On the evolution by duality of domains on manifolds
Sur l'évolution par dualité de domaines dans des variétés

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Abstract

On a manifold, consider an elliptic diffusion X admitting an invariant measure μ. The goal of this paper is to introduce and investigate the first properties of stochastic domain evolutions $(D_t)_{t\in[0,\tau]}$ which are intertwining dual processes for X (where τ is an appropriate positive stopping time before the potential emergence of singularities). They provide an extension of Pitman's theorem, as it turns out that $(\mu(D_t))_{t\in[0,\tau]}$ is a Bessel-3 process, up to a natural time-change. When X is a Brownian motion on a Riemannian manifold, the dual domain-valued process is a stochastic modification of the mean curvature flow to which is added an isoperimetric ratio drift to prevent it from collapsing into singletons.

Résumé

Sur une variété, considérons une diffusion elliptique X de mesure invariante μ. Le but de ce papier est d'introduire et d'étudier les premières propriétés d'évolutions stochastiques de domaines $(D_t)_{t\in[0,\tau]}$ qui sont des processus duaux par entrelacement de X (où τ est un temps d'arrêt strictement positif précédant l'apparition éventuelle de singularités). Il s'agit d'une extension du théorème de Pitman, puisqu'il ressort que $(\mu(D_t))_{t\in[0,\tau]}$ est un processus de Bessel-3, à un changement naturel de temps près. Quand X est un mouvement brownien sur une variété compacte, ce processus dual à valeurs domaines est une modification stochastique du flot par courbure moyenne auquel est ajouté une dérive fournie par un quotient isopérimétrique qui l'empêche de s'effondrer en des singletons.

Keywords: Elliptic diffusions, Riemannian manifolds, stochastic domain dynamics, duality by intertwining, Bessel processes, modified mean curvature flows, constant curvature spaces.
Mots clés : Diffusions elliptiques, variétés riemanniennes, dynamiques stochastiques de domaines, dualité par entrelacement, processus de Bessel, flots par courbure moyenne modifiés, espaces à courbure constante.

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1 Introduction

In the finite state space framework, Diaconis and Fill [5] have shown that ergodic Markov chains can be intertwined with Markov chains living on the set of non-empty subsets of the state space and ending up being absorbed at the full state space. This result enabled them to construct strong stationary times for ergodic Markov chains, leading to quantitative bounds on their convergence to equilibrium, in the separation discrepancy and in the total variation distance. In [19], this point of view was extended to real ergodic diffusion processes, but the one-dimensionality seemed crucial in the method. As noted in this previous paper, it is quite unfortunate, since otherwise it could lead to a new probabilistic approach to the hypoellipticity theorem of Hörmander [11]. Here we make an important step further in this program, by showing that elliptic diffusions on differential manifolds admitting an invariant measure can indeed be intertwined with domain-valued Markov processes. Although the hypoellipticity is not yet entering in the game (but see [18] for a first illustration in dimension 1), the introduced domain-valued processes are already very intriguing and promising for themselves. When dealing with the Brownian motion on a Riemannian manifold, they are natural stochastic modifications of the mean curvature flow. In the more general case, when a gradient drift is added to the Brownian motion, one has to consider some weighted extensions.

Let L be an elliptic diffusion generator on a differentiable manifold V. Here we will not be interested in regularity problems, so V and the coefficients of L are supposed to be smooth. Assume there exists a σ-finite measure μ on V which is invariant for L in the sense that

$$\forall\ f \in \mathcal{C}_{\mathrm{c}}^{\infty}(V), \qquad \mu[L[f]] \;=\; 0$$

where $\mathcal{C}_{\mathrm{c}}^{\infty}(V)$ stands for the space of smooth functions on V with compact support. By ellipticity, the measure μ admits a positive density with respect to the Riemannian measure. Note that in general μ is not unique, even up to a positive factor, e.g. for the generator $\partial^2 + \partial$ on $\mathbb{R}$, all the measures with a density of the form $\mathbb{R} \ni x \mapsto a + b\exp(x)$, with $a, b \geqslant 0$, are invariant. But there is at most one finite invariant measure and in this case it is usual to normalize μ into a probability measure.

Let $\mathcal{D}$ be the set of non-empty, compact and connected domains D in V, which coincide with the closure of their interior and whose boundary $C := \partial D$ is smooth. Denote also $\underline{\mathcal{D}} := \mathcal{D} \sqcup \{\{x\} : x \in V\}$, obtained by adjunction of all the singletons to $\mathcal{D}$, and $\overline{\mathcal{D}}$ the set of all measurable subsets D of V which either satisfy $\mu(V) \in (0, +\infty)$ or are singletons (so that $\mathcal{D} \subset \underline{\mathcal{D}} \subset \overline{\mathcal{D}}$). Consider the Markov kernel Λ from $\overline{\mathcal{D}}$ to V given by

$$\forall\ D \in \overline{\mathcal{D}}, \forall\ A \in \mathcal{B}(V), \qquad \Lambda(D, A) \;:=\; \begin{cases} \dfrac{\mu(A \cap D)}{\mu(D)} & \text{, if } \mu(D) > 0 \\ \delta_x(A) & \text{, if } D = \{x\}, \text{ with } x \in V \end{cases} \tag{1}$$

where $\mathcal{B}(V)$ is the set of measurable subsets of V and δ_x the Dirac mass at x. As usual, such an integral kernel can be seen as an operator transforming bounded (respectively positive) measurable functions on V into finite-valued (resp. $(0, +\infty]$-valued) functions on $\overline{\mathcal{D}}$.

The main goal of this paper is to find a Markov generator $\mathfrak{L}$ with state space $\mathcal{D}$ satisfying, in an appropriate sense, the **intertwining relation**

$$\mathfrak{L}\Lambda \;=\; \Lambda L \tag{2}$$

and for which the singletons are entrance boundaries.

Remark 1 This was done in [19] when $V = \mathbb{R}$ and when $-\infty$ and $+\infty$ were entrance boundaries for L. The latter assumption was needed to insure that the resulting Markov processes on the set of the closed segments (which were not assumed to be compact in [19]) end up being absorbed at the whole state space $\mathbb{R}$, because we were primarily interested in constructing strong stationary times. This is no longer our objective here (even if we should come back to this question in a future work), that is why no assumption is made on the behavior of L at infinity.

Note also that in general there is not a unique Markov generator satisfying the above requirements, since in [19] we constructed a whole family of such operators when $V = \mathbb{R}$. Nevertheless, among them, one was the fastest to be absorbed at $\mathbb{R}$, it is a generalization of this Markov generator that will be considered below.

□

As a consequence of the previous remark, from now on, we assume that the dimension of V is larger or equal to 2. To describe our candidate $\mathfrak{L}$, we need to introduce some notations.

By using the inverse of the matrix diffusion of L to induce a Riemannian structure on V (see e.g. the book [13] of Ikeda and Watanabe for the details), L can be decomposed as $L = \triangle + b$, where $\triangle$ is the **Laplacian operator** associated to the Riemannian structure and b is a vector field (seen as a first order differential operator). We assume that V is complete, endowed with the **Riemannian distance** d. Let λ be the **Riemannian measure** on V. It is well-known that μ is absolutely continuous with respect to λ and that its density is smooth. Let us write $U := \ln(d\mu/d\lambda) \in \mathcal{C}^\infty(V)$ (a priori defined up to an additive constant, except when μ is normalized into a probability measure). The vector field b can written as $\nabla U + \beta$, with the vector field β satisfying $\mathrm{div}(\exp(U)\beta) = 0$; it corresponds to the μ-weighted Hodge decomposition of b. In the previous sentence, ∇ and $\mathrm{div}(\cdot)$ are the **gradient** and **divergence** operators associated to the Riemannian structure. Other Riemannian notions that will be useful for our purpose are the **scalar product** $\langle \cdot, \cdot \rangle$, as well as the **exterior normal vector** ν_C, the "**mean**" **curvature** ρ_C and the $(\dim(V) - 1)$-**Hausdorff measure** σ_C, all the last three objects being defined on the boundary C of an element $D \in \mathcal{D}$. The mean curvature is signed with respect to our choice of the orientation of ν_C and it is not really a mean, since it is the trace (without renormalization) of the second fundamental form. A priori the orientation of ν_C and the sign of ρ_C require to know on which side of C is the interior of D (except when V is not compact, then the mapping $\mathcal{D} \ni D \mapsto C$ is one-to-one, otherwise it is two-to-one), but $\rho_C \nu_C$ depends only on C.

Let us first describe heuristically the type of stochastic evolution $(D_t)_{t \in [0,\tau)}$ in $\mathcal{D}$ we want to consider. The positive stopping time τ is earlier than the exit time from $\mathcal{D}$, typically due to the apparition of singularities on the boundary $C_t := \partial D_t$. We want, as long as $t \in [0, \tau)$, the infinitesimal evolution of any $Y_t \in C_t$ to be given by

$$dY_t \;=\; \left(\sqrt{2}dB_t + \left(2\frac{\sigma_{C_t}(\exp(U))}{\mu(D_t)} + \langle \beta - \nabla U, \nu_{C_t} \rangle (Y_t) - \rho_{C_t}(Y_t)\right) dt\right) \nu_{C_t}(Y_t) \tag{3}$$

where $B := (B_t)_{t \geqslant 0}$ is a standard real Brownian motion. The evolution (3) can be seen as a deterministic and stochastic modification of the mean curvature flow, which corresponds to

$$dy_t \;=\; -\rho_{C_t}(y_t)\nu_{C_t}(y_t)\, dt$$

for the points y_t on the evolving boundary.

The global term $\sigma_{C_t}(\exp(U))/\mu(D_t)$ (it does not depend on the position of Y_t on C_t) in (3) can be seen as an isoperimetric ratio with respect to μ. Indeed, it can be rewritten as $\underline{\mu}(C_t)/\mu(D_t)$, where $\underline{\mu}$ is the $(\dim(V) - 1)$-dimensional measure on C_t admitting $\exp(U)$ as density with respect to σ_{C_t}. So this term explodes as D_t becomes closer and closer to a point. In some sense, it will compensate the trend of the mean curvature flow on compact boundaries to make them smaller and smaller (and rounder and rounder). Though too qualitative to be convincing, this observation is a first hint of why the singletons will be entrance boundaries for the Markov processes determined by (3).

The term $\langle\beta,\nu_{C_t}\rangle(Y_t)\nu_{C_t}(Y_t)$ in (3) could be replaced by $\beta(Y_t)$, since the tangential components in the description of the evolution of the points on boundary can be removed, up to a diffeomorphism of C_t (see e.g. Section 1.3 of Mantegazza [17]). Only the radial component (i.e. the projection on the normal vectors ν_{C_t}) is important, thus an equation such as (3) will be said to be **radial**.

In fact, the radial stochastic differential equation (3) of the points on the boundary is not the most convenient way to work with the process $(D_t)_{t\in[0,\tau)}$. In Markov theory, the martingale problem approach is usually more helpful (for a general introduction and an extensive development of this notion, cf. for instance the book of Ethier and Kurtz [7]). It needs convenient observables on the state space. On $\mathcal{D}$, the role of **elementary observables** is played by the mappings

$$F_f \ : \ \mathcal{D} \ni D \ \ \mapsto \ \ F_f(D) := \int_D f\, d\mu \tag{4}$$

associated to the functions $f \in \mathcal{C}^\infty(V)$, the space of smooth mappings on V.

To proceed in the direction of the definition of the generator $\mathfrak{L}$ on an appropriate algebra $\mathfrak{D}$ of functionals defined on $\mathcal{D}$, we begin by defining the action of $\mathfrak{L}$ on the above elementary observables: for any $f \in \mathcal{C}^\infty(V)$,

$$\forall\ D \in \mathcal{D}, \qquad \mathfrak{L}[F_f](D) \ := \ \int_D L[f]\, d\mu + 2\frac{\mu(C)}{\mu(D)}\int_C f\, d\underline{\mu} \tag{5}$$

Using Stokes formula, we will check in Section 3 that the above r.h.s. can be written as an integral over C only:

$$\forall\ D \in \mathcal{D}, \qquad \mathfrak{L}[F_f](D) \ = \ \int_C \langle \nabla f, \nu_C\rangle + \left(2\frac{\mu(C)}{\mu(D)} + \langle\beta,\nu_C\rangle\right) f\, d\underline{\mu} \tag{6}$$

Furthermore, we introduce a bilinear form $\Gamma_{\mathfrak{L}}$ (which will be the **carré du champs** associated to $\mathfrak{L}$) on such functionals, via

$$\forall\ f,g \in \mathcal{C}^\infty(V),\ \forall\ D \in \mathcal{D}, \qquad \Gamma_{\mathfrak{L}}[F_f,F_g](D) \ := \ \left(\int_C f\, d\underline{\mu}\right)\left(\int_C g\, d\underline{\mu}\right) \tag{7}$$

Since the $\mathcal{D}$-valued Markov processes we are interested in will have continuous sample paths (namely they will be **diffusions**), we are naturally led to the following definitions (see e.g. the book of Bakry, Gentil and Ledoux [3]). Consider $\mathfrak{D}$ the algebra consisting of the functionals of the form $\mathfrak{F} := \mathfrak{f}(F_{f_1},...,F_{f_n})$, where $n \in \mathbb{Z}_+$, $f_1,...,f_n \in \mathcal{C}^\infty(V)$ and $\mathfrak{f} : \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\mathcal{D}$ by $(F_{f_1},...,F_{f_n})$. For such a functional $\mathfrak{F}$, define

$$\mathfrak{L}[\mathfrak{F}] \ := \ \sum_{j\in[\![1,n]\!]} \partial_j\mathfrak{f}(F_{f_1},...,F_{f_n})\mathfrak{L}[F_{f_j}] + \sum_{k,l\in[\![1,n]\!]} \partial_{k,l}\mathfrak{f}(F_{f_1},...,F_{f_n})\Gamma_{\mathfrak{L}}[F_{f_k},F_{f_l}] \tag{8}$$

To two elements of $\mathfrak{D}$, $\mathfrak{F} := \mathfrak{f}(F_{f_1},...,F_{f_n})$ and $\mathfrak{G} := \mathfrak{g}(F_{g_1},...,F_{g_m})$, we also associate

$$\Gamma_{\mathfrak{L}}[\mathfrak{F},\mathfrak{G}] \ := \ \sum_{l\in[\![n]\!],k\in[\![m]\!]} \partial_l\mathfrak{f}(F_{f_1},...,F_{f_n})\partial_k\mathfrak{g}(F_{g_1},...,F_{g_m})\Gamma_{\mathfrak{L}}[F_{f_l},F_{g_k}] \tag{9}$$

Remark 2 A priori the above definitions are ambiguous, since they seem to depend on the writing of $\mathfrak{F} \in \mathfrak{D}$ under the form $\mathfrak{f}(F_{f_1},...,F_{f_n})$ and similarly for $\mathfrak{G}$. To see that they are indeed well-defined, note that

$$\forall\ \mathfrak{F},\mathfrak{G} \in \mathfrak{D}, \qquad \Gamma_{\mathfrak{L}}[\mathfrak{F},\mathfrak{G}] \ = \ \frac{1}{2}\left(\mathfrak{L}[\mathfrak{F}\mathfrak{G}] - \mathfrak{F}\mathfrak{L}[\mathfrak{G}] - \mathfrak{G}\mathfrak{L}[\mathfrak{F}]\right)$$

This property implies that if $\mathfrak{f}$ is a polynomial in n variables, then for any $\mathfrak{F} := \mathfrak{f}(f_1, ..., f_n)$, with $f_1, ..., f_n \in \mathcal{C}^\infty(V)$, the object $\mathfrak{L}[\mathfrak{F}]$ is uniquely defined. Indeed, it relies on an iteration on the degree of $\mathfrak{f}$, starting from (6) and (7). The general case of smooth functions $\mathfrak{f}$ is deduced from their approximation over compact domains by polynomial mappings.

□

Let us come back to the Markov operator Λ defined in (1). For any $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, $\Lambda[f]$ is an element of $\mathfrak{D}$, since it can be written

$$\forall\ D \in \mathcal{D}, \qquad \Lambda[f](D) \;=\; \frac{F_f}{F_{\mathbb{1}}}(D)$$

where $\mathbb{1}$ is the constant function taking the value 1. This relation also leads us to endow $\mathcal{D}$ with the σ-algebra generated by the mappings F_f, for $f \in \mathcal{C}^\infty(V)$, so that Λ is really a Markov kernel from $\mathcal{D}$ to V: for any fixed $A \in \mathcal{B}(V)$, the mapping $\mathcal{D} \ni D \mapsto \Lambda(D, A)$ is measurable. For this mapping to be measurable on $\underline{\mathcal{D}}$, put on the set $\{\delta_x : x \in V\}$ the σ-algebra obtained by identifying it with V (seeing δ_x as x) and consider on $\underline{\mathcal{D}}$ the σ-algebra generated by those on $\mathcal{D}$ and on $\{\delta_x : x \in V\}$. Since we already mentioned continuity of trajectories, we must also endow $\underline{\mathcal{D}}$ with a topology. The simplest way to do so would be to consider the smallest topology such that all the mappings F_f, for $f \in \mathcal{C}^\infty(V)$, are continuous (with the natural extension that the F_f vanish on the singletons). But for our purpose, we will need a stronger topology making continuous the following functionals, for any $f \in \mathcal{C}^\infty(V)$:

$$\underline{\mathcal{D}} \ni D \;\mapsto\; \Lambda[F_f](D) \tag{10}$$

$$\underline{\mathcal{D}} \ni D \;\mapsto\; \int_C f\, d\underline{\mu} \tag{11}$$

with the convention that if D is a singleton, then $C = \emptyset$ (so that the latter r.h.s. is 0). Condition (10) enables us to topologically identify $\{\delta_x : x \in V\}$ with V. The topology on $\mathcal{D}$ will be such that the σ-algebra put on $\underline{\mathcal{D}}$ is the Borelian one. Condition (11) implies that for any $f \in \mathcal{C}^\infty(V)$, $\mathfrak{L}[F_f]$ is continuous on $\mathcal{D}$. For the precise definition of this topology, see Section 3, where $\mathcal{D}$ will furthermore be endowed with an infinite-dimensional differential structure.

After these structural precisions, let us come back to $\mathfrak{L}$, whose main interest is to fulfill our goal (2):

Theorem 3 *For any $f \in \mathcal{C}^\infty(V)$, we have*

$$\forall\ D \in \mathcal{D}, \qquad \mathfrak{L}[\Lambda[f]](D) \;=\; \Lambda[L[f]](D)$$

To go further, we want to construct Markov processes whose generator is $\mathfrak{L}$ and to establish a link with (3).

Let be given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, all subsequent notions from stochastic process theory will be relative to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Consider a stopped continuous and adapted stochastic process $(D_t)_{t\in[0,\tau)}$, taking values in $\underline{\mathcal{D}}$ and where τ is a positive stopping time. It is said to be a solution to **the martingale problem** associated to $(\mathfrak{D}, \mathfrak{L})$, if for all $t \in (0, \tau)$, $D_t \in \mathcal{D}$ and if for any $\mathfrak{F} \in \mathfrak{D}$, the process $M^{\mathfrak{F}} := (M_t^{\mathfrak{F}})_{t\in[0,\tau)}$ defined by

$$\forall\ t \in [0, \tau), \qquad M_t^{\mathfrak{F}} \;:=\; \mathfrak{F}(D_t) - \mathfrak{F}(D_0) - \int_0^t \mathfrak{L}[\mathfrak{F}](D_s)\, ds$$

is a local martingale. More precisely, in this situation we say that $(D_t)_{t\in[0,\tau)}$ is a solution to the martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ and to the initial distribution $\mathcal{L}(D_0)$, the law of D_0, or starting from $D_0 \in \underline{\mathcal{D}}$, when $\mathcal{L}(D_0)$ is a Dirac mass.

One key to the following result is the adaptation of the Doss [6] and Sussman [28] method to the infinite dimensional stochastic differential equation (3).

Theorem 4 *For any $D_0 \in \mathcal{D}$, there is a solution to the martingale problem associated to $(\mathfrak{D}, \mathfrak{L})$ starting from D_0.*

In certain homogeneous spaces, it is possible to start from singletons, because these situations can be brought back to the 1-dimensional setting treated in [19]. Indeed, the processes $(D_t)_{t\geqslant 0}$ end up being balls centered at the point from the initial singleton and it is sufficient to study the evolution of the radius. This is the case of the Laplacian operator on Euclidean, hyperbolic and spheric spaces. The stopping time τ is infinite in the two former situations and corresponds to the hitting time of the whole sphere in the latter one. But in general to consider $\mathcal{D}$ as state space is probably too restrictive. We believe there exists a set $\mathcal{G}$ of subdomains of V, with $\mathcal{D} \subset \mathcal{G} \subset \overline{\mathcal{D}}$, such that $\mathfrak{L}$ can be naturally extended to $\mathcal{G}$, in particular one should be able to define $\underline{\mu}$ and $\nu_{\partial D}$, $\underline{\mu}$-a.e. Heuristically, the set of singular points of the boundary of a domain from $\mathcal{G}$ should be very small. We hope to investigate this question in a future work via the geometric measure theory, but for the moment being, let us assume that we are given such a set $\mathcal{G}$ with Theorem 4 holding up to a positive stopping time earlier than the exit time of $\mathcal{G}$. Still denote by $(D_t)_{t\in[0,\tau)}$ the corresponding Markov processes. Consider

$$\varsigma \;\coloneqq\; 2\int_0^\tau (\underline{\mu}(C_s))^2\, ds \;\in (0, +\infty] \tag{12}$$

and the time change $(\theta_t)_{t\in[0,\varsigma]}$ defined by

$$\forall\, t \in [0, \varsigma], \qquad 2\int_0^{\theta_t} (\underline{\mu}(C_s))^2\, ds \;=\; t. \tag{13}$$

Theorem 5 *The process $(\mu(D_{\theta_{t\wedge\varsigma}}))_{t\geqslant 0}$ is a (possibly stopped) Bessel process of dimension 3.*

By taking into account that 0 is an entrance boundary for the Bessel process of dimension 3, a consequence of Theorem 5 is that the set of singletons is an entrance boundary for the Markov processes associated to $(\mathfrak{D}, \mathfrak{L})$, if we were able to extend Theorem 4 to initial conditions that are singletons. Theorem 5 can be seen as a multidimensional extension of the intertwining relation between the real standard Brownian motion and the Bessel process of dimension 3 by Pitman [25]: it corresponds to (2) when L is the Laplacian on $\mathbb{R}$ (see also Remark 37 in [19]).

Up to now, we did not consider the Markov processes associated to L, whereas their study is the first motivation for the above developments. The martingale problems associated to $(\mathcal{C}^\infty(V), L)$ are well-posed (see e.g. the book of Ikeda and Watanabe [13]), so to any initial distribution on V, we can associate a stopped Markov process $(X_t)_{t\in[0,\tau)}$ where τ is the explosion time (maybe infinite). The conjunction of Theorems 3 and 4 should lead to the following result, which is the reason behind our interest in the relation (2):

Conjecture 6 Assume that the martingale problems associated to $(\mathcal{C}^\infty(V), L)$ are well-posed and defined for all times (no explosion). Let $x_0 \in V$ be given and let $X \coloneqq (X_t)_{t\geqslant 0}$ be a solution starting from $x_0 \in V$ for the martingale problems associated to $(\mathcal{C}^\infty(V), L)$. Up to enlarging the underlying probability space, it is possible to couple the trajectory $(X_t)_{t\geqslant 0}$ with a solution $(D_t)_{t\in[0,\tau]}$ starting from the singleton $\{x_0\}$ to the martingale problem associated to $(\mathfrak{D}, \mathfrak{L})$, such that for any stopping time T with $T \leqslant \tau$, we have for the conditional laws:

$$\mathcal{L}(D_{[0,T]}|X) \;=\; \mathcal{L}(D_{[0,T]}|X_{[0,T]}) \tag{14}$$

$$\mathcal{L}(X_T|D_{[0,T]}) \;=\; \Lambda(D_T, \cdot) \tag{15}$$

□

The difficulty behind the proof of such a result is technical, since conceptually it is an immediate extension of the ideas of Diaconis and Fill [5] in the context of finite Markov chains. Two different approaches to such couplings for diffusions via coalescing stochastic flows have been proposed in Machida [16] and [21], but they would need to be developed further to deal with the generality of our present framework. A related point of view is currently under construction in [2]. Note that Conjecture 6 would enable us to come back to our initial motivation, first by recovering the density theorem for elliptic diffusions:

Corollary 7 *Assume that a coupling of $(X_t)_{t\geqslant 0}$ with $(D_t)_{t\in[0,\tau]}$ can be constructed as in Conjecture 6. Then for any $t > 0$, the restriction to V of the law of X_t is absolutely continuous with respect to the Riemannian measure λ.*

To obtain this result, only the existence of a domain-valued dual process is needed (as well as its coupling with the process X), its uniqueness is irrelevant. The well-posedness of the martingale problems associated to $(\mathfrak{D}, \mathfrak{L})$ is not crucial for this kind of consideration, more important for us would be the possibility for the dual process to start from singletons.

Another interesting stochastic domain evolution is obtained by removing the isoperimetric ratio from the generator, namely corresponding to the generator $(\mathfrak{D}, \widetilde{\mathfrak{L}})$, where (5) is replaced by

$$\forall\ D \in \mathcal{D}, \qquad \widetilde{\mathfrak{L}}[F_f](D) \ := \ \int_D L[f]\, d\mu \tag{16}$$

for its action on the elementary observables (but (7), (8) and (9) remain unchanged). The associated Markov processes are the analogues of the evolving sets considered by Morris and Peres [22] in discrete settings. One downside of the processes $(\widetilde{D}_t)_{t\in[0,\tau)}$ associated to the generator $(\mathfrak{D}, \widetilde{\mathfrak{L}})$ is that they have a strong tendency to collapse in singletons in finite time and they remain singletons when starting from a singleton. The heuristic reason behind this collapse is that $(\mu(\widetilde{D}_t))_{t\in[0,\tau)}$ is a non-negative martingale, due to $\widetilde{\mathfrak{L}}[F_{\mathbb{1}}] = 0$. Thus, assuming for instance that $\tau = +\infty$, $(\mu(\widetilde{D}_t))_{t\in[0,+\infty)}$ must converge in large time, as well as its bracket. It follows that $\liminf_{t\to+\infty} \underline{\mu}(\widetilde{C}_t) = 0$ and appropriate geometric assumptions will enable to conclude that $\widetilde{D}_t$ becomes closer and closer to a singleton, at least along a sequence of diverging times (in the same spirit, an isoperimetric-type inequality between $\underline{\mu}$ and μ will imply that $\lim_{t\to+\infty} \mu(\widetilde{D}_t) = 0$). The convergence toward a singleton can be checked rigorously when starting from a ball in the constant curvature framework of the next section. In fact, taking into account the general theory of Doob transforms (with respect to the mappings $\underline{\mathcal{D}} \ni D \mapsto \mu(D)$), the processes $(D_t)_{t\in[0,\tau)}$ correspond to the process $(\widetilde{D}_t)_{t\in[0,\tau)}$ conditioned not to hit the set of singletons, or more precisely, conditioned so that $(\mu(\widetilde{D}_t))_{t\in[0,\tau)}$ does not hit zero. This property gives an understanding of the emergence of the Bessel-3 process in Theorem 5, seen as the Brownian motion conditioned not to hit 0 (see also the observations at the end of Section 7).

The plan of the paper is as follows. In the next section, we will deal with the simple but illustrative situation of the Euclidean, spheric and hyperbolic Brownian motion starting from a point. In Section 3 we prove Theorem 3 and Theorem 4. Section 4 presents a result on the existence of stochastic modified mean curvature flows, which was required by the proof of Theorem 4. Section 5 comes back to the homogeneous situations of Section 2, pursuing further some computations relative to the mean curvature addressed in Section 3. It will also show some critical differences between two ways of applying the Doss-Sussman method in these homogeneous geometric frameworks. In Section 6, Theorem 5 is proved as well as other properties of the solutions to the martingale problems associated to $(\mathfrak{D}, \mathfrak{L})$. In particular, we will see that if the evolution $(D_t)_{t\geqslant 0}$ is defined for all times, relatively to the usual

Laplacian $L = \triangle$ on the plane, then renormalizing the domains so that their areas is brought back to 1, we get a convergence in large time toward the disk centered at 0 of radius $1/\sqrt{\pi}$. An appendix provides supplementary informations on product situations and alternative dual processes (on domains whose boundaries are naturally non smooth).

2 Homogeneous situations

There are examples where the radial evolution equation (3) can be globally solved by coming back to the one-dimensional situation as it is treated in [19] (see also Fill and Lyzinski [9]). They correspond to spaces V with constant curvature endowed with the Laplacian $\triangle$ and we take $\mu = \lambda$ and $\underline{\mu} = \sigma_C$ (denoted σ, to simplify), for $C = \partial D$ and $D \in \mathcal{D}$, with the notation of the introduction. For them, we investigate solutions $(D_t)_{t\geqslant 0}$ of the form $(B(\mathbf{0}, R_t))_{t\geqslant 0}$, where $\mathbf{0}$ is any fixed point of the state space, $B(\mathbf{0}, r)$ is the closed ball centered at $\mathbf{0}$ of radius $r \geqslant 0$ and $(R_t)_{t\geqslant 0}$ is a $\mathbb{R}_+$-valued diffusion process starting from 0. We will describe separately the three situations of null, negative and positive constant curvature spaces.

2.1 Euclidean spaces

We consider here the Euclidean space $\mathbb{R}^n$, with $n \in \mathbb{N}\backslash\{1\}$. Without loss of generality, we can assume that $\mathbf{0}$ is the point zero from $\mathbb{R}^n$. For $r > 0$, the Lebesgue volume of $B(\mathbf{0}, r)$ is $\lambda(B(\mathbf{0}, r)) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n$ and the corresponding hypersurface volume of the sphere $\partial B(\mathbf{0}, r)$ is $\sigma(\partial B(\mathbf{0}, r)) = n\frac{\pi^{n/2}}{\Gamma(n/2+1)} r^{n-1}$. The mean curvature of any element $x \in \partial B(\mathbf{0}, r)$ is $\rho(x) = (n-1)/r$. Thus a solution $(B(\mathbf{0}, R_t))_{t\geqslant 0}$ of the radial evolution equation (3), is given by

$$\begin{aligned} dR_t &= \sqrt{2}dB_t + \left(\frac{2n}{R_t} - \frac{n-1}{R_t}\right) dt \\ &= \sqrt{2}dB_t + \frac{n+1}{R_t} dt \end{aligned}$$

where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion. Thus $(R_{t/2})_{t\geqslant 0}$ has for generator the operator A given by

$$\forall\ f \in \mathcal{C}^\infty(\mathbb{R}_+),\ \forall\ x \in \mathbb{R}_+, \qquad A[f](x) \ :=\ \frac{1}{2} f''(x) + \frac{n+1}{2x} f'(x)$$

(in the sequel such a generator will be denoted $\frac{1}{2}\partial^2 + \frac{n+1}{2x}\partial$), namely it is a Bessel process of dimension $n+2$. In particular 0 is an entrance boundary for $(R_t)_{t\geqslant 0}$ and we can make it start from 0, i.e. we can let $(B(\mathbf{0}, R_t))_{t\geqslant 0}$ start from $\{\mathbf{0}\}$.

Let us check directly that $(\lambda(B(\mathbf{0}, R_t)))_{t\geqslant 0}$ is a Bessel of process of dimension 3, up to a time change, as announced in Theorem 5. It is sufficient to show that the same is true for $(R_t^n)_{t\geqslant 0}$. We compute

$$\begin{aligned} dR_t^n &= nR_t^{n-1}\left(\sqrt{2}dB_t + \frac{n+1}{R_t}dt\right) + 2\frac{n(n-1)}{2}R_t^{n-2}dt \\ &= \sqrt{2}nR_t^{n-1}dB_t + 2n^2R_t^{n-2}dt \end{aligned}$$

So the generator of $(R_t^n)_{t\geqslant 0}$ is $2n^2x^{2-2/n}[\frac{1}{2}\partial^2 + \frac{1}{x}\partial]$. It follows that $(R_{\theta_t}^n)_{t\geqslant 0}$ is a Bessel process of dimension 3, where the time change $(\theta_t)_{t\geqslant 0}$ is defined by

$$\forall\ t \geqslant 0, \qquad \int_0^{\theta_t} R_s^{2-2n}\, ds \ =\ 2n^2 t$$

2.2 Spherical spaces

We consider now the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, with $n \in \mathbb{N}$. Without loss of generality, we can assume that $\mathbf{0}$ is the point $(1, 0, 0, .., 0)$ from $\mathbb{R}^{n+1}$. For any $r \in [0, \pi]$, $B(\mathbf{0}, r)$ is the closed cap centered at $\mathbf{0}$ of radius r. In particular, we have $B(\mathbf{0}, 0) = \{\mathbf{0}\}$ and $B(\mathbf{0}, \pi) = \mathbb{S}^n$. Let λ be the uniform distribution on $\mathbb{S}^n$ and σ be the corresponding hypersurface volume. The projection of λ on the first coordinate of $\mathbb{R}^{n+1}$ is the measure $Z_n^{-1}(1 - x^2)^{n/2-1}\mathbb{1}_{[-1,1]}(x)\, dx$, where the renormalising factor is given by the Wallis integral

$$\begin{aligned} Z_n &= \int_{-1}^{1} (1 - x^2)^{n/2-1}\, dx \\ &= \int_0^{\pi} \sin^{n-1}(u)\, du \\ &= \sqrt{\pi} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} \end{aligned}$$

The cap $B(\mathbf{0}, r)$ is exactly the set of elements of $\mathbb{S}^n$ whose first coordinate belongs to $[\cos(r), 1]$. So we get

$$\begin{aligned} \lambda(B(\mathbf{0}, r)) &= Z_n^{-1} I(r) \;\coloneqq\; Z_n^{-1} \int_0^r \sin^{n-1}(u)\, du \\ \sigma(\partial B(\mathbf{0}, r)) &= Z_n^{-1} \sin^{n-1}(r) \end{aligned}$$

The mean curvature of any element $x \in \partial B(\mathbf{0}, r)$ is $\rho(x) = (n - 1)\cot(r)$. Indeed, the mean curvature ρ on $\partial B(\mathbf{0}, r)$ is the function such that for any $\mathcal{C}^\infty(\mathbb{S}^n)$, we have

$$\partial_r \int_{\partial B(\mathbf{0},r)} f\, d\sigma = \int_{\partial B(\mathbf{0},r)} \langle \nabla f, \nu \rangle\, d\sigma + \int_{\partial B(\mathbf{0},r)} f \rho\, d\sigma$$

(for more details, see e.g. Lemma 10 in Section 3 below). Due to the symmetries of $\partial B(\mathbf{0}, r)$, one sees that ρ must be constant on $\partial B(\mathbf{0}, r)$. Thus considering $f = \mathbb{1}$ in the above equality, we get

$$\begin{aligned} \rho &= \frac{\partial_r \sigma(\partial B(\mathbf{0}, r))}{\sigma(\partial B(\mathbf{0}, r))} \\ &= (n - 1)\cot(r) \end{aligned}$$

It follows that a solution $(B(\mathbf{0}, R_t))_{t\in[0,\tau)}$ of the radial evolution equation (3), where τ is the hitting time of π by $(R_t)_{t\in[0,\tau)}$, is given by

$$dR_t = \sqrt{2} dB_t + \left(\frac{2\sin^{n-1}(R_t)}{I(R_t)} - (n - 1)\cot(R_t) \right) dt \tag{17}$$

where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion.

As $r \to 0_+$, we have

$$\begin{aligned} \frac{2\sin^{n-1}(r)}{I(R_t)} - (n - 1)\cot(r) &\sim \frac{2r^{n-1}}{\int_0^r u^{n-1}\, du} - \frac{n-1}{r} \\ &= \frac{n+1}{r} \end{aligned}$$

and this enables us to see that 0 is an entrance boundary for $(R_t)_{t\in[0,\tau)}$ and we can make it start from 0, namely we can let $(B(\mathbf{0}, R_t))_{t\in[0,\tau)}$ start from $\{\mathbf{0}\}$.

In general we did not find a nice expression for the drift of (17), but in the case $n = 2$, this evolution equation can be written

$$dR_t = \sqrt{2}dB_t + \frac{2+\cos(R_t)}{\sin(R_t)}dt$$

Similarly to the Euclidean situation, let us check directly Theorem 5, i.e. that $(\lambda(B(\mathbf{0}, R_t)))_{t\in[0,\tau)}$ is a stopped Bessel of process of dimension 3, up to a time change. It is sufficient to show that the same is true for $(I(R_t))_{t\in[0,\tau)}$. We compute

$$\begin{aligned} dI(R_t) &= I'(R_t)\left(\sqrt{2}dB_t + \left(\frac{2\sin^{n-1}(R_t)}{I(R_t)} - (n-1)\cot(R_t)\right)dt\right) + I''(R_t)dt \\ &= \sqrt{2}\sin^{n-1}(R_t)dB_t + \left(\frac{2\sin^{2n-2}(R_t)}{I(R_t)} - (n-1)\sin^{n-1}(R_t)\cot(R_t)\right)dt \\ &\quad +(n-1)\sin^{n-2}(R_t)\cos(R_t)dt \\ &= \sqrt{2}\sin^{n-1}(R_t)dB_t + \frac{2\sin^{2n-2}(R_t)}{I(R_t)}dt \end{aligned}$$

So the generator of $(I(R_t))_{t\in[0,\tau)}$ is $2\sin^{2n-2}(I^{-1}(x))[\frac{1}{2}\partial^2 + \frac{1}{x}\partial]$, where I^{-1} is the inverse mapping of $I : [0, \pi] \to [0, Z_n]$. This shows that $(I(R_{\theta_t}))_{t\in[0,\tau)}$ is a Bessel process of dimension 3 starting from 0 and stopped when it hits Z_n, where the time change $(\theta_t)_{t\in[0,\tau)}$ is defined by

$$\forall\ t \in [0, \tau), \qquad \int_0^{\theta_t} \frac{1}{\sin^{2n-2}(I^{-1}(R_s))}ds = 2t$$

Consider the case where $R_0 = 0$. Then θ_τ has the same law as the first hitting time of Z_n by a Bessel process of dimension 3 starting from 0. It follows that τ is a.s. finite. Thus, starting from $\{\mathbf{0}\}$, the process $(B(\mathbf{0}, R_t))_{t\in[0,\tau]}$ ends up covering the whole sphere $\mathbb{S}^n$ at the (a.s.) finite time τ. According to the theory of strong duality (see e.g. the initial paper of Diaconis and Fill [5] for the principe and Section 7 for its application to the present context), this property leads to the construction of strong stationary times for the Brownian motion on $\mathbb{S}^n$ starting from $\mathbf{0}$ (and more generally for any initial distribution on $\mathbb{S}^n$, by symmetry and conditioning with respect to the initial position of the spheric Brownian motion).

2.3 Hyperbolic spaces

Consider the Poincaré's ball model of the hyperbolic space $\mathbb{H}^n$ of dimension $n \in \mathbb{N}\backslash\{1\}$. For references on the subject, one can consult the book of Anderson [1] and we find the unpublished report of Parkkonen [24] very convenient. As above, the choice of the point $\mathbf{0}$ is irrelevant, let us choose for instance the center of the Euclidean ball on which is imposed the classical hyperbolic metric. Let λ be the Riemannian distribution on $\mathbb{S}^n$ and σ be the corresponding hypersurface volume. Denote by $B(\mathbf{0}, r)$ the closed ball in $\mathbb{H}^n$ centered at $\mathbf{0}$ and of radius $r \geqslant 0$. Up to a factor, we have

$$\lambda(B(\mathbf{0}, r)) = \int_0^r \sinh^{n-1}(u)\,du \tag{18}$$

$$\sigma(\partial B(\mathbf{0}, r)) = \sinh^{n-1}(r) \tag{19}$$

From these formulas (and even only from (19), since (18) is already a consequence of (19)), one can develop the same arguments as in the spherical situation, replacing the trigonometric functions by their hyperbolic counter-parts, to get the following results. A solution $(B(\mathbf{0}, R_t))_{t\geqslant 0}$ of the radial evolution equation (3), is given by

$$dR_t = \sqrt{2}dB_t + \left(\frac{2\sinh^{n-1}(R_t)}{J(R_t)} - (n-1)\coth(R_t)\right)dt$$

where $J : \mathbb{R}_+ \ni r \mapsto \int_0^r \sinh^{n-1}(u)\, du$. In particular, for the hyperbolic plane ($n = 2$), we get

$$dR_t = \sqrt{2}dB_t + \frac{2+\cosh(R_t)}{\sinh(R_t)}dt$$

Again, 0 is an entrance boundary for $(R_t)_{t\geqslant 0}$ and we can make it start from 0, namely we can let $(B(\mathbf{0}, R_t))_{t\geqslant 0}$ start from $\{\mathbf{0}\}$. From this initial point, the process $(\lambda(R_{\theta_t}))_{t\geqslant 0}$ is a Bessel process of dimension 3 starting from 0, where the time change $(\theta_t)_{t\geqslant 0}$ is defined by

$$\forall\, t \geqslant 0, \qquad \int_0^{\theta_t} \frac{1}{\sinh^{2n-2}(J^{-1}(R_s))}\, ds = 2t$$

where J^{-1} is the inverse mapping of $J : \mathbb{R}_+ \to \mathbb{R}_+$. This is obtained through computations similar to those of Subsection 2.2 or as a consequence of Theorem 5.

3 Smooth initial conditions

After proving Theorem 3, we will show how to solve (3) for small times, when the initial domain has a smooth boundary. It will provide a solution of the martingale problem associated to $\mathcal{L}$, thus showing Theorem 4.

As announced, we begin by the

Proof of Theorem 3

Consider $\mathcal{R} \coloneqq \{(x, y) \in \mathbb{R}^2 \,:\, y > 0\}$ and the mapping

$$\mathfrak{f} : \mathcal{R} \ni (x, y) \mapsto \frac{x}{y}$$

For any $f \in \mathcal{C}^\infty(V)$, we have $\Lambda[f] = \mathfrak{f}(F_f, F_{\mathbb{1}})$, so that $\Lambda[f] \in \mathfrak{D}$.

It follows that

$$\mathfrak{L}[\Lambda[f]] = \frac{1}{F_{\mathbb{1}}}\mathfrak{L}[F_f] - \frac{F_f}{F_{\mathbb{1}}^2}\mathfrak{L}[F_{\mathbb{1}}] - \frac{2}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}}[F_f, F_{\mathbb{1}}] + \frac{2F_f}{F_{\mathbb{1}}^3}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}]$$

which can be rewritten under the form

$$F_{\mathbb{1}}\mathfrak{L}[\Lambda[f]] = \mathfrak{L}[F_f] - \frac{2}{F_{\mathbb{1}}}\Gamma_{\mathfrak{L}}[F_f, F_{\mathbb{1}}] + F_f\left(\frac{2}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}] - \frac{1}{F_{\mathbb{1}}}\mathfrak{L}[F_{\mathbb{1}}]\right)$$

We compute, for any $D \in \mathcal{D}$, with $C \coloneqq \partial D$, $\nu \coloneqq \nu_C$ and $\sigma \coloneqq \sigma_C$,

$$\begin{aligned}\mathfrak{L}[F_{\mathbb{1}}](D) &= \int_D L[\mathbb{1}]\, d\mu + 2\frac{\underline{\mu}(C)}{\mu(D)}\int_C \mathbb{1}\, d\underline{\mu} \\ &= 2\frac{\underline{\mu}(C)^2}{\mu(D)}\end{aligned}$$

Furthermore, remark that

$$\begin{aligned}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}](D) &= \left(\int_C \mathbb{1}\, d\underline{\mu}\right)^2 \\ &= \underline{\mu}(C)^2\end{aligned}$$

so taking into account that $F_{\mathbb{1}}(D) = \mu(D)$, we get

$$\frac{2}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}] - \frac{1}{F_{\mathbb{1}}}\mathfrak{L}[F_{\mathbb{1}}] = 0$$

Thus, we have

$$\begin{aligned}F_{\mathbb{1}}\mathfrak{L}[\Lambda[f]](D) &= \mathfrak{L}[F_f](D) - \frac{2}{F_{\mathbb{1}}}\Gamma_{\mathfrak{L}}[F_f, F_{\mathbb{1}}](D) \\ &= \int_D L[f]\, d\mu + 2\frac{\underline{\mu}(C)}{\mu(D)}\int f\, d\underline{\mu} - \frac{2\underline{\mu}(C)}{\mu(D)}\int_C f\, d\underline{\mu} \\ &= \int_D L[f]\, d\mu\end{aligned}$$

and we conclude to the announced intertwining relation

$$\mathfrak{L}[\Lambda[f]] = \frac{F_{L[f]}}{F_{\mathbb{1}}}$$

■

In the above proof, Definition (5) was helpful. Nevertheless to understand the dynamic of the domains generated by $\mathfrak{L}$, it is preferable to resort to (6), so let us show its equivalence with (5). It amounts to check that for any $D \in \mathcal{D}$ and any $f \in \mathcal{C}^\infty(V)$, we have

$$\int_D L[f]\, d\mu = \int_C \langle \nabla f, \nu_C\rangle + \langle \beta, \nu_C\rangle f\, d\underline{\mu} \tag{20}$$

This equality is based on the integration by parts formula (Stokes' theorem), stating that for any smooth vector field v on V, we have

$$\int_D \mathrm{div}(v)\, d\lambda = \int_C \langle v, \nu\rangle\, d\sigma \tag{21}$$

Indeed, we have

$$\begin{aligned}\int_D L[f]\, d\mu &= \int_D (\triangle f + \langle \nabla U + \beta, \nabla f\rangle) \exp(U) d\lambda \\ &= \int_D \mathrm{div}(\exp(U)\nabla f) + \langle \exp(U)\beta, \nabla f\rangle\, d\lambda\end{aligned}$$

By integration by parts formula, we get

$$\begin{aligned}\int_D \mathrm{div}(\exp(U)\nabla f)\, d\lambda &= \int_C \langle \exp(U)\nabla f, \nu\rangle\, d\sigma \\ &= \int_C \langle \nabla f, \nu\rangle\, d\underline{\mu}\end{aligned}$$

Recalling that $\mathrm{div}(\exp(U)\beta) = 0$, we have $\mathrm{div}(\exp(U)f\beta) = \langle \exp(U)\beta, \nabla f\rangle + \mathrm{div}(\exp(U)\beta)f = \langle \exp(U)\beta, \nabla f\rangle$, so another integration by parts gives us

$$\int_D \langle \exp(U)\beta, \nabla f\rangle \, d\lambda \;=\; \int_C \langle \beta, \nu\rangle f \, d\underline{\mu}$$

ending the proof of (20).

