{1}{2}\left(\vartheta_2^6(\tau)\overline{\vartheta}_2^6(\overline{\tau}) + \vartheta_1^6(\tau)\overline{\vartheta}_1^6(\overline{\tau})\right) = \sum_{\substack{x,y \in \mathbb{Z}^6+\frac{1}{2}, \\ x-y \in D_6}} q^{\frac{x^2}{2}} \overline{q}^{\frac{y^2}{2}},
$$

where we have introduced the root lattice $D_6 := \{n \in \mathbb{Z}^6 \mid \sum n_i \equiv 0(2)\}$ of $SO(12)$, and $\frac{1}{2} \in \left(\frac{1}{2}\mathbb{Z}\right)^6$ denotes the vector with all entries given by $\frac{1}{2}$. Using $D_6^* = \mathbb{Z}^6 \cup \left(\mathbb{Z}^6 + \frac{1}{2}\right)$, we find

$$
Z_{SO(12)}(\tau, \overline{\tau}) = \frac{1}{|\eta(\tau)|^{12}} \sum_{\substack{x,y \in D_6^*, \\ x-y \in D_6}} q^{\frac{x^2}{2}} \overline{q}^{\frac{y^2}{2}} = \frac{1}{|\eta(\tau)|^{12}} \sum_{(x,y) \in \Gamma} q^{\frac{x^2}{2}} \overline{q}^{\frac{y^2}{2}}
$$

with

$$
\Gamma := \left\{(x,y) \in \mathbb{R}^{6,6} \mid x, y \in D_6^*, \ x-y \in D_6\right\}. \tag{2.17}
$$

The claim now is that Γ can be brought into the standard Narain form

$$
\Gamma(\Lambda, B) = \left\{(p_L, p_R) = \tfrac{1}{\sqrt{2}}\left(\mu - B\lambda + \lambda, \mu - B\lambda - \lambda\right) \mid \lambda \in \Lambda, \mu \in \Lambda^*\right\} \tag{2.18}
$$

for an appropriate lattice $\Lambda \subset \mathbb{R}^6$ with dual $\Lambda^* \subset (\mathbb{R}^6)^*$ (using the standard Euclidean scalar product on $\mathbb{R}^6$ to view $\Lambda^* \subset \mathbb{R}^6 \cong (\mathbb{R}^6)^*$), and for an appropriate B-field $B : \Lambda \otimes \mathbb{R} \longrightarrow \Lambda^* \otimes \mathbb{R}$. If for a toroidal CFT $\mathcal{C}$ with central charges $c = \overline{c} = d$ the charge lattice Γ can be brought into the form $\Gamma = \Gamma(\Lambda, B)$ with such Λ, B, then (Λ, B) gives a geometric interpretation of $\mathcal{C}$: The CFT $\mathcal{C}$ is the nonlinear sigma model on $\mathbb{R}^d/\Lambda$ with B-field B. Note that any two B-fields B, $B' = B + \delta B$ yield $\Gamma(\Lambda, B) = \Gamma(\Lambda', B')$ iff $\delta B(\Lambda) \subset \Lambda^*$. For given Λ we say that B, B' are equivalent iff they define the same CFT, i.e. iff $\Gamma(\Lambda, B) = \Gamma(\Lambda', B')$.

¿From (2.17) and (2.18) we directly read off $\Lambda = \frac{1}{\sqrt{2}}D_6$, $\Lambda^* = \sqrt{2}D_6^*$, such that $\Lambda^* \subset \Lambda \subset \frac{1}{2}\Lambda^*$. Since $D_6 \subset D_6^*$, (2.17) tells us that $\Gamma(\Lambda, B)$ contains all vectors of type $(p_L, p_R) = (x, 0)$ and $(p_L, p_R) = (0, y)$ with $x, y \in D_6$. In other words, $(B-1)\Lambda \subset \Lambda^*$ (or equivalently $(B-1)D_6 \subset 2D_6^*$), which is equivalent to $(B+1)\Lambda \subset \Lambda^*$ since $\Lambda \subset \frac{1}{2}\Lambda^*$. In fact, $(B-1)D_6 \subset 2D_6^*$ holds iff all off-diagonal entries of

B are odd, and all such choices of B are equivalent. Without loss of generality we can therefore take $B = B_*$ with

$$
B_* = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1 & 1 & 1 \\
-1 & -1 & 0 & 1 & 1 & 1 \\
-1 & -1 & -1 & 0 & 1 & 1 \\
-1 & -1 & -1 & -1 & 0 & 1 \\
-1 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}. \tag{2.19}
$$

To show that the free fermion model with basis B_{tor} and $\varepsilon'_k = 1$, $\kappa = 0$ agrees with the Narain model as claimed, for instance by using [NW01, Thm. 3.1], we still need to identify their W-algebras and charge lattices with respect to $\mathfrak{u}(1)_L^6 \oplus \mathfrak{u}(1)_R^6$. Since the theory is left-right symmetric, we can focus on the left-handed degrees of freedom. With

$$
\begin{aligned}
j_k &:= i :\! y^k w^k \!:, \quad k \in \{1, \ldots, 6\}, \\
x^k &:= \tfrac{1}{\sqrt{2}}\left(y^k + iw^k\right), \qquad (x^k)^* := \tfrac{1}{\sqrt{2}}\left(y^k - iw^k\right), \quad k \in \{1, \ldots, 6\},
\end{aligned}
$$

in addition to the j_k which generate $\mathfrak{u}(1)^6$ we find 60 further $(1, 0)$ fields in the free fermion model:

$$
i :\! x^k x^l \!: = -i :\! x^l x^k \!:, \qquad i :\! x^k (x^l)^* \!:, \qquad i :\! (x^k)^* (x^l)^* \!: = -i :\! (x^l)^* (x^k)^* \!: \qquad (k \neq l),
$$

with charges with respect to $\vec{j} = (j_1, \ldots, j_6)$ given by

$$
e_k + e_l, \quad e_k - e_l, \quad -e_k - e_l,
$$

where the e_i denote the standard basis vectors in $\mathbb{R}^6$. Hence we can identify these $(1, 0)$ fields with the holomorphic vertex operators $V_{(p_L, 0)}$ of the respective charges $(p_L, 0)$. One checks that this identification is compatible with the OPE, i.e. the free fermion model and our toroidal CFT share the same W-algebra, with zero mode algebra of the generators given by $\mathfrak{so}(12)$. By a similar analysis one identifies all $V_{(p_L, p_R)}$ for $(p_L, p_R) \in \Gamma_{Narain}$ with fields in the free fermion model: Since we have already dealt with the $(1, 0)$ fields, and since in our model $(B \pm 1)\Lambda \subset \Lambda^*$, it suffices to identify the $V_{(p_L, p_R)}$ with $p_L = p_R = \frac{1}{\sqrt{2}}\mu$, $\mu \in \Lambda^*$. Now

$$
i :\! x^k \overline{x}^l \!: \longmapsto V_{(e_k, e_l)} \quad \text{and} \quad \prod_{i=1}^6 |\delta_i\rangle_i \overline{|\delta_i\rangle_i} \longmapsto V_{(\frac{1}{2}\sum_i \delta_i e_i, \frac{1}{2}\sum_i \delta_i e_i)}, \quad \delta_i \in \{\pm\}
$$

gives the desired identification.

