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Decoding the Geometry of Conformal Field Theories

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Abstract

To certain geometries, string theory associates conformal field theories. We discuss techniques to perform the reverse procedure: To recover geometrical data from abstractly defined conformal field theories. This is done by introducing appropriate notions of limits of conformal field theories and their degenerations, and by applying techniques from noncommutative geometry.

This note is a summary of our work [7], aimed to be less technical than the original paper, along with some new calculations confirming our interpretation of the rescaled limiting zero mode of the Virasoro field.

1 Introduction

In string theory, it is believed that one can associate quantum field theories to certain geometries by a so-called non-linear sigma model construction. The corresponding conformal quantum field theories (CFTs) naturally exhibit traces of the original "target space geometry". This connection between geometry and CFT is a continuous driving force both in geometry and in string theory, as is exemplified by the success stories of mirror symmetry and T-duality, to name just two examples. However, in general, quantisation requires renormalisation and thus obscures the underlying geometry in non-linear sigma models. It is therefore difficult to describe how exactly geometry is encoded in CFT. It is nevertheless appealing to investigate this problem, since CFTs can be abstractly defined in more general contexts. Moreover, the applications should be manifold, by the above, for example, in the study of mirror symmetry and T-duality.

An exact correspondence between CFT and geometry is expected in general only in so-called large volume regions of the target space, where quantum effects become small. More precisely, the Riemannian metric of the underlying target space needs to be considered close to a degenerating metric. On the CFT side

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one should therefore expect an analogous degeneration phenomenon. In [7] we introduce a notion of limiting processes for CFTs which we argue leads to precisely such phenomena. Moreover, we show that the limiting data can be used to associate geometric interpretations to degenerate limits of CFTs, and we substantiate our claim by proving that the procedure works for the large level limit of the bosonic unitary Virasoro minimal models of A-type. To achieve this, we employ ideas of Fröhlich and Gawȩdzki [4], using the fact that the energy operator in a semi-classical limit of CFTs should yield a generalised Laplacian on the target space. While its spectrum belongs to the data recovered in the semi-classical limit, in general, one cannot hear the shape of a drum. However, Fröhlich and Gawȩdzki argue that the operator product expansion (OPE) of CFT yields the necessary hearing aid: It promotes a subspace of the Hilbert space (interpreted as space of square integrable functions on the target) to an algebra of (continuous) functions. Altogether, one obtains a triple of data which is closely related to Connes' spectral triples [1–3]. In particular, one can read off a topological space as target space together with two conformally equivalent Riemannian metrics on it, one the target space metric and the other the dilaton-corrected metric.

The present paper serves two main purposes: On the one hand, we wish to give a more digestible summary of our previous work [7], by focussing on the central ideas instead of technical details. On the other hand, thanks to comments by Maxim Kontsevich, we have meanwhile obtained additional evidence for the validity of our geometric interpretation of limiting processes in CFTs, which we present here.

The structure of the paper is as follows: In section 2, we recall the main ideas behind our definition of limiting sequences of CFTs. The section contains a brief review of our abstract definition of such theories, since it is fundamental to the notion of limits. Section 3 begins with a review of those properties of the limiting data which allow us to associate geometric interpretations to them. We then recall the relevant techniques from noncommutative geometry. Finally, we state the so-called seven-term identity, and we prove that this identity can be viewed as an analogue of the Leibniz rule, characterising differential operators of order two without constant term and with symmetric symbol. The seven-term identity is also the central theme of section 4. Here we prove that in fact the operator H^∞, obtained by rescaling the energy operator in a degenerate limit of CFTs, obeys the seven-term identity. In other words, our interpretation of H^∞ as generalised Laplacian without constant term is confirmed. We end this note with some conclusions and an outlook in section 5.

2   Limits of CFTs: The definition

The aim of this work is to find a general procedure to extract geometry from conformal field theory (CFT), using intrinsic CFT data only. Thus we need to work with an abstract definition of CFT. Here, we restrict ourselves to stating the main ingredients of such a definition for the cases of interest to us. Further details can be found in [7] and in the references given there.

A unitary two-dimensional Euclidean conformal field theory (in short a

CFT), roughly, is a unitary representation of two commuting copies $\mathcal{V}\mathrm{ir}_c$, $\overline{\mathcal{V}\mathrm{ir}_{\overline{c}}}$ of the Virasoro algebra on a complex vector space $\mathcal{H}$ with positive definite scalar product $\langle \cdot, \cdot \rangle$ and complex conjugation $*\colon \mathcal{H} \to \mathcal{H}$. It is assumed that the commuting self-adjoint operators L_0, $\overline{L}_0$ given by the zero modes of the left and the right handed Virasoro algebra, respectively, are simultaneously diagonalisable such that

$$\mathcal{H} = \bigoplus_{(h,\overline{h}) \in R} \mathcal{H}_{h,\overline{h}}, \quad R \subset \mathbb{R}^2,$$

where all eigenvalues h, $\overline{h}$ are non-negative, all simultaneous eigenspaces $\mathcal{H}_{h,\overline{h}}$ of $L_0, \overline{L}_0$ are finite dimensional, and such that $\mathcal{H}_{0,0} = \ker(L_0) \cap \ker(\overline{L}_0)$ is one-dimensional. A unit generator $\Omega \in \mathcal{H}_{0,0}$ is chosen, called the vacuum. We introduce the "graded dual"

$$\check{\mathcal{H}}^* := \bigoplus_{(h,\overline{h}) \in R} (\mathcal{H}_{h,\overline{h}})^*$$

of $\mathcal{H}$ as well as the natural map $\mathcal{H} \longrightarrow \check{\mathcal{H}}^*$, $\psi \mapsto \psi^* \in \check{\mathcal{H}}^*$ such that for every $\chi \in \mathcal{H}$ one has $\psi^*(\chi) = \langle \psi, \chi \rangle$.

