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Partial Order Semantics and Read Arcs

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Abstract

We study a new partial order semantics of Petri nets with read arcs, where read arcs model reading without consuming, which is often more adequate than the destructive-read-and-rewrite modelled in ordinary nets. As basic observations we take ST-traces, which are sequences of transition starts and ends. We define processes of our nets and derive two partial orders modelling causality and start precedence. These partial orders are related to observations and system states just as in the ordinary approach the single partial order of a process is related to firing sequences and reachable markings. Our approach also supports a new view of concurrency as captured by steps.

1 Introduction

Describing the runs of a concurrent system by sequences of actions ignores the possible concurrency of these actions, which can be important e.g. for judging the temporal efficiency of the system. Alternatively to this so-called interleaving approach, one can take step sequences, where a step consists of simultaneous actions, or partial orders to describe runs – resulting in a so-called 'true concurrency' semantics. We will use safe Petri nets to model concurrent systems; for these models, the most prominent partial order semantics are so-called processes. A process of a net N is essentially a very simple net consisting of events (transition firings in N) and conditions (tokens in N produced during the run); the process gives a partial order on these events and conditions.

The beauty of the approach is that operationally defined entities of N can now be derived order-theoretically: Each linearization of the events is a firing sequence of N, and vice versa, each firing sequence of N is a linearization of a unique process. We can view the process as a run and its linearizations as observations of the run; essentially by Szpilrajn's Theorem, we can reconstruct the partial order of the events simply as intersection of the total orders given by all these observations. Furthermore, unordered conditions are

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coexisiting tokens, and each slice (maximal set of unordered conditions) is a reachable marking of N; each reachable marking is a slice of some process and each step is a set of unordered events.

Recently, Petri nets with read arcs have found considerable interest [CH93, JK95, MR95, BG95, BP96]; read arcs – as the lines from s in Figure 1 – describe reading without consuming, e.g. reading in a database; consequently, a and b in N_1 can occur concurrently. In ordinary nets, loops (arcs from a to s and from s to a and similarly for b) would be used instead, which describe a destructive-read-and-rewrite and do not allow concurrency; this is certainly often not adequate. [MR95, JK95, BP96] define processes of nets with read arcs and generalize some of the results listed above, taking step sequences as observations. Whereas in Figure 1 [MR95, BP96] allow a step $\{a, b\}$ only for N_1, [JK95] allows this step also for N_2 and N_3; the reason is that [JK95] views these nets as translations from nets with inhibitor arcs and there these steps are intuitively reasonable if we assume that a and b both start and then end some time later. For read arcs, this intuition does not seem so convincing. Also, an undesirable effect is that in N_3 the step reaches a marking that is not reachable by firing sequences. (Correspondingly, [JK95] allows more processes than [MR95, BP96].)

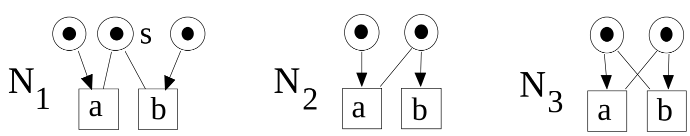

Figure 1

The purpose of the present paper is a partial order semantics under the assumption that activities have durations; consequently, observations of runs are ST-traces [Gla90, Vog92] where we see transitions start and then end. The respective states are ST-markings consisting of marked places and currently firing transitions; hence, ST-markings treat places and transitions on an equal footing just as nets themselves do. An advantage of using ST-traces is that their definition is (hopefully) indisputable: a transition can start if it is enabled; when it starts, it removes a token from each place in the preset and leaves the places in the read set untouched; after the start, it can end and produce a token for each place in the postset. Furthermore, firing and step sequences can be seen as special ST-traces – similarly as firing sequences can be seen as special step sequences; thus, ST-traces give a reference point for a suitable definition of steps for nets with read arcs.

We will show that, for nets with read arcs, the operationally defined ST-traces and ST-markings are interrelated with *spc-structures*, our new partial order semantics, just as in the ordinary approach firing sequences and reachable markings are interrelated with the classical partial order semantics as described above.

If transitions start and end, we have the following phenomenon in N_2 above: when a starts, b remains enabled and can start during the occurrence of a; thus, a and b overlap in time and $\{a, b\}$ is observably a step; note that for a and b both to occur, *a has to start* before b. This view allows more concurrency than that of [MR95, BP96]. In fact, in the latter approach each net with read arcs can be translated to an ordinary net with the same partial order semantics. Such a translation does not exist for N_2 in our setting; $\{a, b\}$ is a step of N_2 but ba is not a firing sequence; this is impossible for ordinary nets. Hence, read arcs really make a difference in our approach, see also [Vog96a]. On the other hand, if in N_3 one of a and b starts, the other is disabled; in general, our approach is a conservative extension of the ordinary setting since steps only reach markings that are also reachable by firing sequences.

Our processes are the same as those in [MR95], but the relational structures we derive from them are new; our spc-structures have two partial orders $\prec$ and $\sqsubset$ modelling causality and start precedence: $e \prec f$ means that e necessarily ends before f starts (causality), while $e \sqsubset f$ means that e necessarily starts before f starts – that this is important is demonstrated by a and b in N_2 above.

In Section 2, we define ST-traces, firing and step sequences for nets with read arcs and relate them to each other. Section 3 studies spc-structures: General spc-structures model general partial-order runs, while sequences, step sequences and ST-traces can be identified with special spc-structures. Thus, analogously to partial orders for ordinary nets, spc-structures give a framework for a variety of behaviour descriptions in the interleaving–'true concurrency' spectrum for nets with read arcs. The main result of this section is a suitable analogue of Szpilrajn's Theorem: each spc-structure is (essentially) the intersection of its so-called ST-linearizations. (Other generalizations of Szpilrajn's Theorem can be found in [JK93], but these cannot be applied here.) In Section 4, we define processes and the spc-structures they induce, and we show: Each order-theoretically derived ST-linearization of a process of some net N is an ST-trace of N; each cut (maximal causally unordered set of events and conditions) is an ST-marking reached along such an ST-trace. Vice versa, for each ST-trace of N we can construct a unique corresponding process, each reachable ST-marking is a cut of some process and each step corresponds to a set of causally unordered events in some process. For ordinary nets without read arcs, our spc-structures coincide with the ordinary partial order semantics based on processes; our results are also of interest in this case, since they study the relation of ST-traces and ST-markings to processes; this is a refinement of the usual results since, as mentioned above, ST-traces generalize step and firing sequences. Finally, we also have a look at so-called lines.

For the results on cuts, it is important that the spc-structures are defined on events and conditions. [JK95] also derives from a process a relational structure with two relations, but these are only defined on events, and they aim at step sequences; consequently, neither the ST-markings nor the ST-traces of a net can be obtained. The paper closes with a more detailed comparison to the existing approaches in Section 5.

2 Petri nets, read arcs, steps and ST-traces

In this section, we introduce safe Petri nets (place/transition-nets) with read arcs, also called positive contexts [MR95], test arcs [CH93] or activator arcs [JK95]. In particular, we will discuss what a suitable notion of step is for such nets, and we will introduce ST-traces which are useful to describe runs where activities have a duration. For general information on ordinary Petri nets, the reader is referred to e.g. [Pet81, Rei85].

We start with some relational notions: a (binary) *relation on* a finite set X is some $R \subseteq X \times X$; we often write xRy in lieu of $(x,y) \in R$ – or sometimes $xy \in R$ if we view R as the directed edges of a graph with vertex set X. Composition of relations on X is defined by $R \circ S = \{(x,z) \mid \exists y \in X : (x,y) \in R \wedge (y,z) \in S\}$; with this notation, R is transitive iff $R \circ R \subseteq R$. We assume that $\circ$ binds stronger than $\cup$, thus e.g. $R \circ (S \cup T) = R \circ S \cup R \circ T$. We write R^+ and R^* for the transitive and the reflexive-transitive closure of R, and R^{-1} for its inverse. If a relation is written $\prec$ or $\sqsubset$, we write $x \preceq y$ for $x \prec y \vee x = y$ and $x \sqsubseteq y$ for $x \sqsubset y \vee x = y$ as usual. Thus, transitivity of $\prec$ means that $\preceq \circ \prec \; = \; \prec \; = \; \prec \circ \preceq$.

Assume $\prec$ is a *partial order* on X, i.e. it is irreflexive and transitive. A *linearization* of $\prec$ is a sequence containing each element of X once such that x occurs before y whenever $x \prec y$; if we speak of a linearization of a set without mentioning a partial order, then we assume the empty partial order. We write $x \; co_{\prec} \; y$ if neither $x \prec y$ nor $y \prec x$. $Y \subseteq X$ is a *$co_{\prec}$-set* if $x \; co_{\prec} \; y$ for all $x, y \in Y$. The set of the *$\prec$-maximal* elements in $Y \subseteq X$ is $max_{\prec}(Y) = \{y \in Y \mid y \prec x \text{ for no } x \in Y\}$; $min_{\prec}(Y)$ is defined analogously. We call Y *left-closed under* $\prec$, if $x \prec y \in Y$ implies $x \in Y$.

A *Petri net with read arcs* $N = (S, T, W, R, M_N)$ (or just a *net* for short) consists of finite disjoint sets S of *places* and T of *transitions*, the (ordinary) *arcs* $W \subseteq S \times T \cup T \times S$ (which all have weight 1), the set of *read arcs* $R \subseteq S \times T$, and the *initial marking* $M_N : S \to \{0,1\}$; we always assume $(R \cup R^{-1}) \cap W = \emptyset$. When we introduce a net N or N_1 etc., then we assume that implicitly this introduces its components S, T, W, ... or S_1, T_1, ..., etc. and similarly for other tuples later on. In general, we will not distinguish isomorphic nets (nor isomorphic partial orders etc.). The tuple (S, T, W, R) is called a *net graph*. A net is called *ordinary*, if $R = \emptyset$.

As usual, we draw transitions as boxes, places as circles and arcs as arrows; read arcs are drawn as lines without arrow heads.

For each $x \in S \cup T$, the *preset* of x is ${}^\bullet x = \{y \mid (y,x) \in W\}$, the *read set* of x is $\hat{x} = \{y \mid (y,x) \in R \cup R^{-1}\}$, and the *postset* of x is $x^\bullet = \{y \mid (x,y) \in W\}$. These notions are extended pointwise to sets, e.g. ${}^\bullet X = \bigcup_{x \in X} {}^\bullet x$. If $x \in {}^\bullet y \cap y^\bullet$, then x and y form a *loop*. A *marking* is a function $S \to I\!N_0$. We sometimes regard sets as characteristic functions, which map the elements of the sets to 1 and are 0 everywhere else; hence, we can e.g. add a marking and a postset of a transition or compare them componentwise. Vice versa, a function with images in $\{0,1\}$ is sometimes regarded as a set such that we can e.g. apply union to it.

We now define the basic firing rule, which extends the firing rule for ordinary nets by regarding the read arcs as loops.

- A transition t is *enabled* under a marking M, denoted by $M[t\rangle$, if ${}^\bullet t \cup \hat{t} \leq M$.

  If $M[t\rangle$ and $M' = M + t^\bullet - {}^\bullet t$, then we denote this by $M[t\rangle M'$ and say that t can *occur* or *fire* under M yielding the marking M'. Thus, when t fires, it checks its pre- and read-set, removes a token from each place in its preset and puts a token onto each place in its postset.

- This definition of enabling and occurrence can be extended to sequences as usual: a sequence w of transitions is *enabled* under a marking M, denoted by $M[w\rangle$, and yields the follower marking M' when *occurring*, denoted by $M[w\rangle M'$, if $w = \lambda$ and $M = M'$ or $w = w't$, $M[w'\rangle M''$ and $M''[t\rangle M'$ for some marking M''. If w is enabled under the initial marking, then it is called a *firing sequence*.

A marking M is called *reachable* if $\exists w \in T^* : M_N[w\rangle M$. The net is *safe* if $M(s) \leq 1$ for all places s and reachable markings M.

General assumption All nets considered in this paper are safe and T-restricted, i.e. each transition has a nonempty preset and a nonempty postset (where we sometimes omit the postsets in figures).