2.5.2 Free fermion model on the square torus

In the previous subsection, we have argued that a free fermion model with basis B_{tor} for $\mathcal{F}$ yields a conformal field theory with geometric interpretation on the torus $\mathbb{R}^6/\Lambda$ with $\Lambda = \frac{1}{\sqrt{2}}D_6$. The lattice $\Lambda' = \sqrt{2}\mathbb{Z}^6$ is a sublattice of Λ of index

2^5. Correspondingly, for the dual lattices we find that $(\Lambda')^*$ is generated by Λ^* and the multiples $\frac{1}{\sqrt{2}}e_i$ of the first five standard basis vectors e_i, $i \in \{1, \ldots, 5\}$. This is evidence for the fact that there is also a free fermion model with geometric interpretation on the square torus $T^6 = \mathbb{R}^6/\sqrt{2}\mathbb{Z}^6$: It should arise by orbifolding with respect to a group of type $(\mathbb{Z}_2)^5$ from the toroidal free fermion model on the $SO(12)$ torus.

Indeed, with the same techniques as in the previous section, one shows: Consider the free fermion model with basis $B_\square = \{s, \zeta_1, \ldots, \zeta_5, \xi_1, \xi_2, \xi_3\}$, where $\xi_3 = 1 + s + \xi_1 + \xi_2$ as before, and ζ_i with $i \in \{1, \ldots, 5\}$ is the vector which has an entry 1 corresponding to the fermions y^i, w^i, $\overline{y}^i$, $\overline{w}^i$ and entries 0 otherwise. For the coefficients C, for all $\beta \in B_\square$, we set $C\begin{bmatrix} s \\ \beta \end{bmatrix} := -1$, while for $\alpha, \beta \in B_\square - \{s\}$, we set $C\begin{bmatrix} \alpha \\ \beta \end{bmatrix} := 1$. The resulting free fermion model has geometric interpretation on the square torus $T^6 = \mathbb{R}^6/\sqrt{2}\mathbb{Z}^6$ with the same B-field B_* as for the previous toroidal model, c.f. (2.19).

This is an important observation with respect to our classification in Section 1. It implies that for all the orbifolds X/G given there, $X \cong T^6$ with the complex structure of a product $E_1 \times E_2 \times E_3$ of three elliptic curves, a free fermion model exists which has geometric interpretation on X/G, provided that the action of G is compatible with the B-field B_* given in (2.19). Compatibility here means that for every $g \in G$, the g-conjugate B-field is equivalent to B_*, which indeed is the case for all groups G discussed in Section 1.

Note that the above orbifolding by $(\mathbb{Z}_2)^5$ is not described in terms of a geometric orbifolding: While a geometric orbifolding would have to lead to a model with geometric orbifold interpretation on a quotient of $\mathbb{R}^6/\Lambda$, the geometric interpretation $(\Lambda', B) = (\sqrt{2}\mathbb{Z}^6, B_*)$ of the orbifold CFT yields an unbranched cover $T^6 = \mathbb{R}^6/\sqrt{2}\mathbb{Z}^6$ of the geometric interpretation $(\Lambda, B) = (\frac{1}{\sqrt{2}}D_6, B_*)$ of the original theory on $\mathbb{R}^6/\frac{1}{\sqrt{2}}D_6$. The reverse of this orbifolding, obtained in the free fermion language by omitting the basis vectors ζ_i, $i \in \{1, \ldots, 5\}$ from the basis $B_\square$, is a geometric orbifolding of type $(\mathbb{Z}_2)^5$ by shifts, and it yields $\mathbb{R}^6/\frac{1}{\sqrt{2}}D_6 = T^6/(\mathbb{Z}_2)^5$.

2.5.3 The NAHE model

As an example of orbifolding by a group which does not act as shift orbifold on the torus, we consider the free fermion model with basis $B_{NAHE^+} = \{1, s, \xi_1, \xi_2, g_1, g_2\}$. This is the geometric part of what Faraggi calls the extended NAHE set [FGKP87, INQ87, AEHN89, FNY90, Fa92, FN93], and in the notation of [DF04] one has $g_k = b_k + s + \xi_2$ and $g_3 = g_1 + g_2 + 1 + \xi_1$. The 8 Dirac fermions $\overline{\varphi}^i$ are renamed into $\overline{\psi}^1, \ldots, \overline{\psi}^5$, $\overline{\eta}^1$, $\overline{\eta}^2$, $\overline{\eta}^3$. Omitting untwisted fermions, for the additional basis

vectors we set

	$\chi^1, \chi^2,$ $\overline{\eta}^1$	$\chi^3, \chi^4,$ $\overline{\eta}^2$	$\chi^5, \chi^6,$ $\overline{\eta}^3$	$y^1, y^2,$ $\overline{y}^1, \overline{y}^2$	$w^1, w^2,$ $\overline{w}^1, \overline{w}^2$	$y^3, y^4,$ $\overline{y}^3, \overline{y}^4$	$w^3, w^4,$ $\overline{w}^3, \overline{w}^4$	$y^5, y^6,$ $\overline{y}^5, \overline{y}^6$	$w^5, w^6,$ $\overline{w}^5, \overline{w}^6$
g_1	0	1	1	0	0	1	0	1	0
g_2	1	0	1	1	0	0	0	0	1
g_3	1	1	0	0	1	0	1	0	0

$$(2.20)$$

The geometric action on the $SO(12)$ torus model with pre-Hilbert space $\mathcal{H}_{SO(12)}$ built on the y^i, w^i; $\overline{y}^i$, $\overline{w}^i$ is left-right symmetric. Translating into the fundamental fields of the toroidal theory we get

