THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par Université Toulouse 1 Capitole

Présentée et soutenue par

Jean-Baptiste LATRE

Le 18 février 2019

Algèbres hypercomplexes pour le Calcul

École doctorale et discipline ou spécialité

EDMITT : Mathématiques et applications

Unité de recherche

IMT CEREMATH - CERFACS

Directrice(s) ou Directeur(s) de Thèse

Pr. Françoise CHATELIN

Jury

Pr. Mario ARIOLI – Università LUM Jean Monnet - Rapporteur

Pr. Serge PETITON – Université des Sciences et Technologies de Lille - Rapporteur

Pr. Françoise CHATELIN – Université Toulouse 1 Capitole - Directrice de Thèse

Pr. Bénédicte ALZIARY-CHASSAT – Université Toulouse 1 Capitole – Examinatrice

Dr. Philippe Ricoux – Massachussets Institute of Technology et Fellow du groupe TOTAL - Invité

Algèbres hypercomplexes pour le Calcul
Hypercomplex algebras for Computation

Jean-Baptiste Latre

Université Toulouse 1 Capitole

IMT-CEREMATH

CERFACS

Résumé :

Dans les domaines mathématique ou applicatif, la multiplication de nombres possède un rôle clef pour le Calcul. En Science et en Ingénierie, la nonlinéarité offre de grands défis de modélisation mais aussi de résolution. Notre approche vise, via la multiplication, l'étude de certains phénomènes non linéaires que l'on retrouve fréquemment dans le domaine de la Science et de l'Industrie.

Pour cela, nous étudions dans cette thèse la multiplication de nombres multidimensionnels, associée à des structures algébriques en dimension finie appelées algèbres hypercomplexes. Nous utilisons la multiplication comme lien entre les divisions apparentes des différents domaines théorique et pratique que nous abordons par une approche transdisciplinaire.

Nous effectuons une analyse comparative entre les algèbres hypercomplexes et les principaux outils de Calcul, approche qui n'est pas développée dans la littérature existante. Nous présentons une synthèse des applications existantes (par ex. robotique, modélisation 3D, électromagnétisme) et des principaux avantages des algèbres hypercomplexes, pour la Science et l'Ingénierie. A partir des conséquences de l'utilisation des structures alternatives (autres que réelles ou complexes), nous proposons une extension nouvelle de la théorie spectrale présentée sous le nom de couplage spectral. Grâce aux algèbres hypercomplexes et à la théorie du couplage spectral, nous présentons des applications inédites à la mécanique et à la chimie ainsi que des perspectives pour le domaine du calcul quantique.

Pour les domaines d'applications présentés, existants ou inédits, nous étudions les aspects de modélisation théorique et aussi d'analyse numérique. Nous montrons que suivant les cas d'étude, les aspects numériques avantageux découlent d'un choix judicieux des modèles et des algèbres hypercomplexes associées. Ces avantages sont principalement dus à la manière de définir la multiplication dans les algèbres concernées.

Dans les domaines applicatifs abordés, une grande partie des modèles théoriques et numériques repose actuellement sur l'utilisations des nombres réels ou complexes ainsi que sur l'algèbre linéaire. Nous montrons dans cette thèse que les algèbres hypercomplexes sont complémentaires des outils algébriques actuellement utilisés et possèdent un vaste potentiel théorique et pratique, grandement sous-exploité pour le Calcul.

Mots-clefs :

Calcul, algèbres hypercomplexes, nonlinearité, multiplication, couplage spectral, modélisation, stabilité numérique

Remerciements

Je tiens à remercier Philippe Ricoux qui depuis le début du travail avec le groupe Qualitative Computing a démontré depuis plus de six ans son soutien fidèle et constant. Sans lui cette thèse n'aurait jamais vu le jour et je le remercie de m'avoir accordé sa confiance et de m'avoir donné l'occasion d'accomplir mon travail de thèse. Malgré toutes ses occupations industrielles et son agenda surchargé, je suis très reconnaissant que, grâce à son ouverture d'esprit et son goût de la Science, il ait choisi de consacrer autant d'énergie aux algèbres hypercomplexes.

Ensuite, je tiens également à remercier la Directrice du CERFACS, Catherine Lambert d'avoir contribué avec Philippe Ricoux à la construction d'une solution qui me permette de travailler sereinement et de réaliser mon projet de thèse. Son engagement a permis au groupe Qualitative Computing de continuer ses recherches pour contribuer à faire perdurer le lien entre mathématiques et applications industrielles.

En tant que laboratoire universitaire de rattachement, je remercie le CEREMATH qui m'a grand ouvert ses portes. Depuis de nombreuses années, Jacqueline Fleckinger et Bénédicte Alziary s'occupent de faire vivre cette équipe de recherche et d'y insuffler leur énergie bienveillante. Avant tout, il y a des êtres humains derrière les mathématiques et sans cette contribution vivante, les mathématiques s'en retrouveraient fort dépouillées.

Je remercie Mario Arioli et Serge Petiton d'avoir accepté de rapporter ma thèse. Chacun à leur manière, ils ont apporté des remarques constructives pour compléter ma réflexion.

Je tiens également a remercier toutes les personnes sont intervenues pour résoudre les nombreuses questions administratives et les situations ubuesques que j'ai pu rencontrer au cours de ces années. Je remercie Chantal Soulé-Dupuy, Directrice de l'EDMITT, qui a débloqué et accompagné ma thèse tout comme les deux secrétaires de l'EDMITT, Agnès Requis et Martine Labruyère qui m'ont aidé tant de fois pour le dossier de thèse. Pour l'UT1, je remercie Viginie Mangion qui connaît elle-même très bien les incohérences administratives et m'a permis de naviguer dans les méandres de l'administration universitaire. Je remercie aussi toute l'équipe administrative du CERFACS qui m'a accompagné au quotidien.

Etant donné le thème particulier de la thèse, même avec internet et les solutions modernes, je n'aurai pas pu avancer sans le travail précieux de Séverine Toulouse à la bibliothèque du CERFACS. La quantité d'articles et de livres en langues et époques variées dont j'ai eu besoin lui ont donné du fil à retordre. Je remercie son investissement et sa gentillesse.

Je remercie les équipes du CEREMATH et du CERFACS avec qui j'ai pu échanger sur de nombreux thèmes et qui ont pris le temps de m'aider à peaufiner mes travaux et la présentation de la thèse.

Je remercie du fond du cœur Françoise Chatelin pour m'avoir fait grandir, scientifiquement et personnellement, tout au long de ces années. Sa passion dévorante pour les nombres et le Calcul est un témoignage vivant que la recherche ne se limite heureusement pas à empiler théorèmes, articles et honneurs en tout genre. Après toutes ces années et une carrière bien fournie, elle continue à calculer et penser pour améliorer sans cesse sa quête d'une compréhension mathématique du monde. Au cours de nos longues discussions, nous avons pu développer les questions mathématiques en relation avec d'autres domaines tels que l'épistémologie, la philosophie et l'Histoire, entre autres. C'est seulement par la transdisciplinarité que les mathématiques prennent tout leur sens. Je la remercie également pour son suivi quotidien et pour ses nombreuses corrections du manuscrit et des présentations. Elle a tempéré mon appétence algébrique qui aurait rendu les travaux de thèse encore plus difficilement transmissibles pour l'application, quoique nécessaire, de l'algèbre au calcul scientifique.

Le Loup et le Chien

Un Loup n'avait que les os et la peau,
Tant les chiens faisaient bonne garde.
Ce Loup rencontre un Dogue aussi puissant que beau,
Gras, poli, qui s'était fourvoyé par mégarde.
L'attaquer, le mettre en quartiers,
Sire Loup l'eût fait volontiers ;
Mais il fallait livrer bataille,
Et le Mâtin était de taille
À se défendre hardiment.
Le Loup donc l'aborde humblement,
Entre en propos, et lui fait compliment
Sur son embonpoint, qu'il admire.
« Il ne tiendra qu'à vous beau sire,
D'être aussi gras que moi, lui repartit le Chien.
Quittez les bois, vous ferez bien :
Vos pareils y sont misérables,
Cancres, hères, et pauvres diables,
Dont la condition est de mourir de faim.
Car quoi ? rien d'assuré : point de franche lippée ;
Tout à la pointe de l'épée.
Suivez-moi : vous aurez un bien meilleur destin. »
Le Loup reprit : « Que me faudra-t-il faire ?
– Presque rien, dit le Chien, donner la chasse aux gens
Portants bâtons, et mendiants ;
Flatter ceux du logis, à son Maître complaire :
Moyennant quoi votre salaire
Sera force reliefs de toutes les façons :
Os de poulets, os de pigeons,
Sans parler de mainte caresse. »
Le Loup déjà se forge une félicité
Qui le fait pleurer de tendresse.
Chemin faisant, il vit le col du Chien pelé.
« Qu'est-ce là ? lui dit-il. – Rien. – Quoi ? rien ? – Peu de chose.
– Mais encore ? – Le collier dont je suis attaché
De ce que vous voyez est peut-être la cause.
– Attaché ? dit le Loup : vous ne courez donc pas
Où vous voulez ? - Pas toujours ; mais qu'importe ?
– Il importe si bien, que de tous vos repas
Je ne veux en aucune sorte,
Et ne voudrais pas même à ce prix un trésor. »
Cela dit, maître Loup s'enfuit, et court encor.

Jean de La Fontaine

Table des matières

Introduction

Le Calcul est bien plus qu'une action mécanique et une réalisation pratique des opérations sur les nombres. Le Calcul est aussi une action de pensée, c'est-à-dire le choix d'outils conceptuels dont on dispose à chaque époque et que l'on s'autorise à utiliser. Pourquoi calculer ? Pour tenter de représenter et d'interpréter de façon relative l'information qui nous entoure. Pour calculer, les nombres et les opérations nous servent à construire la perception mathématique de notre environnement. Cette perception évolue au fil de l'expérience humaine intelligible (ce que l'on peut comprendre) et sensible (ce que l'on peut percevoir et/ou mesurer). Les savoirs développés et hérités depuis les communautés humaines des premiers temps ont été compilés et enrichis pour être utilisés et transmis au cours des siècles.

Depuis l'Antiquité jusqu'à l'époque médiévale, le Calcul a été enseigné en Occident dans ce qui s'appelle les arts libéraux, qui sont à la source de l'enseignement universitaire en Europe. Les arts libéraux (ou *septivium*) désignent sept piliers : trois pour le « pouvoir de la langue » regroupés sous le nom de *trivium* (grammaire, dialectique et rhétorique) , quatre pour le « pouvoir des nombres » appelés le *quadrivium* (arithmétique, géométrie, musique et astronomie). L'acquisition du *trivium* était un prérequis à la maîtrise du *quadrivium* pour le Calcul. L'enseignement des arts libéraux avait pour objectif de développer une ouverture intellectuelle et mettait en avant l'interaction entre le *trivium* et le *quadrivium*, c'est-à-dire la maîtrise de la pensée et des nombres. Les arts libéraux étaient une source d'inspiration importante pour les illustrations médiévales (voir Figure 1).

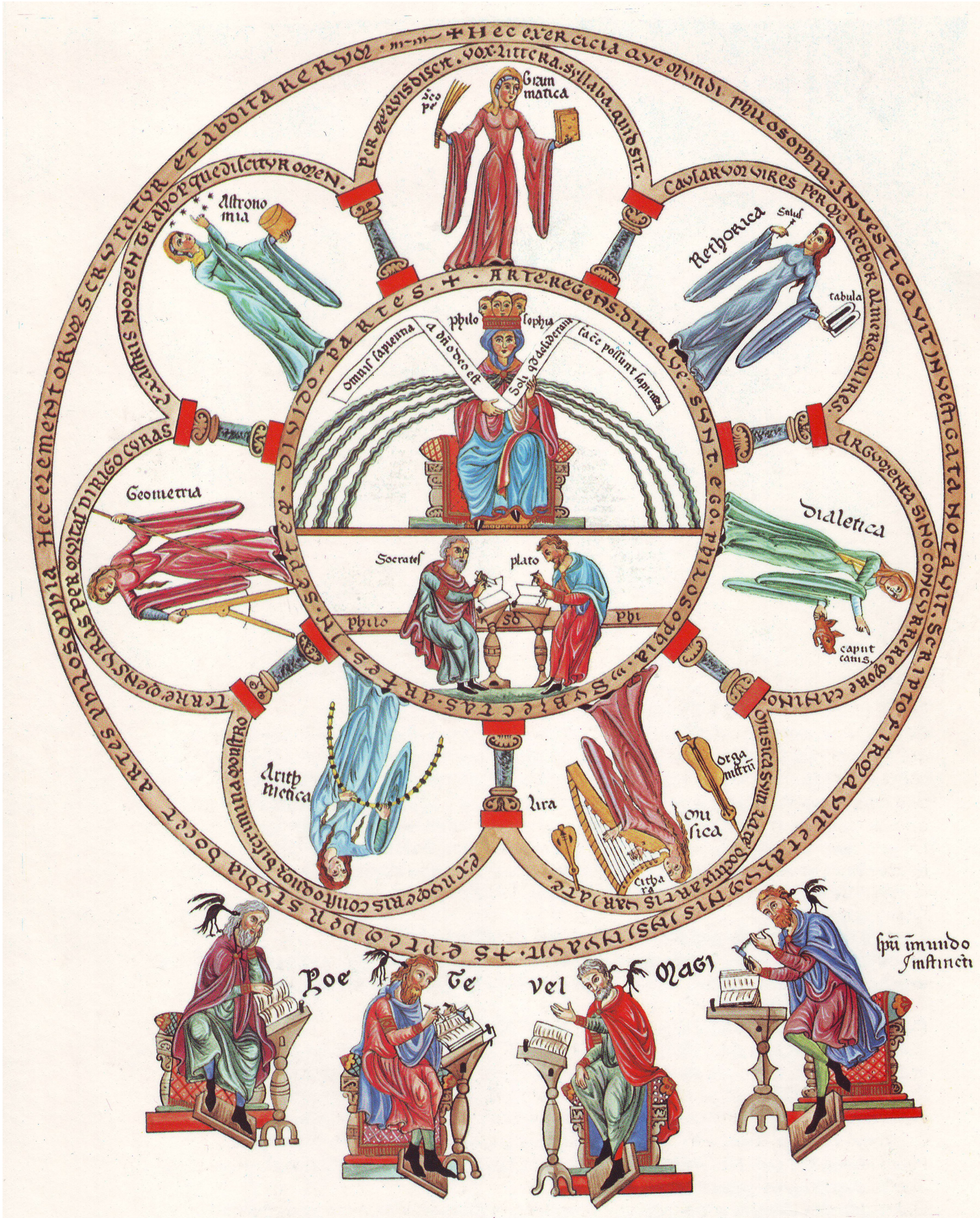


FIGURE 1 – Allégorie des arts libéraux, *Hortus deliciarium*, H. de Landsberg (1180)

Cependant, les arts libéraux étaient séparés des arts mécaniques (*artes mechanicae*). Les arts mécaniques regroupaient les métiers issus des différentes corporations d'artisans pour des applications techniques telles que la construction, les métiers du textile, l'agriculture ou de la logistique du commerce (navigation, transport). Cette séparation constitue ce que nous interprétons ici comme un premier principe de division entre l'intellect et la pratique.

La mise à l'écart de la pensée de l'époque médiévale au cours des siècles suivants ansi que le développement progressif des disciplines et des sciences modernes ont lentement fait évoluer l'enseignement vers une spécialisation, une séparation et un cloisonnement des disciplines. Ce mouvement n'a eu de cesse de s'amplifier, chaque discipline étant elle-même découpée en parties plus ou moins hermétiques. Nous observons donc un second principe de division entre les disciplines et/ou au sein même des disciplines. S'il faut reconnaître que cette séparation a permis des avancées notables dans chacune des disciplines, force est de constater que la communication entre chacune d'entre elles devient complexe voire quasi impossible et nécessite toujours plus de temps et d'efforts (principalement sur le vocabulaire, les outils, les méthodes) pour arriver à faire fructifier leur interaction et leur complémentarité.

Ce constat de divisions etant posé, nous pouvons revenir au Calcul et à la position du travail présenté dans cette thèse. Un objectif idéal aurait été de présenter une approche unifiée du Calcul permettant la synergie des différentes contributions, littéraire, scientifique et technique. Néanmoins, les sujets interdisciplinaires demandent beaucoup de connaissances transverses pour rédiger mais aussi pour lire ce type de recherche. La difficulté d'effectuer une synthèse se répercute alors sur la complexité de lecture d'une telle tentative. Il faut également mesurer l'intérêt de vouloir consacrer un temps conséquent à relier de force des sujets parfois très éloignés, ce qui ne résulte pas nécessairement sur des apports significatifs à la connaissance du Calcul. C'est pourquoi nous restreignons notre champ de recherches. Dans cette thèse, nous présentons une approche plus raisonnable qui met en relation et compare certaines branches des mathématiques et d'autres disciplines applicatives, avec un certain aspect historique.

Le fil d'Ariane de ce manuscrit est la multiplication de nombres multidimensionnels, associée à une structure algébrique en dimension finie. Nous utiliserons la multiplication comme lien entre les divisions apparentes des différents domaines qui seront abordés dans la thèse. Dans le domaine mathématique ou applicatif, la multiplication de nombres possède un rôle clef, souvent implicite ; rôle que nous précisons dans la thèse. Nous nous pencherons particulièrement sur les questions de modélisation en physique et en ingénierie. En particulier, notre approche vise, via la multiplication, l'étude de certains phénomènes non linéaires que l'on retrouve fréquemment dans le domaine de la Science et de l'Industrie. La nonlinéarité offre de grands défis de modélisation mais aussi de résolution. Nous verrons que suivant les cas d'étude, des aspects numériques avantageux découlent d'un choix judicieux des modèles et des structures algébriques associées. Ces avantages sont principalement dus à la manière de définir la multiplication dans les algèbres concernées. Nous verrons que parmi toutes les structures algébriques que l'on peut définir de manière générale, certaines ont plus d'intérêt que d'autres pour calculer.

Nous avertissons le lecteur que le manuscrit suit une trajectoire quelque peu singulière et non linéaire, à l'image des thèmes mathématiques abordés. Les nombreux domaines mathématiques explorés (algèbre, géométrie, modélisation numérique) et les liens avec d'autres disciplines (mécanique, chimie, physique) nécessitent une très large bibliographie qui est disparate dans le temps, les thèmes et le style de rédaction. La mise en commun et la comparaison de différents formalismes nous a amené à faire, à certains moments, des concessions sur la forme et la rigueur, afin de favoriser une prise de recul par rapport aux points purement techniques qui nécessiteraient un développement plus large, mais peu utile pour notre objectif qui est, rappelons-le, d'étudier le rôle de la multiplication pour le Calcul.

La thèse est constituée de deux parties articulées autour du Calcul et de la multiplication. Afin de faciliter l'accès à la bibliographie, nous proposons une bibliographie par chapitre en plus de la bibliographie complète.

-Partie I :

La première partie est une mise en perspective des nombres et structures algébriques utilisés pour calculer. Dans le premier chapitre, nous analysons certaines des inventions mathématiques pour le Calcul. Ces inventions sont des transgressions radicales qui ne peuvent être présentées de manière atemporelle mais qui sont liées au cours de l'Histoire.

Les Chapitres 2 et 3 présentent respectivement les algèbres hypercomplexes et les algèbres de Dickson que nous allons utiliser et en expliciter l'usage dans la seconde partie.

Nous comparons ces structures algébriques dans le quatrième chapitre avec d'autres approches de Calcul (calcul vectoriel et algèbres de Clifford). Nous mettons en avant la différence entre une équivalence théorique descriptive (les isomorphismes de structures) et les conséquences pour le Calcul (modélisation et comportement numérique) afin d'expliquer certains phénomènes applicatifs que nous décrivons dans la seconde partie. Dans cette première partie, nous adoptons volontairement une présentation discursive, non structurée par des théorèmes et des propositions. En effet, les résultats présentés sont techniquement démontrés dans la bibliographie, nous mettons en avant l'interprétation de ces différents travaux et le fil qui les relie, aspects qui, *a contrario*, font défaut dans la littérature actuelle. La nouveauté que nous apportons dans cette partie est une approche comparative des différentes structures utilisées pour calculer. Nous verrons que cette comparaison fait apparaître des différences notables pour le Calcul et des aspects très spécifiques rarement soulevés dans les références bibliographiques, qui se limitent pour la plupart à une présentation isolée d'une structure ou d'un groupe particulier de structures algébriques.

-Partie II :

La seconde partie présente des nouveaux résultats mathématiques et applicatifs.

Le Chapitre 5 se concentre sur les conséquences de l'utilisation des structures alternatives (au sens ordinaire : autres que $\mathbb{R}$ ou $\mathbb{C}$) pour les racines de polynômes.

Ceci permet dans le Chapitre 6 de proposer une extension nouvelle de la théorie spectrale présentée sous le nom de couplage spectral.

Le Chaptitre 7 présente des applications inédites des Chapitres 5 et 6 à la mécanique et à la chimie.

Le Chapitre 8 est une synthèse des applications existantes et des principaux avantages des algèbres que nous étudions, pour la Science et l'Ingénierie. Les structures algébriques étudiées offrent également des perspectives pour le domaine du Calcul Quantique.

Au long de notre recherche, nous allons développer une certaine vision du Calcul pour montrer des aspects méconnus, mais remarquables, que nous offre la multiplication de nombres.

Première partie

Approche comparative pour le Calcul : un état des lieux

Chapitre 1

Des chiffres et des nombres à travers les âges

Dans ce chapitre, nous présentons quelques éléments de l'Histoire des nombres pour le Calcul. Cette approche qui peut paraître n'être qu'un détail, un point très particulier de l'Histoire des mathématiques, est révélatrice de la lente évolution de la notion de Calcul au cours des âges. Il n'est pas ici question de développer l'ensemble immense des connaissances mathématiques, mais de se concentrer sur le Calcul et ses deux aspects numérique et algébrique. Par numérique, nous entendons relatif au concept de nombre relié à des phénomènes de Calcul, ce qui diffère énormément du terme moderne concernant les progrès techniques associés à l'informatique. Les avancées majeures, liées aux découvertes survenues au cours des siècles, sont des exemples préparatoires portant sur les nombres, qui vont introduire notre démarche d'algèbres alternatives pour le Calcul. Nous réintroduisons l'aspect historique des mathématiques lié au développement de certaines idées ainsi qu'à leur résistible acceptation. Les exemples historiques que nous présentons constituent des ruptures **radicales** par rapport à l'idée de ce qui était considéré comme un nombre suivant l'époque. En tant que nouveauté qui intrigue et qui dérange, ces ruptures ont fait face à une résistance conceptuelle et idéologique dont il ne faut pas sous-estimer le rôle dans le développement actuel des mathématiques. Les résistances face à ces nouvelles idées perdurent jusqu'à nos jours. Nous verrons en effet que certaines inventions associées aux structures algébriques étudiées dans cette thèse ne sont pas encores pleinement considérées, malgré leur intérêt mathématique ainsi que leur apport dans les disciplines applicatives.

1.1 Une succession de transgressions créatives

Le concept de nombre évolue au cours du temps. Loin d'une approche monolithique où les mathématiques peuvent apparaître comme une séquence logique de découvertes menant aux mathématiques du XXI^{me} siècle, les nombres qui sont l'outil de base du Calcul ont évolué par une succession de ruptures idéologiques ancrées dans l'histoire de leur temps. On peut regretter que l'Histoire des mathématiques ne soit pas enseignée conjointement aux aspects techniques, au moins pour les personnes qui suivent des études scientifiques. Nous ne pouvons qu'encourager la lecture des références sur l'Histoire des nombres ou des mathématiques qui permettent à chacun de construire son opinion à partir de faits historiques. Nous verrons dans la suite que ces aspects sont indissociables de notre approche mathématique. Parmi les nombreuses possibilités plus ou moins connues telles que [Weil, 1984, van der Waerden, 1985, Conway and Guy, 1996], nous citons en particulier [Ebbinghaus et al., 1998] qui présente dans leur contexte historique certains résultats peu développés ailleurs.

Il nous semble trop réducteur de ne voir dans les mathématiques qu'une collection de théories et de théorèmes à démontrer de façon purement logico-technique. Sans l'aide d'une interprétation dans un contexte historique et philosophique, il semble hors de portée de comprendre les enjeux et la profondeur des concepts mis à jour au cours des siècles.

Plutôt que de suivre la présentation classique des mathématiques selon une logique atemporelle, nous rappelons le rôle de l'Histoire dans les inventions des mathématiciens. De grands savants reconnus ont marqué leur temps, mais les réputations ou titres prestigieux associés à l'idéologie d'une époque ne suffisent pas à garantir la valeur d'une découverte. Nous verrons qu'un certain nombre de résultats dont la richesse dépasse toute distinction honorifique, sont relégués aux marges ou même oubliés par l'histoire officielle. Ces résultats dus aussi bien à des amateurs qu'à des mathématiciens accomplis sont fondamentaux pour notre étude sur les nombres et le Calcul.

Il est important de mesurer la dimension temporelle qui caractérise l'Histoire des nombres. Le temps écoulé entre les découvertes, qui sont par nature des transgressions des règles communément acceptées, et l'acceptation de leur validité peut durer des siècles. Le faible nombre de mathématiciens ou les moyens de communication de l'époque ne sont pas les seules raisons de la lenteur de ces développement. Il ne faut pas négliger la résistance idéologique ou conceptuelle que les théories nouvelles peuvent rencontrer initialement. Pourtant ces inventions ne sont pas le résultat d'errances mathématiques mais la réponse créative face à la nécessité imposée le plus souvent par le Calcul. Ces différentes inventions ne sont pas non plus une simple reformulation ou une synthèse des connaissances existantes, mais correspondent à la production d'un concept qui n'existait pas auparavant.

1.2 Elements de contexte numérique

Nous rappelons en préambule la confusion commune qu'il faut éviter entre les chiffres et les nombres. Les chiffres sont associés à la question de l'écriture (sytèmes sexagésimal, décimal, binaire, chiffres romains...). Les nombres sont en lien avec la notion de Calcul. Quelle que soit l'écriture des nombres à l'aide de chiffres, cela ne change pas leur rôle calculatoire. Certaines notions de nombres nous sont parvenues sous une forme discursive sans aucune représentation chiffrée, en particulier pour l'Antiquité. Confondre les chiffres et les nombres revient à confondre l'écriture et le concept.

Nous présentons ci-dessous trois concepts de nombres particulièrement marquants apparus au cours de l'Histoire (les nombres irrationnels, le zéro et les nombres négatifs). Chacun à leur façon, ils ont créé une rupture dans la définition communément admise du concept de nombre à leur époque respective. Ces trois exemples illustrent l'**émergence** d'un nouveau concept en réponse à un besoin de calcul. Ils nous préparent à l'arrivée de ruptures algébriques qui seront l'objet des prochains chapitres. Les quelques éléments qui suivent respectent le sens chronologique de l'Histoire. Depuis l'époque sumérienne, l'Antiquité grecque et les civilisations d'Orient (Inde, Chine, Bagdad) puis l'Occident, les idées pour le Calcul se sont développées et enrichies. Ces quelques éléments offrent une perpective sommaire et succinte que l'on peut retrouver de manière plus détaillée dans les ouvrages cités plus haut.

1.2.1 Sumer

Les Sumériens, il y a plus de 4000 ans avaient des connaissances avancées de Calcul dans de nombreux domaines. Le travail toujours en cours des archéologues permet de retracer quelques éléments de leur vaste savoir, depuis les tables de multiplication, calculs de taux d'intérêts jusqu'aux nombreuses études géométriques, en particulier concernant les mesures astronomiques. Les Sumériens calculaient en base soixante, notre mesure actuelle du temps (secondes, minutes et heures) et des angles en est l'héritage direct. Ils étaient capables de résoudre certaines équations du second degré revêtues d'un habillage pratique, par exemple pour certains problèmes liés à l'agriculture. Mais il ne faut pas réduire les connaissances sumériennes à quelques calculs pratiques. La maîtrise mathématique sumérienne, telle que nous la connaissons aujourd'hui, montre une compréhension remarquable de certains concepts profonds. Parmi les traces écrites retouvées, la tablette Plimpton 322 contient une liste de triplets pythagoriciens. La tablette YBC 7289 donne quant à elle une approximation de $\sqrt{2}$ en base 60 à environ 10^{-6} près ! Encore aujourd'hui, il n'y a aucune explication argumentée de l'algorithmitique et des motivations liées à ces connaissances, remarquables pour l'époque. Malgré les découvertes des historiens et des archéologues, les origines de la maîtrise mathématique de l'époque sumérienne reste encore au niveau des suppositions. A partir des ces premières traces écrites, les connaissances mathématiques se sont enrichies en traversant les civilisations et les époques dans les plus grands empires, royaumes et cités du monde connu.

1.2.2 Grèce antique

La découverte de l'irrationnalité de $\sqrt{2}$ a profondément heurté la philosophie mathématique grecque. Il n'y a pas de trace directe comme les tablettes babyloniennes, seulement des copies de discours. Le concept philosophique de nombre dans l'antiquité hellénique est avant tout fondé sur les nombres rationnels. Les philosophes grecs tentent de comprendre le monde qui les entoure par le concept de *cosmos*. Le cosmos représente l'idée d'ordre et d'harmonie du monde dont tous les éléments doivent obéir à des lois physiques et mathématiques par un rapport de quantités commensurables. C'est la raison pour laquelle $\sqrt{2}$, qui défie ces lois rationnelles, n'est pas reconnu par les Grecs comme un nombre, bien qu'il mesure la diagonale du carré unité ! N'oublions pas que *ratio* signifie aussi bien raison que rapport.

1.2.3 Inde

Quelques siècles plus tard, en Inde, le zéro qui n'était qu'un chiffre, a émergé de manière explicite en tant que nombre. Il était déjà utilisé pour effectuer des calculs pratiques depuis l'époque sumérienne mais c'est en tant que concept que le zéro interroge les mathématiciens. En effet, naturellement lié à l'addition, le zéro est à la fois un défi philosophique et mathématique [Chatelin, 2012]. Le nombre zéro qui symbolise la non-existence peut-il lui même exister ? C'est également grâce à la culture mathématique indienne qu'a été développée la numération décimale, représentée par ce qui est plus connu en Occident sous le nom de chiffres arabes. L'usage de ces chiffres permet d'écrire plus facilement certaines opérations usuelles (multiplications, fractions), par comparaison avec les chiffres romains utilisés dans une grande partie de l'Europe médiévale.

1.2.4 Orient (Inde, Chine, Bagdad)

Nous nous intéressons maintenant au cas des nombres négatifs, qui sont directement liés au zéro. Les nombres négatifs, en tant qu'outils de calcul, sont connus depuis environ le premier siècle avant notre ère en Chine. Dans *Les Neuf Chapitres sur l'art mathématique*, livre écrit comme compilation anonyme des résultats de cette époque, les nombres positifs y sont représentés par des baguettes rouges et les nombres négatifs par des baguettes noires, dans le but de résoudre des problèmes d'échanges commerciaux. Au VII$^{\text{ème}}$ siècle, en Inde, Brahmagupta, après avoir introduit le zéro donnait des règles de calcul sur les nombres négatifs, sans aucune justification sur leur sens : « Une dette retranchée du néant devient un bien, un bien retranché du néant devient une dette ». Le cas d'application et le vocabulaire sont là encore limités aux échanges commerciaux, où les quantités négatives sont plus aisément concevables. Les connaissances mathématiques grecque et indienne ont été par la suite rassemblées et développées par Al-Khwarizmi, né dans la région du Khwarezm (actuel Ouzbékistan) dans une famille perse qui a travaillé ensuite à Bagdad (IX$^{\text{ème}}$ siècle). C'est du nom de cet érudit que découle le mot *algoriste* apparu au XIII$^{\text{ème}}$ siècle en Italie pour qualifier un calculateur. Puis au XVIII$^{\text{ème}}$ siècle, le terme algorithme désigna une ordonnance de calculs arithmétiques.

1.2.5 Europe et Occident

Le socle de savoirs algébriques du monde arabophone a été apporté en Europe et développé principalement par Gerbert d'Aurillac, le futur Pape Sylvestre II (999-1003), mais aussi Fibonacci à Pise en Italie (v.1175-v.1250). Le manuscrit de Pamiers, rédigé en 1430 dans la langue vernaculaire du comté de Foix (Ariège), est un traité d'arithmétique commerciale qui est une compilation des connaissances mathématiques indienne et grecque. Il contient ce qui semble être la première solution négative à une équation en Occident, **explicitement** écrite ($-10\ \frac{3}{4}$).

Cependant, l'interprétation des quantités négatives en tant que nombres est toujours un problème. En 1637, dans *La géometrie*, Descartes écrit : « Mais souvent il arrive, que quelques-unes de ces racines sont fausses, ou moindre que rien, comme si on suppose que x désigne aussi le défaut d'une quantité, qui soit 5, on a $x + 5 = 0$ ». Descartes poursuit, et à propos d'un certain polynôme lié au calcul précédent, écrit : « il y a quatre racines, à savoir trois vraies qui sont 2, 3, 4 et une fausse qui est 5 ». La difficulté d'interpréter les nombres négatifs se maintient pendant plusieurs siècles. A titre d'exemple, nous mentionnons la position peu amène de Maseres, mathématicien anglais, à propos des racines négatives : « Elles servent seulement pour autant que je sois capable d'en juger, à obscurcir la doctrine tout entière des équations et à rendre ténébreuses des choses qui sont dans leur nature excessivement évidentes et simples. Il eût été souhaitable en conséquence que les racines négatives n'aient jamais été admises dans l'algèbre ou qu'elles en aient été écartées » [Maseres, 1758].

Il est intéressant de constater que la difficulté d'interprétation des nombres négatifs est naturellement liée à celle de l'interprétation du zéro, en tant que quantité absolue ou relative. Lazare Carnot (le père de Sadi Carnot), mathématicien reconnu de l'Académie des Sciences écrit : « Pour obtenir réellement une quantité négative isolée, il faudrait retrancher une quantité effective de zéro, ôter quelque chose de rien : opération impossible. Comment donc concevoir une quantité négative isolée ? » [Carnot, 1803].

Pour résumér la situation, nous nous appuyons sur l'analyse de Jean-Robert Argand, souvent considéré comme mathématicien amateur, qui a pourtant apporté une interprétation rigoureuse des nombres négatifs. Nous verrons dans la Section 1.3 qu'Argand joue un rôle important pour ses contributions sur les nombres complexes. Concernant les nombres négatifs, il écrit que « Ces idées sont très élémentaires ; néanmoins, il n'est pas si aisé qu'il pourrait le paraître d'abord de les établir d'une manière bien lumineuse, et d'y donner cette généralité que demande leur application aux calculs. On ne peut d'ailleurs douter de la difficulté du sujet, si l'on réfléchit que les sciences exactes aient été cultivées pendant un grand nombre de siècles, et qu'elles avaient fait de grands progrès avant qu'on eût acquis les véritables notions des quantités négatives, et qu'on eût conçu la manière générale de les employer » [Argand, 1806].

Il faudra attendre le XIX^{me} siècle afin que les nombres négatifs commencent à être pleinement acceptés par les savants de cette époque, sous l'influence de mathématiciens faisant autorité qui ont repris à leur compte, ou à qui ont été attribués les résultats de mathématiciens moins renommés.

1.2.6 Résumé

Chacun de ces trois exemples, les nombres irrationnels, le zéro et les nombres négatifs présente un développement similaire dans l'Histoire du Calcul que nous pouvons découper en trois étapes. D'abord, ce sont des inventions utilisées comme un outil pratique de Calcul sans interprétation. Ensuite, la formalisation de cette idée en tant que concept mathématique implique une remise en question des savoirs établis. Selon un temps plus ou moins long, grâce à l'influence de personnes reconnues ou à leur intérêt applicatif, ces ruptures radicales sont progressivement acceptées. Plus que les nombres en eux-mêmes, introduits pour une nécessité pratique de calcul, c'est leur portée philosophique qui vient heurter les certitudes et qui, jusqu'à nos jours, met en avant la « déraisonnable efficacité des mathématiques » [Wigner, 1960] pour représenter le monde qui nous entoure. Le développement de l'algèbre au XIX^{me} siècle va contribuer à l'étape de formalisation des exemples présentés dans ce chapitre.

Nous verrons dans la partie suivante en quoi la découverte des nombres complexes constitue le point de départ de notre étude sur les structures algébriques de nombres en dimension strictement plus grande que 1.

1.3 Autour des nombres complexes

L'histoire des nombres complexes est un cas d'école de la conception des nouvelles idées et des ses conséquences pour le Calcul. Les nombres complexes ont été introduits par Cardan comme un outil pour trouver les solutions réelles de certaines équations du troisème degré [Cardano, 1545]. Cette nouveauté s'est confrontée à une résistance notable (au premier chef de Cardan lui-même très troublé par ses propres inventions), en tant que transgression du cadre bien connu donc sécurisant des nombres réels où seuls les nombres positifs ont une racine carrée. Presque trois siècles après, ils ont reçu une interprétation géométrique dans le plan numérique grâce à Wessel [Wessel, 1799] et aussi Argand [Argand, 1806], l'auteur de la première démonstration complète et rigoureuse du théorème de d'Alembert (« Fundamental Theorem of Algebra (FTA) ») [Argand, 1814] qui parachève les tentatives incomplètes de d'Alembert (1746) et de Gauss (1799).

L'invention de l'unité imaginaire $\sqrt{-1}$, notée i par Euler, est une réponse créative à la nécessité de dépasser le cadre limité de l'époque (nombres réels de carrés positifs) car ils peuvent servir d'outil pour calculer des solutions réelles de certaines équations polynomiales à coefficients réels. Cette idée qui ne semble si évidente aujourd'hui que parce qu'elle nous est familière, a fini par profondément changer la vision du Calcul au XIX^{me} siècle. Elle a ouvert des voies fécondes qui ont mené à des théories fondamentales telles que l'anaylse complexe, la théorie de Galois et bien d'autres. L'invention des nombres complexes est une rupture radicale qui a contribué au développement de l'algèbre. Une autre conséquence est l'association entre une structure algébrique et une géométrie donnée du plan. Les nombres complexes représentent un saut conceptuel pour l'interprétation du plan réel, passant de l'espace vectoriel $\mathbb{R} \times \mathbb{R}$ à l'algèbre $\mathbb{C}$. C'est pourquoi il n'est pas exagéré de dire que les nombres complexes sont une révolution pour l'algèbre, la géométrie et l'analyse.

1.4 Des résolutions numériques aux structures algébriques

Les nombres complexes, fondés sur une transgression du Calcul (existence de racines carrés de nombres négatifs), ont été progressivement acceptés en Mathématiques principalement après qu'ils eurent démontré leur intérêt en Physique au XIX^{eme} siècle. La nécessité qui s'impose par le Calcul de considérer des racines carrées de nombres négatifs ouvre la porte à l'idée de multiplier des nombres en deux dimensions. L'idée de i est une transformation radicale d'une multiplication 1D sur la droite réelle, le produit $aa' \in \mathbb{R}$, en une multiplication 2D dans le plan, $(a+ib)(a'+ib') = aa'-bb'+i(ab'+ba')$, qui mène à $\mathbb{C}$.

D'un point de vue historique et épistémologique, le rôle implicite de la multiplication en 2D est le fil d'Ariane qui relie les travaux de Cardan à l'intepértation géométrique d'Argand. Il est intéressant de noter que dans le cadre de l'interprétation géométrique des nombres complexes, Argand a également beaucoup travaillé sur les nombres négatifs. Ceci met en évidence le lien étroit entre l'idée des nombres négatifs et celle des nombres complexes, idées qui posaient toutes deux encore problème à l'époque d'Argand, pour la géométrie et le développement de l'algèbre.

Les nombres complexes sont donc un pont entre les découvertes mathématiques liées aux nombres, sur la droite réelle, et celles associées à l'algèbre, aux structures de nombres qui sortent de la dimension réelle et proposent des perspectives nouvelles pour la multiplication. Les nombres complexes constituent un exemple fondateur pour notre travail car le recul historique dont nous disposons aujourd'hui, nous permet de mesurer l'enjeu et l'apport dans les disciplines scientifiques de ruptures radicales de Calcul.

Le fil directeur avec la suite de notre développement est celui du Calcul et de la diversification des outils algébriques utiles à cette fin. Pour cela, nous aurons besoin de détailler plus amplement les propriétés mathématiques associées aux opérations classiques du Calcul et principalement de la **multiplication**. Nous pourrons alors introduire plus précisément les structures algébriques et celles de leurs applications que nous allons étudier dans la thèse.