Now we will define ST-traces, see e.g. [Gla90, Vog92], a suitable behaviour notion if we assume that the firing of a transition takes time. (Using ST-traces and partial orders, [Vog95] studies durational transitions for ordinary nets.) The key idea is that the firing of a transition t consists of a beginning t^+ and an end t^-; t^+ checks the enabledness of t and consumes the input of t, and t^- produces the output. We will need the following general notions, where the notion ST-sequence will not be applied to transitions, but – in the next section – to events, i.e. transition firings.

- For a finite set X, $X^{\pm}$ denotes the union of two disjoint copies of X; for $x \in X$, the copies of x are denoted by x^+, called the *start* of x, and x^-, the *end* of x. A sequence over $X^{\pm}$ is *closed*, if it contains each x^+ as often as the respective x^-.

  An *ST-sequence* over X is a sequence containing each x^+ once and each x^- at most once and only after the corresponding x^+. It is closed, if it contains each x^- once.

If transitions have a beginning and an end, a system state cannot adequately be described by a marking alone; instead, it consists of a marking together with some transitions that have started, but have not finished yet. We call such a system state an ST-marking (S = Stellen, T = Transitionen (German)); ST-markings were introduced in [GV87] in a slightly different version.

- An *ST-marking* of a net N is a pair $Q = (M, C)$, where M is a marking of N and $C \subseteq T$; C is the set of *currently firing* transitions. The *initial ST-marking* is $Q_N = (M_N, \emptyset)$.

- The elements of $T^{\pm}$ are called *transition parts*. For an ST-marking $Q = (M, C)$, a transition start t^+ is *enabled* under Q, $Q[t^+\rangle$, if $M[t\rangle$; a transition end t^- is *enabled* under Q, $Q[t^-\rangle$, if $t \in C$. *Firing* yields a *follower ST-marking* given by $Q[t^+\rangle(M - {}^\bullet t, C \cup \{t\})$ and $Q[t^-\rangle(M + t^\bullet, C - \{t\})$.

- We extend this definition to sequences, and if we have $Q_N[w\rangle Q$ for a sequence w of transition parts, then w is an *ST-trace* and Q a *reachable* ST-marking of N.

We have the following observations, which show in particular that ST-traces are a fairly conservative, refined version of firing sequences; in particular, i) shows that we can view a firing sequence as a special ST-trace. Observe that by the last part of ii), it is adequate to consider a set (instead of a multiset) of currently firing transitions.

Proposition 2.1 *Let N be a net.*

i) For a reachable marking M and transitions $t_1, \ldots, t_n$, we have $M[t_1\rangle M_1 \ldots [t_n\rangle M_n$ iff $(M,\emptyset)[t_1^+t_1^-\rangle(M_1,\emptyset)\ldots[t_n^+t_n^-\rangle(M_n,\emptyset)$.

ii) If (M,C) is a reachable ST-marking, then $M+\sum_{t\in C} t^\bullet$ is a reachable marking; in particular, we have $M \cap t^{\bullet\prime} = \emptyset$ for $t' \in C$, and $(M,C)[t'^+\rangle$ implies $t' \notin C$.

iii) A marking M is reachable iff $(M,\emptyset)$ is a reachable ST-marking.

iv) If w is an ST-trace, then t^+ and t^- occur alternatingly in w starting with t^+ for each $t \in T$. If w' is obtained from w by moving some t^- to an earlier position that is still after the preceding t^+, then w' is an ST-trace as well and reaches the same ST-marking. (In particular, w' could be obtained by replacing some $t_1^-t_2^-$ in w by $t_2^-t_1^-$.)

Proof: Part i) is obvious. Part ii) can be shown by induction on the length of the respective ST-trace using the safety of N, see [Vog92] for details (in the case that N is ordinary); to see why $t' \notin C$, assume to the contrary and take some $s \in t'^\bullet$: since $M + \sum_{t\in C} t^\bullet$ is reachable and has one token on s, we have $M(s) = 0$ and thus $s \notin {}^\bullet t'$ by $M[t'^+\rangle$; firing t' under $M+\sum_{t\in C} t^\bullet$ violates the safety for s. Now iii) follows from i) and ii).

To prove the first statement of iv), we apply the last statement of ii). To see the second statement, observe that along w' we simply have more tokens than along w since they are produced earlier, but at the end the same tokens have been produced and consumed in w and w'. □

While the definitions of firing sequence and ST-trace are quite unquestionable, there are at least two different definitions of a step for nets with read arcs, and we define a third one. Our notion is more general than the one of [MR95]; it is more restrictive than the one of [JK95] and also more conservative, because our steps only reach markings that are reachable by firing sequences as well.

A step is meant to be a set of transitions that can fire concurrently. We have already argued in the introduction that, in N_2 shown in Figure 1, firing a does not disable b, since a does not take away any tokens needed by b; therefore, b should be able to fire concurrently to a and $\{a, b\}$ should be a step. Generalizing this idea, we get the following definition.

Definition 2.2 A transition t of a net N can *fire concurrently* to a set $G \subseteq T$ under a marking M, if $(M - {}^\bullet G)[t\rangle$. A set G with $\emptyset \neq G \subseteq T$ is a *step enabled under* a marking M if for some linearization $t_1t_2\ldots t_n$ of G (equipped with the empty partial order) we have for $i = 1, \ldots, n$ that t_i can fire concurrently to $\{t_1, \ldots, t_{i-1}\}$ under M; $t_1 \ldots t_n$ is a *generation ordering* for G under M. The marking M' reached by firing G is $M - {}^\bullet G + G^\bullet$ and we write $M[G\rangle M'$; this is generalized to *step sequences* as usual. □

In the net N_2 above, $\{a, b\}$ is a step, but ba is not a firing sequence. This cannot happen with the usual definition of a step for ordinary nets. This shows that, with our step definition, nets with read arcs cannot be simulated by ordinary nets – in contrast with results in [CH93, MR95]. That our definition is nevertheless a conservative extension is demonstrated in the following theorem in parts iv) and v) (and in Corollary 2.6 below). Part ii) establishes that steps and, thus, step sequences can be seen as special ST-traces; part iii) shows that steps are sets of transitions that can appear as currently firing (and thus concurrent) transitions in reachable ST-markings. Part i) shows that steps should be sets and not multisets in our setting of safe nets.

Theorem 2.3 *Let N be a net and M be a reachable marking.*

i) If t can fire concurrently to G under M, then $t \notin G$.

ii) Let $G \neq \emptyset$ have a linearization $t_1 \dots t_n$. Then G is a step under M with generation ordering $t_1 \dots t_n$ iff $(M, \emptyset)[t_1^+ \dots t_n^+\rangle$. In this case, ${}^\bullet G \cup \hat{G} \subseteq M$ and the ${}^\bullet t_i$ are disjoint; furthermore for M' with $M[G\rangle M'$, we have $(M, \emptyset)[t_1^+ \dots t_n^+\rangle(M - {}^\bullet G, G)[t_1^- \dots t_n^-\rangle(M', \emptyset)$.

iii) If G is a step under M, then there exists a reachable ST-marking (M', G).

iv) If G is a step with $M[G\rangle M'$ and generation ordering w, then $M[w\rangle M'$.

v) The markings reachable by step sequences are exactly the reachable markings.

Proof: i) If $t \in G$ and $s \in {}^\bullet t$, then s is empty under $M - {}^\bullet G$.

ii) For the following, observe that $M(s) \leq 1$ for all $s \in S$ by safety of N. G is a step as required iff, for $i = 1, \dots, n$, we have ${}^\bullet t_i \cup \hat{t_i} \subseteq M - {}^\bullet\{t_1, \dots, t_{i-1}\}$. This implies the desired inclusion and disjointness, and it is equivalent to $(M - {}^\bullet\{t_1, \dots, t_{i-1}\}, \{t_1, \dots, t_{i-1}\})[t_i^+\rangle(M - {}^\bullet\{t_1, \dots, t_i\}, \{t_1, \dots, t_i\})$ for $i = 1, \dots, n$, which in turn is equivalent to $(M, \emptyset)[t_1^+ \dots t_n^+\rangle$. The ST-marking reached after this sequence is $(M - {}^\bullet G, G)$ and obviously $(M - {}^\bullet G, G)[t_1^- \dots t_n^-\rangle(M', \emptyset)$.

iii) follows from ii).

iv) If $w = t_1 \dots t_n$, then by ii) $(M, \emptyset)[t_1^+ \dots t_n^+ t_1^- \dots t_n^-\rangle(M', \emptyset)$. Now we can rearrange $t_1^+ \dots t_n^+ t_1^- \dots t_n^-$ to $t_1^+ t_1^- \dots t_n^+ t_n^-$ reaching $(M', \emptyset)$ by Proposition 2.1 iv), and 2.1 i) implies $M[w\rangle M'$.

v) Follows from iv); observe that firing sequences can be seen as special step sequences. □

An interesting question is whether a converse of iii) holds, i.e. whether C is a step whenever (M, C) is a reachable ST-marking. In ordinary nets, this is the case; but in the net N of Figure 2, we can start a, fire b and then start c reaching $(\emptyset, \{a, c\})$; no reachable marking M exists where we can fire the step $\{a, c\}$, i.e. where $(M, \emptyset)[a^+c^+\rangle$ or $(M, \emptyset)[c^+a^+\rangle$ – compare ii) above.

The definition of a step requires a suitable linearization; the next theorem describes how such a linearization can be found, and it prepares our partial order approach. Observe that in N_2 of Figure 1, a has to start before b because b takes the token a has to read, i.e. because $a(R_2^{-1} \circ W_2)b$.

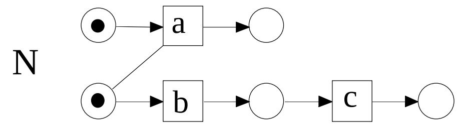

Figure 2

Theorem 2.4 *Let N be a net, M a reachable marking and $\emptyset \neq G \subseteq T$. Then G is a step under M if and only if*

i) $^\bullet G \cup \hat{G} \subseteq M$

ii) For all $t_i, t_j \in G$, $t_i \neq t_j$ implies $^\bullet t_i \cap {}^\bullet t_j = \emptyset$

iii) The relation $R^{-1} \circ W$ is acyclic on G, i.e. $(R^{-1} \circ W)^+$ is irreflexive and thus a partial order.

If G is a step, then the linearizations of $(R^{-1} \circ W)^+$ on G are exactly the generation orderings.

Proof: Let G be a step under M with generation ordering $t_1 \dots t_n$. Conditions i) and ii) are satisfied by Theorem 2.3 ii). For $i < j$ we have $\hat{t_j} \subseteq M - {}^\bullet\{t_1, \dots, t_{j-1}\}$, i.e. $^\bullet t_i \cap \hat{t_j} = \emptyset$. Thus, we can only have $t_k (R^{-1} \circ W)^+ t_l$ if $k < l$, hence iii); it also implies that a generation ordering must be a linearization of $(R^{-1} \circ W)^+$ on G.

Now assume that G satisfies i)-iii) and $t_1 \dots t_n$ is a linearization of $(R^{-1} \circ W)^+$ on G, which exists by iii). We are done once we have shown that $t_1 \dots t_n$ is a generation ordering for G under M. We show $^\bullet t_i \cup \hat{t_i} \subseteq M - {}^\bullet\{t_1, \dots, t_{i-1}\}$ for $i = 1, \dots, n$. Since $^\bullet G \subseteq M$, ii) implies $^\bullet t_i \subseteq M - {}^\bullet\{t_1, \dots, t_{i-1}\}$. If we had $s \in \hat{t_i}$ and $s \notin M - {}^\bullet\{t_1, \dots, t_{i-1}\}$, then by $\hat{G} \subseteq M$ we would find some $j < i$ with $s \in {}^\bullet t_j$, i.e. $t_i (R^{-1} \circ W) t_j$ and $t_1 \dots t_n$ is not a linearization as required. □

Corollary 2.5 *Let N be a net, G a step under a reachable marking M, and $\emptyset \neq G' \subseteq G$. Then G' is a step under M.*

The last corollary of this section shows that our definition of a step restricted to ordinary nets coincides with the usual definition.