■

Now that we know that $\mathfrak{L}$ satisfies the wanted intertwining relation with L, given $D_0 \in \mathcal{D}$, we would like to construct a Markov process $(D_t)_{t\in[0,\tau)}$ starting from D_0 and whose generator is $\mathfrak{L}$, where τ will be a positive stopping time, in a first step. To do so, we come back to the radial evolution equation (3) that we reinterpret under the heuristic $\mathcal{D}$-valued stochastic differential equation

$$dD_t \;=\; \mathfrak{V}_1(D_t)\left(\sqrt{2}dB_t + 2\frac{\underline{\mu}(C_t)}{\mu(D_t)}dt\right) + \mathfrak{V}_2(D_t)\,dt \tag{22}$$

where $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are “vector fields” on $\mathcal{D}$. This formulation will enable us to adapt the Doss-Sussman method [6, 28] to this infinite dimensional setting to construct a solution to the martingale problem associated to the generator $\mathfrak{L}$ and to the initial position D_0, at least for small times.

Before explaining in general what we mean by a vector fields on $\mathcal{D}$, we study the flow generated by $\mathfrak{V}_1$, which is very simple to describe. For any $r \in \mathbb{R}$, denote

$$\Psi(D,r) \;\coloneqq\; \begin{cases} \{x\in V \,:\, d(x,D)\leqslant r\} & ,\ \text{if } r>0\\ D & ,\ \text{if } r=0\\ \{x\in D \,:\, d(x,D^{\mathrm{c}})\geqslant -r\} & ,\ \text{if } r<0\end{cases} \tag{23}$$

where we recall that for any subset $A \subset D$ and $x \in V$,

$$d(x,A) \;\coloneqq\; \inf\{d(x,y) \,:\, y\in A\}$$

with d the Riemannian distance on V.

It is easy to realize that the family $(\Psi(D,r))_{r\in\mathbb{R}}$ does not behave well for some $r \in \mathbb{R}$: it does not stay in $\mathcal{D}$ and does not satisfy the flow property (see Remark 9 below). So we are going to restrict the parameter r to a convenient open segment containing 0.

For any $x \in V$ and $v \in T_xV$, let $(\exp_x(rv))_{r\in\mathbb{R}}$ stand for the geodesic flow whose position and speed at time 0 are x and v. By our assumption of completeness on V, these geodesic flows are defined for all times. For any $r \in \mathbb{R}$, define the mapping

$$\psi_{C,r} \,:\, C \ni x \mapsto \exp_x(r\nu_C(x)) \tag{24}$$

Define

$$R_+(D) \;=\; \inf\{r\in(0,+\infty) \,:\, \psi_{C,r} \text{ is not a diffeomorphism on its image}\} \tag{25}$$

$$R_-(D) \;=\; -\inf\{r\in(0,+\infty) \,:\, \psi_{C,-r} \text{ is not a diffeomorphism on its image}\} \tag{26}$$

Due to the existence of a normal tubular neighborhood around the compact set C, we have that $R_+(D) > 0$ and $R_-(D) < 0$. The interest of the segment $(R_-(D), R_+(D))$ is summarized as follows:

Proposition 8 *Let $D \in \mathcal{D}$ be given. For any $r \in (R_-(D), R_+(D))$, we have*

$$\begin{aligned} \partial\Psi(D,r) &= \psi_{C,r}(C) \\ &= \begin{cases} \{x \in D^{\mathrm{c}} : d(D,x) = r\} & , \text{ if } r > 0 \\ C & , \text{ if } r = 0 \\ \{x \in D : d(D^{\mathrm{c}},x) = -r\} & , \text{ if } r < 0 \end{cases} \end{aligned} \tag{27}$$

showing that $\Psi(D,r) \in \mathcal{D}$.

Furthermore, for any $r, r' \in (R_-(D), R_+(D))$ such that $r + r' \in (R_-(D), R_+(D))$, the "semi-group property" holds:

$$\Psi(D, r + r') = \Psi(\Psi(D,r), r') = \Psi(\Psi(D,r'), r)$$

Proof

The above result is certainly standard, even we were not able to find a corresponding reference.

For the first assertion, we begin by considering the case $r \in (0, R_+(D))$. For any $x \in \Psi(D,r)\backslash D$, there exists $y \in C$ such that $d(x,y) = d(x,D) \in (0,r]$. Let us check that $x = \psi_{C,d(x,y)}(y)$. Denote $(\gamma(s))_{s\in[0,d(x,y)]}$ a unitary minimizing geodesic going from y to x. There exists $v \in T_yV$ with $\|v\| = 1$ such that $\gamma(s) = \exp_y(sv)$ for all $s \in [0, d(x,y)]$. If v is not orthogonal to T_yC, then for small $s > 0$, we could find $y_s \in C$ with $d(y_s, \gamma(s)) < d(y, \gamma(s))$, contradicting the minimizing property of y, since we would get $d(x,y) = d(y,\gamma(s)) + d(\gamma(s),x) > d(y_s,\gamma(s)) + d(\gamma(s),x) \geqslant d(x,y_s)$. If v was directed toward the interior of D, we would also end up with a contradiction, by considering the last time $s \in (0, d(x,y))$ such that $\gamma(s) \in D$. It follows that $v = \nu_C(y)$, showing that $x = \psi_{C,d(x,y)}(y)$. We furthermore get such a point $y \in C$ is unique, otherwise we would be in contradiction with the fact that $\psi_{C,d(x,D)}$ is injective. Conversely, if $s \in (0,r]$ and $y \in C$, then $x \coloneqq \psi_{C,s}(y) \in \Psi(D,s)$, with $d(x,D) \leqslant d(x,y) \leqslant s$. Thus we have the description

$$\forall\ r \in (0, R_+(D)), \qquad \Psi(D,r) = D \bigcup_{s\in(0,r]} \psi_{C,s}(C)$$

Let us show that all the sets of the r.h.s. are disjoint. First we prove by contradiction that

$$\forall\ s \in (0,r], \qquad D \cap \psi_{C,s}(C) = \varnothing \tag{28}$$

So assume that $\psi_{C,s}(x) \in D$, for some $x \in C$. Replacing s by $\inf\{t > 0 : \psi_{C,t}(x) \in D\}$, which is still positive, because $\psi_{C,t}(x)$ does not belong to D for $t > 0$ small enough, we can assume that $\psi_{C,s}(x) \in C$. Consider the mapping $\phi : [0,s] \ni t \mapsto d(\psi_{C,t}(x), C)$. We have seen above that for $t > 0$ small enough, we have $\phi(t) = t$. Since $\phi(s) = 0$, let $u \coloneqq \inf\{t > 0 : \phi(t) \neq t\}$, which belongs to $(0,s)$. Note that for $t \in [0,u)$, the directing normal vector $\frac{d}{dt}\psi_{C,t}(x)$ is orthogonal to the tangent space of $\psi_{C,t}(C)$ at $\psi_{C,t}(x)$, otherwise for $v \in (t,u)$, we could find a shortest way from $\psi_{C,v}(x)$ to $\psi_{C,t}(C)$ than the one given by the geodesic $(\psi_{C,w}(x))_{w\in[t,v]}$ and it would follow that $d(\psi_{C,v}(x), C) < v$. The tangent space of $\psi_{C,t}(C)$ at $\psi_{C,t}(x)$ coincides with the image of T_xC by $T\psi_{C,t}(x)$, by the fact that $\psi_{C,t}$ is a diffeomorphism on its image. Letting t go to u, we get the directing normal vector $\frac{d}{dt}\psi_{C,t}(x)\big|_{t=u}$ is still orthogonal to the tangent space of $\psi_{C,u}(C)$ at $\psi_{C,u}(x)$. As above, this property insures us that for $\epsilon > 0$ small enough,

$$d(\psi_{C,u+\epsilon}(x), \psi_{C,u}(C)) = \epsilon \tag{29}$$

namely either $d(\psi_{C,u+\epsilon}(x), C) = u + \epsilon$ or $d(\psi_{C,u+\epsilon}(x), C) = u - \epsilon$. The first alternative is forbidden by the definition of u. For the second alternative, we get, for $\epsilon > 0$ small enough, $\psi_{C,u+\epsilon}(x) \neq \psi_{C,u-\epsilon}(x)$ belongs to $\psi_{C,u-\epsilon}(C)$, thus we can find $y \in C\backslash\{x\}$ with $\psi_{C,u+\epsilon}(x) = \psi_{C,u-\epsilon}(y)$. If follows from (29) that

$\psi_{C,u}(x) = \psi_{C,u}(y)$, in contradiction with the injectivity of $\psi_{C,u}$. This ends the proof of (28). The proof that for $s \neq s' \in (0, r]$, we have $\psi_{C,s}(C) \cap \psi_{C,s'}(C) = \emptyset$ is similar. Indeed, if this equality was not true, then one would be able to find again $x \in C$ and $t \in (0, r]$ such that $d(\psi_{C,t}(x), C) > t$. We end up with the "foliation"

$$\forall\ r \in (0, R_+(D), \qquad \Psi(D, r) \;=\; D \bigsqcup_{s\in(0,r]} \psi_{C,s}(C) \tag{30}$$

From this decomposition and the continuity of $C \times (0, R_+(D)) \ni (x, s) \mapsto \psi_{C,s}(x)$, we deduce that for $r \in (0, R_+(D))$,

$$\begin{aligned}\partial\Psi(D, r) &= \psi_{C,r}(C)\\ &= \{x \in D^c : d(D, x) = r\}\end{aligned}$$

The analogous relations when $r \in (R_-(D), 0)$ are obtained in a similar way, taking into account that

$$\forall\ r \in (R_-(D), 0), \qquad \Psi(D, r) \;=\; D\backslash \left(\bigsqcup_{s\in[r,0)} \psi_{C,s}(C)\right) \tag{31}$$

The semigroup property is also a consequence of (30) and (31), taking into account that for r, r' as in the above proposition, we have

$$\psi_{C,r+r'} \;=\; \psi_{C,r} \circ \psi_{C,r'} \;=\; \psi_{C,r'} \circ \psi_{C,r}$$

(remarking that for any $x \in C$ and $r \in (R_-(D), R_+(D))$, we have $T_x\psi_{C,r}[\nu_C(x)] = \nu_{\Psi(C,r)}(\psi_{C,r}(x))$).

■

Remark 9 The semi-group property of Corollary 8 is no longer necessarily true if the conditions on $r, r' \in \mathbb{R}$ are not satisfied. Consider first the following (non connected) example: let D be the union of the open balls $B((0, 0), 3)$ and $B((0, 5), 1)$. Then we have $\Psi(D, -2) = B((0, 0), 1)$ and $\Psi(B((0, 0), 1), 2) = B((0, 0), 3) \neq D$. This example can be modified into a connected one by joining $B((0, 0), 3)$ and $B((0, 5), 1)$ through the open rectangle $[0, 5] \times [-1, 1]$. The boundary of the resulting domain D is not smooth, nevertheless, the definition (23) makes sense. The boundary $\partial\Psi(D, r)$ makes an "irreversible transition" at $r = -1$.

□

From now on, for $r \in (R_-(D), R_+(D))$, denote by $\Psi(C, r)$ the set described in (27). For given $D \in \mathcal{D}$, the family $(\Psi(C, r))_{r\in(R_-(D),R_+(D))}$ is the solution of the **normal flow equation**, which can be written under the radial form

$$\begin{cases} & \Psi(C, 0) = C\\ \forall\ r \in (R_-(D), R_+(D)), \forall\ x \in \Psi(C, r), & \partial_r x = \nu_{\Psi(C,r)}(x)\end{cases} \tag{32}$$

where the points of the boundaries are pushed according to the outward normal.

For our purposes, it is convenient to look at this set-valued evolution through our elementary observables:

Lemma 10 *Let $D \in \mathcal{D}$ and $f \in \mathcal{C}^\infty(V)$ be fixed. The mapping $(R_-(D), R_+(D)) \ni r \mapsto F_f(\Psi(D, r)) \in \mathbb{R}$ is $\mathcal{C}^2$ and for any $r \in (R_-(D), R_+(D))$, we have*

$$\begin{aligned}\partial_r F_f(\Psi(D, r)) &= \int_{\Psi(C,r)} f\, d\underline{\mu}\\ \partial_r^2 F_f(\Psi(D, r)) &= \int_{\Psi(C,r)} \left\langle \nabla f, \nu_{\Psi(C,r)}\right\rangle d\underline{\mu} + \int_{\Psi(C,r)} (\left\langle \nabla U, \nu_{\Psi(C,r)}\right\rangle + \rho_{\Psi(C,r)}) f\, d\underline{\mu}\end{aligned}$$

To simplify the notation, when the set C will be clear from the context (e.g. coming from the domain of integration), we will write σ, ν and ρ instead of σ_C, ν_C and ρ_C, convention which was already adopted for $\underline{\mu}$. So that the last r.h.s. admits the more readable expression

$$\int_{\Psi(C,r)} \langle \nabla f, \nu \rangle \, d\underline{\mu} + \int_{\Psi(C,r)} f(\langle \nabla U, \nu \rangle + \rho) \, d\underline{\mu}$$

Proof

The first differentiation is a classical result. It can also be deduced from the disintegration of μ with respect to (30) and (31). For instance for $r \in [0, R_+(D))$, we have

$$F_f(\Psi(D,r)) \quad = \quad F_f(D) + \int_0^r \int_{\Psi(C,s)} f \, d\underline{\mu} \, ds$$

and the r.h.s. is easily differentiated with respect to r.

For the second differentiation, first write

$$\int_{\Psi(C,r)} f \, d\underline{\mu} \quad = \quad \int_{\Psi(C,r)} f \exp(U) \, d\sigma$$

To differentiate with respect to r the r.h.s., one has to adapt the arguments of Section 1.2 of the book of Mantegazza [17], to get

$$\begin{aligned} \partial_r \int_{\Psi(C,r)} f \exp(U) \, d\sigma &= \int_{\Psi(C,r)} \langle \nabla (f \exp(U)), \nu \rangle \, d\sigma + \int_{\Psi(C,r)} f \exp(U) \rho \, d\sigma \\ &= \int_{\Psi(C,r)} \langle \nabla f, \nu \rangle \, d\underline{\mu} + \int_{\Psi(C,r)} (\langle \nabla U, \nu \rangle + \rho) f \, d\underline{\mu} \end{aligned}$$

■

We will also need to differentiate Ψ with respect to the first variable $D \in \mathcal{D}$. We must first give a meaning to the underlying notion of differentiation in $\mathcal{D}$.

Consider a family $(G_s)_{s \in [0,S_+)}$ taking values in $\mathcal{D}$, for a real number $S_+ > 0$. We say this family is **strongly continuous in a neighborhood** of $s \in [0, S_+)$ if there exist a neighborhood N_s of s in $[0, S_+)$ and a continuous mapping $\varphi_s : N_s \times \partial G_s \to V$ such that for any $u \in N_s$, the function $\partial G_s \ni x \mapsto \varphi_s(u, x)$ is a homeomorphism between ∂G_s and ∂G_u and if $\varphi_s(s, \cdot)$ is the identity mapping. In this statement, the boundaries ∂G_s, for $s \in [0, S_+)$ are endowed with the topology inherited from that of V. Similarly, these boundaries will be endowed below with the smooth differentiable structure inherited from V as smooth submanifolds. The family $(G_s)_{s \in [0,S_+)}$ is said to be **strongly continuous** on $[0, S_+)$, if for any $s \in [0, S_+)$, it is strongly continuous in a neighborhood of s.

Remark 11 Let $\mathfrak{d}$ be the Hausdorff metric on the compact subsets of V. It endows $\mathcal{D}$ with a metric structure. The strong continuity defined above implies the continuity for the Hausdorff metric, but the converse is not always true, as it is illustrated by the following picture:

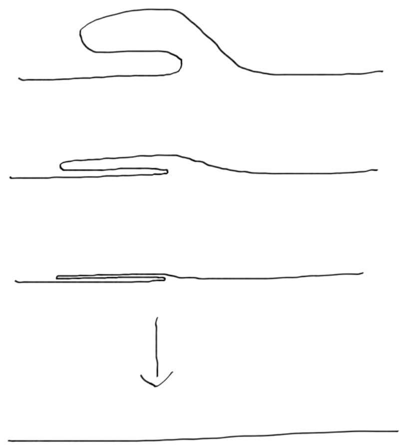

Figure 1: convergence in the Hausdorff topology, not in the strong sense

□

Note that the restrictions to $\mathcal{D}$ of the mappings defined in (10) and (11) are strongly continuous.

By analogy, we say the family $(G_s)_{s\in[0,S_+)}$ is **strongly smooth in a neighborhood** of $s \in [0, S_+)$ if there exist a neighborhood N_s of s in $[0, S_+)$ and a smooth mapping $\varphi_s : N_s \times \partial G_s \to V$ such that for any $u \in N_s$, the function $\partial G_s \ni x \mapsto \varphi_s(u, x)$ is a diffeomorphism between ∂G_s and ∂G_u and $\varphi_s(s, \cdot)$ is the identity mapping. The family $(G_s)_{s\in[0,S_+)}$ is then said to be **strongly smooth** if it is strongly smooth in the neighborhood of any $s \in [0, S_+)$. For such a family, consider for any $s \in [0, S_+)$ and $x \in \partial G_s$, the vector

$$X_{\partial G_s}(x) \;\coloneqq\; \partial_u \varphi_s(u, x)|_{u=s}$$

The TV-valued vector field $X_{\partial G_s}$ on ∂G_s enables to describe the infinitesimal evolution of G_s via a formula similar to (32)

$$\forall\ s \in [0, S_+),\ \forall\ x \in \partial G_s, \qquad \partial_s x \;=\; X_{\partial G_s}(x)$$

This description is not unique, because the mappings $\varphi_s(u, \cdot)$ are not unique: they can be composed by diffeomorphisms of ∂G_u, depending on s and (smoothly) on u. Indeed, as already mentioned, the discussion of Section 1.3 of Mantegazza [17] shows that for $x \in \partial G_s$, only the radial part $\alpha_{\partial G_s}(x) \coloneqq \langle X_{\partial G_s}(x), \nu_{\partial G_s}(x)\rangle$ is unique. Furthermore, it is possible to choose the mappings φ_s in such a way so that

$$\forall\ s \in [0, S_+),\ \forall\ x \in \partial G_s, \qquad X_{\partial G_s}(x) \;=\; \alpha_{\partial G_s}(x)\nu_{\partial G_s}(x)$$

and the function α is **continuous** in the sense that if the sequences $(s_n)_{n\in\mathbb{N}}$ in $[0, S_+)$ and $(x_n)_{n\in\mathbb{N}}$, taking values respectively in $(\partial G_{s_n})_{n\in\mathbb{N}}$, are converging toward $s \in [0, S_+)$ and $x \in \partial G_s$, then $\lim_{n\to\infty} \alpha_{\partial G_{s_n}}(x_n) = \alpha_{\partial G_s}(x)$.

The family $(G_s)_{s\in[0,S_+)}$ can thus be described more intrinsically as a solution of the **radial equation equation**

$$\forall\ s \in [0, S_+),\ \forall\ x \in \partial G_s, \qquad \partial_s x \;=\; \alpha_{\partial G_s}(x)\nu_{\partial G_s}(x) \tag{33}$$

This formula enables us to identify the "**tangent space**" $T_D\mathcal{D}$ at $D \in \mathcal{D}$ with the space $\mathcal{C}^\infty(C)$ of real smooth functions on C (of the form α_C with the above notation). At least it appears that

$T_D\mathcal{D} \subset \mathcal{C}^\infty(C)$. Conversely, given $\alpha \in \mathcal{C}^\infty(C)$, we will see in Remark 16 how to construct a strongly smooth family $(G_s)_{s\in[0,S_+)}$ such that

$$\begin{cases} & G_0 &=& D \\ \forall\ x \in \partial G_0, & \partial_s x|_{s=0} &=& \alpha(x)\nu_{\partial G_0}(x) \end{cases} \tag{34}$$

This shows that $\mathcal{C}^\infty(C) \subset T_D\mathcal{D}$.

Following the traditional definition in differential geometry, we say that a mapping $\Phi : \mathcal{D} \to \mathcal{D}$ is **strongly smooth** if any strongly smooth family $(G_s)_{s\in[0,S_+)}$ is transformed by Φ into a strongly smooth family, i.e. $(\Phi(G_s))_{s\in[0,S_+)}$ is smooth (to simplify the terminology, from now on, smooth means strongly smooth). Then there exists a vector field $\widetilde{\alpha}$ on $(\Phi(G_s))_{s\in[0,S_+)}$ such that

$$\forall\ s \in [0, S_+),\ \forall\ x \in \partial\Phi(G_s), \qquad \partial_s x \ = \ \widetilde{\alpha}_{\partial\Phi(G_s)}(x)\nu_{\partial\Phi(G_s)}(x)$$

Fix $s \in [0, S_+)$. It is not difficult to see that the function $\widetilde{\alpha}_{\partial\Phi(G_s)}$ depends on α satisfying (33) only through $\alpha_{\partial G_s}$. For fixed $D \in \mathcal{D}$, consider any smooth and $\mathcal{D}$-valued family $(G_s)_{s\in[0,S_+)}$ with $0 \in [0, S_+)$ and $G_0 = D$. Let α be associated with $(G_s)_{s\in[0,S_+)}$ as in (33). The linear functional transforming α_C into $\widetilde{\alpha}_{\partial\Phi(D)}$, as above, is called the **tangent mapping** $T_D\Phi$ of Φ at D.

Remark 12 A natural converse question is: given $D, \widetilde{D} \in \mathcal{D}$ and a linear mapping T from $\mathcal{C}^\infty(C)$ to $\mathcal{C}^\infty(\widetilde{C})$ (with $\widetilde{C} := \partial\widetilde{D}$), is there a smooth function Φ on $\mathcal{D}$ with $\Phi(D) = \widetilde{D}$ and such that $T = T_D\Phi$? The investigation of this kind of general issues is out of the scope of the present paper. Nevertheless, a first step in this direction is as follows. Let $\alpha, \widetilde{\alpha}$ be given in $T_D\mathcal{D}$ and $T_{\widetilde{D}}\mathcal{D}$ respectively. Remark 16 shows how to extend α and $\widetilde{\alpha}$ on $\mathcal{D}$ in order to be able to solve locally in time (34) to get smooth families $(G_s)_{s\in[0,S_+)}$ and $(\widetilde{G}_s)_{s\in[0,\widetilde{S}_+)}$. Replace S_+ by $S_+ \wedge \widetilde{S}_+$. Assuming that α did not vanish identically on ∂D, we can furthermore impose that S_+ is small enough so that $[0, S_+) \ni s \mapsto G_s$ is one-to-one. It enables us to define Φ on $\{G_s : s \in [0, S_+)\}$ via $\Phi(G_s) = \widetilde{G}_s$, for all $s \in [0, S_+)$. Then we get $T_D\Phi[\alpha] = \widetilde{\alpha}$. To go further would require a better understanding of the neighborhood of D in $\mathcal{D}$.

□

With all these preliminaries at our disposal, we can now compute the tangent mapping $T_D\Psi(\cdot, r)$ for $r \in (R_-(D), R_+(D))$. Rigorously, for given $r \in \mathbb{R}$, the mapping $\Psi(\cdot, r)$ is not defined on the whole set $\mathcal{D}$ but only on the subset

$$\mathcal{D}_r \ := \ \{D \in \mathcal{D} : r \in (R_-(D), R_+(D))\} \tag{35}$$

This subset is open for the strong topology alluded before (but not in the Hausdorff topology, see Remark 11), so that the notion of tangent mapping can be extended to this setting (as soon as $\mathcal{D}_r \neq \varnothing$).

The tangent mapping $T_D\Psi(\cdot, r)$ is among the simplest possible ones:

Lemma 13 *Let $D \in \mathcal{D}$ and $r \in (R_-(D), R_+(D))$ be given. For any $\alpha \in \mathcal{C}^\infty(C)$ and $x \in C$, we have*

$$T_D\Psi(\cdot, r)[\alpha](x) \ = \ \alpha(\psi_{C,r}^{-1}(x))$$

where $\psi_{C,r}^{-1} : \Psi(C, r) \to C$ is the inverse mapping of the function $\psi_{C,r}$ defined in (24).

Proof

Let $\alpha \in \mathcal{C}^\infty(C)$ be given, extend it smoothly on V and solve (34) for $\epsilon > 0$ small enough. For $x \in C$ and $s \in (-\epsilon, \epsilon)$, denote $\varphi(x, s) := x_s$ and $A_s := \{\varphi(x, s) : x \in C\}$. According to the previous discussion, to get the wanted result, we just need to check that for any $x \in C$, the part of $\partial_s \psi_{A_s,r}(\varphi(x,s))|_{s=0}$ which is (outwardly) normal to $\Psi(A_s,r)$ is equal to $\alpha(x)$, namely that

$$\forall\ x \in C, \qquad \left\langle \partial_s \psi_{A_s,r}(\varphi(x,s))|_{s=0}, \nu_{\Psi(C,r)}(\psi_{C,r}(x)) \right\rangle_{\psi_{C,r}(x)} \;=\; \alpha(x) \tag{36}$$

Denote

$$\forall\ t \in [0,r], \qquad J_t \;\coloneqq\; \partial_s \psi_{A_s,t}(\varphi(x,s))|_{s=0}$$

so that $(J_t)_{t\in[0,r]}$ is a vector field over the geodesic $(\gamma(t))_{t\in[0,r]} \coloneqq (\psi_{C,t}(x))_{t\in[0,r]}$. For all $s \in (-\epsilon,\epsilon)$, $(\psi_{C,t}(x_s))_{t\in[0,r]}$ is a geodesic, it follows that $(J_t)_{t\in[0,r]}$ is a Jacobi fields (cf. for instance Proposition 3.45 from the book of Gallot, Hulin and Lafontaine [10], whose Chapter 3 serves as a reference for all the following considerations). Thus $(J_t)_{t\in[0,r]}$ is defined by its initial conditions $J(0)$ and $J'(0)$, where the prime corresponds to the covariant derivative with respect to t, and by the evolution $J'' = -R(J,\dot{\gamma})\dot{\gamma}$, where R is the Riemannian curvature tensor. To prove (36) amounts to show that the mapping $[0,r] \ni t \mapsto \langle J(t), \dot{\gamma}(t)\rangle_{\gamma(t)}$ is constant. The covariant derivative is constructed so that the scalar product is left invariant, so that

$$\begin{aligned}\forall\ t \in [0,r], \qquad \frac{d}{dt}\langle J(t), \dot{\gamma}(t)\rangle_{\gamma(t)} &= \langle J'(t), \dot{\gamma}(t)\rangle_{\gamma(t)} + \langle J(t), \dot{\gamma}'(t)\rangle_{\gamma(t)} \\ &= \langle J'(t), \dot{\gamma}(t)\rangle_{\gamma(t)}\end{aligned}$$

since by definition of a geodesic, we have $\dot{\gamma}'(t) = 0$. Differentiating once more, we get

$$\begin{aligned}\frac{d}{dt}\langle J'(t), \dot{\gamma}(t)\rangle_{\gamma(t)} &= \langle J''(t), \dot{\gamma}(t)\rangle_{\gamma(t)} + \langle J'(t), \dot{\gamma}'(t)\rangle_{\gamma(t)} \\ &= \langle J''(t), \dot{\gamma}(t)\rangle_{\gamma(t)} \\ &= -R(J,\dot{\gamma},\dot{\gamma},\dot{\gamma}) \\ &= 0\end{aligned}$$

since the $(0,4)$-curvature tensor R is anti-symmetric in its last two vector variables (as well as in first two vector variables). Thus, to get the wanted result, we just need to check that $J'(0)$ is orthogonal to $\dot{\gamma}(0) = \nu_C(x)$. From the first equality of Proposition 3.29 of Gallot, Hulin and Lafontaine [10] (applied with the commutating vector fields $X = \partial_s$ and $J = \partial_t$ on $(-\epsilon,\epsilon)\times[0,r]$ parametrized by (s,t)), it appears that $J'(0)$ coincides with the covariant derivative with respect to s of the tangent vectors of the geodesic $(\psi_{C,t}(x_s))_{t\in[0,r]}$, at $s=0$ and $t=0$. The latter tangent vectors are unitary, so their covariant derivatives are orthogonal to them. Thus at $s=0$ and $t=0$ we get $\langle J'(0), \dot{\gamma}(0)\rangle_x = 0$, ending the proof of (36).

■

We deduce the differentiation of our favorite observables.

Corollary 14 *In the setting of Lemma 13, let be given $f \in \mathcal{C}^\infty(V)$ and $(G_s)_{s\in[0,S_+)}$ with $G_0 = D$ and $\alpha_{\partial G_0} = \alpha$ (in the sense of (33)). We have*

$$\left.\frac{d}{ds}F_f(\Psi(G_s,r))\right|_{s=0} \;=\; \int_{\Psi(C,r)} f(x)\alpha(\psi_{C,r}^{-1}(x))\,\underline{\mu}(dx)$$

Proof

As in the first part of the proof of Lemma 10, we get

$$\frac{d}{ds}F_f(G_s)\bigg|_{s=0} = \int_C f(x)\alpha(x)\,\underline{\mu}(dx)$$

Taking into account Lemma 13, the announced result follows from this formula, with $(G_s)_{s\in[0,S_+)}$ replaced by $(\Psi(G_s,r))_{s\in[0,S_+)}$.

■

A famous example of radial evolution of the type (33) is the mean curvature flow:

$$\forall\ s\in[0,S_+),\ \forall\ x\in\partial G_s,\qquad \partial_s x = -\rho_{\partial G_s}(x)\nu_{\partial G_s}(x)$$

where $G_0\in\mathcal{D}$ is given and $[0,S_+)$ is the maximum interval on which this flow remains in $\mathcal{D}$ (there are various ways to define the mean curvature flow beyond the times when it gets out of $\mathcal{D}$, see e.g. Chapter 1 of the book of Mantegazza [17]). When $V=\mathbb{R}^2$ endowed with its usual Riemannian structure, it is possible to compute explicitly the image of the mean curvature vector field ρ by the tangent applications to the normal flow Ψ, see Subsection 5.1. In general, it is more difficult (see nevertheless Remark 49 for the usual Riemannian structure on $V=\mathbb{R}^n$), since the curvature of V will enter into the game.

The arguments of Section 1.5 of Mantegazza [17] can be adapted to get existence and uniqueness of the solutions $(G_s)_{s\in[0,S_+)}$ to the radial evolution equations of the form

$$\forall\ s\in[0,S_+),\ \forall\ x\in\partial G_s,\qquad \partial_s x = (-\rho_{\partial G_s}(x)+\langle b(x),\nu_{\partial G_s}(x)\rangle_x + a(x))\nu_{\partial G_s}(x) \tag{37}$$

where $[0,S_+)$ is a small enough interval containing 0, where G_0 is a given element from $\mathcal{D}$ and where a and b are respectively a smooth function and a smooth vector field on V. The obtained solution $(G_s)_{s\in[0,S_+)}$ is a smooth family. The underlying idea is to consider again the parametrization $(r_-,r_+)\times G_0\ni(r,x)\mapsto\psi_{G_0,r}(x)$ of a tubular neighborhood of G_0, where (r_-,r_+) is a small neighborhood of 0. Then one looks for a mapping $[0,S_+)\times G_0\ni(s,x)\mapsto y(s,x)$, whose image is included into the tubular neighborhood $\psi_{G_0,(r_-,r_+)}(G_0)$ and which is such that for any $s\in[0,S_+)$ and any $x\in G_0$

$$\begin{aligned} y(0,x) &= \varphi_0(x)\\ \langle\partial_s y(s,x),\nu_{\partial G_s}(y(s,x))\rangle_{y(s,x)} &= -\rho_{\partial G_s}(y(s,x))+\langle b(y(s,x)),\nu_{\partial G_s}(y(s,x))\rangle_{y(s,x)}+a(y(s,x)),\end{aligned}$$

where $\varphi_0: G_0\to\mathbb{R}^n$ is the inclusion map. Then writing $y(s,x)=\psi_{G_0,f(s,x)}(x)$, for all $(s,x)\in[0,S_+)\times G_0$, we end up with the quasi-linear parabolic equation with respect to f: for any $x\in G_0$,

$$\left\{\begin{aligned} f(0,x) &= 0\\ \forall\ s\in[0,S_+),\ \partial_s f(s,x) &= \triangle_{G_0,s}f(s,x)\\ &\quad +H(x,f(s,x),(\partial_{x_i}f(s,x))_{i\in[\![n-1]\!]},(\partial_{x_i}(\partial_{x_j}f(s,x))^2)_{i,j\in[\![n-1]\!]})\end{aligned}\right. \tag{38}$$

where H is a smooth mapping on $\mathbb{R}_+\times\mathbb{R}^{n-1}\times\mathbb{R}^{(n-1)^2}$ and where $\triangle_{G_0,s}$ is the Laplacian relatively to the Riemannian structure on G_0 obtained by pulling back through the diffeomorphism $G_0\ni x\mapsto\psi_{G_0,f(s,x)}(x)$ the Riemannian structure on G_s inherited from that of V. Note that H will be independent from the chart in which we compute $\partial_{x_i}f(s,x)$ and $\partial_{x_i}\partial_{x_j}f(s,x)$.

Before going further, let us explain how to get (38) from (37), when $V=\mathbb{R}^n$. In this case we have

$$y(s,x) = \psi_{G_0,f(s,x)}(x) = \varphi_0(x)+f(s,x)\nu_0(x)$$

and note that for f small and smooth, $y(s,.)$ will be a diffeomorphism, with $f(0,x)=0$ and $\partial_{x_i}f(0,x)=0$ and $\partial_{x_i}\partial_{x_j}f(0,x)=0$. We compute the equation satisfied by $f(s,x)$ such that $y(s,x)$ is a solution of

(37). Taking into account Corollary 1.3.5 in Mantegazza [17], up to a reparametrization, the evolution of $G_s = y(s, G_0)$ is characterized by it's normal evolution, namely $\langle \partial_s y(s,x), \nu_{\partial G_s}(y(s,x))\rangle_{y(s,x)}$. Let us compute the pullback metric at $x \in G_0$, $g(s,x) = y(s,.)_* g_{|\partial G_s}$, where g is the canonical metric in V. In a local chart of G_0, $(x_i)_{i\in[\![n-1]\!]}$ at $x \in G_0$, we have:

$$
\begin{aligned}
g_{i,j}(s,x) &= \langle \partial_{x_i} y(s,x), \partial_{x_j} y(s,x)\rangle \\
&= \langle \partial_{x_i}\varphi_0 + \partial_{x_i} f(s,x)\nu_0(x) + f(s,x)\partial_{x_i}\nu_0(x), \partial_{x_j}\varphi_0 + \partial_{x_j} f(s,x)\nu_0(x) + f(s,x)\partial_{x_j}\nu_0(x)\rangle \\
&= \langle \partial_{x_i}\varphi_0 + f(s,x)\partial_{x_i}\nu_0(x), \partial_{x_j}\varphi_0 + f(s,x)\partial_{x_j}\nu_0(x)\rangle + \partial_{x_i} f(s,x)\partial_{x_j} f(s,x)
\end{aligned}
$$

Using Gauss-Weingarten equation, namely:

$$
\partial_{x_i}\nu_0(x) = h_{i,l}(0,x) g^{l,k}(0,x)\partial_{x_k}\varphi_0(x)
$$

where $h_{i,l}(0,x)$ is the second fundamental form of G_0 at x, and $(g^{l,k}(0,x))_{l,k}$ is the inverse of the metric $(g_{l,k}(0,x))_{l,k}$, and we use the convention that every repeated lower indices and upper indices are considered as a sum, as in the whole paper. We get,

$$
g_{i,j}(s,x) = g_{i,j}(0,x) + 2f(s,x)h_{i,j}(0,x) + f^2(s,x)h_{i,l}g^{l,m}h_{j,m}(0,x) + \partial_{x_i} f(s,x)\partial_{x_j} f(s,x)
$$

Using again Gauss-Weingarten equation, and since $\langle \nu(s,x), \partial_{x_i} y(s,x)\rangle = 0$ we have

$$
\begin{aligned}
h_{i,j}(s,x) &= -\langle \nu(s,x), \partial_{x_i}\partial_{x_j} y(s,x)\rangle \\
&= -\langle \nu(s,x), \partial_{x_i}\partial_{x_j}\varphi_0 + \partial_{x_i}\partial_{x_j} f\nu_0(x) + \partial_{x_j} f\partial_{x_i}\nu_0(x) + \partial_{x_i} f\partial_{x_j}\nu_0 + f\partial_{x_i}\partial_{x_j}\nu_0(x)\rangle \\
&= -\partial_{x_i}\partial_{x_j} f\langle \nu(s,x), \nu_0(x)\rangle + \widehat{H}_{i,j}(x, f(s,x), (\partial_{x_l} f(s,x))_{l\in[\![n-1]\!]})
\end{aligned}
$$

where $\nu(s,x)$ is the exterior normal vector of G_s at the point $y(s,x)$, and $h_{i,j}(s,x)$ is the second fundamental form of G_s at $y(s,x)$ in the basis $(\partial_{x_i} y(s,x))_{i\in[\![n-1]\!]}$ of it's tangent space and $\widehat{H}$ is a smooth function when the two last argument are small enough. We also have

$$
\partial_s y(s,x) = \partial_s f(s,x)\nu_0(x)
$$

and $\rho_{\partial G_s}(y(s,x)) = g^{i,j}(s,x)h_{i,j}(s,x)$, note that this quantity is independent of the chart. If we write :

$$
\begin{aligned}
&\rho_{\partial G_s}(y(s,x)) \\
&= g^{i,j}(s,x)h_{i,j}(s,x) \\
&= -g^{i,j}(s,x)\partial_{x_i}\partial_{x_j} f\langle \nu(s,x), \nu_0(x)\rangle + g^{i,j}(s,x)\widehat{H}_{i,j}(x, f(s,x), (\partial_{x_l} f(s,x))_{l\in[\![n-1]\!]}) \\
&= -g^{i,j}(s,x)\big(\partial_{x_i}\partial_{x_j} f - \Gamma^k_{i,j}(s,x)\partial_{x_k} f(s,x)\big)\langle \nu(s,x), \nu_0(x)\rangle \\
&+ g^{i,j}(s,x)\big(\widehat{H}_{i,j}(x, f(s,x), (\partial_{x_l} f(s,x)_{l\in[\![n-1]\!]}) - \Gamma^k_{i,j}(s,x)\partial_{x_k} f(s,x)\langle \nu(s,x), \nu_0(x)\rangle)\big) \\
&= -\triangle_{G_0,s} f\langle \nu(s,x), \nu_0(x)\rangle + \check{H}(x, f(s,x), (\partial_{x_l} f(s,x))_{l\in[\![n-1]\!]}, (\partial_{x_l}(\partial_{x_k} f(s,x))^2)_{l,k\in[\![n-1]\!]})
\end{aligned}
$$

where $\Gamma^k_{i,j}(s,x)$ is the Christoffel for the metric $g(s)$, this quantity depends on the derivative of g and thus on the second derivative of f, but only via $(\partial_{x_l}(\partial_{x_k} f(s,x))^2)_{l,k\in[\![n-1]\!]}$. Furthermore, as we can see below, $\nu(s,x)$ depends on the derivative of f up to order one. Note that since $\rho_{\partial G_s}(s,x)$ and $\triangle_{G_0,s} f$ are independent on the choice of the chart, the same is true for $\check{H}(x, f(s,x), (\partial_{x_l} f(s,x))_{l\in[\![n-1]\!]}, (\partial_{x_l}(\partial_{x_k} f(s,x))^2)_{l,k\in[\![n-1]\!]})$.

So if $y(s,x)$ is a solution of (37) then after taking bracket with the normal vector we get:

$$
\begin{aligned}
&\partial_s f(s,x)\langle \nu(s,x), \nu_0(x)\rangle \\
&= \langle \partial_s y(s,x), \nu_{\partial G_s}(y(s,x))\rangle_{y(s,x)} \\
&= -\rho_{\partial G_s}(y(s,x)) + \langle b(y(s,x)), \nu_{\partial G_s}(y(s,x))\rangle_{y(s,x)} + a(y(s,x)) \\
&= \triangle_{G_0,s} f\langle \nu(s,x), \nu_0(x)\rangle \\
&\quad + \widetilde{H}(x, a, b, f(s,x), (\partial_{x_l} f(s,x))_{l\in[\![n-1]\!]}, (\partial_{x_l}(\partial_{x_k} f(s,x))^2)_{l,k\in[\![n-1]\!]})
\end{aligned} \tag{39}
$$

where $\widetilde{H}$ is independent on the choice of the chart.

Furthermore, since $(v_i(s,x))_{i\in[\![n-1]\!]} := (\sqrt{g}^{i,l}(s,x)\partial_{x_l}y(s,x))_{i\in[\![n-1]\!]}$ is an orthonormal basis of the tangent space of G_s at $y(s,x)$, the vector $\nu_0(x) - \sum_{i\in[\![n-1]\!]}\langle\nu_0(x), v_i(s,x)\rangle v_i(s,x)$ is orthogonal to this tangent space. Let us compute it, taking into account that $\langle\nu_0(x), \partial_{x_l}\varphi_0(x)\rangle = 0$ and $\langle\nu_0(x), \partial_{x_l}\nu_0(x)\rangle = 0$:

$$\begin{aligned}
&\nu_0(x) - \sum_{i\in[\![n-1]\!]}\langle\nu_0(x), v_i(s,x)\rangle v_i(s,x) \\
&= \nu_0(x) - \sum_{i\in[\![n-1]\!]}\left\langle \nu_0(x), \sum_{l\in[\![n-1]\!]}\sqrt{g}^{i,l}(s,x)\partial_{x_l}y(s,x)\right\rangle \sum_{k\in[\![n-1]\!]}\sqrt{g}^{i,k}(s,x)\partial_{x_k}y(s,x) \\
&= \nu_0(x) - \sum_{i\in[\![n-1]\!]}\sum_{l\in[\![n-1]\!]}\sqrt{g}^{i,l}(s,x)\langle\nu_0(x), \partial_{x_l}y(s,x)\rangle \sum_{k\in[\![n-1]\!]}\sqrt{g}^{i,k}(s,x)\partial_{x_k}y(s,x) \\
&= \nu_0(x) - \sum_{i\in[\![n-1]\!]}\sum_{l\in[\![n-1]\!]}\sqrt{g}^{i,l}(s,x)\partial_{x_l}f(s,x) \sum_{k\in[\![n-1]\!]}\sqrt{g}^{i,k}(s,x)\partial_{x_k}y(s,x) \\
&= \nu_0(x) - \sum_{l\in[\![n-1]\!]}\sum_{k\in[\![n-1]\!]}\sum_{i\in[\![n-1]\!]}\sqrt{g}^{i,l}(s,x)\sqrt{g}^{i,k}(s,x)\partial_{x_l}f(s,x)\partial_{x_k}y(s,x) \\
&= \nu_0(x) - \sum_{l\in[\![n-1]\!]}\sum_{k\in[\![n-1]\!]}g^{kl}(s,x)\partial_{x_l}f(s,x)\partial_{x_k}y(s,x)
\end{aligned}$$

In particular, this vector is different to zero for f and ∇f small enough and we get then

$$\nu(s,x) = \frac{\nu_0(x) - \partial_{x_i}f(s,x)g^{i,j}(s,x)\partial_{x_j}(\varphi_0(x) + f(s,x)\nu_0(x))}{\|\nu_0(x) - \partial_{x_i}f(s,x)g^{i,j}(s,x)\partial_{x_j}(\varphi_0(x) + f(s,x)\nu_0(x))\|}$$

It follows equally that $\langle\nu(s,x), \nu_0(x)\rangle$ is different to zero for f and ∇f small enough, and thus dividing (39) by $\langle\nu(s,x), \nu_0(x)\rangle$, we get (38) for a smooth function H deduced from the previous computations.

When s, $f(s,x)$ and $\nabla_{G_0,s}f(s,x)$ are small, the implicit function theorem enables us to write (38) under the form considered in Appendix A of Mantegazza [17], due to the strict ellipticity of the operator $\triangle_{G_0,s}$ on G_0 and to the fact that

$$(\partial_{x_i}(\partial_{x_j}f(s,x))^2)_{i,j\in[\![n-1]\!]} = 2(\partial_{x_j}f(s,x)(\partial_{x_i}\partial_{x_j}f(s,x)))_{i,j\in[\![n-1]\!]}$$

As shown by Appendix A of Mantegazza [17], such quasi-linear parabolic equations admit a unique solution on a small time interval containing 0, so this existence and uniqueness result holds for (38). It would also be possible to put in front of the term $\rho_{\partial G_s}(x)$ of (37) a positive quantity depending smoothly on x.