In addition to this structure, the vector space $\mathcal{H}$ in a CFT carries so-called n-point functions which in turn encode the OPE. The n-point functions are multilinear, symmetric maps from $\mathcal{H}^{\otimes n}$ into functions in n (holomorphic) complex coordinates $z_1, \dots, z_n$ on a Riemann surface, which are real analytic outside the partial diagonals and with prescribed pole behaviour near the partial diagonals. These n-point functions "transform covariantly under infinitesimal conformal transformations", a notion which can be made precise in terms of the representation theory of $\mathcal{V}\mathrm{ir}_c \oplus \overline{\mathcal{V}\mathrm{ir}_{\overline{c}}}$. In particular, insertion of the vacuum into the ith component of an n-point function yields a function which is independent of z_i and which can be viewed as an $(n-1)$-point function. There are various additional consistency conditions, the most important of which, from our point of view, ensures that all n-point functions can be recovered from the structure constants of the three-point functions, that is from certain maps

$$C \colon \check{\mathcal{H}}^* \otimes \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{C}$$

which obey

$$\forall \, \psi, \chi \in \mathcal{H}: \qquad C(\psi^*, \Omega, \chi) = \langle \psi, \chi \rangle. \tag{1}$$

The expert will notice that C is related to the usual three-point functions by

$$C(\psi^*, \varphi, \chi) = \langle \psi^\dagger(\overline{x}^{-1})\, \varphi(z)\, \chi(w) \rangle_{|x=w=0, z=1}.$$

In fact, the n-point functions can be recovered from the restriction of C to the space of all highest weight vectors $\mathcal{H}^{\mathcal{W}}$ (and its graded dual) with respect to an appropriate choice of W-algebra $\mathcal{W}$ such that $\mathcal{V}\mathrm{ir}_c \oplus \overline{\mathcal{V}\mathrm{ir}_{\overline{c}}} \subset \mathcal{W} \subset \ker(L_0) \oplus \ker(\overline{L}_0)$. The elements of $\mathcal{H}^{\mathcal{W}}$ are the so-called primaries with respect to $\mathcal{W}$. In

what follows, it will be more convenient to encode C in the following way: For every $\varphi \in \mathcal{H}^{\mathcal{W}}$, we define $A_\varphi \in \mathrm{End}(\mathcal{H}^{\mathcal{W}})$ such that

$$\forall\, \psi, \chi \in \mathcal{H}^{\mathcal{W}}: \quad C(\psi^*, \varphi, \chi) = \langle \psi, A_\varphi(\chi) \rangle.$$

The axioms of CFT ensure that this weak definition of A_φ indeed gives a well-defined linear operator A_φ for every $\varphi \in \mathcal{H}^{\mathcal{W}}$. Due to (1) we immediately find $A_\Omega = \mathrm{id}$. The expression $A_\varphi(\chi)$ can be viewed as a truncation to $\mathcal{H}^{\mathcal{W}}$ of the OPE between φ and χ, where $\varphi, \chi \in \mathcal{H}^{\mathcal{W}}$. This implies $A_\varphi \circ A_\chi \neq A_{A_\varphi(\chi)}$ in general, yielding the algebra $\mathcal{A} \subset \mathrm{End}(\mathcal{H}^{\mathcal{W}})$ which is formally generated by all operators A_φ with $\varphi \in \mathcal{H}^{\mathcal{W}}$ rather complicated. In fact, $A_\varphi \circ A_\chi$ may not be of the form A_ψ for any $\psi \in \mathcal{H}^{\mathcal{W}}$.

In what follows, of all these structures we will mainly use:

- The W-algebra $\mathcal{W} \subset \ker L_0 \oplus \ker \overline{L}_0$ with $\mathcal{W} \supset \mathcal{V}\mathrm{ir}_c \oplus \overline{\mathcal{V}\mathrm{ir}_{\overline{c}}}$, and its associated space of primaries $\mathcal{H}^{\mathcal{W}}$.

- The action of the so-called energy operator $H := L_0 + \overline{L}_0$ on $\mathcal{H}^{\mathcal{W}}$.

- The algebra $\mathcal{A} \subset \mathrm{End}(\mathcal{H}^{\mathcal{W}})$ generated by the operators A_φ with $\varphi \in \mathcal{H}^{\mathcal{W}}$, where the map $\varphi \mapsto A_\varphi$ is obtained from C as above, and C encodes the n-point functions.

For later convenience, we list the resulting structures in the case of the CFT which describes a free boson compactified on a circle of radius $R \in \mathbb{R}^+$:

- $\mathcal{W} = \mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$, and the space $\mathcal{H}_R^{\mathcal{W}}$ of corresponding primaries has a basis $\{|m, n\rangle_R,\ m, n \in \mathbb{Z}\}$ indexed by pairs of integers.