Corollary 2.6 *Let N be an ordinary net, $\emptyset \neq G \subseteq T$, and M be a reachable marking. Then G is a step under M iff for all $t, t' \in G$ we have $M[t\rangle$ and $t \neq t' \Rightarrow {}^\bullet t \cap {}^\bullet t' = \emptyset$; in this case, $M[w\rangle$ for each linearization w of G.*

Proof: Follows from Theorem 2.4 since $(R^{-1} \circ W)^+$ is empty. □

3 Structures for causality and start precedences

Usually, a partial order description of a system run is a set of events (and possibly conditions) ordered by some partial order $\prec$, where $\prec$ models causality; i.e. for events e and f, $e \prec f$ means that e necessarily ends before f starts. As argued in the introduction, we also have to consider for some events e and f that e necessarily *starts* before f starts; we will write $e \sqsubset f$ in this case. It is clear that $\sqsubset$ should be a partial order, too. Furthermore, if e ends before f starts, then it also starts before f; finally, if e ends before f starts and f starts before g starts, then e ends before g starts. Hence, $\prec$ and $\sqsubset$ should satisfy the requirements of the following definition.

Definition 3.1 An *spc-order* $p = (E, \prec, \sqsubset)$ consists of a finite set E, whose elements we call *events* in this section, and two partial orders $\prec$ and $\sqsubset$ on E such that

i) $\prec \subseteq \sqsubset$

ii) $\prec \circ \sqsubset \subseteq \prec$ (i.e. $e \prec f \sqsubset g$ implies $e \prec g$ for all $e, f, g \in E$) or equivalently $\prec \circ \sqsubseteq = \prec$

An *spc-structure* is a labelled spc-order $p = (E, \prec, \sqsubset, l)$ where $(E, \prec, \sqsubset)$ is an spc-order and $l : E \to X$ some function, the *labelling*, and such that $e \; co_{\prec} \; f \wedge e \neq f$ implies $l(e) \neq l(f)$ for all $e, f \in E$.

By this label requirement, the events with a given label x are totally ordered by $\prec$ and we can speak of the i-th event with label x; p is *canoncial*, if $E \subseteq X \times I\!N$ and each $(x, i) \in E$ is the i-th event with label x. □

We will see that the requirements for spc-orders are complete in the sense that for each spc-order p there exists a run of some net which is modelled using p. In this paper, we will only need labelled spc-orders that satisfy the label requirement. Obviously, each spc-structure is isomorphic to a canonical spc-structure, i.e. we can restrict attention to canonical spc-structures whenever this seems to be an advantage. The next proposition gives some first useful properties.

Proposition 3.2 *Let E be a finite set with a partial order $\sqsubset$ and an arbitrary relation $\prec \subseteq \sqsubset$ satisfying $\prec \circ \sqsubset \subseteq \prec$.*

i) $\prec$ is a partial order, i.e. $(E, \prec, \sqsubset)$ is an spc-order.

ii) If $e \prec f$, then $\neg f \sqsubset e$.

Proof: i) Since $\prec \subseteq \sqsubset$, irreflexivity of $\prec$ is implied by that of $\sqsubset$. Furthermore, $\prec \circ \prec \subseteq \prec \circ \sqsubset \subseteq \prec$.

ii) $e \prec f$ implies $e \sqsubset f$ and $\sqsubset$ is irreflexive. □

Graphically, we present an spc-order by writing down the events of E and connect e and f by an arrow if $e \prec f$ and by a dashed arrow if $e \sqsubset f$. (For spc-structures, we replace the events of E by their labels.) Arrows implied by Definition 3.1 i) and ii) are often omitted, in particular we never draw an ordinary *and* a dashed arrow from e to f. If the arrows of such a drawing seen as arcs of a directed graph are acyclic, then the drawing represents an spc-order, which is described in the next proposition.

Proposition 3.3 *Let E be a finite set with relations R_1 and R_2 such that $R_1 \cup R_2$ is acyclic, i.e. $\sqsubset := (R_1 \cup R_2)^+$ is irreflexive. Then $p = (E, \prec, \sqsubset)$ is an spc-order, where $\prec$ is defined as $R_1 \circ \sqsubseteq$.*

For all spc-orders p' with $R_1 \subseteq \prec'$ and $R_2 \subseteq \sqsubset'$, we have $\prec \subseteq \prec'$ and $\sqsubset \subseteq \sqsubset'$.

Proof: Since $\sqsubset$ is a partial order, we only have to show $\prec \subseteq \sqsubset$ and $\prec \circ \sqsubset \subseteq \prec$ and apply 3.2 i) to get the first claim. Since $R_1 \subseteq \sqsubset$, we get $\prec = R_1 \circ \sqsubseteq \subseteq \sqsubset \circ \sqsubseteq = \sqsubset$. Furthermore, $\prec \circ \sqsubset = R_1 \circ \sqsubseteq \circ \sqsubset = R_1 \circ \sqsubset \subseteq \prec$.

For the second claim, observe that $R_2 \subseteq \sqsubset'$ and $R_1 \subseteq \prec' \subseteq \sqsubset'$ implies $\sqsubset = (R_1 \cup R_2)^+ \subseteq \sqsubset'$ by transitivity of $\sqsubset'$. Hence, $\prec = R_1 \circ \sqsubseteq \subseteq \prec' \circ \sqsubseteq' \subseteq \prec'$. □

If we regard the ordinary arrows of an acyclic drawing as discussed above as R_1 and the dashed arrows as R_2, then the p of this proposition contains just all the orderings implied by the arrows; we call p the spc-order *induced* by the arrows.

Definition 3.4 Let E be a finite set with relations R_1 and R_2 such that $R_1 \cup R_2$ is acyclic; then we call the spc-order p according to Proposition 3.3 *induced* by R_1 and R_2. □

From a partial order, we can derive its augmentations (or extensions) to total orders; total orders obviously represent sequences and vice versa; the derived sequences are called linearizations. Similarly, one can order-theoretically define the derived step-sequences. This shows that various behaviour descriptions in the interleaving–'true concurrency' spectrum can be studied in the partial-order framework. From the set of derived sequences, one can reconstruct the partial order as the intersection of the respective total orders.

In the case of spc-orders, we will analogously define which spc-orders correspond to sequences, step-sequences and ST-sequences; then, from a given spc-order, we can again derive sequences etc. order-theoretically as augmentations. Finally, as the main result of this section, we show that an spc-order can be reconstructed from the collection of its corresponding ST-sequences. First, we identify the spc-orders that correspond to sequences, step sequences and – more or less – to ST-sequences.

Definition 3.5 Let p be an spc-order. Then, p is an *spc-sequence*, if $\prec$ is a total order; the obvious linearization w of E is the *corresponding sequence*. If $co_\prec$ is an equivalence relation, p is an *spc-step-sequence*; the obvious sequence w of the equivalence classes ordered according to $\prec$ is the *corresponding step-sequence*. For an spc-structure p, p is analogously an *spc-trace* or an *spc-step-trace*; if w is the corresponding (step) sequence of $(E, \prec, \sqsubset)$, then replacing each $e \in E$ in w by its label gives the *corresponding trace* or *step-trace* with w as *underlying* sequence or step-sequence.

Finally, if $\sqsubset$ is total, p is an *interval-spc-order*. An ST-sequence w over E is a *corresponding ST-sequence* if it satisfies for all $e, f \in E$: e^+ occurs before f^+ if and only if $e \sqsubset f$; e^- occurs before f^+ if and only if $e \prec f$. As above, we derive from this the definitions of *interval-spc-structure*, *corresponding ST-trace* (a sequence over $X^\pm$) and *underlying* ST-sequence. □

The definitions of the first part are straightforward generalizations from the case of partial orders. Note that the labelling l of an spc-structure is injective on the equivalence classes of $co_{\prec}$ by the label requirement; hence, the corresponding step trace is a sequence of sets (and not multisets).

The second part needs more explanations. A partial order $\prec$ on E is an *interval order*, if for all $e, e', f, f' \in E$ we have: if $e \prec e'$ and $f \prec f'$, then $e \prec f'$ or $f \prec e'$; in this case, we can associate each $e \in E$ with an interval of real numbers such that $e \prec f$ iff the interval of e lies completely before that of f; a basic reference for interval orders is [Fis85, Chapter 2]. The following result explains the name interval-spc-order.

Proposition 3.6 *If p is an interval-spc-order, then $\prec$ is an interval order.*

Proof: Let $e \prec e'$ and $f \prec f'$. If $\{e, e'\} \cap \{f, f'\} \neq \emptyset$, then we are done. Otherwise, we have $e' \sqsubset f'$ without loss of generality, which together with $e \prec e'$ implies $e \prec f'$. □

Different from the case of sequences and step-sequences, an interval-spc-order does not have a unique corresponding ST-sequence, but a set of such sequences. The next result shows that these sequences coincide up to simple modifications; by definition, each of the sequences allows to reconstruct the interval-spc-order, i.e. an interval-spc-order is a simple abstraction of an ST-sequence.

Proposition 3.7 *Let p be an interval-spc-order and I the set of its corresponding ST-sequences.*

i) There exists a closed w in I.

ii) I is the set of sequences v that can be obtained from w by repeatedly replacing some $e^- f^-$ by $f^- e^-$ and deleting some e^- at the end of w.

Proof: i) Assume $E = \{e_1, \dots, e_n\}$ and $e_1 \sqsubset e_2 \sqsubset \dots e_n$; thus, w contains $e_1^+, \dots, e_n^+$ in this order. For $i \in \{1, \dots, n\}$ we have that $e_i \prec e_j \sqsubset e_k$ implies $e_i \prec e_k$; hence, $\{j \mid e_i \prec e_j\}$ is some set $\{l+1, \dots, n\}$ with $l \in \{1, \dots, n\}$. We simply have to insert e_i^- somewhere after the respective e_l^+ and, for $l \neq n$, before e_{l+1}^+.

ii) This is clear from the way w has to be constructed in i). □

This proposition immediately carries over to interval-spc-structures, as stated in Corollary 3.9 below. The following lemma tells us that from a corresponding ST-trace of a canonical interval-spc-structure we can determine the underlying ST-sequence; hence, we can additionally reconstruct such a structure from each of its ST-traces by definition of a corresponding ST-sequence.

Lemma 3.8 *Let p be a canonical interval-spc-structure and w be a corresponding ST-trace, x a label. Then, x^+ and x^- alternate in w starting with x^+, and the i-th x^+ and x^- correspond to $(x, i)^+$ and $(x, i)^-$ in the underlying ST-sequence.*

Proof: By definition of an ST-sequence, $(x, i)^+$ occurs before $(x, i)^-$, which occurs before $(x, i+1)^+$ since $(x, i) \prec (x, i+1)$. □

Corollary 3.9 *Let p be a canonical interval-spc-structure and I the set of its corresponding ST-traces.*

i) There exists a closed w in I.

ii) I is the set of sequences v that can be obtained from w by repeatedly replacing some x^-y^- by y^-x^- and deleting some x^- at the end of w.

iii) For an arbitrary $v \in I$, put $(x,i) \prec' (y,j)$ if the i-th x^- occurs in v before the j-th y^+ and put $(x,i) \sqsubset' (y,j)$ if the i-th x^+ occurs in v before the j-th y^+. Then $\prec = \prec'$ and $\sqsubset = \sqsubset'$.

This corollary shows that an interval-spc-structure corresponds to a set of closely related ST-traces, and that it can be reconstructed from each of these up to isomorphism. Thus, interval-spc-structures are a moderate abstraction of ST-traces. Observe that this abstraction is compatible with the application to nets: if w in the above corollary is an ST-trace of a net, then the modifications v according to ii) are also ST-traces of the net by Proposition 2.1 iv) – independently of the net.

Now we will describe how we can order-theoretically derive sequences etc. from an arbitrary spc-order.

Definition 3.10 An spc-order $p' = (E, \prec', \sqsubset')$ is an *augmentation* of an spc-order p, if $\prec \subseteq \prec'$ and $\sqsubset \subseteq \sqsubset'$. If p' is additionally an spc-sequence, an spc-step-sequence or an interval-spc-order, then it is called a *linear*, *step* or *interval augmentation.*

A *linearization* of p is the corresponding sequence of a linear augmentation of p. Analogously, a *step linearization* and an *ST-linearization* correspond to a step and an interval augmentation of p.

This definition carries over to spc-structures; note that augmenting $\prec$ cannot violate the label requirement. Linearizations etc. are analogously defined as corresponding traces, step traces and ST-traces with underlying sequences as in Definition 3.5. □

The following theorem shows how to read off the ST-linearizations etc. directly; this demonstrates how $\prec$ and $\sqsubset$ describe relationships between starts and ends of the events in E.