$$g_1: \quad j_k \mapsto -j_k \quad \text{for} \quad k \in \{3,\ldots,6\}, \quad x^k \longleftrightarrow -(x^k)^* \quad \text{for} \quad k \in \{3,\ldots,6\},$$

$$g_2: \quad j_k \mapsto -j_k \quad \text{for} \quad k \in \{1,2,5,6\}, \quad x^k \longleftrightarrow -(x^k)^* \quad \text{for} \quad k \in \{1,2\},$$

$$x^k \longleftrightarrow (x^k)^* \quad \text{for} \quad k \in \{5,6\},$$

$$g_3: \quad j_k \mapsto -j_k \quad \text{for} \quad k \in \{1,\ldots,4\}, \quad x^k \longleftrightarrow (x^k)^* \quad \text{for} \quad k \in \{1,\ldots,4\},$$

and analogously for the right-handed fields. In geometric language with real coordinates $v_1, \ldots, v_6$ this corresponds to

$$\left.\begin{array}{ll} g_1: & v_k \mapsto -v_k \quad \text{for} \quad k \in \{3,\ldots,6\}, \text{ up to a shift on the charge} \\ & \text{lattice by } \delta = \frac{1}{2}\,(1,1,0,0,0,0;1,1,0,0,0,0), \\ g_2: & v_k \mapsto -v_k \quad \text{for} \quad k \in \{1,2,5,6\}, \text{ up to a shift on the charge} \\ & \text{lattice by } \delta = \frac{1}{2}\,(1,1,0,0,0,0;1,1,0,0,0,0), \\ g_3: & v_k \mapsto -v_k \quad \text{for} \quad k \in \{1,\ldots,4\}, \text{ i.e. } g_3 = g_1 \circ g_2. \end{array}\right\} \quad (2.21)$$

The claim is that the shifts involved in g_1, g_2 can be ignored, i.e. that g_1, g_2, g_3 act geometrically as the three non-trivial elements of the Kleinian $(\mathbb{Z}_2)^2$ twist group T_0.

To see this, let us assume that the g_k act as claimed in the geometric interpretation and derive (2.20) from this assumption. By the above, we only need to confirm the choices between placing the 1's in the $\{y^k\}$ instead of the $\{w^k\}$ columns in (2.20) for g_1, g_2. First note that for the construction of the untwisted sector of the orbifold these choices are irrelevant. Namely, the additional sign in $x^k \leftrightarrow -(x^k)^*$ merely results in a choice of, say, $:x^1 x^3: - :x^1 (x^3)^*:$ instead of $:x^1 x^3: + :x^1 (x^3)^*:$ as invariant field under g_1, with no consequence on the OPE. In accord with this, all the contributions to the partition function

$$\substack{g_k \boxed{} \\ \,^1}(\tau) = \operatorname{tr}_{\mathcal{H}_{SO(12)}} \left(g_k q^{L_0 - 6/24} \overline{q}^{\overline{L}_0 - 6/24} \right) = \frac{1}{2} \left\{ \left|\frac{\vartheta_4}{\eta}\right|^4 \left|\frac{\vartheta_3}{\eta}\right|^8 + \left|\frac{\vartheta_3}{\eta}\right|^4 \left|\frac{\vartheta_4}{\eta}\right|^8 \right\}$$

$$(2.22)$$

for $k \in \{1, 2, 3\}$ agree, where the factors raised to the fourth power in each summand come from the action of the twist on four of the real fermions. Similarly the traces over the full g_k twisted sectors of the orbifold are

$$_1\boxed{}_{g_k}(\tau) = {}_{g_k}\boxed{}_1(-\tfrac{1}{\tau}) = \tfrac{1}{2}\left\{\left|\frac{\vartheta_2}{\eta}\right|^4\left|\frac{\vartheta_3}{\eta}\right|^8 + \left|\frac{\vartheta_3}{\eta}\right|^4\left|\frac{\vartheta_2}{\eta}\right|^8\right\}, \qquad (2.23)$$

where it again should be kept in mind that the factors raised to the fourth power in each summand come from the action of the twist on four of the fermions. The choice of placing the 1's in the $\{y^k\}$ instead of the $\{w^k\}$ columns in (2.20) hence only enters into the encoding of the g_i action on the g_k twisted sector with $i \neq k$. In terms of the geometric interpretation of the toroidal theory equivalently to (2.22) and (2.23) we write

$$\begin{aligned}
{}_{g_k}\boxed{}_1(\tau) &= \left|\frac{2\eta}{\vartheta_2}\right|^4 \cdot \tfrac{1}{2}\left\{\left|\frac{\vartheta_3}{\eta}\right|^4 + \left|\frac{\vartheta_4}{\eta}\right|^4\right\}, \\
{}_1\boxed{}_{g_k}(\tau) &= \left|\frac{2\eta}{\vartheta_4}\right|^4 \cdot \tfrac{1}{2}\left\{\left|\frac{\vartheta_3}{\eta}\right|^4 + \left|\frac{\vartheta_2}{\eta}\right|^4\right\},
\end{aligned} \qquad (2.24)$$

where the first factor in each case accounts for the contributions from the twisted states in four real coordinate directions, whereas the second factor comes from the trace over states left invariant by g_k. Hence in the usual $(\mathbb{Z}_2)^2$ orbifold, when g_i with $i \neq k$ acts on the g_k twisted sector, it must leave a factor $\left|\frac{2\eta}{\vartheta_4}\right|^2$ invariant, and act by the usual $\mathbb{Z}_2$ twist on a second factor $\left|\frac{2\eta}{\vartheta_4}\right|^2$ transforming it into $\left|\frac{2\eta}{\vartheta_3}\right|^2$, while it introduces the usual factor $\left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^2 = \left|\frac{2\eta}{\vartheta_2}\right|^2$ for a twisted sector in the directions which are left invariant by g_k but not by g_i. All in all we get

$$\text{for } i \neq k: \qquad {}_{g_i}\boxed{}_{g_k}(\tau) = \left|\frac{2\eta}{\vartheta_4}\right|^2\left|\frac{2\eta}{\vartheta_3}\right|^2\left|\frac{2\eta}{\vartheta_2}\right|^2 \stackrel{(A.1)}{=} 2^4. \qquad (2.25)$$