- The basis of $\mathcal{H}^{\mathcal{W}}$ given above is an eigenbasis for the energy operator H, namely
$$H\left(|m, n\rangle_R\right) = \frac{1}{2}\left(\frac{m^2}{R^2} + n^2 R^2\right)|m, n\rangle_R.$$

- The algebra $\mathcal{A} \subset \mathrm{End}(\mathcal{H}^{\mathcal{W}})$ is generated by the operators $A_{|m,n\rangle_R}$, $m, n \in \mathbb{Z}$, with
$$A_{|m,n\rangle_R}\left(|m', n'\rangle_R\right) = (-1)^{mn'}|m + m', n + n'\rangle_R.$$

Indeed, the above CFTs form a family parametrised by $R \in \mathbb{R}^+$, and it is natural to study their behaviour when R converges to 0 or ∞. To be able to address such problems a notion of convergent sequences of CFTs is needed. The limit of a convergent sequence of CFTs should exhibit those properties which are common to almost all of its members. To extract such common properties, we use the notion of direct limits of vector spaces, and we impose conditions on the sequence of CFTs which allow us to define some of the usual structures of a CFT on such a direct limit. It is crucial, however, that a limit of a sequence of CFTs according to our definitions need not be a full-fledged CFT itself.

In more detail, we define a sequence of CFTs to consist of a family of CFTs whose underlying vector spaces $\left\{\mathcal{H}^i\right\}_{i \in I}$ form a direct system. In other words, I is an ordered index set, and vector space homomorphisms $f_i^j \colon \mathcal{H}^i \longrightarrow \mathcal{H}^j$ are given for all $i \leq j$ such that $f_i^i = \mathrm{id}$, $f_j^k \circ f_i^j = f_i^k$ for $i \leq j \leq k$. We additionally assume that $f_i^j * = * f_i^j$ and that the f_i^j map the most fundamental states in our CFTs to one another, that is the vacua and the states created by the holomorphic and the antiholomorphic Virasoro fields. These conditions suffice to define a vector space $\mathcal{K}^\infty$ obtained as direct limit $\mathcal{K}^\infty = \varinjlim \mathcal{H}^i$. Elements of $\mathcal{K}^\infty$ hence are equivalence classes of elements of the $\mathcal{H}^i$, whose representatives form sequences $\{f_i^j(\psi_i)\}_{j \geq i}$ with $\psi_i \in \mathcal{H}^i$.

In [7] we give a detailed definition for the notion of convergence of a sequence of CFTs with underlying vector spaces $\widetilde{\mathcal{H}}^i$. This definition is quite technical, such that here we prefer to focus on a summary. Convergence roughly means that the direct system $\widetilde{\mathcal{H}}^i$ has a direct subsystem $\mathcal{H}^i \subset \widetilde{\mathcal{H}}^i$ such that after restriction to the $\mathcal{H}^i$ one has:

1. For all $\psi, \varphi, \chi, \rho \in \mathcal{H}^i$, the four-point function

$$z \longmapsto \langle f_i^j(\psi) \mid f_i^j(\varphi)(1)\, f_i^j(\chi)(z) \mid f_i^j(\rho)\rangle$$

   converges for $j \to \infty$ as a real analytic function and with the standard behaviour near the singularities $z \in \{0, 1, \infty\}$.

2. The maps f_i^j respect a collection of chosen W-algebras $\mathcal{W}^i \subset \ker(L_0^i) \oplus \ker(\overline{L}_0^i)$ in the following sense: The f_i^j respect an appropriate decomposition of the $\mathcal{H}^i$ into common L_0^i and $\overline{L}_0^i$ eigenspaces, primaries with respect to $\mathcal{W}^i$ are mapped to primaries with respect to $\mathcal{W}^j$, and none of these primaries represents zero in $\mathcal{K}^\infty$.

3. The convergence of three-point functions is uniform in an appropriate sense, as is the convergence of four-point functions involving primaries. Moreover, only finitely many homogeneous components $\varphi^i \in \mathcal{H}^i_{h_\varphi^i, \overline{h}_\varphi^i}$ of each truncated OPE represent elements of $\mathcal{K}^\infty$ with

$$\lim_{i \to \infty} C^i((\varphi^i)^*, \Omega^i, \varphi^i) \neq 0 \quad \text{and} \quad \lim_{i \to \infty} \left(h_\varphi^i + \overline{h}_\varphi^i\right) = 0.$$

In [7] we show that our definition of limits of CFTs allows us to obtain the following structures from a convergent sequence of CFTs:

A unique vector $\Omega \in \mathcal{K}^\infty$ exists which is represented by the vacua in the $\mathcal{H}^i$. There is a direct limit $(\check{\mathcal{K}}^\infty)^*$ of the $(\check{\mathcal{H}}^i)^*$ and an induced isomorphism $\psi \mapsto \psi^*$ between $\mathcal{K}^\infty$ and $(\check{\mathcal{K}}^\infty)^*$. The limiting structure constants C^∞ of three-point functions are well-defined on $\mathcal{K}^\infty$, and they induce a sesqui-linear form $\langle \psi, \chi \rangle := C^\infty(\psi^*, \Omega, \chi)$ on $\mathcal{K}^\infty$ which however need not be positive definite. Setting

$$\mathcal{N}^\infty := \{\nu \in \mathcal{K}^\infty \mid C^\infty(\nu^*, \Omega, \nu) = 0\},$$