Theorem 3.11 *Let p be an spc-order.*

i) w is an ST-linearization of p iff it is an ST-sequence over E such that e^+ occurs before f^+ if $e \sqsubset f$ and e^- occurs before f^+ if $e \prec f$.

ii) w is a linearization of p iff it is a linearization of $(E, \emptyset)$ such that $e \sqsubset f$ implies that e occurs before f.

iii) w is a step linearization of p iff it is a sequence of sets that form a partition of E with the following two properties: $e \prec f$ implies that the set containing e occurs before the set containing f; $e \sqsubset f$ implies that the set containing e does not occur later than the set containing f.

Proof: i) For the only-if case, let w be an ST-linearization and p' be a respective interval augmentation of p. Then $e \sqsubset f$ implies $e \sqsubset' f$ and this implies that e^+ occurs before f^+ in w; the case of $\prec$ is similar.

For the if-case, let w be given with the required properties and define $\prec'$ and $\sqsubset'$ such that they make w a corresponding ST-sequence to $p' = (E, \prec', \sqsubset')$ according to Definition 3.5. We show that p' is an interval-spc-order; then it is clearly an interval-augmentation. First, $\sqsubset'$ is a total order. Now observe that $e \prec' f$ implies that e^+, e^- and f^+ occur in w in this order; thus, $\prec'$ is contained in $\sqsubset'$ and, furthermore, $e \prec' f \sqsubset' g$ implies $e \prec' g$.

ii) Let w be a linearization of $(E, \emptyset)$ and $\prec'$ be the respective total order on E. Then, w is a linearization of p iff $(E, \prec', \prec')$ is an augmentation of p iff $\prec \subseteq \sqsubset \subseteq \prec'$ iff $e \sqsubset f$ (and thus $e \prec f$) implies that e occurs before f.

iii) Let w be a sequence of sets that form a partition of E and $\prec'$ be the respective partial order on E such that the sets are the equivalence classes of $co_{\prec'}$. We observe that

$(*)$ $e \prec' f$ iff the set containing e occurs before the set containing f.

Also, w is a step linearization of p iff

$(**)$ for some $\sqsubset'$ $(E, \prec', \sqsubset')$ is an augmentation of p.

On the one hand, we can conclude from $(**)$ that $e \prec f$ implies $e \prec' f$, hence the first desired ordering of the sets by $(*)$. Furthermore, if the set containing e occurs later than the set containing f, then $f \prec' e$ and $\neg e \sqsubset' f$ by Proposition 3.2, which implies $\neg e \sqsubset f$; this shows the second desired implication.

On the other hand, given the two implications for the ordering of sets in w, we define $\sqsubset' = \sqsubset \cup \prec'$. As a union of two irreflexive relations, $\sqsubset'$ is also irreflexive. From the implication for $\sqsubset$ and $(*)$, we conclude that $\sqsubset \circ \prec' \subseteq \prec'$ and $\prec' \circ \sqsubset \subseteq \prec'$. On the one hand, this implies that $\sqsubset' \circ \sqsubset' \subseteq \sqsubset'$, i.e. $\sqsubset'$ is transitive. On the other hand, it also implies $\prec' \circ \sqsubset' \subseteq \prec'$. Clearly, $\prec' \subseteq \sqsubset'$ by definition, hence $p' = (E, \prec', \sqsubset')$ is an spc-order.

Furthermore, $\sqsubset \subseteq \sqsubset'$ by definition. We have that $e \prec f$ implies that the set containing e occurs before the set containing f, which implies $e \prec' f$ by $(*)$. Thus, $(**)$ is satisfied. □

[JK95] also studies relational structures with two relations to describe system runs; these are tuned to obtain a result as Theorem 3.11 iii). We discuss at the end of this section why step sequences are not expressive enough for some purposes.

Theorem 3.11 also tells us how to read off the ST-linearizations etc. of an spc-structure p: we simply read off the ST-linearizations of the spc-order $(E, \prec, \sqsubset)$ and apply the labelling. The next theorem implies that ST-linearizations are all we need, since they have (more or less) linearizations and step linearizations as special cases.

Theorem 3.12 *a) Let p be an spc-order, $e_i \in E$ and $\emptyset \neq E_i \subseteq E$ for $i = 1, \ldots, n$.*

i) $e_1 \ldots e_n$ is a linearization of p iff $e_1^+ e_1^- \ldots e_n^+ e_n^-$ is an ST-linearization of p.

ii) $E_1 \ldots E_n$ is a step linearization of p iff for some indexing $E_i = \{e_{i1}, \ldots, e_{im_i}\}$ for $i = 1, \ldots, n$ and $e_{11}^+ \ldots e_{1m_1}^+ e_{11}^- \ldots e_{1m_1}^- \ldots e_{n1}^- \ldots e_{nm_n}^-$ is an ST-linearization of p.

b) Let p be an spc-structure, $x_i \in X$ and $\emptyset \neq X_i \subseteq X$ for $i = 1, \ldots, n$. Then:

i) $x_1 \ldots x_n$ is a linearization of p iff $x_1^+ x_1^- \ldots x_n^+ x_n^-$ is an ST-linearization of p.

ii) $X_1 \ldots X_n$ is a step linearization of p iff for some indexing $X_i = \{x_{i1}, \ldots, x_{im_i}\}$ for $i = 1, \ldots, n$ and $x_{11}^+ \ldots x_{1m_1}^+ x_{11}^- \ldots x_{1m_1}^- \ldots x_{n1}^- \ldots x_{nm_n}^-$ is an ST-linearization of p.

Proof: a) i) $e_1 \ldots e_n$ is a linearization of p iff there exists a linear augmentation p' of p where $e_1 \prec' e_2 \ldots \prec' e_n$. Since in this case $\sqsubset' = \prec'$ is also total, this is equivalent to: $e_1^+ e_1^- \ldots e_n^+ e_n^-$ is an ST-linearization of p.

ii) If $E_1 \ldots E_n$ is a step linearization of p, then for the respective step augmentation p' $\sqsubset'$ is a partial order on each E_i and for all $i < j$, $e \in E_i$ and $f \in E_j$ we have $e \sqsubset' f$. We can augment $\sqsubset'$ to a total order $\sqsubset''$. Then we have for all $e, f, g \in E$ with $e \prec' f \sqsubset'' g$ that $e \in E_i$, $f \in E_j$, $g \in E_k$ for $i < j \leq k$, i.e. $e \prec' g$. Thus, $(E, \prec', \sqsubset'')$ is a step augmentation giving rise to the same step sequence and an interval augmentation.

Thus, $E_1 \ldots E_n$ is a step linearization of p iff there exists a suitable step augmentation p' that is an interval augmentation as well – i.e. for some indexing we have $E_i = \{e_{i1}, \ldots, e_{im_i}\}$ and $e_{i1} \sqsubset' \ldots \sqsubset' e_{im_i}$ for $i = 1, \ldots, n$. This clearly implies that for some indexing $E_i = \{e_{i1}, \ldots, e_{im_i}\}$ for $i = 1, \ldots, n$ and $e_{11}^+ \ldots e_{1m_1}^+ e_{11}^- \ldots e_{1m_1}^- \ldots e_{n1}^- \ldots e_{nm_n}^-$ is an ST-linearization of p. Vice versa, given such an indexing, ST-linearization and suitable interval augmentation p' of p, we see that $e_{ik} \prec' e_{jl}$ iff $i < j$, thus $e_{ik}\ co_{\prec'}\ e_{jl}$ iff $i = j$; therefore p' is also a step augmentation in this case, suitable for $E_1 \ldots E_n$.

b) follows now from a). □

Observe that Theorem 3.12 fits Proposition 2.1 i) and Theorem 2.3 ii): if we have an spc-structure p and a net N such that all ST-linearizations of p are ST-traces of N, then all linearizations (step linearizations) of p are firing sequences (step sequences) of N; vice versa, if we can find for each ST-trace w of N an spc-structure p of a certain type such that w is an ST-linearization of p, then we can also find for each firing sequence or step sequence w of N an spc-structure p of this type such that w is a linearization or step linearization of p. Hence, if we want to study the behaviour of nets using spc-structures, it is enough to relate such spc-structures to ST-traces of nets – the relationship to firing and step sequences is then immediate.

Clearly, for each spc-order p, we can extend $\sqsubset$ to a total order $\sqsubset'$ and put $\prec' := \sqsubset'$; hence, linear augmentations exist, and they are also step and interval augmentations. To construct more interesting interval augmentations we give three lemmata.

Lemma 3.13 *Given an spc-order p and a partial order $\sqsubset'$ with $\sqsubset \subseteq \sqsubset'$; define $\prec'$ as $\prec \circ \sqsubseteq'$. Then $p' = (E, \prec', \sqsubset')$ is an spc-order and an augmentation of p.*

Proof: By Proposition 3.3 with $\prec$ as R_1 and $\sqsubset'$ as R_2 (and hence as $\sqsubset$) of 3.3, p' is an spc-order and obviously an augmentation of p. □

Lemma 3.14 *Let $(E, \sqsubset)$ be a partial order with different e and f in E such that $\neg f \sqsubset e$. Then there exists a linearization $(E, \sqsubset')$ of $(E, \sqsubset)$ where $e \sqsubset' f$ and $\{g \mid e \sqsubset g \sqsubset f\} = \{g \mid e \sqsubset' g \sqsubset' f\}$.*

Proof: If $\neg e \sqsubset f$, we can extend $\sqsubset$ as a first step to the partial order $(\sqsubset \cup \{(e,f)\})^+$, which satisfies the desired equality. Hence, we can assume that $e \sqsubset f$.

Define a partition of E by $E_1 = \{g \mid \neg e \sqsubset g\}$, $E_2 = \{g \mid e \sqsubseteq g \sqsubseteq f\}$ and $E_3 = \{g \mid e \sqsubset g \wedge \neg g \sqsubseteq f\}$. Obviously, e and f are the minimum and the maximum of E_2; if we have for $i,j = 1,2,3$ that $g_i \in E_i$ and $g_i \sqsubset g_j$, then $i \leq j$. (E.g. $i = 2$ and $j = 1$ would give $e \sqsubseteq g_2 \sqsubset g_1$, i.e. $g_1 \notin E_1$.) Hence, $\sqsubset \cup E_1 \times (E_2 \cup E_3) \cup E_2 \times E_3$ is a partial order extending $\sqsubset$, it satisfies the desired equality, and so does any linearization of it. □

Lemma 3.15 *Let p be an spc-order and $e,f \in E$ be different.*

i) If $\neg e \sqsubset f$, then there exists an interval augmentation p' with $\neg e \sqsubset' f$ and $\neg e \prec' f$.

ii) If $e \sqsubset f$ and $\neg e \prec f$, then there exists an interval augmentation p' with $\neg e \prec' f$.

Proof: i) Define $\sqsubset'$ as a linearization of the partial order $(\sqsubset \cup \{(f,e)\})^+$, and apply Lemma 3.13; observe that $\sqsubset'$ is irreflexive and $\prec' \subseteq \sqsubset'$.

ii) By $\neg f \sqsubset e$, we can apply Lemma 3.14 to get $\sqsubset'$ and then apply Lemma 3.13 to define the interval augmentation p'. If we had $e \prec' f$, then there would be g with $e \prec g \sqsubset' f$, i.e. $e \sqsubset' g \sqsubset' f$ and $e \sqsubset g \sqsubset f$; this would imply $e \prec g \sqsubset f$ and $e \prec f$. □

We will use spc-orders to model system runs; an ST-sequence is an observation and, as we have seen, an interval-spc-order is a moderate abstraction of an observation; such abstract observations can be derived order-theoretically from a run: they are the interval augmentations. The following theorem shows that we can reconstruct a run from the set of its abstract observations.

Theorem 3.16 *Let p be an spc-order and I the set of its interval augmentations. Then $\prec = \bigcap_{p' \in I} \prec'$ and $\sqsubset = \bigcap_{p' \in I} \sqsubset'$.*

Proof: The inclusion is in both cases obvious. For the reverse inclusion, we can apply Lemma 3.15 i) for $\sqsubset$; for $\prec$ and $\neg e \prec f$, we either have $\neg e \sqsubset f$ and apply 3.15 i), or we have $e \sqsubset f$ and apply 3.15 ii). □

Again, this result carries over to spc-structures. The resulting corollary is the most important result of this section; it is an analogue to Szpilrajn's Theorem and will be applied in Corollary 4.14.