Bibliographie

[Argand, 1806] Argand, J.-R. (1806). *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques.* chez Madame Veuve Blanc, Horloger, Paris.

[Argand, 1814] Argand, J. R. (1814). Réflexions sur la nouvelle théorie des imaginaires, suivies d'une application à la démonstration d'un théoreme d'analyse. *Annales de Mathématiques Pures et Appliquées*, 5 :1814–1815.

[Cardano, 1545] Cardano, G. (1545). *Ars Magna or the rules of algebra (english translation 1968).* Dover NY.

[Carnot, 1803] Carnot, L. (1803). *Géométrie de position.* Imprimerie de Crapelet.

[Chatelin, 2012] Chatelin, F. (2012). *Qualitative Computing : a computational journey into nonlinearity.* World Scientific, Singapore.

[Conway and Guy, 1996] Conway, J. H. and Guy, R. K. (1996). *The book of numbers.* Springer Verlag NY.

[Ebbinghaus et al., 1998] Ebbinghaus, H., Hermes, H., and Hirzebruch, F. (1998). *Les nombres.* Vuibert, Paris.

[Maseres, 1758] Maseres, F. (1758). *A Dissertation on the Use of the Negative Sign in Algebra.* Samuel Richardson ; and sold by Thomas Payne.

[van der Waerden, 1985] van der Waerden, B. L. (1985). *A history of algebra.* Springer.

[Weil, 1984] Weil, A. (1984). *Number theory, an approach through history from Hammurapi to Legendre.* Birkhauser.

[Wessel, 1799] Wessel, Caspar, . (1799). *On the analytical representation of direction : an attempt applied chiefly to solving plane and spherical polygons.* Memoir Royal Acad. Denmark.

[Wigner, 1960] Wigner, E. (1960). The unreasonable effectiveness of Mathematics in the natural sciences. *Comm. Pure and Appl. Math*, 13 :1–14.

Chapter 2

Let us multiply vectors

In this chapter, we will focus on some general algebraic principles that underlie the specific algebraic constructions and applications to Computation to be presented in the following chapters. First, we will recall some basic elements of algebraic structures and then the properties of operations classically used in Computing. A special attention will be given to multplication, and especially multiplication of vectors. This chapter contains basic concepts that are well known in abstract algebra but more rarely used in a numerical context, that is the use of numbers to Compute. It is not written in a standard algebraic way (several results or definitions are quickly presented on purpose), because the goal is to use these properties for Computation and not only for a descriptive comprehension.

2.1 General properties

2.1.1 Basic notations and definitions

As a preliminary warning we mention that the following definitions are not to be considered for their axiomatic aspects but for their role in Computation. In particular, the classical definition for rings and algebras assumes associativity of multiplication. When the multiplication is no longer associative (see below), these structures are often called *non associative rings or algebras* [Raffin, 1951, Schafer, 1995] and we will see some examples later. We will specify it for the related structures, but we will keep the usual terms algebra and ring, even though it is a slight abuse of classical definition.

We will follow the definition in [Schafer, 1995], a *ring* $(R, +, \times)$ is equipped with an addition $+$, an internal multiplication $\times$ such that multiplication $\times$ is distributive over addition. We suppose by definition that a ring contains the neutral element for multiplication, denoted 1. Multiplication is *not* supposed to be associative. As a convention, we suppose that rings are not reduced to zero $(0 \neq 1)$.

In what follows, A is a real algebra of finite dimension equipped with the standard operation of addition $+$ (performed componentwise), an external multiplication by a scalar $\bullet$ and an internal multiplication $\times$. We recall that a real algebra relies on these three properties:

1. $(A, +, \bullet)$ is a vector space over $\mathbb{R}$, of finite dimension $n \in \mathbb{N}^*$,
2. $(A, +, \times)$ is a ring,
3. $\forall \lambda \in \mathbb{R},\ \forall (x, y) \in A^2,\ \lambda \bullet (x \times y) = (\lambda \bullet x) \times y = x \times (\lambda \bullet y)$.

A very important remark is that multiplication by an element a, distributive over addition, can be considered as a *linear* application $x \mapsto a \times x$ as for a, x, y in A, α, β in $\mathbb{R}$, $a \times (\alpha \bullet x + \beta \bullet y) = \alpha \bullet a \times x + \beta \bullet a \times y$.

Remark 2.1.1. *In what follows, we will leave aside the symbol of external multiplication* $\bullet$*. The multiplication symbol* $\times$ *between real or complex quantities will be omitted in general. The notation* $\times$ *will only be used for multiplication of the ring. We will explain later why we restrict this symbol to multiplication and do not use it in linear algebra for the cross product. The cross product, as it is* *known in linear algebra, will be described as a consequence of multiplication in composition algebras (see Chapter 3).*

In this thesis, internal multiplication $\times$ will be our main concern. We recall the following properties:

1. *Commutativity*: $\forall (x, y) \in A^2, x \times y = y \times x$,
2. *Associativity*: $\forall (x, y, z) \in A^3, (x \times y) \times z = x \times (y \times z)$,
3. *Alternativity*: $\forall (x, y) \in A^2, x \times (x \times y) = (x \times x) \times y$ and $(x \times y) \times y = x \times (y \times y)$,
4. *Flexibility*: $\forall (x, y) \in A^2, (x \times y) \times x = x \times (y \times x)$,
5. *Power associativity*: $\forall x \in A, \forall (i, j) \in \mathbb{N}^2, x^0 = x, x^{i+1} = x \times x^i, x^i \times x^j = x^{i+j}$.

From associativity to power associativity, each property implies the followings in a logical sequence ($2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$). Notice that there exist commutative algebras that are nonassociative (e.g. Jordan algebras). In the most common frame for Computation, multiplication is often commutative (real or complex analysis) or associative (matrices over $\mathbb{R}$ or $\mathbb{C}$). It seems that Hamilton was the first to introduce the name associativity [Ebbinghaus et al., 1998] and to discover the nonassociativity of the octonions (see Chapter 3).

In the algebraic structures that we will use later, we will be interested in the consequences of the existence or absence of these properties for Computation. In particular, it is quite instructive to study the absence of a given property to understand its added incentive or, its necessity. We will develop later why the properties 1, 2 and 3 (commutativity, associativity and alternativity) can be relaxed without hindering computation. A suprising consequence is that this can be turned into an advantage in the computational possibilites and the modelling of Physics [Chatelin, 2012b]. This choice could be considered as a trade-off process, to give up something to gain in return a more adequate Computational tool for given applications.

In order to estimate the effect of these properties on multliplication between vectors, we will use three different quantitites for elements in A. In relation to the properties 1, 2 and 3, we introduce the *commutator* of two elements $x, y \in A$, $[x, y] = x \times y - y \times x$, and the *associator* of x, y, z, $[x, y, z] = (x \times y) \times z - x \times (y \times z)$. The commutator (resp. associator) is identically zero iff the algebra is commutative (resp. associative). We will see that the commutator plays a central role in the definition of the cross product in linear algebra. We call *alternator* of two elements $x, y \in A$, the associator $[x, x, y] = (x \times x) \times y - x \times (x \times y)$ of x, x and y. These three concepts can be seen as a measure of a distance to the relaxed property (commutativity, associativity, alternativity).

We intend consider these definitions in relation with Computation. The purpose of Computation is twofold. Given a phenomenon, Computation can be used as a tool, either to quantify the phenomenon, or to model it. Whereas the first goal can be greatly helped by machine computers, the second one only relies purely on mathematical skill. It is important to keep in mind this dual purpose because the usefulness of an algebra may depend on the chosen goal. Moreover our comparative study will confirm that the loss of algebraic properties entails a loss of universality. As a sartorial metaphor, considering algebras as a way to dress for Computation, $\mathbb{R}$ and $\mathbb{C}$ "fit all" like off the rack items whereas nonalternative algebras are bespoke, "tailor-made".

2.1.2 Specific concepts related to multiplication: inverses and zerodivisors

2.1.2.1 Invertible elements

Concerning multiplication, one of the first natural questions to ask is the existence of an inverse. We recall that in the case of number algebras used for Computation, there will always exist the unit 1 as a neutral element. We say that an element $x \in A$ is invertible iff there exists $y, z \in A$, such that $x \times y = 1$ (right inverse) and $z \times x = 1$ (left inverse). In the case where $y = z$, this element is called the *inverse* of x.

The existence of a unique inverse is strongly related to the property of associativity. In an alternative ring, in case of existence the inverse is also unique. This is a consequence of a theorem due to Artin saying that in an alternative ring, any subring generated by two elements is associative. In the most general case of non associative rings, there is no general result [Raffin, 1951] . When a structure with an identity element is non associative but has unique right and left inverses for every element, it is called a *loop*.

A ring where all nonzero elements are invertible is a *division ring.* Another English-speaking denomination that can be found in the litterature is *skew field.* We will not use this last term which contains the implicit assumption that fields are supposed to be commutative which is not a necessity. In all the following, we follow the continental concept of a *field* whether multiplication is commutative or not.

We will see that the field structure is a very specific property in algebraic structures that we can use to compute. As we explained before for the properties of multiplication (absence or existence), we will keep the same idea to compare the complementary cases for the existence of invertible elements and the notion of zerodivisors in the next section.

2.1.2.2 Zerodivisors

An element $x \in A$ is a *zerodivisor* if $x \neq 0$ and $\exists y \neq 0$, $x \times y = 0$. It seems that the term zerodivisor is due to Weierstrass in 1883 [Cartan, 1908, p.374, note 125]. As for inverses, there exists the notion of left and right zerodivisors but this will not be necessary in our case. In associative structures, an element cannot be at the same time invertible and a zerodivisor ($x \times y = 0$ and x invertible $\Rightarrow x^{-1} \times x \times y = 0$ and thus $y = 0$).

For an element $x \in A$, we introduce two notions that are important for the algebras that will be of interest for Computation :

— *idempotence*: $x^2 = x$,

— *nilpotence*: $\exists\, n \geq 2, x^n = 0$ (we will use in particular the case $n = 2$).

For a nilpotent element x, we call the *degree or index* of nilpotence the smallest integer i such that $x^i = 0$ and $x^{i-1} \neq 0$. Both idempotent and nilpotent elements are zerodivisors. If x is idempotent, $x \times (x - 1) = 0$ and for y nilpotent of index n, $y \times y^{n-1} = 0$. In associative algebras, all the multiples of these elements are also zerodivisors. In [Lam, 1999, pp. 320–322], a noncommutative ring in which every element is either invertible or a zerodivisor is called a *classical* ring. It seems that the litterature on this topic is quite rare. The fact that an element is either invertible or a zerodivisor will be widely used in the associative algebras that we will consider. For non associative algebras, this distinction is not valid anymore [Raffin, 1951] .

Idempotents and nilpotents are two specific categories of zerodivisors that will be often encountered in the following. Zerodivisors, in general, will play a pivotal role in the examples that we will study.

2.1.2.3 Examples with matrix algebras of order 2

Matrices are widely used in Linear Algebra and, when they are square, they offer a good example of the results presented in this chapter. Even if square matrices are primarily considered as linear operators, we will see that considered as numbers, they illustrate some computational features that remain more or less implicit in their traditional role in linear algebra.

Let us consider some examples of the properties presented in this section with the ring of real matrices of order 2.

The multiplication between matrices is associative and non commutative. Every matrix M such that $\det M \neq 0$ is invertible. If $\det M = 0$, then $M^2 - (trM)M = 0$, rewritten as $M(M - (trM)I_2) = 0$. M is thus a zerodivisor. Every matrix is either invertible or zerodivisor.

Let us now consider the matrices $E_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $E_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ which satisfy $E_+^2 = E_+$ and $E_-^2 = E_-$. This gives us two examples of idempotent elements as specific zerodivisors.

The matrix $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the most basic example of nilpotent matrix as $N^2 = 0$. This kind of structure is basic for the block decomposition of Jordan in linear algebra.

These simple examples have to be kept in mind as they will be useful in Chapter 4 when we will compare the isomorphic formulations between algebraic structures. After this sequence of preparatory algebraic properties, we will present the constructions of algebraic strutures starting with the fundamental example of the numerical plane.

2.2 Quadratic algebras of the numerical plane

In the first chapter, we have presented some elements concerning complex numbers as a first example of 2D-multiplication. However, complex multiplication is not the only way to define a multiplication of vectors in $\mathbb{R}^2$. For example, there exist two other nonreal numbers $u = \sqrt{1} \neq \pm 1$, $u^2 = 1$, [Cockle, 1848] and $n = \sqrt{0} \neq 0$, $n^2 = 0$, [Clifford, 1873]. The three numbers i, u, n are the *generators* of the three *canonical* algebras $\mathbb{R} \oplus \mathbb{R}g_c$, $g_c \in \{i, u, n\}$ that can equip $\mathbb{R}^2$ [Capelli, 1941, Kantor and Solodovnikov, 1973]. We denote ${}^2\mathbb{R} = \mathbb{R} \oplus \mathbb{R}u$ the algebra of *bireal numbers* and $\mathbb{D} = \mathbb{R} \oplus \mathbb{R}n$ the algebra of *dual numbers*. Notice that the dual unit is nilpotent $n^2 = 0$ and that the bireal numbers $e_+ = \frac{1}{2}(1+u)$, $e_- = \frac{1}{2}(1-u)$ are idempotent $e_\pm^2 = e_\pm$ and zerodivisors $e_+e_- = e_-e_+ = 0$. The general case presented by Capelli replaces g_c by an arbitrary nonreal *quadratic* number g called *generator*. In order to ensure the closure of the plane by multiplication, it is necessary that $g^2 = \gamma + 2\beta g$, $\gamma, \beta \in \mathbb{R}$. The two real parameters γ, β define a 2D algebra over $\mathbb{R}$ in the basis $\{1, g\}$ so that every number of this algebra can be written $z = x + yg$, $x, y \in \mathbb{R}$. The quadratic equation for the generator g can be rewritten $(g - \beta)^2 = \gamma + \beta^2 = \delta$. Depending on the sign of δ, g can be taken of the form:

- $\beta \pm i\sqrt{-\delta}$, $\delta < 0$,
- $\beta + na$, $\forall a \in \mathbb{R}$, $\delta = 0$,
- $\beta \pm u\sqrt{\delta}$, $\delta > 0$.

We define the *conjugate* of g as $g^* = 2\beta - g$. With the conjugation $z \mapsto z^* = x + yg^* = x + 2\beta y - yg$, every algebra of this kind is *involutive*. The quantity $\mu(z) = zz^*$ is called the *magnitude* of z. It is fundamental to remark that μ is a quadratic form: $\mu(z) = (x+yg)(x+2\beta y - yg) = x^2 + 2\beta xy - \gamma y^2 = z^T S z$, where $S = \begin{pmatrix} 1 & \beta \\ \beta & -\gamma \end{pmatrix}$. The determinant $detS = -\delta$ and the three above forms for the generator g depending on the sign of δ ($\delta < 0$, $\delta = 0$, $\delta > 0$) are related respectively to the possible types for the quadratic form μ (elliptic, parabolic, hyperbolic); μ is positive definite iff $\delta < 0$. Complex numbers, bireal numbers and dual numbers respectively correspond to the cases $\beta = 0$ and ($\gamma = -1$, $\gamma = 1$, $\gamma = 0$).

The algebraic interest is to equip a vector space with a multiplication *and* a conjugation. The structures $\mathbb{R} \oplus \mathbb{R}g$ are basic examples of quadratic *ring extension*. The operation of 2D-addition is performed componentwise using 1D-addition only. The specificity of an algebraic structure is contained in the properties of its multiplication. The shift of focus from addition to multiplication, from linear to nonlinear is at the root of all algebras of higher dimensions [Cartan, 1908].

2.3 Algebraic diversification beyond the numerical plane

The first chapter has put in evidence the transition between purely numerical inventions and algebraic inventions, from isolated numbers to algebraic structures. The case of complex numbers has illustrated this transition with $\sqrt{-1}$ considered first as a number and then as a generator for the complex field. We will now stay on the track of agebraic inventions and we will have a decriptive look at the fundamental work of Hamilton and its consequences. The technical description of the algebras presented in this section will be done in the Chapter 3.

The quaternions $\mathbb{H}$, a 4-D algebra invented by Hamilton in 1843 [Hamilton, 1844], are the first cornerstone in the modelling of 3D-space by a multiplicative structure (see Chapters 3 and 4, $\mathcal{I}\mathbb{H}$, the imaginary part of $\mathbb{H}$). From this date, during several decades, an outburst of algebraic concepts occured. Three months after the discovery of quaternions, J. T. Graves, Hamilton's friend and Irish compatriot, mentionned in a letter his invention of 8D- octonions. Hamilton quickly discovered that the multiplication invented by Graves was not associative. This started a sequence of inventions,that came out during some decades after Hamilton's idea, such as Clifford algebras and Dickson algebras that will be described in the next chapter as number structures for Computation.

Like complex numbers at their time, some of these new algebraic structures faced strong resistance coming from trendsetting scientists. Even though Maxwell used quaternions to write a first version of electromagnetism, this did not prevent them from being harshly criticised. Maxwell was aware at the same time of the algebraic potential and of the difficulty for quaternions to be accepted as computational tools. He wrote in 1870 in *Application of Quaternions to Electromagnetism*: "The limited use which has up to the present time been made of Quaternions must be attributed partly to the repugnance of most mature minds to new methods involving the expenditure of thought" [Maxwell, 1870]. He adds later in his famous book [Maxwell, 1891]: "I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of Quaternions, will be of great use to us in the study of all parts of our subject, and especially in electrodynamics, where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few words of Hamilton's than by the ordinary equations".

Quaternions have been replaced in electromagnetism by Vector Calculus [Crowe, 1994] under the influence of Gibbs, a chemist, and Heaviside, a physicist, that transformed Maxwell's equations. We will illustrate in Chapters 3 and 4, the differences between a Vector Calculus approach and quaternions with their consequences for Computation. We will see that Vector Calculus splits the quaternions $\mathbb{H} = \mathbb{R} \oplus \mathcal{I}\mathbb{H}$ into the real part with scalars in $\mathbb{R}$ and the imaginary part $\mathcal{I}\mathbb{H}$ isomorphic to $\mathbb{R}^3$.

Beyond electromagnetism, Vector Calculus techniques quickly spread and linear algebra became a main topic for researchers. The vector space structure lead to great advances during the 20^{th} century especially for Scientific Computing. The resolution of linear systems and spectral analysis represented a large part of the results in applied research during the last decades such as Krylov methods and the QR algorithm. For more information, see [Chatelin, 2012a].

The short summary presented above illustrates two main ideas. The first one is that the pioneering work of Hamilton had the effect of a dam break. It gave birth to a real algebraic explosion of concepts. The second point to keep in mind and meditate is the strong delay in time, from creation to acceptation. Almost two centuries after their birth and in spite of all their fruitful consequences, a large part of these revolutionary ideas remains quite marginal.

2.4 Algebras in any dimension

After decades of intense algebraic inventions, a natural question that has been explored at the end of the 19^{th} and the beginning of the 20^{th} century was: how to relate all these algebraic strutures ? Some of the great mathematicians of that time, amongst them Frobenius, Study, Cartan, Wedderburn, Dickson and Noether, investigated this question. Each of them gave contributions that are now gathered under the name of *hypercomplex numbers*.

2.4.1 A landmark: Cartan's article (1908)

The fundamental paper of Elie Cartan [Cartan, 1908], developing the work of Study, presents the most general way to define a multiplication over a vector space in any finite dimension $n \geq 2$. It starts with complex numbers and then summarises the number structures known at that time. This includes Clifford algebras and goes further with the doubling process for Dickson algebras (non associativity and beyond) and with a general framework for hypercomplex numbers. The special case of dimension 2 described in section 2.2 is a basic brick for the algebras that are of interest for Computation amongst all possible ones. We will now describe how to equip a n-dimensional vector space with a multiplication following the work of Cartan.

Let K be equal to $\mathbb{R}$ or $\mathbb{C}$, then the linear vector space K^n, $n \geq 3$ with basis $\{e_i\}$, $i = 1$ to n can be transformed into an algebra by defining the multiplication

$$e_i \times e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k, \quad 1 \leq i, j, \leq n$$

which is distributive with respect to addition, we denote this algebra by A. The *structural cube* is formed by the n^3 scalars $\gamma_{ijk} \in K$ and is represented in Figure 2.1.

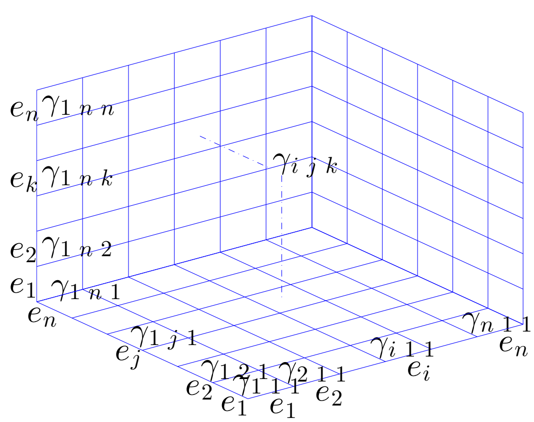


Figure 2.1 – Structural cube $\gamma_{ijk} \in K$

The associativity property $((e_i \times e_j) \times e_l = e_i \times (e_j \times e_l))$ is guaranteed iff the scalars $\gamma_{ijk} \in K$

satisfy the condition

$$\sum_{k=1}^{n} \gamma_{ijk}\gamma_{klm} = \sum_{k=1}^{n} \gamma_{ikm}\gamma_{jlk}, \quad 1 \leq i, j, l, m \leq n, \ \ 1 \leq k \leq n.$$

The commutativity condition $e_i \times e_j = e_j \times e_i$ is satisfied iff

$$\gamma_{ijk} = \gamma_{jik}, \quad 1 \leq i, j, k \leq n.$$

The notion of structural cube is not contained in the work of Cartan. This is an interpretation of the relations expressed with the coefficients γ_{ijk}. If the hypercomplex algebra is associative and commutative, it consists of what we call *analytic numbers* and is denoted by AN_n because the classical differential and integral calculus may be performed [Scheffers, 1893]. For more information about the history and the technique related to hypercomplex numbers see [van der Waerden, 1985] and [Chatelin, 2018].

There is an interesting consequence to this presentation of hypercomplex numbers. In the litterature, it can sometimes be found that Hamilton invented the quaternions because he could not multiply triplets. With no more precision, this sentence is incomplete. It appears that with the notion of hypercomplex number and Cartan's work, there is no restriction to the dimension $n \geq 2$. In particular, it is possible to define a multiplication in a 3D-space. The implicit assumption of Hamilton at that time was to maintain the property $||x \times y|| = ||x|| \, ||y||$ and to avoid zerodivisors. The theorem of Frobenius (1877) proved in fact that the only associative fields over the real numbers are $\mathbb{R}$, $\mathbb{C}$ and the quaternions $\mathbb{H}$. The notion of zerodivisor came after the invention of quaternions and the concept was still hard to accept at that time, even by Hamilton himself[1]. Of course, this remark is not made to criticise Hamilton's exceptional invention, but to correct misleading opinions that can be encountered until today in some historical accounts. The next section will give a definition of hypercomplex numbers which is more restrictive than the one given by Cartan but is taken as a basis in the current international litterature.

2.4.2 Hypercomplex numbers

The litterature on the subject is highly scattered and we will make a choice and take as definition the one found in [Kantor and Solodovnikov, 1973], mainly used in current papers. A hypercomplex algebra is a finite dimensional algebra over the real numbers. Elements are expressed on a basis $(1, i_1, ..., i_n)$ with real coordinates where $n+1$ is the dimension of the algebra, 1 the real unit and for practical reasons, it is conventional to choose the vectors $(i_1, .., i_n)$ such that $i_k^2 \in \{-1, 0, 1\}$. Notice that the multiplication is only required to be distributive with respect to addition and not necessarily associative. Wedderbun showed in 1908 that associative hypercomplex numbers could be represented by matrices, or direct sums of systems of matrices [Wedderburn, 1908] . Two remarks can be made on this point. First, we will see in Chapter 4 that a descriptive equivalence is not a computational equivalence so that hypercomplex numbers cannot be described only by matrices. The second point is that hypercomplex numbers also include non associative structures such as the octonions that cannot fully be described by associative structures.

1. In a letter to Graves on the 17th October 1843 (Math. Papers 3,pp.106—110) he wrote:"Behold me therefore tempted for a moment to fancy that $ij = 0$. But this seemed odd and uncomfortable, and I perceived that the same supression of the term which was *de trop* might be attained by assuming what seemed to me less harsh, namely that $ji = -ij$"

This general frame contains a lot of algebraic structures. The algebras of the numerical plane that we presented in section 2.2 are simple examples of hypercomplex numbers. In the next chapter, we will give details about specific constructions when the dimension of the algebra is $n = 2^k$.

We will now make the link between hypercomplex numbers and square matrices. This is another example of how to see differently standard computational tools.

2.4.3 Square matrices as hypercomplex numbers

Matrices over $\mathbb{R}$ or $\mathbb{C}$ are intensively used in Scientific Computing in the framework of Linear Algebra. Contemporary to the algebraic explosion of the 19^{th} century, the concept of matrices became a full-blown topic that lead to great advances in Science and its applications. Nevertheless, the relation between matrices and hypercomplex numbers is rarely mentionned in Linear Algebra textbooks.

We have already put in evidence some examples for matrices of order 2 in section 2.1.2.3, but these are general properties shared by squares matrices. A square matrix of order p is a hypercomplex number of dimension $n = p^2$. The algebras of square matrices are associative and non commutative. As was said for matrices of order 2, square matrices are invertible ($\det A \neq 0$) or zerodivisors ($\det A = 0$).

The description of matrices as hypercomplex numbers is not founded on purely aesthetic algebraic reasons. Indeed, we will see in Chapter 4 and in the applications of Chapter 7 how some properties, usually presented in a matrix formulation, can be explained with hypercomplex numbers. For practical uses, we will also illustrate the difference between these formulations and why matrices, compared to hypercomplex numbers, may not offer the most efficient way to compute in finite precision.

2.5 Bibliographical notes

The topic of nonassociative algebras is rarely addressed in the general textbooks, safe for abstract algebra matters. The book [Schafer, 1995] is a very precious reference for non associative algebras.

Concerning hypercomplex numbers, it is quite remarkable that this very rich content is almost never taught to students. The different algebraic possibilities for the numerical plane remain quite unknown in most western countries (outside Russia) safe for some algebraists. It is always instructive to note the cultural differences concerning education. The reference [Kantor and Solodovnikov, 1973] which is a good introduction to hypercomplex numbers is used in Russia as a textbook for highschool students.

Beyond its high mathematical value, Cartan's article is also interesting on several other aspects. This article remains untranslated in English language up to now and this may explain why this article is not a classical reference. In the last conference on Clifford algebras in 2017 (ICCA11), a large majority of researchers were unaware of the fact that Cartan discovered the 8-periodicity of Clifford algebras, usually attributed to Bott [Baez, 2002]. We will see other examples with different languages and we must not forget that a large part of Science and Mathematics before World War II was not written in English. Some results tend now to fall into oblivion because their original language is not English. Cartan's article is also an example of the intensive cooperation between european resarchers and countries (France and Germany in this case). As mentionned earlier, the article is an augmented translation of the work of Study. This cooperation was suspended at the beginning of World War I. The two world wars of the 20^{th} century slayed, amongst all the victims, many world class mathematicians and deeply affected the evolution of mathematical ideas.

Bibliography

[Baez, 2002] Baez, J. (2002). The octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205.

[Capelli, 1941] Capelli, P. F. (1941). Sur le nombre complexe binaire. *Bulletin of the American Mathematical Society*, 47(8):585–595.

[Cartan, 1908] Cartan, E. (1908). Nombres complexes. in *Encycl. Sc. Math.,* (J. Molk ed.) d'après l'article allemand de Study, I(1-5):329–468 (Gauthier–Villars, Paris).

[Chatelin, 2012a] Chatelin, F. (2012a). *Eigenvalues of Matrices: Revised Edition.* Classics in Applied Mathematics, volume 71. SIAM.

[Chatelin, 2012b] Chatelin, F. (2012b). *Qualitative Computing: a computational journey into nonlinearity.* World Scientific, Singapore.

[Chatelin, 2018] Chatelin, F. (2018). *Numbers in Mind: the transformative ways of Multiplication.* Book in preparation to be published by World Scientific.

[Clifford, 1873] Clifford, W. K. (1873). Preliminary sketch of bi-quaternions. *Proc. London Math. Soc*, 4(381-395):157.

[Cockle, 1848] Cockle, J. (1848). III. On Certain Functions Resembling Quaternions, and on a New Imaginary Algebra. *Phil. Mag. (3)*, 33:435–439.

[Crowe, 1994] Crowe, M. J. (1994). *A history of vector analysis: The evolution of the idea of a vectorial system.* Dover, NY.

[Ebbinghaus et al., 1998] Ebbinghaus, H., Hermes, H., and Hirzebruch, F. (1998). *Les nombres.* Vuibert, Paris.

[Hamilton, 1844] Hamilton, W. R. (1844). On quaternions; or on a new system of imaginaries in algebra. *Philosophical Magazine Series 3*, 25(163):10–13.

[Kantor and Solodovnikov, 1973] Kantor, I. L. and Solodovnikov, A. S. (1973). *Giperkompleksnye chisla (in Russian), Moscow, Nauka. English translation. Hypercomplex numbers: an elementary introduction to algebras, 1989. Springer.*

[Lam, 1999] Lam, T. Y. (1999). *Lectures on ring and modules.* Graduate Texts in Mathematics.

[Maxwell, 1870] Maxwell, J. C. (1870). On the application of the ideas of the calculus of quaternions to electromagnetic phenomena. *Maxwell's Scientific Papers* (Sir W. Niven ed.), 2:570–576, Dover Publ. New York (1890).

[Maxwell, 1891] Maxwell, J. C. (1891). *A treatise on electricity and magnetism.* 3^{rd} edition (J.J. Thomson ed.), Dover republication (1954).

[Raffin, 1951] Raffin, R. (1951). Anneaux non associatifs. *Séminaire Dubreil, Algèbre et théorie des nombres*, 4:11–16.

[Schafer, 1995] Schafer, R. (1995). *An Introduction to Nonassociative Algebras (first edition 1966).* Dover, New York.

[Scheffers, 1893] Scheffers, M. (1893). Sur la généralisation des fonctions analytiques. *CR Acad. Sc*, 116:1114.

[van der Waerden, 1985] van der Waerden, B. L. (1985). *A history of algebra.* Springer.

[Wedderburn, 1908] Wedderburn, J. (1908). On hypercomplex numbers. *Proceedings of the London Mathematical Society*, 2(1):77–118.

Chapter 3

Dickson algebras

In Chapter 2, we presented the three types of algebras for the plane and the extension to hypercomplex numbers in any dimension. For computational purposes, we will now introduce a specific type of algebras called Dickson (or Cayley-Dickson) algebras. First, we will give some mathematical and conceptual precisions on the quaternions and the octonions. We will see that the real numbers, the complex, the quaternions and octonions share some crucial properties for measure. In addition to common features, these four algebras of dimension $1, 2, 4, 8$ are the first four instances of the duplication (or doubling) process that generates an infinite sequence of algebras in dimension 2^k, $k \in \mathbb{N}$ through an iterative definition of multiplication and conjugation by Dickson [Dickson, 1919].

3.1 At the origin of Dickson algebras: quaternions and octonions

3.1.1 Quaternions

As mentionned in Chapter 2, quaternions have been introduced by Hamilton in 1843 as an answer to his original quest to multiply triplets representing geometrical points in the real 3D-space. Quaternions are a non commutative field.

The algebra of quaternions is defined in the orthonormal basis $1, i, j, k$ as:

$$\mathbb{H} = \{q = a + bi + cj + dk, (a, b, c, d) \in \mathbb{R}^4\}.$$

where i and j (the two non real generators) and $k = i \times j$ are linked by the relation that Hamilton cut on a stone of Brougham (Broom) bridge in Dublin:

$$i^2 = j^2 = k^2 = ijk = -1.$$

These relations came to his mind in 1843 in Dublin, when Hamilton was on his way to the Royal Irish Academy. As he walked along the towpath of the Royal Canal with his wife, he suddenly had the idea of the fundamental formulae of quaternions.

The multiplication table defines the rules to multiply quaternions:

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

For a quaternion $q = a + bi + cj + dk$, $a \in \mathbb{R}$ was called by Hamilton the *scalar part* and $bi + cj + dk$ the *vector part* (or *pure* or *imaginary part*). These concepts have later been used by Gibbs and Heaviside to restrict $\mathbb{H}$ to vector calculus by splitting the 4D-space of quaternions $\mathbb{H}$ into $\mathbb{R} \oplus \mathbb{R}^3$ with no multiplicative link between the scalar and the vector parts. The version of Gibbs and Heaviside ignores the connection which is explicit in quaternions through the definition of their 4D-multiplication.

Following the formulation of complex numbers from real numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$, we can write the quaternion as $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} \times j$, $q = a + bi + cj + dk = a + bi + (c + di) \times j$. The quaternion q is thus represented by complex components and can be written $q = (a + bi, c + di)$. We will see in Section 3.3 that this form is obtained from $\mathbb{C}$ by the doubling process of Dickson.

Like Cardano's complex numbers, quaternions were far too advanced an idea at the time when Hamilton invented them. After its invention, quaternionic calculus was soon eclipsed by vector calculus, designed to be easier to grasp, it only enjoyed a revival in the second half of the 20^{th} century. Quaternions are now the tool of choice used in practical applications in Science and Industry to represent rotations in a more efficient way than Euler angles or matrices for various fields of research, from spatial engineering to 3D-modelling for computers and also molecular simulation in Chemistry (Chapters 4, 8).

3.1.2 Octonions

Octonions have been first invented by Graves in 1843 and independently rediscovered by Cayley two years later [Graves, 1845, P.S. p.320] [Cayley, 1845]. As observed by Hamilton [Ebbinghaus et al., 1998], they are an 8D non associative division algebra. Unlike quaternions' table, the multiplication table for octonions is not unique. Non associativity implies the existence of 480 tables essentially disctinct up to a homeomorphism [Coxeter et al., 1946]. Only two are generally considered, the one of Cartan and Schouten for Physics [Cartan and Schouten, 1926] with the Fano plane [Baez, 2002] and the other one given by Graves who defines an octonion as a pair of quaternions. The table of Cartan is related to geometric aspects whereas the table of Graves is based on the recursive definition of multiplication of octonions as pairs of quaternions, which is the starting point of Dickson's work. In what follows, the algebra of octonions will be denoted $\mathbb{G}$ to underlie Graves' pioneering invention.

Despite the fact that they are the first historical example of a non associative algebra, octonions have not been a main focus of attention for researchers. Due to non associativity, there is no equivalent matrix description. However nonassociativity in the form of **alternativity** can be turned into an asset: alternative structures have remarkable aspects that are used in particle and high energy physics to describe fundamental interactions. Non associativity is often presented as an irretrievable loss, but it can be of added value for geometry and computation such as with the seven dimensional cross product to be presented in Section 3.5.

3.2 Composition algebras

In 1877, Frobenius discovered that the algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only associative fields over the real numbers. In relaxing associativity, this result has been extended by Hurwitz to show that the only finite dimensional division algebras over $\mathbb{R}$ with multiplicative norm ($||x \times y|| = ||x|| \, ||y||$) are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and the octonions $\mathbb{G}$ [Hurwitz, 1898]. Zorn also found a similar result, replacing the hypothesis of multiplicative norm by that of alternativity of multiplication [Zorn, 1931]. The logical link between norm multiplicativity and alternativity is given at the end of Section3.5.1. More generally, a *composition algebra* A is an alternative algebra over a field K equipped with a quadratic form N such that

$$\forall x, y \in A \ , \ N(x \times y) = N(x)N(y)$$

.

The only possible dimensions for composition algebras over a field K are 1, 2 , 4 and 8. For example, over $\mathbb{R}$, the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{G}$ are composition algebras associated with $N = ||.||^2$, the square of the euclidean norm.

3.3 Dickson algebras

3.3.1 Definition

Dickson algebras $(\mathbb{A}_k)_{k\in\mathbb{N}}$ are a sequence of embedded hypercomplex algebras over $\mathbb{R}$ obtained by a duplication process from $\mathbb{A}_0 = \mathbb{R}$, $\mathbb{A}_{k-1} \to \mathbb{A}_k = \mathbb{A}_{k-1} \oplus \mathbb{A}_{k-1} \times g_k$, where the g_k are nonreal orthonormal generators such that $g_k^2 = -1$, $k \geq 1$ [Dickson, 1919].

The elements of $\mathbb{A}_k$, $k \geq 1$ are obtained as pairs of elements in $\mathbb{A}_{k-1}$:

$$\mathbb{A}_k = \{z = x + y \times g_k,\ (x, y) \in \mathbb{A}_{k-1}^2\}.$$

By construction, $\mathbb{A}_k$ is a vector space over $\mathbb{R}$. The addition and external multiplication are defined from the ones in $\mathbb{A}_{k-1}$. Therefore, $dim(\mathbb{A}_k) = 2dim(\mathbb{A}_{k-1}) = 2^k$ and for $z = x+y\times g_k$, $z' = x'+y'\times g_k \in \mathbb{A}_k$, and $\lambda \in \mathbb{R}$ we have $z + z' = (x + x') + (y + y') \times g_k$ and $\lambda z = \lambda x + \lambda y \times g_k$. For $z = x + y \times g_k$, we will also use the notation $z = (x, y)$.

Conjugation and multiplication are defined recursively in $\mathbb{A}_k$ by:

— $\overline{(x, y)} = (\overline{x}, -y)$,

— $(x, y) \times (x', y') = (x \times x' - \overline{y'} \times y,\ y' \times x + y \times \overline{x'})$.

Conjugation is used to define two other real-valued applications for $z \in \mathbb{A}_k \to \mathbb{R}$:

— *trace* : $z \mapsto \tau(z) = z + \overline{z} \in \mathbb{R}$,

— *magnitude*: $z \mapsto \mu(z) = z \times \overline{z} \in \mathbb{R}^+$.

We define for an element $z \in \mathbb{A}_k$ the real part $Re(z) = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}\tau(z)$ and the imaginary part $Im(z) = \frac{1}{2}(z - \overline{z})$ so that $z = Re(z) + Im(z)$. We denote $\mathcal{R}e(\mathbb{A}_k) = \{z \in \mathbb{A}_k,\, z = \overline{z} \in \mathbb{R}\}$ and $\mathcal{I}(\mathbb{A}_k) = \{z \in \mathbb{A}_k,\, \tau(z) = 0\}$, the imaginary part of $\mathbb{A}_k$, i.e. the set of *imaginary* or *pure vectors*. For $z \in \mathcal{I}(\mathbb{A}_k)$, $\overline{z} = -z$.

In the case of Dickson algebras, the magnitude has a simple form. For $z = (x, y) \in \mathbb{A}_k$, $k \geq 1$, the magnitude can be written as the sum of the squares of the euclidean norms of the parts in $\mathbb{A}_{k-1}$, $\mu(z) = ||x||^2 + ||y||^2$. Hence it equals the sum of the square of all its components. This is a reason why we use the term magnitude instead of norm to avoid an abuse of language (in some papers the word "norm" is incorrectly used for the square of a norm).

Magnitude $\mu : z \mapsto \mu(z) = z \times \overline{z}$ is a quadratic form already seen for 2D-algebras; for the trace of an element z we have $\tau(z) = 2Re(z)$.

$\forall k \geq 1$ Dickson algebras $\mathbb{A}_k$ are **quadratic**:

$$\forall z \in \mathbb{A}_k \text{ , } z^2 = (z + \overline{z}) \times z - z \times \overline{z} = \tau(z)z - \mu(z).$$

This quadratic relation is **essential** as z^2 can be expressed as a linear combination of 1 and z with real coefficients that are the trace $\tau(z)$ and the opposite of magnitude $-\mu(z)$ respectively . This implies that the evolution of multiplicative powers for an element z remains in the plane $\{1, z\}$, $z \notin \mathbb{R}$, with consequences for polynomials and exponential functions.

As a consequence, we have a general formula for the inverse of z with nonzero magnitude:

$$\forall z, \mu(z) \neq 0, z^{-1} = \frac{\overline{z}}{\mu(z)}$$

.