Remark 15 We have written in a natural way the leading term of $\rho_{\partial G_s}$ in terms of the Laplacian for the metric $g(s)$. Unfortunately the equation we will need will not be exactly of this form, because we will have an additional stochastic term, carefully studied in Section 4. For the short time existence, we will prefer to write this leading term in terms of a fixed manifold with a fixed metric as in (62) in Subsection 4.1 .

□

Remark 16 Let us come back to the search of a smooth family $(G_s)_{s\in[0,S_+)}$ satisfying (34), where $\alpha \in \mathcal{C}^\infty(G_0)$ is given. First extend $\rho_{\partial G_0} + \alpha$ from ∂G_0 to V, to obtain a smooth function $a \in \mathcal{C}^\infty(V)$ coinciding with $\rho_{\partial G_0} + \alpha$ on ∂G_0. Next define for any $D \in \mathcal{D}$,

$$\forall\ x \in C, \qquad \alpha_C(x) = -\rho_C(x) + a(x)$$

The radial evolution equation

$$\forall\ s \in [0, S_+),\ \forall\ x \in \partial G_s, \qquad \partial_s x \;=\; \alpha_{\partial G_s}(x)\nu_{\partial G_s}(x) \tag{40}$$

is of the form (37) and so admits a unique solution for small enough intervals $[0, S_+)$. Restricting the above equation to $s = 0$ shows that $(G_s)_{s\in[0,S_+)}$ solves (34).

This construction seems particularly cumbersome, it would be more natural to extend α from ∂G_0 to V to get a smooth function $c \in \mathcal{C}^\infty(V)$ and to solve the radial evolution equation

$$\forall\ s \in [0, S_+),\ \forall\ x \in \partial G_s, \qquad \partial_s x \;=\; c(x)\nu_{\partial G_s}(x) \tag{41}$$

Unfortunately, doing so, we end up with a Hamilton-Jacobi equation (see e.g. Chapter 3 of Evans [8]) instead of the quasi-linear parabolic equation (38). One would then be led to investigate if the usual conditions for existence and uniqueness of the solutions to the Hamilton-Jacobi equations are satisfied and thus to describe more precisely the function H appearing in (38), but this is not so nice.

The normal flow equation (32), corresponding to $c = \mathbb{1}$, was simple to solve (in both direction of the time, contrary to the above quasi-linear parabolic equations), because the normal vectors are transported in a parallel way by the geodesic flows directed by these normal vectors.

□

Equations of the type (37) are adapted to our purposes: only considering the last vector field in (22), i.e. the heuristic $\mathcal{D}$-valued "ordinary" differential equation $dD_t = \mathfrak{V}_2(D_t)dt$, amounts to solve the following modification of the mean curvature flow:

$$\forall\ s \in [0, S_+),\ \forall\ x \in \partial G_s, \qquad \partial_s x \;=\; -\rho^b_{\partial G_s}(x)\nu_{\partial G_s}(x) \tag{42}$$

where

$$\forall\ D \in \mathcal{D},\ \forall\ x \in C, \qquad \rho^b_C(x) \;\coloneqq\; \rho_C(x) + \langle \nabla U(x) - \beta(x), \nu_C(x)\rangle_x \tag{43}$$

(despite the b in supscript, remember that $b = \nabla U + \beta$ and not $\nabla U - \beta$, as the above formula could suggest).

Let $D_0 \in \mathcal{D}$ be given, as well as $(B_t)_{t\geqslant 0}$ a standard (one-dimensional) Brownian motion starting from 0. To solve (22), we are looking for a stochastic $\mathcal{D}$-valued evolution $(D_t)_{t\in[0,\tau)}$, where $\tau > 0$ is a stopping time (wrt. to the filtration generated by the Brownian motion), such that

$$\forall\ t \in [0, \tau),\ \forall\ x \in C_t, \qquad dx \;=\; \left(\sqrt{2}dB_t + 2\frac{\underline{\mu}(C_t)}{\mu(D_t)}dt - \rho^b_{C_t}(x)dt\right)\nu_{C_t}(x) \tag{44}$$

where $C_t \coloneqq \partial D_t$.

To explain the Doss [6] and Sussman [28] approach to such stochastic differential equations, it is helpful to first replace $\sqrt{2}\,dB_t + 2\underline{\mu}(C_t)/\mu(D_t)\,dt$ by $d\xi_t = \xi'_t\,dt$, where $\xi : \mathbb{R}_+ \to \mathbb{R}$ is a given $\mathcal{C}^1$ function with $\xi_0 = 0$. Still starting from D_0, we would like to solve the radial evolution equation

$$\forall\ t \in [0, \epsilon),\ \forall\ x \in C_t, \qquad \partial_t x \;=\; \left(\xi'_t - \rho^b_{C_t}(x)\right)\nu_{C_t}(x) \tag{45}$$

for some $\epsilon > 0$, without using the derivative $(\xi'_t)_{t\in[0,\epsilon]}$. To do so, we begin by solving another radial evolution equation

$$\begin{cases} G_0 \;=\; D_0 \\ \forall\ t \in [0, \widetilde{\epsilon}),\ \forall\ x \in \partial G_t, \qquad \partial_t x \;=\; \alpha_{\partial G_t, \xi_t}(x)\nu_{\partial G_t}(x) \end{cases} \tag{46}$$

for some $\widetilde{\epsilon} > 0$ small enough, where α is defined by

$$\forall\ r > 0,\ \forall\ D \in \mathcal{D}_r,\ \forall\ x \in C, \qquad \alpha_{C,r}(x) \ \coloneqq\ -\rho^b_{\Psi(C,r)}(\psi_{C,r}(x)) \tag{47}$$

where $\Psi(C, r)$ was defined after Remark 9, taking into account (24), (25), (26), (35). Next, consider

$$\epsilon \ \coloneqq\ \inf\{t \in [0, \widetilde{\epsilon}) : G_t \notin \mathcal{D}_{\xi_t}\} \ >\ 0$$

(with the usual convention that $\epsilon = \widetilde{\epsilon}$ if the set in r.h.s. is empty) and define

$$\forall\ t \in [0, \epsilon), \qquad D_t \ \coloneqq\ \Psi(G_t, \xi_t)$$

Let us check that this is indeed a solution of (45). First, we have $\Psi(G_0, \xi_0) = \Psi(D, 0) = D_0$. Concerning the evolution, differentiate with respect to the first and second variables of Ψ to find

$$\begin{aligned}\forall\ t \in [0, \epsilon),\ \forall\ x \in C_t, \qquad \partial_t x \ &=\ (T_{G_t}\Psi(\cdot, \xi_t)[\alpha_{\partial G_t, \xi_t}](x) + \xi'_t)\, \nu_{C_t}(x) \\ &=\ \left(-\rho^b_{C_t}(x) + \xi'_t\right) \nu_{C_t}(x)\end{aligned}$$

as wanted, where we used Lemma 13. Denote h the mapping defined on $\mathcal{D}$ by

$$\forall\ D \in \mathcal{D}, \qquad h(D) \ =\ 2\frac{\mu(C)}{\mu(D)}$$

For given $D_0 \in \mathcal{D}$ and a $\mathcal{C}^1$ function $\zeta : \mathbb{R}_+ \to \mathbb{R}$, we are now looking for a solution, starting from D_0, to the radial evolution equation

$$\forall\ t \in [0, \epsilon),\ \forall\ x \in C_t, \qquad \partial_t x \ =\ \left(\zeta'_t + h(D_t) - \rho^b_{C_t}(x)\right) \nu_{C_t}(x) \tag{48}$$

for some $\epsilon > 0$. Following computations similar to those presented above, we get a solution by taking, for $t > 0$ small enough,

$$D_t \ \coloneqq\ \Psi(G_t, \zeta_t + \theta_t) \tag{49}$$

where the $\mathbb{R}_+ \times \mathcal{D}$-valued family $(\theta_t, G_t)_{t\in[0,\epsilon)}$, for $\epsilon > 0$ small enough, is a solution of the system starting from $(\theta_0, G_0) = (0, D_0)$ and satisfying

$$\forall\ t \in [0, \epsilon), \qquad \begin{cases} \frac{d}{dt}\theta_t \ =\ h(\Psi(G_t, \zeta_t + \theta_t)) \\ \forall\ x \in \partial G_t, \quad \partial_t x \ =\ \alpha_{\partial G_t, \zeta_t+\theta_t}(x)\nu_{\partial G_t}(x) \end{cases} \tag{50}$$

The formulations (49) and (50) do not require that the function ζ is differentiable.

These remarks suggest to solve (44) by replacing $(\zeta_t)_{t\geqslant 0}$ by $(\sqrt{2}B_t)_{t\geqslant 0}$ in (49) and (50), up to the random time τ these constructions are allowed: τ will be a stopping time with respect to the filtration generated by the Brownian motion $(B_t)_{t\geqslant 0}$. This is the Doss [6] and Sussman [28] method, adapted to our evolving domain framework.

So given $D_0 \in \mathcal{D}$, we are led to consider the following stochastic radial evolution equation system with respect to $(\theta_t, G_t)_{t\in[0,\epsilon)}$, starting with $(\theta_0, G_0) = (0, D_0)$:

$$\forall\ t \in [0, \epsilon), \qquad \begin{cases} \frac{d}{dt}\theta_t \ =\ h(\Psi(G_t, \sqrt{2}B_t + \theta_t)) \\ \forall\ x \in \partial G_t, \quad \partial_t x \ =\ \alpha_{\partial G_t, \sqrt{2}B_t+\theta_t}(x)\nu_{\partial G_t}(x) \end{cases} \tag{51}$$

In Section 4, we show the existence of a solution of (46), where $(\xi_t)_{t\geqslant 0} = (\sqrt{2}B_t)_{t\geqslant 0}$ and the existence of a solution of (51). There, we will only consider the case $V = \mathbb{R}^{n+1}$, the situation of a general manifold

V is similar up to some modifications, which are straightforward from a conceptual point of view, but induce complicated notations.

Once (51) is solved, define as in (49),

$$\forall\, t \in [0, \tau), \qquad D_t \;\coloneqq\; \Psi(G_t, \sqrt{2}B_t + \theta_t). \tag{52}$$

up to the stopping time τ until which this construction is permitted.

Let us now check that (52) provides a solution to the martingale problem presented in the introduction:

Theorem 17 *The stopped stochastic process $(D_t)_{t\in[0,\tau)}$, defined on the natural filtered probability space of the standard Brownian motion $(B_t)_{t\geqslant 0}$, is a solution to the stopped martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ and to the starting domain D.*

Proof

Fix some $f \in \mathcal{C}^\infty(V)$. On the set $I \coloneqq \{(s, r) \in \mathbb{R}_+ \times \mathbb{R} : G_s \in \mathcal{D}_r\}$, consider the mapping

$$(s, r) \;\mapsto F_f(\Psi(G_s, r)) \tag{53}$$

According to Lemma 10, this mapping is $\mathcal{C}^2$ in the second variable. Concerning the first variable, note that for $(s, r) \in I$, we have

$$\forall\, x \in \partial G_s, \qquad \partial_s x \;=\; -\rho^b_{\Psi(\partial G_s, r)}(\psi_{\partial G_s, r}(x))\nu_{\partial G_s}(x) \tag{54}$$

From Lemma 13, we deduce that

$$\forall\, x \in \Psi(\partial G_s, r), \qquad \partial_s x \;=\; -\rho^b_{\Psi(\partial G_s, r)}(x)\nu_{\Psi(\partial G_s, r)}(x) \tag{55}$$

and from Lemma 14, that for any $f \in \mathcal{C}^\infty(V)$,

$$\frac{d}{ds} F_f(\Psi(G_s, r)) \;=\; -\int_{\Psi(\partial G_s, r)} f(x)\rho^b(x)\,\underline{\mu}(dx) \tag{56}$$

In particular, the mapping defined in (53) is $\mathcal{C}^1$ in the first variable.

These observations enable us to apply Itô's formula to $[0, \tau) \ni t \mapsto F_f(\Psi(G_t, \sqrt{2}B_t + \theta_t))$ to get its stochastic evolution:

$$\begin{aligned}
& dF_f(\Psi(G_t, \sqrt{2}B_t + \theta_t)) \\
&= -\left(\int_{\partial\Psi(G_t, \sqrt{2}B_t+\theta_t)} f\rho^b\, d\underline{\mu}\right) dt + \left(\int_{\partial\Psi(G_t, \sqrt{2}B_t+\theta_t)} f\, d\underline{\mu}\right)(\sqrt{2}dB_t + \partial_t\theta_t dt) \\
&\quad + \left(\int_{\partial\Psi(G_t, \sqrt{2}B_t+\theta_t)} \langle \nabla f, \nu\rangle\, d\underline{\mu} + \int_{\partial\Psi(G_t, \sqrt{2}B_t+\theta_t)} f(\rho + \langle \nabla U, \nu\rangle)\, d\underline{\mu}\right) dt \\
&= \left(\int_{\partial\Psi(G_t, \sqrt{2}B_t+\theta_t)} \langle \nabla f, \nu\rangle + f(h(\Psi(G_t, \sqrt{2}B_t + \theta_t)) + \langle \beta, \nu\rangle)\, d\underline{\mu}\right) dt \\
&\quad + \sqrt{2}\left(\int_{\partial\Psi(G_t, \sqrt{2}B_t+\theta_t)} f\, d\underline{\mu}\right) dB_t \\
&= \mathfrak{L}[F_f](D_t)\, dt + dM_t
\end{aligned}$$

where we used (6) and where $(M_t)_{t\in[0,\tau)}$ is a local martingale whose bracket is given by

$$\forall\, t \in [0, \tau), \qquad \langle M\rangle_t \;=\; 2\int_0^t \Gamma_{\mathfrak{L}}[F_f, F_g](D_s)\, ds$$

This description and the continuity of the trajectories $[0, \tau) \ni t \mapsto F_f(D_t)$ imply that $(D_t)_{t\in[0,\tau)}$ is a solution to the martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ (see e.g. the book of Bakry, Gentil and Ledoux [3]). Since $D_0 = D$, we conclude to the wanted result.

■

Remark 18 There are potentially other ways to use the Doss-Sussman approach. For instance, Equation (22) can be rewritten under the form

$$dD_t = \sqrt{2}\,\mathfrak{V}_1(D_t)\,dB_t + \widetilde{\mathfrak{V}}_2(D_t)\,dt \tag{57}$$

where $\widetilde{\mathfrak{V}}_2(D) := 2h(D)\mathfrak{V}_1(D) + \mathfrak{V}_2(D)$ for any $D \in \mathcal{D}$. Similarly to (43) and (47), define

$$\begin{aligned} \forall\ D \in \mathcal{D},\ \forall\ x \in C, \qquad \widetilde{\rho}^b_C(x) &:= \rho_C(x) + \langle \nabla U(x) - \beta(x), \nu_C(x)\rangle_x - h(D) \\ \forall\ r > 0,\ \forall\ D \in \mathcal{D}_r,\ \forall\ x \in C, \qquad \widetilde{\alpha}_{C,r}(x) &:= -\widetilde{\rho}^b_{\Psi(C,r)}(\psi_{C,r}(x)) \end{aligned}$$

Next try to construct a family $(\widetilde{G}_t)_{t\in[0,\epsilon)}$ (where $\epsilon > 0$ is a stopping time) such that

$$\forall\ t \in [0, \epsilon),\ \forall\ x \in \partial\widetilde{G}_t, \qquad \partial_t x = \widetilde{\alpha}_{\partial G_t,\sqrt{2}B_t}(x)\nu_{\partial G_t}(x)$$

Contrary to (51), no auxilliary $(\theta_t)_{t\in[0,\epsilon)}$ is needed here, but the above equation is not really of the type (37), due to the isoperimetric ratio. Nevertheless, it should be possible to adapt to this situation the fixed point approach presented in Section 4.

Once $(\widetilde{G}_t)_{t\in[0,\epsilon)}$ has been constructed, consider

$$\forall\ t \in [0, \tau), \qquad D_t := \Psi(\widetilde{G}_t, \sqrt{2}B_t)$$

with

$$\tau := \inf\{t \in [0, \epsilon)\ :\ \widetilde{G}_t \notin \mathcal{D}_{\sqrt{2}B_t}\}$$

Then the stopped stochastic process $(D_t)_{t\in[0,\tau)}$, defined on the natural filtered probability space of the standard Brownian motion $(B_t)_{t\geqslant 0}$, is a solution to the martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ and to the starting domain G_0.

We preferred to present how to solve (22), because the flows associated to $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are quite famous (at least when $\nabla U = \beta = 0$) and well-investigated. But maybe the flow associated to the radial equation

$$\forall\ x \in C_t, \qquad \partial_t x = (h(D_t) - \rho_{C_t}(x))\,\nu_{C_t}(x)$$

is also a natural object to study. In Subsection 5.2, we will check in the homogeneous setting of Section 2 that this alternative Doss-Sussman approach should be preferred to the one considered in the proof of Theorem 17.

□

4 Existence of a stochastic modified mean curvature flow

This section presents the quite technical proofs of the existence of regular solutions to (46) and (51), respectively the following subsections. As announced before Theorem 17, we only deal with $V = \mathbb{R}^{n+1}$ to avoid complicated notations.

We begin by recollecting our notations: $\mathcal{D}$ is the set of non-empty, compact and connected domains D in V, which coincide with the closure of their interior and whose boundary $C \coloneqq \partial D$ is smooth. The exterior normal vector ν_C and the mean curvature ρ_C are defined on C. Recall we were given a function $U \in \mathcal{C}^\infty(V)$ and a smooth vector field β satisfying $\mathrm{div}(\exp(U)\beta) = 0$, to which is associated the smooth vector field $b \coloneqq \nabla U + \beta$. Denote $\mu \coloneqq \exp(U)\lambda$, the measure admitting the density $\exp(U)$ with respect to the Riemannian measure λ (when μ gives a finite weight to V, it is normalized into a probability measure, which amounts to add a constant to U). The interest of μ is to be reversible for the operator $L \coloneqq \triangle + b$. We associate to the boundary C the $(\dim(V)-1)$-Hausdorff measure $\underline{\mu}_C$ coming from μ, namely admitting the density $\exp(U)$ with respect to the usual Riemannian $(\dim(V)-1)$-Hausdorff measure. We also distort ρ_C by introducing the modified mean curvature ρ_C^b defined by

$$\forall\, x \in C, \qquad \rho_C^b(x) \;\coloneqq\; \rho_C(x) + \langle \nabla U(x) - \beta(x), \nu_C(x)\rangle_x$$

Let $D_0 \in \mathcal{D}$ be given, as well as $(B_t)_{t\geqslant 0}$ a standard real Brownian motion starting from 0. We are looking for a stochastic $\mathcal{D}$-valued evolution $(D_t)_{t\in[0,\tau)}$, where $\tau > 0$ is a stopping time, such that

$$\forall\, t \in [0,\tau),\ \forall\, x \in C_t, \qquad dx \;=\; \left(\sqrt{2}dB_t + 2h(D_t)dt - \rho_{C_t}^b(x)dt\right)\nu_{C_t}(x) \tag{44}$$

where

$$\forall\, D \in \mathcal{D}, \qquad h(D) \;\coloneqq\; 2\frac{\underline{\mu}(C)}{\mu(D)}$$

Resorting to the Doss [6] and Sussman [28] method, we are led to solve consecutively:

- The deterministic radial equation in $(G_t)_{t\in[0,\widetilde{\epsilon})}$:

$$\left\{\begin{array}{rcl} G_0 &=& D_0 \\ \forall\, t \in [0,\widetilde{\epsilon}),\ \forall\, x \in \partial G_t, \qquad \partial_t x &=& \alpha_{\partial G_t,\xi_t}(x)\nu_{\partial G_t}(x)\end{array}\right. \tag{46}$$

  where $\mathbb{R}_+ \ni t \mapsto \xi_t \in \mathbb{R}$ is assumed to be α-Hölder regular with $\alpha \in (0,1/2)$, $\widetilde{\epsilon}$ is small enough and

$$\forall\, r > 0,\ \forall\, D \in \mathcal{D}_r,\ \forall\, x \in C, \qquad \alpha_{C,r}(x) \;\coloneqq\; -\rho^b_{\Psi(C,r)}(\psi_{C,r}(x))$$

  with for any $r \in \mathbb{R}$,

$$\begin{array}{rcl} \psi_{C,r} &:& C \ni x \mapsto \exp_x(r\nu_C(x)) \in V \\ \Psi(C,r) &\coloneqq& \{\psi_{C,r}(x) \,:\, x \in C\} \\ \mathcal{D}_r &\coloneqq& \{D \in \mathcal{D} \,:\, r \in (R_-(D), R_+(D))\} \\ R_+(D) &\coloneqq& \inf\{r \in (0,+\infty) \,:\, \psi_{C,r} \text{ is not a diffeomorphism on its image}\} \\ R_-(D) &\coloneqq& -\inf\{r \in (0,+\infty) \,:\, \psi_{C,-r} \text{ is not a diffeomorphism on its image}\}\end{array}$$

- The radial system in $(\theta_t, G_t)_{t\in[0,\epsilon)}$:

$$\forall\, t \in [0,\epsilon), \qquad \left\{\begin{array}{rcl} \frac{d}{dt}\theta_t &=& h(\Psi(G_t,\sqrt{2}\zeta_t + \theta_t)) \\ \forall\, x \in \partial G_t, \quad \partial_t x &=& \alpha_{\partial G_t,\sqrt{2}\zeta_t+\theta_t}(x)\nu_{\partial G_t}(x)\end{array}\right. \tag{50}$$

  where $\mathbb{R}_+ \ni t \mapsto \zeta_t \in \mathbb{R}$ is assumed to be α-Hölder regular with $\alpha \in (0,1/2)$, ϵ is small enough and

$$\forall\, r \in \mathbb{R},\ \forall\, D \in \mathcal{D}, \qquad \Psi(D,r) \;\coloneqq\; \bigcup_{s\in(-\infty,r]} \Psi(C,s)$$

The interest of these manipulations is that a solution of (44) will be given by

$$\forall\, t \in [0,\tau), \qquad D_t \;\coloneqq\; \Psi(G_t, \sqrt{2}B_t + \theta_t)$$

where in (50) we take $(\zeta_t)_{t\geqslant 0} = (B_t)_{t\geqslant 0}$ and where τ is the corresponding ϵ, which ends up being a stopping time with respect to the filtration generated by $(B_t)_{t\geqslant 0}$.

4.1 Local existence of a pushed mean curvature flow

Let $F_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of an n-dimensional manifold M such that $F_0(M) = C$. Let $r : t \in [0, \infty) \mapsto r(t) \in \mathbb{R}$ be a real continuous function. Consider the following equation, which is similar to (46) (i.e. $\partial G_t = F(t, M)$), taking into account the remark made before Lemma 13:

$$\begin{cases} \forall\, x \in M, \quad \left\langle \frac{\partial}{\partial t} F(t, x), \nu^F(t, x) \right\rangle &= -\rho^b_{\Psi(F(t,M),r(t))}(\psi_{F(t,M),r(t)}(x)) \\ F(0, x) &= F_0(x), \end{cases} \tag{58}$$

where $\nu^F(t, x)$ is the normal vector of the hypersurface $F(t, M)$ at $F(t, x)$. The goal of this section is to show existence in small time of solution of (58) with enough regularity in space and time, under the hypotheses that $r(0)$ is small enough and that r is $\alpha/2$-Hölder regular, for some $\alpha \in (0, 1)$.

To get a small time existence of equation (58) we will convert the problem in terms of a quasi-parabolic equation. We will study the linearisation of this equation, it turns to be linear and strictly parabolic for small time, with $C^{\alpha/2,\alpha}([0, T] \times M)$ coefficients when M is a $C^{2+\alpha}$ manifold. We will resort to an existing result on the existence and regularity of the solution of such a linear equation. Then we will use the inverse function theorem to get a solution of the original equation (58).

Let $C = F_0(M)$, we will suppose that M is a $C^{3+\alpha}$ manifold and F_0 is a $C^{3+\alpha}$ diffeomorphism (in general we will denote by $\mathrm{reg}(M)$ the manifold regularity of M), so that C is also a $C^{3+\alpha}$ manifold. Small perturbations in time of C under (58) live in a small tubular neighborhood of C, and as in Mantegazza [17], a useful way to obtain a quasi-linear equation from (58) is to represent the solution as graphs over the fixed hypersurface C. The underlying idea is to consider again the parametrization $(r_-, r_+) \times C \ni (r, y) \mapsto \psi_{C,r}(y)$ of a tubular neighborhood of C, where (r_-, r_+) is a small neighborhood of 0. Let $x \in M$, and $\nu_0(x)$ be the unit outward normal of the hypersurface $C = F_0(M)$ at the point $F_0(x)$. Then one looks at the function $f(t, .) : M \to \mathbb{R}$, with enough regularity, whose image is included into (r_-, r_+) and which satisfies

$$F(t, x) = \psi_{C,f(t,x)}(F_0(x)) = F_0(x) + f(t, x)\nu_0(x),$$

for all $(t, x) \in [0, S_+) \times M$, with S_+ small enough, i.e. we represent $F(t, M)$ as a graph over C, since $C = F_0(M)$ we have $f(0, .) = 0$ and the existence of S_+ is due to the regularity of f and the compactness of M.

Let x_i be a local chart of M, $g_{i,j}(0, x) = \langle \partial_i F_0, \partial_j F_0 \rangle$ the Riemannian metric at x in this chart, $g^{i,j}(0, x)$ its inverse, $h_{i,j}(0, x) := \Pi(\partial_i F_0, \partial_j F_0) = \langle \nabla_{\partial_i F_0(x)} \nu_0(x), \partial_j F_0(x) \rangle$ where Π in the second fundamental form of C at $F_0(x)$ and define $S_{i,j}(0, x) = h_{i,k} g^{k,l} h_{lj}(0, x)$, where the convention that every repeated lower indices and upper indices is considered as a sum is enforced, as in the whole paper. We end up with the quasi-linear parabolic equation with respect to f in order that $F(t, .)$ satisfies (58), after taking care that we have some dilation term $r(t)$ in the equation. We have for all $i, j \in [\![n]\!]$, $t \in [0, S_+)$ and $x \in M$,

$$\begin{aligned}
\tfrac{\partial}{\partial t} F(t, x) &= \partial_t f(t, x)\nu_0(x) \\
\partial_i \nu_0(x) &= h_{i,k} g^{k,l}(0, x) \partial_l F_0(x) \\
\partial_i F(t, x) &= \partial_i F_0(x) + f(t, x) h_{i,k} g^{k,l}(0, x) \partial_l F_0(x) + \partial_i f(t, x)\nu_0(x) \\
g_{i,j}(t, x) &= \langle \partial_i F(t, x), \partial_j F(t, x) \rangle \\
&= g_{i,j}(0, x) + 2 f(t, x) h_{i,j}(0, x) + f^2(t, x) S_{i,j}(0, x) + \partial_i f(t, x) \partial_j f(t, x) \\
&=: G_{i,j}(t, x, f, \nabla f) \\
\nu(t, x) &= \frac{\nu_0(x) - \partial_i f(t,x) g^{i,j}(t,x) \partial_j (F_0(x) + f(t,x)\nu_0(x))}{\|\nu_0(x) - \partial_i f(t,x) g^{i,j}(t,x) \partial_j (F_0(x) + f(t,x)\nu_0(x))\|} \\
h_{i,j}(t, x) &= -\langle \nu(t, x), \partial_i \partial_j F(t, x) \rangle \\
&= -\langle \nu(t, x), \partial_i \partial_j f(t, x)\nu_0(x) + \partial_i \partial_j F_0(x) + \partial_i f(t, x) \partial_j \nu_0(x) + \partial_j f(t, x) \partial_i \nu_0(x) \\
&\quad + f(t, x) \partial_i \partial_j \nu_0(x) \rangle =: H_{i,j}(t, x, f, \nabla f, \nabla\nabla f)
\end{aligned} \tag{59}$$

where the second equality is the Gauss-Weingarten formula, where $\nu(t,x)$ is the unit normal of the hypersurface $F(t,M)$ at $F(t,x)$, and where we used the Gram-Schmidt procedure in the computation of $\nu(t,x)$ (taking into account that $\left(v_i := \sqrt{g}^{i,l}(t,x)\partial_l F(t,x)\right)_{i\in[\![n]\!]}$ is an orthonormal basis of $T_{F(t,x)}F(t,M)$). To simplify the notations, denote $G := (G_{i,j}(t,x,f,\nabla f))_{i,j\in[\![n]\!]}$ and $H := (H_{i,j}(t,x,f,\nabla f,\nabla\nabla f))_{i,j\in[\![n]\!]}$, which take values in $S^{n\times n}$, the space of symmetric matrices. Note that G does not depend on $\nabla\nabla f$ and that H has regularity $\mathrm{reg}(M)-3$ in x (due to the term $\partial_i\partial_j\nu_0(x)$ in $H_{i,j}(t,x,f,\nabla f,\nabla\nabla f)$).

To manage the right hand side of (58), define

$$\begin{aligned}\widetilde{M}_t &:= \Psi(F(t,M),r(t))\\ \widetilde{F}(t,x) &:= \Psi_{F(t,M),r(t)}(F(t,x)) = F(t,x)+r(t)\nu(t,x)\end{aligned}$$

and denote all the quantities that depend on $\widetilde{M}_t = \widetilde{F}(t,M)$ by the same letter as for $F(t,M)$ with a tilde. So by the same computation as above we have for all $i,j\in[\![n]\!]$, $t\in[0,S_+)$ and $x\in M$:

$$\begin{aligned}\partial_i\widetilde{F}(t,x) &= \partial_i F(t,x)+r(t)\partial_i\nu(t,x)\\ &= \partial_i F(t,x)+r(t)h_{i,k}g^{k,l}(t,x)\partial_l F(t,x)\\ \widetilde{g}_{i,j}(t,x) &= g_{i,j}(t,x)+2r(t)h_{i,j}(t,x)+r(t)^2S_{i,j}(t,x)\\ &= (G(\mathrm{Id}+2r(t)G^{-1}H+r(t)^2G^{-1}HG^{-1}H))_{i,j}\\ &= \left(G\left(\mathrm{Id}+r(t)G^{-1}H\right)^2\right)_{i,j} =: \widetilde{G}_{i,j}(t,x,f,\nabla f,\nabla\nabla f)\\ \widetilde{\nu}(t,x) &= \nu(t,x)\\ \widetilde{h}_{i,j}(t,x) &= -\langle\nu(t,x),\partial_i\partial_j\widetilde{F}(t,x)\rangle = H_{i,j}-r(t)\langle\nu(t,x),\partial_i\partial_j\nu(t,x)\rangle\\ &= H_{i,j}+r(t)\langle\partial_i\nu(t,x),\partial_j\nu(t,x)\rangle\\ &= H_{i,j}+r(t)S_{i,j}(t,x) = (H+r(t)HG^{-1}H)_{i,j} = (H(\mathrm{Id}+r(t)G^{-1}H))_{i,j}\\ &=: \widetilde{H}_{i,j}(t,x,\nabla f,\nabla\nabla f)\end{aligned} \tag{60}$$

As usual, denote $\widetilde{G} := (\widetilde{G}_{i,j}(t,x,f,\nabla f,\nabla\nabla f)_{i,j\in[\![n]\!]}$, $\widetilde{H} := (\widetilde{H}_{i,j}(t,x,f,\nabla f,\nabla\nabla f))_{i,j\in[\![n]\!]}$ and $(\widetilde{G}^{i,j}(t,x,f,\nabla f,\nabla\nabla f))_{i,j\in[\![n]\!]} := \widetilde{G}^{-1}$, all taking values in $S^{n\times n}$, so that we have for the mean curvature

$$\begin{aligned}-\rho_{\Psi(F(t,M),r(t))}(\psi_{F(t,M),r(t)}(x)) &= -\widetilde{G}^{i,j}\widetilde{H}_{i,j} = -\mathrm{tr}(\widetilde{G}^{-1}\widetilde{H})\\ &= -\mathrm{tr}\left(\left(\mathrm{Id}+r(t)G^{-1}H\right)^{-2}G^{-1}H(\mathrm{Id}+r(t)G^{-1}H)\right)\\ &= -\mathrm{tr}\left(\left(\mathrm{Id}+r(t)G^{-1}H\right)^{-1}G^{-1}H\right)\\ &=: \hat{\Phi}_1(t,x,f,\nabla f,\nabla\nabla f).\end{aligned}$$

for some mapping $\hat{\Phi}_1$. Note that in the above formula only H depends on $\nabla\nabla f$. Furthermore consider the mapping $\hat{\Phi}_2$ such that

$$\begin{aligned}\left\langle\nabla U-\beta,\nu_{\Psi(F(t,M),r(t))}\right\rangle_{\psi_{F(t,M),r(t)}(x)} &= \left\langle\nabla U(\widetilde{F}(t,x))-\beta(\widetilde{F}(t,x)),\widetilde{\nu}(t,x)\right\rangle_{\widetilde{F}(t,x)}\\ &= \left\langle\nabla U(\widetilde{F}(t,x))-\beta(\widetilde{F}(t,x)),\nu(t,x)\right\rangle_{\widetilde{F}(t,x)}\\ &=: \hat{\Phi}_2(t,x,f,\nabla f)\end{aligned}$$

Remark the above expression does not depend on $\nabla\nabla f$. Define

$$\hat{\Phi}(t,x,f,\nabla f,\nabla\nabla f) := \hat{\Phi}_1(t,x,f,\nabla f,\nabla\nabla f)+\hat{\Phi}_2(t,x,f,\nabla f) \tag{61}$$

so that Equation (58) becomes the following non-linear parabolic equation

$$\left\{\begin{array}{lll} \partial_t f(t,x) & = & \frac{1}{\langle \nu_0(x),\nu(t,x)\rangle}\hat{\Phi}(t,x,f,\nabla f,\nabla\nabla f) \\ & =: & \Phi(t,x,f,\nabla f,\nabla\nabla f) \\ f(0,x) & = & 0 \end{array}\right. \tag{62}$$

Note that at time $t=0$ we have $f(0,x)=0$, $\nabla f(0,x)=0$, $\nabla\nabla f(0,x)=0$.

The application Φ defined above will be considered with the following argument $\Phi(t,x,z,v,q)$, where $(t,x)\in M_T := [0,T]\times M$, $z\in\mathbb{R}$, $v\in T_xM$ and q is a symmetric matrix in $T_x^*M\odot T_x^*M$. Since r is continuous and $r(0)=0$ (or small enough), for small T, Φ is smooth in three last variables in a neighborhood $(0,0,0)$ and have at least the regularity $\mathrm{reg}(M)-3$ in x, and the same Hölder regularity in time as r (i.e. it is enough to have G invertible and $\|r(t)G^{-1}H\|<1$). More precisely we have the following proposition.

Proposition 19 *There exist $T>0$ and $R_0>0$ such that*

- *the mapping*

$$\begin{array}{rcl} \Phi\ :\ [0,T]\times M\times B_{(0_{\mathbb{R}},0_{\mathbb{R}^n},0_{S^{n\times n}})}(R_0) & \to & \mathbb{R} \\ (t,x,z,v,q) & \mapsto & \Phi(t,x,z,v,q) \end{array} \tag{63}$$

  is smooth in the three last components,

- *the mapping $t\mapsto\Phi(t,x,z,v,q)$ have the same Hölder regularity in time as r,*

- *the mapping $x\mapsto\Phi(t,x,z,v,q)$ have at least the regularity* $\mathrm{reg}(M)-3$.

Proof

Recall that

$$\begin{array}{rcl} G(t,x,z,v) & = & G(0,x)+2zH(0,x)+z^2S(0,x)+v\otimes v, \\ \nu(t,x,z,v) & = & \frac{\nu_0(x)-v_ig^{i,j}(t,x,z,v)\left(\partial_jF_0(x)+zh_{j,k}g^{k,l}(0,x)\partial_lF_0(x)+v_j\nu_0(x)\right)}{\|\nu_0(x)-v_ig^{i,j}(t,x,z,v)\left(\partial_jF_0(x)+zh_{j,k}g^{k,l}(0,x)\partial_lF_0(x)+v_j\nu_0(x)\right)\|}, \\ H_{i,j}(t,x,z,v,q) & = & -\langle\nu(t,x,z,q),\nu_0(x)\rangle q_{i,j}-\langle\nu(t,x,z,q),\partial_i\partial_jF_0(x)\rangle \\ & & -\langle\nu(t,x,z,q),v_i\partial_j\nu_0(x)+v_j\partial_i\nu_0(x)+z\partial_i\partial_j\nu_0(x)\rangle \end{array} \tag{64}$$

Since $G(0,x)$ is invertible and M is compact, there exist $R_0,C_1,C_2>0$ such that for $|z|,\|v\|,\|q\|\leqslant R_0$,

$$G(t,x,z,v) \text{ is invertible for all } x\in M,$$

$$\|G^{-1}(t,x,z,v)\|\leqslant C_1,$$

$$\|\nu(t,x,z,v)-\nu_0(x)\|\leqslant\frac{1}{2},$$

$$\|H(t,x,z,v,q)\|\leqslant C_2.$$

Thus, since r is continuous and $r(0)=0$ (or small enough), take $T>0$ such that

$$T\ :=\ \sup\left\{u\geqslant 0\ :\ \sup_{s\in[0,u]}|r(s)|\leqslant\frac{1}{2C_1C_2}\right\} \tag{65}$$

Then $\|r(t)G^{-1}H\|\ \leqslant\ \frac{1}{2}$, and $(\mathrm{Id}+r(t)G^{-1}H)$ is invertible for all $(t,x,z,v,q)\in[0,T]\times M\times B_{(0_{\mathbb{R}},0_{\mathbb{R}^n},0_{S^{n\times n}})}(R_0)$, and the wanted conclusions easily follow.

■

Lemma 20 *Let T be given by (65). For all $(t,x,z,v,q) \in [0,T] \times M \times B_{(0_{\mathbb{R}},0_{\mathbb{R}^n},0_{S^{n\times n}})}(R_0)$, we have:*

$$\partial_{q_{i,j}}\Phi(t,x,z,v,q) \quad = \quad (G - 2r(t)H + r(t)^2 HG^{-1}H)^{-1}_{i,j}$$

Furthermore, $\partial_q\Phi(t,x,z,v,q) := (\partial_{q_{i,j}}\Phi(t,x,z,v,q))_{i,j\in[\![n]\!]}$ is uniformly elliptic.

Proof

Let us write H as

$$\begin{aligned} H(q) \quad &:= \quad H(t,x,z,v,q) \\ &=: \quad -\langle \nu(t,x,z,v), \nu_0(x)\rangle q - H_1(t,x,z,v) \end{aligned}$$

and recall that $\nu(t,x)$ and G do not depend on $\nabla\nabla f$, ie are constant in q. Consider $\psi(q) := -G^{-1}H(q)$, so

$$\hat{\Phi}(t,x,z,v,q) = \mathrm{tr}\left((\mathrm{Id} - r(t)\psi(q))^{-1}\psi(q)\right).$$

Let $M \in M^{n\times n}$, $X \in M^{n\times n}$ small and $u \in \mathbb{R}$ such that $\|u(M+X)\| < 1$ then

$$\begin{aligned} (\mathrm{Id} - u(M+X))^{-1}(M+X) \quad &= \quad \textstyle\sum_{n\in\mathbb{Z}_+}(u(M+X))^n(M+X) \\ &= \quad \textstyle\sum_{n\in\mathbb{Z}_+}\left(u^n M^{n+1} + u^n \sum_{m\in[\![0,n]\!]} M^m X M^{n-m}\right) + o(X) \\ &= \quad (\mathrm{Id} - uM)^{-1}M + \textstyle\sum_{n\in\mathbb{Z}_+,\, m\in[\![0,n]\!]} u^n \sum M^m X M^{n-m} + o(X) \end{aligned} \tag{66}$$

so $d[(\mathrm{Id} - uM)^{-1}M](X) = \sum_{n\in\mathbb{Z}_+,\, m\in[\![0,n]\!]} u^n \sum M^m X M^{n-m}$. Hence

$$d_q((\mathrm{Id} - u\psi(q))^{-1})(X) = \sum_{n\in\mathbb{Z}_+,\, m\in[\![0,n]\!]} u^n \psi(q)^m d\psi(q)(X)\psi(q)^{n-m}$$

Thus using the trace property

$$\begin{aligned} &d_q\mathrm{tr}((\mathrm{Id} - u\psi(q))^{-1}\psi(q))(X) \\ &\quad = \quad \mathrm{tr}\left(\sum_{n\in\mathbb{Z}_+,\, m\in[\![0,n]\!]} u^n \psi(q)^m d\psi(q)(X)\psi(q)^{n-m}\right) \\ &\quad = \quad \sum_{n\in\mathbb{Z}_+,\, m\in[\![0,n]\!]} u^n \mathrm{tr}\left(\psi(q)^m d\psi(q)(X)\psi(q)^{n-m}\right) \\ &\quad = \quad \sum_{n\in\mathbb{Z}_+,\, m\in[\![0,n]\!]} u^n \mathrm{tr}\left(\psi(q)^n d\psi(q)(X)\right) \\ &\quad = \quad \mathrm{tr}\left(\sum_{n\in\mathbb{Z}_+} (n+1)u^n\psi(q)^n d\psi(q)(X)\right) \\ &\quad = \quad \mathrm{tr}\left((\mathrm{Id} - u\psi(q))^{-2} d\psi(q)(X)\right) \end{aligned} \tag{67}$$

Thus we have

$$\begin{aligned} d_q\hat{\Phi}(t,x,z,v,q)(X) \quad &= \quad d_q\hat{\Phi}_1(t,x,z,v,q)(X) \\ &= \quad \langle \nu(t,x,z,v), \nu_0(x)\rangle \mathrm{tr}((\mathrm{Id} - r(t)G^{-1}H(q))^{-2}G^{-1}X) \end{aligned}$$

so for any $i,j \in [\![n]\!]$,

$$\begin{aligned} \partial_{q_{i,j}}\hat{\Phi}(t,x,z,v,q) \quad &= \quad \langle \nu(t,x,z,v), \nu_0(x)\rangle((\mathrm{Id} - r(t)G^{-1}H(q))^{-2}G^{-1}X)_{j,i} \\ &= \quad \langle \nu(t,x,z,v), \nu_0(x)\rangle(G - 2r(t)H + r(t)^2HG^{-1}H)^{-1}_{i,j} \end{aligned}$$

where $G + 2r(t)H - r(t)^2 HG^{-1}H \in S^{n\times n}$. For the last point of the lemma, use Proposition 19, and the choice of T in its proof, to get

$$(G + 2r(t)H + r(t)^2 HG^{-1}H) = G(\mathrm{Id} + r(t)G^{-1}H)^2$$

is invertible for all $t \in [0,T]$, and is continuous as function of t, so its spectrum remains positive as the spectrum at time 0, when $r(0) = 0$.

■

To show the existence result with sufficient regularity in time and space of Equation (58), we will show the existence result of the equivalent equation (62) up to a parametrization as in Proposition 1.3.4 in [17]. We will intensively use the existence and regularity result of the linearised equation exposed in Lunardi [15]. Let us recall briefly this result that appears as Theorem 5.1.10 of Lunardi [15] and whose extension to the compact Riemannian manifold could be find e.g. as Theorem 2.3 of Huang [12] (with the bundle $E = M \times \mathbb{R}$).

For $\alpha \in (0,1)$ and $T > 0$ let

$$\begin{aligned} C^{\alpha,0}([0,T]\times M) \;&:=\; \Big\{ f \in C([0,T]\times M) \,:\, f(\cdot,x) \in C^{\alpha}([0,T]),\ \forall x \in M, \\ &\qquad \text{and such that } \|f\|_{C^{\alpha,0}} := \sup_{x\in M}\{\|f(\cdot,x)\|_{C^{\alpha}([0,T])} < \infty \Big\} \end{aligned}$$

where for any function $f : [0,T] \to \mathbb{R}$,

$$\|f\|_{C^{\alpha}([0,T])} \;:=\; \|f\|_{\infty,[0,T]} + \langle f\rangle_{C^{\alpha}([0,T])} \tag{68}$$

$$\langle f\rangle_{C^{\alpha}([0,T])} \;:=\; \sup\left\{ \frac{|f(t)-f(s)|}{|t-s|^{\alpha}},\ s \neq t \in [0,T] \right\} \tag{69}$$

Similarly, we define

$$\begin{aligned} C^{0,\alpha}([0,T]\times M) \;&:=\; \Big\{ f \in C([0,T]\times M) \,:\, f(t,.) \in C^{\alpha}(M),\ \forall t \in [0,T], \\ &\qquad \text{and such that } \|f\|_{C^{0,\alpha}} := \sup_{t\in[0,T]}\{\|f(t,\cdot)\|_{C^{\alpha}(M)} < \infty \Big\} \end{aligned}$$

where the norm $\|\cdot\|_{C^{\alpha}(M)}$ is defined as in (68) and (69), with $[0,T]$ replaced by M.