Let us now translate the g_i action on the g_k twisted sector back into the language of the free fermion model (2.23). We already know that in (2.23) a global factor $\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^2$ must remain invariant, coming from the directions twisted by g_k but not by g_i. A factor $\left|\frac{\vartheta_3}{\eta}\right|^4$ in the first summand is transformed into $\left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^2$, and a factor $\left|\frac{\vartheta_2}{\eta}\right|^4$ in the second summand is transformed into $\left|\frac{\vartheta_2\vartheta_1}{\eta^2}\right|^2 = 0$, each coming from directions twisted by g_i but not by g_k. Since by (2.25) the final result of the transformation must be $\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^2 \cdot \left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^2 \cdot \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^2$, we find the remaining factor coming from directions twisted by both g_i and g_k, namely $\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^2$ in the first summand is transformed into $\left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^2$. Hence the twist is applied to fermions previously yielding a ϑ_3 contribution,

in other words to fermions which had been untwisted so far. Altogether this indeed leads to the data listed in (2.20) for the y^i, w^i; $\overline{y}^i$, $\overline{w}^i$.

In the construction of a semi-realistic free fermion model [FNY90, INQ87], the authors also use three further $\mathbb{Z}_2$ actions, α, β, γ, where again we only list fermions that are in fact twisted:

	$y^1, y^5,$ $w^1, w^5,$ $\overline{w}^1, \overline{y}^5,$	$\overline{y}^1,$ $\overline{w}^5$	$y^2, y^4,$ w^2, w^4 $\overline{y}^2, \overline{y}^4$	$\overline{w}^2,$ $\overline{w}^4$	$y^3, y^6,$ w^3, w^6 $\overline{w}^3, \overline{w}^6$	$\overline{y}^3,$ $\overline{y}^6$	$\overline{\psi}^{1,...,5}$	$\overline{\eta}^i$	$\overline{\phi}^{1,...,8}$
α	0	1	0	1	1	0	1 1 1 0 0	0	1 1 1 1 0 0 0 0
β	1	0	0	1	0	1	1 1 1 0 0	0	1 1 1 1 0 0 0 0
γ	0	1	1	0	0	1	$\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}$	0	$\frac{1}{2}$ 0 1 1 $\frac{1}{2}\frac{1}{2}\frac{1}{2}$ 0

Note that α, β, γ share the property of leaving all the j_k invariant and multiplying all the $\overline{j}_k$ by -1. This is hard to interpret geometrically, since the left-right coupling of the local coordinate functions corresponding to the pairs $y^i + \overline{y}^i, w^i + \overline{w}^i$ is broken. The difference in sign between the action on the holomorphic j_k and the antiholomorphic $\overline{j}_k$ is reminiscent of some type of mirror symmmetry. The actions of $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ on the y^i, w^i; $\overline{y}^i$, $\overline{w}^i$, however, have a geometric interpretation:

	$y^1, y^5, w^1, \overline{y}^1,$ $w^5, \overline{y}^5, \overline{w}^1, \overline{w}^5$	$y^2, y^4, w^2, \overline{w}^2,$ $\overline{y}^2, \overline{y}^4, w^4, \overline{w}^4$	$y^3, w^3, w^6, \overline{y}^3,$ $y^6, \overline{w}^3, \overline{w}^6, \overline{y}^6$
$\alpha\beta$	1	0	1
$\beta\gamma$	1	1	0
$\gamma\alpha$	0	1	1

Each of these actions is left-right symmetric and acts trivially on all j_k, $\overline{j}_k$. As such, they are shift orbifolds, namely by

$$\alpha\beta: \quad \tfrac{1}{2}\left(0,1,0,1,0,0;0,1,0,1,0,0\right),$$
$$\beta\gamma: \quad \tfrac{1}{2}\left(0,0,1,0,0,1;0,0,1,0,0,1\right),$$
$$\gamma\alpha: \quad \tfrac{1}{2}\left(1,0,0,0,1,0;1,0,0,0,1,0\right).$$

Faraggi shows that his model is a three generation model [Fa92], which is a necessary requirement for a theory to be viewed as "semi-realistic". It is natural to ask whether there exists an underlying geometric orbifold with Hodge numbers $(h^{1,1}, h^{2,1})$ yielding three generations $3 = h^{1,1} - h^{2,1}$. In [DF04], this question was answered to the negative, however without a complete classification of all possible orbifolds. Our classfication, summarized in Table 1 of Section 1.6, completes this task, and again answers the question to the negative. The numbers of generations that can be produced by purely geometric methods, according to the results of Section 1.6, are 48, 24, 12, 6, or 0. It is interesting that precisely the number 3 is lacking in this list.

3 Special models within our classification

In this section, we discuss some special cases of the orbifolds that we have classified in Section 1. More precisely, we identify some of the resulting Calabi-Yau threefolds as degenerate cases of so-called Borcea-Voisin threefolds and Schoen threefolds or their orbifolds. All these particular Calabi-Yau threefolds have been widely discussed in the literature, either in relation to mirror symmetry or to model building in heterotic string theory. Since the results of Section 2 in particular imply that for every Calabi-Yau threefold listed in Section 1.6 there exists a free fermion model of an associated CFT, we automatically obtain free fermion constructions for theories associated to certain Borcea-Voisin threefolds, Schoen threefolds, and their orbifolds. This may eventually yield further insight into the geometry of these threefolds, and it may simplify some of the existing string theory constructions, since free fermion models are constructed using very simple mathematical tools.

3.1 The Vafa-Witten and NAHE models

As was briefly mentioned at the end of our discussion of Table 1 in Section 1.6, our model $(0-1)$ agrees with the $(\mathbb{Z}_2)^2$ orbifold which was extensively studied by Vafa and Witten in their seminal work [VW95] on discrete torsion and mirror symmetry. Since the Vafa-Witten model is indeed obtained as orbifold of the product of three elliptic curves by the group T_0 of ordinary twists, agreement with our model $(0-1)$ is immediate.

