Naudts-like duality and the extreme Fisher information principle

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We show that using Frieden and Soffer's extreme information principle (EPI) [1] with a Fisher measure constructed with escort probabilities [2], the concomitant solutions obey a type of Naudts' duality [3] for nonextensive ensembles [4].

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I. INTRODUCTION

We are going to be concerned in what follows with the workings of two information measures that have received much attention lately, those of Fisher [1,5] and Tsallis [4,6,7], respectively. Our goal is to show that their interplay naturally yields a type of Naudts' duality [3].

Fisher's information measure (FIM) [1,5] was advanced already in the twenties, well before the advent of Information Theory (IT), being conventionally designed with the symbol I [5] (see Eq. (2.1) below for the pertinent definition). Much interesting work has been devoted to the physical applications of FIM in recent times (see, for instance, [1,5,8,9] and references therein). Frieden and Soffer [1] have shown that Fisher's information measure provides one with a powerful variational principle, the extreme physical information (EPI) one, that yields the canonical Lagrangians of theoretical physics [1,5]. Additionally, I has been shown to provide an interesting characterization of the "arrow of time" alternative to the one associated with Shannon's S [8–10].

Tsallis' measure is a generalization of Shannon's one. Notice that IT was created by Shannon in the forties [11,12]. One of its fundamental tenets is that of assigning an information content (Shannon's measure) to any normalized probability distribution. The whole of statistical mechanics can be elegantly re-formulated by extremization of this measure, subject to the constraints imposed by the *a priori* information one may possess concerning the system of interest [12]. It is shown in [4,6,7] that a parallel process can be undertaken with reference to Tsallis' one, giving rise to what is called Tsallis' thermostatistics, responsible for the successful description of an ample variety of phenomena that cannot be explained by appeal to the conventional one (that of Boltzmann-Gibbs-Shannon) [4,6,7].

II. A BRIEF FISHER PRIMER

Fisher's information measure I is of the form

$$I = \int dx\, f(x,\theta) \left[\frac{1}{f(x,\theta)} \frac{\partial f}{\partial \theta}\right]^2, \tag{2.1}$$

where x is a stochastic variable and θ a parameter on which the probability distribution $f(x,\theta)$ depends. The Fisher information measure provides a lower bound for the mean-square error associated with the estimation of the parameter θ. No matter what specific procedure we chose in order to determine it, the associated mean square error e^2 has to be larger or equal than the inverse of the Fisher measure [5]. This result, i.e., $e^2 \geq \frac{1}{I}$, is referred to as the Cramer-Rao bound, and constitutes a very powerful statistical result [5].

The special case of *translation families* deserves special mention. These are mono parametric families of distributions of the form $f(x-\theta)$ which are known up to the shift parameter θ. Following Mach's principle, all members of the family possess identical shape (there are no absolute origins), and here Fisher's information measure adopts the appearance

$$I = \int dx\, \frac{1}{f} \left[\frac{\partial f}{\partial x}\right]^2. \tag{2.2}$$

The parameter θ has dropped out. $I = I[f]$ becomes then a functional of f.

At this point we introduce the useful concept of escort probabilities (see [2] and references therein), that one defines in the fashion

$$F_q(x) = \frac{f(x)^q}{\int f(x)^q dx}, \tag{2.3}$$

q being any real parameter, $\int F_q(x)dx = 1$, and, of course, for $q=1$ we have $F_1 \equiv f$.

The concomitant "escort-FIM" becomes

$$I[F_q] \;=\; \int dx\, F_q(x) \left[\frac{1}{F_q(x)} \frac{\partial F_q(x)}{\partial x}\right]^2, \qquad (2.4)$$

that, in terms of the original $f(x)$ acquires the aspect

$$I[F_q] = q^2 \frac{\int dx\, f(x)^{q-2} \left[\frac{\partial f(x)}{\partial x}\right]^2}{\int dx\, f(x)^q}. \qquad (2.5)$$

We shall denote with I_q the new "escort-FIM"

$$I_q = \frac{\int dx\, f(x)^{q-2} \left[\frac{\partial f(x)}{\partial x}\right]^2}{\int dx\, f(x)^q}. \qquad (2.6)$$

(Notice that for $q = 0$ the integration range must be finite in order to avoid divergences in the denominator.)