our conditions on convergence ensure that C^∞ descends to well-defined structure constants on $\mathcal{H}^\infty := \mathcal{K}^\infty / \mathcal{N}^\infty$, where $\langle \cdot, \cdot \rangle$ is positive definite. On $\mathcal{H}^\infty$

there is a well-defined action of a limiting W-algebra $\mathcal{W}$ and thus in particular of a limiting Virasoro algebra with zero modes L_0, $\overline{L}_0$, as well as a notion of primaries. In fact, $\mathcal{H}^\infty$ decomposes into a direct sum of simultaneous eigenspaces $\mathcal{H}^\infty_{h,\overline{h}}$ of the operators L_0, $\overline{L}_0$. The limiting energy operator is denoted by $H := L_0 + \overline{L}_0$. Furthermore, a truncated OPE $A_\varphi(\chi)$ can be defined between any two primaries φ, χ in $\mathcal{H}^\infty$.

We thus in particular obtain the following data from any limit of a convergent sequence of CFTs, where for simplicity we assume that the elements of the sequence are indexed by some $I \subset \mathbb{R}$:

- A vector space $\mathbb{H}^\infty := \ker H \subset \mathcal{H}^\infty$ with positive definite scalar product $\langle \cdot, \cdot \rangle$.

- A limiting energy operator H^∞ on $\mathbb{H}^\infty$ obtained by rescaling H appropriately: If there is a function $r : I \to \mathbb{R}$ with $\underrightarrow{\lim} r = 0$ such that for all $\varphi \in \mathbb{H}^\infty$ represented by homogeneous vectors $(\varphi^i \in \mathcal{H}^i_{h_i, \overline{h}_i})_i$ the rescaled energies $\frac{1}{r(i)}(h_i + \overline{h}_i)$ remain finite in the limit, then define $H^\infty \varphi = E\varphi$ with $E := \underrightarrow{\lim} \frac{1}{r(i)}(h_i + \overline{h}_i)$.

- A limiting zero mode algebra $\mathcal{A}^\infty \subset \text{End}\left((\mathcal{H}^\infty)^{\mathcal{W}}\right)$ generated by the operators A_φ with $\varphi \in \mathbb{H}^\infty$,

$$C^\infty(\psi^*, \varphi, \chi) = \langle \psi, A_\varphi(\chi) \rangle \quad \text{for all} \quad \psi, \chi \in (\mathcal{H}^\infty)^{\mathcal{W}}.$$

It is important to keep in mind that the structure found in the limit of a convergent sequence of CFT severely depends on the choice of the direct system, *i.e.* the maps f_i^j. In [7], we give several instructive examples to this effect. However, in many cases there are natural choices for the maps f_i^j. For instance for the free boson on a circle, a natural choice is induced by $f_R^{R'} : |m, n\rangle_R \mapsto |m, n\rangle_{R'}$ for all $m, n \in \mathbb{Z}$. It is not hard to check convergence and that then the following structures arise in the limit $R \to \infty$:

- Each primary $|m, 0\rangle_R \in \mathcal{H}_R^{\mathcal{W}}$ with $m \in \mathbb{Z}$ represents a primary $|m, 0\rangle_\infty \in \mathcal{H}^\infty$, and in fact $\mathbb{H}^\infty$ has orthonormal basis $\{|m, 0\rangle_\infty, m \in \mathbb{Z}\}$. The highest weight states $|m, n\rangle_R$ with $n \neq 0$ and their descendants do not contribute to the limit, because in the limit $R \to \infty$ their energies diverge and they decouple from the rest of the theory.

- The energy eigenvalues of representatives of elements in $\mathbb{H}^\infty$ can be rescaled by a factor of $\frac{1}{r(R)} = R^2$, so $H^\infty \left(|m, 0\rangle_\infty\right) = \frac{m^2}{2}|m, 0\rangle_\infty$.

- The limiting zero mode algebra $\mathcal{A}^\infty \subset \text{End}\left((\mathcal{H}^\infty)^{\mathcal{W}}\right)$ is generated by the operators $A_{|m,0\rangle_\infty}$ with $m \in \mathbb{Z}$, and

$$A_{|m,0\rangle_\infty}\left(|m', 0\rangle_\infty\right) = |m + m', 0\rangle_\infty. \tag{2}$$

In [7] we also show that the sequence of diagonal bosonic unitary Virasoro minimal models can be given the structure of a sequence of CFTs, which converges. We will recall the interpretation of its limit below in somewhat greater detail.

3 Limits of CFTs: Extracting geometric interpretations

The notion of convergence of sequences of CFTs given in the last section allows for certain degeneration phenomena to occur. We will argue that precisely these degeneration phenomena allow us to extract geometric data from limits of CFTs.