Let us now discuss the two models $(1-1)$ and $(2-9)$ in our list, both of which have Hodge numbers $(27,3)$. They are not equivalent as topological spaces, since they can be distinguished by their fundamental groups $C = \mathbb{Z}_2$ and 0, respectively. However, there seems to have been some confusion between these two models, which we now wish to lift. Clearly, $(1-1)$ is obtained as $\mathbb{Z}_2$-orbifold of the Vafa-Witten model. On the other hand, we claim that $(2-9)$ agrees with the Calabi-Yau threefold Y which is obtained by orbifolding an $SO(12)$ torus by the orbifolding group T_0. This follows using the ideas described at the end of Section 2.5.2: The $SO(12)$ torus can be obtained from the product X of three elliptic curves by a shift orbifold using the group $\widetilde{G}_S := (\mathbb{Z}_2)^5$ with generators

$$(\tau, 0, 0), (0, \tau, 0), (0, 0, \tau), (0, 1, 1), (1, 0, 1).$$

Hence Y is topologically equivalent to $X/\widetilde{G}$, where $\widetilde{G} = \widetilde{G}_S \times T_0$. However, the group $\widetilde{G}$ is redundant, since shifts by the first three vectors listed above are redundant. Hence Y is also topologically equivalent to X/G with G generated by T_0 and the shifts $(0, 1, 1), (1, 0, 1)$. This is precisely our model $(2-9)$.

It now follows that the free fermion model with basis B_{NAHE^+} discussed in Section 2.5.3 gives a CFT with geometric interpretation on our threefold $(2-9)$: In Section 2.5.3 we have described this free fermion model as a $(\mathbb{Z}_2)^2$-orbifold of the toroidal model on the $SO(12)$ torus, and (2.21) identifies the relevant action of $(\mathbb{Z}_2)^2$ with T_0. By the above, this gives a geometric interpretation on $(2-9)$. It also means

that the NAHE free fermion model with basis B_{NAHE^+} does *not* have a geometric interpretation on the $\mathbb{Z}_2$-shift orbifold $(1-1)$ of the Vafa-Witten model, as is sometimes claimed. Using the techniques described so far, one also checks that the free fermion model with basis $B_{NAHE^+} \cup \{\alpha\beta, \beta\gamma, \gamma\alpha\}$ (see Section 2.5.3 for notations) has geometric interpretation on our Calabi-Yau threefold $(4-1)$ with Hodge numbers $(15, 3)$. Faraggi, on the other hand, constructs a semi-realistic free fermion model with chiral spectrum $(6, 3)$ [FNY90, Fa92]. As can be seen from our classification in Section 1.6, there is no geometric orbifold of the appropriate type with these Hodge numbers.

3.2 Borcea-Voisin threefolds

Within our list of orbifolds tabulated in Section 1.6, there are several examples of Borcea-Voisin threefolds [Bor97, Voi93]. Namely, let $BV(r, a, \delta)$ denote a connected component of the moduli space of Borcea-Voisin threefolds obtained by a $\mathbb{Z}_2$ orbifolding procedure from the product of a $K3$-surface and an elliptic curve, $(K3 \times E_3)/(\iota, -1)$, where ι acts as antisymplectic automorphism on $K3$. Here, $(r, a, \delta) \in \mathbb{N}^3$ are the parameters from Nikulin's classification of $K3$ surfaces with such automorphisms [Nik79]. These parameters uniquely specify the topological invariants of each element in $BV(r, a, \delta)$, and there are precisely 75 possible triples (r, a, δ). One finds that the Hodge numbers of the resulting Borcea-Voisin threefolds are

$$h^{1,1} = 5 + 3r - 2a, \quad h^{2,1} = 65 - 3r - 2a,$$

except for $(r, a, \delta) = (10, 10, 0)$ where $h^{1,1} = h^{2,1} = 11$, see e.g. [Bor97, Voi93]. A related set of invariants describes the components of the fixed locus of the involution ι. In all cases except $(10, 10, 0)$ and $(10, 8, 0)$, this set consists of $k+1$ curves, one of which has genus g while the others are rational. In the exceptional case $(10, 10, 0)$ the fixed locus is empty, while in case $(10, 8, 0)$ it consists of two elliptic curves. In the remaining cases, these invariants are related to r, a by:

$$2g = 22 - r - a, \quad 2k = r - a.$$

We claim that seven of the orbifolds listed in Section 1.6 are among the Borcea-Voisin families of threefolds:

$$
\begin{aligned}
(0-1) &\in BV(18, 4, 0), & (h^{1,1}, h^{2,1}) &= (51, 3), \\
(0-2) &\in BV(10, 8, 0), & (h^{1,1}, h^{2,1}) &= (19, 19), \\
(0-3), (1-10) &\in BV(10, 10, 0), & (h^{1,1}, h^{2,1}) &= (11, 11), \\
(1-6) &\in BV(14, 8, 1), & (h^{1,1}, h^{2,1}) &= (31, 7), \\
(1-8) &\in BV(10, 10, 1), & (h^{1,1}, h^{2,1}) &= (15, 15), \\
(2-13) &\in BV(12, 10, 1), & (h^{1,1}, h^{2,1}) &= (21, 9).
\end{aligned}
$$

In general, any automorphism of a two-dimensional abelian variety that commutes with the (-1) involution permutes its 16 fixed points and induces an isomorphism

between the tangent spaces at corresponding points. It hence lifts to an automorphism of the $K3$ surface obtained by resolving the Kummer surface. The symplectic form must be mapped to some multiple of itself, and that multiple can be evaluated at any point of the resulting $K3$ surface. We may therefore safely ignore the fixed points and work on the torus $E_1 \times E_2 \times E_3$.

We write these seven quotients in the form $(E_1 \times K3)/(-1, \iota)$, where -1 sends $x \mapsto -x$ while ι is induced (as above) from an involution (still denoted ι) of $E_2 \times E_3$. The twist part of ι will always be $(y, z) \mapsto (y, -z)$.

In each case we write:

- The group acting on $E_1 \times E_2 \times E_3$ (in a couple of cases we need a permutation of what we have in Section 1).

- The subgroup G_0 fixing E_1 and acting only on $E_2 \times E_3$.

- The involution ι on $E_2 \times E_3$.

- The fixed curves of ι and its composites with G_0 in $E_2 \times E_3$ and their image in the $K3$ surface, i.e. mod G_0, that is the ramification curve of the $K3$ involution.