The most important functional spaces for our analysis will be, still for given $0 < \alpha < 1$,

$$\begin{aligned} C^{\alpha/2,\alpha}([0,T]\times M) \;&:=\; C^{\alpha/2,0}([0,T]\times M) \cap C^{0,\alpha}([0,T]\times M) \\ C^{1+\alpha/2,2+\alpha}([0,T]\times M) \;&:=\; \{ f \in C^{1,2}([0,T]\times M) \,:\, \partial_t f, \partial_i\partial_j f \in C^{\alpha/2,\alpha}([0,T]\times M),\ \forall i,j \in [\![n]\!] \} \end{aligned}$$

respectively endowed with the norms

$$\begin{aligned} \|f\|_{C^{\alpha/2,\alpha}} \;&:=\; \|f\|_{C^{\alpha/2,0}} + \|f\|_{C^{0,\alpha}} \\ \|f\|_{C^{1+\alpha/2,2+\alpha}} \;&:=\; \|f\|_{\infty} + \sum_{i=1}^{n} \|\partial_i f\|_{\infty} + \|\partial_t f\|_{C^{\alpha/2,\alpha}} + \sum_{i,j=1}^{n} \|\partial_i\partial_j f\|_{C^{\alpha/2,\alpha}} \end{aligned}$$

As in Lemma 5.1.1 in Lunardi [15], there exists a uniform constant $C_\alpha > 0$ such that for all $f \in C^{1+\alpha/2,2+\alpha}$:

$$\|\partial_i f\|_{C^{(1+\alpha)/2,1+\alpha}} \;\leqslant\; C_\alpha \|f\|_{C^{1+\alpha/2,2+\alpha}} \tag{70}$$

Consider the following linear equation:

$$\begin{cases} \partial_t f(t,x) &= \widetilde{g}^{i,j}(t,x)\partial_i\partial_j f(t,x) + \sum_i \widetilde{H}_{1,i}(t,x)\partial_i f(t,x) + \widetilde{H}_0(t,x)f(t,x) + q(t,x) \\ f(0,x) &= f_0(x) \end{cases} \tag{71}$$

where $\widetilde{g} := (\widetilde{g}^{i,j})_{i,j\in[\![n]\!]}$, $\widetilde{H}_1 := (\widetilde{H}_{1,i})_{i\in[\![n]\!]}$, $\widetilde{H}_0$ and q (respectively f_0) are some given mappings on $M_T := [0,T] \times M$ (resp. M). As usual we will say that Equation (71) is uniformly elliptic in M_T when there exists an ellipticity coefficient $\lambda > 0$ such that for all $t \in [0,T]$ and all $\xi_1, ..., \xi_n \in \mathbb{R}$, we have:

$$\widetilde{g}^{i,j}(t,x)\xi_i\xi_j \geqslant \lambda\|\xi\|^2 \tag{72}$$

We recall the following theorem:

Theorem 21 (Th 5.1.10 Lunardi [15], Th 2.5 Huong [12]) *Let $\widetilde{g}$, $\widetilde{H}_1$, $\widetilde{H}_0$ and q belong to $C^{\alpha/2,\alpha}([0,T] \times M)$, with $0 < \alpha < 1$ and let $f(0,.) \in C^{2+\alpha}$. Assume moreover that (71) is uniformly elliptic, i.e. (72) holds. Then there exists a quantity $C > 0$, depending on the norms of $\widetilde{g}$, $\widetilde{H}_{1,i}$ and $\widetilde{H}_0$, as well as on the ellipticity coefficient of $\widetilde{g}$, such that Equation (71) has a unique solution $f \in C^{1+\alpha/2,2+\alpha}([0,T] \times M)$ and we have the Schauder estimate:*

$$\|f\|_{C^{1+\alpha/2,2+\alpha}} \leqslant C\big(\|f_0\|_{C^{2+\alpha}} + \|q\|_{C^{\alpha/2,\alpha}}\big)$$

Let us come back to the original equation i.e. (62), we will consider the following space $M_{t_0} = [0,t_0] \times M$ where the constant $0 < t_0 \leqslant T$ is to be chosen later, and let

$$\mathfrak{X} := \{u \in C^{1+\alpha/2,2+\alpha}(M_{t_0}) \,:\, u(0,.) = 0, \max(\|u\|_{\infty,M_{t_0}}, \|\nabla u\|_{\infty,M_{t_0}}, \|\nabla\nabla u\|_{\infty,M_{t_0}}) \leqslant R_0\}$$

We define the map:

$$\begin{array}{llll} \mathcal{S} \,:\, \mathfrak{X} & \to & C^{\alpha/2,\alpha}(M_{t_0}) \\ u & \mapsto & \partial_t u - \Phi(t,x,u,\nabla u,\nabla\nabla u). \end{array} \tag{73}$$

This is clearly a continuously differentiable map.

We have the following theorem.

Theorem 22 *Let M be a $\mathcal{C}^{5+\alpha}$ manifold, for some fixed $\alpha \in (0,1)$. If $t \mapsto r(t)$ is $\alpha/2$-Hölder and $r(0) = 0$ then there exists $t_0 > 0$ such that equation (62) has a unique solution defined in M_{t_0} with regularity $C^{1+\alpha/2,2+\alpha}(M_{t_0})$.*

Proof

The above theorem is a consequence of inverse function theorem around a specific function. Let $u_0(t,x) := \int_0^t \Phi(s,x,0,0,0)\,ds$ and note that $u_0 \in C^{1+\alpha/2,2+\alpha}$ by the assumption on the regularity of M. The Fréchet derivative of $\mathcal{S}$ at u_0 it is given by

$$d\mathcal{S}(u_0)u = \partial_t u - \big(\frac{\partial\Phi}{\partial q_{i,j}}\partial_{i,j}u + \frac{\partial\Phi}{\partial v_i}\partial_i u + \frac{\partial\Phi}{\partial z}u\big),$$

where the coefficients are all evaluated at u_0, for instance, $\frac{\partial\Phi}{\partial q_{i,j}}$ stands for $\frac{\partial\Phi}{\partial q_{i,j}}(t,x,u_0,\nabla u_0,\nabla\nabla u_0)$. By definition of u_0, there exists $0 < t_1 \leqslant T$ such that for all $0 \leqslant t \leqslant t_1$, $(u_0,\nabla u_0,\nabla\nabla u_0)(t,x) \in B_{(0_{\mathbb{R}},0_{\mathbb{R}^n},0_{\mathcal{S}^{n\times n}})}(R_0/2)$, so $u_0 \in \mathfrak{X}$. Lemma 20 yields $\frac{\partial\Phi}{\partial q}(t,x,u_0,\nabla u_0,\nabla\nabla u_0)$ is strongly elliptic in M_{t_1} and is in $C^{\alpha/2,\alpha}(M_{t_1})$. Using Theorem 21, for the linearisation of (62), we get $d\mathcal{S}(u_0)$ is locally invertible, and its inverse is continuous. By the inverse function theorem there exist $\epsilon > 0, \delta_1 > 0$ such that for all

$0 \leqslant t \leqslant t_1$ and for all g satisfying $\|g - \mathcal{S}(u_0)\|_{C^{\alpha/2,\alpha}(M_t)} < \epsilon$, there exists an unique $f \in C^{1+\alpha/2,2+\alpha}(M_t)$ satisfying $\|f - u_0\|_{C^{1+\alpha/2,2+\alpha}(M_t)} < \delta_1$ such that $\mathcal{S}(f) = g$. For f such that $\|f - u_0\|_{C^{1+\alpha/2,2+\alpha}(M_t)} < \delta_1$, since $f(0,x) = u_0(0,x) = 0$ and using (70), we get

$$\|f - u_0\|_{\infty,M_t} + \|\nabla(f - u_0)\|_{\infty,M_t} + \|\nabla\nabla(f - u_0)\|_{\infty,M_t} \leqslant (t + C_\alpha t^{(\alpha+1)/2} + t^{\alpha/2})\delta_1 \quad (74)$$

where C_α is the constant appearing in (70). So for t sufficiently small such that $(t+C_\alpha t^{(\alpha+1)/2}+t^{\alpha/2})\delta_1 \leqslant R_0/2$, we deduce $f \in \mathfrak{X}$ for $0 < t_0 \leqslant t$.

Let us show that with respect to the $C^{\alpha/2,\alpha}(M_t)$ norm, $\mathcal{S}(u_0)$ tends to 0 as t goes to 0_+. We will first show that $\|\mathcal{S}(u_0)(t,x) - \mathcal{S}(u_0)(s,x)\| \leqslant C_1(\delta)|t-s|^{\alpha/2}$, for all $s,t \in [0,\delta]$ and $x \in M$, and with $C_1(\delta)$ tending to 0 as δ tends to 0.

Let $\sigma \in [0,1]$ and

$$\zeta_\sigma(t,x) = \sigma(u_0(t,x), \nabla u_0(t,x), \nabla\nabla u_0(t,x)),$$

by definition of u_0, there exists a constant $C_1 > 0$ such that

$$|\zeta_\sigma(t,x) - \zeta_\sigma(s,x)| \leqslant C_1|t-s|,$$

$$|\zeta_\sigma(t,x) - \zeta_\sigma(t,y)| \leqslant C_1|x-y|^\alpha.$$

Let $\vec{u_0}(t,x) := (u_0(t,x), \nabla u_0(t,x), \nabla\nabla u_0(t,x))$, we have:

$$\begin{aligned}\mathcal{S}(u_0)(t,x) &= \Phi(t,x,0,0,0) - \Phi(t,x,u_0(t,x),\nabla u_0(t,x),\nabla\nabla u_0(t,x)) \\ &= -\int_0^1 d_3\Phi(t,x,\zeta_\sigma(t,x))(\vec{u_0}(t,x))\,d\sigma\end{aligned}$$

hence

$$\begin{aligned}|\mathcal{S}(u_0)(t,x) - \mathcal{S}(u_0)(s,x)| &= \Big|\int_0^1 \big(d_3\Phi(t,x,\zeta_\sigma(t,x))(\vec{u_0}(t,x)) - d_3\Phi(s,x,\zeta_\sigma(s,x))(\vec{u_0}(s,x))\big)\,d\sigma\Big| \\ &= \Big|\int_0^1 \big(d_3\Phi(t,x,\zeta_\sigma(t,x)) - d_3\Phi(s,x,\zeta_\sigma(s,x))\big)(\vec{u_0}(t,x))\,d\sigma \\ &\quad + \int_0^1 \big(d_3\Phi(s,x,\zeta_\sigma(s,x))\big)(\vec{u_0}(t,x) - \vec{u_0}(s,x))\,d\sigma\Big| \\ &\leqslant \int_0^1 |\big(d_3\Phi(t,x,\zeta_\sigma(t,x)) - d_3\Phi(s,x,\zeta_\sigma(s,x))\big)(\vec{u_0}(t,x))|\,d\sigma \\ &\quad + \int_0^1 |\big(d_3\Phi(s,x,\zeta_\sigma(s,x))\big)(\vec{u_0}(t,x) - \vec{u_0}(s,x))|\,d\sigma.\end{aligned}$$

We have, since M is compact and Φ is regular in the three last variables:

$$\begin{aligned}&|d_3\Phi(t,x,\zeta_\sigma(t,x)) - d_3\Phi(s,x,\zeta_\sigma(s,x))| \\ &\leqslant |d_3\Phi(t,x,\zeta_\sigma(t,x)) - d_3\Phi(t,x,\zeta_\sigma(s,x))| + |d_3\Phi(t,x,\zeta_\sigma(s,x)) - d_3\Phi(s,x,\zeta_\sigma(s,x))| \\ &\leqslant C_1|\zeta_\sigma(t,x) - \zeta_\sigma(s,x)| + C_2|t-s|^{\alpha/2} \\ &\leqslant (C_1\delta^{1-\alpha/2} + C_2)|t-s|^{\alpha/2}\end{aligned}.$$

where C is a constant whose value can change from one line to the other (also below). Also we have $|\vec{u_0}(t,x)| \leqslant Ct \leqslant C\delta$. On the other hand we have:

$$|d_3\Phi(s,x,\zeta_\sigma(s,x)| \leqslant C$$

and

$$|\vec{u_0}(t,x) - \vec{u_0}(s,x)| \leqslant C|t-s|.$$

Putting all things together we get:

$$|\mathcal{S}(u_0)(t,x) - \mathcal{S}(u_0)(s,x)| \leqslant C(\delta)|t-s|^{\alpha/2}$$

with $C(\delta)$ tending to 0 as δ tends to 0.

Let us show that:

$$|\mathcal{S}(u_0)(t,x) - \mathcal{S}(u_0)(t,y)| \leqslant C(\delta)|x-y|^{\alpha}$$

with $C(\delta)$ tending to 0 as δ tends to 0. With the same computation as above, we have:

$$\begin{aligned}|\mathcal{S}(u_0)(t,x) - \mathcal{S}(u_0)(t,y)| \;&\leqslant\; \int_0^1 |\big(d_3\Phi(t,x,\zeta_\sigma(t,x)) - d_3\Phi(t,y,\zeta_\sigma(t,y))\big)(\vec{u_0}(t,x))|\, d\sigma \\ &\quad + \int_0^1 |\big(d_3\Phi(t,y,\zeta_\sigma(t,y))\big)(\vec{u_0}(t,x) - \vec{u_0}(t,y))|\, d\sigma\end{aligned}$$

We also have, since M is compact:

$$\begin{aligned}&|d_3\Phi(t,x,\zeta_\sigma(t,x)) - d_3\Phi(t,y,\zeta_\sigma(t,y))| \\ &\leqslant\; |d_3\Phi(t,x,\zeta_\sigma(t,x)) - d_3\Phi(t,y,\zeta_\sigma(t,x))| + |d_3\Phi(t,y,\zeta_\sigma(t,x)) - d_3\Phi(t,y,\zeta_\sigma(t,y))| \\ &\leqslant\; C_2|x-y|^\alpha + C_1|\zeta_\sigma(t,x) - \zeta_\sigma(t,y)| \\ &\leqslant\; (C_1+C_2)|x-y|^\alpha,\end{aligned}$$

as well as

$$\begin{aligned}|\vec{u_0}(t,x)| \;&\leqslant\; C\delta \\ |d_3\Phi(t,y,\zeta_\sigma(t,y))| \;&\leqslant\; C\end{aligned}$$

Moreover

$$|u_0(t,x) - u_0(t,y)| \leqslant \int_0^t |\Phi(s,x,0,0,0) - \Phi(s,y,0,0,0)|\, ds \;\leqslant\; C\delta|x-y|^\alpha$$

and in the same way, using the regularity of $\Phi(s,x,0,0,0)$ in terms of x, we get:

$$|\vec{u_0}(t,x) - \vec{u_0}(t,y)| \;\leqslant\; C\delta|x-y|^\alpha$$

We deduce that:

$$|\mathcal{S}(u_0)(t,x) - \mathcal{S}(u_0)(t,y)| \;\leqslant\; C(\delta)|x-y|^\alpha$$

Hence $\|\mathcal{S}(u_0)\|_{C^{\alpha/2,\alpha}(M_t)}$ tends to 0 as t tends to 0.

So there exist $0 < t_2$ such that $\|\mathcal{S}(u_0)\|_{C^{\alpha/2,\alpha}(M_{t_2})} < \epsilon$. Let $t_0 = t_1 \wedge t_2$, we get by inverse function theorem that $\mathcal{S}f = 0$ has a solution $f \in C^{1+\alpha/2,2+\alpha}(M_{t_0})$, this is actually a solution of equation (62).

For the uniqueness, let f be the solution of (62) constructed above on M_{t_0}. Consider another solution g of (62) on M_{t_0}, in particular g starts with the same initial condition $g_0 = f_0 = 0$. Since $g \in C^{1+\alpha/2,2+\alpha}$, let $t_3 \in (0,t_0]$ be the maximum value of t such that

$$\|g\|_{\infty,M_t}, \|\nabla g\|_{\infty,M_t}, \|\nabla\nabla g\|_{\infty,M_t} \leqslant R_0$$

By construction of f, we have

$$\|f\|_{\infty,M_t},\ \|\nabla f\|_{\infty,M_t},\ \|\nabla\nabla f\|_{\infty,M_t} \leqslant R_0$$

for any $t \in [0,t_0]$ and in particular for $t \in [0,t_3]$.

Let $u = f - g$, then u satisfies the following linear equation:

$$\begin{aligned}
\partial_t u &= \Phi(t,x,f,\nabla f,\nabla\nabla f) - \Phi(t,x,g,\nabla g,\nabla\nabla g)\\
&= \int_0^1 \frac{\partial}{\partial\sigma}\Phi(t,x,\sigma\vec{f} + (1-\sigma)\vec{g})\,d\sigma\\
&= \int_0^1 \frac{\partial\Phi}{\partial q_{ij}}(t,x,\sigma\vec{f} + (1-\sigma)\vec{g})\partial_{i,j}(f-g)\,d\sigma + \int_0^1 \frac{\partial\Phi}{\partial v_i}(t,x,\sigma\vec{f} + (1-\sigma)\vec{g})\partial_i(f-g)\,d\sigma\\
&+ \int_0^1 \frac{\partial\Phi}{\partial z}(t,x,\sigma\vec{f} + (1-\sigma)\vec{g})(f-g)\,d\sigma\\
&= A_{ij}(t,x)u_{ij} + B_i(t,x)u_i + C(t,x)u,
\end{aligned}$$

where

$$\begin{aligned}
A_{ij}(t,x) &= \int_0^1 \frac{\partial\Phi}{\partial q_{ij}}(t,x,\sigma\vec{f} + (1-\sigma)\vec{g})\,d\sigma,\\
B_i(t,x) &= \int_0^1 \frac{\partial\Phi}{\partial v_i}(t,x,\sigma\vec{f} + (1-\sigma)\vec{g})\,d\sigma,\\
C(t,x) &= \int_0^1 \frac{\partial\Phi}{\partial z}(t,x,\sigma\vec{f} + (1-\sigma)\vec{g})\,d\sigma.
\end{aligned}$$

According to Lemma 20, A_{ij} is uniformly elliptic. Let $\lambda < -\|C\|_{M_{t_3}}$, and $W := e^{\lambda t}u$ then we have:

$$\partial_t W = A_{i,j}(t,x)\partial_i\partial_j W + B_i(t,x)W_i + (C+\lambda)W.$$

The proof of uniqueness will be done by contradiction, suppose $f \neq g$ then there exists for example $\beta > 0$ (the negative possibility will be done in a similar way) and $(t,x) \in [0,t_3] \times M$ such that $W(t,x) = \beta$. Consider the first time t_0 such that there exist $x_0 \in M$ such that $W(t_0,x_0) = \beta$, clearly $t_0 > 0$. By definition $W(t_0,x_0) = \max\{W(t,x),(t,x) \in [0,t_0] \times M\}$, and

$$\begin{aligned}
\partial_t W(t_0,x_0) &\geqslant 0\\
\mathrm{Hess}(W)(t_0,x_0) &\leqslant 0\\
\nabla W(t_0,x_0) &= 0
\end{aligned}$$

We have at (t_0,x_0)

$$0 \leqslant \partial_t W = A_{ij}(t_0,x_0)\partial_i\partial_j W + (C+\lambda)\beta < A_{ij}(t_0,x)\partial_i\partial_j W \leqslant 0$$

where the last inequality come from $A_{ij}(t_0,x_0)\partial_i\partial_j W = \mathrm{tr}(A\,\mathrm{Hess}W) \leqslant 0$, and this is a contradiction, so $W \leqslant 0$. We do the same thing to get $W \geqslant 0$ and so $f = g$ for all $t \in [0,t_3]$. It follows in fact that $t_3 = t_0$.

■

Remark 23 From the above proof, we see there exist two quantities $\eta_1, \eta_2 > 0$, only depending on some bounds on the geometry of C, such that t_0 can be expressed as

$$t_0 \;\; := \;\; \eta_1 \wedge \inf\{s \geqslant 0 \,:\, |r(s)| \geqslant \eta_2\}$$

□

Remark 24 Using the $\alpha/2$-Hölder regularity of the Brownian motion, for all $0 < \alpha < 1$, we get the existence and the regularity of the equation, similar to (44), corresponding to the stochastic modified mean curvature flow:

$$\begin{cases} & D_0 \;=\; D \\ \forall\ t \in [0, \tau),\ \forall\ x \in C_t, & dx \;=\; \left(\sqrt{2}dB_t - \rho^b_{C_t}(x)dt\right) \nu_{C_t}(x) \end{cases} \tag{75}$$

where $C_t := \partial D_t$. The solution of this equation is obtained as above, first we solve equation (58) and we obtain G_t and then $D_t := \Psi(G_t, \sqrt{2}B_t)$.

□

Remark 25 Note that in the above proof we only need that $r(0)$ is small enough, such that $\|r(0)G^{-1}H(0,\cdot)\| < 1$, so starting the same procedure at time t_0, we have a notion of maximal solution of equation (62). A slight modification of the proof of Theorem 22 also yields existence and uniqueness of solution of (62) for f_0 small enough, as well as all its derivatives up to order 2.

□

Using the strong maximum principle instead of the maximum principle in the proof of Theorem 22, we have the following corollary:

Corollary 26 *Let $U, \hat{U} \in \mathcal{D}$ with $\mathcal{C}^{5+\alpha}$ boundaries, $\alpha \in (0,1)$, and $C = \partial U, \hat{C} = \partial \hat{U}$. Suppose that*

$$\hat{U} \subset U, \quad \hat{C} \neq C$$

and that $\hat{C}$ belongs to an open tubular neighborhood of C. Let $(\partial G_t)_{t\in[0,\tau_C)}$ (resp. $(\partial \hat{G}_t)_{t\in[0,\tau_{\hat{C}})}$) be a solution of (58) with $r(t) = \sqrt{2}B_t$ started at C (resp. $\hat{C}$), then there exist a positive stopping time $\tau_{C,\hat{C}} > 0$ (a priori smaller than $\tau_C \wedge \tau_{\hat{C}}$ because we want ∂G_t to remain in an open tubular neighborhood of $\partial \hat{G}_t$), such that

$$\forall\ t \in (0, \tau_{C,\hat{C}}), \quad \hat{G}_t \subset G_t, \text{ and } \quad \partial G_t \cap \partial \hat{G}_t \;=\; \varnothing$$

The above corollary shows that even if the initial hypersurfaces are equal in a large portion, it is sufficient they are different somewhere for the flow to detach them instantaneously, at least when one of them lives in a tubular neighborhood of the other. When the latter condition is not fulfilled, we have to impose that the initial boundaries are disjoint:

Corollary 27 *Let $U, \hat{U} \in \mathcal{D}$ with $C^{5+\alpha}$ boundaries, and $C = \partial U, \hat{C} = \partial \hat{U}$. Suppose that*

$$\hat{U} \subset U, \quad \hat{C} \cap C = \varnothing$$

Let $(\partial G_t)_{t\in[0,\tau_C)}$ (resp. $(\partial \hat{G}_t)_{t\in[0,\tau_{\hat{C}})}$) be a solution of (58) with $r(t) = \sqrt{2}B_t$ started at C (resp. $\hat{C}$), then for a positive stopping time $\tau_{C,\hat{C}} > 0$, we have

$$\forall\ t \in [0, \tau_{C,\hat{C}}), \quad \hat{G}_t \subset G_t, \text{ and } \quad \partial G_t \cap \partial \hat{G}_t \;=\; \varnothing$$

Proof
Since C and $\hat{C}$ are compact, and $\hat{C} \cap C = \varnothing$ we have $\delta = d(C, \hat{C}) > 0$. Using the continuity of the solution of (58), we get the existence of $0 < T_C < \tau_C$ (resp. $0 < T_{\hat{C}} < \tau_{\hat{C}}$) such that for all $t \in [0, T_C]$, we have $d(C, \partial G_t) \leqslant \frac{\delta}{4}$ (resp. for all $t \leqslant T_{\hat{C}}$ we have $d(\hat{C}, \partial \hat{G}_t) \leqslant \frac{\delta}{4}$). Take $\tau_{C,\hat{C}} = T_C \wedge T_{\hat{C}}$.

■

Consider the following stochastic mean curvature evolution starting from $C_0 = \partial D_0$

$$dx \quad = \quad \left(\sqrt{2} dB_t - \rho_{C_t}(x) dt\right) \nu_{C_t}(x) \tag{76}$$

According to the Doss and Sussman approach, a solution of (76) is given by $(\Psi(G_t, \sqrt{2}B_t))_{t\in[0,\tau)}$ where $(\partial G_t)_{t\in[0,\tau)}$ is a solution of (58) with $r(t) = \sqrt{2}B_t$. Equation (76) is a particular case of equation (75) with $b = 0$.

Corollary 28 *Let $D, \hat{D} \in \mathcal{D}$ with $\mathcal{C}^{5+\alpha}$ boundaries, $\alpha \in (0,1)$, and $C = \partial D, \hat{C} = \partial \hat{D}$. Suppose that*

$$\hat{D} \subset D, \quad \hat{C} \cap C = \varnothing$$

Let $(\partial D_t)_{t\in[0,\tau_C)}$ (resp. $(\partial \hat{D}_t)_{t\in[0,\tau_{\hat{C}})}$) be a solution of (76) started at C (resp. $\hat{C}$) then for a positive stopping time $\tau_{C,\hat{C}} > 0$ we have:

$$\forall\ t \in [0, \tau_{C,\hat{C}}), \quad \hat{D}_t \subset D_t \ \text{and} \ \ \partial D_t \cap \partial \hat{D}_t \quad = \quad \varnothing$$

Proof
Use Corollary 27 we get there exist $\tau_{C,\hat{C}} > 0$ such that

$$\forall\ t \in [0, \tau_{C,\hat{C}}), \quad \partial G_t \cap \partial \hat{G}_t \quad = \quad \varnothing$$

We have $\partial D_t = \Psi(\partial G_t, \sqrt{2}B_t)$ for $t \in [0, \tau_C)$ (resp. $\partial \hat{D}_t = \Psi(\partial \hat{G}_t, \sqrt{2}B_t)$ for $t \in [0, \tau_{\hat{C}})$). For $t \in [0, \tau_{C,\hat{C}})$, $\Psi(., \sqrt{2}B_t)$ is a diffeomorphism between ∂G_t and its image ∂D_t (resp. between $\partial \hat{G}_t$ and its image $\partial \hat{D}_t$). The proof of the corollary will be done by contradiction, suppose that there exists a time $0 < t < \tau_{C,\hat{C}}$ such that $\Psi(\partial G_t, \sqrt{2}B_t) \cap \Psi(\partial \hat{G}_t, \sqrt{2}B_t) \neq \varnothing$. Then there exist $x \in G_t$ and $\hat{x} \in \hat{G}_t$ such that $\Psi_{\partial G_t}(x, \sqrt{2}B_t) = \Psi_{\partial \hat{G}_t}(\hat{x}, \sqrt{2}B_t)$. We have

$$d_{\partial G_t}(\Psi_{\partial G_t}(x, \sqrt{2}B_t)) \quad = \quad \sqrt{2}|B_t| \quad = \quad d_{\partial \hat{G}_t}(\Psi_{\partial \hat{G}_t}(\hat{x}, \sqrt{2}B_t))$$

where $d_{\partial G_t}(\cdot)$ stands for the distance to ∂G_t. If $B_t > 0$, then $\Psi_{\partial G_t}(x, \sqrt{2}B_t) \in G_t^c$ so the geodesic curve $r \mapsto \Psi_{\partial \hat{G}_t}(\hat{x}, r)$ has to cross ∂G_t at time $r_0 \in (0, \sqrt{2}B_t]$ (since $\partial G_t \cap \partial \hat{G}_t = \varnothing$ and $\hat{G}_t \subset G_t$). Hence

$$\sqrt{2}|B_t| \quad = \quad d_{\partial G_t}(\Psi_{\partial G_t}(x, \sqrt{2}B_t)) \quad \leqslant \quad d(\Psi_{\partial \hat{G}_t}(\hat{x}, r_0), \Psi_{\partial \hat{G}_t}(\hat{x}, \sqrt{2}B_t)) \quad \leqslant \quad \sqrt{2}|B_t| - r_0$$

so we get a contradiction.

The case $B_t = 0$ is clear.

If $B_t < 0$ namely $\Psi_{\partial \hat{G}_t}(\hat{x}, \sqrt{2}B_t) \in \mathrm{Int}(\hat{G}_t)$, the interior of $\hat{G}_t$, and the geodesic $\Psi_{\partial G_t}(x, -r)$ have to cross $\partial \hat{G}_t$ at time $r_0 \in (0, \sqrt{2}|B_t|]$, so

$$\sqrt{2}|B_t| \quad = \quad d_{\partial \hat{G}_t}(\Psi(\hat{x}, \sqrt{2}B_t)) \quad \leqslant \quad d(\Psi_{\partial G_t}(x, -r_0), \Psi_{\partial G_t}(x, \sqrt{2}B_t)) \quad \leqslant \quad \sqrt{2}|B_t| - r_0$$

and we get a contradiction.

■

We want to control the distance between to different hypersurface evolving by the stochastic mean curvature by quantities that only depend on the ambient curvature.

Lemma 29 *Let $D, \hat{D} \in \mathcal{D}$ with C^2 boundaries in a d-dimensional manifold V, $C = \partial D$, $\hat{C} = \partial \hat{D}$, $\hat{D} \subset D$ and $\hat{C} \cap C = \varnothing$. Suppose that there exists $k \in \mathbb{R}$ such that $Ric \geqslant (d-1)kg$, then at points $(p,q) \in C \times \hat{C}$ such that $d(p,q) = d(C,\hat{C})$ (or local minimizers of the distance function restricted to $C \times \hat{C}$) we have:*

(i) *if $k > 0$, and p is not conjugate to q then*

$$2(d-1)\sqrt{k}\frac{\left(1-\cos(\sqrt{k}d(p,q))\right)}{\sin(\sqrt{k}d(p,q))} \quad \leqslant \quad \rho_{\hat{C}}(q) - \rho_C(p)$$

(ii) *if $k \leqslant 0$, and p is not conjugate to q then*

$$2(d-1)\sqrt{|k|}\frac{\left(1-\cosh(\sqrt{|k|}d(p,q))\right)}{\sinh(\sqrt{|k|}d(p,q))} \quad \leqslant \quad \rho_{\hat{C}}(q) - \rho_C(p)$$

(iii) *In particular for all k, if p is not conjugate to q then we have:*

$$(d-1)kd(p,q) \quad \leqslant \quad \rho_{\hat{C}}(q) - \rho_C(p)$$

(iv) *If $V = \mathbb{R}^d$ then*

$$0 \quad \leqslant \quad \rho_{\hat{C}}(q) - \rho_C(p)$$

Proof

Let $(p,q) \in C \times \hat{C}$ such that $d(p,q) = d(C,\hat{C})$. Using the first variation formula, we get there exists an unit speed geodesic γ in V such that $\gamma(0) = q$, $\gamma(d(p,q)) = p$, $\dot{\gamma}(0)$ is orthogonal to $T_q\widehat{C}$ and $\dot{\gamma}(d(p,q))$ is orthogonal to T_pC. Let $(e_i)_{i \in [\![1,d-1]\!]}$ be a orthonormal basis of $T_q\hat{C}$. Let $\gamma_{1,i}(t)$ be a geodesic in $\hat{C}$ such that $\gamma_{1,i}(0) = q$ and $\dot{\gamma}_{1,i}(0) = e_i$. Let $\gamma_{2,i}(t)$ be a geodesic in C such that $\gamma_{2,i}(0) = p$ and $\dot{\gamma}_{1,i}(0) = /\!/_{d(p,q)}e_i$, where $/\!/$ is the parallel transport along the geodesic γ. We have $0 = \langle e_i, \dot{\gamma}(0)\rangle = \langle /\!/_{d(p,q)}e_i, \dot{\gamma}(d(p,q))\rangle$. Since $(p,q) \in C \times \hat{C}$ is a local minimizer of the distance function restricted to $C \times \hat{C}$, we have that

$$0 \quad \leqslant \quad \left.\frac{d^2}{dt^2}\right|_{t=0} d(\gamma_{1,i}(t), \gamma_{2,i}(t)).$$

Let Y_i be the Jacobi field along γ obtained by the variation of geodesic connecting $\gamma_{1,i}(t)$ to $\gamma_{2,i}(t)$, we have: $Y_i(0) = e_i$, $Y_i(d(p,q)) = /\!/_{d(p,q)}e_i$. Using second variation formula, the fact that $\dot{\gamma}(0)$ is the exterior normal vector of $\hat{C}$ at q and $\dot{\gamma}(d(p,q))$ is the exterior normal vector of C at p we get:

$$\begin{aligned}
0 \quad &\leqslant \quad \frac{d^2}{d^2t} d(\gamma_{1,i}(t), \gamma_{2,i}(t))|_{t=0} \\
&= \quad \left[\langle \nabla_{t=0}\dot{\gamma}_{2,i}(t), \dot{\gamma}(d(p,q))\rangle - \langle \nabla_{t=0}\dot{\gamma}_{1,i}(t), \dot{\gamma}(0)\rangle\right] + I(Y_i,Y_i) \\
&= \quad \left[\langle \nabla_{t=0}\dot{\gamma}_{2,i}(t), \nu_C(p)\rangle - \langle \nabla_{t=0}\dot{\gamma}_{1,i}(t), \nu_{\hat{C}}(q)\rangle\right] + I(Y_i,Y_i) \\
&= \quad \left[-\langle \dot{\gamma}_{2,i}(0), \nabla_{\dot{\gamma}_{2,i}(0)}\nu_C\rangle + \langle \dot{\gamma}_{1,i}(0), \nabla_{\dot{\gamma}_{1,i}(0)}\nu_{\hat{C}}\rangle\right] + I(Y_i,Y_i) \\
&= \quad -\Pi_C(/\!/_{d(p,q)}e_i, /\!/_{d(p,q)}e_i) + \Pi_{\hat{C}}(e_i,e_i) + I(Y_i,Y_i)
\end{aligned} \tag{77}$$

where $I(Y_i,Y_i)$ is the index of the Jacobi field Y_i along γ, and Π_C (resp. $\Pi_{\hat{C}}$) is the second fundamental form of C (resp. $\hat{C}$). Let $X_i(s) = f(s)/\!/_s e_i$, be a vector field along γ such that $f(0) = f(d(p,q)) = 1$ and $f'' = -kf$, using the Index Lemma since p and q are not conjugate along γ, we have for all $i \in [\![1, d-1]\!]$

$$I(Y_i,Y_i) \quad \leqslant \quad I(X_i,X_i)$$

Taking the sum in (77) we get:

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{d-1} \big(-\Pi_C(/\!/_{d(p,q)}e_i, /\!/_{d(p,q)}e_i) + \Pi_{\hat{C}}(e_i, e_i) + I(Y_i, Y_i) \big) \\
&\leqslant \rho_{\hat{C}}(q) - \rho_C(p) + \sum_{i=1}^{d-1} I(X_i, X_i) \\
&= \rho_{\hat{C}}(q) - \rho_C(p) + \sum_{i=1}^{d-1} \int_0^{d(p,q)} \|\nabla_s X_i\|^2 - \langle R(X_i, \dot\gamma)X_i, \dot\gamma\rangle \, ds \\
&= \rho_{\hat{C}}(q) - \rho_C(p) + \sum_{i=1}^{d-1} \int_0^{d(p,q)} |f'|^2 - f^2 \langle R(/\!/_s e_i, \dot\gamma)/\!/_s e_i, \dot\gamma\rangle \, ds \\
&= \rho_{\hat{C}}(q) - \rho_C(p) + \int_0^{d(p,q)} (d-1)|f'|^2 - f^2 Ric(\dot\gamma, \dot\gamma) \, ds \\
&\leqslant \rho_{\hat{C}}(q) - \rho_C(p) + (d-1) \int_0^{d(p,q)} (f')^2 - f^2 k \, ds \\
&= \rho_{\hat{C}}(q) - \rho_C(p) + (d-1)(f'(d(p,q)) - f'(0)).
\end{aligned}
$$

After computations of f, we get the result. For the particular case, we could take $X_i = /\!/_s e_i$ in the above computation and directly get the result.

■

Proposition 30 *Let $D, \hat{D} \in \mathcal{D}$ with C^1 boundaries in a d-dimensional manifold V, and $C = \partial D, \hat{C} = \partial \hat{D}$. Suppose that*

$$\hat{D} \subset D$$

For $r \in \mathbb{R}$ such that $\Psi_C(., r)$ (resp. $\Psi_{\hat{C}}(., r)$) is diffeomorphism onto its image $\Psi_C(C, r)$ (resp. $\Psi_{\hat{C}}(\hat{C}, r)$) then

$$d(\Psi(C, r), \Psi(\hat{C}, r)) = d(C, \hat{C})$$

Proof

Let $(p, q) \in C \times \hat{C}$ such that

$$d(p, q) = d(C, \hat{C})$$

If $d(C, \hat{C}) > 0$, using Gauss Lemma, and the fact that $\hat{D} \subset D$, we get the exterior normal vector of C at p is the parallel transport, along the geodesic γ that connects q to p, of the exterior normal vector of $\hat{C}$ at q. Hence by definition of Ψ we have $d(\Psi_C(p, r), \Psi_{\hat{C}}(q, r)) = d(p, q)$

We get

$$d(\Psi(C, r), \Psi(\hat{C}, r)) \leqslant d(\Psi_C(p, r), \Psi_{\hat{C}}(q, r)) = d(p, q) = d(C, \hat{C})$$

So $d(\Psi(C, r), \Psi(\hat{C}, r)) \leqslant d(C, \hat{C})$.

In a similar way let $(p, q) \in \Psi(C, r) \times \Psi(\hat{C}, r)$ such that

$$d(p, q) = d(\Psi(C, r), \Psi(\hat{C}, r))$$

we have since $\Psi_C(.,r)$ (resp. $\Psi_{\hat{C}}(.,r)$) is a diffeomorphism onto their respective image,

$$d(C,\hat{C}) \leqslant d(\Psi_{\Psi(C,r)}(p,-r),\Psi_{\Psi(\hat{C},r)}(q,-r)) = d(p,q) = d(\Psi(C,r),\Psi(\hat{C},r))$$

Putting all things together we get

$$d(\Psi(C,r),\Psi(\hat{C},r)) = d(C,\hat{C}).$$

If $d(p,q) = d(C,\hat{C}) = 0$, since $\hat{D} \subset D$ then $\nu_{\hat{C}}(q) = \nu_C(p)$ and the result follows as above.

■

Remark 31 The above proposition also gives an alternative proof of Corollary 28.

□

Let

$$\iota_V := \inf_{(p,v)\in V\times T_pV\,:\,\|v\|=1} \inf\{t>0, \gamma_v(t) \text{ is conjugate to } \gamma_v(0)=p\}$$

where γ_v is a geodesic starting at $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.

Lemma 32 *Let $D,\hat{D} \in \mathcal{D}$ with $\mathcal{C}^{5+\alpha}$ boundaries, $\alpha \in (0,1)$, and $C = \partial D, \hat{C} = \partial\hat{D}$. Suppose that there exists $k \leqslant 0$ such that $Ric \geqslant (d-1)kg$, $\iota_V = \infty$ (for example if V have non-positive sectional curvature) and*

$$\hat{D} \subset D, \quad \hat{C} \cap C = \varnothing.$$

Let $(\partial D_t)_{t\in[0,\tau_C)}$ (resp. $(\partial\hat{D}_t)_{t\in[0,\tau_{\hat{C}})}$) be a solution of (76) started at C (resp. $\hat{C}$) then:

(i) *The mapping $t \mapsto d(\partial D_t,\partial\hat{D}_t)$ is locally Lipschitz in $[0,\tau_C\wedge\tau_{\hat{C}})$*

(ii) *For all $t \in [0,\tau_C\wedge\tau_{\hat{C}})$*

$$d(C,\hat{C})e^{k(d-1)t} \leqslant d(\partial D_t,\partial\hat{D}_t)$$

(iii) *We have $D_t \cap \hat{D}_t = \varnothing$ for all $t \in [0,\tau_C\wedge\tau_{\hat{C}})$.*

(iv) *In particular, if $V = \mathbb{R}^d$ then $t \mapsto d(\partial D_t,\partial\hat{D}_t)$ is non decreasing in $[0,\tau_C\wedge\tau_{\hat{C}})$.*

Proof

We have

$$\begin{aligned} D_t &= \Psi(G_t,\sqrt{2}B_t), \text{ for } t<\tau_C \\ \hat{D}_t &= \Psi(\hat{G}_t,\sqrt{2}B_t), \text{ for } t<\tau_{\hat{C}} \end{aligned}$$

where ∂G_t and $\partial\hat{G}_t$ are solutions of (58) with $r(t) = \sqrt{2}B_t$ and $\partial G_0 = C$ respectively $\partial\hat{G}_0 = \hat{C}$. Let

$$\tau := \inf\{t \geqslant \tau_{C,\hat{C}}, \text{ s.t. } \partial D_t \cap \partial\hat{D}_t \neq \varnothing\} \wedge \tau_C \wedge \tau_{\hat{C}}$$

Using Proposition 30 and Corollary 28, we have

$$\forall t \in [0,\tau), \qquad d(\partial D_t,\partial\hat{D}_t)) = d(\partial G_t,\partial\hat{G}_t)$$

Recall that $G_t = \{F_0(x) + f_C(t,x)\nu_0^C(x), x \in M\}$ with $F_0(M) = C$, and $f_C(t,x)$ the solution of (62). We have the same construction for $\hat{G}$. We recall that $f_C \in C^{1+\alpha/2,2+\alpha}(M_{\tau_C})$ and $f_{\hat{C}} \in C^{1+\alpha/2,2+\alpha}(\hat{M}_{\tau_{\hat{C}}})$. So by definition, for $0 \leqslant t < \tau$,

$$d(\partial G_t, \partial \hat{G}_t) = \inf_{(x,y)\in M\times \hat{M}} d(F_C(t,x), F_{\hat{C}}(t,y))$$

where $F_C(t,x) = F_0(x) + f_C(t,x)\nu_0^C(x)$ and $F_{\hat{C}}(t,y) = \hat{F}_0(y) + f_{\hat{C}}(t,y)\nu_0^{\hat{C}}(y)$. Also $t \mapsto F_C(t,x)$ and $t \mapsto F_{\hat{C}}(t,y)$ are uniformly Lipschitz on any compact $[0,T] \subset [0,\tau)$. Hence $t \mapsto d(\partial G_t, \partial \hat{G}_t) = d(\partial D_t, \partial \hat{D}_t))$ is Lipschitz on $[0,T]$, hence almost everywhere differentiable on $[0,T]$ and absolutely continuous. At differentiability time $t \in [0,T]$ we have

$$\begin{aligned}
&\frac{d}{dt} d(\partial D_t, \partial \hat{D}_t)\\
&= \frac{d}{dt}\Big(\inf_{(x_t,y_t)\in \partial G_t\times\partial\hat{G}_t\,:\,d(x_t,y_t)=d(\partial G_t,\partial\hat{G}_t)} d(x_t,y_t)\Big)\\
&= \inf_{(x_t,y_t)\in \partial G_t\times\partial\hat{G}_t\,:\,d(x_t,y_t)=d(\partial G_t,\partial\hat{G}_t)} \frac{d}{dt} d(x_t,y_t)\\
&= \inf_{(x_t,y_t)\in \partial G_t\times\partial\hat{G}_t\,:\,d(x_t,y_t)=d(\partial G_t,\partial\hat{G}_t)} \langle \frac{d}{dt}x_t, \nu^{\partial G_t}(x_t)\rangle - \langle \frac{d}{dt}y_t, \nu^{\partial \hat{G}_t}(y_t)\rangle\\
&= \inf_{(x_t,y_t)\in \partial G_t\times\partial\hat{G}_t\,:\,d(x_t,y_t)=d(\partial G_t,\partial\hat{G}_t)} -\rho_{\Psi(\partial G_t,\sqrt{2}B_t)}(\psi_{\partial G_t,\sqrt{2}B_t}(x_t)) + \rho_{\Psi(\partial \hat{G}_t,\sqrt{2}B_t)}(\psi_{\partial \hat{G}_t,\sqrt{2}B_t}(y_t))\\
&= \inf_{(x_t,y_t)\in \partial D_t\times\partial\hat{D}_t\,:\,d(x_t,y_t)=d(\partial D_t,\partial\hat{D}_t)} -\rho_{\partial D_t}(x_t) + \rho_{\partial \hat{D}_t}(y_t)\\
&\geqslant (d-1)k d(\partial D_t, \partial \hat{D}_t)
\end{aligned}$$

where in the second equality we use the usual Lagrange Theorem, in the third one we use the first variation formula, and in the last one we use Lemma 29. Since $t \mapsto d(\partial D_t, \partial \hat{D}_t)$ is absolutely continuous we can integrate the above inequality. Hence, using Gronwall's lemma, we get the conclusions (i), (ii), (iii) and (iv) of the lemma, at least on $[0,\tau)$. Since $d(C,\hat{C}) > 0$, we easily deduce that $\tau = \tau_C \wedge \tau_{\hat{C}}$.

■

Remark 33 If the $\hat{D} \subset D^c$ and $C \cap \hat{C} = \varnothing$ for all reasonable r, we have $d(\Psi_C(p,r), \Psi_{\hat{C}}(q,r)) = d(p,q) - 2r$ and we could get a similar kind of result.