The parameter q can be identified with Tsallis' nonextensivity index [13–15], which allows one to speak of "Fisher measures in a nonextensive context". Their main properties have been discussed in [16].

III. THE EXTREME PHYSICAL INFORMATION PRINCIPLE (EPI)

The Principle of Extreme Physical Information (EPI) is an overall physical theory that is able to unify several sub-disciplines of Physics [1,5]. In Ref. [1] Frieden and Soffer (FS) show that the Lagrangians in Physics arise out of a mathematical game between an intelligent observer and Nature (that FS personalize in the appealing figure of a "demon", reminiscent of the celebrated Maxwell's one). The game's payoff introduces the EPI variational principle, which determines simultaneously the Lagrangian *and* the physical ingredients of the concomitant scenario.

FS [1] envision the following situation, involving Fisher's information for translation families: some physical phenomenon is being investigated so as to gather suitable, pertinent data. Measurements must be performed. Any measurement of physical parameters appropriate to the task at hand initiates a relay of information I (or I_q in a non-extensive environment) from Nature (the demon) into the data. The observer acquires information, in this fashion, that is precisely I (or I_q). FS assume that this information can be elicited via a pertinent experiment. Nature's information is called, say, J [1,5].

Assume now that, due to the measuring process, the system is perturbed, which in turn induces a change δJ. It is natural to ask ourselves how the data information I_q will be affected. Enters here FS's EPI: *in its relay from the phenomenon to the data no loss of information should take place.* The ensuing new Conservation Law states that $\delta J = \delta I_q$, or, rephrasing it

$$\delta(I_q \,-\, J) \,=\, 0, \qquad (3.1)$$

so that, defining an action $\mathcal{A}_q$

$$\mathcal{A}_q \,=\, I_q \,-\, J, \qquad (3.2)$$

EPI asserts that the whole process described above extremizes $\mathcal{A}_q$. FS [1,5] conclude that the Lagrangian for a given physical environment is not just an *ad-hoc* construct that yields a suitable differential equation. It possesses an intrinsic meaning. Its integral represents the physical information $\mathcal{A}_q$ for the physical scenario. On such a basis some of the most important equations of Physics can be derived for $q = 1$ [1,5]. For an interesting Quantum Mechanical derivation see [17]. A cosmological application of the nonextensive ($q \neq 1$) conservation law (3.1) is reported in [18]. Mechanical analogs that can be built up using this law are discussed in [19]. Notice, however, that the last two references use an old Tsallis' normalization procedure (advanced in [13,14]), that cannot be assimilated within the framework of the escort distribution concept.

IV. SOLUTIONS TO THE VARIATIONAL PROBLEM

According to EPI, J is fixed by the physical scenario [5]. We adopt here a more modest posture by assuming that J embodies only the normalization constraint, and say nothing regarding a specific physical scenario. J is just

$$J \,=\, \lambda \int \; f(x)\; dx, \qquad (4.1)$$

where λ is the pertinent Lagrange multiplier. Such a J has been successfully employed in [17] with reference to a quantum mechanical problem. Playing the Frieden-Soffer game, i.e., performing the variation (3.1), leads then to

$$2f\;\ddot{f} + (q-2)\;\dot{f}^2 + q\;I_q\;f^2 + \lambda\;Q\;f^{3-q} = 0 \qquad (4.2)$$

a q-dependent, non-linear differential equation that should yield our "optimal" probability distribution f (we set $Q = \int f^q dx$). Now, one should demand that, for $q = 1$, (4.2) become identical to the differential equation that arises in such circumstances (see that equation in [17], for instance, and call $\lambda^{'}$ the concomitant Lagrange multiplier used there). This requirement is fulfilled if we set $\lambda = \lambda^{'} - qI_q$. The $q = 1$-expression becomes then

$$2f\;\ddot{f} -\;\dot{f}^2 + \lambda'\;f^2 = 0, \qquad (4.3)$$

where, of course, one has $Q = 1$. The solution of Eq. (4.3) is of the form

$$f_{q=1}(x) = A^2 \cos^2\; k(x - x_0) \qquad (4.4)$$

where k is a constant to be determined below and A, x_0 are arbitrary integration constants.