- The invariants g, k when they make sense (i.e. except in cases $(10, 10, 0)$ and $(10, 8, 0)$, when the fixed locus is empty or two elliptic curves, respectively), and (r, a, δ).

model	group	G_0	ι	Fix($\iota \cdot G_0$) $\rightarrow$ ramif. curve	(k, g), (r, a, δ)
$(0-1)$	$(0+, 0-, 0-)$, $(0-, 0+, 0-)$	$(0-, 0-)$	$(0+, 0-)$	8 elliptics: $\{2y = 0\} \cup \{2z = 0\}$ $\rightarrow$ 8 rationals	$(0, 7)$, $(18, 4, 0)$
$(0-2)$	$(0+, 0-, 0-)$, $(0-, 0+, 1-)$	$(0-, 0-)$	$(0+, 1-)$	4 elliptics: $\{2z = 1\}$ $\rightarrow$ 2 elliptics	$(10, 8, 0)$
$(0-3)$	$(0+, 0-, 0-)$, $(0-, 1+, 1-)$	$(0-, 0-)$	$(1+, 1-)$	empty $\rightarrow$ empty	$(10, 10, 0)$
$(1-6)$	$(0+, 0-, 0-)$, $(0-, 0+, 0-)$, $(0, t, t)$	$(0-, 0-)$, (t, t)	$(0+, 0-)$	8 elliptics: $\{2y = 0\} \cup \{2z = 0\}$ $\rightarrow$ 4 rationals	$(0, 3)$, $(14, 8, 1)$
$(1-8)$	$(0+, 0-, 1-)$, $(0-, 0+, 0-)$, $(0, t, t)$	$(0-, 1-)$, (t, t)	$(0+, 0-)$	4 elliptics: $\{2z = 0\}$ $\rightarrow$ 1 elliptic	$(1, 0)$, $(10, 10, 1)$
$(1-10)$	$(0+, 0-, 0-)$, $(0-, 1+, 1-)$, $(0, t, t)$	$(0-, 0-)$, (t, t)	$(1+, 1-)$	empty $\rightarrow$ empty	$(10, 10, 0)$
$(2-13)$	$(0+, 0-, 0-)$, $(0-, 0+, 0-)$, $(0, 1, 1)$, $(0, t, t)$	$(0-, 0-)$, $(1, 1)$, (t, t)	$(0+, 0-)$	8 elliptics: $\{2y = 0\} \cup \{2z = 0\}$ $\rightarrow$ 2 rationals	$(0, 1)$, $(12, 10, 1)$

The above argument shows that $(1-10)$ is in the same family as $(0-3)$. More precisely, these are two distinct three-parameter subfamilies of the eleven dimensional family of Borcea-Voisin threefolds of type $(10, 10, 0)$. In each case, the three parameters arise as the modulus of the elliptic curve plus two moduli for Kummer-like $K3$ surfaces, but these are two different two-parameter families of the latter.

As to the determination of the invariants (r, a, δ), the above calculations give us the fixed divisor in the orbifolding, hence by standard formulas also r and a. To obtain δ, in case $(0-1)$ we check explicitly that the class of the ramification divisor is even, basically because it has even multiplicity (namely, two) at each of the 16 blown up points. It follows that $\delta = 0$ in this case. In all other cases δ is uniquely determined, either because only one possibility occurs in Nikulin's list, or because the fixed divisor is either empty or it consists of two elliptic curves, which means that these yield cases $(10, 10, 0)$ and $(10, 8, 0)$, respectively.

It is curious that all the examples of Borcea-Voisin threefolds which occur in our list either have Hodge numbers $h^{1,1} = h^{2,1}$ or do not have Borcea-Voisin mirror partners since they have parameters (r, a, δ) where $(20 - r, a, \delta)$ does not belong to the list of 75 possible triples found by Nikulin [Nik79]. Again, the most prominent example of this type is the model $(0-1) \in BV(18, 4, 0)$ discussed by Vafa and Witten in [VW95]. For each of these models, it seems that discrete torsion allows the construction of a mirror partner. Using our results, one even has free fermion constructions for examples of CFTs associated to these "exceptional" Borcea-Voisin threefolds.

3.3 The Schoen threefold and its descendants

We remark that our orbifold $(0-2)$, with Hodge numbers $(19, 19)$, can be identified with Schoen's threefold [Sch88]. This may be of importance for the study of semi-realistic heterotic string theories, as we shall explain below. Let us first argue why $(0-2)$ does indeed agree with Schoen's threefold [Sch88] which is obtained as the fiber product over $\mathbb{P}^1$ of two rational elliptic surfaces S_1, S_2.

To this end note first that Schoen's threefold has Hodge numbers $(19, 19)$ in agreement with our claim. Namely, the complex structure of each rational elliptic surface depends on 8 (complex) parameters, and three more parameters are needed to fix an isomorphism between the two $\mathbb{P}^1$ bases, resulting in $8 + 8 + 3 = 19$ parameters in all. We claim that our orbifolds $(0-2)$ form a 3 dimensional subfamily of the family of Schoen threefolds. The rational elliptic surface, which generically has 12 degenerate fibers of type I_1, specializes here to an isotrivial one, having two degenerate fibers of type I_0^* and all other fibers having a fixed value of the j-invariant. These surfaces depend on a single complex parameter, the fixed value of j. Since these surfaces have automorphisms acting non trivially on the $\mathbb{P}^1$ bases, we get only one additional parameter for matching the bases, for a total of $1 + 1 + 1 = 3$ parameters, accounting for the moduli of our three elliptic curves E_i.

To finally identify our threefolds of type $(0-2)$ with Schoen's threefold, note that

our orbifolds can be written in the form:

$$Y = S_1 \times_{\mathbb{P}^1} S_2,$$

where in the obvious notation:

$$
\begin{aligned}
S_1 &:= (E_1 \times E_3)/\langle (0+, 0-), (0-, 1-) \rangle, \\
S_2 &:= (E_2 \times E_3)/\langle (0-, 0-), (0+, 1-) \rangle, \\
\mathbb{P}^1 &:= E_3/\langle (0-), (1-) \rangle = (E_3/\langle (1+) \rangle)/\langle (0-) \rangle.
\end{aligned}
$$

Each S_i maps to this $\mathbb{P}^1$, with constant fiber E_i except over two points of $\mathbb{P}^1$ where the fiber degenerates.

The various quotients of our orbifold $(0 - 2)$ can be similarly identified with quotients of special cases of the Schoen threefolds. Of greatest immediate interest is orbifold $(1 - 3)$. This was studied in [DOPW02] in an attempt to construct heterotic string compactifications with the low energy spectrum of the Standard Model of particle physics. This attempt succeeded through the construction of a different heterotic vector bundle on the same threefold, in [BD06], some of whose physical properties were further investigated in [BCD06]. Note that our identification of $(1 - 3)$ with the threefold used in these works implies that free fermion constructions may suffice to construct the associated string theories. This would dramatically simplify the rather technical approach of [DOPW02, BD06].