Corollary 3.17 *Let p be an spc-structure and I the set of its interval augmentations. Then $\prec = \bigcap_{p' \in I} \prec'$ and $\sqsubset = \bigcap_{p' \in I} \sqsubset'$.*

We have seen in Proposition 3.7 and Corollary 3.9 that we can reconstruct an interval-spc-order or -structure (up to isomorphism) from each of its ST-sequences or -traces. Furthermore, we can reconstruct an spc-order or -structure from the set of its interval augmentations and, hence, (up to isomorphism) from the set of its ST-linearizations by Theorem 3.16 and Corollary 3.17.

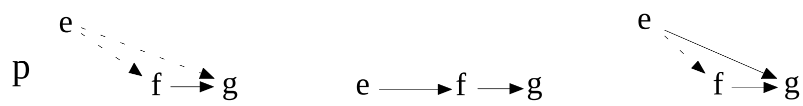

Figure 3

Corollary 3.18 *If spc-orders p and p′ have the same set of ST-linearizations, then they are equal. If spc-structures p and p′ have the same set of ST-linearizations, then they are isomorphic; i.e. an spc-structure can be reconstructed (up to isomorphism) from its set of ST-linearizations.*

The above results that lead to this corollary do not hold for step sequences. Figure 3 shows on the right an spc-step-sequence, where we cannot derive $e \sqsubset f$ from its corresponding step sequence $\binom{e}{f}g$. The spc-order p on the left cannot be reconstructed from its two step augmentations – which are also shown –, because we cannot derive that $\neg e \prec g$.

If we are only interested in step sequences, it is irrelevant whether $e \prec g$ or only $e \sqsubset g$. But if we are interested in the durations of events and runs, this difference is important: assume e.g. that e has duration 3 and f and g have duration 1 in p; then e can start time 1 before f and later carry on in parallel with g, such that the whole run p takes time 3. If we had $e \prec g$, the whole run would take at least time 4. The relation between partial order semantics and temporal efficiency of ordinary nets where events have durations has been explored in [Vog95].

4 Processes of nets with read arcs

A process is essentially a so-called occurrence net describing one run of another net N. Transitions of occurrence nets are called events and model the firings of transitions of N, places of occurrence nets are called conditions and model tokens, i.e. they correspond to statements 's is marked' that hold at some stage of a run. We will extend the definition of processes to nets with read arcs, essentially following [MR95]. Occurrence nets are usually very simple: they are acyclic, i.e. give a partial order on their elements, and conditions are unbranched; here, these requirements are a little more difficult to define, since read arcs allow some sort of branching and since we deal with two partial orders. We will explain the following definition below.

Definition 4.1 For a T-restricted net graph $O = (B, E, F, A)$, we define two relations on $B \cup E$: $\sqsubset$ is $(F \cup A \cup A^{-1} \circ F)^+$ and $\prec$ is $F \circ \sqsubseteq$.

O is an *occurrence net* if

i) $|{}^\bullet b|, |b^\bullet| \leq 1$ for all $b \in B$, i.e. conditions are unbranched;

ii) $F \cup A \cup A^{-1} \circ F$ is acyclic, i.e. $\sqsubset$ is a partial order.

The spc-order $(B \cup E, \prec, \sqsubset)$ induced by F and $A \cup A^{-1} \circ F$ according to Definition 3.4 is denoted by $spc(O)$. We call the places $b \in B$ *conditions*, the transitions $e \in E$ *events*. (F stands for flow, A for activator arcs as read arcs are called in [JK95].) We denote $min_{\prec}(B \cup E)$ by ${}^\bullet O$, $max_{\prec}(B \cup E)$ by $O^\bullet$ and the restrictions of $\prec$ and $\sqsubset$ to E by $\prec_E$ and $\sqsubset_E$.

We also consider a graph with vertices $B \cup E$ and (directed) edges $F \cup A \cup A^{-1} \circ F$. We have $x \sqsubset y$ iff $x \neq y$ and there exists a path in this graph from x to y, and $x \prec y$ iff $x \neq y$ and there exists such a path starting with an edge in F; we call such a path *justifying* for $x \sqsubset y$, $x \prec y$ resp. □

As usual, there is (at most) one event that produces a token and (at most) one event that consumes it; in this sense, conditions are unbranched in an occurrence net, but additionally a condition might be incident to some read arcs.

For events e, f and a condition b, eFb means that e produces b, i.e. the firing e starts and ends before b starts holding; similarly, bFe means that e consumes b, i.e. the holding of b starts and ends before the firing e starts. In the case bAe e reads b, i.e. the holding of b starts before the firing e starts. Actually, e has to start before the end of b in this case, which is not modelled in $spc(O)$; modelling this would make the theory much more clumsy, and the omission creates almost no problems. Finally, we have already discussed in the introduction (using N_2 of Figure 1) that bAe and bFf enforce that e starts before f. Thus, it is intuitively clear that $\sqsubset$ gives an ordering of starts and should be acyclic, and that according to Proposition 3.3 $spc(O)$ should model the necessary relations between starts and ends of conditions and events in the run described by O; in our graphical notation for spc-orders, F gives the ordinary arrows while $A \cup A^{-1} \circ F$ gives the dashed arrows.

Observe that for $A = \emptyset$ O is an occurrence net according to the classical definition and that $\prec = \sqsubset = F^+$.

Lemma 4.2 *Let O be an occurrence net, c a condition and $x \in B \cup E$.*

i) $x \sqsubset c$ *implies that there is an event* e *with* $x \sqsubseteq e$ *and* $e \in {}^\bullet c$.

ii) $x \prec c$ *implies that there is an event* e *with* $x \preceq e$ *and* $e \in {}^\bullet c$.

iii) There is some $b \in {}^\bullet O$ *with* bF^*x.

iv) There is some $b \in O^\bullet$ *with* xF^*b.

v) ${}^\bullet O = \{b \in B \mid {}^\bullet b = \emptyset\} = min_{\sqsubset}(B \cup E)$

vi) $O^\bullet = \{b \in B \mid b^\bullet = \emptyset\}$

vii) $(E, \prec_E, \sqsubset_E)$ *is the spc-order induced by* $F \circ F \cup F \circ A$ *and* $A^{-1} \circ F$.

Proof: i) and ii) The last edge of a justifying path cannot be in A or in $A^{-1} \circ F$, hence it must be in F.

iii) One can construct a path with edges in F backwards from x. If we have found y with yF^*x, then either y is an event and by T-restrictedness there is some z with $z \in {}^\bullet y$, i.e. zFy; or y is a condition, which is in ${}^\bullet O$ giving the claim or has some f with $f \in {}^\bullet y$ by ii), i.e. fFy. Since F is acyclic and $B \cup E$ is finite, the construction stops at some condition $b \in {}^\bullet O$.

iv) is shown analogously, where this time each condition $y \notin O^\bullet$ must have an outgoing edge in F, since a justifying path for $y \prec y'$ must start with such an edge.

v) We have seen in the proof of iii) that ${}^\bullet O \subseteq B$. If $b \in {}^\bullet O$ and there is some $e \in {}^\bullet b$, then $e \prec b$, a contradiction; this gives the first inclusion. If $b \in B$ and for some x $x \sqsubset b$, then we can by i) find some $e \in {}^\bullet b$, which shows the second inclusion. The third set is contained in the first one, since $\prec \subseteq \sqsubset$.

vi) We have already seen in the proof of iv) that $O^\bullet \subseteq B$. If $e \in b^\bullet$, then $b \prec e$, hence inclusion holds. For the reverse inclusion, observe that $b^\bullet = \emptyset$ implies $b \in O^\bullet$ by iv).

vii) For events e and f, $e \sqsubset f$ iff there is a path from e to f with edges in $F \cup A \cup A^{-1} \circ F$. The first edge can be in F reaching a condition c; then the next edge can be in F or in A reaching an event. If the first edge is not in F, then it must be in $A^{-1} \circ F$ reaching an event. Thus, the path consists of portions in $F \circ (F \cup A) \cup A^{-1} \circ F$, and stringing such portions together always gives a suitable path.

Similarly, $e \prec f$ iff there is a path as above starting with an edge in F, i.e. with a portion in $F \circ (F \cup A)$ reaching some event g with $g \sqsubseteq f$. In other words, $\prec_E = F \circ (F \cup A) \circ \sqsubseteq_E$. □

Part vii) of the above lemma shows that we can directly construct the spc-order on events we will mostly be interested in; this result also makes the comparison to other approaches in the literature easier.

Now we define a process of a net N as in the classical setting, i.e. as an occurrence net O whose events correspond to transitions of N and whose conditions to places of N; ${}^\bullet O$ corresponds to the initial marking of N, and F and A correspond to the arcs and read arcs of N.

Definition 4.3 A *process* $\pi = (O, l)$ of a net N consists of an occurrence net O and a *labelling* $l : B \cup E \rightarrow S \cup T$ such that

i) $l(B) \subseteq S$, $l(E) \subseteq T$

ii) l is injective on ${}^\bullet O$ and $l({}^\bullet O) = M_N$

iii) For all $e \in E$, l is injective on ${}^\bullet e$, $e^\bullet$ and $\hat{e}$ with $l({}^\bullet e) = {}^\bullet l(e)$, $l(e^\bullet) = l(e)^\bullet$, $l(\hat{e}) = \widehat{l(e)}$.

We put ${}^\bullet \pi = {}^\bullet O$ and $\pi^\bullet = O^\bullet$. We call $fspc(\pi) = (B \cup E, \prec, \sqsubset, l)$ the *full spc-structure* of π and its restriction $spc(\pi)$ to E (in all components) the *spc-structure* of π. An *ST-linearization* of π is an ST-linearization of $spc(\pi)$ and similarly for (step) linearizations.

A *cut* of π is a maximal $co_\prec$-set of $fspc(\pi)$; a *slice* is a cut $D \subseteq B$. A cut D *corresponds* to the ST-marking $(\{l(b) \mid b \in B \cap D\}, \{l(e) \mid e \in E \cap D\})$ and a slice D *corresponds* to the marking $l(D)$. □

The last sentence of this definition makes sense, because it will turn out that l is injective on all cuts. Observe that Definition 4.3 coincides for ordinary nets with the usual definition of a process.

Figure 4 shows a net, which is its own process π (if we remove the marking and add the identity as labelling); also $spc(\pi)$ is shown.

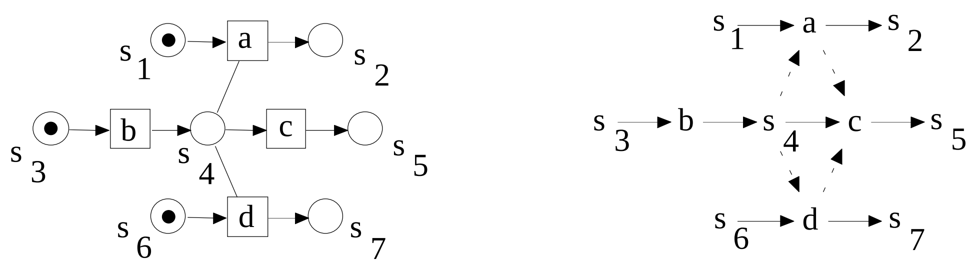

Figure 4

We want to show that ST-linearizations of π correspond to ST-traces of N, that cuts correspond to ST-markings reached along such an ST-trace and similarly for slices. We start with a technical lemma.

Lemma 4.4 *Let N be a net, π one of its processes. Let $P \subseteq B \cup E$ satisfy*

a) $^\bullet\pi \subseteq P$*;*

b) P *is left-closed under* $\sqsubset$ *;*

c) for all events e, $e^\bullet \cap P \neq \emptyset$ implies $e^\bullet \subseteq P$.

Then we have for $D = max_\prec(P)$ that

i) for all $x \in P$, we have $x \in D$ or $x^\bullet \subseteq P$;

ii) D is a cut.

Proof: i) If $x \in P - D$, then $x \prec d$ for some $d \in D$. By definition of $\prec$, there exists some y with $xFy \sqsubseteq d$ and by b) we have $y \in P$. If x is a condition, then $x^\bullet = \{y\} \subseteq P$. If x is an event, c) implies the claim.

ii) Clearly, D is a $co_\prec$-set. So consider some $z \notin D$. If $z \in P$, then $z \preceq d$ for some $d \in D$ by definition of D. If $z \notin P$, then by a) and Lemma 4.2 iii), there is some $b \in P$ with a path from b to z with edges in F; this path must leave P somewhere, say with xy, justifying $x \prec z$; but now $x \in D$ by i). Hence, D is a maximal $co_\prec$-set. □

As a corollary, we see that $^\bullet\pi$ and $\pi^\bullet$ are slices; since these should correspond to the markings where π starts and ends, this result is a first indication that slices indeed correspond to markings.