□

Theorem 34 *Let $D, \hat{D} \in \mathcal{D}$ with $\mathcal{C}^{5+\alpha}$ boundaries, $\alpha \in (0,1)$, and $C = \partial D, \hat{C} = \partial \hat{D}$. Suppose that there exists $k \in \mathbb{R}$ such that Ric $\geqslant (d-1)kg$ and $\iota_V > 0$ (for example if the sectional curvature is bounded above by a^2 then $\iota_V \geqslant \frac{\pi}{a}$, see e.g. [10] page 159) and*

$$\hat{D} \subset D, \quad \hat{C} \cap C = \varnothing$$

Let $(\partial D_t)_{t\in)0,\tau_C)}$ (resp. $(\partial \hat{D}_t)_{t\in[0,\tau_{\hat{C}})}$) be a solution of (76) *started at C (resp. $\hat{C}$) then*

(i) *The mapping $t \mapsto d(\partial D_t, \partial \hat{D}_t)$ is locally Lipschitz on $[0, \tau_C \wedge \tau_{\hat{C}})$.*

(ii) *If $k \geqslant 0$ then for all $t \in [0, \tau_C \wedge \tau_{\hat{C}})$,*

$$(d(C,\hat{C})e^{k(d-1)t}) \wedge \iota_V \leqslant d(\partial D_t, \partial \hat{D}_t)$$

(iii) *If $k \leqslant 0$ then for all $t \in [0, \tau_C \wedge \tau_{\hat{C}})$,*

$$(d(C, \hat{C}) \wedge \iota_V)e^{k(d-1)t} \quad \leqslant \quad d(\partial D_t, \partial \hat{D}_t)$$

(iv) *We have $D_t \cap \hat{D}_t = \varnothing$ for $t \in [0, \tau_C \wedge \tau_{\hat{C}})$.*

Proof

The proof is similar to the proof of Lemma 32. Using (iii) in Lemma 29, we have:

$$d(\partial D_t, \partial \hat{D}_t) < \iota_V \quad \Longrightarrow \quad \frac{d}{dt} d(\partial D_t, \partial \hat{D}_t) \geqslant (d-1)kd(\partial D_t, \partial \hat{D}_t)$$

We deduce that, if $k \geqslant 0$ then for all $t \in [0, \tau_C \wedge \tau_{\hat{C}})$

$$(d(C, \hat{C})e^{k(d-1)t}) \wedge \iota_V \quad \leqslant \quad d(\partial D_t, \partial \hat{D}_t)$$

since after being above ι_V, $d(\partial D_t, \partial \hat{D}_t)$ cannot go below ι_V again.
Similarly, if $k \leqslant 0$ then for all $t \in [0, \tau_C \wedge \tau_{\hat{C}})$

$$(d(C, \hat{C}) \wedge \iota_V)e^{k(d-1)t} \quad \leqslant \quad d(\partial D_t, \partial \hat{D}_t)$$

■

As a consequence of Theorem 34, we can extend Corollary 26 under an assumption relaxing the requirement that one of the initial boundaries must be in a tubular neighborhood of the other initial boundary:

Proposition 35 *Let $D, \hat{D} \in \mathcal{D}$ with $\mathcal{C}^{5+\alpha}$ boundaries, $\alpha \in (0,1)$, and $C = \partial D, \hat{C} = \partial \hat{D}$. Suppose that*

$$\hat{D} \subset D \text{ and } C \neq \hat{C}$$

Let $(\partial D_t)_{t \in [0,\tau_C)}$ (resp. $(\partial \hat{D}_t)_{t \in [0,\tau_{\hat{C}})}$) be a solution of (76) started at C (resp. $\hat{C}$). Suppose that there exists $k \in \mathbb{R}$ such that $Ric \geqslant (d-1)kg$, $\iota_V > 0$, and
(H): it is possible to interpolate between C and $\hat{C}$ by a family of $\mathcal{C}^{5+\alpha}$ hypersurfaces $(C_i)_{i \in [\![0,n]\!]}$ such that $C_i = \partial D_i$ with $D_i \in \mathcal{D}$, C_i is in a tubular neighborhood of C_{i+1}, and $\hat{D} \subset D_{i+1} \subsetneq D_i \subset D$, for $i \in [\![0, n-1]\!]$, $C_0 = C$ and $C_n = \hat{C}$. Then

(i) *The mapping $t \mapsto d(\partial D_t, \partial \hat{D}_t)$ is locally Lipschitz on $[0, \tau_C \wedge \tau_{\hat{C}})$.*

(ii) *$\partial D_t \cap \partial \hat{D}_t = \varnothing$, for $t \in (0, \tau_C \wedge \tau_{\hat{C}})$.*

Proof

We can use Corollary 26 with initial conditions C_i and C_{i+1}, and extend this corollary without the hypothesis that $\hat{C}$ belongs to an open tubular neighborhood of C, up to the time $\tau_{C,\hat{C}} := \inf_{i \in [\![1,n-1]\!]} \tau_{C_i, C_{i+1}}$. Hence for all $t \in (0, \tau_{C,\hat{C}})$ and all $i \in [\![1, n-1]\!]$ we have

$$(G_{i+1})_t \subset (G_i)_t \quad \text{and } \partial(G_{i+1})_t \cap \partial(G_i)_t = \varnothing$$

so for all $t \in (0, \tau_{C,\hat{C}})$ we have

$$\hat{G}_t \subset G_t \quad \text{and } \partial \hat{G}_t \cap \partial G_t = \varnothing \tag{78}$$

Let

$$\tau \ := \ \inf\{t \geqslant \tau_{C,\hat{C}}, \text{ s.t. } \partial D_t \cap \partial \hat{D}_t \neq \emptyset\} \wedge \tau_C \wedge \tau_{\hat{C}}$$

Using the same reasoning as the proof of Theorem 34, since $\partial D_t = \Psi(\partial G_t, \sqrt{2}B_t)$ and $\partial \hat{D}_t = \Psi(\partial \hat{G}_t, \sqrt{2}B_t)$ for all $t \in [0,\tau)$, we get

$$\begin{array}{rcl} \forall t \in [0,\tau), \qquad d(\partial D_t, \partial \hat{D}_t)) & = & d(\partial G_t, \partial \hat{G}_t) \\ \text{and } t \mapsto d(\partial D_t, \partial \hat{D}_t) \text{ is locally Lipschitz on } [0,\tau) \end{array}$$

Hence using (78) we get

$$\forall\ t \in (0,\tau), \quad \hat{D}_t \subset D_t \quad \text{and} \quad \partial D_t \cap \partial \hat{D}_t \ = \ \emptyset$$

Let $t_0 = \frac{\tau_{C,\hat{C}}}{2}$, since $\hat{D}_{t_0} \subset D_{t_0}$ and $d(\partial D_{t_0}, \partial \hat{D}_{t_0}) > 0$ we apply (ii) or (iii) of Theorem 34 to $\hat{D}_{t_0} \subset D_{t_0}$. We get, independently of the sign of the constant k,

$$\forall\ t \in [t_0, \tau_{C_{t_0}} \wedge \tau_{\hat{C}_{t_0}}), \qquad d(\partial D_t, \partial \hat{D}_t) \ > \ 0$$

since $\tau_{C_{t_0}} = \tau_C - t_0$ and $\tau_{\hat{C}_{t_0}} = \tau_{\hat{C}} - t_0$ we have $\tau = \tau_C \wedge \tau_{\hat{C}}$.

■

Remark 36 In the above proposition, Hypothesis (H) seems to be satisfied for all $D, \hat{D} \in \mathcal{D}$ with $\hat{D} \subset D$, even if $\partial D \cap \partial \hat{D} \neq \emptyset$, but for the moment we do not have a complete proof of this fact.

□

4.2 Local existence of (51)

In this subsection, we will show the existence of a solution to the system of equations (51). As the basic principle described in the paragraph following (16), a solution of (51) could be obtained as a solution of (58) conditioned not to collapse. Unfortunately, to develop this approach, we would need a solution of (58) defined for all times up to this collapsing. Since we have not been able to find such a maximal solution, we will directly work on (51), inspired by the previous subsection.

We recall the notations:

$$\begin{array}{rcl} \forall\ D \in \mathcal{D}, C = \partial D \qquad h(D) & = & 2\dfrac{\mu(C)}{\mu(D)} \\ \forall\ r > 0,\ \forall\ D \in \mathcal{D}_r,\ \forall\ x \in C, \qquad \alpha_{C,r}(x) & := & -\rho^b_{\Psi(C,r)}(\psi_{C,r}(x)). \end{array}$$

For given $D_0 \in \mathcal{D}$, we are interested in the system of equations:

$$\left\{ \begin{array}{rcl} \frac{d}{dt}\theta_t & = & h(\Psi(G_t, \sqrt{2}B_t + \theta_t)) \\ \forall\ x \in \partial G_t, \quad \partial_t x & = & \alpha_{\partial G_t, \sqrt{2}B_t + \theta_t}(x) \nu_{\partial G_t}(x) \\ (\theta_0, G_0) & = & (0, D_0) \end{array} \right. \tag{79}$$

To prove the existence of a solution to the above system of equations, we consider the equation described below. Let $g : [0,+\infty) \ni t \mapsto g(t) \in \mathbb{R}$ be a real $\frac{\alpha}{2}$-Hölder function, such that $g(0) = 0$ (or

small enough), and $0 < \alpha < 1$.
The goal of this first step is to show the existence of real numbers $t_0 > 0$ and $\delta > 0$, such that for all $g \in B_{C^{\alpha/2}}(0, \delta)$ and $g(0) = 0$, there exists a family $(G_t^g)_{t\in[0,t_0]}$ solution of

$$\begin{cases} \forall\, t \in [0, t_0],\, \forall\, x \in \partial G_t^g, & \partial_t x = \alpha_{\partial G_t^g, \sqrt{2}B_t+g(t)}(x)\nu_{\partial G_t^g}(x) \\ & G_0^g = D_0 \end{cases} \tag{80}$$

We adopt the same strategy as in the previous section, in order to deal with the quasi-parabolic equation, and we adopt the same notation, let $\partial D_0 = F_0(M)$.

We consider the following equation.

$$\begin{cases} \left\langle \frac{\partial}{\partial t} F^g(t,x), \nu^{F^g}(t,x) \right\rangle = -\rho^b_{\Psi(F^g(t,M),\sqrt{2}B_t+g(t))}(\psi_{F^g(t,M),\sqrt{2}B_t+g(t)}(x)) \\ F(0,x) = F_0(x), \end{cases} \tag{81}$$

As before we represent the solution as graphs over the fixed hypersurface $C = F_0(M)$, and we write the solution as:

$$F^g(t,x) = \psi_{C,f^g(t,x)}(F_0(x)) = F_0(x) + f^g(t,x)\nu_0(x)$$

for a function f^g with enough regularity and $f^g(0,.) = 0$. With similar computations as in the above section, F^g is a solution of (81) (with $r(t) = \sqrt{2}B_t + g(t)$ for any $t \geqslant 0$) if f^g satisfy the following non linear parabolic equation:

$$\begin{cases} \partial_t f^g(t,x) = \Phi^g(t, x, f^g, \nabla f^g, \nabla\nabla f^g) \\ f^g(0,x) = 0, \end{cases} \tag{82}$$

where Φ^g have the same definition as Φ in Proposition 19, but with $r(t) = \sqrt{2}B_t + g(t)$, for all $t \geqslant 0$. Taking into account that C is smooth, Theorem 22 leads to:

Proposition 37 *Take $g = g_0 \equiv 0$. There exists $0 < t_0 \leqslant T$ (where T comes from Proposition 19) such that (82) admits a solution f^{g_0} belonging to*

$$\mathfrak{X}(t_0) := \{u \in C^{1+\alpha/2,2+\alpha}(M_{t_0}) \;:\; u(0,.) = 0, \max(\|u\|_{\infty,M_{t_0}}, \|\nabla u\|_{\infty,M_{t_0}}, \|\nabla\nabla u\|_{\infty,M_{t_0}}) \leqslant R_0\}$$

We deduce:

Proposition 38 *With the same notation as in the above proposition, there exist two real $\delta_0, \delta_1 > 0$ and a continuously differentiable map*

$$\begin{aligned} \Theta \;:\; B_{C^{\frac{\alpha}{2}}([0,t_0])}(g_0, \delta_0) &\rightarrow B_{C^{1+\frac{\alpha}{2},2+\alpha}(M_{t_0})}(f^{g_0}, \delta_1) \\ g &\mapsto f^g, \end{aligned} \tag{83}$$

where f^g is a solution of (82). Moreover Θ is uniformly Lipschitz in $B_{C^{\frac{\alpha}{2}}([0,t_0])}(g_0, \delta_0)$.

Proof

Consider the mapping

$$\begin{aligned} S \;:\; \mathfrak{X}(t_0) \times C^{\frac{\alpha}{2}}([0,t_0]) &\rightarrow C^{\frac{\alpha}{2},\alpha}(M_{t_0}) \\ (u,g) &\mapsto \partial_t u - \Phi^g(t, x, u, \nabla u, \nabla\nabla u) \end{aligned} \tag{84}$$

It is continuously differentiable, at least when g belongs to a small ball. Note that from Proposition 37, there exists $(f^{g_0}, g_0) \in \mathfrak{X}(t_0) \times C^{\frac{\alpha}{2}}([0,t_0])$ such that $S(f^{g_0}, g_0) = 0$. Also

$$dS_u(f^{g_0}, g_0)(v) \;=\; d\mathcal{S}(f^{g_0})(v)$$

where $\mathcal{S}$ is defined before the proof of Theorem 22 (with $r(t) := \sqrt{2}B_t$). Since f^{g_0} is in $\mathfrak{X}$, $d\mathcal{S}(f^{g_0})$ is invertible with continuous inverse, according to Lemma 20 and Theorem 21. The result follows from implicit function theorem.

■

We will show the existence of solution of (79) by using a fixed point theorem. For $g \in B_{C^{\frac{\alpha}{2}}([0,t_0])}(g_0, \delta_0)$, define

$$F^g(t,x) \;:=\; F_0(x) + f^g(t,x)\nu_0(x)$$

and consider the family of hypersurfaces

$$\partial G_t^g \;:=\; F^g(t, M)$$

note that $G_0^g = D_0$.

Proposition 39 *There exist* $0 < t_1 \leqslant t_0$ *and a mapping*

$$\Gamma \,:\, B_{C^{\frac{\alpha}{2}}([0,t_1])}(g_0,\delta_0) \cap \{g \in C^{\frac{\alpha}{2}} | g(0) = 0\} \;\to\; B_{C^{\frac{\alpha}{2}}([0,t_1])}(g_0,\delta_0) \cap \{g \in C^{\frac{\alpha}{2}} \,:\, g(0) = 0\}$$

such that

$$\begin{cases} \forall\, t \in [0,t_1], & \frac{d}{dt}\Gamma(g)(t) \;=\; h(\Psi(G_t^g, \sqrt{2}B_t + g(t))), \\ & \Gamma(g)(0) \;=\; 0. \end{cases} \tag{85}$$

Moreover Γ *is a contraction and there exists an unique fixed point for* Γ *in* $B_{C^{\frac{\alpha}{2}}([0,t_1])}(g_0,\delta_0) \cap \{g \in C^{\frac{\alpha}{2}}([0,t_1]) \,:\, g(0) = 0\}$.

Proof

Take δ_0 such that by Proposition 38, Θ is uniformly Lipschitz in $B_{C^{\frac{\alpha}{2}}([0,t_0])}(g_0,\delta_0)$. Let $g \in B_{C^{\frac{\alpha}{2}}([0,t_0])}(g_0,\delta_0)$, $r \in \mathbb{R}$ and $f^g = \Theta(g)$, define for all $x \in M$:

$$\begin{aligned} F_\psi^g(t,x,r) &= F^g(t,x) + r\nu^{F^g}(t,x) \\ &= F_0(x) + f^g(t,x)\nu_0(x) + r\nu^{F^g}(t,x) \\ &= F_0(x) + f^g(t,x)\nu_0(x) + r\nu(t,x,f^g(t,\cdot),\nabla f^g(t,\cdot)) \end{aligned}$$

then we have

$$\Psi(\partial G_t^g, r(t)) = \{F_0(x) + f^g(t,x)\nu_0(x) + r(t)\nu(t,x,f^g(t,\cdot),\nabla f^g(t,\cdot)) \,:\, x \in M\},$$

and

$$h(\Psi(G_t^g, \sqrt{2}B_t + g(t))) = 2\frac{\underline{\mu}(\Psi(\partial G_t^g, \sqrt{2}B_t + g(t)))}{\mu(\Psi(G_t^g, \sqrt{2}B_t + g(t)))}.$$

We have the following formula for the n-volume of the boundary:

$$\begin{aligned} \underline{\mu}(\Psi(\partial G_t^g, \sqrt{2}B_t + g(t))) &= \int_{F_\psi^g(t,M,\sqrt{2}B_t+g(t))} d\underline{\mu}_{F_\psi^g(t,M,\sqrt{2}B_t+g(t))} \\ &= \int_M \det[\nu^{F_\psi^g}(t,x), d_xF_\psi^g(t,x,\sqrt{2}B_t + g(t))]\, d\mu_M. \end{aligned} \tag{86}$$

In the above formula, $d\mu_M$ is a Riemannian measure for a fixed metric in M and $d_x F^g_\psi(t,x,\sqrt{2}B_t+g(t))$ is evaluated in an orthonormal basis for this metric. Let

$$d^g(t,x) := \det[\nu^{F^g_\psi}(t,x), d_x F^g_\psi(t,x,\sqrt{2}B_t+g(t))] =: V(x,\sqrt{2}B_t+g(t), f^g(t,x), \nabla f^g(t,x), \nabla\nabla f^g(t,x)),$$

where V is a function regular in the four last components. It follows that there exists a constant $C>0$ such that

$$\langle (t,x)\mapsto d^g(t,x)\rangle_{\alpha/2} \leqslant C(\|\sqrt{2}B_.\|_{C^{\alpha/2}} + \|g\|_{C^{\alpha/2}} + \|f^g\|_{C^{1+\alpha/2,2+\alpha}})$$

with the semi-norm

$$\langle (t,x)\mapsto d^g(t,x)\rangle_{\alpha/2} := \sup\left\{\frac{|f(t)-f(s)|}{|t-s|^\alpha},\ s\neq t\in[0,t_0],\ x\in M\right\}$$

We deduce there exists C_{δ_0,δ_1}, depending on δ_0, δ_1 and on the random quantity $\|\sqrt{2}B_.\|_{C^{\alpha/2}}$, such that

$$\langle t\mapsto d^g(t,x)\rangle_{\alpha/2} \leqslant C_{\delta_0,\delta_1}$$

and thus

$$\|t\mapsto d^g(t,x)\|_{C^{\alpha/2}[0,t_0]} \leqslant C_{\delta_1,\delta_0}(t_0^{\alpha/2}+1)+K$$

with $K=\|d^g(0,.)\|_\infty$ not depending on g.

Hence $t\mapsto \underline{\mu}(\Psi(\partial G^g_t,\sqrt{2}B_t+g(t)))$ is in $C^{\alpha/2}$ and

$$\|t\mapsto \underline{\mu}(\Psi(\partial G^g_t,\sqrt{2}B_t+g(t)))\|_{C^{\alpha/2}} \leqslant (C_{\delta_1,\delta_0}(t_0^{\alpha/2}+1)+K)\mu(M), \tag{87}$$

Using Stoke's Theorem we have that the volume of $\mu(\Psi(G^g_t,\sqrt{2}B_t+g(t)))$ enclosed by the hypersurface $\Psi(\partial G^g_t,\sqrt{2}B_t+g(t))$ is

$$\begin{aligned}\mu(\Psi(G^g_t,\sqrt{2}B_t+g(t))) &= \tfrac{1}{n+1}\textstyle\int_{F^g_\psi(t,M)}\langle\vec{x},\nu^{F^g_\psi}\rangle\, d\underline{\mu}_{F^g_\psi(t,M)}(\vec{x})\\ &= \tfrac{1}{n+1}\textstyle\int_M\langle F^g_\psi(t,x,\sqrt{2}B_t+g(t)),\nu^{F^g_\psi}(t,x)\rangle d^g(t,x)\, d\mu_M(x).\end{aligned} \tag{88}$$

As before, we get, for some $C'_{\delta_1,\delta_0}>0$ and $K'>0$ of the same nature as $C_{\delta_1,\delta_0}>0$ and $K>0$, that

$$\|t\mapsto \mu(\Psi(G^g_t,\sqrt{2}B_t+g(t)))\|_{C^{\alpha/2}} \leqslant (C'_{\delta_1,\delta_0}(t_0^{\alpha/2}+1)+K')\mu(M) \tag{89}$$

As a quotient, it follows that $t\mapsto h(\Psi(G^g_t,\sqrt{2}B_t+g(t))$ is in $C^{\alpha/2}$, as long as the domain $\Psi(G^g_t,\sqrt{2}B_t+g(t))$ keeps a positive mass, which may lead us to replace t_0 by a smaller value, and we deduce that

$$\|t\mapsto h(\Psi(G^g_t,\sqrt{2}B_t+g(t)))\|_{C^{\alpha/2}([0,t_0])} \leqslant C \tag{90}$$

for a constant C that only depends on δ_0 ,δ_1, t_0 and $\|\sqrt{2}B_.\|_{C^{\alpha/2}}$. So $\Gamma(g)\in C^{1+\alpha/2}$. We have for $0<s,t<t_1\leqslant t_0$

$$|\Gamma(g)(t)-\Gamma(g)(s)| \leqslant |t-s|C \leqslant Ct_1^{1-\alpha/2}|t-s|^{\alpha/2},$$

since $\Gamma(g)(0)=0$ we have that:

$$\|\Gamma(g)\|_{C^{\alpha/2}[0,t_1]} \leqslant Ct_1 + Ct_1^{1-\alpha/2}.$$

Take $0<t_1\leqslant t_0$ sufficiently small such that $Ct_1+Ct_1^{1-\alpha/2}\leqslant\delta_0$ we have Γ maps $B_{C^{\frac{\alpha}{2}}([0,t_1])}(g_0,\delta_0)\cap\{g\in C^{\frac{\alpha}{2}}|g(0)=0\}$ into himself.

Let us show that Γ is a contraction.

Let $g_1, g_2 \in B_{C^{\frac{\alpha}{2}}([0,t_1])}(g_0, \delta_0)$, and $f^{g_1} = \Theta(g_1)$, $f^{g_2} = \Theta(g_2)$ then

$$\begin{array}{l} \underline{\mu}(\Psi(\partial G_t^{g_1}, \sqrt{2}B_t + g_1(t))) - \underline{\mu}(\Psi(\partial G_t^{g_2}, \sqrt{2}B_t + g_2(t))) \\ = \int_M V(x, \sqrt{2}B_t + g_1(t), f^{g_1}(t,x), \nabla f^{g_1}(t,x), \nabla\nabla f^{g_1}(t,x)) \\ \quad -V(x, \sqrt{2}B_t + g_2(t), f^{g_2}(t,x), \nabla f^{g_2}(t,x), \nabla\nabla f^{g_2}(t,x))\, \mu_M(dx). \end{array} \tag{91}$$

We want to control the norm of the above function in $C^{\alpha/2}$. Since it vanishes at time 0, we have only to control its semi-norm $\langle\cdot\rangle_{\alpha/2}$.

We write for simplicity $\vec{f^g}(t,x) := (f^g(t,x), \nabla f^g(t,x), \nabla\nabla f^g(t,x))$, and let

$$J(t,x) \;:=\; V(x, \sqrt{2}B_t + g_1(t), \vec{f^{g_1}}(t,x)) - V(x, \sqrt{2}B_t + g_2(t), \vec{f^{g_2}}(t,x))$$

Let $\sigma \in [0,1]$ and

$$\zeta_\sigma(t,x) \;:=\; \sigma(\sqrt{2}B_t + g_1(t), \vec{f^{g_1}}(t,x)) + (1-\sigma)(\sqrt{2}B_t + g_2(t), \vec{f^{g_2}}(t,x))$$

we have, for all $0 \leqslant s, t \leqslant t_1$,

$$\begin{aligned} |\zeta_\sigma(t,x) - \zeta_\sigma(s,x)| &\leqslant |t-s|^{\alpha/2}(2\sqrt{2}\|B.\|_{C^{\alpha/2}} + 2\delta_0 + 2\delta_1) \\ &\leqslant C_{\delta_0,\delta_1}|t-s|^{\alpha/2} \end{aligned}$$

Also using the regularity of V in the four last variables we have

$$J(t,x) \;=\; \int_0^1 d_3V(x, \zeta_\sigma(t,x))((g_1(t), \vec{f^{g_1}}(t,x)) - (g_2(t), \vec{f^{g_2}}(t,x)))\, d\sigma$$

Hence,

$$\begin{aligned} &|J(t,x) - J(s,x)| \\ &= |\int_0^1 d_3V(x, \zeta_\sigma(t,x))(g_1(t) - g_2(t), \vec{f^{g_1}}(t,x) - \vec{f^{g_2}}(t,x)) \\ &\quad -d_3V(x, \zeta_\sigma(s,x))(g_1(s) - g_2(s), \vec{f^{g_1}}(s,x) - \vec{f^{g_2}}(s,x)) d\sigma| \\ &\leqslant \int_0^1 |\big(d_3V(x, \zeta_\sigma(t,x)) - d_3V(x, \zeta_\sigma(s,x))\big)(g_1(t) - g_2(t), \vec{f^{g_1}}(t,x) - \vec{f^{g_2}}(t,x))|\, d\sigma \\ &\quad + \int_0^1 \big|d_3V(x, \zeta_\sigma(s,x))\big((g_1(t) - g_2(t), \vec{f^{g_1}}(t,x) - \vec{f^{g_2}}(t,x)) \\ &\qquad\qquad -(g_1(s) - g_2(s), \vec{f^{g_1}}(s,x) - \vec{f^{g_2}}(s,x))\big)\big|\, d\sigma \end{aligned}$$

Since $d_3V(x, \zeta_\sigma(s,x))$ is bounded we have (again the constant C can change from one line to the other),

$$\begin{aligned} &|d_3V(x, \zeta_\sigma(s,x))\big((g_1(t) - g_2(t), \vec{f^{g_1}}(t,x) - \vec{f^{g_2}}(t,x)) - (g_1(s) - g_2(s), \vec{f^{g_1}}(s,x) - \vec{f^{g_2}}(s,x))\big) \\ &\quad\leqslant C|t-s|^{\alpha/2}(\|g_1 - g_2\|_{C^{\alpha/2}} + \|\vec{f^{g_1}} - \vec{f^{g_2}}\|_{C^{\alpha/2,\alpha}}) \\ &\quad\leqslant C|t-s|^{\alpha/2}(\|g_1 - g_2\|_{C^{\alpha/2}} + \|\Theta(g_1) - \Theta(g_2)\|_{C^{1+\alpha/2,2+\alpha}}) \\ &\quad\leqslant C|t-s|^{\alpha/2}(1 + \|\Theta\|_{\mathrm{Lip}})\|g_1 - g_2\|_{C^{\alpha/2}} \end{aligned}$$

where in the last line we use Proposition 38. Using that $d_3V(x,.)$ is Lipschitz in the last variable:

$$\begin{aligned}|d_3V(x,\zeta_\sigma(t,x)) - d_3V(x,\zeta_\sigma(s,x)) &\leqslant C|\zeta_\sigma(t,x)-\zeta_\sigma(s,x)| \\ &\leqslant CC_{\delta_0,\delta_1}|t-s|^{\alpha/2}\end{aligned}$$

Since $(g_1(0),\vec{f}^{g_1}(0,x)) = \vec{0} = (g_2(0),\vec{f}^{g_2}(0,x))$ we have:

$$|(g_1(t)-g_2(t),\vec{f}^{g_1}(t,x)-\vec{f}^{g_2}(t,x))| \leqslant Ct^{\alpha/2}(1+\|\Theta\|_{\text{Lip}})\|g_1-g_2\|_{C^{\alpha/2}}$$

Putting all things together we get $\langle t\mapsto J(t,x)\rangle_{C^{\alpha/2}} \leqslant C\|g_1-g_2\|_{C^{\alpha/2}}$ and since $J(0,x)=0$,

$$\|t\mapsto J(t,x)\|_{C^{\alpha/2}} \leqslant C\|g_1-g_2\|_{C^{\alpha/2}}$$

Hence

$$\|t\mapsto \underline{\mu}(\Psi(\partial G_t^{g_1},\sqrt{2}B_t+g_1(t))) - \underline{\mu}(\Psi(\partial G_t^{g_2},\sqrt{2}B_t+g_2(t)))\|_{C^{\alpha/2}} \leqslant C\|g_1-g_2\|_{C^{\alpha/2}} \tag{92}$$

With the same proof we also have:

$$\|t\mapsto \mu(\Psi(G_t^{g_1},\sqrt{2}B_t+g_1(t))) - \mu(\Psi(G_t^{g_2},\sqrt{2}B_t+g_2(t)))\|_{C^{\alpha/2}} \leqslant C\|g_1-g_2\|_{C^{\alpha/2}} \tag{93}$$

Let $\mu(g)(t) := \mu(\Psi(G_t^g,\sqrt{2}B_t+g(t)))$ and $\underline{\mu}(g)(t) := \underline{\mu}(\Psi(\partial G_t^g,\sqrt{2}B_t+g(t)))$

$$\begin{aligned}\frac{d}{dt}\Big(\Gamma(g_1)-\Gamma(g_2)\Big) &= 2\Big(\frac{\underline{\mu}(g_1)}{\mu(g_1)} - \frac{\underline{\mu}(g_2)}{\mu(g_2)}\Big) \\ &= 2\Big(\frac{\underline{\mu}(g_1)\mu(g_2)-\underline{\mu}(g_2)\mu(g_1)}{\mu(g_1)\mu(g_2)}\Big) \\ &= 2\Big(\frac{\underline{\mu}(g_1)(\mu(g_2)-\mu(g_1)) - \mu(g_1)(\underline{\mu}(g_2)-\underline{\mu}(g_1))}{\mu(g_1)\mu(g_2)}\Big)\end{aligned}$$

Hence using (87), (89), (92) and (93),

$$\left\|\frac{d}{dt}\Big(\Gamma(g_1)-\Gamma(g_2)\Big)\right\|_{C^{\alpha/2}([0,t_1])} \leqslant C\|g_1-g_2\|_{C^{\alpha/2}}$$

and so

$$\left\|\frac{d}{dt}\Big(\Gamma(g_1)-\Gamma(g_2)\Big)\right\|_{C^{0}([0,t_1])} \leqslant C\|g_1-g_2\|_{C^{\alpha/2}}$$

Since $\Gamma(g_1)(0) = 0 = \Gamma(g_2)(0)$,

$$\|\Gamma(g_1)-\Gamma(g_2)\|_{C^{\alpha/2}([0,t_1])} \leqslant (t_1+t_1^{1-\alpha/2})C\|g_1-g_2\|_{C^{\alpha/2}}$$

Reducing t_1 such that $(t_1+t_1^{1-\alpha/2})C \leqslant \frac{1}{2}$, we get:

$$\|\Gamma(g_1)-\Gamma(g_2)\|_{C^{\alpha/2}([0,t_1])} \leqslant \frac{1}{2}\|g_1-g_2\|_{C^{\alpha/2}([0,t_1])}$$

Hence Γ have a unique fixed point in $B_{C^{\frac{\alpha}{2}}([0,t_1])}(g_0,\delta_0) \cap \{g\in C^{\frac{\alpha}{2}} | g(0)=0\}$.

■

Theorem 40 *Let $D_0 \in \mathcal{D}$, then there exists $0 < t_1$ such that the system of equations (79) has a unique solution.*

Proof

Let θ be the fixed point of Γ, and $f^\theta = \Theta(\theta)$ then $F^\theta(t,x) = F_0(x) + f^\theta(t,x)\nu_0(x)$ solves

$$\begin{cases} \frac{\partial}{\partial t}F^\theta(t,x) &= \left(-\rho^b_{\Psi(F^\theta(t,M),\sqrt{2}B_t+\theta(t))}(\psi_{F^\theta(t,M),\sqrt{2}B_t+\theta(t)}(x))\right)\nu^{F^\theta}(t,x) \\ F(0,x) &= F_0(x). \end{cases}$$

and so

$$\begin{cases} \forall\ t \in [0,t_1],\ \forall\ x \in \partial G^\theta_t, & \partial_t x = \alpha_{\partial G^\theta_t,\sqrt{2}B_t+\theta(t)}(x)\nu_{\partial G^\theta_t}(x) \\ & G^\theta_0 = D_0 \end{cases}$$

Also

$$\begin{cases} \frac{d}{dt}\theta(t) &= \frac{d}{dt}\Gamma(\theta)(t) \\ &= h(\Psi(G^\theta_t, \sqrt{2}B_t + \theta(t)), \\ \Gamma(\theta)(0) &= 0. \end{cases}$$

■

Let $D \in \mathcal{D}$, $C = \partial D$ with $\mathcal{C}^{5+\alpha}$ boundaries, $\alpha \in (0,1)$, in a d-dimensional Riemannian manifold V, and $(\theta_t, G_t)_{0\leqslant t<\tau}$ be a solution of (79) given by Theorem 40. As in the beginning of this section, the solution of

$$\forall\ t \in [0,\tau),\ \forall\ x \in C_t := D_t, \qquad dx = \left(\sqrt{2}dB_t + 2\frac{\mu(C_t)}{\mu(D_t)}dt - \rho_{C_t}(x)dt\right)\nu_{C_t}(x) \tag{94}$$

is given by $(D_t)_{t\in[0,\tau)}$, where

$$\forall\ t \in [0,\tau), \qquad D_t := \Psi(G_t, \sqrt{2}B_t + \theta_t)$$

(as a special case of (44)).

Proposition 42 below will give a control of the extrinsic diameter of C_t defined by

$$\mathrm{diam}(C_t) := \sup_{(x,y)\in C_t^2} d(x,y)$$

where $d(\cdot,\cdot)$ is the Riemannian distance in V. First we need the following proposition bounding the sum of the mean curvature at points that realize the diameter, in terms of the extrinsic curvature (by extrinsic we mean in the ambient manifold V, i.e. not intrinsic in the hypersurface). For all $b \in \mathbb{R}$, we denote by $V^b(d)$ the d-dimensional manifold with constant curvature b. Let $\iota_{V^b(d)}$ defined before Lemma 32. We have:

$$\begin{cases} \iota_{V^b(d)} = \infty &, \text{if} \quad b \leqslant 0 \\ \iota_{V^b(d)} = \frac{\pi}{\sqrt{b}} &, \text{if} \quad b > 0 \end{cases}$$

Proposition 41 *Let $D \in \mathcal{D}$ with a C^2 boundary in a d-dimensional manifold V, and $C = \partial D$. Suppose that there exists $b \in \mathbb{R}$ such that the sectional curvature K_V of V is bounded above by b, i.e. $K_V \leqslant b$. For all $(p,q) \in C^2$ such that $d(p,q) = \mathrm{diam}(C)$ and $d(p,q) < \iota_{V^b(d)}$, we have*

1. *if* $b \leqslant 0$ *then* $-\rho_C(p) - \rho_C(q) \leqslant 2(d-1)\sqrt{|b|}\big(\frac{1-\cosh(\sqrt{|b|}d(p,q))}{\sinh(\sqrt{|b|}d(p,q))}\big) \leqslant 0,$

2. *if* $b > 0$ *then* $-\rho_C(p) - \rho_C(q) \leqslant 2(d-1)\sqrt{b}\big(\frac{1-\cos(\sqrt{b}d(p,q))}{\sin(\sqrt{b}d(p,q))}\big).$

Proof

As in the proof of Lemme 29, consider $(p,q) \in C^2$ such that $d(p,q) = \mathrm{diam}(C)$. Using the first variation formula, we get there exists an unit speed geodesic γ in V such that $\gamma(0) = q$, $\gamma(d(p,q)) = p$, $\dot\gamma(0) = -\nu_C(q)$ and $\dot\gamma(d(p,q)) = \nu_C(p)$. Let $(e_i)_{i\in[\![1,d-1]\!]}$ be a orthonormal basis of T_qC. For $i \in [\![1, d-1]\!]$, let $\gamma_{1,i}(t)$ be a geodesic in C such that $\gamma_{1,i}(0) = q$ and $\dot\gamma_{1,i}(0) = e_i$. Let $\gamma_{2,i}(t)$ be a geodesic in C such that $\gamma_{2,i}(0) = p$ and $\dot\gamma_{1,i}(0) = /\!/_{d(p,q)}e_i$, where $/\!/$ is the parallel transport along the geodesic γ. Since $(p,q) \in C^2$ is a local maximum of the distance function restricted to $C \times C$, we have that

$$\frac{d^2}{dt^2}\bigg|_{t=0} d(\gamma_{1,i}(t), \gamma_{2,i}(t)) \leqslant 0.$$

Let Y_i be the Jacobi field along γ obtained by the variation of geodesic connecting $\gamma_{1,i}(t)$ to $\gamma_{2,i}(t)$, we have: $Y_i(0) = e_i$, $Y_i(d(p,q)) = /\!/_{d(p,q)}e_i$. Using second variation formula, we get:

$$\begin{aligned}
\frac{d^2}{d^2t}d(\gamma_{1,i}(t), \gamma_{2,i}(t))|_{t=0} &= \big[\langle \nabla_{t=0}\dot\gamma_{2,i}(t), \dot\gamma(d(p,q))\rangle - \langle \nabla_{t=0}\dot\gamma_{1,i}(t), \dot\gamma(0)\rangle\big] + I(Y_i, Y_i) \\
&= \big[\langle \nabla_{t=0}\dot\gamma_{2,i}(t), \nu_C(p)\rangle - \langle \nabla_{t=0}\dot\gamma_{1,i}(t), -\nu_C(q)\rangle\big] + I(Y_i, Y_i) \\
&= -\Pi_C(/\!/_{d(p,q)}e_i, /\!/_{d(p,q)}e_i) - \Pi_C(e_i, e_i) + I(Y_i, Y_i)
\end{aligned}$$

Put the above two computations together and take the sum to get:

$$-\rho_C(q) - \rho_C(p) \leqslant -\sum_{i=1}^{d-1} I(Y_i, Y_i).$$

We have to bound from below the index of the normal Jacobi field Y_i for all i. Since Y_i is a normal Jacobi field, there exist real functions f_i^j for $j \in [\![1, d-1]\!]$ such that $Y_i(t) = \sum_{j=1}^{d-1} f_i^j(t)/\!/_t e_j$. By construction of Y_i, we have $f_i^j(0) = f_i^j(d(p,q)) = \delta_i^j$. Consider $\tilde\gamma(t)_{t\in[0,d(p,q)]}$ a geodesic in $V^b(d)$ with same length as γ, take $(\tilde e_i)_{i\in[\![1,d-1]\!]}$ an orthonormal basis of $\dot{\tilde\gamma}(0)^\perp$ in $T_{\tilde\gamma(0)}V^b(d)$, and denote by $\tilde{/\!/}$ the parallel transport along $\tilde\gamma$. Let $\tilde X_i(t) = \sum_{j=1}^{d-1} f_i^j(t)\tilde{/\!/}_t\tilde e_j$, be a vector field along $\tilde\gamma$, note that $\tilde X_i(0) = \tilde e_i$ and $\tilde X_i(d(p,q)) = \tilde{/\!/}\tilde e_i$. Let $\tilde Y_i$ be the Jacobi field in $V^b(d)$ along $\tilde\gamma$ such that $\tilde Y_i(0) = \tilde e_i$ and $\tilde Y_i(d(p,q)) = \tilde{/\!/}_{d(p,q)}\tilde e_i$. We have by definition:

$$\begin{aligned}
I(Y_i, Y_i) &= \int_0^{d(p,q)} \|\nabla_t Y_i\|^2 - \langle R(Y_i, \dot\gamma)Y_i, \dot\gamma\rangle\, dt \\
&\geqslant \int_0^{d(p,q)} \|\nabla_t Y_i\|^2 - b\|Y_i\|^2\, dt \\
&= \int_0^{d(p,q)} \|\nabla_t \tilde X_i\|^2 - b\|\tilde X_i\|^2\, dt \\
&\geqslant \int_0^{d(p,q)} \|\nabla_t \tilde Y_i\|^2 - b\|\tilde Y_i\|^2\, dt
\end{aligned}$$

where in the last inequality we used again the Index Lemma, since $d(p,q) < \iota_{V^b(d)}$. So $\tilde{\gamma}(0)$ and $\tilde{\gamma}(d(p,q))$ are not conjugate in $V^b(d)$. Since $\tilde{Y}_i(t) = f_b(t)\tilde{/\!/}_t \tilde{e}_i$ with $f_b'' = -bf_b$, and $f_b(0) = f_b(d(p,q)) = 1$, we get

$$\begin{aligned} I(Y_i, Y_i) &\geqslant \int_0^{d(p,q)} (f_b')^2 - bf_b^2\, dt \\ &= (f_b'(d(p,q)) - f_b'(0)) \end{aligned}$$

Hence

$$-\rho_C(q) - \rho_C(p) \leqslant -(d-1)(f_b'(d(p,q)) - f_b'(0))$$

and the result follows by explicit computation of f_b in different cases.

∎

Proposition 42 *Let $D \in \mathcal{D}$ with a $\mathcal{C}^{5+\alpha}$ boundary $C := \partial D$ in a d-dimensional manifold V, for some fixed $\alpha \in (0,1)$. Suppose there exists $b \in \mathbb{R}$ such that the sectional curvature of V satisfies $K_V \leqslant b$. Then the evolution of the diameter of the solution $(C_t)_{t\in[0,\tau)}$ of (94) started at C is controlled by:*

(i) *If $b \leqslant 0$, we get for all $0 \leqslant t < \tau$,*

$$d\,\mathrm{diam}(C_t) \leqslant 2(\sqrt{2}dB_t + h(D_t)dt) + 2(d-1)\sqrt{|b|}\Big(\frac{1-\cosh(\sqrt{|b|}\,\mathrm{diam}(C_t))}{\sinh(\sqrt{|b|}\,\mathrm{diam}(C_t))}\Big)dt$$

(ii) *If $b > 0$, we get, for all $0 \leqslant t < \tau \wedge \tau_{\frac{\pi}{\sqrt{b}}}(\mathrm{diam}(C_.))$,*

$$d\,\mathrm{diam}(C_t) \leqslant 2(\sqrt{2}dB_t + h(D_t)dt) + 2(d-1)\sqrt{b}\Big(\frac{1-\cos(\sqrt{b}\,\mathrm{diam}(C_t))}{\sin(\sqrt{b}\,\mathrm{diam}(C_t))}\Big)dt$$

where $\tau_{\frac{\pi}{\sqrt{b}}}(\mathrm{diam}(C_.)) := \inf\{t \geqslant 0 \,:\, \mathrm{diam}(C_t) \geqslant \frac{\pi}{\sqrt{b}}\}$.

Proof

Using the construction of $(D_t)_{t\in[0,\tau)}$, we get, for $0 \leqslant t < \tau$,

$$\begin{aligned} \mathrm{diam}(C_t) &= \sup_{(x,y)\in\partial G_t^2} d(\Psi_{G_t}(x, \sqrt{2}B_t + \theta_t), \Psi_{G_t}(y, \sqrt{2}B_t + \theta_t)) \\ &= 2(\sqrt{2}B_t + \theta_t) + \sup_{(x,y)\in\partial G_t^2} d(x,y) \end{aligned}$$

where in the second equality, we used that for $0 \leqslant t < \tau$, $\Psi_{G_t}(., \sqrt{2}B_t + \theta_t)$ is a diffeomorphism onto its image, and a reasoning similar to the proof of Proposition 30. Also since

$$\sup_{(x,y)\in\partial G_t^2} d(x,y) = \sup_{x,y\in M^2} d(F^\theta(t,x), F^\theta(t,y)),$$

and the mappings $t \mapsto F^\theta(t,x)$ are uniformly Lipschitz on any compact $[0,T] \subset [0,\tau)$, we deduce that

$$t \mapsto \sup_{(x,y)\in\partial G_t^2} d(x,y)$$

is Lipschitz on $[0,T]$, hence almost everywhere differentiable on $[0,T]$ and absolutely continuous.

At a differentiability time $t \in [0, T]$, we have, as in the proof of Proposition 30,

$$
\begin{aligned}
&\frac{d}{dt} \sup_{(x,y)\in\partial G_t^2} d(x,y) \\
&= \frac{d}{dt} \sup_{(x_t,y_t)\in\partial G_t^2\,:\,d(x_t,y_t)=\mathrm{diam}(\partial G_t)} d(x_t,y_t) \\
&= \sup_{(x_t,y_t)\in\partial G_t^2\,:\,d(x_t,y_t)=\mathrm{diam}(\partial G_t)} \frac{d}{dt} d(x_t,y_t) \\
&= \sup_{(x_t,y_t)\in\partial G_t^2\,:\,d(x_t,y_t)=\mathrm{diam}(\partial G_t)} \langle \frac{d}{dt}x_t, \nu^{\partial G_t}(x_t)\rangle + \langle \frac{d}{dt}y_t, \nu^{\partial G_t}(y_t)\rangle \\
&= \sup_{(x_t,y_t)\in\partial G_t^2\,:\,d(x_t,y_t)=\mathrm{diam}(\partial G_t)} -\rho_{\Psi(\partial G_t,\sqrt{2}B_t+\theta_t)}(\psi_{\partial G_t,\sqrt{2}B_t+\theta_t}(x_t)) - \rho_{\Psi(\partial G_t,\sqrt{2}B_t+\theta_t)}(\psi_{\partial G_t,\sqrt{2}B_t+\theta_t}(y_t)) \\
&= \sup_{(x_t,y_t)\in\partial D_t^2\,:\,d(x_t,y_t)=\mathrm{diam}(\partial D_t)} -\rho_{\partial D_t}(x_t) - \rho_{\partial D_t}(y_t)
\end{aligned}
$$

Taking into account Proposition 41, we obtain the wanted points (i) and (ii).