All free group actions on Schoen threefolds were analyzed in [BD07], where they are tabulated in Table 11. The last two, with fundamental group $\mathbb{Z}_2$, correspond to our models $(1 - 3)$ and $(1 - 7)$. In [BD07] they are distinguished by the invariants $m = 2$ and $m = 1$, respectively. The two quotients with fundamental group $(\mathbb{Z}_2)^2$ correspond to our models $(2 - 5)$ and $(2 - 14)$, corresponding again to $m = 2$ and $m = 1$, respectively.

Let us argue that the invariant m in Table 11 of [BD07] can indeed be used to distinguish our families. Let Y be a Schoen quotient, and $\pi: \widetilde{Y} \to Y$ its universal cover, of degree n. The Schoen quotient Y has a fibration $f: Y \to \mathbb{P}^1$. The composition $\widetilde{f} := f \circ \pi$ is the original abelian surface fibration of the Schoen threefold $\widetilde{Y}$. The generic fiber $A = E_1 \times E_2$ of $\widetilde{f}$ is the product of two elliptic curves E_1, E_2, and the generic fiber of f is its quotient by a finite subgroup. The invariant m is defined so that the size of this subgroup is n/m: the covering map π has degree n/m along the fibers of f and degree m along the base $\mathbb{P}^1$. So π^{-1} of a generic abelian surface fiber splits into m disconnected components, each an abelian surface. In other words, m can be recovered from the topology of Y plus the fibration f. So if we know that the fibration f is unique, it follows that m can be used to distinguish threefolds.

To recover f for the generic member Y in each of our families, we assume that $\widetilde{Y}$ is the fiber product of two rational elliptic surfaces S_1, S_2, and that there exists a point of $\mathbb{P}^1$ such that the two elliptic fibers E_1, E_2 over it are not isogenous. This can be arranged since by moving in the moduli space of Y we can vary the j-function continuously. Then the generic fiber $A = E_1 \times E_2$ of $\widetilde{f} := f \circ \pi$ is the product of two non isogenous elliptic curves. The only line bundles on such an A

are products of pullbacks from the two components. Any map $A \to \mathbb{P}^1$ is given by such a line bundle of self-intersection 0, hence the line bundle must be a pullback from a single E_i, and the map must factor through that E_i. Therefore any map $\widetilde{f}' : \widetilde{Y} \to \mathbb{P}^1$ must factor through an elliptic fibration on one of the rational elliptic surfaces S_i. But the elliptic fibration on S_i is unique, and is given by E_i in the anticanonical system: the connected component C of the general fiber of any other fibration on S_i has positive intersection number with E_i, so by adjunction it has to be rational rather than elliptic. This proves that the fibration f is unique. It follows that Schoen quotients with distinct invariants m are non isomorphic as algebraic varieties. Since each family of Schoen quotients dominates its complex structure moduli space, it also follows that Schoen quotients with distinct invariants m are not deformation equivalent.

A   Jacobi theta functions and their properties

We use the following functions of $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, $\mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ and $y = e^{2\pi i z}$, $z \in \mathbb{C}$,

$$
\begin{aligned}
\vartheta_1(\tau, z) = -\vartheta_{11}(\tau, z) \ &:= \ i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} \\[2mm]
&= \ i q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1} y)(1 - q^n y^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_2(\tau, z) = \ \vartheta_{10}(\tau, z) \ &:= \ \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} \\[2mm]
&= \ q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1} y)(1 + q^n y^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_3(\tau, z) = \ \vartheta_{00}(\tau, z) \ &:= \ \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n \\[2mm]
&= \ \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}} y)(1 + q^{n-\frac{1}{2}} y^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_4(\tau, z) = \ \vartheta_{01}(\tau, z) \ &:= \ \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n \\[2mm]
&= \ \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}} y)(1 - q^{n-\frac{1}{2}} y^{-1}).
\end{aligned}
$$

The functions ϑ_1, ϑ_2, ϑ_3, ϑ_4 are commonly known as Jacobi theta functions. We frequently denote $\vartheta_k(\tau) := \vartheta_k(\tau, 0)$ or even $\vartheta_k := \vartheta_k(\tau, 0)$, so in particular since

$\vartheta_1(\tau, z)$ is an odd function in z, $\vartheta_1 = 0$.

The following transformation laws are obtained directly from the definition or by Poisson resummation:

Operation	$\vartheta_1(\tau)$	$\vartheta_2(\tau)$	$\vartheta_3(\tau)$	$\vartheta_4(\tau)$
$\tau \mapsto \tau + 1$	$e^{\frac{2\pi i}{8}}\vartheta_1(\tau, z)$	$e^{\frac{2\pi i}{8}}\vartheta_2(\tau, z)$	$\vartheta_4(\tau, z)$	$\vartheta_3(\tau, z)$
$\tau \mapsto -\frac{1}{\tau},$ $z \mapsto \frac{z}{\tau}$	$(-i)(-i\tau)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\tau}}\cdot$ $\cdot\vartheta_1(\tau, z)$	$(-i\tau)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\tau}}\cdot$ $\cdot\vartheta_4(\tau, z)$	$(-i\tau)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\tau}}\cdot$ $\cdot\vartheta_3(\tau, z)$	$(-i\tau)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\tau}}\cdot$ $\cdot\vartheta_2(\tau, z)$

We also use the Dedekind eta function

$$\eta = \eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}\left(1 - q^n\right).$$

Under modular transformations, it obeys

$$\eta(\tau + 1) = e^{2\pi i/24}\eta(\tau), \quad \eta(-\tfrac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau).$$

By using the Jacobi triple identity one can prove the following product formulas:

$$\begin{aligned} \vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) &= 2\eta(\tau)^3 \\ \vartheta_2(\tau)^4 - \vartheta_3(\tau)^4 + \vartheta_4(\tau)^4 &= 0. \end{aligned} \tag{A.1}$$