Corollary 4.5 *For a process π, $^\bullet\pi$ and $\pi^\bullet$ are slices.*

Proof: Use Lemma 4.4 with $P = {}^\bullet\pi$ (left-closed under $\sqsubset$ by Lemma 4.2 v)) and $P = B \cup E$. $\square$

The next lemma shows that ST-linearizations are ST-traces and that the ST-markings reached along such an ST-trace correspond to cuts.

Lemma 4.6 *Let $w = \varepsilon_1 \dots \varepsilon_m$ with $\varepsilon_i \in E^{\pm}$ be an ST-sequence underlying an ST-linearization v of a process π. Define for $i = 0, \dots, m$*

$P_i = \{x \mid$ some e^+ with $x \sqsubseteq e$ occurs in $\varepsilon_1 \dots \varepsilon_i\} \cup \{e^\bullet \mid e^-$ occurs in $\varepsilon_1 \dots \varepsilon_i\} \cup {}^\bullet\pi$ and

$D_i = max_{\prec}(P_i)$.

Then each P_i satisfies Lemma 4.4, each D_i is a cut, l is injective on D_i, and for the ST-markings Q_i corresponding to the D_i we have: $Q_N = Q_0[l(\varepsilon_1)\rangle Q_1[l(\varepsilon_2)\rangle \dots Q_m$.

Proof: For this proof, we use the characterization of ST-linearizations given in Theorem 3.11.

(i) Each P_i is left-closed under $\sqsubset$.

Proof of (i): Clearly, the first and the third set constituting P_i are left-closed under $\sqsubset$, see Lemma 4.2 v). If $b \in e^\bullet$ with e^- in $\varepsilon_1 \dots \varepsilon_i$ and $x \sqsubset b$, then $x \sqsubseteq e$ by Lemma 4.2 i) since b is unbranched; hence, x is in P_i since e^+ is in $\varepsilon_1 \dots \varepsilon_i$, too. $\square$

(ii) If for events e and f we have $e \prec f \in P_i$, then e^- occurs in $\varepsilon_1 \dots \varepsilon_i$.

Proof of (ii): If $e \prec f$, then e^- occurs before f^+ in w. If $f \in P_i$, then there exists some event g with $f \sqsubseteq g$ – i.e. f^+ occurs not later than g^+ – and g^+ occurs in $\varepsilon_1 \dots \varepsilon_i$. $\square$

(iii) For an event e, $e^\bullet \cap P_i \neq \emptyset$ iff $e^\bullet \subseteq P_i$ iff e^- is in $\varepsilon_1 \dots \varepsilon_i$.

Proof of (iii): The if-cases being clear by T-restrictedness and definition of P_i, take some $b \in e^\bullet \cap P_i$; b is not in the third set of P_i, and if it is in the second we are done. Hence, assume $b \sqsubset f$ and f^+ occurs in $\varepsilon_1 \dots \varepsilon_i$, i.e. $f \in P_i$. Then $e \prec b \sqsubset f$, i.e. $e \prec f \in P_i$, and we can apply (ii) to see that b is in the second set as well. $\square$

Thus, each P_i satisfies Lemma 4.4: precondition a) is clear, b) is (i) and c) is contained in (iii); this implies that each D_i is a cut.

(iv) For an event e, $e \in P_i$ iff e^+ occurs in $\varepsilon_1 \dots \varepsilon_i$.

Proof of (iv): The if-case is clear, hence take $e \in P_i$; e can only be in the first set of P_i, hence $e \sqsubseteq g$ with g^+ occurring in $\varepsilon_1 \dots \varepsilon_i$. Then, e^+ occurs in w not later than g^+ and we are done. $\square$

We show the remaining claims by induction on i, where l is injective on $P_0 = D_0 = {}^\bullet\pi$ and ${}^\bullet\pi$ corresponds to Q_N by definition of π. Assume the statements for $i-1$; then there are two cases:

a) $\varepsilon_i = e^+$

We check the enabledness of $l(e^+)$ under Q_{i-1}; consider $b \in {}^\bullet e \cup \hat{e}$. Either $b \in {}^\bullet\pi \subseteq P_{i-1}$ or there is some event f with $b \in f^\bullet$. In the latter case, $f \prec e \in P_i$, and (ii) implies $b \in P_{i-1}$ in this case, too. Assume $b \prec x \in P_{i-1}$. By definition of $\prec$, there exists some event g with $bFg \sqsubseteq x$. Now either $e = g$ or $e(A^{-1} \circ F)g$, hence $e \sqsubseteq x \in P_{i-1}$ and $e \in P_{i-1}$ by (i). By (iv) e^+ already occurs in $\varepsilon_1 \dots \varepsilon_{i-1}$, a contradiction to being in case a). Hence, $b \in D_{i-1}$ and $Q_{i-1}[l(e^+)\rangle$.

(v) This also implies by Proposition 2.1 ii) that no event in D_{i-1} is labelled $l(e)$ in case a).

We will now show that $D_i = D_{i-1} - {}^\bullet e \cup \{e\}$. Consider $x \in P_i - P_{i-1}$; from the definition of P_i, we must have $x \sqsubseteq e$. If $x \in E$, then $x = e$ – or x^+ occurs before e^+, i.e. in $\varepsilon_1 \ldots \varepsilon_{i-1}$, a contradiction to $x \notin P_{i-1}$. If $x \in B$, then the path justifying $x \sqsubset e$ contains some $y \in {}^\bullet e \cup \hat{e} \subseteq P_{i-1}$ or some event y with $y \sqsubset e$; in the latter case, y^+ occurs in $\varepsilon_1 \ldots \varepsilon_{i-1}$ since w is an ST-linearization, i.e. $y \in P_{i-1}$ in any case. Since P_{i-1} is left-closed under $\sqsubset$, we get $x \in P_{i-1}$, a contradiction.

We conclude that $P_i - P_{i-1} = \{e\}$ and therefore $D_i \subseteq D_{i-1} \cup \{e\}$; since ${}^\bullet e \cap D_i = \emptyset$, we even get $D_i \subseteq D_{i-1} - {}^\bullet e \cup \{e\}$. This shows by induction and (v) that l is injective on D_i.

Since P_{i-1} is left-closed under $\sqsubset$, hence under $\prec$, e is $\prec$-maximal in P_i. To show $D_i = D_{i-1} - {}^\bullet e \cup \{e\}$, we only have to prove that $\neg x \prec e$ for $x \in D_{i-1} - {}^\bullet e$. But $x \prec e$ implies for some y that $xFy \sqsubseteq e$; since $x \notin {}^\bullet e$, this gives $y \sqsubset e$ and $y \in P_i - \{e\} = P_{i-1}$; since $x \prec y$, this is a contradiction to $x \in D_{i-1}$.

This equality for D_i shows $Q_{i-1}[l(e^+)\rangle Q_i$.

b) $\varepsilon_i = e^-$

We check the enabledness of $l(e^-)$ under Q_{i-1}, i.e. $e \in D_{i-1}$. Obviously, e is in the first set of P_{i-1}; hence, assume $e \prec x \in P_{i-1}$, i.e. $eFb \sqsubseteq x \in P_{i-1}$ for some b. By (i) $b \in P_{i-1}$, which gives a contradiction to (iii). Hence $Q_{i-1}[l(e^-)\rangle$ and by Proposition 2.1 ii)

(vi) no condition in D_{i-1} has a label in $l(e)^\bullet = l(e^\bullet)$.

From the definition and (iii), P_i is the disjoint union of P_{i-1} and $e^\bullet$. Thus, $D_i \subseteq D_{i-1} - \{e\} \cup e^\bullet$ and l is injective on D_i by induction and (vi).

We prove $D_i = D_{i-1} - \{e\} \cup e^\bullet$. Assume $c \in e^\bullet$ and $c \prec y \in P_i$; since P_{i-1} is left-closed under $\sqsubset$ and $\prec$, but $c \notin P_{i-1}$, we get $y \in e^\bullet$. With Lemma 4.2 ii) (for c and condition y) we get $c \prec e \prec c$, a contradiction to $\prec$ being irreflexive.

Now assume $x \in D_{i-1} - \{e\}$ and $x \prec y \in P_i$; by definition of D_{i-1}, we get again $y \in e^\bullet$, and again by Lemma 4.2 ii) we find $x \preceq e$. Since $x \neq e \in D_{i-1}$, this contradicts the definition of D_{i-1}.

Thus, $D_i = D_{i-1} - \{e\} \cup e^\bullet$ and $Q_{i-1}[l(e^-)\rangle Q_i$. □

Lemma 4.7 *Let D be a cut of a process π, and let $X = \{x \in B \cup E \mid \exists d \in D : x \preceq d\}$. Then:*

i) $X = \{x \in B \cup E \mid \exists d \in D : x \sqsubseteq d\}$; thus, X is left-closed under $\sqsubset$.

ii) $max_\prec(X) = D$

iii) X satisfies the conditions of Lemma 4.4.

Furthermore, l is injective on D and D corresponds to a reachable ST-marking which is reached along an ST-linearization of π.

Proof: i) Inclusion follows from $\prec \subseteq \sqsubset$. For the reverse inclusion, observe that $D \subseteq X$; hence, we consider $x \notin D$ and $d \in D$ with $x \sqsubset d$. Since D is a cut, we have some $d' \in D$ with $x \prec d'$, i.e. $x \in X$, or we have some $d' \in D$ with $d' \prec x$, which gives $d' \prec d$, a contradiction.

ii) Inclusion is immediate from the definition of X; if some $d \in D$ were not $\prec$-maximal in X, then $d \prec x$ for some $x \in X$, i.e. $d \prec x \preceq d'$ for some $d' \in D$, a contradiction.

iii) Since $b \in {}^\bullet\pi$ is $\prec$-minimal and D a cut, there is some $d \in D$ with $b \preceq d$, i.e. ${}^\bullet\pi \subseteq X$. X is left-closed under $\sqsubset$ by i). Finally, take $b, c \in e^\bullet$ and $b \in X$, which implies $e \prec b$ and thus $e \in X - D$ by ii). Assume $c \notin X$. Then there is some $d \in D$ with $d \prec c$; by Lemma 4.2 ii), this gives $d \preceq e \prec b$ and $d \notin max_\prec(X)$, a contradiction to ii).

For the remaining claims, we will construct an ST-sequence underlying a suitable ST-linearization of π and apply Lemma 4.6. Let $E_1 = X \cap E$, $E_2 = E - E_1$, and let $v_i \in E_i^*$ be a linearization of $(E_i, \sqsubset_{E_i})$, $i = 1, 2$. By i), $v_1 v_2$ is a linearization of $(E, \sqsubset_E)$; let v_2' be a linearization of $E \cap D$. We construct w_1 from v_1 by replacing each $e \in E_1 - D$ by e^+e^- and each $e \in E \cap D$ $(= E_1 \cap D)$ by e^+; we construct w_2 from $v_2' v_2$ by replacing each $e \in E \cap D$ by e^- and each $e \in E_2$ by e^+e^-. We will show that $w_1 w_2$ is an ST-sequence underlying an ST-linearization of π.

Take events e and f; If $e \sqsubset f$, then e starts before f in $w_1 w_2$ because $v_1 v_2$ is a linearization of $(E, \sqsubset_E)$. If $e \prec f$, then e starts before f in $w_1 w_2$ and, by construction, e ends before f starts provided $e \notin D$. If $e \in D$, then $f \in E_2$ by definition of X and since D is a $co_\prec$-set; hence, e ends in the first part of w_2 (corresponding to v_2') before f starts in the second part of w_2 (corresponding to v_2). Thus, w is an ST-sequence underlying an ST-linearization as desired.

We now show that X is the P_i from Lemma 4.6 corresponding to $\varepsilon_1 \ldots \varepsilon_i = w_1$. By iii), ${}^\bullet\pi \subseteq X$. If e^+ occurs in w_1 and $x \sqsubset e$, then $e \in X$ and X is left-closed under $\sqsubset$, hence $x \in X$. If e^- occurs in w_1, then $e \in E_1 - D$; a path justifying $e \prec d$ with $d \in D$ shows that some $b \in e^\bullet$ is in X, hence $e^\bullet \subseteq X$ by iii). We conclude that $P_i \subseteq X$.