The first four Dickson algebras are the real division algebras $\mathbb{A}_0 = \mathbb{R}$, $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$ and $\mathbb{A}_3 = \mathbb{G}$. They are alternative composition algebras. The doubling process of Dickson is a constructive link between the four division algebras. We will se below, from Section 3.3.3 to 3.3.5, ways to generalise Dickson algebras.

3.3.2 Generators, basis and dimension

In addition to the real unit 1, the algebra $\mathbb{A}_k$ of dimension 2^k, $k \geq 1$ has k non real generators $(g_i)_{1\leq i\leq k}$. They are the generators that give, by multiplication, a basis for the vector space A_k. Each new generator g_k is orthogonal to the set of the other generators.

For example in A_1, the field of complex numbers $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$, 1 and $g = i$ are the generators of $\mathbb{C}$ and form the 2D-basis $\{e_0 = (1,0) = 1,\ e_1 = (0,1) = i\}$. $\mathbb{C}$ is a special case because the basis vectors are exactly the generators (the possible products between generators are limited to $1 \times i = i$).

A_2, the algebra of quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} \times j$, has 3 generators 1, $g_1 = i$, $g_2 = j$ and form the $(2^2 = 4)$D-basis $\{e_0 = (1,0,0,0) = 1,\ e_1 = (0,1,0,0) = i,\ e_2 = (0,0,1,0) = j,\ e_3 = (0,0,0,1) = k\}$. Notice that $k = i \times j$ is a basis vector but not a generator, k is the product of the two generators i and j.

For $k \geq 3$, as multiplication is no longer associative, the order of multiplication should be respected to mark the difference between the two basis vectors $(g_1 \times g_2) \times g_3 \neq g_1 \times (g_2 \times g_3)$ fo example. The distinction between the k generators and the 2^k basis vectors, obtained as products of generators is fundamental for Dickson algebras. We will see in Chapter 4 that this is a key point for the comparison with Clifford algebras.

The increase in dimension should be contrasted with the one occuring in numerical linear algebra. For matrix problems, the dimension is often a consequence of mesh refinement for finite elements or from the number of nodes in graph theory, among other possiblities. The notion of dimension impacts the complexity of resolution for the resulting linear systems, but computation rules and physical meaning remain unchanged whatever the size of the matrices.

As a consequence of multiplication, one of the main differences between linear algebra formulations and the duplication process for Dickson algebras is that the increase of dimension implies a modification of the associated algebraic properties and therefore of the physical interpretation that can be done. Dimension in Dickson algebras is linked to modelling, not to discretisation. Each new duplication step creates a change in the computing rules: as $k \to k+1$ the loss of some properties in $\mathbb{A}_k$ can be balanced by the emergence of new ones in $\mathbb{A}_{k+1}$.

3.3.3 Generalised Dickson algebras

From the classical definition of multiplication and conjugation in the Dickson process, a more general form for $\mathbb{A}_k$ can be obtained by replacing in the definition of multiplication -1 by the square of the generators $g_i^2 = \gamma_i \in \mathbb{R}^*$, $0 \leq i \leq k$ as $(x, y) \times (x', y') = (x \times x' + \gamma_i \overline{y'} \times y,\ y' \times x + y \times \overline{x'})$.

This generalisation is due to Albert [Albert, 1942], but we will be interested in this thesis, by the particular case that we present in the next section. The original Dickson algebras with $g_i^2 = -1$ are often called standard.

3.3.4 Split-Dickson algebras

The case $\gamma_i = -1$, $i < k$ and $\gamma_k = 1$ gives a sequence of algebras called *split algebras* denoted $\not{\mathbb{A}}_k$, as a reference to the signature of the magnitude as a quadratic form (equal numbers of + and −) [Schafer, 1995]. We have already seen that the term "norm" is abusive in the standard Dickson process as the magnitude is the *square* of the euclidean norm but it is even more confusing in split-Dickson algebras where magnitude is a real quantity, with no fixed sign (for $z = (x, y), \mu(z) = ||x||^2 - ||y||^2 \in \mathbb{R}$).

The real numbers $\mathbb{R}$, the complexes $\mathbb{C}$, the quaternions $\mathbb{H}$, the octonions $\mathbb{G}$ and their associated split forms (split-complexes $\not{\mathbb{C}} = {}^2\mathbb{R}$, split-quaternions $\not{\mathbb{H}}$, split-octonions $\not{\mathbb{G}}$) are the seven composition algebras that exist over $\mathbb{R}$ [Jacobson, 1958].

3.3.5 Extension of scalars

Another generalisation concerns the extension of scalars for Dickson algebras from $\mathbb{R}$ to fields or rings. In [Albert, 1942], the Dickson process is presented over a field, not supposed to be $\mathbb{R}$. An interesting possiblity is to use the complex field $\mathbb{C}$ for scalars in Dickson algebras. Hamilton already considered the case with his *biquaternions* denoted $\mathbb{H}(\mathbb{C})$, quaternions with complex coefficients that are use in Quantum Mechanics and Quantum Computing under the name of Pauli algebra (Chapter 8). It is also possible to use one of the two others algebras in the plane (dual or bireal numbers) , which are no longer fields but rings. In [Clifford, 1873], Clifford used the term "biquaternions" to denote $\mathbb{H}({}^2\mathbb{R})$, quaternions with bireal components. The biquaternions of Hamilton should not be confused with those of Clifford. Hamilton's biquaternions have received more attention than Clifford's biquaternions due to their applications and we will restrict the term biquaternions to the work of Hamilton. Using dual numbers as scalars, is a very effective way to compute in rigid body mechanics. In particular, we can mention the use of dual quaternions $\mathbb{H}(\mathbb{D})$ (Chapter 8). Notice that the extension of scalars deeply transforms the properties of the original real algebras. For example, quaternions are a field but complex quaternions form a ring with zerodivisors (nilpotents and idempotents).

We made a detailed comparative classification of possible algebras based on Dickson algebras that could be used for Computation [Latre, 2013]. Notice that the required framework for a correct mathematical description would be module theory. We will not go here at this level of generality as the point of this thesis is not to present algebras for the sake of descriptive or algebraic purpose, but rather to focus on some specific structures for their remarkable consequences in practical computations.

3.3.6 Zerodivisors in non alternative Dickson algebras, $k \geq 4$

In the case of Dickson algebras, we have already seen that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{G}$ are the only division algebras over $\mathbb{R}$. The bifurcation between $\mathbb{A}_3$ and $\mathbb{A}_4$ is quite remarkable and implies deep consequences for the notion of measure as we will see below. For Dickson algebras that are not alternative ($\mathbb{A}_k, k \geq 4$), there exist zerodivisors and for $z \in \mathbb{A}_k$, $z \neq 0$, we consider its set of zerodivisors $Zer(z) = \{a \in \mathbb{A}_k,\, a \times z = 0, a \neq 0\}$.

In the case of Dickson algebras, there is no need to consider the difference between left and right zerodivisors, due to the properties of magnitude ($||a \times x||^2 = ||x \times a||^2$). It is quite remarkable that

$Zer(\mathbb{A}_4)$, the set of zerodivisors in the *sedenions* $\mathbb{A}_4$, is isomorphic to the exceptional compact Lie group $\mathcal{G}_2$ [Khalil and Yiu, 1997, Moreno, 1998]. For more information on the matter of zerodivisors in Dickson algebras, see [Chatelin, 2012].

It is also fundamental to notice the consequences of mutliplication in $\mathbb{A}_k$ when we try to measure the product $x \times y$. For x, y in $\mathbb{A}_k$ considered as the vector space $\mathbb{R}^{2^k}$, the two vectors x and y can be described thanks to their real components $x = (x_i)$, $y = (y_i)$. A direct calculation shows that the classical euclidean scalar product in $\mathbb{R}^{2^k}$, $< x|y >= \sum_i x_i y_i \in \mathbb{R}$, can be written $< x|y >= \frac{1}{2}(\mathcal{R}e(\overline{x} \times y)) = \frac{1}{2}(x \times \overline{y} + y \times \overline{x})$. From the geometric intuition in the 3D-space, the length, that is the euclidean norm of a single vector $x \in \mathbb{A}_k$, is obtained from the scalar product: $\mu(x) = ||x||^2 =< x|x >= \overline{x} \times x = x \times \overline{x}$.

For the product $x \times y$, we have $||x \times y||^2 =< x \times y|x \times y >=< y|\overline{x} \times (x \times y) >$ and, for a given vector x, we consider the linear map $F_x : y \mapsto \overline{x} \times (x \times y)$. For $0 \leq k \leq 3$, $F_x = ||x||^2 I_{\mathbb{R}^{2^k}}$, where $I_{\mathbb{R}^{2^k}}$ is the identity map of $\mathbb{R}^{2^k}$. This result corresponds to the case of composition algebras where the norm is multiplicative $||x \times y|| = ||x|| \, ||y||$. For $k \geq 4$, this simple expression for F_x is no longer true, F_x becomes a nondiagonal positive semi definite operator with nontrivial spectral decomposition. $||x||^2$ is still an eigenvalue of F_x, with multiplicity $4p$, $p \geq 2$ and there are up to $2^{k-2} - 2 \geq 2$ distinct eigenvalues of multiplicity $4p$, $p \geq 1$ [Moreno, 2005] whose arithmetic mean is $||x||^2$ [Chatelin, 2012]. The norm of $x \times y$ could be either equal to $||x \times y|| = ||x|| \, ||y||$ if y is in the eigenspace associated to $||x||^2$, or $||x \times y|| = \sqrt{\lambda} \, ||y||$ if y is in the eigenspace for λ, where λ is another eigenvalue of F_x, such that $0 \leq \lambda < ||x||^2$ or $\lambda > ||x||^2$. If $\lambda = 0$ is an eigenvalue of F_x, $||x \times y||^2 = 0 \Rightarrow x \times y = 0$ and x,y are zerodivisors (but $||x|| \, ||y|| \neq 0$).

These results must be considered seriously. First, we have to remember that nonalternativity for

$\mathbb{A}_k$, $k \geq 4$ cannot be interpreted thanks to the usual computational and geometrical habits. This huge difference is revealed when we try to measure a product and not a single vector. For a single vector, its magnitude (squared euclidean norm) is a ("static") isolated measure. But the magnitude of a product measures the ("dynamic") interaction between two vectors. Considering multiplication with the linear map F_x, the information relative to a product is a local result that reflects the nonisotropy of the space $\mathbb{R}^{2^k}$ equipped with Dickson multiplication, in the presence of the multiplicative action of x on y.

3.4 Automorphisms of Dickson algebras

The group of automorphisms $Aut(A)$, applications that are isomorphisms over an algebra A, preserves the properties of the structure A, and in particular, multiplication. Given our specific attention to multiplication, it is quite natural a question to study the automorphisms in Dickson algebras, that have a specific structure.

3.4.1 Dickson algebras in dimensions $1, 2, 4, 8$

Over $\mathbb{R}$, the only automorphism is the identity. Over $\mathbb{C}$, the identity and the conjugation are the two elements of $Aut(\mathbb{C})$. We only consider continous maps and do not take into account the wild automorphisms of $\mathbb{C}$ that require the axiom of choice to exist. For the quaternions, all the automorphims are *inner*: $Aut(\mathbb{H}) = \{x \mapsto q \times x \times q^{-1},\, q \in \mathbb{H},\, q \neq 0\}$. We will see in the next chapter the computational role of $Aut(\mathbb{H})$ for spatial rotations. Concerning octonions, it is remarkable that $Aut(\mathbb{G})$ is isomorphic to $\mathcal{G}_2$. This makes a tight connection between the automorphisms of octonions and the zerodivisors of the sedenions (section 3.3.6) through the compact form of the exceptional Lie group $\mathcal{G}_2$, $\mathcal{G}_2^c \simeq Aut(\mathbb{G}) \simeq Zer(\mathbb{A}_4)$ [Khalil and Yiu, 1997].

3.4.2 Composition algebras

From the Dickson algebras process, Jacobson described the automorphisms related to composition algebras [Jacobson, 1958]. This includes the automorphisms presented in the previous sections for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{G}$ and also their split form (split-complexes $\not\mathbb{C}$, split-quaternions $\not\mathbb{H}$, split-octonions $\not\mathbb{G}$). It is interesting to notice the similarity between the Dickson algebras and their related split-form. The only automorphism of split-complex numbers is the conjugation. For split-quaternions, all the automorphims are *inner*: $Aut(\not\mathbb{H}) = \{x \mapsto q \times x \times q^{-1}, q \in \not\mathbb{H}, q \neq 0\}$. As the automorphisms of $\mathbb{H}$ are related to euclidean rotations in 3D, $Aut(\not\mathbb{H})$ is linked to the hyperbolic rotations of the Poincaré disk model [Karzel and Kist, 1985]. $Aut(\not\mathbb{G})$ is isomorphic to the non-compact form of $\mathcal{G}_2^{nc}$.

3.4.3 Nonalternative Dickson algebras $\mathbb{A}_k$, $k \geq 4$

The general case of automorphisms of Dickson algebras has been conjectured and proved for the sedenions $\mathbb{A}_4$ in [Brown, 1967]. The complete proof of this conjecture appeared in [Eakin and Sathaye, 1990]. What is remarkable in the formula given by Brown is that there is a clear break point for Dickson algebras $\mathbb{A}_k$, $k \geq 4$. There is no way to relate the automorphisms of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{G}$. But, for $k \geq 4$, there is an iterative formula to construct the automorphisms of $\mathbb{A}_k$ from the automorphisms of $\mathbb{A}_{k-1}$. It is very interesting that the loss of alternativity in the sedenions $\mathbb{A}_4$ implies a radical transformation for the zerodivisors and the automorphisms for $\mathbb{A}_k$, $k \geq 4$. We have seen in Section 3.3.6 that zerodivisors are associated to the question of measure. Automorphisms are often considered because they preserve the structure and are associated to conservative aspects. As a consequence, computational properties related to measure and conservative operations must be addressed with caution, depending whether they concern the algebras $\mathbb{A}_k$ for $k \leq 3$ or $k \geq 4$.

The automorphisms of Dickson algebras are a very wide and dense topic of which we only presented a very small part useful for our purpose.

3.5 About cross/vector and dot/scalar products

3.5.1 Scalar and Vector product in Dickson algebras

Let us consider $x = x_0 + X$ (resp. $y = y_0 + Y$) $\in \mathbb{A}_k$, where $x_0 = Re(x)$ (resp. $y_0 = Re(y)$) and $X = Im(x)$ (resp. $Y = Im(y)$). We define in Dickson algebras the *scalar product* of x and y, by $< x|y >= \frac{1}{2}(Re(\overline{x} \times y)) = \frac{1}{2}(x \times \overline{y} + y \times \overline{x})$. This corresponds to the classical euclidean scalar product $< x|y >= \sum_i x_i y_i \in \mathbb{R}$. In the case of an imaginary vector $X \in \mathcal{I}(\mathbb{A}_k)$, $\overline{X} = -X$ (Section 3.3), thus we have $< X|Y >= -\frac{1}{2}(X \times Y + Y \times X)$. We also define for two imaginary vectors of $\mathcal{I}(\mathbb{A}_k)$, the *vector product* $X \& Y = \frac{1}{2}(X \times Y - Y \times X) = \frac{1}{2}[X, Y] \in \mathcal{I}(\mathbb{A}_k)$.

With these definitions in $\mathbb{A}_k$, the product of two imaginary vectors $X \times Y$ can be simply written as :

$$X \times Y = - < X|Y > + X \& Y.$$

For $\mathbb{A}_0 = \mathbb{R}$ and $\mathbb{A}_1 = \mathbb{C}$, the vector product is identically 0 because multiplication is commutative. For the quaternions $\mathbb{A}_2 = \mathbb{H}$ and the octonions $\mathbb{A}_3 = \mathbb{G}$ the product & corresponds to a specific vector product known as cross product in linear algebra. We will use the notation wedge $\wedge$ originally used by Clifford for the cross product. As stated in Chapter 2, Remark 1.1, the symbol $\times$ is only used for multiplication in our presentation. The notation wedge is now classically attached to the exterior product. But is the exterior product meaningful for Comptutation ? We will address this question in Section 3.5.2 and also in the next chapter when we will compare the hypercomplex formulation of Clifford algebras.

In an oriented euclidean space of dimension 3 (historically $\mathbb{R}^3$), the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the vector $\mathbf{a} \wedge \mathbf{b}$ othogonal to $\mathbf{a}$ and $\mathbf{b}$ such that the basis $(\mathbf{a}, \mathbf{b}, \mathbf{a} \wedge \mathbf{b})$ respects right-hand rule (direct basis) and that $||\mathbf{a} \wedge \mathbf{b}|| = ||\mathbf{a}||\ ||\mathbf{b}||\ |sin(\angle(\mathbf{a}, \mathbf{b}))|$ (parallelogram rule). The operation $\wedge$ is distributive and anticommutative and was first defined by Clifford, then borrowed by Gibbs for his vector calculus.

If we identify the canonical basis of $\mathbb{R}^3$ with the three vectors i, j, k in the imaginary part of the quaternions $\mathbb{H} = \mathbb{A}_2$, we can link a vector $a = (a_1, a_2, a_3)$ (resp. $b = (b_1, b_2, b_3)$) to the imaginary quaternion $q = a_1 i + a_2 j + a_3 k$ (resp. $q' = b_1 i + b_2 j + b_3 k$) in $\mathcal{I}(\mathbb{H}) \simeq \mathbb{R}^3$. By direct calculations of the components it appears that the multiplication of the quaternions q and q' gives $Re(q \times q') = - < \mathbf{a}|\mathbf{b} >$ and $q \& q' = Im(q \times q') = \mathbf{a} \wedge \mathbf{b}$, where $< \mathbf{a}|\mathbf{b} >$ is the dot/scalar product. The cross/vector product is the imaginary part of the product of two pure quaternions.

In standard linear algebra, the cross product is only defined in 3D euclidean space. It is less known that the classical cross product is also defined in 7D, identifying $\mathbb{R}^7$ with the imaginary part of the octonions. In 7D, the choice of geometric direction for the orthogonal cross product is in $\mathbb{R}^5$ instead of $\mathbb{R}$ in 3D but the parallelogram rule is still valid. In fact, it can be shown that the only dimensions where a vector product *always* respects this rule for its length are $0, 1, 3$ and 7 [Brown and Gray, 1967, Darpö, 2009] and these dimensions correspond to the imaginary parts of the alternative composition algebras that we redefined in Section 3.2. In dimension 0 and 1, the vector product is identically zero but in the cases of 3D and 7D four algebras can be equipped with the classical notion of vector product (standard and split forms of quaternions and octonions).

The definition of the vector product as $\frac{1}{2}[X, Y]$ is still valid in all Dickson algebras but the property that the length is the area of the parallelogram built on X, Y is no longer valid in general. For X and Y in $\mathcal{I}(\mathbb{A}_k)$, $k \geq 4$, the vector product & satisfies the parallelogram rule when $||X \times Y|| = ||X||\ ||Y|| \Leftrightarrow < [X, X, Y], Y >= 0 \Leftrightarrow < [Y, Y, X], X >= 0 \Leftarrow [X, X, Y] = 0$ or $[Y, Y, X] = 0$ (see [Chatelin, 2012, Prop 2.3.2 and Prop 2.7.1]. Due to non alternativity for $k \geq 4$, this is only a **local** property which does not concern the whole set $\mathcal{I}(\mathbb{A}_k)$.

3.5.2 Alternative extensions for the vector product

Should the cross product be only a geometric property? Or could it be a notion related to Computation which has, in the particular case of composition algebras a very specific geometric interpretation ? We should always be careful about geometric intuition beyond dimension 3 and we may ask the following question: does the definition of the cross product inherited from 3D-geometric intuition should obey the limitations of its geometric origin? Could it not, rather, betray its computational origin? The 3D-cross product is a **consequence** of multiplication in quaternions, even if the link between both is often presented as a coincidence. The distinction in Physics between the scalar part and vectors is reunited in the framework of 4D-quaternions. Dot product and cross product are linked through multiplication. To ignore this connection may lead to additional hypotheses such as the intoduction of specific gauges in Physics, especially in electromagnetism [Chatelin, 2015].

We will develop these remarks in the next chapter as we compare the geometrical generalisation of the cross product in exterior algebra and its computational generalisation in nonalternative Dickson alebras. The idea of Gibbs and Heaviside to linearly split $\mathbb{H}$ into the real part and the vector part as $\mathbb{R}\oplus\mathbb{R}^3$ was a pragmatic way to avoid to have to deal with an abstract 4D-superstructure [Gibbs, 1893]. Nevertheless, the multiplicative link between both parts which is implicit in their formulation may have tangible consequences, see [Chatelin, 2019].

All of this sheds light on interesting differences between a bottom-up and top-down view. They are complementary hence bring complementary insights. The example of $\mathbb{H}$ considered in linear algebra and Physics as $\mathbb{R} \oplus \mathbb{R}^3$ is quite instructive. The cross product can be viewed as (i) a direct computational consequence of multiplication restricted to $\mathcal{I}(\mathbb{H})$, or (ii) as a constructed geometric 3D-object. The first option (i) maintains a global multiplication $\times$ in dimension 2^k and relaxes length (in dimension 2^{k-1}). The second option (ii) maintains length (in dimension $n \in \mathbb{N}$) and downplays multiplication. The idea of bifurcation that we introduced for numbers and structures, applies equally for the cross product. The dominant influence of Grassmann, Clifford and Cartan in geometry has implicitly determined the future of the concept of cross product in more than 3D, in fostering the second option (ii) which is geometric (length by parallelogram rule) rather than the first (i) which is computational (existence of multiplication in a superstructure).

3.5.3 Differences in vocabulary and notations

With few exceptions, the birth of the algebraic tools that are now especially used in linear algebra took place in Europe during the first 70 years of the 19^{th} century. Multiplication was denoted $\times$, scalar product $< .|. >$ and vector product $\wedge$, where names were coined by Clifford. A major disruption occured in the 1880's in the english spaking world due to the invention of vector calculus by Gibbs in the USA and then transmitted to Heaviside in the UK. The break impacted notations and vocabulary in the following way:

Europe	English speaking world
product $x \times y$	xy
scalar product $< x\|y >$	dot product $x.y$
vector product $x \wedge y$	cross product $x \times y$

We observe that the right column is largely dominant in most of countries. Notations and writing habits substantially vary between different countries around the World and correspond to several ways to write Mathematics. The cultural differences impact the approach of a given topic (from the notations to te mathematical properties) and as we have seen earlier (Bibliographical notes in Chapter 2), this is really noticeable in textbooks and teaching methods.

3.6 Summary

Dickson algebras provide an instructive example of a family of algebras $\mathbb{A}_k$ whose evolutive structure is recursively defined from $\mathbb{A}_0 = \mathbb{R}$ by three operations (addition, multiplication and conjugation). The loss of associativity when k evolves from 2 to 3 entails no severe disruption because $\mathbb{G}$ retains alternativity as a weak form of associativity on two vectors, one of them repeated. By comparison, it appears that the step from $k = 3$ to $k = 4$ marks a fundamental frontier on many accounts. First, on the one hand, there is the loss of alternativity, of norm multiplicativity, of general invertibility and of isotropy. Second, on the other hand automorhpisms and derivations evolve iteratively. The pay-off for the increased freedom in behaviour is a decrease in predictability. High dimensional Dickson algebras remain largely *terra incognita*.

3.7 Bibliographical notes

The main remark concerns the name that we choose for Dickson algebras. In the litterature, the usual name is Cayley-Dickson algebras. On the one hand, Cayley discovered the octonions 2 years later than Graves but octonions are still often called Cayley numbers, a chronology can be found in [Baez, 2002]. On the other hand, there is no sign in Cayley's work that indicates how to go from the quaternions to the octonions (which is the way Graves found the octonions in December 1843), nor any indication of the general duplication process found by Dickson [Dickson, 1919]. The fact that Cayley was considered as a professional mathematicican whereas Graves was a lawyer learnt in Mathematics may explain this unfortunate attribution. In a whole, it is not a lack of consideration to say that Cayley did not play a central role in these algebras, and that Graves, often presented as an amateur, succeded in finding the remarkable idea of octonions. The real father and central figure of recursive definition was Dickson who discovered the doubling process by generalising Graves' construction from $\mathbb{H}$ to $\mathbb{G}$. He collaborated with Wedderburn who worked on hypercomplex numbers and Jacobson was a student of Wedderburn. The duplication process obtained by Dickson has been generalised by his student Albert [Albert, 1942]. A detailed study of the suprising specificities of standard Dickson algebras can be found in [Chatelin, 2012].

Bibliography

[Albert, 1942] Albert, A. A. (1942). Quadratic forms permitting composition. *Annals of Mathematics*, pages 161–177.

[Baez, 2002] Baez, J. (2002). The octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205.

[Brown, 1967] Brown, R. (1967). On generalized Cayley-Dickson algebras. *Pacific Journal of Mathematics*, 20(3):415–422.

[Brown and Gray, 1967] Brown, R. B. and Gray, A. (1967). Vector cross products. *Commentarii Mathematici Helvetici*, 42(1):222–236.

[Cartan and Schouten, 1926] Cartan, É. and Schouten, J. A. (1926). *On riemannian Geometries Admitting an Absolute Parallelism*. Koninklijke Akademie van Wetenschappen te Amsterdam.

[Cayley, 1845] Cayley, A. (1845). On Jacobi's elliptic functions, in reply to the rev. Brice Bronwin; and on quaternions: To the editors of the Philosophical Magazine and Journal. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 26(172):208–211.

[Chatelin, 2012] Chatelin, F. (2012). *Qualitative Computing: a computational journey into nonlinearity*. World Scientific, Singapore.

[Chatelin, 2015] Chatelin, F. (2015). Differential Information Processing in the light of quaternions. Technical report, (CERFACS TR/PA/15/07).

[Chatelin, 2019] Chatelin, F. (2019). Differential Calculus à la Hamilton and Maxwell in non alternative Dickson algebras (in preparation). Technical report, CERFACS.

[Clifford, 1873] Clifford, W. K. (1873). Preliminary sketch of bi-quaternions. *Proc. London Math. Soc*, 4(381-395):157.

[Coxeter et al., 1946] Coxeter, H. et al. (1946). Integral Cayley numbers. *Duke Mathematical Journal*, 13(4):561–578.

[Darpö, 2009] Darpö, E. (2009). Vector product algebras. *Bulletin of the London Mathematical Society*, 41(5):898–902.

[Dickson, 1919] Dickson, L. E. (1919). On quaternions and their generalization and the history of the eight square theorem. *Annals of Mathematics*, pages 155–171.

[Eakin and Sathaye, 1990] Eakin, P. and Sathaye, A. (1990). On automorphisms and derivations of Cayley-Dickson algebras. *Journal of Algebra*, 129(2):263–278.

[Ebbinghaus et al., 1998] Ebbinghaus, H., Hermes, H., and Hirzebruch, F. (1998). *Les nombres*. Vuibert, Paris.

[Gibbs, 1893] Gibbs, J. W. (1893). Quaternions and vector analysis. *Nature*, 48(1242):364–367.

[Graves, 1845] Graves, J. T. (1845). On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 26(173):315–320.

[Hurwitz, 1898] Hurwitz, A. (1898). Ueber die composition der quadratischen formen von belibig vielen variablen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1898:309–316.

[Jacobson, 1958] Jacobson, N. (1958). Composition algebras and their automorphisms. *Rendiconti del Circolo Matematico di Palermo*, 7(1):55–80.

[Karzel and Kist, 1985] Karzel, H. and Kist, G. (1985). Kinematic algebras and their geometries. In *Rings and Geometry*, pages 437–509. Springer.

[Khalil and Yiu, 1997] Khalil, S. and Yiu, P. (1997). The Cayley-Dickson algebras, a theorem of a. Hurwitz, and quaternions. *Bull. Soc. Sci. Lett. Lódz Sér. Rech. Déform*, 24:117–169.

[Latre, 2013] Latre, J.-B. (2013). Sur quelques structures algébriques utiles au traitement de l'information en robotique, informatique et physique. Technical report, Rapport de Master, 25/03/13-06/09/13, CERFACS WN-PA-13-105.

[Moreno, 1998] Moreno, G. (1998). The zero divisors of the Cayley-Dickson algebras over the real numbers. *Boletín de la Sociedad Matemática Mexicana: Tercera Serie*, 4(1):13–28.

[Moreno, 2005] Moreno, G. (2005). Constructing zero divisors in the higher dimensional Cayley-Dickson algebras. *arXiv preprint math/0512517*.

[Schafer, 1995] Schafer, R. (1995). *An Introduction to Nonassociative Algebras (first edition 1966)*. Dover, New York.

[Zorn, 1931] Zorn, M. (1931). Theorie der alternativen ringe. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 8(1):123–147.

Chapter 4

Isomorphic formulations and Modelling

In this chapter, we will be interested in comparing some isomorphic algebraic formulations, not of resolution methods for a given equation but of equations modelling a given phenomenon. Just as there exists a variety of resolution methods to choose from, there exists equally a large collection of mathematical tools that can be considered to approach and build a representation of physical phenomena. Rather than adapting a physical problem to an a priori fixed algebraic frame, our objective is to look for an algebra that matches as best as possible the physical properties. For associative algebras, there exist several other formulations than the ones we presented in the previous chapters and we will put in evidence that the notion of descriptive equivalence is different from that of computational equivalence. We will mainly focus on the relations beween the three other formulations, namely matrices, Clifford algebras and tensors that are widely used in Science and Engineering.

4.1 Matrix representations of associative structures

As a preliminary remark, we first mention that because their product is associative, square matrices cannot faithfully represent non associative algebras as we explained in Chapter 3 with the (alternative) octonions, used for example in Particle Physics. This means that a matrix formulation is not sufficient to describe the algebras that we consider, in particular Dickson algebras in dimension ≥ 8.

4.1.1 2D-algebras

We recall that an *isomorphism* is a bijective application that preserves the algebraic structure (group, ring, algebra). Two algebraic structures are *isomorphic* if there exists an isomorphism between them. For isomorphic structures there can be several distinct applications that describe the isomorphism between these structures. In Chapter 2, we described the canonical algebras of the plane, the complex numbers $\mathbb{C}$ with $i^2 = -1$, the bireal numbers ${}^2\mathbb{R}$, with $u^2 = -1$ and the dual numbers $\mathbb{D}$, with $n^2 = 0$. There exists isomorphisms $z \mapsto M_z$ between these three algebras and three subrings of square matrices of order 2. This means that for z in one of the three algebras of the plane and α, β in $\mathbb{R}$, the isomorphism preserves addition, internal multiplication and multiplication by scalars: $M_{\alpha z+\beta z'} = \alpha M_z + \beta M_{z'}$ and $M_{zz'} = M_z M_{z'}$. We present below in parallel for each of the three algebras, the explicit form of M_z, the matrix representation I_2 of the real unit 1 and the matrix version of each generator i, u, n in the corresponding algebra. In what follows, x, y are real components.

$\mathbb{C},\ i^2 = -1$ | ${}^2\mathbb{R},\ u^2 = 1$ | $\mathbb{D},\ n^2 = 0$

$z = x + iy$ | $z = x + uy$ | $z = x + ny$

$$z \mapsto M_z = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \qquad z \mapsto M_z = \begin{bmatrix} x & y \\ y & x \end{bmatrix} \qquad z \mapsto M_z = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{I}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = M_i \qquad \tilde{\mathbf{U}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M_u \qquad \tilde{\mathbf{N}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = M_n$$

For complex numbers, the isomorphism can also be represented by $z \mapsto M'_z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = M_z^T$.

M_z represents the linear map $a \mapsto z \times a = (xb - yc, yb + xc)$ considering the complex number $a = b + ic$ as a 2D-real vector $a = \begin{pmatrix} b \\ c \end{pmatrix}$, that is $M_z a = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} xb - yc \\ yb + xc \end{pmatrix}$.

The matrix $M'_z = M_{\overline{z}}$ represents, as a linear map, the multiplication by $\overline{z}$, $a \mapsto \overline{z}a$, that is equal to M_z only if z is real ($y = 0$). The two matrices M_z and $M'_z = M_z^T$ define *two* isomorphisms between complex numbers and matrices of order 2, but they do not represent the same multiplication if they are considered as linear maps. It is important to make the difference between the isomorphism between algebras and the matrix formulation of multiplication as a linear application for vectors in $\mathbb{R}^2$.

In Section 1.2.3 of Chapter 2, we introduced examples of zerodivisors for matrices. For dual numbers, the nilpotent matrix $\tilde{\mathbf{N}}$, such that $\tilde{\mathbf{N}}^2 = 0$ is a representation of the nilptotent unit n.

Notice that we can also use the matrix $\tilde{\mathbf{N}}^T$ (there is no descriptive distinction in taking $\tilde{\mathbf{N}}^T$ instead of $\tilde{\mathbf{N}}$).

For the bireal numbers, the two matrices $M_{e_+} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $M_{e_-} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ that have been respectively introduced in Chapter 2 as E_+ and E_- correspond to the bireal numbers $e_+ = \frac{1}{2}(1+u)$ and $e_- = \frac{1}{2}(1-u)$ such that $e_\pm^2 = e_\pm$, $e_+e_- = e_-e_+ = 0$ correspond to $M_{e_\pm}^2 = M_{e_\pm} = E_\pm$ and $E_+E_- = M_{e_+}M_{e_-} = 0$. Notice that $||e_\pm|| = \frac{1}{\sqrt{2}}$

Let $z \in {}^2\mathbb{R}$, $z = x + yu = Xe_+ + Ye_-$, $X = x + y, Y = x - y$. The bireal numbers e_+ and e_- form an *idemptotent basis* of ${}^2\mathbb{R} = \mathbb{R}e_+ \oplus \mathbb{R}e_-$. In this basis, addition *and* multiplication are performed **componentwise**: because $e_+e_- = e_-e_+ = 0$, $zz' = XX'e_+ + YY'e_-$. This natural parallelism in ${}^2\mathbb{R}$ has interesting consequences for multiplanar numbers (Section 4.1.2) and for practical numerical computations (see Chapter 8). If we represent in Figure 4.1 the bireal plane with the axes $1, u$ as the original basis and the idempotent basis $\{e_+, e_-\}$, the change of basis represents a geometric transformation combining a rotation centred at the origin with angle $-\dfrac{\pi}{4}$ and a scale change of the axes, as the basis $\{e_+, e_-\}$ is no longer orthonormal in the sense of euclidean geometry ($||e_\pm|| = \frac{1}{\sqrt{2}}$).

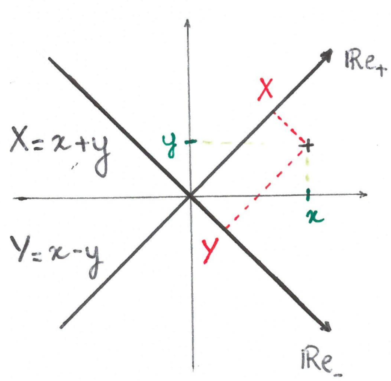


Figure 4.1 – The idempotent basis $\{e_+, e_-\}$ for ${}^2\mathbb{R}$

The above change of basis is represented by the matrix $P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, where $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is known as a *Hadamard matrix*. We recall that Hadamard matrices are square matrices with ± 1 as entries and orthogonal column vectors [Hadamard, 1893]. The link between idempotent basis and Hadamard matrix will be developped in the next section and also in Chapters 7 and 8 where we address applications in Mechanics and Quantum Computing.

4.1.2 Muliplanar numbers

4.1.2.1 Definition and origin

An iterative construction based on the bireal structure allows us to generate a sequence of algebras in dimension 2^k, $k \geq 2$ in order to obtain the *multiplanar numbers*. Let $R_1 \in \{\mathbb{C},\ {}^2\mathbb{R}, \mathbb{D}\}$ be a basis ring of dimension 2. Given R_{k-1} and a nonreal generator $\mathbf{u}_k \notin R_{k-1}$, $\mathbf{u}_k^2 = 1$, then

$$R_k = R_{k-1}[\mathbf{u}_k] = R_{k-1} \oplus R_{k-1}\mathbf{u}_k$$

is a commutative ring where $\mathbf{u}_k$ is the k^{th} nonreal generator such that $dim R_k = 2^k$, $k \geq 2$.

Addition is performed componentwise and for $z = x + y\mathbf{u}_k, z' = x' + y'\mathbf{u}_k \in R_k$, $z \times z' = x \times x' + y \times y' + (x \times y' + y \times x')\mathbf{u}_k$.

In setting $R_1 =\ {}^2\mathbb{R}$, one can see that this is equivalent to start with $R_0 = \mathbb{R}$ and $\mathbf{u}_1 = u$ the bireal unit. This adds one step to the process and shows why in this case the family of algebras $\{R_k\}_{k\geq 0}$ is called *multireals*.

The original idea of this sequence of algebras goes back to the italian algebraist C. Segre who defined in[Segre, 1892] the *multicomplex numbers* using imaginary units $\mathbf{i}_k$, $\mathbf{i}_k^2 = -1$ for duplication instead of the $\mathbf{u}_k$. Multicomplex numbers are defined inductively such that $\mathbf{C}_0 = \mathbb{R}$ and $\mathbf{C}_{n+1} = \{z = x + y\mathbf{i}_{n+1}$ $, x, y \in \mathbf{C}_n\}$, $n \geq 1$, where $(\mathbf{i}_k)_{1 \leq k \leq n}$ is a sequence of genererators such that $\mathbf{i}_k^2 = -1$. The properties of mutliplication are similar at each level k to complex multiplication using the fact that by definition generators commute, $\forall k, l \geq 1$, $\mathbf{i}_k \times \mathbf{i}_l = \mathbf{i}_l \times \mathbf{i}_k$. $\mathbf{C}_1$ is the set of complex numbers and elements of $\mathbf{C}_2 = {}^2\mathbb{C}$ are called *bicomplex numbers*. The algebra $\mathbf{C}_n$ is of dimension 2^n over $\mathbb{R}$. The algebraic study has been fully performed by Segre and the properties of these structures as Banach algebras for analysis have been investigated in [Price, 1991].

As generators commute, we notice that $(\mathbf{i}_k \times \mathbf{i}_l)^2 = 1$ plays a role equivalent to that of $\mathbf{u}_k$, this is why the definition of multicomplex numbers by Segre is contained into the multiplanar one by choosing $R_1 = \mathbb{C}$. This can be checked for example with the bicomplex numbers $\mathbf{C}_2$ between the version of Segre (two imaginary units i_1, i_2) and the one with the imaginary unit i and the bireal unit u. The two basis $(1, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_1 \times \mathbf{i}_2)$ and $(1, i, u \times i, u)$ can be easily related, taking $i = \mathbf{i}_1$, $u \times i = i_2$ and $u = -\mathbf{i}_1 \times \mathbf{i}_2$. This last version of bicomplex numbers goes back to Cockle who called them *tessarines* some 44 years before Segre [Cockle, 1848]. Inspired by Hamilton, he invented the nonreal numbers $\mathbf{u}_1 = u$ and $\mathbf{u}_2$ which square to 1 in respective dimensions 2 and 4. The two forms have distinct properties especially concerning their matrix formulations [Chatelin, 2018, Chap.5]: in Segre's formulation, the matrix reduction of multicomplex numbers does not present the same block structure as we will see in the next section. This is why we will keep the formulation with the units $\mathbf{u}_k$.

Multiplanar algebras are very different from the Dickson algebras presented in Chapter 3. Even if they share a common dimension 2^k resulting from a duplication process, the definition of multiplication in multiplanar algebras implies no novelty in the algebraic properties for multiplanar numbers. By contrast, only Dickson algebras display a creative power by induction. For multiplanar algebras, the induction simply produces replication of the properties of the initial algebra. We will see in the next section a consequence in this replication, with the specific matrix formulation of multireal numbers.

4.1.2.2 Recursive construction and reduction for hierarchical parallelism

Let Z be a multireal number in R_k, we can expand the components of Z in the successively embedded algebras:

$$\begin{array}{ccccccc} Z & = & A & + & B\ \mathbf{u_k} & \in R_k & ,\ A, B \in R_{k-1} \\ & & \swarrow\searrow & & \swarrow\searrow & & \\ & & A_1 + B_1\mathbf{u_{k-1}} & & A_2 + B_2\mathbf{u_{k-1}} & \in R_{k-1} & ,\ A_1, A_2, B_1, B_2 \in R_{k-2} \end{array}$$

which corresponds in matrix representation to:

$$M_Z = \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \rightarrow \left(\begin{array}{c|c|c|c} A_1 & B_1 & A_2 & B_2 \\ \hline B_1 & A_1 & B_2 & A_2 \\ \hline A_2 & B_2 & A_1 & B_1 \\ \hline B_2 & A_2 & B_1 & A_1 \end{array}\right) = \left(\begin{array}{c|c} M_A & M_B \\ \hline M_B & M_A \end{array}\right)$$

The matrix that represents the multireal number Z has a 2×2 block circulant structure and each block is itself circulant.