As emphasized above, the limit of a convergent sequence of CFTs is not necessarily a full-fledged conformal field theory. Indeed, our definition of convergence assures the existence of limiting correlation functions on the sphere which satisfy the appropriate conditions, but correlation functions on higher genus surfaces can diverge due to degenerations of the spectra of L_0 and $\overline{L}_0$. In particular, it may happen that the space $\mathbb{H}^\infty$ of states whose energy converges to zero, in the limit has infinite dimension. This is already the case for the sequence of CFTs associated to the free boson on the circle, as we have seen above. However, our notion of convergence ensures that this space and its associated zero mode algebra $\mathcal{A}^\infty$ have surprisingly simple properties:

The vanishing of the conformal weights for every $\varphi \in \mathbb{H}^\infty$ ensures

Lemma: *In a convergent family of CFTs, for every $\varphi \in \mathbb{H}^\infty$, the corresponding zero mode $A_\varphi \in \mathcal{A}^\infty$ acts as an endomorphism of each simultaneous eigenspace $\mathcal{H}^\infty_{h,\overline{h}}$ of $L_0, \overline{L}_0$. In particular, the zero mode algebra $\mathcal{A}^\infty$ acts on $\mathbb{H}^\infty$.*

Moreover, crossing symmetry of four-point functions on the sphere implies

Proposition: *The zero mode algebra $\mathcal{A}^\infty$ associated to a convergent family of CFTs is commutative, $A_\varphi \circ A_\chi = A_{A_\varphi(\chi)}$ with $A_\varphi(\chi) \in \mathbb{H}^\infty$ for all $\varphi, \chi \in \mathbb{H}^\infty$.*

Therefore, by means of the Gelfand-Naimark theorem to every such degenerate limit of CFTs, one can associate a topological space $X = \mathrm{Spec}(\overline{\mathcal{A}^\infty})$ with $\overline{\mathcal{A}^\infty}$ an appropriate completion of $\mathcal{A}^\infty$. In other words, the zero mode algebra $\mathcal{A}^\infty$ can be interpreted to generate the algebra of continuous functions on a topological space X. The limiting Hilbert space $\mathcal{H}^\infty$ can be regarded as space of sections $\mathcal{H}^\infty = \Gamma(\mathcal{S})$ of a sheaf $\mathcal{S}$ of vector spaces on X.

For example, for the limit $R \to \infty$ of the sequence of CFTs describing a free boson on a circle of radius R, we easily identify $|m,0\rangle_\infty$ with the continuous function $x \mapsto \exp(\sqrt{-1}\,m\,x)$ on $X = \mathbb{S}^1 = \mathbb{R}^1/(x \sim x + 2\pi)$, obeying the multiplication law (2). The sheaf $\mathcal{S}$ is a trivial vector bundle on X with fibers given by the vacuum representation of $\mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$.

In general the structure of $\mathcal{S}$ is more interesting. For the $\mathbb{Z}_2$-orbifold of the free boson for instance, the twisted sectors contribute to $\mathcal{S}$ as sections of skyscraper sheaves over the orbifold fixed points, *c.f.* [7]. For brevity we will refrain from discussing the structure of $\mathcal{S}$ any further, instead we will argue how the limiting data $(\mathbb{H}^\infty, H^\infty, \mathcal{A}^\infty)$ obtained above can yield a geometric interpretation on a space X equipped with a Riemannian metric (up to scaling). Here we use ideas from Connes' noncommutative geometry [1–3], following [4, 6]. Namely, apart from the identification of $\overline{\mathcal{A}^\infty}$ as the algebra of continuous functions on X, the vector space $\mathbb{H}^\infty$ should be interpreted as a space of square integrable functions on X, where H^∞ is densely defined and acts as generalised Laplacian. This will put $(\mathbb{H}^\infty, H^\infty, \mathcal{A}^\infty)$ into the context of Connes' spectral triples. In more detail, in favourable cases there exist conformally equivalent

Riemannian metrics g and $\widetilde{g}$ on X such that $\mathbb{H}^\infty$ generates $L^2(X, \mathrm{dvol}_g)$, and H^∞ is a generalised Laplacian associated to $\widetilde{g}$. Writing $\mathrm{dvol}_g = e^{2\Phi}\mathrm{dvol}_{\widetilde{g}}$ with $\Phi \in C^\infty(X)$, in coordinates on X, the linear operator $2H^\infty$ corresponds to the generalised Laplacian

$$\Delta_{\widetilde{g}}^{\Phi} = -e^{-2\Phi}\sqrt{\det(\widetilde{g}^{-1})} \sum_{i,j} \partial_i\, e^{2\Phi}\sqrt{\det(\widetilde{g})}\widetilde{g}^{ij}\partial_j\,. \qquad (3)$$

The function Φ is called a dilaton field.

For the infinite radius limit of CFTs describing a free boson on a circle we indeed easily identify $2H^\infty$ with the ordinary Laplacian $-\frac{d^2}{dx^2}$, such that g is the flat metric on $\mathbb{S}^1$ and the dilaton $\Phi = 0$ is trivial. In general, the ideas described above are in accord with the aim to find a reversal of non-linear sigma model constructions. As is explained in more detail in [4, 7], the ansatz $(\overline{\mathbb{H}^\infty}, 2H^\infty, \overline{\mathcal{A}^\infty}) = (L^2(X, \mathrm{dvol}_g), \Delta_{\widetilde{g}}^{\Phi}, C^0(X))$ gives a mathematical formulation of a semiclassical limit. The degeneration of CFTs yields a degenerating family of operators $r(i)2H^\infty$, which gives rise to families of degenerating metrics $\frac{g}{r(i)}, \frac{\widetilde{g}}{r(i)}$. The operator $2H^\infty$ and the metrics $g, \widetilde{g}$ are only defined up to an overall scale and serve as a reference for the degeneration. For the infinite radius limit of the free boson, for instance, one obtains the degenerating family of (generalised) Laplacians $-\frac{1}{R^2}\frac{d^2}{dx^2}$ associated to the family of flat metrics on the circle of radius R, as expected.