Vice versa, take $x \in X$; if $x \in E$, then $x \in E_1$ starts in w_1, hence $x \in P_i$. Now consider $x \in X \cap B$. If $x \in {}^\bullet\pi$, then $x \in P_i$; thus, consider some event e with $x \in e^\bullet$. Then we have $e \prec x \preceq d$ for some $d \in D$, hence $e \in E_1 - D$ ends in w_1 and, also in this case, $x \in P_i$.

We now have $P_i = X$ and $D = max_\prec(P_i)$ by ii), which gives the result with Lemma 4.6. □

We now come to the first main result of this section, which shows that the order-theoretically derived ST-linearizations, (step) linearizations, cuts and slices of a process are ST-traces, firing (or step) sequences, reachable ST-markings and markings, which are behaviourally defined.

Theorem 4.8 *Let π be a process of some net N. Then all ST-linearizations of π are ST-traces of N, all (step) linearizations are firing (or step) sequences of N. The labelling l is injective on all cuts. Cuts correspond exactly to those ST-markings that can be reached along ST-linearizations of π, slices correspond exactly to those markings that can be reached along (step) linearizations of π.*

Proof: The 'ST-statements' follow from Lemma 4.6 and 4.7, the other statements then follow with Theorem 3.12 b), Proposition 2.1 and Theorem 2.3. □

The next main result is a converse to 4.8; it shows that *all* the operationally defined entities can also be derived order-theoretically. For this result, we need a lemma. We have defined processes in such a way that they start and end with slices, i.e. with markings; alternatively, one could define them such that they end with an arbitrary reachable ST-marking. The following lemma deals with those cuts that could serve as a final ST-marking in such an alternative definition.

Lemma 4.9 *Let π be a process and $D \subseteq B \cup E$ such that $D = B_1 \cup E_1$, where E_1 consists of events that are $\prec$-maximal in E and $B_1 = \pi^\bullet - \bigcup_{e \in E_1} e^\bullet$. Then*

i) for all $e \in E_1$ and $b \in e^\bullet$, we have $b^\bullet \cup \hat{b} = \emptyset$ and in particular $b \in \pi^\bullet$;

ii) D is a cut.

Proof: i) Take some suitable e and b and assume that $f \in b^\bullet \cup \hat{b}$. Then we would have $eFb(F \cup A)f$, i.e. $e \prec f$, a contradiction to the choice of E_1. Observe Lemma 4.2 vi).

ii) E_1 is a $co_\prec$-set by definition and so is $\pi^\bullet$, hence B_1. For $e \in E_1$ and $b \in B_1$, we cannot have $b \prec e$ by definition of $\pi^\bullet$; so assume $e \prec b$. By Lemma 4.2 ii), this gives an event f with $e \preceq f$ and $b \in f^\bullet$; since $b \in B_1$, we have $e \neq f$, a contradiction to $e \in E_1$. Thus D is a $co_\prec$-set.

Consider $x \in E \cup B - D$. If $x \in \pi^\bullet$, then $e \prec x$ for some $e \in E_1$. If $x \notin \pi^\bullet$, take a path with edges in F from x to $\pi^\bullet$ according to Lemma 4.2 iv); this path passes through E_1 or reaches B_1. Hence, $x \prec d$ for some $d \in D$. We conclude that D is a cut. □

Theorem 4.10 *Let N be a net.*

i) For each ST-trace v of N, there is a process π of N which has v as ST-linearization.

ii) For each firing (or step) sequence v of N, there is a process π of N which has v as (step) linearization.

iii) For each reachable ST-marking Q of N, there is a process π of N with a cut that corresponds to Q.

iv) For each reachable marking M of N, there is a process π of N with a slice that corresponds to M.

Proof: ii) follows from i) by Proposition 2.1 i) and Theorem 3.12 b) i) and by Theorem 2.3 ii) and Theorem 3.12 b) ii); iii) follows from i) and Lemma 4.6, and then iv) follows from iii) and Proposition 2.1 iii). Thus, we only have to show i). In a way, we will read the proof of Lemma 4.6 as a construction.

For each ST-trace $v = \delta_1 \dots \delta_m$, $\delta_i \in T^\pm$, with $Q_N[v\rangle Q$ we construct by induction on m a process π and a set D such that

– v is an ST-linearization of π;

– D corresponds to Q;

– $D = B_1 \cup E_1$, where E_1 consists of events that are $\prec$-maximal in E and $B_1 = \pi^\bullet - \bigcup_{e \in E_1} e^\bullet$.

Then, by Lemma 4.9, D is a cut and, by Lemma 4.7, l is injective on D.

For $v = \lambda$ we take the initial process π_0, i.e. the unique process π with $^\bullet\pi = \pi^\bullet$, and $D = B_1 = \pi_0^\bullet$. Assume now that π and D for v with underlying ST-sequence w are given and $Q[\delta\rangle Q'$.

a) $\delta = t^+$ for some t with $M[t\rangle$.

Since D corresponds to Q, there is a unique set B_e of conditions in D labelled with $^\bullet t \cup \hat{t}$. We add a new event e with label t, arcs and read arcs from B_e to e, new conditions to represent $e^\bullet$ via the labelling and arcs from e to these. Adding e and its ingoing arcs, we add to the edges in $F \cup A \cup A^{-1} \circ F$ only edges going to e, since in π we have $b^\bullet = \emptyset$ for all $b \in D$; thus $F \cup A \cup A^{-1} \circ F$ remains acyclic. Then, the same argument applies for the new conditions. Now it is easy to see that the new π' is a process; also,

(∗) the relations $\prec'$ and $\sqsubset'$ for π' coincide with $\prec$ and $\sqsubset$ for the events and conditions of π.

Hence, $w' = we^+$ is an ST-sequence underlying the ST-linearization $v\delta$ of π'.

We put $D' = D - {}^\bullet e \cup \{e\}$. To see that D' is a suitable union, observe that e is by the above certainly $\prec'$-maximal in E'. The conditions in $B_1 - {}^\bullet e$ still have an empty postset, hence are in $\pi'^\bullet$. It remains to check that the events in E_1 are still $\prec'$-maximal, i.e. by (∗) not less than e. A justifying path for $f \prec' e$ with $f \in E_1$ would start with an edge $fb \in F$, but such a b has in π no outgoing edge in $F \cup A$ by Lemma 4.9, and it has none in π' since $b \notin B_1$. D' obviously corresponds to Q'.

b) $\delta = t^-$ for some $t \in C$.

Since D corresponds to Q, there is a unique $e \in D$ with $l(e) = t$. We leave π unchanged, define D' by $E_1' = E_1 - \{e\}$ and $B_1' = B_1 \cup e^\bullet$ and add e^- to w to get $w' = we^-$.

Obviously, w' is an ST-sequence underlying an ST-linearization of π just as w, this ST-linearization is $l(w') = v\delta$, and D' corresponds to Q'. Since π is unchanged, the events in E_1' are $\prec$-maximal in E and by Lemma 4.9 $B_1' \subseteq \pi^\bullet$. Thus, D' is a suitable union. □

Our results so far also imply that steps of a net give sets of concurrent events in some process.

Corollary 4.11 *Let N be a net and G a step under a reachable marking. Then there exists a process π and a co$_\prec$-set $E' \subseteq E$, such that l is injective on E' and $l(E') = C$.*

Proof: Apply Theorem 2.3 iii) and Theorem 4.10 iii); l is injective by Theorem 4.8. □

We will now sharpen Theorem 4.10; this time, it seems more convenient to prove our result for firing sequences first.

Theorem 4.12 *For each firing sequence v of a net N, there is (up to isomorphism) a unique process π of N which has v as linearization.*

Proof: Existence of π follows from Theorem 4.10. Hence, we only have to show uniqueness by induction on the length of v, the case $v = \lambda$ being clear. Take a firing sequence vt,

$t \in T$, the unique process π for v and a process π' for vt. Then, π' must have a $\sqsubset'$-maximal event e with $l'(e) = t$ by Theorem 3.11 ii).

If $b \in e^\bullet$, then $b^\bullet$ and $\hat{b}$ are empty, since otherwise e would not be $\sqsubset'$-maximal. Thus, removing e and $e^\bullet$ from π' gives a process with linearization v; by induction, this process is π (up to isomorphism).

If $b \in {}^\bullet e$ in π', then clearly $b^\bullet$ is empty in π; furthermore, if $b \in \hat{e}$, then $b^\bullet$ is also empty in π, since otherwise for $f \in b^\bullet$ we would have $e(A'^{-1} \circ F')f$, i.e. $e \sqsubset' f$, a contradiction. Thus, ${}^\bullet e \cup \hat{e}$ is a subset of $\pi^\bullet$, i.e. l is injective on ${}^\bullet e \cup \hat{e}$ by Corollary 4.5 and Lemma 4.7.

We see that π' can be obtained from π in two stages: first, add a new t-labelled event, say e, and add arcs and read arcs from suitable conditions in $\pi^\bullet$ to e, which are uniquely determined by the injective labelling l; then, add new conditions corresponding to $t^\bullet$ and add arcs from e to these. This construction is unique up to the names of the new event and the new conditions. Thus, π' is unique up to isomorphism. □

Corollary 4.13 *For each ST-trace (step sequence) w of a net N, there is (up to isomorphism) a unique process π of N which has w as ST-linearization (step linearization).*

Proof: Existence of π for an ST-trace w follows from Theorem 4.10. Let some process π with ST-linearization w be given. Obtain v from w by replacing each t^+ by t and by deleting all t^-; this can be seen as moving the t^- forward in w and contracting t^+t^-, hence v is a firing sequence by Proposition 2.1. Also, v is a linearization of π by Theorem 3.11 i) and ii). Hence, π is unique (up to isomorphism) by Theorem 4.12.

By Theorem 2.3 ii), a step sequence of N can be seen as an ST-trace and, by Theorem 3.12, a step linearization can be seen as an ST-linearization in the same way; hence the ST-case carries over to the step-case. □

Corollary 4.14 *Let N be a net; denote by $STLin(\pi)$ the set of ST-linearizations of a process π. Then the family of sets $STLin(\pi)$ with π a process of N is a partition of the ST-traces of N. Similarly, processes induce a partition of the set of firing sequences and the set of step sequences of N.*

From a set $STLin(\pi)$ the spc-structure $spc(\pi)$ can be determined (up to isomorphism) without knowledge of N.

Proof: The first claims follow from Theorems 4.8 and 4.12, Corollary 4.13 and the fact that each process has an ST-linearization etc. as argued before Lemma 3.13. The last claim follows from Corollary 3.18. □

In processes of ordinary nets, a *line* is usually defined as a maximal subset of $B \cup E$ that is totally ordered by causality; intuitively, it is the worldline of a pointlike object or the trajectory of a signal in space and time. A cut or slice is a global state of the system seen by some observer. From the intuition, it is to be expected that each line meets each cut in exactly one element, and this is indeed true for the processes of ordinary nets; that the intersection has at most one element is trivial from the definitions, that it is nonempty is the more interesting part.

We now discuss how lines can be defined in our setting. In our discussion, we will use the process in Figure 4 as an example; let D_1 be the slice $\{s_2, s_4, s_6\}$. First observe that s_4 has to start holding before s_2, although the two conditions can coexist. For this reason, it might happen that a sensibly defined line could meet this slice in more than one element; for example, $L_1 = \{s_3, b, s_4, a, s_2\}$ looks like it should be such a line. Hence, we will aim for a definition of a line such that each line meets each cut, but not necessarily in just one element.

If we define a line as usual to be a maximal subset of $B \cup E$ that is totally ordered by causality, i.e. by $\prec$, then L_1 would not be a line. Furthermore, $\{s_3, b, d, s_7\}$ would be a line that does not meet D_1; this line misses s_4, which establishes the link between b and d. This example indicates that a line should rather be related to $\sqsubset$.

If we define a line as a maximal subset of $B \cup E$ that is totally ordered by $\sqsubset$, then $\{s_1, a, c, s_5\}$ would be a line that does not meet D_1; again, this line misses s_4, which establishes the link between a and c. This time, the reason is that actually the end of s_4 is between the starts of a and c, but we have not modelled this in our relations; compare the discussion after Definition 4.1. An alternative would have been to derive from each of our processes a partial order on $B^{\pm} \cup E^{\pm}$ or maybe even on $B \cup E \cup B^{\pm} \cup E^{\pm}$, a severe deviation from the ordinary setting; see e.g. [Mur93] for a variant of event structures where each event has an explicitly modeled start and end. We have chosen to stay closer to the classical approach.