■

When (94) is replaced by (76), the previous proof leads to a similar result:

Proposition 43 *Let $D \in \mathcal{D}$ with a $\mathcal{C}^{5+\alpha}$ boundary $C := \partial D$ in a d-dimensional manifold V, for some fixed $\alpha \in (0,1)$. Suppose there exists $b \in \mathbb{R}$ such that the sectional curvature of V satisfies $K_V \leqslant b$. Then the evolution of the diameter of the solution $(C_t)_{t\in[0,\tau)}$ of (76) started at C is controlled by:*

(i) *If $b \leqslant 0$, we get, for all $0 \leqslant t < \tau$,*

$$
d\,\mathrm{diam}(C_t) \;\leqslant\; 2\sqrt{2}dB_t + 2(d-1)\sqrt{|b|}\Big(\frac{1-\cosh(\sqrt{|b|}\,\mathrm{diam}(C_t))}{\sinh(\sqrt{|b|}\,\mathrm{diam}(C_t))}\Big)dt
$$

(ii) *If $b > 0$, we get ,for all $0 \leqslant t < \tau \wedge \tau_{\frac{\pi}{\sqrt{b}}}(\mathrm{diam}(C_.))$,*

$$
d\,\mathrm{diam}(C_t) \;\leqslant\; 2\sqrt{2}dB_t + 2(d-1)\sqrt{b}\Big(\frac{1-\cos(\sqrt{b}\,\mathrm{diam}(C_t))}{\sin(\sqrt{b}\,\mathrm{diam}(C_t))}\Big)dt,
$$

where $\tau_{\frac{\pi}{\sqrt{b}}}(\mathrm{diam}(C_.)) := \inf\{t \geqslant 0 \,:\, \mathrm{diam}(C_t) \geqslant \frac{\pi}{\sqrt{b}}\}$.

Remark 44 Proposition 43 may seem simpler than Proposition 42, since it does not require to deal with the tricky term $h(D_t)$. For instance when $K_V \leqslant 0$, we have for all $0 \leqslant t < \tau$:

$$
\mathrm{diam}(C_t) \;\leqslant\; 2\sqrt{2}(B_t - B_0) + \mathrm{diam}(C_0)
$$

It follows that $\tau \leqslant \tau_{-\frac{\mathrm{diam}(C_0)}{2\sqrt{2}}}(B_.)$ a.s. But the supplementary term $h(D_t)$ in Proposition 42 should prevent this collapsing in finite time.

□

5 Back to the homogeneous situations

Here we return to the situations encountered in Section 2, where V has a constant curvature and is endowed with the Laplacian $L \coloneqq \Delta$. This section has two main goals developed in the following subsections:

- When V is an Euclidean space, it is possible to go further in the considerations of Section 3. In particular when $V = \mathbb{R}^2$, it is possible to compute explicitly the image of the mean curvature vector field by the tangent mappings to the normal flow.

- When $D_0 = B(x_0, r_0)$ with $x_0 \in V$ and $r_0 > 0$ (small enough in the spherical case), the Doss-Sussman approach can be described explicitly (more generally this is also true when V is rotationally symmetric and x_0 is a center of symmetry). It is then possible to compare the Doss-Sussman methods in the two decompositions (22) and (57), concerning their respective time-domains and to see that the method suggested in Remark 18 is stable when we let r_0 go to zero, namely when we try an approximation of the initial conditions consisting of singletons.

5.1 About the Euclidean and constant curvature spaces

We begin by bringing some precisions about the quantities defined in (25) and (26). They can always be written

$$R_-(D) \;=\; \widetilde{R}_-(D) \vee \widehat{R}_-(D) \quad \text{and} \quad R_+(D) \;=\; \widetilde{R}_+(D) \wedge \widehat{R}_+(D)$$

where

$$\begin{aligned}
\widetilde{R}_+(D) &\coloneqq \inf\{r \in (0, +\infty) \,:\, \psi_{C,r} \text{ is not an immersion}\} \\
\widetilde{R}_-(D) &\coloneqq -\inf\{r \in (0, +\infty) \,:\, \psi_{C,-r} \text{ is not an immersion}\} \\
\widehat{R}_+(D) &\coloneqq \inf\{r \in (0, +\infty) \,:\, \psi_{C,r} \text{ is not one-to-one}\} \\
\widehat{R}_-(D) &\coloneqq -\inf\{r \in (0, +\infty) \,:\, \psi_{C,-r} \text{ is not one-to-one}\}
\end{aligned}$$

(with the usual convention $\inf \varnothing = +\infty$).

Consider the Euclidean case:

Lemma 45 *When $V = \mathbb{R}^n$, with $n \geqslant 2$ and endowed with its Euclidean structure, we have*

$$\begin{aligned}
\widetilde{R}_-(D) &= \frac{1}{\min(0_-, \min\{-\lambda_{n-1,C}(x) \,:\, x \in C\})} \in [-\infty, 0) \\
\widetilde{R}_+(D) &= \frac{1}{\max(0_+, \max\{-\lambda_{1,C}(x) \,:\, x \in C\})} \in (0, +\infty]
\end{aligned}$$

where $\lambda_{1,C}(x) \leqslant \cdots \leqslant \lambda_{n-1,C}(x)$ are the eigenvalues of the second fundamental form (defined with respect to ν_C) at $x \in C$. The notations 0_- and 0_+ just indicate that $1/0_- = -\infty$ and $1/0_+ = +\infty$.

Proof

Recall that the tangent mapping $d\nu_C$ associated to the mapping $C \ni x \mapsto \nu_C(x)$ can be seen as a linear mapping from T_xC (the tangent space of C at x) to itself, and that the second fundamental form is given at $x \in C$ by

$$T_xC \times T_xC \ni (v, w) \;\mapsto\; \langle v, d\nu_C[w]\rangle$$

We deduce that for $r \in \mathbb{R}$, the tangent mapping $d\psi_{C,r}$ satisfies

$$\forall\ v, w \in T_xC, \qquad \langle v, d\psi_{C,r}[w]\rangle \;=\; \langle v, w\rangle + r\,\langle v, d\nu_C[w]\rangle$$

It follows that if r is such that all the quantities $1 + r\lambda_{C,1}(x)$, ..., $1 + r\lambda_{C,n-1}(x)$ are either all positive or all negative, then the tangent mapping $d\psi_{C,r}$ is not degenerate at x. As a consequence, for $r \in (\widetilde{R}_-(D), \widetilde{R}_+(D))$, $d\psi_{C,r}$ is not degenerate on C. More precisely, $(\widetilde{R}_-(D), \widetilde{R}_+(D))$ is the largest interval I containing 0 on which the tangent mapping $d\psi_{C,r}$ is not degenerate on C for all $r \in I$. Indeed, when for some $x \in C$ and $r \in \mathbb{R}$, the values $1 + r\lambda_{C,1}(x)$, ..., $1 + r\lambda_{C,n-1}(x)$ are not of the same sign, we can find $r' \in (-|r|, |r|)$ such that $1 + r'\lambda_{C,1}(x) = 0$, so that $d\psi_{C,r'}$ is degenerate at x.

■

Remark 46 Consider the case where $V = \mathbb{R}^2$ endowed with its usual Riemannian structure (coming from its Euclidean structure). The following picture (where the boundary of the C in black stands for C, while the line in red is a portion of its image by $\psi_{C,r}$, for some positive element $r \in (\widetilde{R}_-(D), \widetilde{R}_+(D))$), shows that in general the mapping $\psi_{C,r}$ is not an embedding of C in the plane.

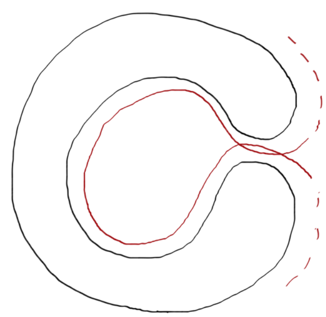

Figure 2: example of a non injective mapping $\psi_{C,r}$

In this picture, if r is reduced a little to be equal to $\widehat{R}_+(D)$ and if $x \neq x' \in C$ are such that $\psi_{C,r}(x) = \psi_{C,r}(x')$, it appears that $\nu_C(x) = -\nu_C(x')$ and x' belongs to the line passing by x and directed by $\nu_C(x)$.

□

The last observation of the above remark corresponds to a general phenomenon that we now describe, coming back to the situation of an abstract Riemannian manifold V.

For any $D \in \mathcal{D}$ and $x \in C$, consider

$$\begin{aligned}\breve{R}_+(x) &\coloneqq \frac{1}{2}\inf\left\{r > 0\ :\ \exp_x(r\nu_C(x)) \in C \text{ and } \nu_C(\exp_x(r\nu_C(x))) = -\frac{d}{dr}\exp_x(r\nu_C(x))\right\}\\ \breve{R}_+(D) &\coloneqq \inf\{\breve{R}_+(x)\ :\ x \in C\}\end{aligned}$$

Similarly, let

$$\begin{aligned}\breve{R}_-(x) &\coloneqq \frac{1}{2}\sup\left\{r < 0\ :\ \exp_x(r\nu_C(x))) \in C \text{ and } \nu_C(\exp_x(r\nu_C(x))) = \frac{d}{dr}\exp_x(r\nu_C(x))\right\}\\ \breve{R}_-(D) &\coloneqq \sup\{\breve{R}_-(x)\ :\ x \in C\}\end{aligned}$$

The interest of these quantities is:

Lemma 47 *When $\widehat{R}_+(D) < \widetilde{R}_+(D)$, it means that $R_+(D) = \widehat{R}_+(D) = \breve{R}_+(D) > 0$. Similarly, we always have $R_-(D) = \widetilde{R}_-(D) \vee \breve{R}_-(D) < 0$.*

Proof

We only prove the first assertion, since the second one can be shown in the same way, by reversing the time (or, when V is compact, by replacing D by D^c).

We begin by remarking that for any $x \in C$, we can find a neighborhood U of x such that the intersection of $U \cap C$ and $U \cap \exp_x([-\epsilon, \epsilon]\nu_C(x))$ is reduced to x for $\epsilon > 0$ small enough (this is a consequence of the assumption that C is a smooth submanifold of V). It follows that the set $\{r > 0 : \exp_x(r\nu_C(x)) \in C \text{ and } \nu_C(\exp_x(r\nu_C(x))) = -\frac{d}{dr}\exp_x(r\nu_C(x))\}$ does not contain 0 as an accumulation point. Since it is also closed, for any $x \in C$, the infimum defining $\breve{R}_+(x)$ is either attained and positive or infinite. Assume that $\breve{R}_+(D) < +\infty$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of C such that $\breve{R}_+(x_n)$ converges toward $\breve{R}_+(D)$. By compactness, we can assume that $(x_n)_{n\in\mathbb{N}}$ converges toward some $x \in C$. Passing to the limit in $\nu_C(\exp_{x_n}(2\breve{R}_+(x_n)\nu_C(x_n))) = -\frac{d}{dr}\exp_{x_n}(r\nu_C(x_n))|_{r=2\breve{R}_+(x_n)}$, we obtain $\nu_C(\exp_x(2\breve{R}_+(D)\nu_C(x))) = -\frac{d}{dr}\exp_x(r\nu_C(x))|_{r=2\breve{R}_+(D)}$. In particular $\breve{R}_+(D) > 0$, otherwise we would end up with $\nu_C(x) = -\nu_C(x)$. As a consequence, we get $\breve{R}_+(x) \leqslant \breve{R}_+(D)$ and finally $\breve{R}_+(D) = \breve{R}_+(x)$, namely the infimum defining $\breve{R}_+(D)$ is attained and is positive. Then the mapping $\psi_{C,\breve{R}_+(D)}$ is not injective, since

$$\psi_{C,\breve{R}_+(D)}(x) \;=\; \exp_x(\breve{R}_+(D)\nu_C(x)) \;=\; \psi_{C,\breve{R}_+(D)}(\exp_x(2\breve{R}_+(D)\nu_C(x)))$$

where x is still a minimizer in the definition of $\breve{R}_+(D)$. Thus we get $\widehat{R}_+(D) \leqslant \breve{R}_+(D)$.

Next, assuming that $\widehat{R}_+(D) < \widetilde{R}_+(D)$, let us show conversely that $\widehat{R}_+(D) \geqslant \breve{R}_+(D)$. Indeed, we can find distinct $x, x' \in C$ and $r \in (0, \widetilde{R}_+(D))$ such that $\psi_{C,r}(x) = \psi_{C,r}(x')$. Since $r \in (0, \widetilde{R}_+(D))$, we can find a neighborhood A of x (respectively A' of x', disjoint from A) in C such that $\psi_{C,r}$ is a diffeomorphism of A (resp. A') on its image. If the tangent space $T_{\psi_{C,r}(x)}\psi_{C,r}(A)$ of $\psi_{C,r}(A)$ at $\psi_{C,r}(x)$ is not equal to the tangent space $T_{\psi_{C,r}(x')}\psi_{C,r}(A')$ of $\psi_{C,r}(A')$ at $\psi_{C,r}(x')$, then $\psi_{C,r}(A)$ and $\psi_{C,r}(A')$ are crossing each other at $\psi_{C,r}(x)$. Then by decreasing a little r into $r' < r$, $\psi_{C,r'}(A)$ and $\psi_{C,r'}(A')$ are still crossing each other. One can then find $y \in A$ and $y' \in A'$ such that $\psi_{C,r'}(y) = \psi_{C,r'}(y') \in \psi_{C,r}(A) \cap \psi_{C,r'}(A')$. This is in contradiction with the definition of $\widehat{R}_+(D)$. Thus we get $T_{\psi_{C,r}(x)}\psi_{C,r}(A) = T_{\psi_{C,r}(x')}\psi_{C,r}(A')$. Note that by parallel transport along the geodesic, $\frac{d}{dr}\exp_x(r\nu_C(x))$ is orthogonal to $T_{\psi_{C,r}(x)}\psi_{C,r}(A)$ and similarly for $\frac{d}{dr}\exp_{x'}(r\nu_C(x'))$. It follows that the two unit vectors $\frac{d}{dr}\exp_x(r\nu_C(x))$ and $\frac{d}{dr}\exp_{x'}(r\nu_C(x'))$ are proportional. They cannot be equal, otherwise by reversing time in the geodesics, we would end up with $x = x'$. So $\frac{d}{dr}\exp_x(r\nu_C(x)) = -\frac{d}{dr}\exp_{x'}(r\nu_C(x'))$ and by considering the geodesic starting from $\psi_{C,r}(x)$ with speed $\frac{d}{dr}\exp_x(r\nu_C(x))$ and its reversed time geodesic, we get $\exp_x(2r\nu_C(x)) = x'$ and $\frac{d}{ds}\exp_x(s\nu_C(x))|_{s=2r} = -\nu_C(x')$, namely $r \geqslant \breve{R}_+(D)$ and as a consequence, $\widehat{R}_+(D) \geqslant \breve{R}_+(D)$, i.e. $\widehat{R}_+(D) = \breve{R}_+(D)$.

■

We now come to the specific situation of the Euclidean plane.

Lemma 48 *Assume that $V = \mathbb{R}^2$, endowed with its usual Euclidean structure. For any $D \in \mathcal{D}$ and $r \in (R_-(D), R_+(D))$, we have*

$$\forall\, x \in C, \qquad \rho_{\Psi(C,r)}(\psi_{C,r}(x)) \;=\; \frac{\rho_C(x)}{1 + r\rho_C(x)}$$

In the context of Lemma 13, if α is given by

$$\forall\, x \in C, \qquad \alpha(x) \;=\; \frac{\rho_C(x)}{1 - r\rho_C(x)}$$

then we have

$$\forall\, x \in \Psi(C,r), \qquad T_D\Psi(\cdot,r)[\alpha](x) \;=\; \rho_{\Psi(C,r)}(x)$$

Proof

One way to compute the curvature $\rho_{\Psi(C,r)}(\psi_{C,r}(x))$, for $x \in C$, is to consider a parametrization $(y(s))_s$ of $\Psi(C,r)$ by its length such that $y(0) = \psi_{C,r}(x)$. The quantity $\rho_{\Psi(C,r)}(\psi_{C,r}(x))$ is then obtained by specializing the following formula at $s = 0$,

$$\partial_s \tau_{\Psi(C,r)}(y(s)) \;=\; -\rho_{\Psi(C,r)}(y(s))\nu_{\Psi(C,r)}(y(s))$$

where $\tau_{\Psi(C,r)}(y(s))$ is the unit vector $\partial_s y(s)$.

Let $(x(s))_s$ be a parametrization of C by its length, with $x(0) = x$. A parametrization of $\Psi(C,r)$ is then given by $(\psi_{C,r}(x(s)))_s$, but it is not by its length, due to the relation

$$\partial_s \psi_{C,r}(x(s)) \;=\; (1 + r\rho_C(x(s)))\tau_C(x(s))$$

To get a parametrization by the length, consider the time change $(\theta_s)_s$ given by

$$\int_0^{\theta_s} 1 + r\rho_C(\psi_{C,r}(x(u)))\, du \;=\; s$$

and define $y(s) = \psi_{C,r}(x(\theta_s))$. We compute

$$\begin{aligned} \partial_s y(s) &= T\psi_{C,r}[\tau_C(x(\theta_s))]\partial_s\theta_s \\ &= \tau_C(x(\theta_s)) \end{aligned}$$

which is a unitary vector. We are thus led to differentiate

$$\begin{aligned} \partial_s \tau_C(x(\theta_s)) &= -\rho_C(x(\theta_s))\nu_C(x(\theta_s))\partial_s\theta_s \\ &= -\frac{\rho_C(x(\theta_s))}{1 + r\rho_C(\psi_{C,r}(x(s)))}\nu_C(x(\theta_s)) \end{aligned}$$

This computation proves that

$$\rho_{\Psi(C,r)}(y(s)) \;=\; \frac{\rho_C(x(\theta_s))}{1 + r\rho_C(\psi_{C,r}(x(s)))}$$

(and that $\nu_{\Psi(C,r)}(y(s)) = \nu_C(x(\theta_s))$, but that was already clear), which at $s = 0$ is the first assertion of the above lemma.

For the second one, note that for any $D \in \mathcal{D}$ and $r \in (R_-(D), R_+(D))$, we have

$$\forall\, x \in \Psi(C,r), \qquad \psi_{C,r}^{-1}(x) \;=\; \psi_{\Psi(C,r),-r}(x)$$

(note that $r \in (R_-(D), R_+(D))$ implies that $-r \in (R_-(\Psi(D,r)), R_+(\Psi(D,r)))$). It follows that for $x \in C$,

$$\begin{aligned} \alpha(\psi_{C,r}^{-1}(x)) &= \frac{\rho_C(\psi_{C,r}^{-1}(x))}{1 - r\rho_C(\psi_{C,r}^{-1}(x))} \\ &= \frac{\rho_C(\psi_{\Psi(C,r),-r}(x))}{1 - r\rho_C(\psi_{\Psi(C,r),-r}(x))} \\ &= \frac{\frac{\rho_{\Psi(C,r)}(x)}{1+r\rho_{\Psi(C,r)}(x)}}{1 - r\frac{\rho_{\Psi(C,r)}(x)}{1+r\rho_{\Psi(C,r)}(x)}} \\ &= \rho_{\Psi(C,r)}(x) \end{aligned}$$

So Lemma 13 leads to the announced result.

■

Remark 49 Lemma 48 is only valid in dimension 2. If $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$, with $n > 2$, recall that the mean curvature $\rho(x)$ at a point x from $C \coloneqq \partial D$, where D is a non-empty, open, bounded, connected domain with smooth boundary, is given by $\lambda_{1,C}(x) + \cdots + \lambda_{n-1,C}(x)$ (with the notation introduced in Lemma 45). Extending in the natural way the previous notions, it appears that

$$\forall\ x \in C,\ \forall\ m \in [\![n-1]\!], \qquad \lambda_{m,\Psi(C,r)}(\psi_{C,r}(x)) \ = \ \frac{\lambda_{m,C}(x)}{1 + r\lambda_{m,C}(x)}$$

(as long as $r \in \mathbb{R}$ is such that $\min_{x \in C} 1 + r\lambda_{1,C}(x) > 0$). Thus to recover the mean curvature vector through the tangent mapping of $\Psi(\cdot, r)$, one must consider the vector α above D given by

$$\forall\ x \in C, \qquad \alpha(x) \ = \ \sum_{m \in [\![n-1]\!]} \frac{\lambda_{m,C}(x)}{1 - r\lambda_{m,C}(x)}$$

(as long as $r \in \mathbb{R}$ is such that $\min_{x \in C} 1 - r\lambda_{n-1,C}(x) > 0$).

□

Lemma 50 *Assume that V is a surface of constant curvature K, $D \in \mathcal{D}$ and $r \in (R_-, R_+)$ then we have:*

- *if $K > 0$ and $x \in C$,*

$$\rho_{\Psi(C,r)}(\psi_{C,r}(x)) \ = \ \frac{(\rho_C^2(x) - K)\frac{\sin(2\sqrt{K}r)}{2\sqrt{K}} + \rho_C(x)\cos(2\sqrt{K}r)}{\left(\cos(\sqrt{K}r) + \frac{\sin(\sqrt{K}r)}{\sqrt{K}}\rho_C(x)\right)^2}$$

- *if $K < 0$ and $x \in C$,*

$$\rho_{\Psi(C,r)}(\psi_{C,r}(x)) \ = \ \frac{(\rho_C^2(x) - K)\frac{\sinh(2\sqrt{-K}r)}{2\sqrt{-K}} + \rho_C(x)\cosh(2\sqrt{-K}r)}{\left(\cosh(\sqrt{-K}r) + \frac{\sinh(\sqrt{-K}r)}{\sqrt{-K}}\rho_C(x)\right)^2}$$

By letting K go to zero in both cases, we recover Lemma 48.

Proof

We only give the proof when $K > 0$, the case $K < 0$ can be deduced by similar computations. For $x \in C$, let $(\gamma_x(s))_s$ be a curve parametrized by its arc length with values in C and $\gamma_x(0) = x$. Denote $\tau(s) \coloneqq \dot{\gamma}_x(s)$ its unitary tangent vectors. Consider for any t, s,

$$\begin{aligned} \gamma(s,t) &\coloneqq \exp_{\gamma_x(s)}(t\nu(\gamma_x(s))) \\ J_s(t) &\coloneqq \partial_s(\gamma(s,t)) \end{aligned}$$

As a variation of a geodesic (for all the following Riemannian geometry notions, see e.g. the book of Gallot, Hulin and Lafontaine [10]), $(J_s(t))_t$ is a Jacobi field. We have $J_s(0) = \tau(s)$ and $\dot{J}_s(0) = \nabla_{\partial_s}\nu(\gamma_x(s)) = \rho_C(\gamma_x(s))\tau(s)$. So there exist $\alpha, \beta \in \mathbb{R}$ such that $J_s(t) = (\alpha\cos(\sqrt{K}t) + \beta\sin(\sqrt{K}t))/\!/_{t\mapsto\gamma(s,t)}\tau(s)$, where $/\!/_{t\mapsto\gamma(s,t)}$ is the parallel transport above the curve $t \mapsto \gamma(s,t)$. Adjusting with the initial condition, we get:

$$J_s(t) \ = \ \left(\cos(\sqrt{K}t) + \frac{\rho_C(\gamma_x(s))}{\sqrt{K}}\sin(\sqrt{K}t)\right)/\!/_{t\mapsto\gamma(s,t)}\tau(s)$$

For fixed and small enough t, to get the arc length parametrization of $s \mapsto \gamma(s,t)$, let us consider the time-change solution of the following equation:

$$\begin{cases} \theta_0^{(t)} &= 0 \\ \frac{d}{ds}\theta_s^{(t)} &= \Big(\cos(\sqrt{K}t) + \frac{\rho_C(\gamma_x(\theta_s^{(t)}))}{\sqrt{K}}\sin(\sqrt{K}t)\Big)^{-1} \end{cases}$$

Let us denote $\tilde\gamma(s,u) \coloneqq \gamma(\theta_s^{(t)}, u)$, we have

$$\begin{aligned} -\rho_{\Psi(C,t)}(\psi_{C,t}(\gamma_x(\theta_s^{(t)}))) &= \langle \nabla_{\partial_s}\frac{\partial}{\partial_s}\tilde\gamma(s,t), \nu_{\Psi(C,t)}(\tilde\gamma(s,t)\rangle \\ &= \langle \nabla_{\partial_s}\frac{\partial}{\partial_s}\tilde\gamma(s,t), \frac{\partial}{\partial_u}\big|_{u=t}\tilde\gamma(s,u)\rangle \\ &= \langle \nabla_{\partial_s}\frac{\partial}{\partial_s}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(s,u)\rangle\big|_{u=t} \end{aligned}$$

Then

$$\begin{aligned} -\rho_{\Psi(C,t)}(\psi_{C,t}(x)) &= \langle \nabla_{\partial_s|_{s=0}}\frac{\partial}{\partial_s}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\rangle\big|_{u=t} \\ &= \int_0^t \nabla_{\partial_u}\langle \nabla_{\partial_s|_{s=0}}\frac{\partial}{\partial_s}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\rangle\, du + \langle \nabla_{\partial_s|_{s=0}}\frac{\partial}{\partial_s}\tilde\gamma(s,0), \partial_u|_{u=0}\tilde\gamma(0,u)\rangle \end{aligned}$$

Recall that

$$\begin{aligned} \langle \nabla_{\partial_s|_{s=0}}\frac{\partial}{\partial_s}\tilde\gamma(s,0), \partial_u|_{u=0}\tilde\gamma(0,u)\rangle &= \langle \nabla_{\partial_s|_{s=0}}\frac{\partial}{\partial_s}(\theta_s^{(t)})\tau(\theta_s^{(t)}), \nu_C(x)\rangle \\ &= -\rho_C(x)(\frac{\partial}{\partial_s}\big|_{s=0}(\theta_s^{(t)}))^2 \\ &= -\frac{\rho_C(x)}{\Big(\cos(\sqrt{K}t) + \frac{\rho_C(x)}{\sqrt{K}}\sin(\sqrt{K}t)\Big)^2} \end{aligned}$$

On the other hand, let $J_{\theta_s^{(t)}}(u) = \frac{\partial}{\partial_s}\tilde\gamma(s,u)$ and let $R(\cdot,\cdot)$ be the curvature tensor, since $u \mapsto \tilde\gamma(s,u)$ is a geodesic, we have

$$\begin{aligned} &\nabla_{\partial_u}\langle \nabla_{\partial_s|_{s=0}}\frac{\partial}{\partial_s}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(0,u))\rangle \\ &= \langle \nabla_{\partial_u}\nabla_{\partial_s|_{s=0}}\frac{\partial}{\partial_s}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\rangle \\ &= \langle \nabla_{\partial_s|_{s=0}}\nabla_{\partial_u}\frac{\partial}{\partial_s}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\rangle + \langle R\big(\frac{\partial}{\partial_s}\big|_{s=0}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\big)\frac{\partial}{\partial_s}\big|_{s=0}\tilde\gamma(s,u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\rangle \\ &= \langle \nabla_{\partial_s|_{s=0}}\nabla_{\partial_u}J_{\theta_s^{(t)}}(u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\rangle + \langle R\big(J_{\theta_0^{(t)}}(u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\big)J_{\theta_0^{(t)}}(u), \frac{\partial}{\partial_u}\tilde\gamma(0,u)\rangle \\ &= -\langle \nabla_{\partial_u}J_{\theta_0^{(t)}}(u), \nabla_{\partial_u}J_{\theta_0^{(t)}}(u)\rangle + K\langle J_{\theta_0^{(t)}}(u), J_{\theta_0^{(t)}}(u)\rangle \end{aligned}$$

where in the last equality, we took into account that $\nabla_{\partial_s}\langle \nabla_{\partial_u}J_{\theta_s^{(t)}}(u), \frac{\partial}{\partial_u}\tilde\gamma(s,u)\rangle = 0$. Since

$$\begin{aligned} J_{\theta_0^{(t)}}(u) &= \frac{\partial}{\partial_s}|_{s=0}(\theta_s^{(t)})(\cos(\sqrt{K}u) + \frac{\rho_C(x)}{\sqrt{K}}\sin(\sqrt{K}u))/\!/_{u\mapsto\tilde\gamma(0,u)}\tau(x) \\ \nabla_{\partial_u}J_{\theta_0^{(t)}}(u) &= \frac{\partial}{\partial_s}|_{s=0}(\theta_s^{(t)})(-\sqrt{K}\sin(\sqrt{K}u) + \rho_C(x)\cos(\sqrt{K}u))/\!/_{u\mapsto\tilde\gamma(0,u)}\tau(x) \\ \frac{d}{ds}|_{s=0}\theta_s^{(t)} &= \frac{1}{\cos(\sqrt{K}t) + \frac{\rho_C(x)}{\sqrt{K}}\sin(\sqrt{K}t)} \end{aligned}$$

we deduce:

$$\begin{aligned}
\rho_{\Psi(C,t)}(\psi_{C,t}(x)) &= \int_0^t \|\nabla_{\partial_u} J_{\theta_0^{(t)}}(u)\|^2 - K\|J_{\theta_0^{(t)}}(u)\|^2\,du + \frac{\rho_C(x)}{\left(\cos(\sqrt{K}t) + \frac{\rho_C(x)}{\sqrt{K}}\sin(\sqrt{K}t)\right)^2} \\
&= \frac{1}{(\cos(\sqrt{K}t) + \frac{\rho_C(x)}{\sqrt{K}}\sin(\sqrt{K}t))^2}\left(\int_0^t \left(-\sqrt{K}\sin(\sqrt{K}u) + \rho_C(x)\cos(\sqrt{K}u)\right)^2\right. \\
&\quad \left. -K\left(\cos(\sqrt{K}u) + \frac{\rho_C(x)}{\sqrt{K}}\sin(\sqrt{K}u)\right)^2 du + \rho_C(x)\right) \\
&= \frac{(\rho_C^2(x) - K)\frac{\sin(2\sqrt{K}t)}{2\sqrt{K}} + \rho_C(x)\cos(2\sqrt{K}t)}{\left(\cos(\sqrt{K}t) + \frac{\sin(\sqrt{K}t)}{\sqrt{K}}\rho_C(x)\right)^2}.
\end{aligned}$$

When the curvature is negative $K < 0$, except for the sign change in the second order differential equation for the Jacobi field, all the computations are similar.

■

Remark 51 In the context of the above lemma, let V be a $(n+1)$-dimensional manifold with constant curvature $K > 0$, $D \in \mathcal{D}$, $r \in (R_-, R_+)$ and $\lambda_{C,1}(x) \leqslant \ldots \leqslant \lambda_{C,n}(x)$ be the principal curvatures of C. It is not so clear how to control the principal curvatures of $\Psi(C,r)$ at the point $\psi_{C,r}(x)$, but for the mean curvature we have:

$$\rho_{\Psi(C,r)}(\psi_{C,r}(x)) = \sum_{l=1}^n \frac{(\lambda_{C,l}^2(x) - K)\frac{\sin(2\sqrt{K}r)}{2\sqrt{K}} + \lambda_{C,l}(x)\cos(2\sqrt{K}r)}{\left(\cos(\sqrt{K}r) + \frac{\sin(\sqrt{K}r)}{\sqrt{K}}\lambda_{C,l}(x)\right)^2}$$

A similar formula holds for $K < 0$.

□

5.2 Comparison of two Doss-Sussman approaches

Consider the Doss-Sussman method corresponding to the decomposition (57) of Remark 18. Similarly to (43) and (47), define in the present Riemannian Brownian setting,

$$\begin{aligned}
\forall\, D \in \mathcal{D},\ \forall\, x \in C, \qquad \widetilde{\rho}_C(x) &:= \rho_C(x) - h(D) \\
\forall\, r > 0,\ \forall\, D \in \mathcal{D}_r,\ \forall\, x \in C, \qquad \widetilde{\alpha}_{C,r}(x) &:= -\widetilde{\rho}_{\Psi(C,r)}(\psi_{C,r}(x))
\end{aligned}$$

We are interested in constructing a family $(\widetilde{G}_t)_{t\in[0,\tau)}$ such that

$$\left\{\begin{aligned}
&& \widetilde{G}_0 &= B(x, r_0) \\
&\forall\, t \in [0,\tau),\ \forall\, x \in \partial\widetilde{G}_t, & \partial_t x &= \widetilde{\alpha}_{\partial\widetilde{G}_t,\sqrt{2}B_t}(x)\nu_{\partial\widetilde{G}_t}(x)
\end{aligned}\right. \tag{95}$$

since the process $(D_t)_{t\in[0,\tau)}$ obtained by a particular composition of the normal flow Ψ and of the flow (95), namely

$$\forall\, t \in [0,\tau), \qquad D_t := \Psi(\widetilde{G}_t, \sqrt{2}B_t) \tag{96}$$

will provide a solution to the martingale problem associated to $(\mathfrak{D}, \mathfrak{L})$, as in Theorem 17.

In the following subsections we reformulate the results of Section 2, using this Doss-Sussman approach.

5.2.1 Euclidean spaces

Let $V = \mathbb{R}^n$, fix $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ and consider the initial condition $\widetilde{G}_0 = B(x_0, r_0)$ and $C_0 = \partial\widetilde{G}_0$. According to Lemma 48 (also by direct computation) we have for all $r > -r_0$,

$$\begin{aligned} \rho_{\Psi(C_0,r)}(\psi_{C_0,r}(x)) &= (n-1)\frac{\frac{1}{r_0}}{1+\frac{r}{r_0}} = \frac{n-1}{r+r_0} \\ h(\Psi(D_0,r)) &= \frac{2n}{r+r_0} \end{aligned}$$

so

$$\forall\, x \in C_0, \qquad \widetilde{\alpha}_{C_0,r}(x) = \frac{n+1}{r+r_0}$$

Since the above quantity does not depend on x, the solution of (95) is radial and $\widetilde{G}_t = B(x, \widetilde{R}_t)$. According to (95), the radius starts with $\widetilde{R}_0 = r_0$ and its evolution is described by

$$\forall\, t \in [0, \tau), \qquad d\widetilde{R}_t = \frac{n+1}{\widetilde{R}_t + \sqrt{2}B_t}dt \tag{97}$$

this equation being well-defined up to the stopping time

$$\tau := \inf\{t \geqslant 0 : \widetilde{R}_t = -\sqrt{2}B_t \quad \text{or} \quad \widetilde{R}_t = 0\}$$

The condition $\widetilde{R}_t > 0$ comes from the fact that the normal flow $\Psi(C, r)$ is not defined when C is reduced to a singleton, and the condition $\widetilde{R}_t > -\sqrt{2}B_t$ comes from the fact that the normal flow $\Psi(\partial B(x_0, \widetilde{R}_t), r)$ is well-defined only for $r > -\widetilde{R}_t$.

We get the following equation:

$$\forall\, t \in [0, \tau), \qquad d(\widetilde{R}_t + \sqrt{2}B_t) = \frac{n+1}{\widetilde{R}_t + \sqrt{2}B_t}dt + \sqrt{2}dB_t$$

so $(\widetilde{R}_t + \sqrt{2}B_t)_{t\geqslant 0} = (\mathrm{Bes}^{(n+2)}_{2t}(r_0))_{t\geqslant 0}$, where $\mathrm{Bes}^{(n+2)}(r_0) := (\mathrm{Bes}^{(n+2)}_t(r_0))_{t\geqslant 0}$ is a Bessel process of dimension $n+2 \geqslant 2$ starting from $r_0 > 0$. For all $t \geqslant 0$, $\widetilde{R}_t + \sqrt{2}B_t > 0$, so $(d\widetilde{R}_t)/(dt) > 0$ and $\widetilde{R}_t \geqslant r_0 > 0$, hence Equation (97) is well-defined for all times, i.e. $\tau = \infty$, and

$$\forall\, t \geqslant 0, \qquad D_t = \Psi(\widetilde{G}_t, \sqrt{2}B_t) = B(x_0, \widetilde{R}_t + \sqrt{2}B_t)$$

Since 0 is a entrance boundary for the Bessel process of dimension $n+2$, it is possible to solve the martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ and to the initial singleton condition $D_0 = \{x_0\}$ as follow: let $\mathrm{Bes}^{(n+2)}(0)$ be a Bessel process of dimension $n+2$ starting at 0, and $(B_t)_{t\geqslant 0}$ be the associated Brownian motion, namely such that

$$\begin{aligned} \forall\, t \geqslant 0, \qquad \mathrm{Bes}^{(n+2)}_{2t} &= \sqrt{2}B_t + \int_0^{2t} \frac{n+1}{2\mathrm{Bes}^{(n+2)}_s(0)}\,ds \\ &= \sqrt{2}B_t + \int_0^{t} \frac{n+1}{\mathrm{Bes}^{(n+2)}_{2s}(0)}\,ds \end{aligned}$$

Consider for any $t \geqslant 0$,

$$\begin{aligned} D_t &:= B(x_0, \mathrm{Bes}^{(n+2)}_{2t}) \\ \widetilde{G}_t &:= \Psi^{-1}(D_t, \sqrt{2}B_t) \end{aligned}$$

where the latter is well-defined since $\mathrm{Bes}_{2t}^{(n+2)} > \sqrt{2}B_t$ for all $t > 0$. It appears that

$$\forall\ t \geqslant 0, \qquad \mathrm{Bes}_{2t}^{(n+2)} \;=\; \sqrt{2}B_t + \int_0^t h(D_s) - \rho_{\partial D_s}\, ds$$

hence

$$\begin{aligned}\forall\ x \in \partial\widetilde{G}_t, \qquad \partial_t x &= (h(D_t) - \rho_{\Psi(\partial G_t, \sqrt{2}B_t)})\nu_{\partial G_t}(x) \\ &= \widetilde{\alpha}_{\partial G_t, \sqrt{2}B_t}(x)\nu_{\partial G_t}(x)\end{aligned}$$

According to Lemma 10 and (56), we have for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$,

$$\begin{aligned} dF_f(D_t) &= dF_f(\Psi(G_t, \sqrt{2}B_t)) \\ &= \left(\int_{\partial D_t} f(h(D_t) - \rho_{\partial D_t})\, d\underline{\mu}\right) dt + \left(\int_{\partial D_t} f\, d\underline{\mu}\right)(\sqrt{2}dB_t) \\ &\quad + \left(\int_{\partial D_t} \langle \nabla f, \nu_{\partial D_t}\rangle\, d\underline{\mu} + \int_{\partial D_t} f\rho_{\partial D_t}\, d\underline{\mu}\right) dt \\ &= \left(\int_{\partial D_t} \langle \nabla f, \nu_{\partial D_t}\rangle + fh(D_t))\, d\underline{\mu}\right) dt \\ &\quad + \sqrt{2}\left(\int_{\partial D_t} f\, d\underline{\mu}\right) dB_t \\ &= \mathfrak{L}[F_f](D_t)\, dt + dM_t \end{aligned}$$

where

$$(M_t)_{t\geqslant 0} \;\coloneqq\; \left(\sqrt{2}\int_0^t \left(\int_{\partial D_s} f\, d\underline{\mu}\right) dB_s\right)_{t\geqslant 0}$$

is a martingale. We get for all $t \geqslant s > 0$,

$$F_f(D_t) - F_f(D_s) \;=\; \int_s^t \mathfrak{L}[F_f](D_u)\, du + M_t - M_s \tag{98}$$

Since a.s.

$$\lim_{s\to 0_+} F_f(D_s) \;=\; 0$$

and

$$\lim_{s\to 0} \mathfrak{L}[F_f](D_s) \;=\; \begin{cases} 0 & \text{, if } n \geqslant 3 \\ 8\pi f(x_0) & \text{, if } n = 2 \end{cases}$$

we can pass to the limit in (98) to get $(D_t)_{t\geqslant 0}$ solves the martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ and to the singleton initial condition $D_0 = \{x_0\}$.

Let us now consider the Doss-Sussman method relative to the decomposition (22), for simplicity only in the illustrative Euclidean plane $V = \mathbb{R}^2$. For $x_0 \in \mathbb{R}^2$ and $r_0 > 0$, we are interested in the initial condition $D_0 \coloneqq B(x_0, r_0)$. Starting with $(\theta_0, G_0) = (0, D_0)$, we solve the evolution equation system (51) with respect to $(\theta_t, G_t)_{t\in[0,\tau_{r_0})}$. The solution $(G_t)_{t\in[0,\tau_{r_0})}$ remains radial, so let us write it as $G_t = B(x, \hat{R}_t)$ for all $t \in [0, \tau_{r_0})$. Equation (51) becomes:

$$\forall\ t \in [0, \tau_{r_0}), \qquad \left\{ \begin{array}{rcl} d\hat{R}_t & = & -\frac{1}{\hat{R}_t+\sqrt{2}B_t+\theta_t}dt, \quad \hat{R}_0 = r_0 \\ d\theta_t & = & \frac{4}{\hat{R}_t+\sqrt{2}B_t+\theta_t}dt, \quad \theta_0 = 0 \end{array} \right. \tag{99}$$

where

$$\tau_{r_0} = \inf\{t \geqslant 0, \hat{R}_t = 0 \quad \text{or} \quad \sqrt{2}B_t + \theta_t = -\hat{R}_t\}$$

It follows that $(\hat{R}_t + \sqrt{2}B_t + \theta_t)_{t\in[0,\tau_{r_0})} = (\mathrm{Bes}^{(4)}_{2t}(r_0))_{t\in[0,\tau_{r_0})}$ where $\mathrm{Bes}^{(4)}(r_0)$ is a Bessel process of dimension 4 starting from r_0 We deduce that $\tau_{r_0} = \inf\{t \geqslant 0, \hat{R}_t = 0\}$ and for any $t \in [0, \tau_{r_0})$,

$$\begin{array}{rcl} \hat{R}_t - r_0 & = & -\displaystyle\int_0^t \frac{1}{\hat{R}_s + \sqrt{2}B_s + \theta_s}\, ds \\ & = & -\displaystyle\frac{1}{2}\int_0^{2t} \frac{1}{\mathrm{Bes}^{(4)}_s(r_0)}\, ds \end{array}$$

Using the iterated logarithm law for the Bessel process for large times, we get

$$\int_0^{+\infty} \frac{1}{\mathrm{Bes}^{(4)}_s(r_0)}\, ds \quad = \quad +\infty$$

It follows that necessarily, a.s. $\tau_{r_0} < \infty$ and more precisely that

$$2r_0 \quad = \quad \int_0^{2\tau_{r_0}} \frac{1}{\mathrm{Bes}^{(4)}_s(r_0)}\, ds \tag{100}$$

Taking into account that for any $t > 0$, we have (a.s.)

$$\int_0^t \frac{1}{\mathrm{Bes}^{(4)}_s(0)}\, ds \quad \in \quad (0, +\infty)$$

we can let r_0 go to 0_+ in (100) to see that

$$\lim_{r_0 \to 0_+} \tau_{r_0} \quad = \quad 0$$

Thus, the Doss-Sussman method relative to the decomposition (22) does not enable to define the dual process for all times nor permits approximations of singleton initial condition, contrary to the Doss-Sussman method associated to the decomposition (57).

Remark 52 It may be surprising at first view that several decompositions of a generator lead to solutions defined on different time domains. This is due to the fact that the flows associated to the corresponding vector fields may not be defined for all times. To get a simple example on $\mathbb{R}_+$, consider the case $n = 1$ in this subsection.

□

5.2.2 Hyperbolic spaces

Let $V = \mathbb{H}^n$ be the hyperbolic space of dimension n. Fix some $x_0 \in \mathbb{H}^n$ and $r_0 > 0$ and consider the initial condition $\widetilde{G}_0 = D_0 \coloneqq B(x_0, r_0)$, and $C_0 = \partial \widetilde{G}_0$. We have for any $r > -r_0$,

$$\begin{aligned}
\rho_{\Psi(C_0,r)}(\psi_{C_0,r}(x)) &= (n-1)\coth(r+r_0) \\
h(\Psi(D_0,r)) &= 2\frac{\sinh^{n-1}(r+r_0)}{J(r+r_0)}
\end{aligned}$$

hence

$$\forall\ x \in C_0, \qquad \widetilde{\alpha}_{C_0,r}(x) = 2\frac{\sinh^{n-1}(r+r_0)}{J(r+r_0)} - (n-1)\coth(r+r_0)$$

where

$$\forall\ r \geqslant 0, \qquad J(r) = \int_0^r \sinh^{n-1}(u)\,du$$

The solution of (95) is radial, say $\widetilde{G}_t = B(x, \widetilde{R}_t)$, and we have, starting with $\widetilde{R}_0 = r_0$:

$$\forall\ t \in [0, \tau), \qquad d\widetilde{R}_t = \left(2\frac{\sinh^{n-1}(\widetilde{R}_t + \sqrt{2}B_t)}{J(\widetilde{R}_t + \sqrt{2}B_t)} - (n-1)\coth(\widetilde{R}_t + \sqrt{2}B_t)\right) dt \tag{101}$$

where,

$$\tau = \inf\{t \geqslant 0 \,:\, \widetilde{R}_t = -\sqrt{2}B_t \quad \text{or} \quad \widetilde{R}_t = 0\}$$

We get, for all $t \in [0, \tau)$,

$$d(\widetilde{R}_t + \sqrt{2}B_t) = \left(2\frac{\sinh^{n-1}(\widetilde{R}_t + \sqrt{2}B_t)}{J(\widetilde{R}_t + \sqrt{2}B_t)} - (n-1)\coth(\widetilde{R}_t + \sqrt{2}B_t)\right) dt + \sqrt{2}dB_t$$

Note that as $r > 0$ goes to zero,

$$2\frac{\sinh^{n-1}(r)}{J(r)} - (n-1)\coth(r) \sim \frac{n+1}{r}$$

This behavior is sufficient to insure that 0 is an entrance boundary for the diffusion $(\widetilde{R}_t + \sqrt{2}B_t)_{t\geqslant 0}$ (see for instance the classical computations of Chapter 15 of Karlin and Taylor [14]). In particular, since $(\widetilde{R}_t + \sqrt{2}B_t)_{t\geqslant 0}$ starts from $r_0 > 0$, it will never reach 0 (a.s.). Furthermore, let us check that the radius process $(\widetilde{R}_t)_{t\geqslant 0}$ of $(\widetilde{G}_t)_{t\geqslant 0}$ is non-decreasing. Indeed, after an integration by parts, we obtain for all $r \geqslant 0$:

$$\begin{aligned}
\int_0^r \sinh^{n-1}(u)\,du &= \int_0^{\sinh(r)} \frac{v^{n-1}}{\sqrt{1+v^2}}\,dv \\
&= \frac{\sinh^n(r)}{n\cosh(r)} + \int_0^{\sinh(r)} \frac{v^{n+1}}{n\sqrt{1+v^2}(1+v^2)}\,dv \\
&\leqslant \frac{\sinh^n(r)}{n\cosh(r)} + \frac{1}{n}\int_0^{\sinh(r)} \frac{v^{n-1}}{\sqrt{1+v^2}}\,dv.
\end{aligned}$$

Hence we have for any $r \geqslant 0$,

$$\int_0^r \sinh^{n-1}(u)\,du \quad \leqslant \quad \frac{\sinh^n(r)}{(n-1)\cosh(r)}$$

namely

$$\frac{\sinh^{n-1}(r)}{J(r)} \quad \geqslant \quad (n-1)\coth(r)$$

and

$$\frac{2\sinh^{n-1}(r)}{J(r)} - (n-1)\coth(r) \quad \geqslant \quad \frac{\sinh^{n-1}(r)}{J(r)} \quad \geqslant \quad 0$$

This non-negativity and (101) show that $(\widetilde{R}_t)_{t\geqslant 0}$ is non-decreasing.