The matrix that represents the multireal number Z can be reduced to a diagonal form, by using the idempotent representation, recursively at each level ($k \rightarrow k-1 \rightarrow .. \rightarrow 1$), as was done for the bireal numbers in Section 4.1.1 with the idempotent basis. From

$$Z = A + Bu_k = (A + B)e_{k+} + (A - B)e_{k-}$$

and then with $A_1 + B_1\mathbf{u_{k-1}}$ and B$=A_2 + B_2\mathbf{u_{k-1}}$, we have

$$\begin{aligned}
Z &= \left[(A_1 + B_1\mathbf{u_{k-1}}) + (A_2 + B_2\mathbf{u_{k-1}})\right] e_{k+} + \left[(A_1 + B_1\mathbf{u_{k-1}}) - (A_2 + B_2\mathbf{u_{k-1}})\right] e_{k-} \\
&= \left[(A_1 + A_2) + (B_1 + B_2)\mathbf{u_{k-1}}\right] e_{k+} + \left[(A_1 - A_2) + (B_1 - B_2)\mathbf{u_{k-1}}\right] e_{k-} \\
&= \big[\left[(A_1 + A_2) + (B_1 + B_2)\right] e_{(k-1)+} + \left[(A_1 + A_2) - (B_1 + B_2)\right] e_{(k-1)-}\big] e_{k+} \\
&\quad + \big[\left[(A_1 - A_2) + (B_1 - B_2)\right] e_{(k-1)+} + \left[(A_1 - A_2) - (B_1 - B_2)\right] e_{(k-1)-}\big] e_{k-} \\
&= \left[(A_1 + A_2) + (B_1 + B_2)\right] e_{(k-1)+}e_{k+} + \left[(A_1 + A_2) - (B_1 + B_2)\right] e_{(k-1)-}e_{k+} \\
&\quad + \left[(A_1 - A_2) + (B_1 - B_2)\right] e_{(k-1)+}e_{k-} + \left[(A_1 - A_2) - (B_1 - B_2)\right] e_{(k-1)-}e_{k-}.
\end{aligned}$$

We can thus recursively construct an idempotent 2^k basis whose vectors are products of idempotents of each level R_m, $1 \leq m \leq k$ which corresponds in matrix representation to:

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \rightarrow \left(\begin{array}{c|c} A+B & 0 \\ \hline 0 & A-B \end{array}\right) \rightarrow \left(\begin{array}{c|c} \begin{array}{c|c} \bullet & 0 \\ \hline 0 & \bullet \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} \bullet & 0 \\ \hline 0 & \bullet \end{array} \end{array}\right) \rightarrow \dots \rightarrow \begin{pmatrix} \cdot & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \end{pmatrix}$$

This process reflects the arborescent structure to obtain a diagonal form thanks to idempotent 2^k basis. This is a structural form due to the algebra and not to computation. The interesting part of this reduction concerns the matrix for the change of basis:

$k \rightarrow k-1$	$k \rightarrow 1$
$\frac{1}{2}\left(\begin{array}{c\|c} I & I \\ \hline I & -I \end{array}\right)$	$P_1 = \frac{1}{2}H_1 = \frac{1}{2}\left(\begin{array}{c\|c} 1 & 1 \\ \hline 1 & -1 \end{array}\right)$, $P_k = \frac{1}{2}H_k = \frac{1}{2}\left(\begin{array}{c\|c} H_{k-1} & H_{k-1} \\ \hline H_{k-1} & -H_{k-1} \end{array}\right)$

Up to a coefficient (the idempotent basis is not orthonormal), the sequence $\{2P_k = H_k\}_{k\geq 1}$ is a particular case of Hadamard matrices [Hadamard, 1893]. This particular sequence of Hadamard matrices was known to Sylverster (1867). It plays a key role in the Walsh-Hadamard transform in Discrete Fourier Transform (DFT) used for data compression in signal and image processing [Ahmed et al., 1974]. The relation between the structure of the multireals and the method of Walsh-Hadamard transform is a remarkable connection which is not widely appreciated [Alfsmann et al., 2007].

The reduction principle for multireals relies on dichotomy and on the uniqueness of the idempotent basis that gives a recursive construction of naturally embedded blocks. Notice that with $R_1 = \mathbb{C}$ or $R_1 = \mathbb{D}$, we cannot achieve a fully diagonal form but a 2×2 block diagonal one. The 2×2 blocks are the matrices M_z of the form given in Section 4.1.1 for complex or dual components.

We will now go back to Dickson algebras with quaternions and we will see the specific consequences of an evolutive definition of multplication for their matrix formulations, which contrast with the replication of algebraic properties in multiplanar algebras.

4.1.3 Quaternions

Let $q = a + bi + cj + dk \in \mathbb{H}$. Quaternions have several representations over $\mathbb{R}$, one of the common form found in the litterature is given by

$$M_{\mathbb{R}}(q) = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} = \begin{bmatrix} Z^T & T^T \\ -T & Z \end{bmatrix},$$

where $Z = M_{a+ib} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $T = M_{c+id} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$.

We can represent the left (resp. right) multiplication of a quaternion p by a quaternion q as matrix L_q (resp. R_q) acting as a linear map over $\mathbb{R}^4$, where the quaternion $p = w + xi + yj + zk$ is considered as a vector of $\mathbb{R}^4$, $p = (w, x, y, z)$:

$$L_q : p \mapsto q \times p,\ L_q = \begin{bmatrix} Z & -V \\ V & Z \end{bmatrix},\ \text{with } Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ and } V = \begin{bmatrix} c & d \\ d & -c \end{bmatrix}$$

$$\text{such that } q \times p = L_q \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = L_q p \text{ and } R_q : p \mapsto p \times q,\ R_q = \begin{bmatrix} Z & -T^T \\ T & Z^T \end{bmatrix} = M_{\mathbb{R}}(q)^T.$$

The matrices that represent left and right multiplications by the quaternion q as the linear maps L_q , R_q are unique, but the matrix representation of the quaternion q itself is not. The standard formulation $M_{\mathbb{R}}(q)$ generaly found in textbooks is the transposed matrix of the right multiplication matrix R_q. As there exist *several* distinct representations of quaternions, "the matrix form" of a quaternion is an ambiguous term. If we represent quaternions by matrices, we introduce a potential misunderstanding that **does not** exist with Hamilton's unique formulation of the quaternions. As for complex numbers, this shows the difference between isomorphic structures and multiplication as a linear map.

Notice also that as $\mathbb{H}$ is associative, $\forall\ q\ ,\ q' \in \mathbb{H}\ ,\ [L_q R_{q'}] = L_q R_{q'} - R_{q'} L_q = 0$ that is $\forall x \in \mathbb{H}\ , q \times (x \times q') = (q \times x) \times q'$. This result will be used for the representation of rotations in Section 4.3.1.

Quaternions also have a matrix representation over $\mathbb{C}$, rewriting $q = (a+bi)+(c+di)\times j = z+t\times j$, $z, t \in \mathbb{C}$, we get back the form obtained by the Dickson doubling process.

Let $M_{\mathbb{C}}(q) = \begin{bmatrix} z & t \\ -\bar{t} & \bar{z} \end{bmatrix}$, that is $M_{\mathbb{C}}(q) = aI_2 + b\mathbf{I} + c\mathbf{J} + d\mathbf{K}$,

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{I} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$.

$M_{\mathbb{C}}$ is an isomorphism between quaternions and 2×2 complex matrices [Dickson, 1924]. Matrices **I**,**J** and **K** are closely related to Pauli matrices in Quantum mechanics (see Chapter 8).

Unlike for real matrices, there is some limitation to the matrix representation over $\mathbb{C}$. **Only** *right* multiplication can be defined for $p = x + yj$, $q = z + tj \in \mathbb{H}$, $x, y, z, t \in \mathbb{C}$. If $p = (x, y)$ is considered as a vector of $\mathbb{C}^2$ and $R_q = \begin{bmatrix} z & -\bar{t} \\ t & \bar{z} \end{bmatrix} = M_{\mathbb{C}}(q)^T$ then :

$R_q : p \mapsto p \times q = (xz - \bar{t}y, tx + y\bar{z}) = R_q \begin{pmatrix} x \\ y \end{pmatrix} = R_q p.$

The left multiplication operator $L_q : p \mapsto q \times p$ cannot be a linear map due to conjugation. The product $q \times p = (zx - \overline{y}t, yz + t\overline{x})$ expressed as a matrix-vector product would involve the two conjugate vectors $\begin{pmatrix} x \\ \overline{y} \end{pmatrix}$ and $\begin{pmatrix} \overline{x} \\ y \end{pmatrix}$ but **not** the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ corresponding to p.

The real matrix forms put in evidence an ambiguity due to nonuniqueness in the formulation. The presentation of a quaternion as a complex matrix is coherent only if we consider matrices as numbers, thanks to the isomorphism between structures, and not as linear applications. The difference is that mutliplication is either an operation between two elements of the same nature (two quaternions) or a matrix-vector product between two distinct mathematical objects to represent a quaternion (a matrix and a vector).

The real formulation of quaternions is the only one that can represent the two linear maps corresponding to left and right multiplications. Once again, this is due to the difference between isomorphic structures and the formulation of left and right multiplication as linear maps.

4.2 Clifford algebras

4.2.1 Preliminary warning

The topic of Clifford algebras requires an advanced technical knowledge in algebra. We will give in this section a quick overview of the main features of Clifford algebras. These algebras are not the main concern of our work and we try to sum up in the next sentences, for the reader not acquainted with this topic, the most important aspects related to our work. A more technical introduction is presented in the Section 4.2.2.

Clifford algebras are **associative** structures and there exist descriptive isomorphisms between associative hypercomplex algebras and Clifford algebras. By definition, given a vector space V of dimension n, the product of vectors is not a vector in V but is an element of the Clifford algebra of dimension 2^n, derived from V. The geometric aspect for $n = 3$ is at the basis of the theory of Clifford algebras with $dimV = n > 3$ along with the notions of exterior algebra due to Grassmann. Clifford algebras represent the choice to favour a geometric perspective inherited from 3D-geometry instead of a computational one centered on multiplication which is the backbone this thesis.

With hypercomplex and Dickson algebras, we focus on an inner multiplication of vectors (a product of vectors is a vector) and geometry is a consequence of Computation. In Clifford algebras, the choice is made to extend as much as possible the 3D geometric behaviour into the n-dimensional case $n \geq 3$. Both formulations are, of course, correct and we can characterise Clifford algebras vs. our hypercomplex algebras as (3D)geometry-driven vs computation-driven approaches. We can also contrast the bottom-up (3D to nD in Clifford algebras) geometric version and the top-down computational perpsective, not limited to associativity, in analogy with the ideas developed at the end of Chapter 3 for the extension of vector/cross product.

4.2.2 Introduction to Clifford algebras

Clifford algebras are a family of unital associative algebras generated by a vector space V of dimension n over a field $K = \mathbb{R}$ or $\mathbb{C}$ and a *nondegenerate* quadratic form $Q : x \in V \mapsto Q(x) \in \mathbb{R}$. In what follows we consider the case $K = \mathbb{R}$ and we mention the differences for the case $K = \mathbb{C}$. We denote by B the symmetric (hermitian for $K = \mathbb{C}$) bilinear form associated to Q such that $\forall x, y \in V, B(x, y) = Q(x + y) - Q(x) - Q(y)$. Notice that even if $K = \mathbb{C}$, as B is hermitian, $B(y, x) = \overline{B(x, y)}$ and $Q(x) = B(x, x) \in \mathbb{R}$. Let F_Q be the symmetric (hermitian for $K = \mathbb{C}$) invertible matrix associated with Q such that, $\forall x, y \in V,\ Q(x) = x^T F_Q x$, then $B(x, y) = x^T F_Q y$. The Clifford algebra $\mathcal{C}l(V, Q)$ of dimension 2^n is the algebra generated by elements of V with the only condition for the associative multiplication in $\mathcal{C}l(V, Q)$ to verify $v^2 = Q(v)$, $\forall v \in V$ or equivalently $uv + vu = 2B(u, v)$. If $V = \mathbb{R}^n$ (resp. $\mathbb{C}^n$), with the euclidean inner product $\langle .|. \rangle$ (in this case $F_Q = I_n$) and the corresponding algebra are denoted $\mathcal{C}l_n(\mathbb{R})$ (resp. $\mathcal{C}l_n(\mathbb{C})$).

The spectrum of the matrix F_Q associated with Q can be split in two parts, p stricly positive and q striclty negative eigenvalues , $p + q = n$ with $p, q \in \mathbb{N}$, and their associated eigenvectors form an orthogonal basis of V $(e_1, .., e_p, e_{p+1}, .., e_{p+q})$. The quadratic form has *signature* (p, q). Notice that depending on English-speaking or continental notations, the role of p and q may be exchanged in the litterature. The Clifford algebra $\mathcal{C}l(V,Q)$ is also denoted $\mathcal{C}l_{p,q}(K)$.

Let us consider an **orthonormal** eigenbasis $(e_1, e_p, e_q, .., e_n)$ for Q in V, such that, for $1 \leq i \leq p$, $e_i^2 = Q(e_i) = 1$, and for $q = n - p \leq i \leq n$, $e_i^2 = Q(e_i) = -1$.

The set $\{e_{i_1}e_{i_2}..e_{i_k},\ 1 \leq i_1 < i_2 < .. < i_k \leq n,\ 0 \leq k \leq n\}$ is a basis of the Clifford algebra with the convention that the case $k = 0$ corresponds to the multiplicative unit 1. This is related to the concept of free algebra and corresponds to all the distinct products obtained by multiplying the vectors $(e_1, e_p, e_q, .., e_n)$ of V taking into account the relations implied by Q: since the basis $(e_1, .., e_n)$ of V is orthonormal, $2B(e_i, e_j) = e_ie_j + e_je_i = 0$ such that for $i \neq j$, e_i and e_j anticommute: $e_ie_j = -e_je_i$. . The dimension of the Clifford algebra is 2^n. For example, let $dimV = 3$ with an orthonormal basis (e_1, e_2, e_3) and Q a quadratic form of signature $(3, 0)$. A basis of the Clifford algebra $\mathcal{C}l(V,Q) = \mathcal{C}l_{3,0}(\mathbb{R})$ of dimension $2^3 = 8$ is $(1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_1e_3, e_1e_2e_3)$. The other products obtained from (e_1, e_2, e_3) can be reduced, up to a sign, to an element of the basis of $\mathcal{C}l(V,Q)$, where $e_1^2 = e_2^2 = e_3^2 = 1$ and e_1, e_2 and e_3 anticommute (e.g. $e_1e_2e_1e_3e_2 = e_1(-e_1e_2)(-e_2e_3) = e_3$). If we compare this to the associative hypercomplex algebras, the orthornomal basis of V plays the role of external generators for the Clifford algebra (Section 3.3.2 in Chapter 3), the basis of $\mathcal{C}l(V,Q)$ is obtained by multiplication of the basis in V associated to the constraints of the quadratic form (anticommutativity and real square). Another important difference is that because multiplication in Clifford algebras is associative, it cannot bring as much evolution when n increases, than what is possible in the case of nonassociative Dickson algebras ($n \geq 3$).

As expected in Clifford algebras, the product of two vectors a, b in V denoted ab is not a vector of V but a vector of the Clifford algebra $\mathcal{C}l(V,Q)$. The antisymmetric part $\frac{1}{2}(ab-ba)=\frac{1}{2}[a,b]$ is called a *bivector* and is denoted $a \wedge b$ (exterior product of Grassmann). The subalgebra generated by the scalars and the bivectors is called the *even subalgebra* denoted $\mathcal{C}l^{+}_{p,q}(K)$. The notion of bivector is linked with geometry, generalised from the 3D-case that we develop now. For $n=3$, the bivector is interpreted as the "oriented area" defined by the two vectors a and b. The geometric case in 3D for V is at the origin of the generalisation of bivectors in higher dimensional Clifford algebras.

Let us detail the case $n=3$ where $K=\mathbb{R}$, V has the orthonormal basis (e_1,e_2,e_3) and Q a quadratic form of signature $(3,0)$ with $S=I_3$. Let $a=a_1e_1+a_2e_2+a_3e_3$ and $b=b_1e_1+b_2e_2+b_3e_3$ in V, with $ab=(a_1e_1+a_2e_2+a_3e_3)(b_1e_1+b_2e_2+b_3e_3)\in\mathcal{C}l(V,Q)=\mathcal{C}l_3(\mathbb{R})$. After computation the exterior product of a and b is $a\wedge b=\frac{1}{2}(ab-ba)=(a_1b_2-a_2b_1)e_1e_2+(a_2b_3-a_3b_2)e_2e_3+(a_3b_1-a_1b_3)e_3e_1$. **The exterior product of two 3D-vectors has the same components in the even subalgebra $\mathcal{C}l^{+}_{3,0}(\mathbb{R})$ as the classical 3D-cross product in $\mathbb{H}$.** $\mathcal{C}l^{+}_{3,0}(\mathbb{R})$ of dimension 4 with the basis $(1,e_2e_3,e_3e_1,e_1e_2)$ is isomorphic to $\mathbb{H}$. For $l\neq m$, $(e_le_m)^2=(e_le_m)(e_le_m)=e_l(-e_le_m)e_m=-e_l^2e_m^2$. As $e_1^2=e_2^2=e_3^2=1$, e_2e_3, e_1e_2, e_3e_1 square to -1, anticommute and verify the same multiplicative rules as the quaternions ($i^2=j^2=k^2=-1$, $i\times j=k$, $j\times k=i$, $k\times i=j$), if we take $i\simeq e_2e_3$, $j\simeq e_1e_2$ and $k\simeq e_3e_1$ (e.g. $(e_2e_3)(e_1e_2)=e_3e_1$ is isomorphic to $i\times j=k$). Notice that as the basis of V is orthonormal, $e_ie_j=e_i\wedge e_j$, $i\neq j$. The subalgebra $\mathcal{C}l^{+}_{3,0}(\mathbb{R})$ is generated by products of generators in V. By comparison in $\mathbb{H}$, the generators are directly given.

It is not clear that Clifford algebras bring any additional computational information compared to the formulation of associative hypercomplex numbers. Indeed, as matrix representations, Clifford algebras introduce ambiguity, because practical computations rely on the coefficients of the basis but several Clifford algebras can describe the same hypercomplex algebra using different vector spaces as generators, and different choices of basis. Isomorphic description could use Clifford algebras or even subalgebras. For example, $\mathcal{C}l_{0,2}(\mathbb{R})$ has four dimensions and is spanned by $(1, e_1, e_2, e_1e_2)$, where e_1, e_2, e_1e_2 square to -1 and anticommute, making $\mathcal{C}l_{0,2}(\mathbb{R})$ isomorphic to $\mathbb{H}$. This brings a lot of confusion because there is no standard form to describe the algebra. On the contrary, in associative hypercomplex algebras, one can exhibit an explicit multiplication table with the generators and associated basis. The generalisation to dimension $n > 3$ designed for geometry does not concern our computational purposes and is unable to fully represent the nonassociative structures, that are used in Physics (particle and high energy Physics) [Baez, 2002, Catto et al., 2016]. Like for matrix algebras, one must distinguish between the **descriptive** use of Clifford algebras in Physics to classify elements (symmetry groups,..) and the practical role for **computational** purpose.

Under the influence of the physicist David Hestenes, Clifford algebras over $\mathbb{R}$ considered primarily for their geometric aspects (rotations, inversions in n dimensions) are called *geometric algebras* [Hestenes and Sobczyk, 2012] and enjoy great popularity. For more on this topic, one of the most fundamental references amongst the existing litterature is once again [Lounesto, 2001].

We give below some of the most used equivalences between Clifford algebras and our formulation. $\mathcal{C}l_{0,0}(\mathbb{R})$ is isomorphic to $\mathbb{R}$. $\mathcal{C}l_{0,1}(\mathbb{R})$ is of dimension 2 and generated by e_1 that squares to -1, so that $\mathcal{C}l_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$. ${}^2\mathbb{R}$ is isomorphic to $\mathcal{C}l_{1,0}(\mathbb{R})$ but also to $\mathcal{C}l^+_{1,1}(\mathbb{R})$ the even part of $\mathcal{C}l_{1,1}(\mathbb{R})$. For the complex case, $\mathcal{C}l_0(\mathbb{C})$ is isomorphic to the complex numbers, $\mathcal{C}l_1(\mathbb{C})$ is isomorphic to the bicomplex numbers $\mathbf{C}_2 = {}^2\mathbb{C}$ (Section 4.1.2.1) and $\mathcal{C}l_1(\mathbb{C})$ is isomorphic to the complex quaternions $\mathbb{H}(\mathbb{C})$ (Section 3.5 in Chapter 3). Let $A \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, it is established that all Clifford algebras can be expressed as one of the three following forms: A, the direct sum $A \oplus A$ or the square matrices over A, $\mathcal{M}_n(A)$. This result is related to the Artin-Wedderburn theorem for matrix formulation and to the periodicity of Clifford algebras discovered by Cartan [Cartan, 1908] and usually attributed to Bott as in [Baez, 2002] (See Bibliographical notes in Chapter 2).

4.3 Rotations

Rotations are an interesting example to compare the different formulations of a given problem. We will put in evidence that some misunderstandings are due to the conceptual approach that is associated with each version. In order to possess all the necessary elements to study rotations in the 3D-space, we will first study the 4D case and the relation with quaternions.

4.3.1 4D-rotations

In 4D, a rotation can be expressed as a matrix of the special orthogonal group, $S \in SO_4(\mathbb{R})$, $SS^T = I_4$, $\det S = 1$. Since Hamilton and Cayley, a 4D-rotation can be also represented as a product of quaternions. Let x, l, $r \in \mathbb{H}$, l and r with unit euclidean norm ($||l|| = ||r|| = 1$), the rotation of x is represented by the map $x \mapsto l \times x \times r$. According to the real matrix form of left and right multiplication as linear maps presented in Section 4.3.1, we can represent the two quaternions l and r by their respective matrices L_l and R_r. For simplicity we write $L_l = L$ and $R_r = R$. As $||l|| = ||r|| = 1$, L and R are orthogonal matrices. If x is considered as a vector of $\mathbb{R}^4$, we can write the rotation of x as $Sx = LRx$. As L and R commute (associativity of $\mathbb{H}$, see Section), we have $S = LR = LR$ and this decomposition is unique. There exist two ways to find from S the components of L and R. This is done either by direct calculation in [van Elfrinkhof, 1897] or with a constructive method in [Rosen, 1930]. It is interesting to notice that Rosen [1] found this decomposition in the case of the relation between quaternions and Lorentz transforms. We introduce in the next section the notion of isoclinic rotations which allows us to link the geometric aspects of rotations angles with quaternions.

4.3.2 Isoclinic rotations in 4D

We recall the standard reduction of orthogonal matrices into a block diagonal form. In the case of a rotation in $\mathbb{R}^4$, a rotation matrix can be reduced as:

$$\begin{bmatrix} C(\alpha_1) & 0 \\ 0 & C(\alpha_2) \end{bmatrix}, \; C(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

with α_1 and α_2 the angles of the two orthogonal rotations corresponding to the 4 complex eigenvalues $e^{\pm i\alpha_1}, e^{\pm i\alpha_2}$ (2 pairs of conjugate eigenvalues), with unit modulus.

In the particular case of 4D, a rotation of $\mathbb{R}^4$ such that $\alpha_1 = \pm\alpha_2$ is said to be *isoclinic*. By convention we distinguish *left isoclinic* ($\alpha_1 = \alpha_2$) and *right isoclinic* ($\alpha_1 = -\alpha_2$) rotations. The decomposition of a general rotation in 4D as the product of 2 rotations (left and right isoclinic) in 2 orthogonal planes in $\mathbb{R}^4$ is due to [Goursat, 1889] for the pure geometric point of view, with no mention of quaternions.

The two references [Coxeter, 1946] and [Weiner and Wilkens, 2005], put in evidence the link between isoclinic rotations and quaternions. For $q \in \mathbb{H}$, $||q|| = 1$, it can be shown that L_q, $R_q \in SO_4$ have a pair of double eigenvalues $e^{\pm i\alpha}$ ($\alpha = \alpha_1 = \pm\alpha_2$). L_q corresponds to a left isoclinic rotation ($\alpha_1 = \alpha_2$) and R_q corresponds to a right isoclinic one ($\alpha_1 = -\alpha_2$).

1. He is the R of the EPR experiment in Quantum Mechanics.

The matrix L_q and R_q are respectively similar to $\begin{bmatrix} C(\alpha) & 0 \\ 0 & C(\alpha) \end{bmatrix}$, and $\begin{bmatrix} C(\alpha) & 0 \\ 0 & C(-\alpha) \end{bmatrix}$.

Going back to the matrix $S = LR = L_l R_r$, if we denote α_l and α_r the angles corresponding to the reduced forms of L_l and R_r, we have:

$$S = \begin{bmatrix} C(\alpha_l) & 0 \\ 0 & C(\alpha_l) \end{bmatrix} \begin{bmatrix} C(\alpha_r) & 0 \\ 0 & C(-\alpha_r) \end{bmatrix} = \begin{bmatrix} C(\alpha_l + \alpha_r) & 0 \\ 0 & C(\alpha_l - \alpha_r) \end{bmatrix}$$

In summary, we can say that a general rotation in 4D is the product of a left and a right isoclinic rotation respectively represented by the left and right multiplications of two unit quaternions.

4.3.3 An alternative view for 3D-rotations

We are now able to explicit the role of quaternions for 3D spatial rotations compared to other formulations. The quaternionic formulation of 3D-rotations can be seen as a particular case of the 4D-rotations.

Let $q = a + bi + cj + dk \in \mathbb{H}$ a unitary quaternion $||q||^2 = a^2 + b^2 + c^2 + d^2 = 1$, and $w = 0 + xi + yj + zk \in \mathcal{I}\mathbb{H}$, an imaginary (=pure) quaternion. The rotation of w in $\mathcal{I}\mathbb{H}$ is given by Hamilton: $w \longmapsto q \times w \times q^{-1} = q \times w \times \overline{q}$. This is an automorphism of quaternions as presented in (Section 4.2 Chapter 3). The result of the rotation $w \longmapsto q \times w \times q^{-1}$ is a pure quaternion.

If we now represent rotations with orthogonal matrices of order 3 and matrix-vector product, corresponding to the quaternionic multiplication, we have the Euler-Rodrigues formula:

$$O = \begin{bmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{bmatrix}, \; w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; in \ \mathbb{R}^3$$

O is a rotation matrix ($OO^T = I_3$, $\det O = 1$) and the rotation of w considered as a vector in $\mathbb{R}^3$ is the vector Ow.

We will now be more specific about the angle of the rotation in 3D which is a source of controversy in the litterature. If we consider the rotation angle θ with the rotation axis $u = (r, s, t)$ a unit vector of $\mathbb{R}^3$, then the rotation is expressed by the 4D-quaternion $q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(ri + sj + tk)$. In [Altmann, 1989], Altmann gives all credit for rotations in 3D to Olinde Rodrigues who actually gave the matrix version and a coordinate free formula [Rodrigues, 1840]. He goes as far as to suggest that Hamilton did not understand properly his own invention of quaternions! His two arguments are that 1) since the rotation of quaternions is not of the form $w \mapsto q \times w$, "it is no longer possible to say that the quaternion operates on a vector transforming it into another vector" and 2) the angle in the quaternionic formula is not the vector angle θ but half of it. Both arguments display a profound misunderstanding of quaternionic computation and its geometric interpretation in 3 and 4D. This misguided point of view is quite popular in the Geometric Algebra community (the real Clifford algebras presented in Section 4.2.2) and forms the basic reason for the definition of general rotations in Clifford algebras' formulation. These two claims of Altmann put in evidence that he only considers imaginary quaternions to represent 3D-rotations. There is no mention in his work of quaternions used for 4D-rotations and that 3D-rotations with quaternions are a consequence of the 4D-case, the full dimension of quaternions. As we explained, quaternions act as 4D-rotations so that the left multiplication corresponds to a left isoclinic rotation in 4D... and not in 3D. Once again, this shows the reluctance to consider a quaternion as a number that can be multiplied. The result of the multiplication is a quaternion not limited to be a pure quaternion (assimilated to a 3D-vector). The quaternionic multiplication that represents 3D-rotations is the composition of two rotations, the left isoclinic with $q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(ri + sj + tk)$ and a right isoclinic with $q^{-1} = \overline{q} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)(ri + sj + tk)$. The resulting geometrical effect of these two rotations in 3D is an invariant axis which is the rotation axis and a plane with a total rotation angle of $\frac{\theta}{2} + \frac{\theta}{2} = \theta$ according to the decomposition into isoclinic rotations. Another important remark is that there is no trace of a multiplication and an algebraic structure in the papers of Rodrigues. His results are absolutely correct but do not reveal the computational role of multiplication blindingly present in Hamilton's work. All of these discussions present two radically distinct points of view. Researchers ignoring the real part of 4D-quaternions tend to force them to behave like 3D-vectors. Following Hamilton, we present a different point of view, based on $w \mapsto q \times w \times q^{-1}$, where $w \in \mathcal{I}(\mathbb{H})$, $q \in \mathbb{H}$.

The example of 3D-rotations and quaternions is highly instructive. We have seen that it is a bifurcation point between different conceptions. In Physics, with a vectorial approach, there is no multiplication of 4D-vectors: scalars in $\mathbb{R}$ and vectors in $\mathcal{I}(\mathbb{H})$ are independent components of $\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}^3$. In Clifford algebras the result of multiplication is **not** a vector. The idea that underlies our work with hypercomplex algebras is based on computational reasons and centred on multiplication. We will see in Chapter 8, that the quaternionic version of 3D-rotations is intensively used in applicative fields for their algorithmic and numerical advantages.

After matrices and Cifford algebras, we will now have a look at tensors formulations. This is an isolated aspect of our work for comparison of structure but we will use it for practical applications in continuum mechanics and quantum computing.

4.4 A few words about tensors

Tensors are used in a large range of applications in Science. We will not discuss their role as multilinear operators in data analysis or in differential geometry since it bears no connection with the goal of this thesis.

In our work, the role of a tensor formulation is reduced to be that of a descriptive tool to classify algebras, encountered for example with complex quaternions $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ but with no computational use. Indeed, concerning practical computations for the two main applications that we will develop in Chapters 7 and 8, we will see that either a tensor formulation does not bring any additional information, or its properties are actually not used in computation. In particular, notice that the notion of eigenvalue or eigenvector of a general tensor has no clear meaning. In the case of continuum mechanics, the properties of the symmetric stress tensor (see Chapter 7) in mechanicist parlance is all due to the spectral theory of matrices.

For Quantum Computing, we will see that the description of quantum gates presented as tensor products can be reduced in practice to matrix block products (Kronecker product) and we will provide another formulation with hypercomplex algebras.

Summing up this chapter, we have seen that there exist different formulations of the same mathematical objects that represent historical and conceptual choices originating in Physics, Geometry and Computation. One key point for Computation is whether the multiplication of vectors is explicitly considered.

We have indicated that isomorphisms with matrices and Clifford algebras can only be used for associative structures, a fact which is a strong limitation compared to the hypercomplex formulation.

Concerning matrices, the isomorphisms with 2D algebras and quaternions are not unique and thus ambiguous if the isomorphism used is not clearly specified. Complex matrices are only able to represent right quaternionic multiplications as linear operators. The real form of left and right quaternionic multiplication is related to the description of general rotations in 4D. The notion of isomorphism is relative: it is valid for formal description where the different formulations reveal different aspects. We will see in the applications (Chapters 7 and 8) that there is not only a fundamental conceptual difference between isomorphic structures but that it also implies important consequences about algorithmic and numerical aspects.

Bibliography

[Ahmed et al., 1974] Ahmed, N., Natarajan, T., and Rao, K. R. (1974). Discrete cosine transform. *IEEE transactions on Computers*, 100(1):90–93.

[Alfsmann et al., 2007] Alfsmann, D., Göckler, H. G., Sangwine, S. J., and Ell, T. A. (2007). Hypercomplex algebras in digital signal processing: Benefits and drawbacks. In *Proceedings of EUSIPCO*, pages 1322–6.

[Altmann, 1989] Altmann, S. L. (1989). Hamilton, Rodrigues, and the quaternion scandal. *Mathematics Magazine*, 62(5):291–308.

[Baez, 2002] Baez, J. (2002). The octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205.

[Cartan, 1908] Cartan, E. (1908). Nombres complexes. in *Encycl. Sc. Math.,* (J. Molk ed.) d'après l'article allemand de Study, I(1-5):329–468 (Gauthier–Villars, Paris).

[Catto et al., 2016] Catto, S., Gürcan, Y., Khalfan, A., and Kurt, L. (2016). Unifying ancient and modern geometries through octonions. In *Journal of Physics: Conference Series*, volume 670, pages 12–16. IOP Publishing.

[Chatelin, 2018] Chatelin, F. (2018). *Numbers in Mind: the transformative ways of Multiplication.* Book in preparation to be published by World Scientific.

[Cockle, 1848] Cockle, J. (1848). III. On Certain Functions Resembling Quaternions, and on a New Imaginary Algebra. *Phil. Mag. (3)*, 33:435–439.

[Coxeter, 1946] Coxeter, H. (1946). Quaternions and reflections. *The American Mathematical Monthly*, 53(3):136–146.

[Dickson, 1924] Dickson, L. E. (1924). Algebras and their arithmetics. *Bulletin of the American Mathematical Society*, 30(5-6):247–257.

[Goursat, 1889] Goursat, E. (1889). Sur les substitutions orthogonales et les divisions régulières de l'espace. *Ann. sci. ecole norm.*, 3:9–102.

[Hadamard, 1893] Hadamard, J. (1893). Résolution d'une question relative aux déterminants. *Bull. des sciences math.*, 2:240–246.

[Hestenes and Sobczyk, 2012] Hestenes, D. and Sobczyk, G. (2012). *Clifford algebra to geometric calculus: a unified language for mathematics and physics*, volume 5. Springer Science & Business Media.

[Lounesto, 2001] Lounesto, P. (2001). *Clifford algebras and spinors*, volume 286. Cambridge university press.

[Price, 1991] Price, G. B. (1991). *An introduction to multicomplex spaces and functions*, volume 140. CRC Press.

[Rodrigues, 1840] Rodrigues, O. (1840). Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace: et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire. *Journal Math. Pures. et App. de Liouville*, 5:380–440.

[Rosen, 1930] Rosen, N. (1930). Note on the general Lorentz transformation. *Journal of Mathematics and Physics*, 9(1-4):181–187.

[Segre, 1892] Segre, C. (1892). Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. *Math. Ann*, 40:413–467.

[van Elfrinkhof, 1897] van Elfrinkhof, L. (1897). Eene eigenschap van de orthogonale substitutie van de vierde orde. In *Handelingen van het zesde Nederlandsch Natuuren Geneeskundig Congres*, pages 237–240.

[Weiner and Wilkens, 2005] Weiner, J. L. and Wilkens, G. R. (2005). Quaternions and rotations in $\mathbb{E}^4$. *The American Mathematical Monthly*, 112(1):69–76.

Deuxième partie

Nouveaux résultats et applications

READING KEYS FOR THE SECOND PART

We present in this part a novel vision and new results on already known fields of application. More than the technical aspects, what really matters in this part is relating disconnected theoretical and applicative fields with common mathematical principles.

"C'est même des hypothèses simples dont il faut le plus se défier, parce que ce sont celles qui ont le plus de chances de passer inaperçues."

Henri Poincaré (1892)

"It is the simple hypotheses of which one must be much wary, because these are the ones that have the most chances of passing unnoticed".

Chapter 5

Hypercomplex roots of real polynomials

In this chapter, we will focus on the alternative solutions of polynomials with **real coefficients**. In particular, we will question the implicit assumption that solutions should be limited to real or complex numbers for practical computations. The result for complex numbers in the Fundamental Theorem of Algebra (FTA) is reconsidered when we relax some properties of classical multiplication, like the nonexistence of zerodivisors. We will mainly focus on the exsitence of roots in the three algebras of the plane, $\mathbb{C}$, ${}^2\mathbb{R}$ and $\mathbb{D}$, but this alternative view for roots of a polynomial with real coefficients can be extended to other hypercomplex and Dickson algebras.

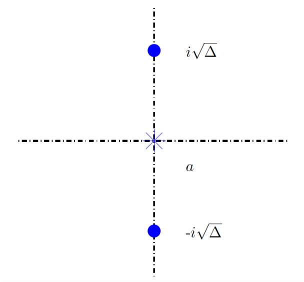
5.1 Quadratic polynomials and algebras in the plane $\mathbb{R}^2$

Let us consider the following quadratic equation

$$X^2 - 2aX + b = 0,\ a, b \in \mathbb{R}.$$

The classic resolution introduces the discriminant and the above equation can be rewritten as $(X - a)^2 = a^2 - b = \Delta \in \mathbb{R}$. If $\Delta < 0$, there are two complex conjugate solutions $\xi = a + i\sqrt{-\Delta}$ and $\overline{\xi} = a - i\sqrt{-\Delta}$. If $\Delta > 0$, the two real solutions $r_1 = a + \sqrt{\Delta}$ and $r_2 = a - \sqrt{\Delta}$ are known since Sumerian times, four millennia ago. The introduction of the bireal unit u naturally adds two bireal conjugate solutions $a \pm u\sqrt{\Delta}$ which satisfy $((a \pm u\sqrt{\Delta}) - a)^2 = u^2\Delta = \Delta$. If $\Delta = 0$, the double real solution $r = a$ is complemented by an infinity of dual roots to yield $a + \mathbb{R}n$ such that $\forall \lambda \in \mathbb{R}$, $((a + \lambda n) - a)^2 = \lambda^2 n^2 = 0$. We summarise in parallel these three cases and represent for each case the 2D-roots in the respective algebras $\mathbb{C}$, $^2\mathbb{R}$ and $\mathbb{D}$ as presented in [Chatelin et al., 2014, Latre et al., 2015].

Basis $(1, i)$

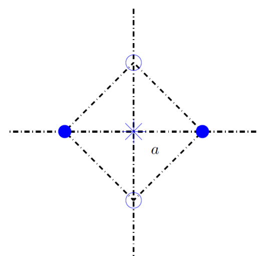


$\Delta < 0$

2 complex nonreal roots

$a \pm i\sqrt{|\Delta|}$

Basis $(1, u)$



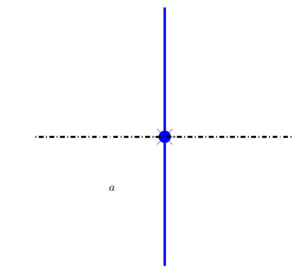
$\Delta > 0$

2 real roots $r_1 \neq r_2$,

+2 bireal nonreal roots

$\frac{r_1 + r_2}{2} \pm \frac{r_1 - r_2}{2}u$

Basis $(1, n)$



$\Delta = 0$

1 double real root $r = a$,

+ infinity of dual nonreal

roots $r + \mathbb{R}n$

The fact that an equation of degree 2 may have four bireal solutions (2 real and 2 nonreal) or even an infinity of dual roots goes beyond what is true (2 roots only) under the *implicit* assumption that limits the set of possible solutions to be $\mathbb{R}$ or $\mathbb{C}$. As bireal roots (resp. dual roots) lie in the ring $^2\mathbb{R}$ (resp. $\mathbb{D}$) and not in the field $\mathbb{C}$, such a limitation is not to be considered. When this is generalised to a polynomial of degree n with real coefficients, there can exist up to n^2 bireal roots and an infinity of dual roots, depending on the multiplicity of the real solutions [Latre, 2013]. In the next section, we will focus on the specific properties of bireal roots.

5.2 Bireal roots of polynomials with real coefficients

The results concerning the existence of bireal roots is a particular case of the theory in [Segre, 1892] for bicomplex numbers (see Multicomplex numbers in Section 1.2 of Chapter 3). Even if it is written in the outdated vocabulary of its time, the paper resonates with modern algebra. It presents the ring structure, the existence of the two ideals of zerodivisors (called "nullifici della prima e seconda schiera", §29 p. 458) and the complete theorem for bicomplex roots of polynomials with bicomplex coefficients (§31, p.462-463). This seminal article remains today untranslated in English. As a result, the solution for bicomplex polynomials has been periodically rediscovered.