As a way out, we recall that a line can just as well be defined as a path from ${}^\bullet\pi$ to $\pi^\bullet$ in the ordinary setting; hence, we will define a line graph-theoretically on the process. We will define two variants of a line, where in the more general variant we try to stay close to $\sqsubset$. In particular, to allow a line going from a to c and including s_4 in the situation just discussed, we allow to use a read arc backwards if we use an arc immediately afterwards. Lines defined this way are in fact close to maximal subsets of $B \cup E$ totally ordered by $\sqsubset$, but the relation is subtle, and it does not seem worth the effort to work it out.

Definition 4.15 A *line* of a process π is a path from ${}^\bullet\pi$ to $\pi^\bullet$ with edges in $F \cup A \cup A^{-1}$, where each edge in A^{-1} is immediately followed by an edge in F.

A line is an *F-line* if it only uses edges in F. □

Observe that each vertex is allowed to appear at most once on a path. This excludes the possibility to use the same edge in A forward and backward; this exclusion seems more natural to me. Furthermore, observe that each path starting in ${}^\bullet\pi$ can be extended to a line by Lemma 4.2 iv).

Theorem 4.16 *Let π be a process of a net N, L a line and D a cut of π. Then $L \cap D \neq \emptyset$.*

Proof: Consider $X = \{x \in B \cup E \mid \exists d \in D : x \preceq d\}$ as in Lemma 4.7. By Lemma 4.7 iii), L starts in X; if it never leaves X, then its last condition is $\prec$-maximal in $B \cup E$ and hence in $max_{\prec}(X) = D$.

Otherwise, L leaves X, say with the edge xy. Assume that $x \notin D$, i.e. there is some $d \in D$ with $x \prec d$. Then, the edge xy is not in F by Lemma 4.7 iii) and Lemma 4.4 i).

First, consider $xy \in A$, i.e. x is a condition read by the event y. Since $x \prec d$, there is some event e with $xFe \sqsubseteq d$. Hence, $ye \in A^{-1} \circ F$, i.e. $y \sqsubset e \sqsubseteq d$ and $y \in X$ by Lemma 4.7

i), a contradiction to the choice of xy. Second, consider $xy \in A^{-1}$; in this case, we have directly $y \sqsubset x \sqsubseteq d$ and hence $y \in X$, a contradiction. We conclude that $x \in D \cap L$. □

Corollary 4.17 *Let π be a process of a net N, L an F-line and D a cut of π. Then L is a maximal subset of $B \cup E$ that is totally ordered by causality, i.e. by $\prec$. Furthermore, $L \cap D$ consists of exactly one element.*

Proof: Obviously, each F-line is totally ordered by causality. Hence, assume L is an F-line, $x \notin L$ and $L \cup \{x\}$ is totally ordered by $\prec$. Since L contains a $\prec$-minimal and a $\prec$-maximal element of the process, x partitions L into two sets with maximal element y and minimal element z resp. such that $y \prec x \prec z$ and $yz \in F$. A justifying path for $y \prec x$ starts with an edge $yy' \in F$, and since $y' \sqsubset x$ but not $z \sqsubset x$, we have $y' \neq z$. Therefore, y is an event and y' and z are conditions. If z is a condition, then a justifying path for $x \prec z$ must end with an edge in F; this is a contradiction to z being unbranched, since y cannot be on this path.

Now the second claim follows from the last theorem and the definition of a cut. □

In the discussion above, we have already mentioned that in Figure 4 $\{s_3, b, d, s_7\}$ is a maximal subset of $B \cup E$ that is totally ordered by causality; observe that this is not an F-line.

We close this section by a result already announced in Section 3: each spc-order appears in the spc-structure of some process of some net.

Theorem 4.18 *Let $p = (E, \prec, \sqsubset)$ be an spc-order. Then there exists a net N and a process π such that $spc(\pi) = (E, \prec, \sqsubset, l)$.*

Proof: We take E as the set of transitions of N and give each transition a marked place for its preset and an empty place for its postset; this guarantees T-restrictedness. Whenever $e \prec f$, we introduce a new empty place in $e^\bullet \cap {}^\bullet f$. Whenever $e \sqsubset f$, we introduce a new marked place in $\hat{e} \cap {}^\bullet f$. Clearly, this net is its own process (if we delete the marking and take the identity as labelling) and this process satisfies the desired equation. Observe Lemma 4.2 vii) and that $F \circ A$ is empty in our case. □

Of course, it is enough in this construction to consider, instead of $\prec$ and $\sqsubset$, relations that induce p. Even then, the result can often be optimized by omitting some of the places introduced to enforce T-restrictedness.

Figure 5

After optimization, p shown in Figure 3 leads to the net N_1 of Figure 5; giving e, f and g the durations discussed at the end of Section 3, we see that N_1 can be completed within time 3, while N_2 needs time at least 4 although it has the same step sequences.

5 Related literature

We have introduced spc-structures to describe system runs and interval-spc-structures as abstract observations of these runs; the latter abstract from ST-traces, the concrete observations, in a way that is compatible with ST-traces of nets: ST-traces that differ only by the ordering of transition ends are identified. Then we have shown a suitable analogue of Szpilrajn's Theorem: each spc-structure is the intersection of its interval augmentations. Similar results are shown in [JK93], but there interval orders are taken as abstract observations; these abstract also from the ordering of transition starts in ST-traces (see [Vog96b]), an abstraction that is not reasonable for nets with read arcs.

We have defined processes axiomatically and we have shown how to construct a corresponding process from a given ST-trace; the same is done (with step or firing sequences instead of ST-traces) in [MR95, BP96] and a construction of processes from step sequences without an axiomatic definition is given in [JK95]. These constructions give the same processes in all approaches except that [JK95] allows some additional processes. The axiomatic definitions in [MR95, BP96] are different from ours. The recent report [JK96], a refined version of [JK95], gives an axiomatic definition similar to ours (the definition of an occurrence net is different); this report gives a process semantics to nets with priorities and to nets with generalized inhibitor arcs (so called branch inhibitor arcs) essentially by translating these nets to nets with read arcs.

[MR95] derives from a process only one relation, which is required to be a partial order and is close but not identical to our $\sqsubset$. In fact, this partial order coincides on events with our $\sqsubset$ such that concurrency (which we define from $\prec$) is somewhat restricted compared to our approach; on conditions, it is identical with our $\prec$ such that the order-theoretically defined slices coincide with ours. It is required in [MR95] that the labelling is injective on *all* slices and that these correspond to reachable markings; we require this only for the initial slice and prove it for the others.

[BP96] essentially extends [MR95] to general S/T-nets that besides read arcs may have inhibitor arcs as well; an inhibitor arc (s, t) allows t to fire only if s is empty. These generalizations naturally lead to complications; but if we restrict [BP96] to safe nets without inhibitor arcs, then the 'linearizability requirement' in [BP96, Definition 9] states simply that the relation derived as in [MR95] is a partial order and it makes requirements 3, 4 and 5 in [BP96, Definition 6] redundant; thus, the processes of [BP96] on this net class are exactly those of [MR95]. [BP96] defines two relations which are not easy to compare to ours; as a consequence, the slices – which are studied similarly as in the present paper – are different from ours: the definition in [BP96] requires that the set X defined in our Lemma 4.7 is linearizable, something we have proven; nevertheless, some slices in [BP96] are not slices in our approach, hence they are not reachable by a linearization. This fits together with the view taken in [BP96] that a process is not really one run: a process π may contain 'possible events', and omitting them gives a different run (contained in π) reaching additional markings.

Finally, [JK95] gives a more general construction for processes. The reason is that [JK95] views nets with read arcs only as translations from nets with inhibitor arcs. For example, the net N_4 in Figure 6 (essentially the net N_3 from Figure 1) is simply the translation of N_5: instead of an inhibitor arc from a place we have in N_4 a read arc from the complementary place. In N_5, it is intuitively convincing that a and b start together at a time where both their postsets are empty; at a later time, they end and fill these postsets. In N_4, such a behaviour is intuitively less convincing, and I believe that the approach of this paper is a convincing alternative.

Figure 6

From a process, a relational structure with two relations – only on the events – is derived in [JK95], and this structure aims at step sequences: one relation says that some event is necessarily in an earlier step than the other, the other relation says that some event is not in a later step than the other, compare Theorem 3.11 iii). We have already explained at the end of Section 3 that step sequences are not sufficient if we are interested e.g. in the durations of runs. Also recall that our results about the correspondence between cuts and ST-markings and between slices and markings rely on the fact that our spc-structures are defined not only on events, but also on conditions.

To deal with inhibitor arcs in the style of the present paper, one could extend spc-structures by a third relation meaning that some event (e.g. a in N_5) has to start before the end of some other event (b). Alternatively, one could also give a process-based partial order semantics to nets with inhibitor arcs by translating them to nets with read arcs as in [JK95] and transporting our semantics for these nets back to the nets with inhibitor arcs.

Lines are mentioned in [MR95], but they are not studied in any of the above papers.

References

[BG95] N. Busi and R. Gorrieri. A Petri net semantics for π-calculus. In L. Insup and S. Smolka, editors, *CONCUR 95*, Lect. Notes Comp. Sci. 962, 145–159. Springer, 1995.

[BP96] N. Busi and M. Pinna. Non-sequential semantics for contextual P/T-nets. In J. Billington and W. Reisig, editors, *Applications and Theory of Petri Nets 1996*, Lect. Notes Comp. Sci. 1091, 113–132. Springer, 1996.

[CH93] S. Christensen and N.D. Hansen. Coloured Petri nets extended with place capacities, test arcs, and inhibitor arcs. In M. Ajmone-Marsan, editor, *Applications and Theory of Petri Nets 1993*, Lect. Notes Comp. Sci. 691, 186–205. Springer, 1993.

[Fis85] P.C. Fishburn. *Interval Orders and Interval Graphs.* J. Wiley, 1985.

[Gla90] R.J. v. Glabbeek. The refinement theorem for ST-bisimulation semantics. In M. Broy and C.B. Jones, editors, *Programming Concepts and Methods, Proc.*

IFIP Working Conference, 27–52. Elsevier Science Publisher(North-Holland), 1990.

[GV87] R.J. v. Glabbeek and F. Vaandrager. Petri net models for algebraic theories of concurrency. In J.W. de Bakker et al., editors, *PARLE Vol. II*, Lect. Notes Comp. Sci. 259, 224–242. Springer, 1987.

[JK93] R. Janicki and M. Koutny. Order structures and generalisations of szpilrajn's theorem. In R. Shyamasundar, editor, *Found. Software Techn. and Theor. Comp. Sci. '93*, Lect. Notes Comp. Sci. 761, 348–357. Springer, 1993.

[JK95] R. Janicki and M. Koutny. Semantics of inhibitor nets. *Information and Computation*, 123:1–16, 1995.

[JK96] R. Janicki and M. Koutny. On causality semantics of nets with priorities. Technical Report 96-03, Dept. Comp. Sci. Sys., McMaster University, Hamilton, Ontario, 1996.

[MR95] U. Montanari and F. Rossi. Contextual nets. *Acta Informatica*, 32:545–596, 1995.

[Mur93] D. Murphy. Time and duration in noninterleaving concurrency. *Fundamentae Informatica*, 19(3), 1993.

[Pet81] J.L. Peterson. *Petri Net Theory.* Prentice-Hall, 1981.

[Rei85] W. Reisig. *Petri Nets.* EATCS Monographs on Theoretical Computer Science 4. Springer, 1985.

[Vog92] W. Vogler. *Modular Construction and Partial Order Semantics of Petri Nets.* Lect. Notes Comp. Sci. 625. Springer, 1992.

[Vog95] W. Vogler. Timed testing of concurrent systems. *Information and Computation*, 121:149–171, 1995.

[Vog96a] W. Vogler. Efficiency of asynchronous systems and read arcs in Petri nets. Technical Report Nr. 352, Inst. f. Mathematik, Univ. Augsburg, 1996. See http://www.math.uni-augsburg.de/~vogler/.

[Vog96b] W. Vogler. The limit of split$_n$-language equivalence. *Information and Computation*, 127:41–61, 1996.