B Representations of the free fermion algebra

A single free fermion ψ can have one of four different spin structures, each characterized by two binaries $\alpha, \beta \in \{0, 1\}$. The fermion ψ is said to belong to the NS (Neveu-Schwarz) sector if $\alpha = 0$, where it has half integer (Fourier) modes on expansion with respect to the parameter $x \in \mathbb{C}^*$ of the field ψ, and otherwise it belongs to the R (Ramond) sector, where it has integer modes. The modes obey

$$\{\psi_a, \psi_b\} = \delta_{a+b,0} \ \text{ for } \ a, b \in \left\{ \begin{array}{ll} \mathbb{Z} + \frac{1}{2} & (\text{NS}) \\ \mathbb{Z} & (\text{R}) \end{array} \right.$$

and thus act as creation or annihilation operators. These modes together with 1 (that is, a central element which in each representation is normalized to act as identity operator) form a vector space basis of the so-called free fermion algebra. Let $\mathcal{H}_0$, $\mathcal{H}_1$ denote the irreducible Fock space representations of the free fermion algebra in the NS and the R sector, respectively, enlarged by $(-1)^F$ with F the worldsheet fermion number, i.e. such that $(-1)^F$ is a non-trivial involution which anticommutes with all ψ_a. Each state in $\mathcal{H}_0$, $\mathcal{H}_1$ is obtained by acting with pairwise distinct fermionic creation operators on a ground state and thereby increasing the conformal dimension by half integer (NS) or integer (R) steps. In the NS sector,

ground states of this Fock space representation of the free fermion algebra have conformal dimension $h = 0$, whereas in the R sector, they have conformal dimension $h = \frac{1}{16}$. In fact, $\mathcal{H}_0$ has a unique ground state (up to scalar multiples) $|0\rangle$, the vacuum, whereas $\mathcal{H}_1$ possesses a two dimensional space of such ground states. The vacuum $|0\rangle$ is a worldsheet boson, i.e. $(-1)^F|0\rangle = |0\rangle$, and in $\mathcal{H}_1$ we choose a basis $|\pm\rangle$ of ground states such that $|+\rangle$ is a worldsheet boson and $|-\rangle$ is a worldsheet fermion, i.e. $(-1)^F|\pm\rangle = \pm|\pm\rangle$. The decomposition of $\mathcal{H}_0$, $\mathcal{H}_1$ into worldsheet bosons and worldsheet fermions,

$$\mathcal{H}_\alpha \cong \mathcal{H}_\alpha^+ \oplus \mathcal{H}_\alpha^-, \quad \alpha \in \{0, 1\}$$

agrees with the decomposition into irreducible representations of the Virasoro algebra at central charge $c = \frac{1}{2}$ which arises from the universal enveloping algebra of the free fermion algebra in either sector. We set

$$\text{for } \alpha, \beta \in \{0, 1\}: \quad Z\begin{bmatrix} \alpha \\ \beta \end{bmatrix} := \sqrt{\frac{\vartheta_{\alpha,\beta}}{\eta}}, \tag{B.1}$$

where the $\vartheta_{\alpha,\beta}$ denote the Jacobi theta functions and η the Dedekind eta function listed in Appendix A. The square root makes sense in terms of the infinite product representations of the $\vartheta_{\alpha,\beta}$ also given there. Then with $q = e^{2\pi i \tau}$ and τ as before, the above discussion together with the explicit product formulas given in Appendix A shows

$$Z\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) = \text{tr}_{\mathcal{H}_0}\left[q^{L_0 - 1/48}\right], \qquad Z\begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau) = \text{tr}_{\mathcal{H}_0}\left[(-1)^F q^{L_0 - 1/48}\right],$$

$$Z\begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) = \tfrac{1}{\sqrt{2}}\text{tr}_{\mathcal{H}_1}\left[q^{L_0 - 1/48}\right], \qquad Z\begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau) = \tfrac{1}{\sqrt{2}}\text{tr}_{\mathcal{H}_1}\left[(-1)^F q^{L_0 - 1/48}\right] = 0.$$

The insertion of $(-1)^F$ in the traces to obtain $Z\begin{bmatrix} \alpha \\ 1 \end{bmatrix}$ from $Z\begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ corresponds in Hamiltonian language to changing the spin structure of the fermion in the imaginary time direction. This means that $Z\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ gives the contribution to the partition function of a free fermion with spin structure specified by α, $\beta \in \{0, 1\}$.

The factors of $\frac{1}{\sqrt{2}}$ in the traces for the R-sector yield $2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$ as coefficient of the leading term $q^{\frac{1}{24}}$ in $Z\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, accounting for the contributions from the space generated by $|+\rangle$ and $|-\rangle$. We obtain integer coefficients as soon as we consider pairs of fermions with coupled spin structures in space direction, which is necessary anyway in order to get pairwise local fields of a well-defined CFT. Given a collection of free fermions, for a tensor product between the R-sectors of the jth and the j'^{th} free fermion, with coupled spin structures, $\mathcal{H}_1^j \otimes \mathcal{H}_1^{j'}$ splits into two isomorphic representations of the free fermion algebras generated by ψ_a^j, $\psi_a^{j'}$ with $a \in \mathbb{Z}$ enlarged by the total worldsheet fermion number operator $(-1)^{F_j + F_{j'}}$, one with ground states

$$|+\rangle \otimes |+\rangle + |-\rangle \otimes |-\rangle, \quad |+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle,$$

the other with ground states

$$|+\rangle \otimes |+\rangle - |-\rangle \otimes |-\rangle, \quad |+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle,$$

respectively. Let pr denote the projection onto one of these two representations. Using $\left(Z\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^2$ as above then gives the trace over $\mathrm{pr}\left(\mathcal{H}_1^j \otimes \mathcal{H}_1^{j'} \right)$, as the coefficients $\sqrt{2}$ conspire correctly to count a two-dimensional space of ground states. Since $\left(Z\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 = 0$, one of the generators is correctly counted as boson, the other as fermion. Summarizing, if ψ^j and $\psi^{j'}$ have coupled spin structures $(\alpha_j, \beta_j) = (\alpha_{j'}, \beta_{j'})$ and pr is extended trivially to $\mathcal{H}_0^j \otimes \mathcal{H}_0^{j'}$, then

$$Z\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} \cdot Z\begin{bmatrix} \alpha_{j'} \\ \beta_{j'} \end{bmatrix} = \mathrm{tr}_{\mathrm{pr}\left(\mathcal{H}_{\alpha_j}^j \otimes \mathcal{H}_{\alpha_{j'}}^{j'} \right)} \left[(-1)^{\beta_j F_j + \beta_{j'} F_{j'}} q^{L_0 - 1/48} \right]. \tag{B.2}$$

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