The development concerning the structure of roots (Figures 5.1 and 5.2) and multiple factorisations of a polynomial with its bireal roots is original. The more general case concerning facorisation of polynomials with bicomplex coefficients is also presented in [Latre, 2013] but the bicomplex case will not be used for practical purposes in this thesis.

5.2.1 Roots description

We consider the polynomial $P = \sum_{i=0}^{n} a_i X^i$ of degree $n > 2$ with real coefficients a_i, $0 \leq i \leq n$ and we consider the zero set of P as the set of bireal roots of P.

As we will only use the result concerning bireal numbers, we will assume that all of the n roots, counted wiht multiplicity, are real (see [Latre, 2013] for the general case with complex roots and bicomplex polynomials). It is useful to keep in mind that nonreal bireal roots exist iff at least 2 *distinct real* roots exist to start with. In what follows, we assume that there are at least two distinct real roots and we denote by d the number of distinct real roots $r_1, .., r_d$ of P, $2 \leq d \leq n$.

Proposition 5.2.1. *There are d^2 roots q_{jk} of P in ${}^2\mathbb{R}$ given by the coupling of real roots r_j and r_k*

such that

$$q_{jk} = \frac{r_j + r_k}{2} + \frac{r_j - r_k}{2}u = r_j\frac{1+u}{2} + r_k\frac{1-u}{2} = r_j e_+ + r_k e_-, 1 \leq j, k \leq n.$$

When $j = k$, we simply get back the d real roots so that there are $d^2 - d = d(d-1)$ *distinct* nonreal bireal roots. If all the roots are simple so that $d = n$, P has n^2 distinct bireal roots, with $2 \leq n \leq n^2$, where n is the number given by the classic FTA. We can represent the bireal roots of P with the table in Figure 5.1 as a root coupling.

$P \backslash P$	r_1	..	r_k	..	r_d
r_1					
$\vdots$			$\vdots$		
r_j		..	q_{jk}	..	
$\vdots$			$\vdots$		
r_d					

Figure 5.1 – $q_{jk} = r_j e_+ + r_k e_- = \frac{r_j+r_k}{2} + \frac{r_j-r_k}{2}u$, d^2 roots of P

If we introduce for the distinct real roots r_i the multiplicities m_i, $1 \leq i \leq d$, the bireal root q_{jk} resulting from the coupling of r_j and r_k is repeated with the multiplicity $m_j m_k$ in the zero set of P (Figure 5.2).

In Chapter 2 we have introduced the conjugation of bireal numbers: for $z = x + uy$, $z^* = x - yu$. Notice that for each bireal root q_{jk}, the bireal conjugate q_{jk}^* is also a root of P and we have $q_{jk}^* = q_{kj}$.

Let us take a simple example with $P = (X-1)(X-2)(X-5)$. Figure 5.3 gives the 9 bireal (3 real and 6 nonreal) solutions of P in ${}^2\mathbb{R}$. Figure 5.4 displays the network of the 6 additionnal roots in the bireal plane $\frac{1}{2}(3 \pm u)$, $\frac{1}{2}(7 \pm 3u)$, $3 \pm 2u$ arising from the 3 real roots of $(X-1)(X-2)(X-5) = 0$.

				m_k		m_d
	$P \backslash P$	r_1	..	$r_k \quad \cdots \quad r_k$	..	$r_d \quad \cdots \quad r_d$
	r_1	q_{11}				$q_{1d} \quad \cdots \quad q_{1d}$
	$\vdots$					
m_j	r_j $\vdots$ r_j			$q_{jk} \quad .. \quad q_{jk}$ $\vdots \quad \ddots \quad \vdots$ $q_{jk} \quad .. \quad q_{jk}$		
	$\vdots$					
m_d	r_d $\vdots$ r_d	q_{d1} $\vdots$ q_{d1}				$q_{dd} \quad \cdots \quad q_{dd}$ $\vdots \quad \ddots \quad \vdots$ $q_{dd} \quad \cdots \quad q_{dd}$

Figure 5.2 – The n^2 bireal roots counted with their multiplicity in the zero set of P

$P \backslash P$	1	2	5
1	1	$\frac{1}{2}(3-u)$	$3-2u$
2	$\frac{1}{2}(3+u)$	2	$\frac{1}{2}(7-3u)$
5	$3+2u$	$\frac{1}{2}(7+3u)$	5

Figure 5.3 – The nine roots of $P = (X-1)(X-2)(X-5)$ in ${}^2\mathbb{R}$

As $(3+2u)(3-2u) = 9-4u^2 = 5$, we remark by direct calculation that $(X-(3+2u))(X-(3-2u)) = X^2-6X+5 = (X-1)(X-5)$, which means that $P = (X-1)(X-2)(X-5) = (X-2)(X-(3+2u))(X-(3-2u))$ so that factorisation of P over ${}^2\mathbb{R}$ is *not unique*. We recall that, because $\mathbb{C}$ is an algebraically closed field, any complex polynomial enjoys a *unique* factorisation over $\mathbb{C}$ (up to order) by means of n factors of degree 1, distinct or not. The most compact representation uses d distinct roots $\rho \in \mathbb{C}$ with multiplicity m, and d factors of the form $(z-\rho)^m$.

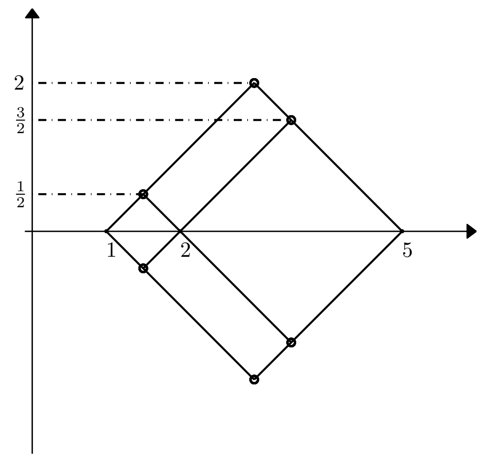


Figure 5.4 – 3 real roots and 6 nonreal roots (circles) in the bireal plane

5.2.2 Multiple distinct factorisations with bireal roots

We define a *system of roots* for the polynomial P to be a set of n roots q_{jk} chosen in the zeroset for P such that

$$P = a_n \prod_{i=1}^{n} (X - q_{jk}). \tag{5.1}$$

If we denote $\rho_1, .., \rho_n$, the n real roots of P without distinction of multiplicity ($\forall s, 1 \leq s \leq n, \exists s', 1 \leq s' \leq d$, such that $\rho_s = r_{s'}$), we have the new result

Proposition 5.2.2. *A system of roots for P is obtained by choosing r_s and r_t so that $\forall s, t 1 \leq s, t \leq n$, each r_s is associated with one and only one r_t to produce q_{st}.*

This corresponds to a set of n bireal roots with exactly one root choosen in each line and column in Figure 5.2. As stated in Section 5.2.1, this result is proven in a more general case in [Latre, 2013].

We denote by F the number of *distinct* factorisations for P (up to order). We have a second new result that shows a key difference between $\mathbb{C}$ and ${}^2\mathbb{R}$:

Proposition 5.2.3. *When the roots of P are simple there are $F = n!$ systems of roots which are all distinct. The presence of multiple roots allows for the existence of identical systems and in that case $F < n!$.*

Going back to the above example, the polynomial $(X-1)(X-2)(X-5)$ admits 3 simple real roots thus $n^2 = 9$ bireal roots which are *simple*. Therefore P admits $3! = 6$ distinct factorisations over ${}^2\mathbb{R}$. In addition to the two already given $(X-1)(X-2)(X-5)$ and $(X-2)(X-(3+2u))(X-(3-2u))$ that correspond respectively to the system of roots $\{1,\ 2,\ 5\}$ and $\{2,\ 3-2u, 3+2u\}$, there are 4 others defined by the 4 systems of roots:

$\{1,\ \frac{1}{2}(7+3u),\ \frac{1}{2}(7-3u)\}, \{\frac{1}{2}(3+u),\ \frac{1}{2}(3-u), 5\}, \{3-2u,\ \frac{1}{2}(3+u),\ \frac{1}{2}(7+3u)\}, \{3+2u,\ \frac{1}{2}(3-u),\ \frac{1}{2}(7-3u)\}$.

One is led to distinguish between two types of "multiplicity" for each distinct root q_{jk} of P. First, the multiplicity of repetition $\mu_{jk} = m_j m_k \geq 1$, $1 \leq j, k \leq d$ such that q_{jk} occurs μ_{jk} times in the zeroset for P. Second, the *algebraic multiplicities* ν_{jk} *relative* to a given factorisation in which q_{jk} can appear. The nonuniqueness of factorisations of polynomials over ${}^2\mathbb{R}$ entails that a bireal root has an algebraic multiplicity **relative to** the factorisation at hand equal to $\nu_{jk} \leq \min(m_j, m_k)$. The multiple root q_{jk} for P fills a block of size $m_j \times m_k$ in Figure 5.2. Multiplicity ν_{jk} in a factorisation for P cannot exceed $\min(m_j, m_k)$, but depends on the chosen factorisation. In particular, it can be 0. We sum up this new result with:

Lemma 5.2.1. $0 \leq \nu_{jk} \leq \min(m_j, m_k)$.

For example with the polynomial $(X-1)(X-2)(X-5)$, all the roots (real and bireal) are distinct and have the muliplicity of repetition 1, but the algebraic multiplicity relative to a factorisation can be 0 or 1 depending on which factorisation is considered: for the bireal root $q_{51} = 3 + 2u$, $\mu_{51} = 1$ but $\nu_{51} = 0$ in the system of roots $\{3 - 2u, \frac{1}{2}(3 + u), \frac{1}{2}(7 + 3u)\}$ and $\nu_{51} = 1$ in the factorisation $(X-2)(X-(3+2u))(X-(3-2u))$.

The general description of distinct factorisations over bicomplex numbers involves combinatorics on multisets which is out of the scope of this thesis. A more specific development about this question and the potential applications will be investigated in [Latre, 2019].

5.3 Summary

We have presented in this chapter some elements concerning roots in the algebras of the plane, but this question is of importance for other hypercomplex algebras. The older example of quaternions $\mathbb{H}$ [Hamilton, 1844] already questions FTA. Every pure unit quaternion $q = xi + yj + zk$, $x^2+y^2+z^2 = 1 = -q^2$, is a solution of $X^2+1 = 0$. The whole imaginary unit sphere is solution of this equation. FTA is **relative** to the fields $\mathbb{R}$ and $\mathbb{C}$. Computation in any other algebraic context exposes the ambiguity of the denomination FTA. We recall that quaternions are a field, thus the relative nature of FTA is not necessarily linked to the existence of zerodivisors. We can put in evidence the existence of other roots of real polynomials in hypercomplex algebras. In this thesis we will only use the results concerning algebras in the plane, but this chapter is an introductive invitation to consider other roots than the results coming from FTA or field extensions. For more about polynomial roots of hypercomplex numbers see [Chatelin, 2018].

The consequences of the existence of alternative roots in hypercomplex algebras do run beyond mere semantics. For example, a natural application of the above study concerns the characteristic polynomial of symmetric matrices (Chapter 6). The classical approach for spectral theory and quadratic forms can thus be expanded from the real line $\mathbb{R}$ to the bireal plane ${}^2\mathbb{R}$. We will see in Chapter 7 and Chapter 8 that these alternative polynomial roots have physical interpretation.

Keep in Mind for Chapter 5

The existence of bireal roots was already known since [Segre, 1892]. New results of Chapter 5 concern root description (tabular form) and multiplicities (Section 5.2.1). The characterisation of factorisations in Section 5.2.1 is fully original. More general results for bicomplex polynomials with bicomplex coefficients are presented in [Latre, 2013] and will be extended in [Latre, 2019].

Bibliography

[Chatelin, 2018] Chatelin, F. (2018). *Numbers in Mind: the transformative ways of Multiplication.* Book in preparation to be published by World Scientific.

[Chatelin et al., 2014] Chatelin, F., Latre, J.-B., Rincon-Camacho, M., and Ricoux, P. (2014). Beyond and behind linear algebra. Technical report, Tech. rep., CERFACS TR/PA/14/80.

[Hamilton, 1844] Hamilton, W. R. (1844). On quaternions; or on a new system of imaginaries in algebra. *Philosophical Magazine Series 3*, 25(163):10–13.

[Latre, 2013] Latre, J.-B. (2013). Sur quelques structures algébriques utiles au traitement de l'information en robotique, informatique et physique. Technical report, Rapport de Master, 25/03/13-06/09/13, CERFACS WN-PA-13-105.

[Latre, 2019] Latre, J.-B. (2019). Multiple factorisations of bicomplex polynomials and their consequences (in preparation). Technical report, CERFACS.

[Latre et al., 2015] Latre, J.-B., Chatelin, F., Rincon-Camacho, M., and Ricoux, P. (2015). Beyond and behind linear algebra. TOTAL Conference Mathias, Paris 28-30 Oct, 2015.

[Segre, 1892] Segre, C. (1892). Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. *Math. Ann*, 40:413–467.

Chapter 6

Spectral coupling for symmetric matrices in the light of $^2\mathbb{R}$

This chapter is a brief introduction to *spectral coupling*, a theory intended to model the notion of coupling in Physics and Life Sciences, intimately connected with *nonlinearity*. The process of pairing two distinct real roots of polynomials in the hyperbolic number plane provides a natural extension of classical spectral theory. One considers vectors which describe an invariant *plane*, rather than being invariant in *direction* (i.e. eigenvectors) under the action of a real symmetric matrix. This chapter recalls some aspects of the interplay between geometry, analysis and algebra taking place in the theory of spectral coupling. The maximal coupling owes much to the more elementary one taking place in the numerical plane $\mathbb{R}^2$ and the algebra of bireal numbers $^2\mathbb{R}$ (Chapter 5).

We mention that spectral coupling in the framework of positive definite hermitian matrices is presented in [Chatelin, 2012]. At the origin of this theory lies the pioneering work of Gustafson started in 1968 [Gustafson, 2012] for operator trigonometry and perturbations for contraction semigroup theory (with application to real positive definite matrices). The theory of spectral coupling is an original extension developed in the Qualitative Computing Group research program which includes this thesis. The chapter focuses on the aspects of spectral theory which betray an implicit connection with $^2\mathbb{R}$. They are our personal contribution. The first original results that relate geometry and analysis of spectral coupling with bireal numbers can be found in [Latre, 2013]. They are incorporated in [Chatelin et al., 2014]. The more recent reference [Chatelin and Rincon-Camacho, 2017] treats in full generality the case of indefinite hermititian matrices.

6.1 About spectral coupling

6.1.1 Pairing distinct real eigenvalues

We consider the coupling of distinct real eigenvalues of a real symmetric matrix, a transformation called *spectral coupling*. Let A be a symmetric matrix of order $n \geq 2$ and $\{\lambda_i\}_{1\leq i\leq n}$ be the n eigenvalues of A which form the spectrum $Sp(A)$. We suppose that there exist at least two distinct eigenvalues in $Sp(A)$, $\lambda, \lambda' \in Sp(A)$, $\lambda < \lambda'$ that are the *real* roots of the quadratic polynomial

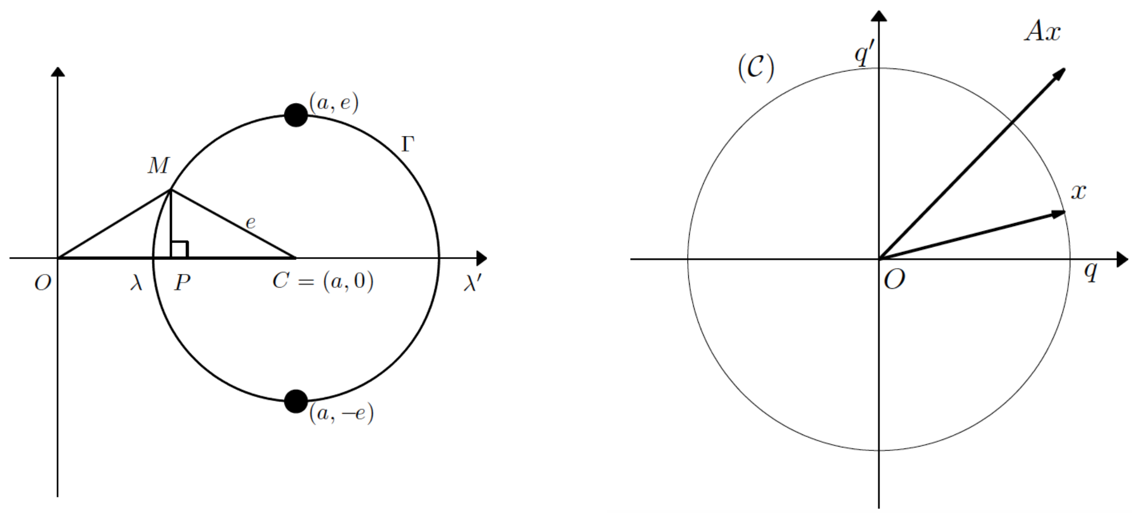
$$(X-\lambda)(X-\lambda') = X^2 - 2aX + b = (X-a)^2 - e^2 = 0, \Delta = a^2 - b > 0 \tag{6.1}$$

where the real numbers a, e and b are defined by $a = \frac{\lambda+\lambda'}{2}$, $e = \sqrt{\Delta} = \frac{\lambda'-\lambda}{2} > 0$ and $b = \lambda\lambda' = a^2 - e^2$. The role of the *bireal* roots of real polynomials, introduced in Chapter 5, is illustrated in this chapter by means of the characteristic polynomial and the eigenvalues of A. We will develop the connection between the bireal roots $a \pm ue$ of Equation 6.1 ($u^2 = 1, u \neq \pm 1$) and the theory of spectral coupling. Observe that $b = \lambda\lambda' = \mu(a \pm eu) \in [-e^2, a^2[$.

Let q and $q' \in \mathbb{R}^n$ be orthonormal eigenvectors associated respectively with λ and λ'. Any $x =$

$x_1q + x_2q', (x_1, x_2) \in \mathbb{R}^2$ in the subspace $\mathbf{M}$ spanned by q and q' is such that $Ax = \lambda x_1 q + \lambda' x_2 q' \in \mathbf{M}$. The subspace $\mathbf{M}$ remains invariant under A, thus it is called *invariant plane*. It has 2 real dimensions and the orthonormal projection of A onto $\mathbf{M}$ defines a 2×2 symmetric matrix $A_{\mathbf{M}}$ whose eigenvalues λ and λ' lie on $\mathbb{R}$ called the *spectral line*. Because $\{\lambda, \lambda'\} = \{a \pm e\}$, $A_{\mathbf{M}}$ is similar to the symmetric matrix $\begin{pmatrix} a & e \\ e & a \end{pmatrix}$. The algebra of bireal numbers appears to be a natural algebraic framework for spectral coupling. Equation 6.1 says that the problem is quadratic in essence. The matrix $A_{\mathbf{M}}$, similar to $\begin{pmatrix} a & e \\ e & a \end{pmatrix}$, is the isomorphic formulation of the bireal number $a + ue$ (denoted M_{a+ue} in Chapter 4) and we will see in Section 6.2 how bireal numbers describe specific configurations in spectral coupling.

Let us draw in $\mathbb{R}^2$ the circle Γ centered at $C = (a, 0)$ with radius $e > 0$: it passes through the real eigenvalues $(\lambda, 0)$ and $(\lambda', 0)$; it also passes through the two bireal ones $a \pm eu$. Hence it realises a geometric link between the two interfering eigenvalues λ and λ' and between the bireal $a \pm eu$, a link drawn in the plane $\mathbb{R}^2$ called the *spectral plane*, which is isomorphic to ${}^2\mathbb{R}$. Assuming that $ae \neq 0$ ($\lambda' \neq \pm\lambda$), we consider M a point lying on Γ and the corresponding triangle OMC. Two of the side lengths are fixed: $OC = |a|$ and $MC = e$, while the third length OM varies with M (Figure 6.1(a)). We denote the three angles of OMC as follows: $\alpha = \angle(OC, OM)$, $\beta = \angle(MC, MO)$ and $\gamma = \angle(CO, CM)$ (Figure 6.2(a)). For future considerations, we also introduce for $0 \leq \gamma \leq \pi$, the half angle $\delta = \frac{\gamma}{2} = \angle(\Lambda'\Lambda, \Lambda' M)$, $0 \leq \delta \leq \frac{\pi}{2}$ ($\Lambda = (\lambda, 0)$, $\Lambda' = (\lambda', 0)$). Unless otherwise stated, we shall assume that $a \neq 0$ ($C \neq O$), so that the triangle OMC is not degenerate into OM. Notice that $\lambda\lambda' = \mu(a \pm eu)$, the magnitude of the bireal eigenvalues $a \pm eu$, is the power of the point O with respect to the circle Γ.



(a) Spectral plane

(b) Invariant plane, $||x|| = 1$

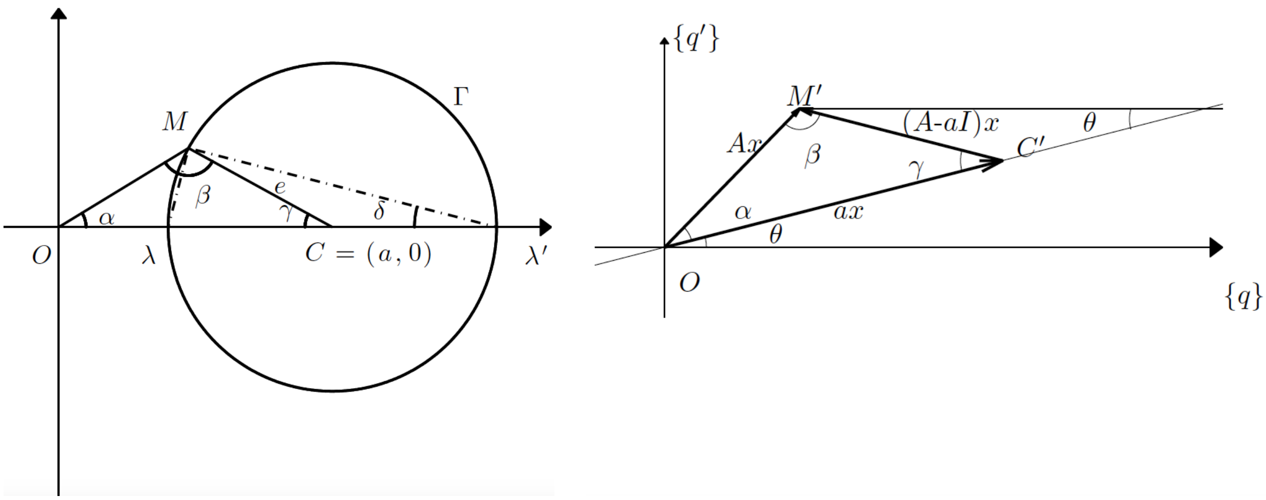
Figure 6.1 – Spectral coupling

The circle Γ is well-known in Continuum Mechanics as Mohr's circle. This circle has been proposed by C.O. Mohr in 1882 as a graphical tool to analyse the dynamics of the Cauchy stress tensor in 2 and 3D. We will see in Chapter 7 how bireal numbers and spectral coupling can offer a novel insight into Continuum Mechanics.

In this chapter we only consider a symmetric real matrix but the use of Γ that is proposed in the general case of spectral coupling is valid whether A is real symmetric or complex hermitian, definite or not. The theory is fully original when A is hermitian and x is a *complex* vector in $\mathbb{C}^n$ (complex angles and toric structure see [Chatelin and Rincon-Camacho, 2017]. Notice also that in the general symmetric case, the position of the circle with respect to the origin depends on the signs of λ and λ' on the real line. In this chapter, the elements of theory concerning spectral coupling in Section 6.2 do not depend on the sign of eigenvalues, which is not true for the complete theory.

6.1.2 Pairing the spectral and invariant planes

We consider $x = (\cos\theta)q + (\sin\theta)q' \in \mathbf{M}$, with unit norm $\|x\| = 1$, $\theta \in [0, 2\pi[$. When θ varies in $[0, 2\pi[$, the vector x describes the unit circle $(\mathcal{C}) \in \mathbf{M}$ centered at O and passing through the eigenvectors $\pm q$, $\pm q'$, see Figures 6.1(b) and 6.2(b). We also consider the action of the matrix A on the vector x, $Ax = (\cos\theta)\lambda q + (\sin\theta)\lambda' q'$.



(a) Spectral plane (b) Invariant plane

Figure 6.2 – $\lambda'\lambda > 0$: triangles are congruent $\simeq \gamma = 2\theta$

When x is not an eigenvector ($\theta \notin \{0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi\}$) and $a \neq 0$, the three vectors ax, Ax and $Ax - ax$ are linearly independent and they form a non degenerate triangle $OM'C'$, see Figure 6.2 (b). If $a = 0$, the triangle $OM'C'$ is degenerate ($C' = O$). We are interested in the relations between the triangle OMC in the spectral plane and the triangle $OM'C'$ in the invariant plane. We introduce therefore a result proved in [Chatelin and Rincon-Camacho, 2017, lemma 2.1]:

Lemma 6.1.1. $\forall x \in (\mathcal{C})$ $C'M' = \|Ax - ax\| = e$.

Proof: $C'M' = ||\cos\theta(\lambda - a)q + \sin\theta(\lambda' - a)q'|| = e|| - q\cos\theta + q'\sin\theta|| = e = CM$.

If we compare the triangle OMC parameterised by δ and the triangle $OM'C'$ parameterised by θ we obtain the following result [Chatelin and Rincon-Camacho, 2017, corollary 2.3]:

Lemma 6.1.2. *The equality $\delta = \theta$ in $]0, \frac{\pi}{2}[$ yields the congruence $OMC = OM'C'$.*

Proof: The triangles have two fixed side lengths $OC = OC' = |a|$ and $CM = C'M' = e$. Each pair of sides envelops the same angle γ if $\delta = \theta$ in $]0, \frac{\pi}{2}[$: $\gamma = 2\theta = 2\delta \in]0, \pi[$.

Therefore we denote the angles of the triangle $OM'C'$ by $\alpha = \angle(x, Ax)$, $\beta = \angle(Ax, Ax - ax)$, $\gamma = \angle(-ax, Ax - ax)$ (Figure 6.2(b)).

When the triangle OMC in the spectral plane and the triangle $OM'C'$ in the invariant plane are *congruent*, a remarkable property is that the spectral information processing in the spectral plane is mirrored in the invariant plane. This connection between geometric information in planes of distinct nature (spectral vs. invariant) is a cornerstone of spectral coupling valid under the assumption $\gamma = 2\theta$.

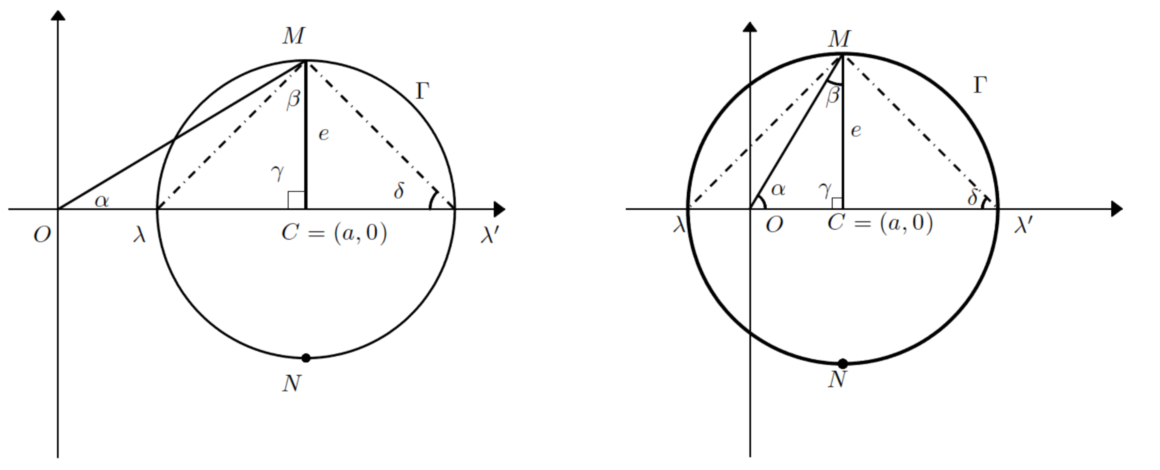
The theory of spectral coupling investigates the key property for symmetric matrices of congruence between the triangle OMC in the spectral plane and the triangle $O'M'C'$ in the invariant plane. The dynamics of eigenvector coupling is reflected in the spectral plane without any loss of information.

In [Chatelin and Rincon-Camacho, 2017], the theory of spectral coupling describes the maximisation of the angles α (when $\lambda\lambda' > 0$) and β (when $\lambda\lambda' < 0$) associated with the above triangles and the related set of vectors x associated to these optimal cases. Remarkably, $\alpha_{max} < \frac{\pi}{2} = \beta$ and $\beta_{max} < \frac{\pi}{2} = \alpha$ respectively: the optimal triangle OMC is right-angled at the vertex M when $\lambda\lambda' > 0$ or at O when $\lambda\lambda' < 0$. The results concerning α and β are linked with variational principles coming from Euler's equations. As shown in this paper as well as in [Gustafson, 2012, Chatelin and Rincon-Camacho, 2015], spectral coupling has fruitful consequences in Physics, Numerical Analysis, Statistics and Econometrics.

6.2 Maximal spectral coupling

6.2.1 Triangle surface optimality and bireal eigenvalues

In this section, we focus on another aspect related to the optimality of the surface of the triangle OMC in the spectral plane and its relation to the algebra of bireal numbers.



(a) $\lambda\lambda' > 0$ (b) $\lambda\lambda' < 0$

Figure 6.3 – Maximum surface of the triangle OMC in the spectral plane

Lemma 6.2.1. *When M describes Γ and $a \neq 0$, the surface of the triangle OMC is maximum and equal to $\Sigma = \frac{1}{2}|a|e$ iff $\gamma = \frac{\pi}{2}$.*

Proof: The surface of OMC is $\Sigma(\gamma) = \frac{1}{2}|a|e \sin\gamma \leq \Sigma = \frac{1}{2}|a|e = \frac{1}{8}|\lambda'^2 - \lambda^2|$.

The maximum Σ is achieved for M at (a, e) so that $OM = \sqrt{a^2 + e^2}$, see Figure 6.3 (a) and (b). Moreover, the real (resp. unreal) part of $(a \pm eu)^2$ is equal to $OM^2 = a^2 + e^2$ (resp. 4Σ). The surface

$\Sigma(\gamma)$ of the triangle is a measure of what we call the *intensity of coupling* when $a \neq 0$. It varies between 0 (eigenvalues) and Σ. The symmetric of M with respect to the horizontal axis $N(a, -e)$ is the second possible case to achieve Σ. If $a = 0$, OMC is degenerate thus $\Sigma(\gamma) = 0$ for all M on Γ: this is not an appropriate measure of coupling when $a = 0$ $\lambda' = -\lambda = e$ It can be replaced by the altitude $PM = e|\sin\gamma| \leq e$. Notice that Lemma 6.2.1 found in [Chatelin and Rincon-Camacho, 2017, lemma 3.5] is valid for $\lambda\lambda' \in \mathbb{R}$. This is not the case in general for the properties of spectral coupling as we stated briefly in Section 6.1.2. More precisely, with the angular (rather than surface) point of view, the optimal triangles are such that $OM^2 = a^2 - e^2 = \lambda\lambda' > 0$ (α_{max}) or $OM^2 = e^2 - a^2 = -\lambda\lambda' > 0$ (β_{max}). One recognizes the hyperbolic magnitudes of $a \pm eu$. They replace the euclidean one $||OM||^2 = a^2 + e^2$ present in the surface point of view given above. This reveals an implicit role of ${}^2\mathbb{R}$ in the angular viewpoint. It appears that the geometric analysis of spectral coupling betrays an interplay between euclidean and hyperbolic geometries. This resonates with Poincaré's opinion: "Une géométrie ne peut être plus vraie qu'une autre; elle peut seulement être plus commode" in [Poincaré, 1902, p.50]. Moreover, if we release the assumption $\lambda < \lambda'$ to consider $e = 0 \simeq \lambda = \lambda' = a$, the third algebra $\mathbb{D}$ enters the game, whenever multiple real eigenvalues exist in $Sp(A)$. In case of the self-interference of λ with itself, the circular link Γ is replaced by the opened straight line $\lambda + \mathbb{R}n$. Going back to the surface point of view, we relate the lemma 6.2.1 with bireal numbers to produce the original result in [Latre, 2013]:

Proposition 6.2.1. *In the spectral plane, equipped with a bireal structure, the maximal surface of the triangle OMC is reached for the points M and N, that are the geometric representation of the bireal conjugate roots $a + eu$ and $a - eu$ of Equation 6.1. These roots are the two nonreal bireal eigenvalues of A.*

It is quite remarkable that the geometric locus of optimality in terms of the surface of OMC in the spectral plane exactly matches the position of bireal roots in the bireal plane (Chapter 5). This configuration represents the *maximal* coupling in terms of surface of the triangles. Under spectral coupling, the spectral plane is thus naturally equipped with an implicit 2D bireal structure which complements the 1D real structure associated with classical spectral analysis.

As $|a|e = \frac{1}{4}(\lambda'^2 - \lambda^2) = \frac{1}{4}(\lambda' + \lambda)(\lambda' - \lambda)$, it is also quite natural to relate the magnitude of the bireal numbers $\lambda + \lambda' u$ and $\lambda' + \lambda u$ in ${}^2\mathbb{R}$ with the measure of the maximal surface Σ.

Corollary 6.2.1. $4|a|e = -\mu(\lambda + \lambda' u) = \mu(\lambda' + \lambda u)$ *and thus,* $8\Sigma = -\mu(\lambda + \lambda' u) = \mu(\lambda' + \lambda u)$.

It is noticeable that the magnitudes $\mu(\lambda + \lambda' u) = -\mu(\lambda' + \lambda u) = -4|a|e = -8\Sigma$ may receive the geometric interpretation in terms of the extremal area $\Sigma = \frac{1}{2}|a|e \in \mathbb{R}$ of the triangle OMC as M describes Γ.

6.2.2 Structural optimality in the invariant plane: midvectors

We define the set of four *midvectors*

$$V = \left\{ v = \frac{1}{\sqrt{2}}(\varepsilon q + \varepsilon' q'),\ \varepsilon = \pm 1,\ \varepsilon' = \pm 1 \right\},$$

The optimal configuration in the spectral plane is associated to $\gamma = 2\delta = \frac{\pi}{2}$. Due to the connection between the triangles OMC and $OM'C'$ with the angles δ and θ, in the case of the maximal surface we have $\delta = \theta = \frac{\pi}{4}$. As a consequence of lemma 3.6 and theorem 3.7 in [Chatelin and Rincon-Camacho, 2017], when $a \neq 0$, the 4 triangles $OM'C'$, $v = OC' \in V$, have the maximal surface $\frac{1}{2}|a|e$.

Taking $P = [v_+, v_-] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with $v_\pm = \frac{1}{\sqrt{2}}(q \pm q')$ (see Figure 6.4),

we have $M_{a+ue} = \begin{pmatrix} a & e \\ e & a \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \lambda' + \lambda & \lambda' - \lambda \\ \lambda' - \lambda & \lambda' + \lambda \end{pmatrix} = P^T \begin{pmatrix} \lambda' & 0 \\ 0 & \lambda \end{pmatrix} P$, with $P^T P = I_2$. The orthonormal basis (v_+, v_-) of midvectors relates the matrix A to the matrix M_{a+ue} associated to the bireal algebra.

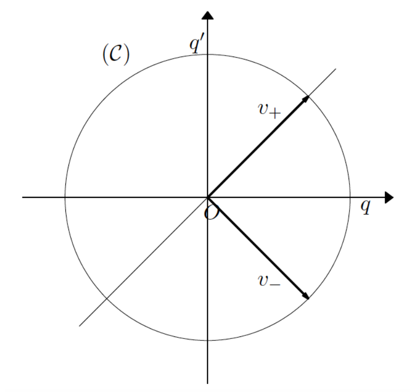


Figure 6.4 – Midvectors v_+ and v_- in $\mathbf{M}$

The vectors associated to the optimal configuration for the triangle surface are the midvectors. The midvectors are the bissectors of the original eigenvectors q and q' and they do not depend of the values of λ and λ' and of the parameters a and e. Midvectors are common to all matrices with a fixed eigenbasis and a varying spectrum. To sum up, in $\mathbf{M}$ the coupling between λ and λ' is *maximum* and *structural* when $A_{\mathbf{M}}$ is represented by $\frac{1}{2} \begin{pmatrix} \lambda' + \lambda & \lambda' - \lambda \\ \lambda' - \lambda & \lambda' + \lambda \end{pmatrix}$ which displays the linear combinations $\lambda' + \lambda$ and $\lambda' - \lambda$. And the eigenbasis is rotated by the angle $\frac{\pi}{4}$ independently of the spectrum.

Remark 6.2.1. The generic concept of a midvector is absent from Gustafson's theory which focuses on α when $\lambda\lambda' > 0$. The notion only appears in a statistical setting under the guise of an "inefficient" vector, see [Chatelin and Rincon-Camacho, 2017, Section 5] and [Gustafson, 2012, p. 190].

6.2.3 Spectral coupling and vector product, $n = 3$ and $n = 7$

We can make a connection between the theory of spectral coupling and vector product that we introduced in Chapter 3. We recall that the vector product in dimension 3 and 7 is related respectively to the imaginary part of the quaternions and the octonions. In dimension 3 and 7, the vector product maintains at the same time the geometric and computational aspects discussed in Chapter 3, Section 5.

Let us consider in $\mathbb{R}^3$ the vector product of $x = (\cos\theta, \sin\theta, 0)^T$ and $\tilde{x} = (-\cos\theta, \sin\theta, 0)^T$: $x \wedge \tilde{x} = (0, 0, 2\sin\theta\cos\theta = \sin 2\theta)^T$. The surface of the triangle OMC is $\Sigma(x) = \frac{1}{2}|a|e\sin\gamma$. The quantity $2\Sigma(x)$ measures the vector product $\mathfrak{K} = ax \wedge (Ax - ax) = ax \wedge Ax = Ax \wedge (Ax - ax) = aex \wedge \tilde{x}$ which lives in $\mathbf{M}^\perp$. The vector $\mathfrak{K}$ is the vector product of any two adjacent sides in $OM'C'$; it represents the action of the coupling *outside* the invariant plane. Its direction is fixed in $\mathbf{M}^\perp$; if $n = 3$, it is but the third eigendirection. This notion is also valid in dimension $n = 7$, but the direction in $\mathbf{M}^\perp$ is in a subspace of dimension $7 - 2 = 5$ instead of $3 - 2 = 1$ in the 3D-case.

The norm $\|\mathfrak{K}\| = |a|e\sin\gamma = 2\Sigma(x)$ is twice the intensity of coupling. It represents the *influence* of x outside its plane of evolution in $\mathbb{R}^3$ or $\mathbb{R}^7$. In other words, the vertices of $OM'C'$ are submitted to an equal torque as the triangle rotates about O. The torque is nonzero when $OM'C'$ is non degenerate ($x \notin \{\pm q, \pm q'\}$ and $ae \neq 0$).

The midvectors have the largest influence $|a|e$ and bireal eigenvalues associated with the maximal surface represent the maximum possible coupling between the real eigenvalues of A.

6.3 Conclusion

The spectral theory and the optimality of quadratic forms are most often treated only in real or complex analysis. The above presentation shows the contribution of a bireal analysis. This should be considered as an invitation to enlarge the algebraic possibilities offered by Computation. Real numbers and complex numbers have become such a natural and implicit choice of frameworks that they may overshadow the potential of other algebras that could be equally useful in Scientific Computing. The case of polynomials with real coefficients is a good example which demonstrates the power of algebraic diversity. It is often forgotten that the context of interpretation is *relative* to the chosen space of solutions (complex numbers, quaternions, bireal numbers among many others..). We will present in the next chapter, a fully original application of this work. It concerns Continuum mechanics and relates Γ to Mohr's circle. It also illustrates the geometric aspect of spectral coupling.