From these observations, we get the solution of (101) is defined for all times, i.e. $\tau = \infty$, and finally

$$\forall\ t \geqslant 0, \qquad D_t \quad = \quad B(x_0, \widetilde{R}_t + \sqrt{2}B_t)$$

provides a solution to the martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ and starting from $B(x_0, r_0)$.

As in the Euclidean case, by letting r_0 go to zero, we solve the martingale problem associated to the generator $(\mathfrak{D}, \mathfrak{L})$ starting from the singleton $\{x_0\}$.

5.2.3 Spherical spaces

Let $V = \mathbb{S}^n$ be the sphere of dimension $n \in \mathbb{N}$. Fix $x_0 \in \mathbb{S}^n$ and $r_0 \in (0, \pi)$, and consider the initial condition $\widetilde{G}_0 = B(x_0, r_0)$, and $C_0 = \partial\widetilde{G}_0$. We have for any $r \in (-r_0, \pi - r_0)$ (note that the normal flow in not well-defined for all positive times):

$$\widetilde{\alpha}_{C_0,r}(x) \quad = \quad 2\frac{\sin^{n-1}(r+r_0)}{I(r+r_0)} - (n-1)\cot(r+r_0)$$

where $I(s) = \int_0^s \sin^{n-1}(u)\,du$, for any $s \in [0, \pi]$.

The solution of (95) is radial, say $\widetilde{G}_t = B(x, \widetilde{R}_t)$. According to (95), starting from $\widetilde{R}_0 = r_0$, we have

$$\forall\ t \in [0, \tau), \qquad d\widetilde{R}_t \quad = \quad \left(2\frac{\sin^{n-1}(\widetilde{R}_t + \sqrt{2}B_t)}{J(\widetilde{R}_t + \sqrt{2}B_t)} - (n-1)\cot(\widetilde{R}_t + \sqrt{2}B_t)\right)dt \tag{102}$$

where

$$\tau \quad = \quad \inf\{t \geqslant 0 \,:\, \widetilde{R}_t = \pi - \sqrt{2}B_t \quad \text{or} \quad \widetilde{R}_t = -\sqrt{2}B_t \quad \text{or} \quad \widetilde{R}_t = 0\}$$

We get

$$\forall\ t \in [0, \tau), \qquad d(\widetilde{R}_t + \sqrt{2}B_t) \quad = \quad \left(2\frac{\sin^{n-1}(\widetilde{R}_t + \sqrt{2}B_t)}{I(\widetilde{R}_t + \sqrt{2}B_t)} - (n-1)\cot(\widetilde{R}_t + \sqrt{2}B_t)\right)dt + \sqrt{2}dB_t$$

Again, we have as r goes to 0_+,

$$2\frac{\sin^{n-1}(r)}{I(r)} - (n-1)\cot(r) \quad \sim \quad \frac{n+1}{r}$$

and this behavior is sufficient to get that 0 is an entrance boundary for the diffusion $(\widetilde{R}_t+\sqrt{2}B_t)_{t\geqslant 0}$. It follows that it never hits 0. To show that $(\widetilde{R}_t)_{t\geqslant 0}$ is non-decreasing, let us check that

$$\forall\, r\in(0,\pi),\qquad 2\frac{\sin^{n-1}(r)}{I(r)}-(n-1)\cot(r)\;\geqslant\;0$$

Observe that it is clearly satisfied for $r\in[\frac{\pi}{2},\pi)$. For $r\in(0,\frac{\pi}{2})$, we have:

$$\begin{aligned}\int_0^r \sin^{n-1}(u)\,du &= \int_0^{\sin(r)}\frac{v^{n-1}}{\sqrt{1-v^2}}\,dv\\ &= \frac{\sin^n(r)}{n\cos(r)}-\int_0^{\sin(r)}\frac{v^{n+1}}{n\sqrt{1-v^2}(1-v^2)}\,dv\\ &\leqslant \frac{\sin^n(r)}{n\cos(r)}\leqslant\frac{\sin^n(r)}{(n-1)\cos(r)}\end{aligned}$$

We deduce that $r\in(0,\frac{\pi}{2})$,

$$2\frac{\sin^{n-1}(r)}{I(r)}-(n-1)\cot(r)\;\geqslant\;\frac{\sin^{n-1}(r)}{I(r)}\;\geqslant\;0$$

From these considerations, it appears that the solution to (102) is well-defined until the (a.s. finite) stopping time

$$\tau\;=\;\inf\{t\geqslant 0\,:\,\widetilde{R}_t+\sqrt{2}B_t=\pi\}$$

and we have

$$\forall\, t\in[0,\tau],\qquad D_t\;=\;B(x,\widetilde{R}_t+\sqrt{2}B_t)$$

In fact τ is the hitting time of the whole sphere $\mathbb{S}^n$ by $(D_t)_{t\in[0,\tau]}$. Since for all $f\in\mathcal{C}^\infty(\mathbb{S}^n)$, we have $\mathfrak{L}[F_f](\mathbb{S}^n)=0$, it is natural to let the latter process be absorbed at $\mathbb{S}^n$, namely to extend it by

$$\forall\, t\geqslant\tau,\qquad D_t\;\coloneqq\;\mathbb{S}^n$$

so that $(D_t)_{t\geqslant 0}$ provides a solution to the martingale problem associated to the generator $(\mathfrak{D},\mathfrak{L})$ and starting from $B(x_0,r_0)$.

As in the Euclidean and hyperbolic cases, the martingale problem associated to the generator $(\mathfrak{D},\mathfrak{L})$ and starting from the singleton $\{x_0\}$ is solved by letting r_0 go to zero.

6 About the martingale problems associated to $\mathfrak{L}$

After proving Theorem 5, we will show that the martingales naturally associated to $\mathfrak{L}$ are directed by a unique Brownian motion, property corresponding to the radial evolution (3). Next, we will enrich the set of elementary observables and see in the particular example of the diffusion X consisting of the Brownian motion in the Euclidean plane how the enriched martingale problem is sufficient to deduce that the dual domain-valued process ends up looking like a big disk, at least if it can be defined for all times.

6.1 Proof of Theorem 5

As explained above Theorem 5, we assume we are given a stochastic process $(D_t)_{t\in[0,\tau)}$ taking values in $\mathcal{G}$ for positive times and solution to the martingale problem associated to $(\mathfrak{D},\mathfrak{L})$, defined as in the introduction, except that the elementary observables are defined on $\mathcal{G}$ instead of $\mathcal{D}$. Despite this generalization, the following arguments are similar to those given in the one-dimensional case treated in [19].

Let a test function $\mathfrak{f}\in\mathcal{C}^\infty(\mathbb{R}_+)$ be given and consider the process $(S_t)_{t\in[0,\tau)}$ defined by

$$\forall\, t\in[0,\tau),\qquad \begin{aligned} S_t &\coloneqq \mathfrak{f}(\mu(D_t)) \\ &= \mathfrak{f}(F_{\mathbb{1}}(D_t)) \end{aligned}$$

Since the mapping $\mathcal{G}\ni D\mapsto \mathfrak{f}(F_{\mathbb{1}}(D))$ belongs to $\mathfrak{D}$, there exists a local martingale $(M_t)_{t\in[0,\tau)}$ such that for all $t\in[0,\tau)$,

$$S_t = S_0 + \int_0^t \mathfrak{L}[\mathfrak{f}\circ F_{\mathbb{1}}](D_s)\,ds + M_t \tag{103}$$

By definition of $\mathfrak{L}$, we have

$$\mathfrak{L}[\mathfrak{f}\circ F_{\mathbb{1}}](D) \;=\; \mathfrak{f}'(F_{\mathbb{1}})\mathfrak{L}[F_{\mathbb{1}}] + \mathfrak{f}''(F_{\mathbb{1}})\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]$$

Recall that in the proof of Theorem 3, we computed, for any $D\in\mathcal{G}$, with $C\coloneqq\partial D$,

$$\mathfrak{L}[F_{\mathbb{1}}](D) \;=\; 2\frac{\underline{\mu}(C)^2}{\mu(D)} \tag{104}$$

$$\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}](D) \;=\; \underline{\mu}(C)^2 \tag{105}$$

so that

$$\begin{aligned} \mathfrak{L}[\mathfrak{f}\circ F_{\mathbb{1}}](D) &= \underline{\mu}(C)^2\left(\mathfrak{f}''(F_{\mathbb{1}}) + 2\frac{\mathfrak{f}'(F_{\mathbb{1}})}{F_{\mathbb{1}}}\right)(D) \\ &= 2\underline{\mu}(C)^2\mathcal{L}[\mathfrak{f}](F_{\mathbb{1}}(D)) \end{aligned}$$

where

$$\forall\, x\in\mathbb{R}_+^*,\qquad \mathcal{L} \;\coloneqq \tfrac{1}{2}\partial^2 + \tfrac{1}{x}\partial$$

is the generator of the Bessel process of dimension 3 on $\mathbb{R}_+$ (see e.g. Chapter 11 of the book [26] of Revuz and Yor). Thus we obtain, for all $t\in[0,\tau)$,

$$S_t = S_0 + 2\int_0^t \underline{\mu}(C_s)^2\mathcal{L}[\mathfrak{f}](\mu(D_s))\,ds + M_t$$

It leads us to introduce the time change described by (12) and (13) and

$$\forall\, t\in[0,\varsigma),\qquad R_t \;\coloneqq\; \mu(D_{\theta(t)})$$

to get $(R_t)_{t\in[0,\varsigma)}$ is a stopped continuous solution to the martingale problem associated to the generator $(\mathcal{C}^\infty(\mathbb{R}_+),\mathcal{L})$. It follows that $(R_t)_{t\in[0,\varsigma)}$ is a stopped Bessel process of dimension 3. For completeness, let us just recall the underlying argument.

Define for $t \in [0, \varsigma)$,

$$W_t \ := \ R_t - R_0 - \int_0^t \frac{1}{R_s}\, ds$$

According to the martingale problem, the process $(W_t)_{t\in[0,\varsigma)}$ is a continuous local martingale whose bracket is given by

$$\forall\ t \in [0, \varsigma), \qquad \langle W \rangle_t \ = \ \int_0^t \Gamma_{\mathcal{L}}[\mathrm{id}, \mathrm{id}](R_s)\, ds$$

where $\Gamma_{\mathcal{L}}$ is the carré du champ operator associated to $\mathcal{L}$ and $\mathrm{id} : \mathbb{R}_+^* \ni x \mapsto x$ is the identity mapping on $\mathbb{R}_+^*$. Since $\Gamma_{\mathcal{L}}[\mathrm{id}, \mathrm{id}] = (\mathrm{id}')^2 \equiv 1$, we get

$$\forall\ t \in [0, \varsigma), \qquad \langle W \rangle_t \ = \ t$$

so Lévy's theorem shows that $(W_t)_{t\in[0,\varsigma)}$ is a stopped Brownian motion. Then $(R_t)_{t\in[0,\varsigma)}$ is solution to the stochastic differential solution

$$\forall\ t \in [0, \varsigma), \qquad dR_t \ = \ dW_t + \frac{1}{R_t} dt$$

which admits a unique strong solution, once R_0 is given. In particular the law of $(R_t)_{t\in[0,\varsigma)}$ is determined by the initial distribution of R_0, it is the Bessel process of dimension 3 with initial law $\mathcal{L}(X_0)$.

■

6.2 The stochastic differential equation associated with the martingale problem

With the notation of the above proof, for $\mathfrak{f} = \mathrm{id}$ in (103), we get $M_{\theta_t} = W_t$ for $t \in [0, \varepsilon)$, or

$$\forall\ t \in [0, \tau), \qquad M_t \ = \ W_{\theta_t^{-1}}$$

where $\theta^{-1} : [0, \tau) \to [0, \varepsilon)$ is the inverse mapping of θ given in (13). In particular, we get

$$\begin{aligned} \forall\ t \in [0, \tau), \qquad \langle M \rangle_t \ &= \ \theta_t^{-1} \\ &= \ 2\int_0^t \underline{\mu}(C_s)^2\, ds \end{aligned}$$

so that we can find a Brownian motion $(B_t)_{t\geqslant 0}$ (up to enlarging the underlying probability space) such that

$$\forall\ t \in [0, \tau), \qquad M_t \ = \ \sqrt{2}\int_0^t \underline{\mu}(C_s)\, dB_s$$

Namely we have

$$\forall\ t \in [0, \tau), \qquad d\mu(D_t) \ = \ 2\frac{\underline{\mu}(C_t)^2}{\mu(D_t)}\, dt + \sqrt{2}\underline{\mu}(C_t)\, dB_t \tag{106}$$

The same Brownian motion $(B_t)_{t\geqslant 0}$ is driving all the $(F_f(D_t))_{t\in[0,\tau)}$, for all $f \in \mathcal{C}^\infty(V)$, and even more:

Proposition 53 *For all $F \in \mathfrak{D}$, we have*

$$\forall\ t \in [0,\tau), \qquad \mathfrak{F}(D_t) \;=\; \mathfrak{F}(D_0) + \int_0^t \mathfrak{L}[\mathfrak{F}](D_s)\,ds + \sqrt{2}\int_0^t \sqrt{\Gamma_{\mathfrak{L}}[\mathfrak{F},\mathfrak{F}]}(D_s)\,dB_s \tag{107}$$

where the determination of the sign of $\sqrt{\Gamma_{\mathfrak{L}}[\mathfrak{F},\mathfrak{F}]}$ is

$$\forall\ D \in \mathcal{D}, \qquad \sqrt{\Gamma_{\mathfrak{L}}[\mathfrak{F},\mathfrak{F}]}(D) \;\coloneqq\; \sum_{l\in[\![n]\!]} \partial_l \mathfrak{f}(F_{f_1},...,F_{f_n})(D)\int_C f_l\,d\sigma$$

when $\mathfrak{F} = \mathfrak{f}(F_{f_1},...,F_{f_n})$, with the notation of the introduction.

Proof

By definition of $(\mathfrak{D},\mathfrak{L})$ and due to the usual rules of continuous stochastic calculus (see for instance the book [26] of Revuz and Yor), it is sufficient to check the above formula on the elementary observables, namely that for all $f \in \mathcal{C}^\infty(V)$,

$$\forall\ t \in [0,\tau), \qquad F_f(D_t) \;=\; F_f(D_0) + \int_0^t \mathfrak{L}[F_f](D_s)\,ds + \sqrt{2}\int_0^t \sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}(D_s)\,dB_s$$

with the determination of sign: $\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]} \coloneqq \int f\,d\sigma$. From the martingale problem, we know that for any $f \in \mathcal{C}^\infty(V)$, the process

$$\forall\ t \in [0,\tau), \qquad M_t^f \;\coloneqq\; F_f(D_t) - F_f(D_0) - \int_0^t \mathfrak{L}[F_f](D_s)\,ds$$

is a local martingale whose bracket is given by

$$\forall\ t \in [0,\tau), \qquad \left\langle M^f\right\rangle_t \;\coloneqq\; 2\int_0^t \Gamma_{\mathfrak{L}}[F_f,F_f](D_s)\,ds$$

So our goal is to check that

$$\forall\ t \in [0,\tau), \qquad M_t^f \;=\; \int_0^t \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}}(D_s)\,dM_s^{\mathbb{1}}$$

Since all the considered martingales start from 0, it is equivalent to show that

$$\forall\ t \in [0,\tau), \qquad \left\langle M_\cdot^f - \int_0^\cdot \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}}(D_s)\,dM_s^{\mathbb{1}}\right\rangle_t \;=\; 0$$

Developing by polarization the l.h.s., we obtain

$$\begin{aligned}
&\left\langle M^f\right\rangle_t + \left\langle \int_0^\cdot \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}}(D_s)\,dM_s^{\mathbb{1}}\right\rangle_t - 2\left\langle M^f, \int_0^\cdot \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}}(D_s)\,dM_s^{\mathbb{1}}\right\rangle_t \\
&= \left\langle M^f\right\rangle_t + \int_0^t \frac{\Gamma_{\mathfrak{L}}[F_f,F_f]}{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}(D_s)\,d\left\langle M^{\mathbb{1}}\right\rangle_s - 2\int_0^t \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}}(D_s)\,d\left\langle M^f, M^{\mathbb{1}}\right\rangle_s \\
&= 2\int_0^t \Gamma_{\mathfrak{L}}[F_f,F_f](D_s)\,ds + 2\int_0^t \frac{\Gamma_{\mathfrak{L}}[F_f,F_f]}{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}(D_s)\,\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}](D_s)ds \\
&\quad -4\int_0^t \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}}(D_s)\,\Gamma_{\mathfrak{L}}[F_f,F_{\mathbb{1}}](D_s)ds \\
&= 4\int_0^t \left(\Gamma_{\mathfrak{L}}[F_f,F_f] - \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f,F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_{\mathbb{1}}]}}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}},F_f]\right)(D_s)\,ds \\
&= 0
\end{aligned}$$

where we used that for any $D \in \mathcal{G}$,

$$\left(\Gamma_{\mathfrak{L}}[F_f, F_f] - \frac{\sqrt{\Gamma_{\mathfrak{L}}[F_f, F_f]}}{\sqrt{\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}]}}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}]\right)(D) = \left(\int_C f\, d\underline{\mu}\right)^2 - \frac{\int_C f\, d\underline{\mu}}{\underline{\mu}(C)}\underline{\mu}(C)\int_C f\, d\underline{\mu} = 0$$

■

Remark 54 The stopped standard Brownian motion $(B_t)_{t\in[0,\tau)}$ in (107) is (a.s.) on the random interval $[0,\tau)$, the same as the one appearing in Theorem 17, when above, one considers the stochastic process $(D_t)_{t\in[0,\tau)}$ constructed in Theorem 17. This is a consequence, on one hand of (106), which enables to recover $(B_t)_{t\in[0,\tau)}$ from $(D_t)_{t\in[0,\tau)}$, since $B_0 = 0$ and $\underline{\mu}(C_t) > 0$ for $t \in [0,\tau)$, and on the other hand of the fact that in the proof of Theorem 17, we have

$$\forall\ t \in [0,\tau), \qquad M_t = \sqrt{2}\int_0^t \left(\int_{C_s} f\, d\underline{\mu}\right) dB_s$$

so by taking $f = \mathbb{1}$, we can recover $(B_t)_{t\in[0,\tau)}$ in the same way.

□

In the same spirit as Theorem 5 and similarly to [19], we also have

Proposition 55 *Under the setting of Theorem 5, the process* $(1/\mu(D_t))_{t\in[0,\tau)}$ *is a positive local martingale. It follows that* $\lim_{t\to\tau-}\mu(D_t)$ *exists a.s. in* $(0,+\infty]$.

Proof

Consider the mapping $\mathfrak{F} : \mathcal{G} \ni D \mapsto 1/\mu(D)$, which belongs to $\mathfrak{D}$. To see that $(1/\mu(D_t))_{t\in[0,\tau)}$ is a local martingale, it is sufficient to check that $\mathfrak{L}[\mathfrak{F}] = 0$. By definition,

$$\begin{aligned}\forall\ D \in \mathcal{G}, \qquad \mathfrak{L}[\mathfrak{F}](D) &= -\frac{1}{F_{\mathbb{1}}^2(D)}\mathfrak{L}[F_{\mathbb{1}}](D) + \frac{2}{F_{\mathbb{1}}^3(D)}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}](D)\\ &= -\frac{1}{F_{\mathbb{1}}^2(D)}\frac{\underline{\mu}(C)^2}{\mu(D)} + \frac{2}{F_{\mathbb{1}}^3(D)}\underline{\mu}(C)^2\\ &= 0\end{aligned}$$

where (104) and (105) were taken into account.

Thus as a positive submartingale $1/\mu(D_t)$, converges a.s. as t goes to τ from below, to a limit belonging to $[0,+\infty)$. By taking the inverse, we get the announced result.

■

6.3 Enrichment of the elementary observables

Up to now, we only considered elementary observables of type (4), since they were sufficient for our purposes, but other functionals are interesting to go further. To simplify the presentation, we restrict ourselves to the situation of the Brownian motion on a Riemannian manifold, namely we take $b = 0$, so that $\mu = \lambda$, $\underline{\mu} = \sigma$ and $\rho^b = \rho$. The general case can be treated similarly (see the manipulations of the proof of Theorem 3).

The first of new elementary observables we would like to add have the following form, for any $f \in \mathcal{C}^\infty(V)$,

$$G_f \,:\, \mathcal{D} \ni D \;\mapsto\; G_f(D) := \int_C f\, d\sigma \tag{108}$$

Indeed, the action (6) of the generator $\mathfrak{L}$ can then be rewritten, taking into account Stokes' theorem (21), as

$$\begin{aligned} \forall\ D \in \mathcal{D}, \qquad \mathfrak{L}[F_f](D) &= \int_C \langle \nabla f, \nu\rangle + 2\frac{\sigma(C)}{\sigma(D)} f\, d\sigma \\ &= F_{\triangle f}(D) + 2\frac{G_{\mathbb{1}}(D) G_f(D)}{F_{\mathbb{1}}(D)} \end{aligned}$$

so it seems natural to study the evolution of $(G_f(D_t))_{t\in[0,\tau)}$, when $(D_t)_{t\in[0,\tau)}$ is a solution to the martingale problem associated to $\mathfrak{L}$.

Unfortunately, it seems difficult to work directly from this martingale problem, while we still don't know if it is well-posed. Our hope is that by enriching the domain of functionals to which it is applied, we should be more able to obtain that it is well-posed. So we rather consider the process $(D_t)_{t\in[0,\tau)}$ given by (52) and construct new martingales for it. More precisely, up to reducing τ (replacing it by its minimum with the first time D_t is no longer included into a nice tubular neighborhood of D_0), we will assume that D_τ is defined and belong to $\mathcal{D}$. Before investigating the functionals of the form (108), we are interested in the composition of the process $(D_t)_{t\in[0,\tau]}$ with the normal flow, which already played a crucial role in the construction of $(D_t)_{t\in[0,\tau]}$. So define

$$\begin{aligned} \mathcal{R} &:= \{r \in \mathbb{R} \,:\, \forall\ t \in [0,\tau],\ D_t \in \mathcal{D}_r\} \\ \forall\ r \in \mathcal{R},\ \forall\ t \in [0,\tau), \qquad D_t^{(r)} &:= \Psi(D_t, r) \\ &= \Psi(G_t, \sqrt{2}B_t + \theta_t + r) \end{aligned} \tag{109}$$

where $(G_t)_{t\in[0,\tau]}$ and $(\theta_t)_{t\in[0,\tau]}$ are defined as in (51). For any $r \in \mathbb{R}$, consider

$$\forall\ D \in \mathcal{D}_r,\ \forall\ x \in C, \qquad \alpha_C^{(r)}(x) := \rho_C(x) - \rho_{\Psi(C,r)}(\psi_{C,r}(x))$$

and the operator $\mathfrak{L}^{(r)}$ acting on $\mathcal{D}_{-r}$ via

$$\begin{aligned} \forall\ f \in \mathcal{C}^\infty(V),\ \forall\ D \in \mathcal{D}_{-r}, \quad \mathfrak{L}^{(r)}[F_f](D) &= \int_C \langle \nabla f, \nu\rangle + \left(2\frac{\sigma(\Psi(C,-r))}{\lambda(\Psi(D,-r))} + \alpha_C^{(-r)}\right) f\, d\sigma \\ &= \int_D \triangle f\, d\lambda + 2\frac{\sigma(\Psi(C,-r))\int_C f\, d\sigma}{\lambda(\Psi(D,-r))} + \int_C \alpha_C^{(-r)} f\, d\sigma \end{aligned} \tag{110}$$

Its interest comes from:

Lemma 56 *For any $f \in \mathcal{C}^\infty(V)$, $t \in [0,\tau]$ and $r \in \mathcal{R}$, we have*

$$F_f(D_t^{(r)}) = F_f(D_0^{(r)}) + \int_0^t \mathfrak{L}^{(r)}[F_f](D_s^{(r)})\, ds + \sqrt{2}\int_0^t G_f(D_s^{(r)})\, dB_s$$

Proof

The arguments are similar to those of the proof of Theorem 17, which lead to

$$\begin{aligned}
&dF_f(\Psi(G_t,\sqrt{2}B_t+\theta_t+r))\\
&= -\left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} \rho_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t)}\circ\psi_{\partial G_t,\sqrt{2}B_t+\theta_t}\circ\psi^{-1}_{\partial G_t,\sqrt{2}B_t+\theta_t+r}\, f\, d\sigma\right) dt\\
&\quad + \left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} f\, d\sigma\right)(\sqrt{2}dB_t+\partial_t\theta_t dt)\\
&\quad + \left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} \langle\nu,\nabla f\rangle\, d\sigma + \int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} \rho f\, d\sigma\right) dt\\
&= -\left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} \rho_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t)}\circ\psi_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r),-r}\, f\, d\sigma\right) dt\\
&\quad + \left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} \langle\nu,\nabla f\rangle + \rho f + \partial_t\theta_t f\, d\sigma\right) dt\\
&\quad +\sqrt{2}\left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} f\, d\sigma\right) dB_t\\
&= \left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} \langle\nu,\nabla f\rangle + \left(h(\Psi(G_t,\sqrt{2}B_t+\theta_t)) + \alpha^{(-r)}_{\partial D_t^{(r)}}\right) f\, d\sigma\right) dt\\
&\quad +\sqrt{2}\left(\int_{\partial\Psi(G_t,\sqrt{2}B_t+\theta_t+r)} f\, d\sigma\right) dB_t\\
&= \mathfrak{L}^{(r)}[F_f](D_t^{(r)})\, dt + \sqrt{2}\left(\int_{\partial D_t^{(r)}} f\, d\sigma\right) dB_t
\end{aligned}$$

■

For any $D\in\mathcal{D}$, define

$$\begin{aligned}
\forall\ x\in C,\qquad \rho_C^{(1)}(x) &:= \partial_r\rho_{\Psi(C,r)}(\psi_{C,r}(x))|_{r=0}\\
&= -\partial_r\alpha_C^{(r)}(x)|_{r=0}\\
&= \partial_r\alpha_C^{(-r)}(x)|_{r=0}
\end{aligned}\tag{111}$$

By differentiation with respect to r at 0 in Lemma 56, we get:

Proposition 57 *For any $f\in\mathcal{C}^\infty(V)$, we have*

$$\forall\ t\in[0,\tau],\qquad G_f(D_t) = G_f(D_0) + \int_0^t \mathfrak{L}[G_f](D_s)\, ds + \sqrt{2}\int_0^t\left(\int_{C_s}\langle\nabla f,\nu\rangle + \rho f\, d\sigma\right) dB_s$$

where

$$\forall\ D\in\mathcal{D},\qquad \mathfrak{L}[G_f](D) := \int_C \triangle f + 2\frac{\sigma(C)}{\lambda(D)}\langle\nu,\nabla f\rangle + \left(2\frac{\sigma(C)}{\lambda(D)}\rho + \rho^{(1)}\right) f\, d\sigma$$

Proof

Consider the evolution described in Lemma 56. Certain terms are very easy to differentiate with respect to r: according to the first part of Lemma 10

$$\forall\ t\in[0,\tau],\qquad \partial_r F_f[D_t^{(r)}]|_{r=0} = G_f[D_t]$$

For the Brownian part, use the second part of Lemma 10:

$$\forall\ t\in[0,\tau],\qquad \partial_r\int_{\partial D_t^{(r)}} f\,d\sigma \;=\; \int_{C_t}\langle\nabla f,\nu\rangle+\rho f\,d\sigma$$

For the remaining term, we decompose the derivative in

$$\partial_r\mathfrak{L}^{(r)}[F_f](D_t^{(r)})|_{r=0} \;=\; (\partial_r\mathfrak{L}^{(r)}|_{r=0})[F_f](D_t)+\partial_r\mathfrak{L}[F_f](D_t^{(r)})|_{r=0}$$

Use (110) for both terms of the r.h.s. For the first one, we get for any $D\in\mathcal{D}$,

$$(\partial_r\mathfrak{L}^{(r)}|_{r=0})[F_f](D) \;=\; -2\frac{\int\rho\,d\sigma\int_C f\,d\sigma}{\lambda(D)}+2\frac{\sigma(C)^2\int_C f\,d\sigma}{\lambda(D)^2}+\int_C\rho^{(1)}f\,d\sigma$$

For the second one, again for any $D\in\mathcal{D}$, taking into account that $\alpha_C^{(0)}\equiv 0$, we have

$$\begin{aligned}\partial_r\mathfrak{L}[F_f](\Psi(D,r))|_{r=0} &= \partial_r\left.\int_{\Psi(D,r)}\triangle f\,d\lambda+2\frac{\sigma(\Psi(C,r))\int_{\Psi(C,r)}f\,d\sigma}{\lambda(\Psi(D,r))}\right|_{r=0}\\ &= \int_C\triangle f\,d\sigma+2\frac{\int\rho\,d\sigma\int_C f\,d\sigma}{\lambda(D)}-2\frac{\sigma(C)^2\int_C f\,d\sigma}{\lambda(D)^2}+2\frac{\sigma(C)\int_C\langle\nu,\nabla f\rangle+\rho f\,d\sigma}{\lambda(D)}\end{aligned}$$

Putting together these computations, we obtain

$$\partial_r\mathfrak{L}^{(r)}[F_f](D_t^{(r)})|_{r=0} \;=\; \int_C\triangle f+\rho^{(1)}f\,d\sigma+2\frac{\sigma(C)}{\lambda(D)}\int_C\langle\nu,\nabla f\rangle+\rho f\,d\sigma$$

which leads to the definition of $L[G_f]$.

■

Note that for any $f\in\mathcal{C}^\infty(V)$ and $D\in\mathcal{D}$, we have

$$\mathfrak{L}[G_f](D) \;=\; G_{\triangle f}(D)+2\frac{G_{\mathbb{1}}(D)}{F_{\mathbb{1}}(D)}F_{\triangle f}(D)+2\frac{G_{\mathbb{1}}(D)}{F_{\mathbb{1}}(D)}\int_C\rho f\,d\sigma+\int_C\rho^{(1)}f\,d\sigma$$

but neither $\int_C\rho f\,d\sigma$ nor $\int_C\rho^{(1)}f\,d\sigma$ are of the form F_g of G_g for some $g\in\mathcal{C}^\infty(V)$. We are thus lead to introduce two new types of elementary observables:

$$\begin{aligned}H_f\ &:\ \mathcal{D}\ni D \;\mapsto\; H_f(D):=\int_C\rho f\,d\sigma\\ H_f^{(1)}\ &:\ \mathcal{D}\ni D \;\mapsto\; H_f(D):=\int_C\rho^{(1)}f\,d\sigma\end{aligned}$$

Investigating the evolution of these observables, one will have to consider more generally for any $l\in\mathbb{Z}_+$

$$H_f^{(l)}\ :\ \mathcal{D}\ni D \;\mapsto\; H_f(D):=\int_C\rho^{(l)}f\,d\sigma \tag{112}$$

where by iteration, for any $n\in\mathbb{Z}_+$,

$$\forall\ x\in C,\qquad \rho_C^{(n+1)}(x) \;:=\; \partial_r\rho^{(n)}_{\Psi(C,r)}(\psi_{C,r}(x))|_{r=0}$$

Probably other functionals will also appear (such as $\mathcal{D}\ni D\mapsto\int_C\rho\langle\nu,\nabla f\rangle\,d\sigma$ or $\mathcal{D}\ni D\mapsto\int_C\rho^2 f\,d\sigma$, see the next lemma), but the study of these iterations, as well as their impact on the well-posedness of the corresponding martingale problems, is left for a future work.

In the same spirit, we remark that the introduction of $\rho^{(1)}$ and $H^{(1)}$ are already needed to consider a third derivative in Lemma 10:

Lemma 58 *For any $f \in \mathcal{C}^\infty(V)$ and $D \in \mathcal{D}$, we have*

$$\partial_r H_f(\Psi(D,r))|_{r=0} = \int_C \rho\langle\nu,\nabla f\rangle + \rho^{(1)}f + \rho^2 f\, d\sigma$$

It follows that

$$\partial_r^3 F_f(D)|_{r=0} = \int_C \triangle f + \rho\langle\nu,\nabla f\rangle + \rho^{(1)}f + \rho^2 f\, d\sigma$$

Proof

The domain $D \in \mathcal{D}$ being fixed, consider a tubular neighborhood T of D such that for any $y \in T$, there exists a unique $r \in \mathbb{R}$ and $x \in C$ such that $y = \psi_{C,r}(x)$. Consider then the mapping $\widetilde{\rho} : T \to \mathbb{R}$ given by $\widetilde{\rho}(y) = \rho_{\Psi(C,r)}(y)$. With this definition, we have for r sufficiently small, $H_f(\Psi(D,r)) = G_{\widetilde{\rho}f}(\Psi(D,r))$. It follows that

$$\begin{aligned}\partial_r H_f(\Psi(D,r))|_{r=0} &= \partial_r G_{\widetilde{\rho}f}(\Psi(D,r))\\ &= \int_C \langle\nu,\nabla(\widetilde{\rho}f)\rangle + \rho\widetilde{\rho}f\, d\sigma\end{aligned}$$

It remains to note that on C, we have

$$\begin{aligned}\langle\nu,\nabla(\widetilde{\rho}f)\rangle &= \widetilde{\rho}\langle\nu,\nabla f\rangle + f\langle\nu,\nabla\widetilde{\rho}\rangle\\ &= \rho\langle\nu,\nabla f\rangle + f\rho^{(1)}\end{aligned}$$

to get the first identity.

The second one comes from the rewriting, in our present context, of the second equality in Lemma 10 as

$$\begin{aligned}\partial_r^2 F_f(\Psi(D,r)) &= \int_{\Psi(D,r)} \triangle f\, d\lambda + \int_{\Psi(C,r)} \rho f\, d\sigma\\ &= F_{\triangle f}(\Psi(D,r)) + H_f(\Psi(D,r))\end{aligned}$$

and by differentiating with respect to r at 0.

■

The case $f = \mathbb{1}$ is particularly interesting, since $G_{\mathbb{1}}(D) = \sigma(C)$ for any $D \in \mathcal{D}$. The quantity $\int_C \rho\, d\sigma$ is called the total mean curvature of C and according to the previous lemma, $\int_C \rho^{(1)} + \rho^2\, d\sigma$ is the derivative of the total mean curvature along the normal radial flow. In the situation of constant curvature in dimension 2, the terms $\rho^{(1)}$ and ρ^2 are in fact comparable:

Lemma 59 *Assume that V is a surface of constant curvature $K \in \mathbb{R}$. Then we have*

$$\forall\ D \in \mathcal{D}, \qquad \rho^{(1)} = -\rho^2 - K$$

Proof

When V is the Euclidean plane, the result follows by differentiating at $r = 0$ the first formula given in Lemma 48. The other null curvature situations (cylinders and flat torus) can be treated similarly, since they can be up-lifted to their locally isometric covering $\mathbb{R}^2$.

For the other constant curvature cases, use instead Lemma 50 of Subsection 5.1.

■

Remark 60 (a) When V is the Euclidean plane, it follows from Lemma 59 that

$$\partial_r \int_{\Psi(C,r)} \rho \, d\sigma \quad = \quad 0$$

namely locally the normal radial flow leaves the total curvature of a smooth curve invariant. This is in fact a consequence of Hopf's Umlaufsatz Theorem, stating that for any piecewise differentiable curve C in $\mathbb{R}^2$

$$\int_C \rho \, d\sigma \quad = \quad 2\pi \tag{113}$$

(with an appropriate convention for the jumps of the tangent vectors, where $\rho \, d\sigma$ has to be seen as the difference of angles times a Dirac mass at the considered singular point). When C is the smooth boundary of a convex domain, this can be obtained by letting r go to $+\infty$ in

$$\int_C \rho \, d\sigma \quad = \quad \int_{\Psi(C,r)} \rho \, d\sigma$$

and by remarking that for large $r > 0$, $\Psi(C, r)$ is quite close to a circle of radius r.

It would be interesting to see if this argument could be adapted to treat the general case.

(b) Consider the Euclidean space (or any null curvature space) of dimension larger than 2. From Remark (49), we deduce that

$$\rho^{(1)}(x) \quad = \quad -\sum_{m \in [\![n-1]\!]} \lambda_m^2(x)$$

More generally, when V has a constant sectional curvature K, we get

$$\rho^{(1)}(x) \quad = \quad -K(n-1) - \sum_{m \in [\![n-1]\!]} \lambda_{m,C}^2(x)$$

Recall that the Gauss curvature at $x \in C$ is given by

$$\kappa_C(x) \quad = \quad \prod_{m \in [\![n-1]\!]} \lambda_{m,C}(x)$$

Similarly to (111), we can introduce

$$\forall\ x \in C, \qquad \kappa_C^{(1)}(x) \quad \coloneqq \quad \partial_r \kappa_{\Psi(C,r)}(\psi_{C,r}(x))|_{r=0}$$

and, if one has indexed in a coherent (e.g. nondecreasing) way the eigenvalues of the second fundamental form,

$$\forall\ x \in C,\ \forall\ m \in [\![n-1]\!], \qquad \lambda_{m,C}^{(1)}(x) \quad \coloneqq \quad \partial_r \lambda_{m,\Psi(C,r)}(\psi_{C,r}(x))|_{r=0}$$

Then we have, at least if none of the eigenvalues vanishes,

$$\forall\ x \in C, \qquad \kappa_C^{(1)}(x) \quad = \quad \kappa_C(x) \sum_{m \in [\![n-1]\!]} \frac{\lambda_{m,C}^{(1)}}{\lambda_{m,C}}(x)$$

As in the proof of Lemma 58, we deduce that

$$\partial_r \int_{\Psi(C,r)} \kappa\, d\sigma\Big|_{r=0} = \int_C \kappa^{(1)} + \rho\kappa\, d\sigma \tag{114}$$

The last two formulas are valid on any Riemannian manifold V of dimension n.

But when V has a constant sectional curvature K, since

$$\forall\, x \in C,\ \forall\, m \in [\![n-1]\!], \qquad \lambda_{m,C}^{(1)}(x) = -K - \lambda_{m,C}^2$$

we obtain that, at least if none of the eigenvalues vanishes,

$$\forall\, x \in C, \qquad \kappa_C^{(1)}(x) = -\left(\rho_C(x) + K \sum_{m\in[\![n-1]\!]} \frac{1}{\lambda_{m,C}(x)}\right)\kappa_C(x)$$

Integrating this relation with respect to σ on C, it follows from (114) that

$$\partial_r \int_{\Psi(C,r)} \kappa_C\, d\sigma\Big|_{r=0} = -K \int_C \sum_{m\in[\![n-1]\!]} \frac{1}{\lambda_{m,C}}\kappa_C\, d\sigma$$

When $n = 3$, we have

$$\begin{aligned}\left(\frac{1}{\lambda_{1,C}} + \frac{1}{\lambda_{2,C}}\right)\kappa_C &= \lambda_{1,C} + \lambda_{2,C}\\ &= \rho_C\end{aligned}$$

thus

$$\begin{aligned}\partial_r \int_{\Psi(C,r)} \kappa_C\, d\sigma\Big|_{r=0} &= -K \int_C \rho_C\, d\sigma\\ &= -K\, \partial_r \lambda(\Psi(D,r))|_{r=0}\end{aligned}$$

Namely the quantity

$$\int_C \kappa_C\, d\sigma + K\lambda(D)$$

is invariant under the normal radial flow (as long as it remains in $\mathcal{D}$). This is a very special case of the Gauss-Bonnet theorem, asserting that the above quantity is equal to 2π times the Euler characteristic of V.

Again, one is left wondering about possible links between the normal radial flow and the generalized Gauss-Bonnet theorem.

(c) It is also natural to ask for a generalization of Lemma 59 when V is a surface whose curvature is not constant.

□

Let us come back to our martingale problem and to Proposition 57. The explicit description of the martingale associated to the evolution of $(G_f(D_t))_{t\in[0,\tau]}$ in terms of the stopped Brownian motion $(B_t)_{t\in[0,\tau]}$, enables us to see that for any $f, g \in \mathcal{C}^\infty(V)$ and $D \in \mathcal{D}$,

$$\begin{aligned}\Gamma_{\mathfrak{L}}[F_f, G_g](D) &= G_f(D)\,(F_{\triangle g}(D) + H_g(D))\\ \Gamma_{\mathfrak{L}}[G_f, G_g](D) &= (F_{\triangle f}(D) + H_f(D))\,(F_{\triangle g}(D) + H_g(D))\end{aligned}$$

These formulas leads to an enrichment of the algebra $\mathfrak{D}$ of the introduction. Indeed, consider the new algebra $\mathfrak{D}$ consisting of the functionals of the form $\mathfrak{F} := \mathfrak{f}(A_1, ..., A_n)$, where $n \in \mathbb{Z}_+$, $A_1, ..., A_n$ are elementary observables of the form (4) or (108) and $\mathfrak{f} : \mathcal{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ mapping, with $\mathcal{R}$ an open subset of $\mathbb{R}^n$ containing the image of $\mathcal{D}$ by $(A_1, ..., A_n)$. For such a functional $\mathfrak{F}$, define

$$\mathfrak{L}[\mathfrak{F}] = \sum_{j \in [\![1,n]\!]} \partial_j \mathfrak{f}(A_1, ..., A_n)\mathfrak{L}[A_j] + \sum_{k,l \in [\![1,n]\!]} \partial_{k,l}\mathfrak{f}(A_1, ..., A_n)\Gamma_{\mathfrak{L}}[A_k, A_l]$$

To two elements of $\mathfrak{D}$, $\mathfrak{F} := \mathfrak{f}(A_1, ..., A_n)$ and $\mathfrak{G} := \mathfrak{g}(\widetilde{A}_1, ..., \widetilde{A}_m)$, we also associate

$$\Gamma_{\mathfrak{L}}[\mathfrak{F}, \mathfrak{G}] := \sum_{l \in [\![n]\!], k \in [\![m]\!]} \partial_l \mathfrak{f}(A_1, ..., A_n)\partial_k \mathfrak{g}(\widetilde{A}_1, ..., \widetilde{A}_m)\Gamma_{\mathfrak{L}}[A_l, \widetilde{A}_k]$$

These formulas can be directly obtained as consequences of Itô's formula applied to the expressions given in (107) and Proposition 57, since the corresponding Brownian motions are the same (cf. Remark 54).

6.4 Asymptotic behavior for large times on the plane

In this last subsection, we present an example of application of the above extension of the domain of $\mathfrak{L}$. We consider the Laplacian $L = \triangle$ on the Euclidean plane $\mathbb{R}^2$. We assume the domain of $\mathfrak{L}$ has been extended to contain all mappings of the forms (4) and (108), defined on $\mathcal{G}$, an extension of $\mathcal{D}$ as described before Theorem 5. Just make the hypothesis that the boundaries of the elements of $\mathcal{G}$ are piecewise differentiable curves.

Theorem 61 *Let $(D_t)_{t\geqslant 0}$ be a solution to the martingale problem associated to $\mathfrak{L}$ defined for all times. Then we have a.s. in the Hausdorff metric,*

$$\lim_{t\to+\infty} \frac{D_t}{\sqrt{\lambda(D_t)}} = B(0, 1/\sqrt{\pi})$$

where $B(0, 1/\sqrt{\pi})$ is the Euclidean ball centered at 0 of radius $1/\sqrt{\pi}$.

Proof

From Theorem 5, we know that for any $t > 0$, $\lambda(D_t) > 0$, namely D_t is not a singleton and belongs to $\mathcal{G}$ by assumption. Up to replacing $(D_t)_{t\geqslant 0}$ by $(D_{1+t})_{t\geqslant 0}$, we assume in this proof that D_t belongs to $\mathcal{G}$ for all $t \geqslant 0$.

In the Euclidean plane, the following isoperimetric inequality holds:

$$\forall\ D \in \mathcal{G}, \qquad \frac{\sigma(C)^2}{\lambda(D)} \geqslant 4\pi \tag{115}$$

with equality if and only if D is a ball.

From Proposition 55 and $\tau = +\infty$, we deduce that

$$\liminf_{t\to+\infty} \sigma(C_t) \geqslant 2 \lim_{t\to+\infty} \sqrt{\pi\lambda(D_t)} > 0$$

Thus in (12) we get $\varsigma = +\infty$ and in (13), $\lim_{t\to+\infty} \theta_t = +\infty$.