It is not hard to explicitly recover the Riemannian metric g as well as the dilaton Φ from $(\overline{\mathbb{H}^\infty}, 2H^\infty, \overline{\mathcal{A}^\infty}) = (L^2(X, \mathrm{dvol}_g), \Delta_{\widetilde{g}}^{\Phi}, C^0(X))$, once X is known, *c.f.* [4]. In [7] we collect further evidence that this procedure works in many cases. In particular, we show that it is valid for an example which does not arise from a non-linear sigma model construction by proving

Proposition: *The family of bosonic unitary Virasoro minimal models of A-type gives a convergent sequence of conformal field theories, whose large level limit has geometric interpretation on the unit interval $X = [0, \pi]$ with flat metric $g \equiv 1$ and dilaton Φ such that* $\exp(2\Phi(x)) = \frac{2}{\pi}\sin^2(x)$ *for $x \in X$.*

This result is in accord with the predictions in [4, §3.3]. Namely, one argues that the Virasoro minimal model at level $k \in \mathbb{N}$ is given by a coset model

$$\left(\mathfrak{su}(2)_{k-1} \oplus \mathfrak{su}(2)_1\right) / \mathfrak{su}(2)_k.$$

This implies that as $k \to \infty$, one approximately has an $\mathfrak{su}(2)$ WZW model at level 1 which is gauged by the adjoint action of $\mathfrak{su}(2)$. The gauge projection is elegantly implemented by $\mathfrak{su}(2) \to U(1)/\mathbb{Z}_2$, $h \mapsto \mathrm{tr}(h)$ with $U(1)/\mathbb{Z}_2 \cong [0, \pi]$ in accord with our proposition. By [8, chapter 4.2], the dilaton $\exp(2\Phi)$ can be obtained from the Lagrangian of a gauged WZW model as the coefficient in front of those contributions which are quadratic in the gauge field. Such contributions have been determined [5, (2.3)], and a straightforward calculation confirms $\exp(2\Phi(x)) \sim \sin^2(x)$.

After completion of [7], Maxim Kontsevich pointed out to us that according to our claim, the rescaled energy operator H^∞ on X corresponds to a second order differential operator without constant term, and that it therefore must satisfy

the following so-called seven-term identity:

Proposition: *For $a, b, c \in \mathcal{A}^\infty$, where by misuse of notation we identify elements $a \in \mathcal{A}$ canonically with $a \cdot \Omega \in \mathbb{H}^\infty$:*

$$H^\infty(abc) - H^\infty(ab)c - aH^\infty(bc) - bH^\infty(ca)$$
$$+ H^\infty(a)bc + aH^\infty(b)c + abH^\infty(c) = 0. \tag{4}$$

Vice versa, (4) can be viewed as an analog of the Leibniz rule for second order differential operators without constant term. Recall that a linear operator D acting on functions $f \colon X \to \mathbb{C}$ on a Riemannian manifold X is called a derivation if D obeys the Leibniz rule

$$\forall f, h \colon \qquad D(fh) = f\,D(h) + h\,D(f).$$

A linear operator acts as a derivation on smooth functions on X if and only if, in local coordinates on X, it is a differential operator of first order without constant term. There is an analogous statement for differential operators of order two:

Proposition: *Consider a Riemannian manifold (X, g) and a linear operator H which is densely defined on $L^2(X, \mathrm{dvol}_g)$. Then H obeys the seven-term identity (4) if and only if H is a second order differential operator without constant term and with symmetric symbol, i.e.*

$$H = \sum_i \frac{1}{2} \left(D_i \circ \widetilde{D}_i + \widetilde{D}_i \circ D_i \right) + D \tag{5}$$

for derivations D_i, $\widetilde{D}_i$, D.

Proof: That H as in (5) with derivations D_i, $\widetilde{D}_i$, D obeys (4) is an immediate consequence of the Leibniz rule for each of the operators D_i, $\widetilde{D}_i$, D.

Conversely, assume that H obeys (4). With $\mathbb{H} \subset L^2(X, \mathrm{dvol}_g)$ its domain of definition, we consider the family $\{D_f\}_{|f \in \mathbb{H}}$ of linear operators defined by

$$\forall f \in \mathbb{H} \colon \qquad D_f \colon \mathbb{H} \longrightarrow \overline{\mathbb{H}}, \quad D_f(h) := H(fh) - f\,H(h) - h\,H(f).$$

Then the seven-term identity (4) implies the Leibniz rule for each D_f. Since $D_f(h) = D_h(f)$ for all $f, h \in \mathbb{H}$, there are derivations D_i, $\widetilde{D}_i$ such that

$$\forall f, h \in \mathbb{H} \colon \qquad D_f(h) = \sum_i \left(D_i(f)\,\widetilde{D}_i(h) + \widetilde{D}_i(f)\,D_i(h) \right).$$

Now consider the linear operator

$$D := H - \sum_i \frac{1}{2} \left(D_i \circ \widetilde{D}_i + \widetilde{D}_i \circ D_i \right).$$

By construction, for all $f, h \in \mathbb{H}$ we have:

$$
\begin{aligned}
D(fh) &= D_f(h) + f\,H(h) + h\,H(f) \\
&\quad - \sum_i \frac{1}{2} D_i \left(f\,\widetilde{D}_i(h) + h\,\widetilde{D}_i(f) \right) - \sum_i \frac{1}{2} \widetilde{D}_i \left(f\,D_i(h) + h\,D_i(f) \right) \\
&= f\,D(h) + h\,D(f).
\end{aligned}
$$

In other words, D is a derivation, as claimed. □

An operator H as in the above proposition which obeys the seven-term identity and in addition is symmetric with respect to the L^2-scalar product on $\mathbb{H}$ can be brought into the form (3). We will therefore prove that (4) holds in general for a limit of CFTs, according to our definition. This is done in the following section and serves as further evidence for the desired interpretation $(\overline{\mathbb{H}^\infty}, 2H^\infty, \overline{\mathcal{A}^\infty}) = (L^2(X, \mathrm{dvol}_g), \Delta_g^\Phi, C^0(X))$ of limiting data of convergent sequences of CFTs.