Keep in Mind for Chapter 6

The theory of spectral coupling has been developed in the Qualitative Computing research Group. The results concerning geometric optimality in spectral coupling are detailed in [Chatelin and Rincon-Camacho, 2017]. The original result of Chapter 6 (Prop. 6.2.1) concerns the relation between spectral coupling and bireal eigenvalues (introduced in Chapter 5). The consequences for Science and Engineering will be developed in the next chapter.

Bibliography

[Chatelin, 2012] Chatelin, F. (2012). *Eigenvalues of Matrices: Revised Edition.* Classics in Applied Mathematics, volume 71. SIAM.

[Chatelin et al., 2014] Chatelin, F., Latre, J.-B., Rincon-Camacho, M., and Ricoux, P. (2014). Beyond and behind linear algebra. Technical report, Tech. rep., CERFACS TR/PA/14/80.

[Chatelin and Rincon-Camacho, 2015] Chatelin, F. and Rincon-Camacho, M. M. (2015). Symmetric and hermitian matrices: a geometric perspective on spectral coupling. Technical report, CERFACS TR/PA/15/56.

[Chatelin and Rincon-Camacho, 2017] Chatelin, F. and Rincon-Camacho, M. M. (2017). Hermitian matrices: Spectral coupling, plane geometry/trigonometry and optimisation. *Linear Algebra and its Applications*, 533(Supplement C):282 – 310.

[Gustafson, 2012] Gustafson, K. (2012). *Antieigenvalue Analysis: With Applications to Numerical Analysis, Wavelets, Statistics, Quantum Mechanics, Finance and Optimization*. World Scientific.

[Latre, 2013] Latre, J.-B. (2013). Sur quelques structures algébriques utiles au traitement de l'information en robotique, informatique et physique. Technical report, Rapport de Master, 25/03/13-06/09/13, CERFACS WN-PA-13-105.

[Poincaré, 1902] Poincaré, H. (1902). *La Science et l'Hypothèse*. Flammarion.

Chapter 7

Applications of bireal numbers and spectral coupling

In this chapter we connect through *multiplication* theories spread along centuries starting from antiquity to present. The results presented in this chapter are fully original and have been presented in major international academic [Latre et al., 2017, Latre, 2017b] and industrial [Latre et al., 2016, Latre, 2017a] conferences. The novel applications of bireal numbers and spectral coupling that we develop in what follows concern structural mechanics, chemistry and in a more general scope, the physical models associated to the harmonic oscillator and a quadratic potential energy.

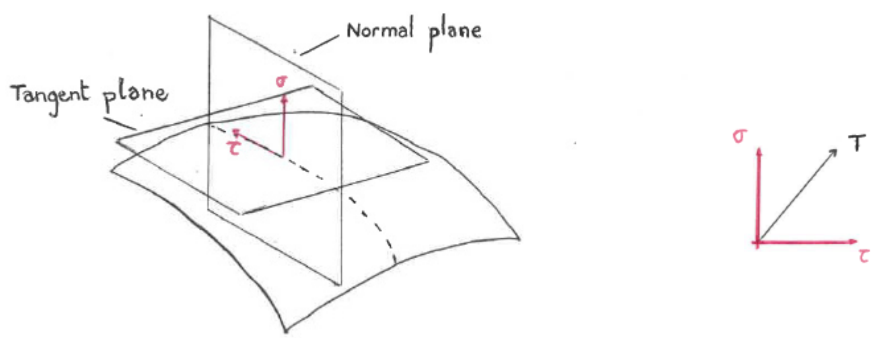
7.1 Application to Continuum Mechanics

7.1.1 Cauchy stress tensor

In order to assess the stress of a mechanical element from the perspective of linear elasticity, a common tool used by engineers in continuum mechanics is the so-called Cauchy stress tensor [Truesdell, 1966], for 3D and 2D stresses. We present the 3D-Cauchy stress tensor in mechanicists parlance. The 3D-Cauchy tensor is a symmetric tensor of order 2 often denoted $\underset{\approx}{\sigma}$ which therefore is isomorphic, in a given orthonormal basis of $\mathbb{R}^3$, to the 3×3 real symmetric matrix $F = \left[\begin{array}{cc|c} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \hline \tau_{xz} & \tau_{yz} & \sigma_z \end{array}\right]$, which we call the *stress matrix*.

A quadratic form Q is associated to the Cauchy stress tensor, that is given the real symmetric matrix F and for a vector $X = (x, y, z)$, we have $Q(X) = X^T F X = \langle X, FX \rangle$ which is an invariant scalar for the tensor [Pernès, 2003, p.64]. For the following, we suppose that F has 3 distinct real eigenvalues. For future reference to the 2D-case, we set $F_1 = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$.

The tensor $\underset{\approx}{\sigma}$ relates a normal unit 3D-vector $\mathbf{n}$ to a 2D-*stress vector* $T = \begin{pmatrix} \sigma \\ \tau \end{pmatrix}$ in the *cutting plane* associated to $\mathbf{n}$ (Figure 7.1 (a)).



(a) (b)

Figure 7.1 – Cutting plane relative to $\mathbf{n}$

The two real components σ and τ of the stress vector T are respectively called *normal* stress and *shear* stress. As shown in Figure 7.1 (b), we have $||T||^2 = ||Fn||^2 = \sigma^2 + \tau^2$. Given F and $\mathbf{n}$, the normal and shear components can be computed thanks to the following formulas:

$$\sigma = \mathbf{n}^T F \mathbf{n} = Q(n) \in \mathbb{R},$$

$$\tau = \sqrt{\mathbf{n}^T F^T F \mathbf{n} - \sigma^2} \in \mathbb{R}.$$

These formulas are expressed in mechanics with $T = \mathbf{n} \odot \underset{\approx}{\sigma}$, where $\odot$ denotes the tensor contraction, such that

$$\sigma = \mathbf{n} \odot \underset{\approx}{\sigma} \odot \mathbf{n} \text{ and } \tau = \sqrt{||\underset{\approx}{\sigma} \odot \mathbf{n}||^2 - \sigma^2}.$$

Notice that the shear stress τ is derived as a consequence of the normal stress σ with pythagorean computation. We give both expressions coming either from tensors or from matrices. Some confusion may arise in the literature when authors do not make the distinction between the tensor $\underset{\approx}{\sigma}$ and its matrix formulation F, the stress matrix that is used in the sequel.

For the 2D-case, the action of the tensor in the $x - y$ plane is represented thanks to its matrix formulation by the 2×2 matrix F_1. The following section will present the graphical technique due to Mohr that is used in continuum mechanics to determine the normal stress σ and the shear stress τ.

7.1.2 Mohr's circle (2D) and the tricircle (3D)

C.O. Mohr proposed in 1882 a geometric technique as a graphical tool to analyse the stress extrema [Timoshenko, 1983]. The so-called *Mohr's circle* limits in the plane of constraints σ (horizontal axis) and τ (vertical axis) the set of admissible stress states induced by the 2D-Cauchy stress tensor [Brannon, 2003]. A derivation from the equilibrium of forces gives the parametrical equations that correspond for a 2D-Cauchy stress to the Mohr's circle (see Figure 7.2). It intersects the σ-axis at the two eigenvalues $\lambda < \lambda'$ of F_1. Thus it plays the role of Γ in Chapter 6 for spectral coupling. Moreover the associated eigenvectors and values for F_1 are the principal axes and stresses of the tensor.

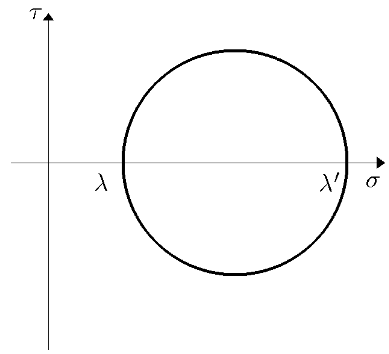
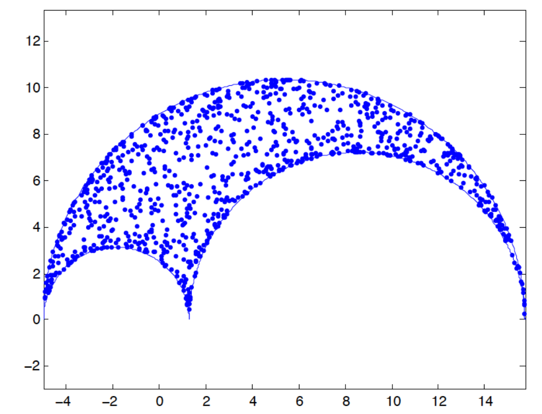


Figure 7.2 – Mohr's circle in the plane $\sigma - \tau$ of constraints

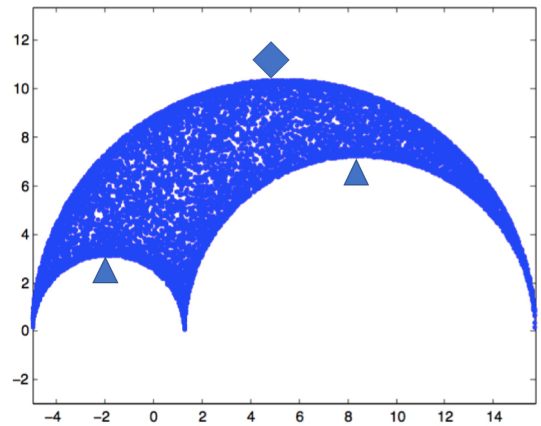
In 3D, F enjoys 3 eigenvalues (supposed to be distinct). Their pairings create 3 circles, the form which involves the corresponding 3 circles is known in mechanics as a *tricircle*. It is based on the determination of the principal axes and the principal stresses of the tensor, which are in matrix language the eigenvectors and eigenvalues of the related stress matrix F. The notion of eigenvector and eigenvalue of a tensor, which is a multilinear operator, has no cristal clear meaning. As mentionned in Chapter 4 about tensor formulations, the critical information of continuum mechanics that we use is contained in spectral form into its matrix formulation F and is not related to any specific property of the tensor $\underset{\approx}{\sigma}$. A numerical example of the Mohr's tricircle for the matrix $F = \begin{bmatrix} 7 & 10 & 0 \\ 10 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ is given in Figure 7.3. The eigenvalues of F are $\Lambda = \begin{pmatrix} 15.7284 \\ 1.2716 \\ -5.0000 \end{pmatrix}$, those of F_1 are $\begin{pmatrix} 15.6119 \\ -4.6119 \end{pmatrix}$.

Notice that the eigenvalues of the stress matrices need not be positive. Negative eigenvalues are interpreted by mechanicists as compression movement.

The numerical simulation for the 3D-stress F represents, for a set of randomly generated normal vectors n_i, the associated normal and shear stresses relative to the corresponding cutting plane. Depending on the number of normal vectors, the whole theoretical area defined by Mohr's tricircle can be filled by the simulation (see Figure 7.3).



(a) 1000 normal vectors



(b) 10000 normal vectors

Figure 7.3 – Numerical example of Mohr's tricircle for F

Figure 7.3 confirms the remarkable and well-known property of the Mohr's circle in 2D and 3D that the intersections of the circles with the horizontal axis are the real eigenvalues of the stress matrices F_1 (2D) and F (3D). The extrema of the normal stress σ are given by the spectral information of the matrix. The extrema for the shear stress τ remain classically *uninterpreted algebraically*. The results in Chapter 5 and 6 suggest that they are the bireal eigenvalues of F_1 and F. They are represented for F in Figure 7.3(b) with triangle tops for local extrema and the lower point of the square for global maximum (apex of the largest circle which connects the extreme eigenvalues).

7.1.3 A global bireal spectral interpretation of the stress extrema σ and τ

We have observed that the position of the shear stress extrema (Figure 7.3) in the plane of constraints is the same as that of the bireal eigenvalues of F in the bireal plane (Chapter 6). In other words, we have the new result

Proposition 7.1.1. *Maximal shear stresses are obtained from the coupling of extremal normal stresses, that are the two extreme eigenvalues λ_{min} and λ_{max} of the 3×3 stress matrix F.*

Eigenvectors for real eigenvalues correspond to the directions for extremal normal stress. Using the correspondence presented in Chapter 6 between the spectral plane (the bireal eigenvalues) and the invariant plane (associated eigenvectors), we have:

Proposition 7.1.2. *The midvectors are the directions of shear stress extrema.*

It is quite remarkable that bireal eigenvalues and their associated eigenvectors and midvectors present a physical meaning in continuum mechanics. We can summarise the aspect of spectral coupling in continuum mechanics:

Theorem 7.1.1. *The extrema of the stress matrix F $(\sigma_{min}, \sigma_{max}, \tau_{min}, \tau_{max})$ are the 4 bireal roots of $(X - \lambda_{min})(X - \lambda_{max})$ that correspond to the spectral coupling of the minimum and maximum real roots of the characteristic polymomial of F. Eigenvectors associated to these bireal eigvenvalues and midvectors correspond respectively to the direction of extremal normal and shear stresses.*

The added-value consists in expressing in a unified way, in the bireal plane, the stress extrema (both normal and shear stresses). They are the solutions (real and bireal) of the characteristic polynomial for the symmetric stress matrix coming from structural mechanics. The geometry of bireal roots (Figure

7.4) is that associated with the extreme spectral coupling (equal to zero for real roots or maximal for nonreal ones), that was treated in Chapter 6, the bireal roots of the real quadratic form associated to the symmetric matrix F. Quadratic forms as well as spectral theory have thus equipped the constraints plane $(\sigma - \tau)$ with the bireal algebraic structure.

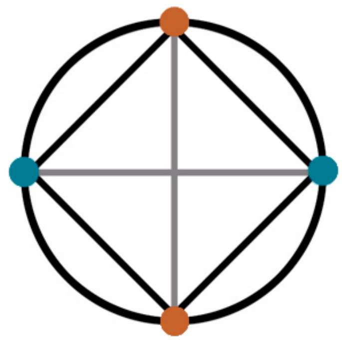


Figure 7.4 – Characteristic geometric configuration for bireal roots and spectral coupling

7.1.4 Mohr's tricircle and Archimedes' arbelos

Figure 7.5 – Arbelos (picture by Thomas Schoch)

The geometric object described by Mohr's tricircle has been known in mathematics since the time of Archimedes (ca. 287- 212 BC) as an *arbelos* [Boas, 2006, Section 7]. The name is that of the ancient tool used by cobblers (leather-cutters) since Antiquity (see Figure 7.5). A detailed study of the arbelos can be found in [Gorin, 2006] and in [Boas, 2006] with 44 historical references.

From 3 seemingly independent topics, bireal numbers in algebra, spectral coupling in geometry and analysis and stress extrema in continuum mechanics, we have shown in this section a tight relation between them in a synthetic way.

We will see in the next section that the role of bireal numbers in spectral coupling is not limited to application in continuum mechanics. We will take a specific example on molecular simulation that can be extended to a large class of problems depending on the harmonic oscillator.

Keep in Mind for Chapter 7

In Continuum Mechanics, it is well known that normal extrema are real eigenvalues and associated eigenvectors are the directions of the normal stress extrema. Shear extrema are obtained with Mohr's circle without algebraic interpretation.

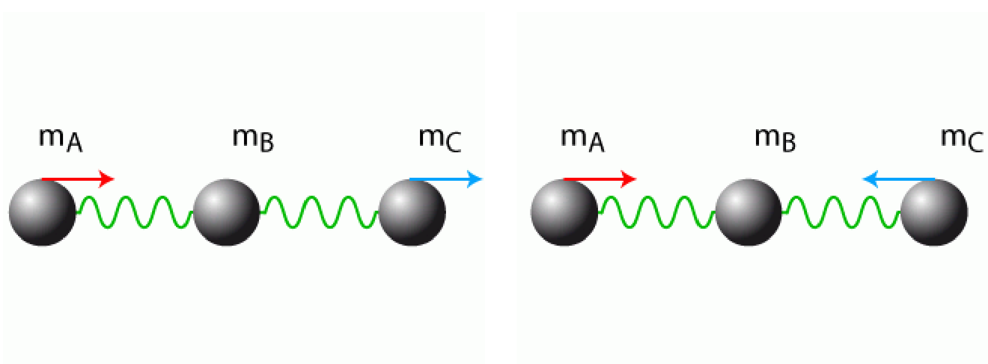
New results of this chapter are obtained thanks to the theory of spectral coupling. The shear extrema are obtained as the coupling of the stress extrema (Prop. 7.1.1) and the corresponding directions are the midvectors (Prop. 7.1.2). Spectral coupling and bireal roots allow to describe in a unified way the different extrema as real and bireal roots of the characteristic polynomial associated to the symmetric stress matrix (Theorem 7.1.1).

7.2 Molecular simulation and harmonic oscillator model

7.2.1 Harmonic oscillator model

A harmonic oscillator is a system, when displaced from its equilibrium position, experiences a restoring force proportional to the displacement (Hooke's law). In the case where there is no damping, a harmonic oscillator has a simple harmonic motion. Simple harmonic motion is typified by the motion of a mass on a spring when it is subject to the linear elastic restoring force given by Hooke's law. One of the key features of the harmonic oscillator is that the potential energy is a **quadratic form**.

The harmonic oscillator model is very important in Physics, because any mass subjected to a force in stable equilibrium acts as a harmonic oscillator for small vibrations. Harmonic oscillators occur widely in Nature and are exploited in many applications such as masses connected to springs (see Figure 7.6) and acoustical systems. In particular, we point out the fields of modal and seismic analysis and the case of molecular vibration presented in Section 7.2.2.



(a) Symmetric motion (b) Asymmetric motion

Figure 7.6 – Example of motions of a harmonic oscillator with 3 masses and 2 springs

When a system is composed of n elements, the equation of motion that concerns our following examples is of the form $\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = AX$, where $X = (x_i)$, $1 \leq i \leq n$ is the real vector of displacement variables and A is a real **symmetric** matrix of size n. The components of A are real constants that depend on the parameters of the system (mass, stiffness of springs).

A *normal mode* of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation. The motion described by the normal modes takes place at the fixed frequencies. These fixed frequencies of the normal modes of a system are known as its *resonant frequencies*. A physical object, such as a building, bridge, or molecule, has a set of normal modes and their resonant frequencies that depend on its structure, materials and boundary conditions.

In mathematical terms, normal modes are obtained from the spectral analysis of the matrix A, with the different eigenvalues relative to resonant frequencies and associated eigenvectors that correspond to normal modes (= eigenmodes). The general solution of the system is a superposition (= linear combination) of its normal modes (eigenvectors). In the general case that we will not present here, the solution is related to the generalised eigenvalue problem [Chatelin, 2012].

7.2.2 Molecular vibration: coupled oscillations in the plane

The point of this thesis is not to focus on Chemistry but rather on the mathematical aspects that undelie the behaviour of atoms and molecules. The brief elements of chemical theory presented in this section and in Section 7.2.3 are discussed in [Wilson et al., 1980, Cotton, 2003] which are the main references in this topic.

A molecular vibration occurs when atoms in a molecule are in periodic motion while the molecule as a whole has constant translational and rotational motion. The frequency of the periodic motion is known as a vibration frequency. As a simple but widely used approximation, the motion in a normal vibration can be described as a harmonic oscillator. In general, a molecule with n atoms has $3n-6$ normal modes of vibration.

Internal coordinates (related to what is called in Chemistry the Eckart conditions) describe the different movements of the molecule (changes in position, angle and length of bonds) [Wilson et al., 1980]. In order to illustrate the link between molecular vibration and spectral coupling, we will take the example of carbon dioxide CO_2, with two double covalent bonds between the atoms of carbon and oxygen $O = C = O$. We are interested in the *stretching* that is, in molecular vibration, a change in the length of a bond of the molecule.

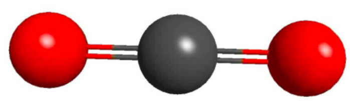


Figure 7.7 – Carbon dioxide molecule with one carbon atom (black) and two oxygen atoms (red)

The carbon dioxide molecule (Figure 7.7) is represented by a system of coupled oscillators in the plane that correspond to the system in Figure 7.6 with m the mass of an oxygen atom and M the mass of the carbon atom. The model for the two C-O bonds is 2 identical springs whose stiffness constant k is set according to the chemical properties of the bond.

As there are two coupled oscillations, the displacement vector X has two components $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

This system of coupled oscillations corresponds to the motion $\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = AX$ where $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and a and b are real constants depending of the masses of the atoms and the stiffness k. Applying Newton's law on the axis of the molecule, one finds after computations that $a = -(1 + \frac{M}{m})\frac{k}{m}$ and $b = -\frac{k}{m}$.

The point is not to focus on the solution of this problem but on the structure of its solution. With the classic assumption that the solution X is of the form $X = X(t) = X_0 e^{i\omega t}$, the equation of motion is reduced to $(A + \omega^2 I)X = 0$ and we have to find the eigenvalues and eigenvectors of A.

The diagonalisation of A is obtained thanks to $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $H^2 = 2I$, $H^{-1} = \frac{1}{2}H$ such that $HAH^{-1} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$.

The two resulting normal modes are $q_1(t) = C_1 \cos(\omega_1 t + \varphi_1)$ $q_2(t) = C_2 \cos(\omega_2 t + \varphi_2)$ where $C_1, C_2, \varphi_1, \varphi_2$ are real constants. The general solution in the original basis is a superposition (=linear combination) of the 2 (eigen)modes $x_1(t) = q_1(t) + q_2(t)$ and $x_2(t) = q_1(t) - q_2(t)$ via H, $X = H \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_1 + q_2 \\ q_1 - q_2 \end{pmatrix}$.

The coupling of oscillators is represented by the characteristic form of the matrix H (Hadamard matrix, see Chapter 4) associated to the idempotent basis of bireal numbers $H = 2\,[e_+, e_-]$ that we used for spectral coupling in Chapter 6. We focus here on the mode coupling and we have the following result:

Proposition 7.2.1. *The general solution for coupled oscillations in the plane is given by the superposition of modes that correspond to midvectors of the symmetric matrix A associated to the system.*

For the CO_2 molecule, the $C = O$ stretches are not independent. There is an $O = C = O$ symmetric stretch that corresponds to x_1, the sum of the two carbon-oxygen stretching coordinates (the two carbon-oxygen bond lengths change by the same value and the carbon atom is stationary), and an $O = C = O$ asymmetric stretch that corresponds to x_2, the difference of the two carbon-oxygen stretching coordinates, one carbon-oxygen bond length increases while the other decreases.

It is important to notice that the resolution is **structural** and that the matrix H does not depend on the 2 parameters a, b of the system. The same eigenvectors are used for varying eigenvalues and the diagonalisation is obtained as as consequence of the real symmetric structure of A of order 2 (see Chapter 6, Section 6.2.2. This model is not limited to CO_2 but to all molecules represented by the harmonic oscillator. In case of more complex molecules, other displacements than streches have to be considered (for example spatial rotations with quaternions, see chapter 8) and this general analysis relies on molecular symmetry that we will briefly introduce in the next section.

The model of coupled harmonic oscillators can be extended to n coupled oscillators in a linear or closed chain. The matrix A, obtained through the equations of motion due to a quadratic potential energy, is still real symmetric. We will not insist on this generalisation and rather focus in the next section on molecular symmetry and quantum mechanics.

7.2.3 Molecular symmetry and quantum mechanics

Molecular symmetry is used to study the symmetry in molecules and the classification of molecules according to their characteristics. Molecular symmetry is a fundamental concept in Chemistry, because it can be used to predict or explain many chemical properties of a molecule (e.g. dipole moment, spectroscopic transitions). Depending on their symmetry, molecules will have different vibration states. The general theory of molecular symmetry requires a lot of group theory which can be found in [Cotton, 2003].

A molecular vibration is excited when the molecule absorbs a quantum of energy E, corresponding to the vibration's frequency ν, as $E = h\nu$ (h is Planck's constant). A fundamental vibration is excited when one such quantum of energy is absorbed by the molecule in its *ground state* (the less excited state). When two quanta are absorbed the first overtone is excited, and so on to higher overtones. The model of the harmonic oscillator is still valid and leads to the resolution of the eigenvalues of the Hamiltonian $\mathbf{H}$ in the stationnary (time-independent) Schrödinger equation

$$\mathbf{H}\Psi = E\Psi,$$

where the eigenvalues E are energy levels associated to eigenstates Ψ called *Molecular Orbitals* (MO). MO describe the wave-like behaviour of an electron in a molecule. A standard approximation is that MO are linear combinations of *Atomic Orbital (AO)*, the functions that describe the behaviour of an electron in the different atoms of the molecule. MO are a key model of wave-particule duality of a molecule.

One of the main objective in molecular symmetry is to determine the symmetry group of the molecule in order to reduce to factored form the characteristic polynomial (called secular equation in Chemistry) and be able to get the different energy levels of the molecule. The energy levels are represented in a 1D-diagram classifying the energy levels in increasing order (the real line of eigenvalues of the Hamiltonian). The most stable state of the molecule, the ground state, corresponds to the lowest energy level. The more excited states that correspond to higher energy levels are represented in the increasing order of the eigenvalues.

7.2.3.1 Natural coupling examples with hydrocarbons

Hydrocarbons are molecules only composed of hydrogen and carbon atoms that are fundamental in all the derivatives of obtained from pretroleum.

Ethylene , C_2H_4, is a such hydrocarbon which is widely used in chemical industry. For hydrocarbons, a standard model is the Hückel method which supposes that molecular orbital Ψ are obtained as linear combinations of the atomic orbitals Φ that constitute the molecule. This leads to a simplification of the Hamiltonian $\mathbf{H} = (\mathbf{H}_{ij})$ in the following way

$$\mathbf{H}_{ij} = \begin{cases} \alpha \text{ , if } i = j \\ \beta \text{ , if the atom } i \text{ is adjacent to } j \\ 0 \text{ else} \end{cases},$$

where α and β are constants and respectively correspond to the energy of an electron in a specific atomic orbital and to the interaction energy between two atomic orbitals (the specific AO is $2p$, see [Cotton, 2003]).

The Hamiltonian of the ethylene obtained from the Hückle method is $\mathbf{H} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, that we now use to relate to spectral coupling.

The two energy levels are $E = \alpha \pm \beta$ and the associated MO are $\Psi_{\pm} = \frac{1}{\sqrt{2}}(\Phi_1 \pm \Phi_2)$ where Φ_1 and Φ_2 are the AO of the ethylene. This corresponds to the spectral coupling of orbitals and we have

Proposition 7.2.2. *Under Hückel method, the energy levels of the ethylene correspond to the spectral coupling of atomic orbitals.*

As the Hamiltonian is symmetric under Hückel assumptions spectral coupling is not limited to ethylene, but can be applied to other hydrocarbons such as benzene that we describe in more details introducing an important phenomenon in Chemistry.

Benzene, C_6H_6, has a very specific structure composed of six carbon atoms joined in a ring with one hydrogen atom attached to each with alternating simple and double bounds between the carbon atoms. In the standard chemical representaion of hydrocarbons, hydrogen atoms are omitted and carbon atoms are the edges of segments that represent bonds (1 line for simple bonds and 2 for double bonds) see Figure 7.8.

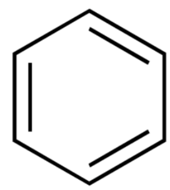


Figure 7.8 – Benzene molecule

The structure of benzene has been given by Kekulé in 1865 after lots of attempts by other chemists. However, the structure of benzene proposed by Kekulé still presented a problem: the observed energy of the benzene does not correspond to the description given by its structure. The phenomem of *resonance* or *delocalisation* has been discovered in 1899 by Thiele as an attempt to explain the stability of benzene. Resonance corresponds to the interaction between two energy states initially close that are split in two equivalent equilibrium states $\frac{1}{2}(\Psi_1 \pm \Psi_2)$. The symmetric combination $\frac{1}{2}(\Psi_1 + \Psi_2)$ gives the ground state while the antisymmetric combination $\frac{1}{2}(\Psi_1 - \Psi_2)$ gives the first excited state as shown in Figure 7.9. The exceptional stability of benzene in chemical reactions is not explained by its structure but by the phenomenon of resonance which is a coupling of equivalent equilibrium energy states. The mathematical formulation of resonance appears to behave according to the theory of spectral coupling. More work has to be done to confirm the link between spectral coupling and resonance.

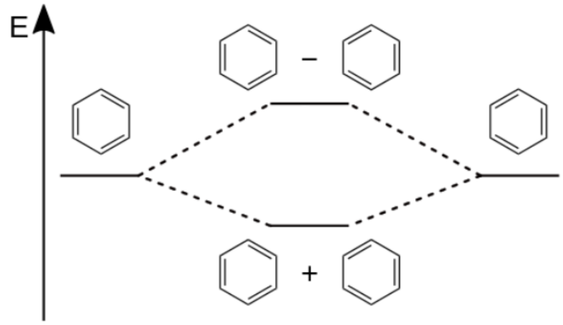


Figure 7.9 – Resonance in the energy levels of benzene

Keep in Mind for Chapter 7

In Chemistry, we use the classical theory of molecular vibration to add the novel interpretation of spectral coupling and bireal numbers in a general solution for coupled oscillations in the plane (Prop. 7.2.1). Bireal structure naturally appears for the given examples of energy levels of molecules as exemplified by ethylene (Prop. 7.2.2). The relation between energy levels of hydrocarbons and spectral coupling requires more investigations.

7.3 Conclusion and perspectives

We have seen in this chapter several novel applications of spectral coupling and potential future research directions. The extrema of the Cauchy stress tensor in continuum mechanics provide a physical meaning to bireal roots. The role of midvectors is also studied in the case of coupled harmonic oscillators that have a great variety of applications.

Molecular vibration is closely linked to quantum mechanics and is a synthesis of mechanical and energetic coupling. We will see in the next chapter that spectral coupling of energy levels may have connections to superposition in quantum mechanics, hence it may not be a surprise to find a link between bireal numbers and Quantum applications. Spectral coupling in the bireal plane is not a method associated to a specific field of application but seems to be a general computational mechanism and could implictly be present in several other domains of Science and Engineering.

7.4 Bibliographical notes

One more example of the difficulty of transdisciplinarity. The remarkable connection between Mohr's tricircle (19^{th} century) and the arbelos (3^{rd} century B.C.) is largely ignored in Continuous Mechanics despite [Bisegna and Podio-guidugli, 1995]. And it remains a piece of lore for the few mathematicians who know about its historical existence (through Apollonius of Perga, Pappus of Alexandria and Pythagorean triangles). H. Boas in 2006 seems to have been the first mathematician aware of the role played by the arbelos in Continuous Mechanics [Boas, 2006, Section 7, pp. 246-247].

Keep in Mind for Chapter 7

> As concluded in this chapter, we repeat and insist on the fact that spectral coupling in the bireal plane is not a method associated to a specific field of application but seems to be a general computational mechanism and could implictly be present in several other domains of Science and Engineering.

Bibliography

[Bisegna and Podio-guidugli, 1995] Bisegna, P. and Podio-guidugli, P. (1995). Mohr's arbelos. *Meccanica*, 30(4):417–424.

[Boas, 2006] Boas, H. P. (2006). Reflections on the Arbelos. *The American Mathematical Monthly*, 113(3):236–249.

[Brannon, 2003] Brannon, R. (2003). Mohr's circle and more circles, Tech. Report, Univ. of Utah.

[Chatelin, 2012] Chatelin, F. (2012). *Eigenvalues of Matrices: Revised Edition.* Classics in Applied Mathematics, volume 71. SIAM.

[Cotton, 2003] Cotton, F. A. (2003). *Chemical applications of group theory.* John Wiley & Sons.

[Gorin, 2006] Gorin, B. (2006). Une étude de l'arbelos, http://baptiste.gorin.pagesperso-orange.fr/Docs/arbelos.pdf.

[Latre, 2017a] Latre, J.-B. (2017a). A comparative study between algebraic structures for a goal-oriented efficiency in Scientific Computing . TOTAL Conference Mathias, Paris 25-27 Oct, 2017.

[Latre, 2017b] Latre, J.-B. (2017b). Spectral coupling: an algebraic perspective running from ancient greek geometry to modern continuum mechanics. 11th International Conference on Clifford Algebras and their applications in mathematical physics, Ghent, 07/08-11/08.

[Latre et al., 2017] Latre, J.-B., Chatelin, F., and Ricoux, P. (2017). Alteralgebras for modelling and computation. SIAM CSE 17, Atlanta (GA), 27/02-03/03.

[Latre et al., 2016] Latre, J.-B., Chatelin, F., Rincon-Camacho, M., and Ricoux, P. (2016). Alternative algebraic structures for Modelling and Computation. TOTAL Conference Mathias, Paris 26-28 Oct, 2016.

[Pernès, 2003] Pernès, P. (2003). *Éléments de calcul tensoriel: introduction à la mécanique des milieux déformables*. Editions Quae.

[Timoshenko, 1983] Timoshenko, S. P. (1983). *History of Strength of Materials*. Dover, New York.

[Truesdell, 1966] Truesdell, C. (1966). *The elements of Continuum Mechanics*. Springer-Verlag, Berlin Heidelberg New York.

[Wilson et al., 1980] Wilson, E. B., Decius, J. C., and Cross, P. C. (1980). *Molecular vibrations: the theory of infrared and Raman vibrational spectra*. Courier Corporation.

Chapter 8

On current and potential uses of hypercomplex numbers in Computational Science, Engineering, Physics and Chemistry

In this chapter, we present the main areas in Science and Engineering for which certain hypercomplex number structures are currently used or show strong promise. Among all the algebras that we defined in the first part of the thesis, the aim is to pinpoint which ones (beyond linear algebra over $\mathbb{R}$ or $\mathbb{C}$) are of interest to compute, for already existing or potential applications, be it from a practical or theoretical point of view.

8.1 Hypercomplex numbers and Numerical Analysis

In this section, we present some aspects of hypercomplex numbers related to numerical analysis and finite precision computations. Quaternions are a fundamental example and maybe the best known algebra among the possible algebraic structures beyond $\mathbb{R}$ or $\mathbb{C}$. The two other examples concerning dual numbers and bireal numbers are more unconventional.

8.1.1 Quaternions for 3D-rotations

Spatial rotations are a key notion for many mechanical systems. Spatial mechanics during the second half of the 20^{th} century put in evidence the advantages of quaternions compared to other representations. Quaternions are a solution to a historical issue related to rotations associated with Euler angles called gimbal lock, that corresponds to singular configurations and the loss of a degree of freedom in the system. Gimbal lock is a dangerous configuration for aircrafts (that occured for example during the Apollo 11 mission), or for robotic systems (called "wrist flip") where it leads to uncontrolled movements and velocities.

In Chapter 4, we have presented the theoretical equivalence between the isomorphic formulations of 3×3 unitary matrices and unit quaternions to represent rotations. However, this does not imply an equivalent numerical behaviour and computational cost.

1) First, only 4 parameters are necessary for quaternions instead of 9 for matrices which means that less memory cost for computations is required.

2) In the different cases of application, the point is not to perform a single rotation but a sequence of successive rotations that represent the evolution of spatial positions in a given time frame. Then, if we compare the number of operations for composing two successive rotations we observe that the product of two rotation matrices requires 27 multiplications and 18 additions whereas the product of two quaternions requires 16 multiplications and 12 additions, outperforming matrix multiplication [Howell and Lafon, 1975, Eberly, 2002].

3)The last point concerns numerical stability. In theory, a sequence of rotations should be a rotation due to the group structure but because of finite precision, we observe a loss of orthogonality for matrices and of the unit norm of quaternions. The renormalisation of a quaternion to adjust for floating point errors (vector norm) is cheaper than the renorthogonalisation of a rotation matrix. Quaternions have a better numerical stability than rotations matrices relative to successive multiplications.

Left dormant for almost a century after their invention by Hamilton because of the rise of vector calculus, quaternions reappeared as main tools to compute for engineers in spatial mechanics and they are now a cornerstone of numerical models for a wide range of applications in industrial and commercial softwares. Beyond their role in Physics, quaternions are widely used in applicative fields such as Chemistry (molecular simulation), robotics and computer graphics see Section 8.2. For a large number of references (more than 1000 !) concerning the role of quaternions in mathematical physics and applications , see [Gsponer and Hurni, 2005a, Gsponer and Hurni, 2005b].

8.1.2 Bireal numbers

8.1.2.1 Idempotent basis and numerical behaviour

Bireal numbers have several algorithmic and numerical attractive features that rely on the idempotent basis. We recall (see Chapters 2 and 4) that in ${}^2\mathbb{R}$, a number $z = x + uy$ has an idempotent representation

$$z = Xe_+ + Ye_-$$

where $X = x + y$, $Y = x - y$ and $\{e_+, e_-\}$ is the orthonormal idempotent basis given by

$$e_\pm = \frac{1}{2}(1 \pm u).$$

Thus, we have the following properties which are interesting for parallel computing. Let $z' = x' + y'u = X'e_+, Y'e_-)$, then

$$\begin{aligned} zz' &= (x + yu)(x' + y'u) \\ &= (Xe_+ + Ye_-)(X'e_+ + Y'e_-) \\ &= XX'e_+ + YY'e_- \end{aligned}$$

that is a natural parallelism. The fact that thanks to the idempotent basis, additions and multiplications are performed componentwise (see Chapter 4) is for example used in signal processing, see Section 8.2.2.

Moreover, there is another point that is not presented in the existing litterature of hypercomplex algebras to the best of our knowlegde. The idempotent representation is found to be helpful for avoiding numerical instability. As an example, we can compute the magnitude $\mu(z)$ previously defined in Chapter 2 as:

$$\mu(z) = x^2 - y^2 \in \mathbb{R}.$$

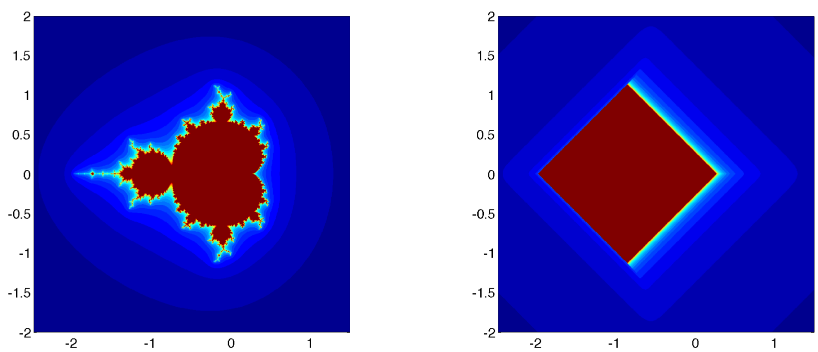
On the other hand, if we use the basis $\{e_+, e_-\}$, the magnitude is equal to

$$\mu(z) = XY.$$

Catastrophic cancellation occurs when operands are subject to rounding errors and the computation of $x^2 - y^2$ is an archetypal example of this phenomenon [Goldberg, 1991]. It is more accurate to evaluate it as $XY = (x+y)(x-y)$. This improved form still has a subtraction, but it is a more benign cancellation of quantities, less catastrophic. Notice that the use of the idempotent basis is not just a rewriting trick but a natural way to transform subtractive operations thanks to bireal multiplication, in order to reduce castrophic cancellation issues.

8.1.2.2 Quadratic iterations in the bireal plane

A compelling example about the differences between the behaviour of complex numbers and bireal numbers is the quadratic iteration $z \mapsto z^2 + c$ and its dynamics in the plane. When $z \in \mathbb{C}$: complex dynamics leads to the well-known Mandelbrot set, when $z \in^2 \mathbb{R}$: bireal dynamics leads to a square! This surprising result is proven in [Metzler, 1994]. The two sets are based on $[-2, \frac{1}{4}]$ horizontally, see Figure 8.1. The case of bireal numbers is presented in [Pavlov et al., 2009a] and the generalisation of quadratic iterations to the quadratic algebras of the plane has been independently done in [Chatelin et al., 2014].