It easy to show that (4.2) has, as a first integral,

$$\dot{f}^2 + I_q\ f^2 + \lambda\ Q\ f^{3-q} = c\ f^{2-q}, \tag{4.5}$$

where c is an integration constant. This equation involves Fisher's generalized information for translation families. We must solve it having (2.6) in mind. In order to establish the consistency between (4.5) and (2.6) we introduce a set of normalized variables

$$z = \int \sqrt{I_q}\, dx, \qquad \bar{\lambda} = \frac{\lambda Q}{I_q}, \qquad \bar{c} = \frac{c}{I_q}, \tag{4.6}$$

(the integral is an indefinite one) in terms of which Eqs. (2.6), (4.1), (4.2), and (4.5) are transformed into

$$1 = \frac{\int f^{q-2} f'^2 dz}{\int f^q dz}, \tag{4.7}$$

$$J_q\ = \bar{\lambda}\, \frac{I_q}{\int f^q dz} \int\ f(z)\ dz, \tag{4.8}$$

(an indefinite integral),

$$2f\ f'' + (q-2)\ f'^2 + q\ f^2 + \bar{\lambda}\ f^{3-q} = 0, \tag{4.9}$$

and

$$f'^2 + f^2 + \bar{\lambda}\ f^{3-q} = \bar{c}\ f^{2-q}. \tag{4.10}$$

Inserting (4.10) into (2.6) we conclude that the integration constant acquires the aspect

$$\bar{c} = \frac{2\ Q + \bar{\lambda}}{x_2 - x_1}, \tag{4.11}$$

where x_2 and x_1 are the integration limits, to be fixed by the remaining parameters of the theory. A quite interesting point is that *the general solution of (4.10) can be given in closed form* as

$$\int^z dz = z - const. = \pm \int \frac{f^{\frac{q}{2}-1}}{\sqrt{\bar{c} - \bar{\lambda} f - f^q}}\, df, \tag{4.12}$$

where the constants $\bar{c}$, $\bar{\lambda}$ must be of such nature that a real f ensues.

V. SYMMETRY PROPERTIES OF THE EPI PROBABILITY DISTRIBUTION

We start by changing variables in (4.9) to

$$u = \frac{f'(z)}{f(z)}\ , \tag{5.1}$$

and obtaining

$$u'' + \alpha\ u\ u' + \beta\ u^3 + \gamma\ u = 0\ , \tag{5.2}$$

with

$$\alpha\ =\ (2\ q - 1),\ \ \beta = \frac{1}{2}\ q\ (q-1),\ \ \gamma = \beta. \tag{5.3}$$

(A complete study of the properties of equation (5.2) is found in [20]). Further, we effect the transformation

$$f \to 1/f, \tag{5.4}$$

so that

$$u \to -u,\ \ u' \to -u',\ \ u'' \to -u''. \tag{5.5}$$

If we require that equation (5.2) be invariant under this transformation, the parameters α, β and γ must change according to $\alpha \to -\alpha$, $\beta \to \beta$ and $\gamma \to \gamma$ respectively. This entails that the parameter q, that characterizes the degree of non-extensivity of the system, transform as $q \to 1-q$. A property of this type has been called "duality" by Naudts [3], although in his case the relationship is of the form $q \to \frac{1}{q}$ (duality between $q > 1$ statistics and $q < 1$ one). In our case, the duality arises between two q-values whose sum adds up to unity.