4 The seven-term identity

Following remarks of Kontsevich, in this section we show that the rescaled energy operator H^∞ acting on the algebra $\mathcal{A}^\infty$ in a degenerate limit of conformal field theories obeys the seven-term identity (4). As argued above, this implies that H^∞ corresponds to a second order differential operator without constant term which in local coordinates can be brought into the form (3).

Consider a one-parameter family of conformal field theories parametrised by $\varepsilon \in (0, 1)$, which yields a convergent sequence and degenerates as $\varepsilon \to 0$. We assume that all correlation functions are analytic in ε at 0. Let $\varphi_i^\varepsilon \in \mathcal{H}_{h_i^\varepsilon, \bar{h}_i^\varepsilon}^\varepsilon$ represent $\varphi_i \in \mathcal{H}^\infty$, such that the respective eigenvalues $h_i^\varepsilon, \bar{h}_i^\varepsilon \to 0$ for $\varepsilon \to 0$. Consider the four-point functions on the sphere

$$F_{1234}^\varepsilon(z, \bar{z}) := \langle \varphi_1^\varepsilon(\infty) \varphi_2^\varepsilon(1) \varphi_3^\varepsilon(z, \bar{z}) \varphi_4^\varepsilon(0) \rangle^\varepsilon = \sum_i C_{34}^i(\varepsilon) C_{2i}^1(\varepsilon) f_i^\varepsilon(z) \bar{f}_i^\varepsilon(\bar{z}).$$

Here C_{ij}^k are OPE coefficients, $C_{34}^i(\varepsilon) = C^\varepsilon((\psi_k^\varepsilon)^*, \varphi_3^\varepsilon, \varphi_4^\varepsilon)$ etc. for an appropriate basis $\{\psi_i^\varepsilon\}$ of $(\mathcal{H}^{\mathcal{W}})^\varepsilon$ for some W-algebra $\mathcal{W}$, and f_i and $\bar{f}_i$ are the corresponding conformal blocks. Since the family is convergent as $\varepsilon \to 0$ and according to our notion of convergence, this factorisation holds with the same summation in an open neighbourhood of $\varepsilon = 0$, to which we will restrict our discussion in the following. In the limit $\varepsilon \to 0$, the conformal blocks behave as $f_i^\varepsilon(z), \bar{f}_i^\varepsilon(\bar{z}) \to 1$. This fact follows from an analysis of $F_{1234}^\varepsilon(z, \bar{z})$ under the conformal transformation $z \mapsto -\frac{1}{z}$ and is the main step in the proof of commutativity of $\mathcal{A}^\infty$, see [7, Prop. 2.2.4]. The analyticity assumption implies

$$F_{1234}^\varepsilon(z, \bar{z}) = \sum_i C_{34}^i(0) C_{2i}^1(0) + \varepsilon \eta(z, \bar{z}) + O(\varepsilon^2),$$

with

$$\eta(z, \bar{z}) = \sum_i \left(\left(C_{34}^i C_{2i}^1 \right)' + C_{34}^i C_{2i}^1 \left(f_i(z) \right)' + C_{34}^i C_{2i}^1 \left(\bar{f}_i(\bar{z}) \right)' \right) \Big|_{\varepsilon=0},$$

where $(\cdot)' = \frac{\partial}{\partial \varepsilon}(\cdot)$. In particular $\bar{\partial}\partial \eta(z, \bar{z}) = 0$, *i.e.* $\partial \eta$ is a meromorphic function on $\overline{\mathbb{C}}$ with singularities in $\{0, 1, \infty\}$. Hence the sum of residues of $\partial \eta$ in the singularities has to vanish:

$$\sum_{z \in \{0, 1, \infty\}} \mathrm{Res}_z \partial \eta(z) = 0. \tag{6}$$

The residues can be easily determined from the OPE-expansion of the four point functions around the singular points $0, 1, \infty$

$$
F_{1234}(z, \bar{z}) = \left\{
\begin{array}{ll}
\sum_\mu C_{34}^\mu C_{2\mu}^1 z^{h_\mu - h_3 - h_4} \bar{z}^{\bar{h}_\mu - \bar{h}_3 - \bar{h}_4} & z \approx 0 \\
\sum_\mu C_{23}^\mu C_{\mu 4}^1 (1-z)^{h_\mu - h_2 - h_3}(1-\bar{z})^{\bar{h}_\mu - \bar{h}_2 - \bar{h}_3} & z \approx 1 \\
\sum_\mu C_{31}^\mu C_{2\mu}^4 z^{h_1 - h_3 - h_\mu} \bar{z}^{\bar{h}_1 - \bar{h}_3 - \bar{h}_\mu} & \frac{1}{z} \approx 0
\end{array}
\right. .
$$