(a) $c, z \in \mathbb{C}$ (b) $c, z \in {}^2\mathbb{R}$

Figure 8.1 – Quadratic iteration $z \mapsto z^2 + c$

8.1.2.3 Numerical computation of magnitudes for quadratic iteration

We can make a link between the computation of bireal magnitude and quadratic iterations in the plane. In [Pavlov et al., 2009b], the analytic solution for Julia set on bireal numbers is presented in the case of the quadratic map $z_{n+1} = z_n + c$, at $c = 0$. In the complex case, Julia set corresponds to points for which the sequence is bounded and in the case $c = 0$ is a circle. In the bireal plane, the analogous case corresponds to the region inside the hyperbolas as presented in Figure 8.2:

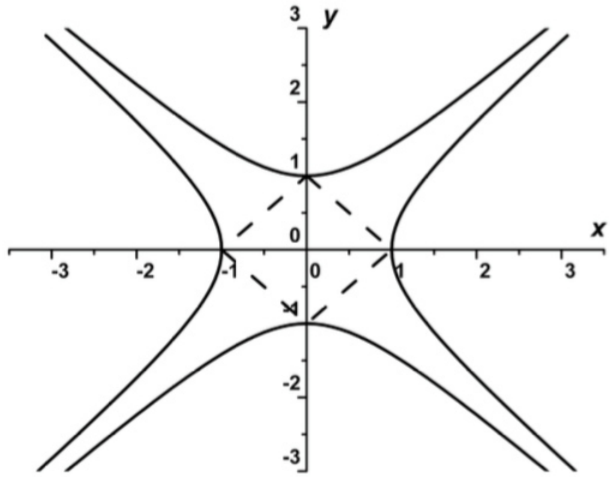


Figure 8.2 – Analytic solution for the bireal Julia set with $c = 0$ [Pavlov et al., 2009b]

In [Pavlov et al., 2009b], the authors point out the problem to compute the correct shape with algorithms. We represent below, for each point of the bireal plane in $[-2, 2]^2$, the computation of the bireal magnitude using $\mu(z) = x^2 - y^2$ in the left column in Figure 8.3(a),(c),(e) and $\mu(z) = XY$ in the right column of Figure 8.3(b),(d),(f). For a given maximum number of iterations ($n = 100$ for (a),(b), 300 for (c),(d) and 800 for (e) and (f)), the colormap at each line of Figure 8.3 represents the number of iterations necessary to reach an absolute value of the magnitude greater than the threshold. As the points oustside of the filled Julia set have unbounded iterations, we make the arbitrary choice of 500 for the threshold. Different values of the threshold do not change the shape of the numerical simulations, only the number of iterations required to get the same behaviour.

We observe a significant difference between the two computations of the bireal magnitude. The computed results in the left column that correspond to $\mu(z) = x^2 - y^2$ are already far from the theoretical solution (Figure 8.2). The correct shape of the filled Julia set is not clearly defined even for a small number of iterations (Figure 8.3(a)). This problem worsens with an increased numbers of iterations (Figure 8.3(c)) and in the last case (Figure 8.3(e)), the exact Julia set is almost indistinguishable. On the contrary, the results obtained using $\mu(z) = XY$ fit with the analytic solution for the different iteration numbers (Figure 8.3(b,d,f)). After 300 iterations, numerical perturbations start to appear in the region where the branches of hyperbolas are close to the asymptotes (Figure 8.3(d)). Unavoidable numerical effects due to finite precision can be seen on the last case (Figure 8.3(f)) but the Julia set is still easily identified.

As pointed out in Section 8.1.2.1, the results on the left column in Figure 8.3 illustrate the poor numerical stability and behaviour related to catastrophic cancellation. Even if it cannot totally avoid the intrinsic issues of computation in finite precision, the computation of the magnitude using the idempotent basis brings a clear improvement in the quality of the results.

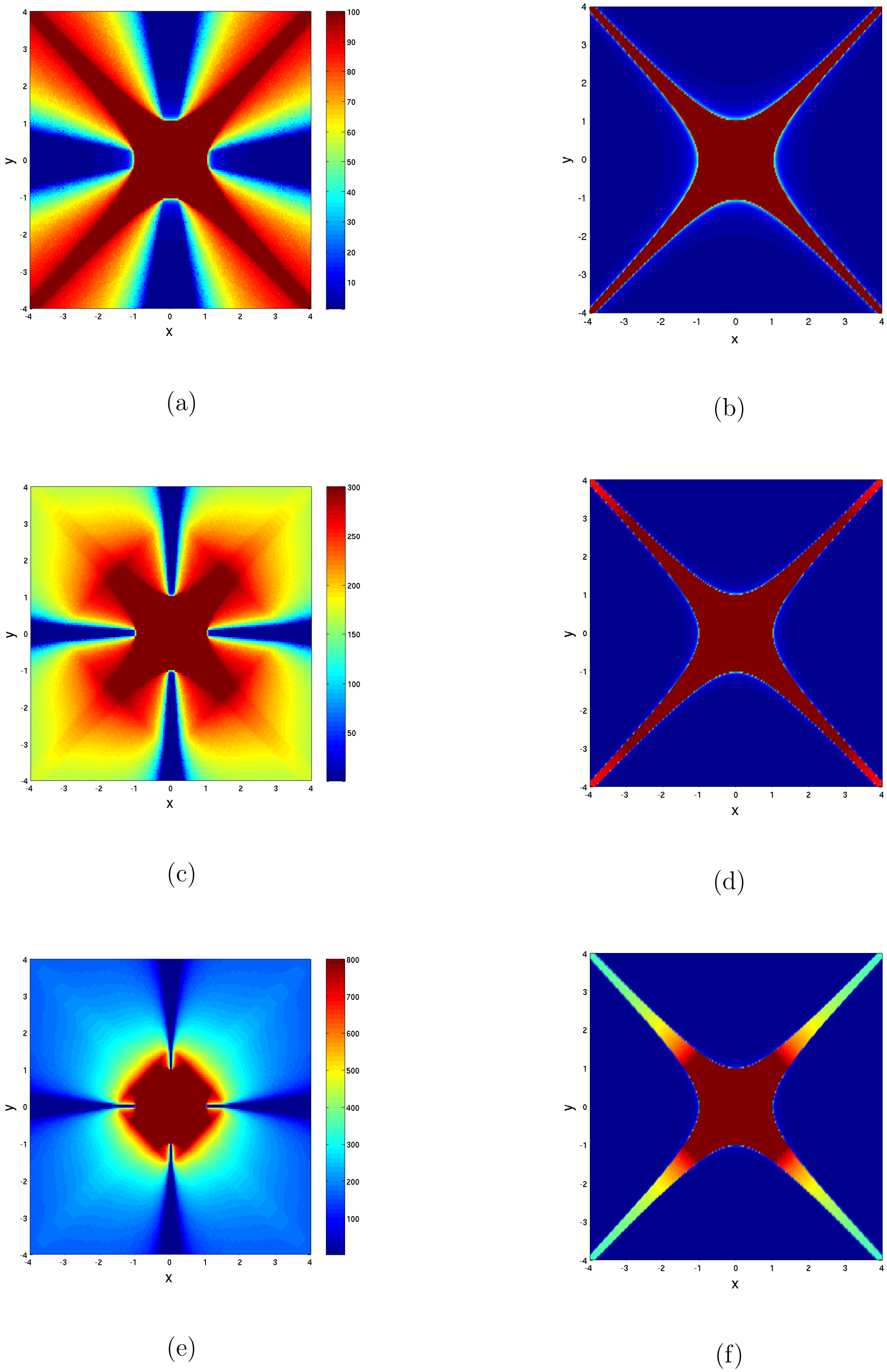


Figure 8.3 – *Numerical simulation of the filled Julia set in* $^2\mathbb{R}$ *for* $c = 0$ *after* 100 *iterations (a,b),* 300 *iterations (c,d),* 800 *iterations (e,f)*

8.1.3 Error free derivatives with dual numbers

We recall that the computation of derivatives is a major topic for Scientific Computing. Algorithmic resolution of PDE's in industry mainly relies on finite differences schemes although more efficient techniques are available. The use of complex numbers for the derivative computation has been already studied in [Martins et al., 2003, Lantoine et al., 2012]. In order to compute the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x, this function is extended over the complex numbers. Numerically this is done by overloading operators and functions in a given programming language. The Taylor expansion around x is as follows for $h \in \mathbb{R}$

$$f(x+hi) = f(x) + f'(x)hi - \tfrac{f''(x)}{2!}h^2 - \tfrac{f'''(x)}{3!}h^3 i + \ldots + \tfrac{f^{(k)}(x)}{k!}h^k i^k + \ldots + o(h^k)$$

with imaginary part $\text{Im}[f(x+hi)] = f'(x)h - \frac{f'''(x)}{3!}h^3 + \frac{f(5)(x)}{5!}h^5 + ... + o(h^k)$

Then, the derivative of f at x is:

$$f'(x) = \tfrac{\text{Im}[f(x+hi)]}{h} + \mathcal{O}(h^2), \tag{8.1}$$

This process can be generalised for higher order derivatives by means of Multicomplex numbers (Chapter 4), see [Lantoine et al., 2012]. Complex steps is a solution for subtractive cancellation error but the truncation error relative to the Taylor expansion remains.

On the contrary, if the dual numbers (Chapter 2) are used as proposed in [Dimentberg, 1965, Dimentberg, 1968], then because $n^2 = 0$, the Taylor expansion around x is simply $f(x + hn) = f(x) + f'(x)hn$, from where we get the nonreal part $\text{Un}[f(x + hn)] = f'(x)h$ and thus

$$f'(x) = \frac{\text{Un}[f(x+hn)]}{h}. \tag{8.2}$$

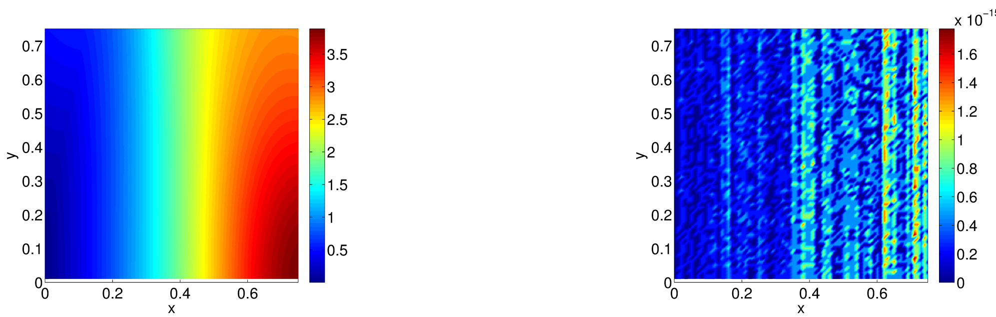
Therefore, there are **no truncation error** and **no subtractive cancellation error**.

In order to test these results, we consider the function

$$f(x) = \frac{e^x}{\sqrt{\sin^3 x + \cos^3 x}}, \quad f'(x) = \frac{e^x(3\cos x + 5\cos 3x + 9\sin x + \sin 3x)}{8(\sin^3 x + \cos^3 x)^{\frac{3}{2}}}$$

which is extended to $\mathbb{C}$ and $\mathbb{D}$ as $f(z) = v(x,y) + w(x,y)g, \;\; g \in \{i, n\}$.

Notice that this could also be done for bireal numbers but we did not find practical applications concerning derivatives. According to (8.1) and (8.2), the approximative derivative is given by $\frac{w(x,y)}{y}$ and the approximation error is $E(x,y) = \|f'(x) - \frac{w(x,y)}{y}\|$. In Figure 8.4, the approximation error



(a) $z \in \mathbb{C}$, $E(x,y) \leq 4$ (b) $z \in \mathbb{D}$, , $E(x,y) \leq 10^{-15}$

Figure 8.4 – Derivative approximation error

is displayed for $(x,y) \in [0, 0.75]^2$. Readily we see that the best option is when the function f is extended to the dual numbers, where the error $E(x,y)$ is of the order of machine epsilon 10^{-15}, Figure 8.4 (b). The reason of this surprising results is due to the Cauchy Riemann (CR) conditions in $\mathbb{D}$ described in [Chatelin et al., 2014]. Let $z = x + yn \in \mathbb{D}$ be in a domain $U \subset \mathbb{R}^2$ and suppose

$f(z) = v(x, y) + w(x, y)n$ to be C^1, i.e. $v, w \in C^1(U)$, thus the (CR) conditions are $v_x = w_y$, $v_y = 0$ and f is of the form $f(x + y\ n) = f(x) + f'(x)y\ n$, for any $y \in \mathbb{R}$, not only small as it is usually supposed in the derivative approximation by Taylor series. Notice that the advantages of this method are limited to functions satisfying these specific conditions. For example, the real functions $\exp(x)$, $\sin(x)$, x^k extended over $\mathbb{D}$ are as follows:

$$\exp(x + yn) = \exp(x) + \exp(x)yn,$$

$$\sin(x + yn) = \sin(x) + \cos(x)yn,$$

$$(x + yn)^k = x^k + kx^{k-1}yn.$$

Moreover, it is possible to recursively compute higher order derivatives by means of the *pluridual* numbers already exploited in the movie industry, [Piponi, 2004]. We denote $\mathbb{D}^{(2)}$ the set of dual numbers with dual components: $z \in \mathbb{D}^{(2)} = \mathbb{D}(\mathbb{D})$, then, if we introduce *two* nilpotent units n_1 and n_2, we get

$$z = y_0 + n_2 y_1, \quad y_0, y_1, \in \mathbb{D} \ \text{ defined by } n_1$$

$$z = (x_0 + x_1 n_1) + n_2(x_2 + x_3 n_1), \quad n_1^2 = 0, \ \ n_2^2 = 0, \ \ (n_1 n_2)^2 = 0$$

and a function $f : \mathbb{D}^{(2)} \mapsto \mathbb{D}^{(2)}$ is given by $f(z) = f(y_0 + n_2 y_1) = f(y_0) + n_2 f'(y_0) y_1$, where $f(y_0) = f(x_0 + x_1 n_1) = f(x_0) + n_1 f'(x_0) x_1$, $\ f'(y_0) = f'(x_0 + x_1 n_1) = f'(x_0) + n_1 f''(x_0) x_1$. Thus, for $z = x_0 + h_1 n_1 + h_2 n_2 \in \mathbb{D}^{(2)}$ it can be proved that:

$$f(x_0 + h_1 n_1 + h_2 n_2) = f(x_0) + h_1 f'(x_0) n_1 + h_2 f'(x_0) n_2 + h_1 h_2 f''(x_0) n_1 n_2.$$

from where we obtain the second order derivative with the component of $f(x_0 + h_1 n_1 + h_2 n_2)$ on $n_1 n_2$ denoted $\text{Un}_{n_1 n_2} f(x_0 + h_1 n_1 + h_2 n_2)$:

$$f''(x_0) = \frac{\text{Un}_{n_1 n_2} f(x_0 + h_1 n_1 + h_2 n_2)}{h_1 h_2}.$$

In [Fike, 2013], this technique is used in the framework of multi-objective optimisation in Computational Fluid Dynamics with computations of second order derivatives for turbulent transonic flow over an airfoil and supersonic flow over a wedge. Computations are more expensive but give better results in terms of precision. This work has been continued in [Brake et al., 2016] and used in uncertainty quantification by developing parameterised reduced order models for geometric perturbations of an ideal model. These parameterisations are based upon Taylor series expansions of the system matrices about the ideal geometry. The numerical derivatives required by the Taylor series expansions are obtained as presented above. In addition to the accuracy of the derivatives up to machine precision, the key advantage is that this method only generates a single mesh for all the different perturbations and associated geometric variations.

8.2 Computer Science and Engineering

8.2.1 Rigid body dynamics

Since their introduction by Clifford, dual numbers presented in Chapter 2 have been applied to constrained movements in mechanics with a specific attention to kinematics of rigid body dynamics through the works of Kotelnikov (Ukraine) and Study (Germany) [Kotelnikov, 1895, Study, 1901].

Dual numbers are used to represent Screw Theory (combined translation and rotation) in a synthetic way. By what roboticians call the transfer principle, the model with real numbers can be written with dual numbers. Dual numbers are used in screw movements to respresent translation along the real part and the rotation angle on the dual part. The generalisation of operators to dual numbers allows engineers to have a compact formulation, with 3 dual equations instead of 6 real equations [Brodsky and Shoham, 1999]. The resolution of kinematic equations is highly improved in term of speed compared to algorithms with real numbers. This is shown with the example of the 4-bar system, which is **ubiquitous** in rigid body dynamics [Pennestrì and Stefanelli, 2007]. For this 4-bar system, dual numbers and dual complex numbers (complex numbers with dual components, see extension of scalars in Chapter 4) are used to represent singularities of the system with both theoretical and numerical examples [Cheng and Thompson, 1996]. In this paper, the infinity of dual roots of polynomials presented in Chapter 5, represent singularities of the displacement equations.

More recently, several approaches have used dual quaternions (8D), quaternions with dual components to combine the respective advantages of each algebra. For example, dual quaternions allow researchers to control robotic arms [Pham et al., 2010] or inertial navigation systems [Wu et al., 2006]. Concerning robots using video cameras to control their movement, dual quaternions are used for the initialisation of relative position and the orientation between the robot and sensors called "hand-eye calibration" . According to the roboticians, dual quaternions are an optimal solution for this kind of problem because there is no redundant information and variables which is not true for matrices. The main advantage of dual quaternions is that they can represent by the same multiplication a translation and a rotation, whereas matrix or vector formulations require to perfom these two operations separately [Daniilidis, 1999]. For a mathematical description, see [Rincon-Camacho and Latre, 2013, Chatelin et al., 2014].

In summary, the different approaches of roboticians concern both analytical and numerical examples and the main reasons in praise of dual quaternions are the compactness of formulation (reduction of the number of equations), a better singularity analysis and in general a more efficient simulation and control of the systems.

8.2.2 Signal and image processing

8.2.2.1 Signal processing

In signal processing, quaternions have been the first alternative algebraic structure to be used in the 90's to define transfer functions and classical transforms (Fourier, Hadamard). Even if effective algorithms have been implemented [Pei et al., 2001], quaternions are not adapted to a large number of signals. Quaternions are limited to 3 distinct signals, one signal for each imaginary axis. On the contrary, more recent works focus on some algebras presented in Chapter 4 especially bireal numbers [Alfsmann and Göckler, 2007], bicomplex numbers [Pei et al., 2004] and the general case in dimension 2^n with multicomplex numbers and multireal numbers [Alfsmann, 2006].

The advantage of structures based on multiplanar numbers is the decomposition in orthogonal components thanks to the idempotent basis (see Section 1.2.2 of Chapter 4). We recall that the idempotent basis relies on the existence of zerodivisors (x and $y \neq 0$ s.t. $x \times y = 0$) which is by definition unreacheable in fields. This specific property due to the definition of multiplication is used to reduce complexity of algorithms with componentwise operations.

8.2.2.2 Image processing

In the case of image processing, bicomplex numbers are used for pattern detection (size, shape,..) [Pei et al., 2004]. Unlike signal processing, quaternions are used in an efficient way for Fourier transform. For a given image using RGB-model (Red, Green, Blue), each point (pixel) of the image can be considered as a single quaternion with data relying on quaternion multiplication and no longer the superpositon of 3 scalar images [Ell and Sangwine, 2007]. As there are only 3 signals (R,G, B) quaternions match the requirements of image processing, which is not the case for general signal processing as presented above.

8.2.2.3 3D-modelling (Mesh, textures, augmented reality)

For several years, quaternions have been widely used in 3D-models (rendering and game engines) to represent rotations [Azuma et al., 1999, DeLoura, 2001]. Another intensive applications are codes for numerical simulation in Chemistry [Karney, 2007]. These applications are directly inherited from the examples in rigid body dynamics. Texture maps (geometric skinning) are both concerned with kinematics of rigid bodies (animated objects) and rotations in 3D-space. As these numerical models are consequences of some aspects of robotics, it is quite natural to find again dual quaternions. The algebra of dual quaternions is an improvement which avoids numerical singularities of meshes due to linear algebra (Figure 8.5, left), keeping an equivalent memory cost and speed [Kavan et al., 2008]. This is not limited to animated characters but concerns, in a more general way, the transformations of 3D-meshes.

The objective of this method using dual quaternions is not to give the most realistic deformations (that requires other techniques [Vaillant et al., 2013]), but to produce a fast algorithm able to handle

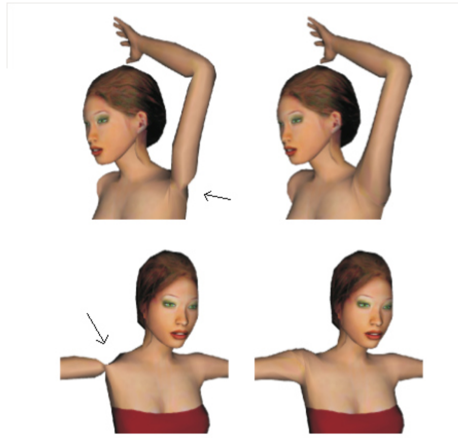


Figure 8.5 – Comparison between a linear model (left) and dual quaternions (right) [Kavan et al., 2008] for textures (geometric skinning).

multiple objects in interaction with a low computational cost.

In 3D computer graphics, there exist specific algorithms concerning lighting technology for the gaming and entertainment industries. Notice that the company Geomerics that produces a code using quaternion compression (with the formalism of Clifford algebras) for this enlighting technique has been bought by ARM, the multinational semiconductor and software design company, in 2013 and is one of the leaders of this segment.

8.3 Physics

The perspectives for the use of hypercomplex numbers in Physics are even more interesting than the previous cases of applications. But we must warn that there is a huge difficulty gap between practical applications in Science or Engineering, and the complexity of hypercomplex models concerning advanced topics of Physics. Hypercomplex algebras, used as alternative tools, are closely linked with major topics of Physics (relativistic mechanics, particle physics, electromagnetism, quantum mechanics). Linear algebra and complex numbers have proven their efficiency for a wide range of topics which can be explicitly formulated, in particular euclidean geometries but in the case of more general phenomena with underlying nonlinear principles, these tools may not be sufficient, they are too restrictive compared to hypercomplex algebras.

A general remark concerning geometry is that the three kinds of nonreal units in the plane that square to -1, 0 or 1 (see Chapter 2) are respectively related to elliptic, parabolic and hyperbolic cases [Kisil, 2012]. A remarkable example concerning bireal numbers and hyperbolic geometry can be found in hydrodynamics, in particular the transition state of the differential equation from subsonic to hypersonic regimes [Lavrentiev and Chabat, 1980].

8.3.1 Electromagnetism

In electromagnetism, we mentionned in Chapter 2 the position of Maxwell concerning quaternions [Maxwell, 1891]. The operator nabla denoted ∇ defined by Hamilton in 1847, whose name was given by P. Tait in a personal correspondence with Maxwell, is the only remaining trace of Hamilton's inheritage. The operator ∇ is often presented as a handy symbol to write the operators gradient, divergence and curl of a function f respectively denoted $gradf$, $divf$ and $rotf$. The original idea of Hamilton to interpret the symbol $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$ as an imaginary quaternion led to write $gradf = \nabla f$, $divf =< \nabla | f >$ and $rotf = \nabla \wedge f$. These identities have been reduced to a mere notation in 3D-vector analysis [Wilson and Gibbs, 1902, Heaviside, 1892]. However, they are well defined formulae in the field of quaternions, thanks to the existence of addition and multiplication in 4D. The modern hypercomplex versions of electromagnetism are using complex quaternions and can represent Maxwell's relations, uniting scalar and vector equations which is impossible in linear algebra [Kravchenko, 2003]. The avantages of complex quaternions are presented in [Khmelnytskaya and Kravchenko, 2009], where analytic and numerical solutions of electromagnetism are compared to those coming from linear algebra. We insist on the fact that the main current version of Gibbs and Heaviside with vector analysis is not the original one of Maxwell, based on quaternionic operations.

It is interesting to notice that famous researchers have followed the original path of Maxwell working at the frontier of Electromagnetism and Special Relativity. In particular, we can mention the work of Silberstein, for the representation of the electromagnetic field with the Riemann-Silberstein vector [Silberstein, 1912]. Another remarkable work comes from Lanczos which is well-known in numerical analysis for his many contributions (Lanczos algorithm, conjugate gradient). Indeed, it is less known that Lanczos started and spent the main part of his career working on mathematical physics and especially Electromagnetism and Relativity since his thesis in 1919 using complex quaternions [Lanczos, 1919]. He also served as assistant to Einstein between 1928 and 1929.

8.3.2 Modern Physics

In relativistic Physics, it is interesting to notice that Lorentz transformations and the Lorentz group have been written in the formalism of complex quaternions. We already pointed out in Chapter 4 the contribution of Rosen (third author of the Einstein-Podolsky-Rosen paradox), to the matrix formulation of quaternions and 4D-rotations in Special Relativity (SR) but the complete description of Lorentz transforms with complex quaternions was given ealier by Klein that uses Noether's work [Klein and Sommerfeld, 1910] and Silberstein [Silberstein, 1912].

After the introduction of spinor theory, particularly in the hands of E. Cartan and then Pauli, the biquaternion (=complex quaternion [Hamilton, 1853]) representation of the Lorentz group was superseded. Nevertheless, it is remarkable to notice that hypercomplex algebras are still implictly present in current theory. For example in Special Relativty (SR), what physicists call the complex lightcone is the set of zerodivisors of $\mathbb{H}(\mathbb{C})$. It is rarely mentionned that, when limited to one space variable, it corresponds to idempotent lines in the bireal plane [Sobczyk, 1995].

Concerning Quantum Mechanics, we can (once again) mention the use of complex quaternions (see Section 8.4) that is common to SR and quantum mechanics, and also the role of octonions and alternativity [Baez, 2002, Dray and Manogue, 2015]. Octonions are especially used in the field of particle physics: there exist, so far, no other way than octonionic multiplication to represent quark transtition states [Catto et al., 2016]. A factual observation is that the more researchers investigate fundamental phenomena and their interactions, the more algebras in higher dimension associated with a specifc multiplication are frequently found useful.

Sir Michael Atyiah (Fields medal 1966, Abel prize 2004) kept all along his career an interest in Clifford and Dickson algebras since his work with Bott [Atiyah et al., 1964] and up to now [Azcarraga, 2018]. His main points of interest concern geometry, spinors and gravitation. In his opinion, non associativity, alternativity and octonions should not be neglected, especially for gravitation.

8.4 Quantum Computing

Over the past years, quantum computing has evolved to appear as an emerging technology with the potential to tackle some scientific problems that cannot be solved by classical computers even with a high level of paralelisation. The point is not to determine if a quantum computer could replace a classical computer but how it could be of great interest for some fields of research such as combinatorics, factorisation, chemistry, quantum sensors for metrology. Quantum computing is a combination between several areas but mainly quantum mechanics, thermodynamics for material sciences, mathematics and algorithmics. We will only briefly present some aspects related to the mathematical context of quantum mechanics.

8.4.1 Elements of quantum principles and their historical background

There exist no fully satisfying theory to explain the phenomena related to quantum mechanics (superposition, entanglement, decoherence,..) and these ideas have been the source of unresolved controveries since their discovery in the beginning of the 20^{th} century. It is interesting to notice that the matrix mechanics formulated by Born, Heisenberg and P. Jordan was for some time opposed to the wave mechanics of Schrödinger. The reconciliation between the two approaches in the framework of Hilbert spaces occured with Dirac. The bra-ket notation of Dirac is just based on complex vectors in Hilbert spaces, but in the early 20^{th} century, matrices and linear algebra were not a common tool for physicists.

P. Jordan, which played a central role in the mathematical aspects of matrix mechanics, extended the algebraic tools to tackle nonassociativity associated with observables, in particular using octonions [Jordan et al., 1934]. Due to the advanced mathematical theory that was required by the path opened by Jordan (non linear algebra, nonassociativity) and also historical reasons that separated Jordan from the Copenhagen group (see Bibliographical notes), most physicists remained on the danish path of linearity and associativity. Jordan's interest for nonassociativity has been taken up by mathematicians (in particular Albert, the student of Dickson, see Chapter 3) and gave birth to what is now called Jordan algebras which are based on a nonassociative multiplication [Schafer, 1995]. After the sixties, octonions made their appearance in quantum mechanics again, but only few physicists kept on the track of nonassociativity. For detailed explanations concerning this point, see [Günaydin et al., 1978].

In quantum mechanics, Pauli described the spin of an electron thanks to the matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \text{ and } \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

that are currently called Pauli matrices. The matrices that Pauli used for spin theory are coming from Lorentz transformations and their complex quaternions formalism, see Section 8.3.2. They are the matrix transcription of the basis vectors of complex quaternions (see Chapter 4, Section 1.3) and we have the isomorphism $\{I_2, i\sigma_3, i\sigma_2, i\sigma_1\} \simeq \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{i} \times \mathbf{j} = \mathbf{k}$ and I_2 the identity matrix of order 2. Considering $p, q, r, s \in \mathbb{C}$, a complex quaternion $p + q\mathbf{i} + r\mathbf{j} + s\mathbf{k} \in \mathbb{H}(\mathbb{C}) = \mathbb{C} \otimes \mathbb{H}$ is isomorphic to the matrix $\begin{bmatrix} p+iq & r+is \\ -r+is & p-iq \end{bmatrix}$.

8.4.2 Quantum Computing and hypercomplex algebras

The model for the quantum analogues of bits for classical computers, called *qubits* "quantum bits ", is based on complex Hilbert spaces and the notation bra-ket of Dirac. The spatial representation of the different states of qubits uses the Bloch sphere, which is a unit sphere of $\mathbb{R}^3$ that describes spatial rotations $(SO(3), SU(2))$. In theory, a quantum computer with n qubits can have up to 2^n superposed states. Computations are performed using quantum gates (represented by unitary hermitian matrices), the quantum equivalent of logic gates.

In the current direction followed by industrials and by the most advanced research works on quantum computing [Bravyi et al., 2018], the main quantum gates are called "Clifford gate set" and correspond to Pauli matrices with notations $I = I_2, X = \sigma_1, Y = \sigma_2, Z = \sigma_3$. Implictly, these quantum gates use complex quaternions computational mechanisms. Another fundamental quantum gate used to model superposition states is the Hadamard gate represented by the matrix $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. The role of this matrix has been largely discussed in Chapters 4, 6 and 7 and correspond to the use of idempotent elements within a bireal structure. A more general relation between quantum gates and hypercomplex algebras can be found in [Vlasov, 1999, Vlasov, 2001]. Fundamental aspects of Quantum Computing are thus naturally related to hypercomplex algebras [Latre, 2018].

8.4.3 Issues for Quantum Computing

We have seen in Chapter 4 that a theoretical isomorphism could lead to misunderstandings and also that two isomorphic formulations do not necessarily have the same numerical behaviour, see Section 8.1. The use of $SO(3)$ for rotations is not numerically efficient . Concerning unitary matrices $SU(2)$ that are generally associated to the tensor product to obtain the dimension 2^n, the first problem is that the formulation is not unique and depends of the choice of coordinates to build matrices. The tensor product, as it is used in quantum computing, has a descriptive role and in practice is reduced to the product of block matrices (Kronecker product). Notice that the role of tensor product in Quantum Computing, limited to linear algebra, should not be confused with the one in differential geometry and also in data analysis, where tensors are used for their multilinear properties.

In Chapter 7, we have presented relations between spectral coupling and quantum chemistry. Chemistry is one of the main targets of quantum computing and the coupling of energy levels obtained from the stationnary Schrödinger equation use the same Hadamard matrix than the superposition of quantum states. Much work has to be done to deepen this connection.

Another key issue concerns the measure problem. Measure is already a difficult question for standard computers. The choice of a specific norm can change a numerical result or the convergence of an algorithm. When it comes to quantum computing, other specific properties (superposition, entanglement) are added which become conflictual with the standard notion of norm. We have seen that complex quaternions or bireal numbers are implicitly present in quantum computations which means that non euclidean metrics and zerodivisors have to be taken into account. Indeed, wee recall (see Chapter 4) that $\mathbb{H}(\mathbb{C})$ cannot be naturally equipped with a norm: there exist idempotent and nilpotent elements that are zerodivisors and thus computational singularities. The use of the algebraic structures presented in this thesis could represent an interesting direction to clarify in an explicit formulaion these crucial points. If we keep in mind the instructive example of electromagnetism and the differences between original quaternionic equations and the modern form in vector analysis we may ask the following question: do complex Hilbert spaces and matrices define the most efficient way to perform quantum computations that are highly nonlinear?

8.5 Summary

In Computational Science and Engineering, many of the algebras presented in this thesis allow researchers to compute with a reduced number of equations and parameters, or with a better numerical stability and efficiency. Bireal and dual numbers offer numerical possibilities that are unreachable with complex numbers even if computations may be more expensive due to the overloading of operators. Concerning the practical implementations, there exist no standard and optimised codes, only custom made examples often designed for a specific field of application. This may explain why these tools are not widespread, since their efficiency is not universal.

Without falling into name dropping, it is quite striking to notice that major mathematicians and physicists of the 19^{th} and 20^{th} centuries have worked with hypercomplex numbers on key parts of Science such as electromagnetism, relativity and quantum mechanics.

We sum up in Table 8.1 the fields of application that we have found, where hypercomplex numbers bring an added value, either theoretical or practical. Concerning split algebras which are not detailed in this thesis, a list of relevant references can be found in [Latre, 2013].

dimension	algebras				
2	$\mathbb{C}$	$\not{\mathbb{C}} \equiv {}^2\mathbb{R}$ 1, 2, 6, **8**		$\mathbb{D}$ 1, 2, 4, 9	
4	$\mathbb{H}$ 1, 2, 5, 7, **8**	$\not{\mathbb{H}}$ 5, 6	${}^2\mathbb{C} = \mathbb{C}({}^2\mathbb{R})$ 1, 2, 5, 7, 9	$\mathbb{C}(\mathbb{D})$ 1, 2, 4	$\mathbb{D}(\mathbb{D})$ 9, 11, 12
8	$\mathbb{G}$ 3, 5, 7	$\not{\mathbb{G}}$ 7	${}^2\mathbb{H} = \mathbb{H}({}^2\mathbb{R})$ 3	$\mathbb{H}(\mathbb{D})$ 1, 2, 4	$\mathbb{H}(\mathbb{C})$ 5,6,7,**8**,10

Table 8.1 – Some of the current theoretical and practical applications for hypercomplex algebras in dimensions 2, 4, 8

1: Signal Processing	2: Image Processing
3: Kinematics	4: Robotics
5: Electromagnetism	6: Special Relativity
7: Quantum Mechanics	8: Quantum Computing
9: Automatic Differentiation	10: High Energy Physics
11: Optimisation	12: CFD

In conclusion of this chapter, we can see that hypercomplex algebras are not limited to specific fields of application, so they should be considered as number structures to compute with great potential for current and future parts of scientific research.

8.6 Bibliographic notes

The bibliographic material is largely scattered due to the many topics addressed, the different communities involved and the long time periods. There is a mix between, on one hand, mathematical publications on the invention of hypercomplex numbers, their algebraic properties and their application to Physics mainly published between the 1840's and 1940's, written in diverse languages, and on the other hand, applications in engineering mostly developed after 1950 with the widespread availability of electronic computers. As pointed in Chapter 2 with the example of Cartan's article, which remains untranslated in English, the facts that hypercomplex numbers are not a common academicly taught topic in Science and the lack of authoritative solid references are major issues which render transdisciplinarity an elusive task.

The mere designation of structures is so diverse that it requires semantic clarification. Depending on the bibliographical references and on the mathematical or engineering communities, the structures appear under multiple names and descriptions. The comparison between algebraic formulations as was done in Chapter 4 is a prerequisite and a first step for a sound communication between researchers in different fields. For example, one can find in current use the following terms (partial list) to refer to the algebra ${}^2\mathbb{R} = \mathbb{R} \oplus \mathbb{R}u$:

— *bireal* numbers [Bencivenga, 1946] , as a subset of bicomplex numbers, [Segre, 1892],

— *double* numbers in plane geometry [Yaglom, 1963],

— *split-complex* numbers, for generalised Dickson and composition algebras [Albert, 1942],

— *the even subalgebra of* $Cl_{1,1}(\mathbb{R})$, for Clifford algebras [Lounesto, 2001],

— *hyperbolic* numbers, for quadratic forms [Deheuvels, 1983], SR in 1D-space [Sobczyk, 1995].

This is just an example for **one** algebraic structure among all the algebras that are presented in the thesis! In all the references since the 1950's that we found, we have to point out significant contributions, both theoretical and numerical, coming from the Russian school. It is quite remarkable to notice, that hypercomplex algebras have received much more attention in Russia than in any other country. As pointed out in Chapter 1, this is partly due to educational differences (hypercomplex algebras are taught in high schools). A large number of russian publications remains unstranlated in English.

Concerning quantum mechanics, we sum up in a few lines some facts concerning P. Jordan found in [Schroer, 2003]. Jordan's case is one more illustration of why Science, as a human construction (reflecting the prejudices of its time, that form the *Zeitgeist*), cannot be separated from History. Jordan joined the NSDAP (the national socialist german party) in 1933 and it appears that, due to his education and personal beliefs, his project was to convince the NSDAP leaders that modern physics with Einstein and the Copenhagen school of quantum theory was the best antidote against the "materialism of the Bolsheviks". His active pre-war collaboration with his Jewish colleagues made him appear less than trustworthy in the german side during the war. Then after WW2, his link with the national-socialist party delayed enormously the recognition of his important scientific contributions. An undeserved double scientific punishment. By comparison, Heisenberg's active involvement in the german war effort hardly tarnished after 1945 his well-established scientific reputation...

Bibliography

[Albert, 1942] Albert, A. A. (1942). Quadratic forms permitting composition. *Annals of Mathematics*, pages 161–177.

[Alfsmann, 2006] Alfsmann, D. (2006). On families of 2^n-dimensional hypercomplex algebras suitable for digital signal processing. In *Proc. European Signal Processing Conf.(EUSIPCO 2006), Florence, Italy*.

[Alfsmann and Göckler, 2007] Alfsmann, D. and Göckler, H. G. (2007). On hyperbolic complex LTI digital systems. In *Proc. EURASIP 15th European Signal Processing Conference (EUSIPCO 2007), Poznan, Poland*, pages 1322–1326.

[Atiyah et al., 1964] Atiyah, M. F., Bott, R., and Shapiro, A. (1964). Clifford modules. *Topology*, 3:3–38.

[Azcarraga, 2018] Azcarraga, J. A. (2018). On mathematics and physics: a conversation with Sir Michael Atiyah. *Spanish journal of Physics*, 32:32–38.

[Azuma et al., 1999] Azuma, R., Hoff, B., Neely III, H., and Sarfaty, R. (1999). A motion-stabilized outdoor augmented reality system. In *Virtual Reality, 1999. Proceedings., IEEE*, pages 252–259. IEEE.

[Baez, 2002] Baez, J. (2002). The octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205.

[Bencivenga, 1946] Bencivenga, U. (1946). Sulla rappresentazione geometrica delle algebre doppie dotate di modulo. *Atti della Reale Accademia delle Scienze e Belle-Lettere di Napoli*, Ser (3) v.2(7).

[Brake et al., 2016] Brake, M., Fike, J. A., and Topping, S. D. (2016). Parameterized reduced order models from a single mesh using hyper-dual numbers. *Journal of Sound and Vibration*, 371:370–392.

[Bravyi et al., 2018] Bravyi, S., Gosset, D., and Koenig, R. (2018). Quantum advantage with shallow circuits. *Science*, 362(6412):308–311.

[Brodsky and Shoham, 1999] Brodsky, V. and Shoham, M. (1999). Dual numbers representation of rigid body dynamics. *Mechanism and machine theory*, 34(5):693–718.

[Catto et al., 2016] Catto, S., Gürcan, Y., Khalfan, A., and Kurt, L. (2016). Unifying ancient and modern geometries through octonions. In *Journal of Physics: Conference Series*, volume 670, pages 12–16. IOP Publishing.

[Chatelin et al., 2014] Chatelin, F., Latre, J.-B., Rincon-Camacho, M., and Ricoux, P. (2014). Beyond and behind linear algebra. Technical report, Tech. rep., CERFACS TR/PA/14/80.

[Cheng and Thompson, 1996] Cheng, H. and Thompson, S. (1996). Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanism. In *Proc. of ASME 24th. Biennial Mechanisms Conference, Irvine, California*, pages 19–22.

[Daniilidis, 1999] Daniilidis, K. (1999). Hand-eye calibration using dual quaternions. *The International Journal of Robotics Research*, 18(3):286–298.

[Deheuvels, 1983] Deheuvels, R. (1983). *Formes quadratiques et groupes classiques*. Presses Universitaires de France.

[DeLoura, 2001] DeLoura, M. A. (2001). *Game Programming: Gems 2*. Cengage learning.

[Dimentberg, 1965] Dimentberg, F. (1965). The screw calculus. *(Book in Russian)*.

[Dimentberg, 1968] Dimentberg, F. M. e. (1968). The screw calculus and its applications in mechanics. Technical report, DTIC Document (english translation of 1965 book).

[Dray and Manogue, 2015] Dray, T. and Manogue, C. A. (2015). *The geometry of the octonions*. World Scientific.