Introducing now into (4.9) the new variable

$$h = \frac{1}{f}, \tag{5.6}$$

we get

$$2hh'' - (q+2)h'^2 - qh^2 - \bar{\lambda} h^{q+1} = 0, \tag{5.7}$$

which under the substitution $q \to 1-q$, becomes

$$2hh'' + (q-3)h'^2 + (q-1)h^2 - \bar{\lambda} h^{2-q} = 0. \tag{5.8}$$

This equation can be rewritten, if we first define

$$w(q) = (-h'^2 - h^2 - \bar{\lambda} h^{2-q} + \bar{c} h^{3-q}), \tag{5.9}$$

as

$$2hh'' + (q-2)h'^2 + qh^2 - \bar{c} h^{3-q} + w(q) = 0, \tag{5.10}$$

where the terms in $w(q)$ correspond to the (transformed) first integral of (4.9)

$$f'^2 + f^2 + \bar{\lambda} f^{3-q} = \bar{c} f^{2-q}, \tag{5.11}$$

which under (5.6) becomes

$$h'^2 + h^2 + \bar{\lambda} h^{2-q} = \bar{c} h^{3-q}. \tag{5.12}$$

As a consequence, $w(q)$ in (5.10) vanishes and the equation (4.9), under the transformation (5.6), turns out to retain its form, changing $q \to 1-q$ and $\bar{c} \to -\bar{\lambda}$. It is convenient at this point to effect a slight change of notation and denote by f_q the solution to (4.9) that obtains when the nonextensivity index is q. The above symmetry argument entails

$$f_q(\bar{c},\bar{\lambda}) \to \frac{1}{f_{1-q}(-\bar{\lambda},-\bar{c})}. \tag{5.13}$$

Using this symmetry property we can re-obtain the probability distribution (4.4) for $q=1$, i.e., the *ordinary, extensive one*, in term of the probability distribution for $q=0$, that can be easily calculated from (4.10)

$$f'^2 = (\bar{c}-1)\; f^2 - \bar{\lambda} f^3, \qquad q=0. \tag{5.14}$$

The solutions are

$$f_0(z) = \frac{\bar{c}-1}{\bar{\lambda}}\; \left\{1-\tanh^2 \frac{\sqrt{\bar{c}-1}}{2}\;(z-z_0)\right\} \qquad \bar{c}>1, \tag{5.15}$$

and

$$f_0(z) = \frac{\bar{c}-1}{\bar{\lambda}}\; \left\{1+\tan^2 \frac{\sqrt{1-\bar{c}}}{2}\;(z-z_0)\right\} \qquad \bar{c}<1, \tag{5.16}$$

where the last solution must be normalized in a finite interval. The symmetry transformation (5.13) yields now the general solution for $q=1$

$$f_1(\bar{c},\bar{\lambda}) \to \frac{1}{f_0(-\bar{\lambda},-\bar{c})}. \tag{5.17}$$

This is to be compared with the result (4.4). We start with (5.16), effect the transformation (5.17) and reach

$$f_1(z) = \frac{\bar{c}}{1+\bar{\lambda}}\;\cos^2 \frac{\sqrt{1+\bar{\lambda}}}{2}\,(z-z_0) \tag{5.18}$$

which, after a little algebra that involves also going back to the x variable adopts indeed the form (4.4) with $A^2 = c/\lambda'$ and $k=\sqrt{\lambda}/2$. A similar analysis can be performed for (5.15).

We have thus found the general solution for the (extensive) EPI variational treatment corresponding to a J that entails just normalization of the probability distribution. Notice that, within the context of Naudts' effort [3], the extensive thermostatistics $q=1$ is self-dual. Instead, according to the present Fisher framework, the self-dual instance obtains for $q=1/2$.

VI. CONCLUSIONS

We have shown that the EPI principle, used in conjunction with a Fisher measure constructed with escort distributions that depend upon the Tsallis index q, renders a probability distribution endowed with a remarkable symmetry: a Naudts'-like duality [3].

Tsallis enthusiasts had thought, before the advent of Naudts work [3], that a different statistics obtains for *each different value of the nonextensivity index q*. The duality concept is then important because it ascribes the same statistics to a given pair of (suitably related) q-values. We have shown here that such a pair can be selected in two distinct manners, i.e., à la Naudts or à la Fisher, and have detailed the prescription corresponding to the latter choice.

Finally, we have also ascertained which is the general (normalized) probability distribution that extremizes the physical information.

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