Hence, the singular part of $\partial\eta$ is given by

$$
(\partial\eta(z))_{\mathrm{sing}} = \left\{
\begin{array}{ll}
\frac{1}{z} \sum_i C_{34}^i C_{2i}^1 \left. (h_i' - h_3' - h_4') \right|_{\varepsilon=0} & z \approx 0 \\
\frac{1}{1-z} \sum_i C_{23}^i C_{i4}^1 \left. (h_i' - h_2' - h_3') \right|_{\varepsilon=0} & z \approx 1 \\
z \sum_i C_{31}^i C_{2i}^4 \left. (h_1' - h_3' - h_i') \right|_{\varepsilon=0} & \frac{1}{z} \approx 0
\end{array}
\right. ,
$$

where the sum reduces to those i with $h_i \to 0$, because for all other i $C_{ab}^i(0) = 0$, $a, b \in \{1, 2, 3, 4\}$. Since $\sum_i C_{34}^i C_{2i}^1|_{\varepsilon=0} = \sum_i C_{23}^i C_{i4}^1|_{\varepsilon=0} = \sum_i C_{31}^i C_{2i}^4|_{\varepsilon=0}$, which is also needed to show commutativity of $\mathcal{A}^\infty$, *c.f.* [7, Lemma 2.2.3], the residue formula (6) turns into

$$
\sum_i \left(\left(C_{34}^i C_{2i}^1 + C_{23}^i C_{i4}^1 + C_{31}^i C_{2i}^4 \right) h_i' \tag{7}
\right.
$$
$$
\left. - C_{34}^i C_{2i}^1 \left(h_1' + h_2' + h_3' + h_4' \right) \right) \Big|_{\varepsilon=0} = 0 .
$$

Because of analyticity, $h_i = \varepsilon h_i'|_{\varepsilon=0} + O(\varepsilon^2)$ with $h_i'|_{\varepsilon=0}$ finite. Therefore, L_0 can be rescaled by $\frac{1}{\varepsilon}$ in the limit $\varepsilon \to 0$ (the rescaling function is $r(\varepsilon) = \varepsilon$), and $h_i'|_{\varepsilon=0} + \overline{h}_i'|_{\varepsilon=0}$ are the eigenvalues of a corresponding rescaled operator H^∞ on $\varphi_i \in \mathbb{H}^\infty$. Equation (7) together with $\sum_i C_{31}^i C_{2i}^4|_{\varepsilon=0} = \sum_i C_{24}^i C_{3i}^1|_{\varepsilon=0}$ implies the 7-term identity

$$
\varphi_2 H^\infty(\varphi_3\varphi_4) + H^\infty(\varphi_2\varphi_3)\varphi_4 + H^\infty(\varphi_4\varphi_2)\varphi_3
$$
$$
- H^\infty(\varphi_2)\varphi_3\varphi_4 - \varphi_2 H^\infty(\varphi_3)\varphi_4 - \varphi_2\varphi_3 H^\infty(\varphi_4) - H^\infty(\varphi_2\varphi_3\varphi_4) = 0,
$$

which holds for all $\varphi_2, \varphi_3, \varphi_4 \in \mathbb{H}^\infty$ and by misuse of notation which identifies φ_i with $A_{\varphi_i} \in \mathcal{A}^\infty$ via $\varphi_i = A_{\varphi_i}(\Omega)$.

5 Conclusions and outlook

In this note, we have summarised the basic ideas behind our work [7], which establishes a new intrinsic notion of limiting processes for unitary CFTs and of degenerate limits. We show how our definition allows to associate geometric interpretations to degenerate limits, by applying techniques from noncommutative geometry, as in [4]. Our definitions ensure that the degenerate limits carry a structure similar to that predicted by Kontsevich/Soibelman in the context of mirror symmetry on tori [6].

The note also contains some new results. For the large level limit of unitary Virasoro minimal models of A-type we had already obtained a sensible geometric interpretation in [7]. We are now able to explain how this result is in accord with predictions in [4], obtained by means of the coset construction for

these models along with general techniques developed for gauged WZW models. Moreover, following ideas of Maxim Kontsevich, we include a discussion of the so-called seven-term identity which we show to be satisfied by the rescaled energy operator in any degenerate limit of CFTs. As we discuss, this identity can be viewed as an analogue of the Leibniz rule, characterising second order differential operators without constant term and with symmetric symbol. This allows us to interpret the energy operator in the limit as a generalised Laplacian, a property which had to be assumed separately in [7].

There are many interesting open questions which should be addressed in the context of degeneration phenomena in CFT. One promising route is the establishment of a larger class of examples for which our techniques can be proved to work explicitly. E.g. for the D-series of unitary Virasoro minimal models one obtains the expected result, as we shall show elsewhere. Certainly, a generalisation of our large level limits to arbitrary WZW models and their cosets should be possible and worthwhile. Another good direction is the inclusion of supersymmetry into the picture. The expected structure of degenerate limits should include a topological space with a Riemannian metric, a dilaton, and a complex structure along with appropriate Dirac operators. Thus the inclusion of supersymmetry is expected to yield two advantages: On the geometric side, we expect to find precisely those data which enter in Connes' noncommutative geometry. On the CFT side, supersymmetry tends to dramatically simplify the structure of the relevant representation theory. These ideas shall be addressed in future work.

Acknowledgments

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