[Eberly, 2002] Eberly, D. (2002). Rotation representations and performance issues. *Magic Software: Chapel Hill, NC, USA*.

[Ell and Sangwine, 2007] Ell, T. A. and Sangwine, S. J. (2007). Hypercomplex fourier transforms of color images. *Image Processing, IEEE Transactions on*, 16(1):22–35.

[Fike, 2013] Fike, J. A. (2013). *Multi-objective optimization using hyper-dual numbers*. PhD thesis, Stanford University.

[Goldberg, 1991] Goldberg, D. (1991). What every computer scientist should know about floating-point arithmetic. *ACM Computing Surveys (CSUR)*, 23(1):5–48.

[Gsponer and Hurni, 2005a] Gsponer, A. and Hurni, J.-P. (2005a). Quaternions in mathematical physics (1): Alphabetical bibliography. *arXiv preprint math-ph/0510059*.

[Gsponer and Hurni, 2005b] Gsponer, A. and Hurni, J.-P. (2005b). Quaternions in mathematical physics (2): Analytical bibliography. *arXiv preprint math-ph/0511092*.

[Günaydin et al., 1978] Günaydin, M., Piron, C., and Ruegg, H. (1978). Moufang plane and octonionic quantum mechanics. *Communications in Mathematical Physics*, 61(1):69–85.

[Hamilton, 1853] Hamilton, W. R. (1853). On the geometrical interpretation of some results obtained by calculation with biquaternions. *Proceedings of the Royal Irish Academy*, 5:388–390.

[Heaviside, 1892] Heaviside, O. (1892). On the forces, stresses, and fluxes of energy in the electromagnetic field. *Proceedings of the Royal Society of London*, 50(302-307):126–129.

[Howell and Lafon, 1975] Howell, T. D. and Lafon, J.-C. (1975). The complexity of the quaternion product. Technical report, Cornell University.

[Jordan et al., 1934] Jordan, P., von Neumann, J., and Wigner, E. P. (1934). On an algebraic generalization of the quantum mechanical formalism. *Annals of mathematics*, 35(1):29–64.

[Karney, 2007] Karney, C. F. (2007). Quaternions in molecular modeling. *Journal of Molecular Graphics and Modelling*, 25(5):595–604.

[Kavan et al., 2008] Kavan, L., Collins, S., Žára, J., and O'Sullivan, C. (2008). Geometric skinning with approximate dual quaternion blending. *ACM Transactions on Graphics (TOG)*, 27(4):105.

[Khmelnytskaya and Kravchenko, 2009] Khmelnytskaya, K. V. and Kravchenko, V. V. (2009). Biquaternions for analytic and numerical solution of equations of electrodynamics. *arXiv preprint arXiv:0902.3490v1*.

[Kisil, 2012] Kisil, V. V. (2012). *Geometry of Möbius Transformations: Elliptic, Parabolic and Hyperbolic Actions of SL2 [real Number]*. World Scientific.

[Klein and Sommerfeld, 1910] Klein, F. and Sommerfeld, A. (1910). Über die theorie des kreisels (4 vols.). *Teubner, Leipzig (1897–1910)*.

[Kotelnikov, 1895] Kotelnikov, A. (1895). Screw calculus and some of its applications to geometry and mechanics. *Annals of Imperial University of Kazan*.

[Kravchenko, 2003] Kravchenko, V. V. (2003). *Applied quaternionic analysis*, volume 28. Heldermann.

[Lanczos, 1919] Lanczos, C. (1919). The relations of the homogeneous Maxwell's equations to the theory of functions, PhD dissertation translated by A. Gsponer and J.-P. Hurni in 2004. *arXiv preprint physics/0408079*.

[Lantoine et al., 2012] Lantoine, G., Russell, R. P., and Dargent, T. (2012). Using multicomplex variables for automatic computation of high-order derivatives. *ACM Transactions on Mathematical Software (TOMS)*, 38(3):16.

[Latre, 2013] Latre, J.-B. (2013). Sur quelques structures algébriques utiles au traitement de l'information en robotique, informatique et physique. Technical report, Rapport de Master, 25/03/13-06/09/13, CERFACS WN-PA-13-105.

[Latre, 2018] Latre, J.-B. (2018). Qualitative Computing for Quantum Computing. TOTAL Conference Mathias, Paris 22-24 Oct, 2018.

[Lavrentiev and Chabat, 1980] Lavrentiev, M. and Chabat, B. (1980). *Effets hydrodynamiques et modèles mathématiques*. Traduit du russe. Mir, Moscou.

[Lounesto, 2001] Lounesto, P. (2001). *Clifford algebras and spinors*, volume 286. Cambridge university press.

[Martins et al., 2003] Martins, J. R., Sturdza, P., and Alonso, J. J. (2003). The complex-step derivative approximation. *ACM Transactions on Mathematical Software (TOMS)*, 29(3):245–262.

[Maxwell, 1891] Maxwell, J. C. (1891). *A treatise on electricity and magnetism.* 3^{rd} edition (J.J. Thomson ed.), Dover republication (1954).

[Metzler, 1994] Metzler, W. (1994). The "mystery" of the quadratic mandelbrot set. *American Journal of Physics*, 62:813–814.

[Pavlov et al., 2009a] Pavlov, D., Panchelyuga, M., Malykhin, V., and Panchelyuga, V. (2009a). On fractality of Mandelbrot and Julia sets on double-numbers plane. *Hypercomplex numbers in geometry and physics*, 6:135–145.

[Pavlov et al., 2009b] Pavlov, D., Panchelyuga, M., and Panchelyuga, V. (2009b). About shape of Julia set at zero parameter on double numbers plane. *Hypercomplex numbers geometry and physics*, 6:146–151.

[Pei et al., 2004] Pei, S.-C., Chang, J.-H., and Ding, J.-J. (2004). Commutative reduced biquaternions and their Fourier transform for signal and image processing applications. *Signal Processing, IEEE Transactions on*, 52(7):2012–2031.

[Pei et al., 2001] Pei, S.-C., Ding, J.-J., and Chang, J.-H. (2001). Efficient implementation of quaternion Fourier transform, convolution, and correlation by 2-D complex FFT. *Signal Processing, IEEE Transactions on*, 49(11):2783–2797.

[Pennestrì and Stefanelli, 2007] Pennestrì, E. and Stefanelli, R. (2007). Linear algebra and numerical algorithms using dual numbers. *Multibody System Dynamics*, 18(3):323–344.

[Pham et al., 2010] Pham, H.-L., Perdereau, V., Adorno, B. V., and Fraisse, P. (2010). Position and orientation control of robot manipulators using dual quaternion feedback. In *Intelligent Robots and Systems (IROS), 2010 IEEE/RSJ International Conference on*, pages 658–663. IEEE.

[Piponi, 2004] Piponi, D. (2004). Automatic differentiation, C++ templates, and photogrammetry. *Journal of Graphics Tools*, 9(4):41–55.

[Rincon-Camacho and Latre, 2013] Rincon-Camacho, M. M. and Latre, J.-B. (2013). Project description: Algebraic structures for information processing associated with Dickson algebras. Technical report, CERFACS TR/PA/13/106.

[Schafer, 1995] Schafer, R. (1995). *An Introduction to Nonassociative Algebras (first edition 1966)*. Dover, New York.

[Schroer, 2003] Schroer, B. (2003). Pascual Jordan, his contributions to quantum mechanics and his legacy in contemporary local quantum physics. *arXiv preprint hep-th/0303241*.

[Segre, 1892] Segre, C. (1892). Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. *Math. Ann*, 40:413–467.

[Silberstein, 1912] Silberstein, L. (1912). Quaternionic form of relativity. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 23(137):790–809.

[Sobczyk, 1995] Sobczyk, G. (1995). The hyperbolic number plane. *The College Mathematics Journal*, 26(4):268–280.

[Study, 1901] Study, E. (1901). *Geometrie der Dynamen*. Teubner, Leipzig.

[Vaillant et al., 2013] Vaillant, R., Barthe, L., Guennebaud, G., Cani, M.-P., Rohmer, D., Wyvill, B., Gourmel, O., and Paulin, M. (2013). Implicit skinning: Real-time skin deformation with contact modeling. *ACM Transaction on Graphics (TOG). Proceedings of ACM SIGGRAPH*.

[Vlasov, 1999] Vlasov, A. Y. (1999). Quantum gates and Clifford algebras. *arXiv preprint quant-ph/9907079*.

[Vlasov, 2001] Vlasov, A. Y. (2001). Clifford algebras and universal sets of quantum gates. *Physical Review A*, 63(5):054302.

[Wilson and Gibbs, 1902] Wilson, E. B. and Gibbs, J. W. (1902). *Vector analysis: a text-book for the use of students of mathematics and physics; founded upon the lectures of J. Willard Gibbs*. Scribner's Sons.

[Wu et al., 2006] Wu, Y., Hu, X., Wu, M., and Hu, D. (2006). Strapdown inertial navigation using dual quaternion algebra: error analysis. *Aerospace and Electronic Systems, IEEE Transactions on*, 42(1):259–266.

[Yaglom, 1963] Yaglom, I. (1963). Kompleksnye chisla i ikh primenenie v geometrii (in russian) Engl. transl. Complex numbers in geometry, 1968 Academic Press, New York.

Conclusion

Nous avons exploré différents aspects de la multiplication de nombres ainsi que les structures algébriques associées qui sont utilisées pour la Science et l'Ingénierie. Le caractère particulier de ces algèbres, lié au rôle de la multiplication, en fait un cadre de calcul potentiellement mieux adapté aux problèmes non linéaires. Cela fait émerger de nouvelles possibilités pour le Calcul, qui restent largement inexplorées à ce jour.

Contrairement à la plupart des références trouvées dans la littérature qui traitent chaque structure ou type de structure comme des cas particuliers isolés, nous avons présenté une approche comparative des différentes algèbres hypercomplexes et du procédé de Dickson tout en regroupant les différentes applications que nous avons recensées à ce jour. C'est justement la **comparaison** de ces structures qui met en évidence que le Calcul revêt par exemple une forme très spécifique dans le corps $\mathbb{C}$, alors qu'il prend une forme différente pour les autres algèbres du plan ${}^2\mathbb{R}$ et $\mathbb{D}$. Nous soulignons notamment l'existence de diviseurs de zéro qui, loin d'être un frein au Calcul, s'avère être un véritable avantage pour les cas que nous avons présentés. Nous avons également vu que, dans une perspective de Calcul, la notion d'isomorphisme est à relativiser. Concernant les apports personnels, la théorie du couplage spectral associée aux nombres biréels permet de présenter des aspects mathématiques novateurs par rapport à l'existant et cela même dans des domaines bien établis comme la théorie spectrale.

L'utilisation des algèbres hypercomplexes semble bien implantée dans certains domaines applicatifs (robotique, informatique) et implicitement présente dans de nombreux exemples de la physique. Nous avons mis en évidence de nouveaux domaines d'applications comme la mécanique des milieux continus ou la chimie, grâce au couplage spectral. Nous mettons en avant trois raisons principales qui, selon nous, légitiment la promotion des algèbres que nous avons étudiées.

1) Premièrement, pour l'aspect scientifique. En effet, il reste aujourd'hui beaucoup à découvrir sur les propriétés théoriques de ces nombres et les aspects numériques qui en découlent. Deux axes de recherche méritent selon nous une attention particulière. Tout d'abord, l'étude du plan numérique 2D demeure incomplète. Les nouveaux résultats que nous avons présentés pour les biréels et le couplage spectral ne sont qu'un début prometteur pour la découverte de nouveaux principes de Calcul qui restent encore implicites. Nous rappelons le travail d'Adrien Douady un demi-millénaire (!) après l'invention des nombres complexes, a approfondi des résultats remarquables sur les itérations quadratiques dans le plan complexe. Cet exemple, très court à l'échelle de temps du Calcul, montre que des résultats profonds peuvent encore surgir grâce à des polynômes de degré 2 et au plan numérique. Le rôle des nombres biréels et celui des duaux (que nous avons à peine éffleuré) pour le Calcul est loin d'être une question résolue, tout comme l'interaction des trois algèbres et géométries du plan. Selon nous, la dimension 2 demeure un défi sous-estimé. Ensuite, les algèbres de Dickson non alternatives (de dimension ≥ 16) jouent un rôle crucial dans le traitement différentiel de l'information de par l'aspect local des calculs et de la notion de mesure [Chatelin, 2019]. Avec l'essor de techniques de traitement des données (machine learning, intelligence artificielle) et de l'émergence du Calcul Quantique, les algèbres hypercomplexes pourraient permettre de compléter utilement les techniques actuelles qui exploitent l'aspect combinatoire et statistique de données discrètes. Cependant, un constat flagrant

concernant les algèbres hypercomplexes et les algèbres de Dickson est que leur visibilité est ralentie par la connaissance encore trop limitée de leur existence et de leurs propriétés algébriques. Du point de vue applicatif, le manque de formation théorique associé à ces structures mais aussi de codes de calculs accessibles et efficaces doit être pris en compte.

2) Dans le prolongement du point 1), un second aspect concerne la nécessité d'un enseignement mathématique associé. En effet, à notre connaissance, la formation actuelle mathématique ou applicative développe principalement le Calcul réel, complexe ou matriciel. La connaissance des différentes possibilités de multiplication, qui est une opération fondamentale, semble souvent limitée aux corps et aux structures associatives, à part pour certains domaines spécifiques de l'algèbre. L'utilisation d'autres structures, au delà et plus encore de l'alternativité, offre une perspective **complémentaire** aux domaines existants.

3) Le dernier point dépasse le simple domaine mathématique, et nous permet de revenir sur une réflexion au delà de la segmentation des disciplines. Il ne faut pas oublier que les mathématiques sont une œuvre humaine, c'est-à-dire une action inscrite dans le temps, qui possède ses propres limites. Nous avons mis en avant la résistance idéologique de la raison humaine face à la nouveauté et aux ruptures radicales. Ce phénomène ne doit pas être négligé ni sous-estimé. Même si ces algèbres ont été découvertes il y a plus d'un siècle par des inconnus ou des mathématiciens de grande renommée, elles sont sujettes d'une manière continue à une forte résistance provenant de la majeure partie de la communauté scientifique.

Au vu de l'intérêt mathématique, de l'emploi varié dans de nombreux domaines de la Science, ainsi que des domaines potentiels comme le Calcul Quantique, il nous semble déterminant d'approfondir dans l'avenir l'étude des structures présentées, afin d'apporter de réelles idées nouvelles pour le Calcul.

Bibliographie

[Ahmed et al., 1974] Ahmed, N., Natarajan, T., and Rao, K. R. (1974). Discrete cosine transform. *IEEE transactions on Computers*, 100(1) :90–93.

[Albert, 1942] Albert, A. A. (1942). Quadratic forms permitting composition. *Annals of Mathematics*, pages 161–177.

[Alfsmann, 2006] Alfsmann, D. (2006). On families of 2^n-dimensional hypercomplex algebras suitable for digital signal processing. In *Proc. European Signal Processing Conf.(EUSIPCO 2006), Florence, Italy*.

[Alfsmann and Göckler, 2007] Alfsmann, D. and Göckler, H. G. (2007). On hyperbolic complex LTI digital systems. In *Proc. EURASIP 15th European Signal Processing Conference (EUSIPCO 2007), Poznan, Poland*, pages 1322–1326.

[Alfsmann et al., 2007] Alfsmann, D., Göckler, H. G., Sangwine, S. J., and Ell, T. A. (2007). Hypercomplex algebras in digital signal processing : Benefits and drawbacks. In *Proceedings of EUSIPCO*, pages 1322–6.

[Altmann, 1989] Altmann, S. L. (1989). Hamilton, Rodrigues, and the quaternion scandal. *Mathematics Magazine*, 62(5) :291–308.

[Argand, 1806] Argand, J.-R. (1806). *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*. chez Madame Veuve Blanc, Horloger, Paris.

[Argand, 1814] Argand, J. R. (1814). Réflexions sur la nouvelle théorie des imaginaires, suivies d'une application à la démonstration d'un théoreme d'analyse. *Annales de Mathématiques Pures et Appliquées*, 5 :1814–1815.

[Atiyah et al., 1964] Atiyah, M. F., Bott, R., and Shapiro, A. (1964). Clifford modules. *Topology*, 3 :3–38.

[Azcarraga, 2018] Azcarraga, J. A. (2018). On mathematics and physics : a conversation with Sir Michael Atiyah. *Spanish journal of Physics*, 32 :32–38.

[Azuma et al., 1999] Azuma, R., Hoff, B., Neely III, H., and Sarfaty, R. (1999). A motion-stabilized outdoor augmented reality system. In *Virtual Reality, 1999. Proceedings., IEEE*, pages 252–259. IEEE.

[Baez, 2002] Baez, J. (2002). The octonions. *Bulletin of the American Mathematical Society*, 39(2) :145–205.

[Bencivenga, 1946] Bencivenga, U. (1946). Sulla rappresentazione geometrica delle algebre doppie dotate di modulo. *Atti della Reale Accademia delle Scienze e Belle-Lettere di Napoli*, Ser (3) v.2(7).

[Bisegna and Podio-guidugli, 1995] Bisegna, P. and Podio-guidugli, P. (1995). Mohr's arbelos. *Meccanica*, 30(4) :417–424.

[Boas, 2006] Boas, H. P. (2006). Reflections on the Arbelos. *The American Mathematical Monthly*, 113(3) :236–249.

[Brake et al., 2016] Brake, M., Fike, J. A., and Topping, S. D. (2016). Parameterized reduced order models from a single mesh using hyper-dual numbers. *Journal of Sound and Vibration*, 371 :370–392.

[Brannon, 2003] Brannon, R. (2003). Mohr's circle and more circles, Tech. Report, Univ. of Utah.

[Bravyi et al., 2018] Bravyi, S., Gosset, D., and Koenig, R. (2018). Quantum advantage with shallow circuits. *Science*, 362(6412) :308–311.

[Brodsky and Shoham, 1999] Brodsky, V. and Shoham, M. (1999). Dual numbers representation of rigid body dynamics. *Mechanism and machine theory*, 34(5) :693–718.

[Brown, 1967] Brown, R. (1967). On generalized Cayley-Dickson algebras. *Pacific Journal of Mathematics*, 20(3) :415–422.

[Brown and Gray, 1967] Brown, R. B. and Gray, A. (1967). Vector cross products. *Commentarii Mathematici Helvetici*, 42(1) :222–236.

[Capelli, 1941] Capelli, P. F. (1941). Sur le nombre complexe binaire. *Bulletin of the American Mathematical Society*, 47(8) :585–595.

[Cardano, 1545] Cardano, G. (1545). *Ars Magna or the rules of algebra (english translation 1968).* Dover NY.

[Carnot, 1803] Carnot, L. (1803). *Géométrie de position.* Imprimerie de Crapelet.

[Cartan, 1908] Cartan, E. (1908). Nombres complexes. in *Encycl. Sc. Math.,* (J. Molk ed.) d'après l'article allemand de Study, I(1-5) :329–468 (Gauthier–Villars, Paris).

[Cartan and Schouten, 1926] Cartan, É. and Schouten, J. A. (1926). *On riemannian Geometries Admitting an Absolute Parallelism.* Koninklijke Akademie van Wetenschappen te Amsterdam.

[Catto et al., 2016] Catto, S., Gürcan, Y., Khalfan, A., and Kurt, L. (2016). Unifying ancient and modern geometries through octonions. In *Journal of Physics : Conference Series*, volume 670, pages 12–16. IOP Publishing.

[Cayley, 1845] Cayley, A. (1845). On Jacobi's elliptic functions, in reply to the rev. Brice Bronwin ; and on quaternions : To the editors of the Philosophical Magazine and Journal. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 26(172) :208–211.

[Chatelin, 2012a] Chatelin, F. (2012a). *Eigenvalues of Matrices : Revised Edition.* Classics in Applied Mathematics, volume 71. SIAM.

[Chatelin, 2012b] Chatelin, F. (2012b). *Qualitative Computing : a computational journey into nonlinearity.* World Scientific, Singapore.

[Chatelin, 2015] Chatelin, F. (2015). Differential Information Processing in the light of quaternions. Technical report, (CERFACS TR/PA/15/07).

[Chatelin, 2018] Chatelin, F. (2018). *Numbers in Mind : the transformative ways of Multiplication.* Book in preparation to be published by World Scientific.

[Chatelin, 2019] Chatelin, F. (2019). Differential Calculus à la Hamilton and Maxwell in non alternative Dickson algebras (in preparation). Technical report, CERFACS.

[Chatelin et al., 2014] Chatelin, F., Latre, J.-B., Rincon-Camacho, M., and Ricoux, P. (2014). Beyond and behind linear algebra. Technical report, Tech. rep., CERFACS TR/PA/14/80.

[Chatelin and Rincon-Camacho, 2015] Chatelin, F. and Rincon-Camacho, M. M. (2015). Symmetric and hermitian matrices : a geometric perspective on spectral coupling. Technical report, CERFACS TR/PA/15/56.

[Chatelin and Rincon-Camacho, 2017] Chatelin, F. and Rincon-Camacho, M. M. (2017). Hermitian matrices : Spectral coupling, plane geometry/trigonometry and optimisation. *Linear Algebra and its Applications*, 533(Supplement C) :282 – 310.

[Cheng and Thompson, 1996] Cheng, H. and Thompson, S. (1996). Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanism. In *Proc. of ASME 24th. Biennial Mechanisms Conference, Irvine, California*, pages 19–22.

[Clifford, 1873] Clifford, W. K. (1873). Preliminary sketch of bi-quaternions. *Proc. London Math. Soc*, 4(381-395) :157.

[Cockle, 1848] Cockle, J. (1848). III. On Certain Functions Resembling Quaternions, and on a New Imaginary Algebra. *Phil. Mag. (3)*, 33 :435–439.

[Conway and Guy, 1996] Conway, J. H. and Guy, R. K. (1996). *The book of numbers*. Springer Verlag NY.

[Cotton, 2003] Cotton, F. A. (2003). *Chemical applications of group theory*. John Wiley & Sons.

[Coxeter, 1946] Coxeter, H. (1946). Quaternions and reflections. *The American Mathematical Monthly*, 53(3) :136–146.

[Coxeter et al., 1946] Coxeter, H. et al. (1946). Integral Cayley numbers. *Duke Mathematical Journal*, 13(4) :561–578.

[Crowe, 1994] Crowe, M. J. (1994). *A history of vector analysis : The evolution of the idea of a vectorial system*. Dover, NY.

[Daniilidis, 1999] Daniilidis, K. (1999). Hand-eye calibration using dual quaternions. *The International Journal of Robotics Research*, 18(3) :286–298.

[Darpö, 2009] Darpö, E. (2009). Vector product algebras. *Bulletin of the London Mathematical Society*, 41(5) :898–902.

[Deheuvels, 1983] Deheuvels, R. (1983). *Formes quadratiques et groupes classiques*. Presses Universitaires de France.

[DeLoura, 2001] DeLoura, M. A. (2001). *Game Programming : Gems 2*. Cengage learning.

[Dickson, 1919] Dickson, L. E. (1919). On quaternions and their generalization and the history of the eight square theorem. *Annals of Mathematics*, pages 155–171.

[Dickson, 1924] Dickson, L. E. (1924). Algebras and their arithmetics. *Bulletin of the American Mathematical Society*, 30(5-6) :247–257.

[Dimentberg, 1965] Dimentberg, F. (1965). The screw calculus. *(Book in Russian)*.

[Dimentberg, 1968] Dimentberg, F. M. e. (1968). The screw calculus and its applications in mechanics. Technical report, DTIC Document (english translation of 1965 book).

[Dray and Manogue, 2015] Dray, T. and Manogue, C. A. (2015). *The geometry of the octonions*. World Scientific.

[Eakin and Sathaye, 1990] Eakin, P. and Sathaye, A. (1990). On automorphisms and derivations of Cayley-Dickson algebras. *Journal of Algebra*, 129(2) :263–278.

[Ebbinghaus et al., 1998] Ebbinghaus, H., Hermes, H., and Hirzebruch, F. (1998). *Les nombres*. Vuibert, Paris.

[Eberly, 2002] Eberly, D. (2002). Rotation representations and performance issues. *Magic Software : Chapel Hill, NC, USA*.

[Ell and Sangwine, 2007] Ell, T. A. and Sangwine, S. J. (2007). Hypercomplex fourier transforms of color images. *Image Processing, IEEE Transactions on*, 16(1) :22–35.

[Fike, 2013] Fike, J. A. (2013). *Multi-objective optimization using hyper-dual numbers*. PhD thesis, Stanford University.

[Gibbs, 1893] Gibbs, J. W. (1893). Quaternions and vector analysis. *Nature*, 48(1242) :364–367.

[Goldberg, 1991] Goldberg, D. (1991). What every computer scientist should know about floating-point arithmetic. *ACM Computing Surveys (CSUR)*, 23(1) :5–48.

[Gorin, 2006] Gorin, B. (2006). Une étude de l'arbelos, http ://baptiste.gorin.pagesperso-orange.fr/Docs/arbelos.pdf.

[Goursat, 1889] Goursat, E. (1889). Sur les substitutions orthogonales et les divisions régulières de l'espace. *Ann. sci. ecole norm.*, 3 :9–102.

[Graves, 1845] Graves, J. T. (1845). On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 26(173) :315–320.

[Gsponer and Hurni, 2005a] Gsponer, A. and Hurni, J.-P. (2005a). Quaternions in mathematical physics (1) : Alphabetical bibliography. *arXiv preprint math-ph/0510059*.

[Gsponer and Hurni, 2005b] Gsponer, A. and Hurni, J.-P. (2005b). Quaternions in mathematical physics (2) : Analytical bibliography. *arXiv preprint math-ph/0511092*.

[Günaydin et al., 1978] Günaydin, M., Piron, C., and Ruegg, H. (1978). Moufang plane and octonionic quantum mechanics. *Communications in Mathematical Physics*, 61(1) :69–85.

[Gustafson, 2012] Gustafson, K. (2012). *Antieigenvalue Analysis : With Applications to Numerical Analysis, Wavelets, Statistics, Quantum Mechanics, Finance and Optimization*. World Scientific.

[Hadamard, 1893] Hadamard, J. (1893). Résolution d'une question relative aux déterminants. *Bull. des sciences math.*, 2 :240–246.

[Hamilton, 1844] Hamilton, W. R. (1844). On quaternions; or on a new system of imaginaries in algebra. *Philosophical Magazine Series 3*, 25(163) :10–13.

[Hamilton, 1853] Hamilton, W. R. (1853). On the geometrical interpretation of some results obtained by calculation with biquaternions. *Proceedings of the Royal Irish Academy*, 5 :388–390.

[Heaviside, 1892] Heaviside, O. (1892). On the forces, stresses, and fluxes of energy in the electromagnetic field. *Proceedings of the Royal Society of London*, 50(302-307) :126–129.

[Hestenes and Sobczyk, 2012] Hestenes, D. and Sobczyk, G. (2012). *Clifford algebra to geometric calculus : a unified language for mathematics and physics*, volume 5. Springer Science & Business Media.

[Howell and Lafon, 1975] Howell, T. D. and Lafon, J.-C. (1975). The complexity of the quaternion product. Technical report, Cornell University.

[Hurwitz, 1898] Hurwitz, A. (1898). Ueber die composition der quadratischen formen von belibig vielen variablen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1898 :309–316.

[Jacobson, 1958] Jacobson, N. (1958). Composition algebras and their automorphisms. *Rendiconti del Circolo Matematico di Palermo*, 7(1) :55–80.

[Jordan et al., 1934] Jordan, P., von Neumann, J., and Wigner, E. P. (1934). On an algebraic generalization of the quantum mechanical formalism. *Annals of mathematics*, 35(1) :29–64.

[Kantor and Solodovnikov, 1973] Kantor, I. L. and Solodovnikov, A. S. (1973). *Giperkompleksnye chisla (in Russian), Moscow, Nauka. English translation. Hypercomplex numbers : an elementary introduction to algebras, 1989. Springer.*

[Karney, 2007] Karney, C. F. (2007). Quaternions in molecular modeling. *Journal of Molecular Graphics and Modelling*, 25(5) :595–604.

[Karzel and Kist, 1985] Karzel, H. and Kist, G. (1985). Kinematic algebras and their geometries. In *Rings and Geometry*, pages 437–509. Springer.

[Kavan et al., 2008] Kavan, L., Collins, S., Žára, J., and O'Sullivan, C. (2008). Geometric skinning with approximate dual quaternion blending. *ACM Transactions on Graphics (TOG)*, 27(4) :105.

[Khalil and Yiu, 1997] Khalil, S. and Yiu, P. (1997). The Cayley-Dickson algebras, a theorem of A. Hurwitz, and quaternions. *Bull. Soc. Sci. Lett. Lódz Sér. Rech. Déform*, 24 :117–169.

[Khmelnytskaya and Kravchenko, 2009] Khmelnytskaya, K. V. and Kravchenko, V. V. (2009). Biquaternions for analytic and numerical solution of equations of electrodynamics. *arXiv preprint arXiv :0902.3490v1.*

[Kisil, 2012] Kisil, V. V. (2012). *Geometry of Möbius Transformations : Elliptic, Parabolic and Hyperbolic Actions of SL2 [real Number].* World Scientific.

[Klein and Sommerfeld, 1910] Klein, F. and Sommerfeld, A. (1910). Über die theorie des kreisels (4 vols.). *Teubner, Leipzig (1897–1910).*

[Kotelnikov, 1895] Kotelnikov, A. (1895). Screw calculus and some of its applications to geometry and mechanics. *Annals of Imperial University of Kazan.*

[Kravchenko, 2003] Kravchenko, V. V. (2003). *Applied quaternionic analysis*, volume 28. Heldermann.

[Lam, 1999] Lam, T. Y. (1999). *Lectures on ring and modules.* Graduate Texts in Mathematics.

[Lanczos, 1919] Lanczos, C. (1919). The relations of the homogeneous Maxwell's equations to the theory of functions, PhD dissertation translated by A. Gsponer and J.-P. Hurni in 2004. *arXiv preprint physics/0408079.*

[Lantoine et al., 2012] Lantoine, G., Russell, R. P., and Dargent, T. (2012). Using multicomplex variables for automatic computation of high-order derivatives. *ACM Transactions on Mathematical Software (TOMS)*, 38(3) :16.

[Latre, 2013] Latre, J.-B. (2013). Sur quelques structures algébriques utiles au traitement de l'information en robotique, informatique et physique. Technical report, Rapport de Master, 25/03/13-06/09/13, CERFACS WN-PA-13-105.

[Latre, 2017a] Latre, J.-B. (2017a). A comparative study between algebraic structures for a goal-oriented efficiency in Scientific Computing . TOTAL Conference Mathias, Paris 25-27 Oct, 2017.

[Latre, 2017b] Latre, J.-B. (2017b). Spectral coupling : an algebraic perspective running from ancient greek geometry to modern continuum mechanics. 11th International Conference on Clifford Algebras and their applications in mathematical physics, Ghent, 07/08-11/08.

[Latre, 2018] Latre, J.-B. (2018). Qualitative Computing for Quantum Computing. TOTAL Conference Mathias, Paris 22-24 Oct, 2018.

[Latre, 2019] Latre, J.-B. (2019). Multiple factorisations of bicomplex polynomials and their consequences (in preparation). Technical report, CERFACS.

[Latre et al., 2017] Latre, J.-B., Chatelin, F., and Ricoux, P. (2017). Alteralgebras for modelling and computation. SIAM CSE 17, Atlanta (GA), 27/02-03/03.

[Latre et al., 2015] Latre, J.-B., Chatelin, F., Rincon-Camacho, M., and Ricoux, P. (2015). Beyond and behind linear algebra. TOTAL Conference Mathias, Paris 28-30 Oct, 2015.

[Latre et al., 2016] Latre, J.-B., Chatelin, F., Rincon-Camacho, M., and Ricoux, P. (2016). Alternative algebraic structures for Modelling and Computation. TOTAL Conference Mathias, Paris 26-28 Oct, 2016.

[Lavrentiev and Chabat, 1980] Lavrentiev, M. and Chabat, B. (1980). *Effets hydrodynamiques et modèles mathématiques*. Traduit du russe. Mir, Moscou.

[Lounesto, 2001] Lounesto, P. (2001). *Clifford algebras and spinors*, volume 286. Cambridge university press.

[Martins et al., 2003] Martins, J. R., Sturdza, P., and Alonso, J. J. (2003). The complex-step derivative approximation. *ACM Transactions on Mathematical Software (TOMS)*, 29(3) :245–262.

[Maseres, 1758] Maseres, F. (1758). *A Dissertation on the Use of the Negative Sign in Algebra*. Samuel Richardson ; and sold by Thomas Payne.

[Maxwell, 1870] Maxwell, J. C. (1870). On the application of the ideas of the calculus of quaternions to electromagnetic phenomena. *Maxwell's Scientific Papers* (Sir W. Niven ed.), 2 :570–576, Dover Publ. New York (1890).

[Maxwell, 1891] Maxwell, J. C. (1891). *A treatise on electricity and magnetism*. 3^{rd} edition (J.J. Thomson ed.), Dover republication (1954).

[Metzler, 1994] Metzler, W. (1994). The "mystery" of the quadratic mandelbrot set. *American Journal of Physics*, 62 :813–814.

[Moreno, 1998] Moreno, G. (1998). The zero divisors of the Cayley-Dickson algebras over the real numbers. *Boletín de la Sociedad Matemática Mexicana : Tercera Serie*, 4(1) :13–28.

[Moreno, 2005] Moreno, G. (2005). Constructing zero divisors in the higher dimensional Cayley-Dickson algebras. *arXiv preprint math/0512517*.

[Pavlov et al., 2009a] Pavlov, D., Panchelyuga, M., Malykhin, V., and Panchelyuga, V. (2009a). On fractality of Mandelbrot and Julia sets on double-numbers plane. *Hypercomplex numbers in geometry and physics*, 6 :135–145.

[Pavlov et al., 2009b] Pavlov, D., Panchelyuga, M., and Panchelyuga, V. (2009b). About shape of Julia set at zero parameter on double numbers plane. *Hypercomplex numbers geometry and physics*, 6 :146–151.

[Pei et al., 2004] Pei, S.-C., Chang, J.-H., and Ding, J.-J. (2004). Commutative reduced biquaternions and their Fourier transform for signal and image processing applications. *Signal Processing, IEEE Transactions on*, 52(7) :2012–2031.

[Pei et al., 2001] Pei, S.-C., Ding, J.-J., and Chang, J.-H. (2001). Efficient implementation of quaternion Fourier transform, convolution, and correlation by 2-D complex FFT. *Signal Processing, IEEE Transactions on*, 49(11) :2783–2797.

[Pennestrì and Stefanelli, 2007] Pennestrì, E. and Stefanelli, R. (2007). Linear algebra and numerical algorithms using dual numbers. *Multibody System Dynamics*, 18(3) :323–344.

[Pernès, 2003] Pernès, P. (2003). *Éléments de calcul tensoriel : introduction à la mécanique des milieux déformables*. Editions Quae.

[Pham et al., 2010] Pham, H.-L., Perdereau, V., Adorno, B. V., and Fraisse, P. (2010). Position and orientation control of robot manipulators using dual quaternion feedback. In *Intelligent Robots and Systems (IROS), 2010 IEEE/RSJ International Conference on*, pages 658–663. IEEE.

[Piponi, 2004] Piponi, D. (2004). Automatic differentiation, C++ templates, and photogrammetry. *Journal of Graphics Tools*, 9(4) :41–55.

[Poincaré, 1902] Poincaré, H. (1902). *La Science et l'Hypothèse*. Flammarion.

[Price, 1991] Price, G. B. (1991). *An introduction to multicomplex spaces and functions*, volume 140. CRC Press.

[Raffin, 1951] Raffin, R. (1951). Anneaux non associatifs. *Séminaire Dubreil, Algèbre et théorie des nombres*, 4 :11–16.

[Rincon-Camacho and Latre, 2013] Rincon-Camacho, M. M. and Latre, J.-B. (2013). Project description : Algebraic structures for information processing associated with Dickson algebras. Technical report, CERFACS TR/PA/13/106.

[Rodrigues, 1840] Rodrigues, O. (1840). Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace : et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire. *Journal Math. Pures. et App. de Liouville*, 5 :380–440.

[Rosen, 1930] Rosen, N. (1930). Note on the general Lorentz transformation. *Journal of Mathematics and Physics*, 9(1-4) :181–187.

[Schafer, 1995] Schafer, R. (1995). *An Introduction to Nonassociative Algebras (first edition 1966)*. Dover, New York.

[Scheffers, 1893] Scheffers, M. (1893). Sur la généralisation des fonctions analytiques. *CR Acad. Sc*, 116 :1114.

[Schroer, 2003] Schroer, B. (2003). Pascual Jordan, his contributions to quantum mechanics and his legacy in contemporary local quantum physics. *arXiv preprint hep-th/0303241*.

[Segre, 1892] Segre, C. (1892). Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. *Math. Ann*, 40 :413–467.

[Silberstein, 1912] Silberstein, L. (1912). Quaternionic form of relativity. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 23(137) :790–809.

[Sobczyk, 1995] Sobczyk, G. (1995). The hyperbolic number plane. *The College Mathematics Journal*, 26(4) :268–280.

[Study, 1901] Study, E. (1901). *Geometrie der Dynamen.* Teubner, Leipzig.

[Timoshenko, 1983] Timoshenko, S. P. (1983). *History of Strength of Materials.* Dover, New York.

[Truesdell, 1966] Truesdell, C. (1966). *The elements of Continuum Mechanics.* Springer-Verlag, Berlin Heidelberg New York.

[Vaillant et al., 2013] Vaillant, R., Barthe, L., Guennebaud, G., Cani, M.-P., Rohmer, D., Wyvill, B., Gourmel, O., and Paulin, M. (2013). Implicit skinning : Real-time skin deformation with contact modeling. *ACM Transaction on Graphics (TOG). Proceedings of ACM SIGGRAPH.*

[van der Waerden, 1985] van der Waerden, B. L. (1985). *A history of algebra.* Springer.

[van Elfrinkhof, 1897] van Elfrinkhof, L. (1897). Eene eigenschap van de orthogonale substitutie van de vierde orde. In *Handelingen van het zesde Nederlandsch Natuuren Geneeskundig Congres*, pages 237–240.

[Vlasov, 1999] Vlasov, A. Y. (1999). Quantum gates and Clifford algebras. *arXiv preprint quant-ph/9907079.*

[Vlasov, 2001] Vlasov, A. Y. (2001). Clifford algebras and universal sets of quantum gates. *Physical Review A*, 63(5) :054302.

[Wedderburn, 1908] Wedderburn, J. (1908). On hypercomplex numbers. *Proceedings of the London Mathematical Society*, 2(1) :77–118.

[Weil, 1984] Weil, A. (1984). *Number theory, an approach through history from Hammurapi to Legendre.* Birkhauser.

[Weiner and Wilkens, 2005] Weiner, J. L. and Wilkens, G. R. (2005). Quaternions and rotations in $\mathbb{E}^4$. *The American Mathematical Monthly*, 112(1) :69–76.

[Wessel, 1799] Wessel, Caspar, . (1799). *On the analytical representation of direction : an attempt applied chiefly to solving plane and spherical polygons.* Memoir Royal Acad. Denmark.

[Wigner, 1960] Wigner, E. (1960). The unreasonable effectiveness of Mathematics in the natural sciences. *Comm. Pure and Appl. Math*, 13 :1–14.

[Wilson et al., 1980] Wilson, E. B., Decius, J. C., and Cross, P. C. (1980). *Molecular vibrations : the theory of infrared and Raman vibrational spectra.* Courier Corporation.

[Wilson and Gibbs, 1902] Wilson, E. B. and Gibbs, J. W. (1902). *Vector analysis : a text-book for the use of students of mathematics and physics ; founded upon the lectures of J. Willard Gibbs.* Scribner's Sons.

[Wu et al., 2006] Wu, Y., Hu, X., Wu, M., and Hu, D. (2006). Strapdown inertial navigation using dual quaternion algebra : error analysis. *Aerospace and Electronic Systems, IEEE Transactions on*, 42(1) :259–266.

[Yaglom, 1963] Yaglom, I. (1963). Kompleksnye chisla i ikh primenenie v geometrii (in russian) Engl. transl. Complex numbers in geometry, 1968 Academic Press, New York.

[Zorn, 1931] Zorn, M. (1931). Theorie der alternativen ringe. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 8(1) :123–147.