In these circumstances, Theorem 5 asserts that $(\lambda(D_{\theta_t}))_{t\geqslant 0}$ is a Bessel process of dimension 3 and in particular

$$\lim_{t\to+\infty} \lambda(D_t) = +\infty$$

We now use Proposition 57. From the relation $G_{\mathbb{1}}(D) = \sigma(C)$, we get in general that

$$d\sigma(C_t) \;=\; \left(\int_{C_t} \rho^{(1)} + 2\frac{\sigma(C_t)}{\lambda(D_t)}\rho\, d\sigma\right) dt + \sqrt{2}\left(\int_{C_t} \rho\, d\sigma\right) dB_t$$

But for the Euclidean space, we have $\rho^{(1)} = -\rho^2$ and $\int \rho\, d\sigma = 2\pi$, according to Lemma 59 and Hopf's Umlaufsatz Theorem (113) (taking into account that the considered boundaries are piecewise differentiable), respectively. Thus we get

$$d\sigma(C_t) \;=\; \left(-\int_{C_t} \rho^2\, d\sigma + 4\pi\frac{\sigma(C_t)}{\lambda(D_t)}\right) dt + 2\sqrt{2}\pi\, dB_t$$

and

$$d\sigma(C_t)^2 \;=\; 2\left(-\int_{C_t} \rho^2\, d\sigma + 4\pi\frac{\sigma(C_t)}{\lambda(D_t)}\right)\sigma(C_t)\, dt + 4\sqrt{2}\pi\sigma(C_t)\, dB_t + 8\pi^2 dt$$

Recall from (106) that

$$d\lambda(D_t) \;=\; 2\frac{\sigma(C_t)^2}{\lambda(D_t)}\, dt + \sqrt{2}\sigma(C_t)\, dB_t$$

Consider the process $Z \coloneqq (Z_t)_{t\geqslant 0}$ defined by

$$\forall\, t \geqslant 0, \qquad Z_t \;\coloneqq\; \sigma(C_t)^2 - 4\pi\lambda(D_t)$$

From the above computations, we deduce that

$$\forall\, t \geqslant 0, \qquad dZ_t \;=\; 2\left(4\pi^2 - \sigma(C_t)\int_{C_t} \rho^2\, d\sigma\right) dt$$

By Cauchy-Schwarz' inequality, we have for any $t \geqslant 0$,

$$4\pi^2 \;=\; \left(\int_{C_t} \rho\, d\sigma\right)^2 \;\leqslant\; \sigma(C_t)\int_{C_t} \rho^2\, d\sigma$$

showing that Z is a.s. non-increasing. Thus we have

$$\forall\, t \geqslant 0, \qquad Z_t \;\leqslant\; Z_0 \tag{116}$$

For any $t \geqslant 0$, denote $\widetilde{D}_t \coloneqq D_t/\sqrt{\lambda(D_t)}$. We have for any $t \geqslant 0$,

$$\begin{aligned}\sigma(\widetilde{C}_t)^2 - 4\pi\lambda(\widetilde{D}_t) \;&=\; \frac{\sigma(C_t)^2 - 4\pi\lambda(D_t)}{\lambda(D_t)}\\ &\leqslant\; \frac{\sigma(C_0)^2 - 4\pi\lambda(D_0)}{\lambda(D_t)}\end{aligned}$$

and the last expression goes to zero as t goes to $+\infty$. From Bonnesen's inequality (see e.g. the book of Burago and Zalgaller [4]), we deduce that as t goes to infinity, $\widetilde{D}_t$ becomes closer and closer, in Hausdorff metric, to a disk of volume 1. To see the announced result, it is sufficient to see that the barycenter of $\widetilde{D}_t$, which is the barycenter of D_t divided by $\sqrt{\lambda(D_t)}$, i.e.

$$\frac{1}{\lambda(D_t)^{3/2}}\int_{D_t} x\, \lambda(dx)$$

converges a.s. to 0 as t goes to $+\infty$. It amounts to see that $F_f/F_{\mathbb{1}}^{3/2}(D_t)$ converges to zero for t large, where f is either the first or the second canonical projection of $\mathbb{R}^2$. So let f be the first coordinate mapping (the second coordinate can be treated similarly, note that a symmetry argument cannot be used here, since the well-posedness is missing). Before investigating the evolution of $\mathbb{R}_+ \ni t \mapsto F_f/F_{\mathbb{1}}^{3/2}(D_t)$, we need a preliminary result.

Lemma 62 *A transition phenomenon occurs:*

$$\forall\ a > 1, \qquad \int_0^{+\infty} \frac{1}{\lambda(D_t)^a}\, ds \quad < \quad +\infty$$

while

$$\forall\ a \leqslant 1, \qquad \int_0^{+\infty} \frac{1}{\lambda(D_t)^a}\, ds \quad = \quad +\infty$$

Furthermore, we have for large $t \geqslant 0$, a.s.,

$$\int_0^t \frac{1}{\lambda(D_s)}\, ds \quad \sim \quad \frac{\ln(F_{\mathbb{1}})(D_t)}{4\pi}$$

Proof

This is based on the fact that $\lambda(D_t)$ goes to infinity as t goes to infinity. More precisely, taking into account (104) and (105), we compute, for any $a > 0$ and any $D \in \mathcal{G}$,

$$\begin{aligned}
\mathfrak{L}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D) &= -\frac{a}{F_{\mathbb{1}}^{a+1}(D)}\mathfrak{L}[F_{\mathbb{1}}](D) + \frac{a(a+1)}{F_{\mathbb{1}}^{a+2}}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}](D) \\
&= a(a-1)\frac{\sigma(C)^2}{\lambda(D)^{a+2}}
\end{aligned}$$

and in the sense of Proposition 53

$$\begin{aligned}
\sqrt{\Gamma_{\mathfrak{L}}}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D) &= -\frac{a}{F_{\mathbb{1}}^{a+1}(D)}G_{\mathbb{1}}(D) \\
&= -\frac{a\sigma(C)}{\lambda(D)^{a+1}}
\end{aligned}$$

where $\sqrt{\Gamma_{\mathfrak{L}}}\,[1/F_{\mathbb{1}}^a]$ stands for $\sqrt{\Gamma_{\mathfrak{L}}\,[1/F_{\mathbb{1}}^a, 1/F_{\mathbb{1}}^a]}$. Since for any $a > 0$, we know that $1/F_{\mathbb{1}}^a(D_t)$ converges to zero as t goes to infinity, we deduce that

$$\begin{aligned}
\frac{1}{F_{\mathbb{1}}^a}(D_t) - \frac{1}{F_{\mathbb{1}}^a}(D_0) &= \int_0^t \mathfrak{L}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D_s)\, ds + \sqrt{2}\int_0^t \sqrt{\Gamma_{\mathfrak{L}}}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D_s)\, dB_s \\
&= a(a-1)\int_0^t \frac{\sigma(C_s)^2}{\lambda(D_s)^{a+2}}\, ds - \sqrt{2}a\int_0^t \frac{\sigma(C_s)}{\lambda(D_s)^{a+1}}\, dB_s \qquad (117)
\end{aligned}$$

converges for large $t \geqslant 0$. By a contradictory argument, assume that

$$\int_0^{+\infty} \frac{\sigma(C_s)^2}{\lambda(D_s)^{2a+2}}\, ds \quad = \quad +\infty$$

which implies in particular that

$$\int_0^{+\infty} \frac{\sigma(C_s)^2}{\lambda(D_s)^{a+2}}\, ds \quad = \quad +\infty \tag{118}$$

since $\lim_{t\to+\infty}\lambda(D_t) = +\infty$. The bracket of the local martingale $(\int_0^t \sqrt{\Gamma_{\mathfrak{L}}}[\frac{1}{F_{\mathbb{1}}^a}](D_s)\, dB_s)_{t\geqslant 0}$ is given for any $t \geqslant 0$ by

$$\begin{aligned}\left\langle \int_0^\cdot \sqrt{\Gamma_{\mathfrak{L}}}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D_s)\, dB_s\right\rangle_t &= \int_0^t \Gamma_{\mathfrak{L}}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D_s)\, ds \\ &= a^2\int_0^t \frac{\sigma(C_s)^2}{\lambda(D_s)^{2a+2}}\, ds\end{aligned}$$

so that the iterated logarithm law for continuous local martingales implies

$$\begin{aligned}\limsup_{t\to+\infty}\int_0^t \sqrt{\Gamma_{\mathfrak{L}}}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D_s)\, dB_s &= +\infty \\ \liminf_{t\to+\infty}\int_0^t \sqrt{\Gamma_{\mathfrak{L}}}\left[\frac{1}{F_{\mathbb{1}}^a}\right](D_s)\, dB_s &= -\infty\end{aligned}$$

In view of (118), it would follow that for large $t \geqslant 0$, the expression in (117) admits $-\infty$ as liminf if $a \leqslant 1$ and $+\infty$ as limsup if $a \geqslant 1$, this is in contradiction with the existence of a finite limit. Thus we get

$$\int_0^{+\infty} \frac{\sigma(C_s)^2}{\lambda(D_s)^{2a+2}}\, ds \quad < \quad +\infty$$

We get the first announced result, remembering that for large $t \geqslant 0$, $\sigma(C_t) \sim 2\sqrt{\pi\lambda(D_t)}$.

For the second result with $a = 1$, rather consider the observable $\ln(1/F_{\mathbb{1}})$. We have for any $D \in \mathcal{G}$,

$$\mathfrak{L}\left[\ln(F_{\mathbb{1}})\right](D) \quad = \quad \frac{1}{F_{\mathbb{1}}(D)}\mathfrak{L}[F_{\mathbb{1}}](D) - \frac{1}{F_{\mathbb{1}}^2(D)}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}](D) \quad = \quad \frac{\sigma(C)^2}{\lambda(D)^2}$$

and

$$\sqrt{\Gamma_{\mathfrak{L}}}\left[\ln(F_{\mathbb{1}})\right](D) \quad = \quad \frac{1}{F_{\mathbb{1}}(D)}G_{\mathbb{1}}(D) \quad = \quad \frac{\sigma(C)}{\lambda(D)}$$

So via similar contradictory arguments as before with

$$\ln(F_{\mathbb{1}})(D_t) - \ln(F_{\mathbb{1}})(D_0) \quad = \quad \int_0^t \frac{\sigma(C_s)^2}{\lambda(D_s)^2}\, ds - \sqrt{2}\int_0^t \frac{\sigma(C_s)}{\lambda(D_s)}\, dB_s \tag{119}$$

which diverges to $+\infty$ as t goes to infinity, we end up with

$$\int_0^{+\infty} \frac{\sigma(C_s)^2}{\lambda(D_s)^2}\, ds \quad = \quad +\infty$$

For the last result, we need to apply more carefully the iterated logarithm law. Let $(M_t)_{t\geqslant 0}$ be the continuous local martingale defined by

$$\forall\ t \geqslant 0, \qquad M_t \quad \coloneqq \quad \int_0^t \frac{\sigma(C_s)}{\lambda(D_s)}\, dB_s$$

Its bracket is given by

$$\forall\ t \geqslant 0, \qquad \langle M \rangle_t \ \coloneqq\ \int_0^t \frac{\sigma(C_s)^2}{\lambda(D_s)^2}\, ds$$

Since $\langle M \rangle_t$ diverges to $+\infty$ for large $t \geqslant 0$, the iterated logarithm law asserts that

$$\limsup_{t\to+\infty} \frac{|M_t|}{\sqrt{\langle M \rangle_t \ln(\ln(\langle M \rangle_t))}} \ =\ 1$$

It follows that for large $t \geqslant 0$,

$$|M_t| \ \ll\ \int_0^{+\infty} \frac{\sigma(C_s)^2}{\lambda(D_s)^2}\, ds$$

and the last statement of the lemma is a direct consequence of (119) and of the fact that $\sigma(C_t)^2 \sim 4\pi\lambda(D_t)$, for large $t \geqslant 0$.

■

Let us come back to our objective to show that ξ_t converges a.s. toward 0, where

$$\forall\ t \geqslant 0, \qquad \xi_t \ \coloneqq\ \frac{F_f(D_t)}{F_{\mathbb{1}}^{3/2}(D_t)}$$

with f the first coordinate mapping of $\mathbb{R}^2$. Instead of applying the martingale problem directly to the composed observable $\mathcal{D} \ni D \mapsto F_f/F_{\mathbb{1}}^{3/2}(D)$, it seems more convenient to decompose ξ_t into $M_t/\sqrt{\lambda(D_t)}$, where $(M_t)_{t\geqslant 0}$ is defined by

$$\forall\ t \geqslant 0, \qquad M_t \ \coloneqq\ \frac{F_f}{F_{\mathbb{1}}}(D_t) \ =\ \Lambda[f](D_t)$$

From Theorem (3), we have

$$\mathfrak{L}[\Lambda[f]](D) \ =\ \Lambda[\triangle[f]] \ =\ 0$$

so it follows that $(M_t)_{t\geqslant 0}$ is a local martingale. More precisely, we get from Proposition 53 that

$$\forall\ t \geqslant 0, \qquad M_t \ =\ M_0 + \int_0^t h_s\, dB_s$$

where for any $s \geqslant 0$,

$$\begin{aligned}
h_s \ &\coloneqq\ \sqrt{\Gamma_{\mathfrak{L}}[F_f/F_{\mathbb{1}}](D_s)}\\
&=\ \frac{G_f}{F_{\mathbb{1}}}(D_s) - \frac{F_f}{F_{\mathbb{1}}^2}(D_s)G_{\mathbb{1}}(D_s)\\
&=\ \frac{G_{\mathbb{1}}}{F_{\mathbb{1}}}(D_s)\left(\frac{G_f}{G_{\mathbb{1}}} - \frac{F_f}{F_{\mathbb{1}}}\right)(D_s)
\end{aligned}$$

When f is replaced by the identity mapping $\mathrm{id}\colon \mathbb{R}^2 \to \mathbb{R}^2$, for any $D \in \mathcal{G}$, the vector $\left(\frac{G_{\mathrm{id}}}{G_{\mathbb{1}}} - \frac{F_{\mathrm{id}}}{F_{\mathbb{1}}}\right)(D)$ is the difference between the barycenter of C and the barycenter of D, so it appears easily that for any $s \geqslant 0$,

$$|h_s| \ \leqslant\ \frac{\sigma(C_s)}{\lambda(D_s)} \left\| \frac{G_{\mathrm{id}}}{G_{\mathbb{1}}}(D_s) - \frac{F_{\mathrm{id}}}{F_{\mathbb{1}}}(D_s) \right\| \ \leqslant\ \frac{\sigma(C_s)^2}{2\lambda(D_s)}$$

More precise computations, separately presented in [20] because they rely on techniques belonging to the field of isoperimetric stability, show that there exists a universal constant $c > 0$ such that for any $D \in \mathcal{G}$ with $\sigma(C)^2 - 4\pi\lambda(D) \leqslant \lambda(D)/\pi$, we have

$$\left\| \frac{G_{\mathrm{id}}}{G_{\mathbb{1}}}(D) - \frac{F_{\mathrm{id}}}{F_{\mathbb{1}}}(D) \right\| \leqslant c\lambda^{1/4}(D)(\sigma(C)^2 - 4\pi\lambda(D))^{1/4}$$

Thus taking into account the decreasing property (116) and the fact that $\lambda(D_s)$ diverges to $+\infty$ as s goes to infinity, we get there exists (a.s.) a random time S and a constant χ (depending on D_0) such that

$$\forall\ s \geqslant S, \qquad |h_s| \leqslant \frac{\chi}{\lambda(D_s)^{1/4}}$$

From the iterated logarithm law, we deduce that as t goes to $+\infty$,

$$|M_t| = \widetilde{\mathcal{O}}\left(\sqrt{\int_0^t \frac{1}{\sqrt{\lambda(D_s)}}\, ds}\right) \tag{120}$$

where the notation $\phi(t) = \widetilde{\mathcal{O}}(\varphi(t))$, for two functions $\phi, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to+\infty} \varphi(t) = +\infty$, means that

$$\limsup_{t\to+\infty} \frac{\phi(t)}{\varphi(t)\ln(\ln(\varphi(t)))} < +\infty$$

Applying the martingale problem to the composed functional $\sqrt{F_{\mathbb{1}}}$, we get that for any $t \geqslant 0$,

$$\sqrt{F_{\mathbb{1}}(D_t)} = \sqrt{F_{\mathbb{1}}(D_0)} + \frac{3}{4}\int_0^t \frac{\sigma(C_s)^2}{\lambda(D_s)^{3/2}}\, ds + \frac{1}{\sqrt{2}}\int_0^t \frac{\sigma(C_s)}{\sqrt{\lambda(D_s)}}\, dB_s$$

Using again, on one hand that $\sigma(C_s)^2$ and $\lambda(D_s)$ are of the same order for large $s \geqslant 0$, and on the other hand the iterated logarithm law, we deduce that for large $t \geqslant 0$,

$$\int_0^t \frac{1}{\sqrt{\lambda(D_s)}}\, ds = \widetilde{\mathcal{O}}(\sqrt{\lambda(D_t)} + \sqrt{t}) \tag{121}$$

Another application of the iterated logarithm law to three independent Brownian motions enables to see that if $(R_t)_{t\geqslant 0}$ is a Bessel process of dimension 3, then a.s.,

$$R_t = \widetilde{\mathcal{O}}(\sqrt{t}) \tag{122}$$

Recall that $(R_t)_{t\geqslant 0} \coloneqq \lambda(D_{\theta_t})_{t\geqslant 0}$ is a Bessel process of dimension 3, according to Theorem 5, where $(\theta_t)_{t\geqslant 0}$ is defined by

$$\forall\ t \geqslant 0, \qquad 2\int_0^{\theta_t} \sigma(C_s)^2\, ds = t$$

The martingale problem applied to $F_{\mathbb{1}}$ shows that for any $t \geqslant 0$,

$$\lambda(D_t) = \lambda(D_0) + 2\int_0^t \frac{\sigma(C_s)^2}{\lambda(D_s)}\, ds + \sqrt{2}\int_0^t \sigma(C_s)\, dB_s$$

Replacing t by θ_t, we deduce that

$$\begin{aligned}
\theta_t &\sim \frac{1}{4\pi}\int_0^{\theta_t}\frac{\sigma(C_s)^2}{\lambda(D_s)}\,ds \\
&= \frac{1}{8\pi}\left(\lambda(D_{\theta_t})-\lambda(D_0)-\sqrt{2}\int_0^{\theta_t}\sigma(C_s)\,dB_s\right) \\
&= \widetilde{\mathcal{O}}\left(\sqrt{t}+\sqrt{\int_0^{\theta_s}\sigma(C_s)^2\,ds}\right) \\
&= \widetilde{\mathcal{O}}(\sqrt{t})
\end{aligned}$$

It follows that

$$\begin{aligned}
t^2 &= \widetilde{\mathcal{O}}(\theta_t^{-1}) \\
&= \widetilde{\mathcal{O}}(R^2_{\theta_t^{-1}}) \\
&= \widetilde{\mathcal{O}}(\lambda(D_t)^2)
\end{aligned}$$

where θ^{-1} stands for the inverse mapping of $\theta : \mathbb{R}_+ \to \mathbb{R}_+$. Finally we obtain

$$\sqrt{t} = \widetilde{\mathcal{O}}(\sqrt{\lambda(D_t)}) \tag{123}$$

and this is sufficient to insure that a.s.

$$\lim_{t\to+\infty}\frac{M_t}{\sqrt{\lambda(D_t)}} = 0$$

in view of (120) and (121).

■

Remark 63 From (123), it appears that

$$\limsup_{t\to+\infty}\frac{\ln(t)}{\ln(\lambda(D_t))} \leqslant 1$$

We believe (in accordance with the beginning of Lemma 62) that

$$\lim_{t\to+\infty}\frac{\ln(\lambda(D_t))}{\ln(t)} = 1$$

but we have not been able to show it, even taking into account a lower bound on the rate of escape for the Bessel process $(R_t)_{t\geqslant 0}$ of dimension 3, stating that for any $a>1$,

$$\liminf_{t\to+\infty}\frac{R_t\ln^a(t)}{\sqrt{t}} = +\infty$$

according to Theorem 3.2 (ii) of Shiga and Watanabe [27], see also Motoo [23] (the part (i) of their theorem extends (122) to any Bessel process with a positive parameter). This implies that

$$\lim_{t\to+\infty}\frac{\ln(R_t)}{\ln(t)} = \frac{1}{2}$$

Furthermore, note that in the above proof we did not use the last part Lemma 62, which also gives an equivalent of $\ln(\lambda(D_t))$ for large $t \geqslant 0$.

These shortcomings are an invitation to study further the asymptotic behavior of the renormalized domains $(D_t/\sqrt{\lambda(D_t)})_{t\geqslant 0}$, in particular their fluctuations around the convergence of Theorem 61.

□

7 Elliptic density theorem revisited

Here we assume that Conjecture 6 is true: not only we can construct a solution $(D_t)_{t\in[0,\tau]}$ to the martingale problem associated to $(\mathfrak{D},\mathfrak{L})$ and starting from any singleton $\{x_0\}\subset V$, but it can be coupled with the primal diffusion X starting from x_0 so that (14) and (15) are satisfied. Let us show how to quickly recover the density theorem for elliptic diffusion from this property.

The proof is based on the following elementary observation:

Lemma 64 *Let $A\subset V$ be a negligible event with respect to μ and denote f its indicator function. For any measurable $D\subset V$ with $\mu(D)>0$ and $s\geqslant 0$, we have*

$$\Lambda[P_s[f]](D) \;=\; 0$$

where $(P_t)_{t\geqslant 0}$ is the Markov semi-group associated to L, seen as a family of Markov kernels.

Proof

Taking into account that μ is invariant for $(P_t)_{t\geqslant 0}$, we have

$$\begin{aligned}\Lambda[P_s[f]](D) &= \frac{\mu[\mathbb{1}_D P_s[f]]}{\mu(D)}\\ &\leqslant \frac{\mu[P_s[f]]}{\mu(D)}\\ &= \frac{\mu[f]}{\mu(D)}\\ &= 0\end{aligned}$$

■

We can now come to the

Proof of Corollary 7

With the notations of the above lemma and Corollary 7, we want to check that for any $x_0\in V$ and any $r>0$, $P_r[f](x_0)\coloneqq \mathbb{E}_{x_0}[f(X_r)]=0$.

For any $t\geqslant 0$, let $\mathcal{F}_t$ be the σ-field generated by $X_{[0,t]}$ and $D_{[0,t\wedge\tau]}$. From (14), we get the diffusion $X\coloneqq (X_t)_{t\geqslant 0}$ is also strongly Markovian with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Remark that $\tau>0$, the stopping time entering into the definition of $(D_t)_{t\in[0,\tau]}$, is also a stopping time with respect to $(\mathcal{F}_t)_{t\geqslant 0}$. It follows that

$$\begin{aligned}\mathbb{E}_{x_0}[f(X_r)] &= \mathbb{E}_{x_0}[\mathbb{E}_{x_0}[f(X_r)|\mathcal{F}_{r\wedge\tau}]]\\ &= \mathbb{E}_{x_0}[P_{r-r\wedge\tau}[f](X_{r\wedge\tau})]\end{aligned}$$

For any $t\geqslant 0$, let $\mathcal{D}_t$ be the σ-field generated by $D_{[0,t\wedge\tau]}$. It follows from (14) with $T=r\wedge\tau$, that

$$\mathbb{E}[h(r\wedge\tau, X_{r\wedge\tau})|\mathcal{D}_{r\wedge\tau}] \;=\; \Lambda(D_{r\wedge\tau}, h(r\wedge\tau,\cdot))$$

for any non-negative measurable mapping $h:\mathbb{R}_+\times V\to\mathbb{R}_+$. We deduce that

$$\begin{aligned}\mathbb{E}_{x_0}[f(X_r)] &= \mathbb{E}_{x_0}[\mathbb{E}_{x_0}[P_{r-r\wedge\tau}[f](X_{r\wedge\tau})|\mathcal{D}_{r\wedge\tau}]]\\ &= \mathbb{E}_{x_0}[\Lambda[P_{r-r\wedge\tau}[f]](D_{r\wedge\tau})]\\ &= 0\end{aligned}$$

according to Lemma 64. Indeed, we took into account Theorem 5, insuring that for any $t \in (0, \tau]$, we have $\mu(D_t) > 0$.

■

With Marc Arnaudon, we are currently working on the existence of a coupling as in Conjecture 6 and some results in this direction will be presented in a future paper.

When the solutions to the martingale problems associated to $(\mathfrak{D}, \mathfrak{L})$ and to initial singleton sets can be defined for all times, there is no need to have such a coupling at our disposal to recover the density theorem for elliptic diffusions. Indeed, assume that for any $x_0 \in V$, we can construct a solution $(D_t)_{t\geqslant 0}$ to the martingale problem associated to $(\mathfrak{D}, \mathfrak{L})$ and starting from the singleton $\{x_0\} \subset V$. First, we remark that we can enrich the martingale problem by adding a temporal component. Let us just sketch the argument: when $\mathfrak{F} \in \mathfrak{D}$ and $f \in \mathcal{C}^1([0,t])$ with $t > 0$ are given, define

$$\forall\ (s, D) \in [0,t] \times \mathcal{D}, \qquad \overline{\mathfrak{L}}[f \otimes \mathfrak{F}](s, D) \ \coloneqq \ \partial_s f(s)\mathfrak{F}(D) + f(s)\mathfrak{L}[\mathfrak{F}](D) \tag{124}$$

A simple computation shows that the process $(M_s^{f\otimes\mathfrak{F}})_{s\in[0,t]}$ given by

$$\forall\ s \in [0,t], \qquad M_s^{f\otimes\mathfrak{F}} \ \coloneqq \ f(s)\mathfrak{F}(D_s) - f(0)\mathfrak{F}(D_0) - \int_0^s \overline{\mathfrak{L}}[f \otimes \mathfrak{F}](u, D_u)\, du$$

is a martingale, whose bracket process is given by

$$\forall\ s \in [0,t], \qquad \left\langle M_\cdot^{f\otimes\mathfrak{F}} \right\rangle_s \ = \ \int_0^s f^2(u)\Gamma_{\mathfrak{L}}[\mathfrak{F},\mathfrak{F}](D_u)\, du$$

By traditional approximations, these considerations can be generalized to more general mappings $\mathfrak{F} : [0,t] \times \mathcal{D} \to \mathbb{R}$, in particular they must be $\mathcal{C}^1$ with respect to the time component so that (124) can be extended to

$$\forall\ (s, D) \in [0,t] \times \mathcal{D}, \qquad \overline{\mathfrak{L}}[\mathfrak{F}](s, D) \ \coloneqq \ \partial_s \mathfrak{F}(s, D) + \mathfrak{L}[\mathfrak{F}(s, \cdot)](D)$$

The fact that the corresponding process defined by

$$\forall\ s \in [0,t], \qquad M_s^{\mathfrak{F}} \ \coloneqq \ \mathfrak{F}(s, D_s) - \mathfrak{F}(0, D_0) - \int_0^s \overline{\mathfrak{L}}[\mathfrak{F}](u, D_u)\, du$$

is a martingale is called the Dynkin's formula.

Fix $g \in \mathcal{C}^\infty(V)$, the above considerations can be applied to the mapping

$$\mathfrak{F}\ :\ [0,t] \times \mathcal{D} \ni (s, D) \ \mapsto \ \Lambda[P_{t-s}[g]](D)$$

for which we compute $\overline{\mathfrak{L}}[\mathfrak{F}] = 0$, due to the intertwining relation of Theorem 3. Taking expectations, it follows that

$$\mathbb{E}_{\{x_0\}}[\Lambda[g](D_t)] \ = \ \Lambda[P_t[g]](\{x_0\})$$

which amounts to intertwining relations at the level of semi-groups:

$$\begin{aligned} \forall\ g \in \mathcal{C}^\infty(V), \qquad \mathfrak{P}_t[\Lambda[g]](\{x_0\}) \ &= \ \Lambda[P_t[g]](\{x_0\}) \\ &= \ P_t[g](x_0) \end{aligned}$$

where $(\mathfrak{P}_t)_{t\geqslant 0}$ is the Markov semi-group associated to $\mathfrak{L}$. Since both the l.h.s. and the r.h.s. can be seen as integration of the mapping g, this relation is extended to any non-negative measurable function g. When we take for g the indicator function of a measurable set negligible with respect to μ, we get

$$\forall\ t > 0, \qquad \mathbb{E}_{\{x_0\}}[\Lambda[g](D_t)] \ = \ 0$$

according to Lemma 64 and due to the fact that $\mu(D_t) > 0$, from Theorem 5. We deduce that $P_t[g](x_0) = 0$, for any $t > 0$ and $x_0 \in V$, as wanted.

An immediate extension is:

Proposition 65 *Assume that there exists $\epsilon > 0$ such that for any $x_0 \in V$, we can construct a solution $(D_t)_{t\in[0,\epsilon]}$ to the martingale problem associated to $(\mathfrak{D}, \mathfrak{L})$ and starting from the singleton $\{x_0\} \subset V$. Then for any $t > 0$ and whatever the initial law $\mathcal{L}(X_0)$, the law of X_t is absolutely continuous with respect to μ.*

Proof

The arguments presented above the statement of this proposition show that for any $s \in (0, \epsilon]$ and any function $f : V \to \mathbb{R}_+$ negligible with respect to μ, we have that $P_s[f] = 0$. By invariance of μ, we also have that for any $u \geqslant 0$, $P_u[f]$ is negligible with respect to μ: $\mu[P_u[f]] = \mu[f] = 0$. We deduce that $P_{s+u}[f] = P_s[P_u[f]] = 0$ and the announced result follows.

■

Of course Corollary 7 and Proposition 65 are well-known in the present elliptic diffusion framework. Nevertheless, we think this new approach can be adapted to more complicated context, as Theorem 5 is quite universal (it was shown to hold also for hypoelliptic diffusions, for the moment in dimension 1, in [18]). We believe it should always be possible to associate to a diffusion some evolving sets (as mentioned in the introduction) whose weights for an invariant measure behave like a continuous martingale. By conditioning the primal diffusion X to remain inside these sets, we would be led to a Bessel-3 process, up to a time-change and at least if the randomness of X is sufficient, as the Brownian motion conditioned to stay positive ends up being a Bessel-3 process.

Another noticeable downside of Corollary 7 is that it requires the a priori knowledge that μ is absolutely continuous with respect to the Riemannian measure. A more general statement would only conclude, at positive times, to the absolute continuity of the time-marginal laws with respect to the invariante measure. In this paper we only considered kernels Λ which are directly related to the invariant measure μ, but it would be instructive to condition with respect to other measures, even time-dependent ones.

A About product situations

As already mentioned in the introduction, there are in general several dual generators intertwined through Λ with a given generator L. We consider in this appendix the product situation, where this multiplicity is particularly obvious.

Let $\widetilde{L}$ and $\widehat{L}$ be two smooth generators on the manifolds $\widetilde{V}$ and $\widehat{V}$ of dimension larger or equal to 1. Consider $V \coloneqq \widetilde{V} \times \widehat{V}$ endowed with $L \coloneqq \widetilde{L} \otimes \widehat{I} + \widetilde{I} \otimes \widehat{L}$ ($\widetilde{I}$ and $\widehat{I}$ are the identity operators acting on $\mathcal{C}^\infty(\widetilde{V})$ and $\mathcal{C}^\infty(\widehat{V})$ respectively). All the notions relative to $\widetilde{L}$ (respectively $\widehat{L}$) will receive a tilde (resp. a hat). Assume that $\widetilde{L}$ admits an invariant Radon measure $\widetilde{\mu}$ and consider on $\widetilde{\mathcal{G}}$, an appropriate set of compact subsets of $\widetilde{V}$ with positive measures, the kernel $\widetilde{\Lambda}$ naturally associated with $\widetilde{\mu}$. Let $\widetilde{\mathfrak{D}}$ be an algebra of observables on $\widetilde{\mathcal{G}}$ on which we are given an operator $\widetilde{\mathfrak{L}}$, intertwined with $\widetilde{L}$ through $\widetilde{\Lambda}$: $\widetilde{\mathfrak{L}}\widetilde{\Lambda} = \widetilde{\Lambda}\widetilde{L}$. Make similar hypotheses for $\widehat{L}$. Next define

$$\begin{aligned} \mathcal{G}_{\text{indep}} &\coloneqq \{\widetilde{D} \times \widehat{D} : \widetilde{D} \in \widetilde{\mathcal{G}} \text{ and } \widehat{D} \in \widehat{\mathcal{G}}\} \\ \mathfrak{D}_{\text{indep}} &\coloneqq \widetilde{\mathfrak{D}} \otimes \widehat{\mathfrak{D}} \\ \mathfrak{L}_{\text{indep}} &\coloneqq \widetilde{\mathfrak{L}} \otimes I_{\widehat{\mathfrak{D}}} + I_{\widetilde{\mathfrak{D}}} \otimes \widehat{\mathfrak{L}} \end{aligned}$$

where $I_{\widetilde{\mathfrak{D}}}$ and $I_{\widehat{\mathfrak{D}}}$ are the identity operators on $\widetilde{\mathfrak{D}}$ and $\widehat{\mathfrak{D}}$ respectively. It is immediate to check that $\mathfrak{L}_{\text{indep}}\Lambda = \Lambda L$, where $\Lambda \coloneqq \widetilde{\Lambda} \otimes \widehat{\Lambda}$ is the natural Markov kernel associated with the measure $\mu \coloneqq \widetilde{\mu} \otimes \widehat{\mu}$, invariant for L. When $(\widetilde{D}_t)_{t\in[0,\widetilde{\tau})}$ and $(\widehat{D}_t)_{t\in[0,\widehat{\tau})}$ are independent processes satisfying the martingale problems associated with $(\widetilde{\mathfrak{D}}, \widetilde{\mathfrak{L}})$ and $(\widehat{\mathfrak{D}}, \widehat{\mathfrak{L}})$ respectively, then $(D_t)_{t\in[0,\tau)}$, defined by

$$\begin{array}{rrcl} & \tau & \coloneqq & \widetilde{\tau} \wedge \widehat{\tau} \\ \forall\ t \in [0, \tau), & D_t & \coloneqq & (\widetilde{D}_t, \widehat{D}_t) \ \in\ \mathcal{G}_{\text{indep}} \end{array}$$

is a solution to the martingale problem associated with $(\mathfrak{D}_{\text{indep}}, \mathfrak{L}_{\text{indep}})$.

It should be clear that such a solution is very different from the one obtained from Theorem 4, due to the fact that the evolutions on $\widetilde{\mathcal{G}}$ and $\widehat{\mathcal{G}}$ are independent. In fact, the state spaces $\mathcal{G}_{\text{indep}}$ and $\mathcal{D}$ are even disjoint. Consider the example where $\widetilde{L} = \widehat{L}$ is the Laplacian on $\mathbb{R}$ and add the singletons to $\mathcal{G}$ and $\mathcal{D}$. Starting from a singleton, the solution associated with $\mathfrak{L}_{\text{indep}}$ evolves as rectangles (centered at the initial point) with independent side-lengths behaving as Bessel processes of dimension 3, while the solution associated with Theorem 4 evolves as disks (centered at the initial point) whose radius are Bessel process of dimension 4 (according to Subsection 2.1). It could be objected that this argument is not really valid, since we did not show uniqueness of the solution to the martingale problem associated with $(\mathfrak{D}, \mathfrak{L})$, or with formal extensions of $(\mathfrak{D}, \mathfrak{L})$, in the sense that exactly the same definitions are applied to more general subsets than those from $\mathcal{D}$. But in Proposition 61, it is proven that a solution to such a martingale problem, which is furthermore defined for all times, ends up looking like a big disk and this is not true for the processes associated with $(\mathfrak{D}_{\text{indep}}, \mathfrak{L}_{\text{indep}})$, since starting from a rectangle, it remains in the set of rectangles.

The fact that under $\mathfrak{L}$ the evolutions of different parts of the boundary of a domain are strongly correlated could suggest to try to couple the evolutions under $\widetilde{\mathfrak{L}}$ and $\widehat{\mathfrak{L}}$. More precisely, assume that $\widetilde{\mathcal{G}} = \widetilde{\mathcal{D}}$ and that $\widetilde{\mathfrak{D}}$ and $\widetilde{\mathfrak{L}}$ are constructed as in the introduction, similarly for $(\widehat{\mathcal{G}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{L}})$. Let $(\widetilde{D}_t)_{t\in[0,\widetilde{\tau})}$ and $(\widehat{D}_t)_{t\in[0,\widehat{\tau})}$ be solutions to the corresponding martingale problems. According to Proposition 53, there exist Brownian motions $(\widetilde{B}_t)_{t\geqslant 0}$ and $(\widehat{B}_t)_{t\geqslant 0}$ such that

$$\begin{array}{rrcl} \forall\ \widetilde{f} \in \mathcal{C}^\infty(\widetilde{V}), \forall\ t \in [0, \widetilde{\tau}), & dF_{\widetilde{f}}(\widetilde{D}_t) & = & \widetilde{\mathfrak{L}}[F_{\widetilde{f}}](\widetilde{D}_t)\, dt + \sqrt{2}\sqrt{\Gamma_{\widetilde{\mathfrak{L}}}}[F_{\widetilde{f}}](\widetilde{D}_t)\, d\widetilde{B}_t \\ & & = & \widetilde{\mathfrak{L}}[F_{\widetilde{f}}](\widetilde{D}_t)\, dt + \sqrt{2}G_{\widetilde{f}}(\widetilde{D}_t)\, d\widetilde{B}_t \end{array}$$

and

$$\begin{array}{rrcl} \forall\ \widehat{f} \in \mathcal{C}^\infty(\widehat{V}), \forall\ t \in [0, \widehat{\tau}), & dF_{\widehat{f}}(\widehat{D}_t) & = & \widehat{\mathfrak{L}}[F_{\widehat{f}}](\widetilde{D}_t)\, dt + \sqrt{2}\sqrt{\Gamma_{\widehat{\mathfrak{L}}}}[F_{\widehat{f}}](\widehat{D}_t)\, d\widehat{B}_t \\ & & = & \widehat{\mathfrak{L}}[F_{\widehat{f}}](\widetilde{D}_t)\, dt + \sqrt{2}G_{\widehat{f}}(\widehat{D}_t)\, d\widehat{B}_t \end{array}$$

In the previous independent framework, $(\widetilde{B}_t)_{t\geqslant 0}$ and $(\widehat{B}_t)_{t\geqslant 0}$ are independent and we end up with the generator $\mathfrak{L}_{\text{indep}}$. Now we would like to couple $(\widetilde{D}_t)_{t\in[0,\widetilde{\tau})}$ with $(\widehat{D}_t)_{t\in[0,\widehat{\tau})}$ by taking $(\widetilde{B}_t)_{t\geqslant 0} = (\widehat{B}_t)_{t\geqslant 0}$, since this is suggested by a naive extension of the radial evolution (3) to the domains belonging to $\mathcal{G}_{\text{indep}}$. But again we end up with a process different from the one obtained from Theorem 4, for the same reason as above: in the case $\widetilde{L} = \widehat{L} = \partial^2$, it will evolve as squares if it is started from a square. It can also be seen on the action of the generators on observables of the form $F_{\widetilde{f}\otimes\widehat{f}}$, where $\widetilde{f} \in \mathcal{C}^\infty(\widetilde{V})$ and $\widehat{f} \in \mathcal{C}^\infty(\widehat{V})$. In the general setting, Itô's formula leads for the above coupling to the generator $\mathfrak{L}_{\text{equal}}$ acting on $\mathcal{G}_{\text{indep}}$ as

$$\mathfrak{L}_{\text{equal}}[F_{\widetilde{f}\otimes\widehat{f}}] \quad = \quad F_{\widehat{f}} \otimes \widetilde{\mathfrak{L}}[F_{\widetilde{f}}] + F_{\widetilde{f}} \otimes \widehat{\mathfrak{L}}[F_{\widehat{f}}] + 2G_{\widetilde{f}} \otimes G_{\widehat{f}}$$

with the notation of Subsection 6.3. But simple computations show that the formal extension of $\mathfrak{L}$ to $\mathcal{G}_{\text{indep}}$ should be given by

$$\mathfrak{L}[F_{\widetilde{f}\otimes\widehat{f}}] = \widetilde{\mathfrak{L}}[F_{\widetilde{f}}]\otimes F_{\widehat{f}} + F_{\widetilde{f}}\otimes\widehat{\mathfrak{L}}[F_{\widehat{f}}] + 2\frac{G_{\widetilde{\mathbb{1}}}}{F_{\widetilde{\mathbb{1}}}}F_{\widetilde{f}}\otimes G_{\widehat{f}} + 2G_{\widetilde{f}}\otimes\frac{G_{\widehat{\mathbb{1}}}}{F_{\widehat{\mathbb{1}}}}F_{\widehat{f}}$$

where $\widetilde{\mathbb{1}}\in\mathcal{C}^\infty(\widetilde{V})$ and $\widehat{\mathbb{1}}\in\mathcal{C}^\infty(\widehat{V})$ are the functions always taking the value 1.

But in both cases, we have the same carré du champs: for any $\widetilde{f}\in\mathcal{C}^\infty(\widetilde{V})$ and $\widehat{f}\in\mathcal{C}^\infty(\widehat{V})$,

$$\begin{aligned}\Gamma_{\mathfrak{L}}[F_{\widetilde{f}\otimes\widehat{f}}] &= \Gamma_{\mathfrak{L}_{\text{equal}}}[F_{\widetilde{f}\otimes\widehat{f}}]\\ &= \left(F_{\widetilde{f}}\otimes G_{\widehat{f}} + G_{\widetilde{f}}\otimes F_{\widehat{f}}\right)^2\end{aligned}$$

which is different from

$$\Gamma_{\mathfrak{L}_{\text{indep}}}[F_{\widetilde{f}\otimes\widehat{f}}] = F_{\widetilde{f}}^2\otimes G_{\widehat{f}}^2 + G_{\widetilde{f}}^2\otimes F_{\widehat{f}}^2$$

Nevertheless, the generator $\mathfrak{L}_{\text{equal}}$ is not intertwined with L through Λ. Indeed, for any $\widetilde{f}\in\mathcal{C}^\infty(\widetilde{V})$ and $\widehat{f}\in\mathcal{C}^\infty(\widehat{V})$, denote

$$\mathfrak{R}_{\widetilde{f}\otimes\widehat{f}} := G_{\widetilde{f}}\otimes G_{\widehat{f}} - \frac{G_{\widetilde{\mathbb{1}}}}{F_{\widetilde{\mathbb{1}}}}F_{\widetilde{f}}\otimes G_{\widehat{f}} - G_{\widetilde{f}}\otimes\frac{G_{\widehat{\mathbb{1}}}}{F_{\widehat{\mathbb{1}}}}F_{\widehat{f}}$$

so that

$$\mathfrak{L}_{\text{equal}}[F_{\widetilde{f}\otimes\widehat{f}}] = \mathfrak{L}[F_{\widetilde{f}\otimes\widehat{f}}] + 2\mathfrak{R}_{\widetilde{f}\otimes\widehat{f}}$$

From the proof of Theorem 3, we have, with $f := \widetilde{f}\otimes\widehat{f}$ and $\mathbb{1} := \widetilde{\mathbb{1}}\otimes\widehat{\mathbb{1}}$,

$$\begin{aligned}F_{\mathbb{1}}\mathfrak{L}_{\text{equal}}[\Lambda[f]] &= \mathfrak{L}_{\text{equal}}[F_f] - \frac{2}{F_{\mathbb{1}}}\Gamma_{\mathfrak{L}_{\text{equal}}}[F_f, F_{\mathbb{1}}] + F_f\left(\frac{2}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}_{\text{equal}}}[F_{\mathbb{1}}, F_{\mathbb{1}}] - \frac{1}{F_{\mathbb{1}}}\mathfrak{L}_{\text{equal}}[F_{\mathbb{1}}]\right)\\ &= \mathfrak{L}_{\text{equal}}[F_f] - \frac{2}{F_{\mathbb{1}}}\Gamma_{\mathfrak{L}}[F_f, F_{\mathbb{1}}] + F_f\left(\frac{2}{F_{\mathbb{1}}^2}\Gamma_{\mathfrak{L}}[F_{\mathbb{1}}, F_{\mathbb{1}}] - \frac{1}{F_{\mathbb{1}}}\mathfrak{L}_{\text{equal}}[F_{\mathbb{1}}]\right)\\ &= F_{\mathbb{1}}\mathfrak{L}[\Lambda[f]] + 2\mathfrak{R}_f - 2\frac{F_f}{F_{\mathbb{1}}}\mathfrak{R}_{\mathbb{1}}\\ &= F_{\mathbb{1}}\Lambda[L[f]] + 2\mathfrak{R}_f - 2\frac{F_f}{F_{\mathbb{1}}}\mathfrak{R}_{\mathbb{1}}\end{aligned}$$

Thus if the generator $\mathfrak{L}_{\text{equal}}$ was to be intertwined with L through Λ, we would have for any $\widetilde{f}\in\mathcal{C}^\infty(\widetilde{V})$ and $\widehat{f}\in\mathcal{C}^\infty(\widehat{V})$,

$$\begin{aligned}\mathfrak{R}_{\widetilde{f}\otimes\widehat{f}} &= \frac{F_{\widetilde{f}\otimes\widehat{f}}}{F_{\mathbb{1}}}\mathfrak{R}_{\mathbb{1}}\\ &= -\frac{F_{\widetilde{f}\otimes\widehat{f}}}{F_{\mathbb{1}}}G_{\widetilde{\mathbb{1}}}\otimes G_{\widehat{\mathbb{1}}}\end{aligned}$$

This equality holds on $\mathcal{G}_{\text{indep}}$, namely for any $\widetilde{D}\in\widetilde{\mathcal{D}}$ and $\widehat{D}\in\widehat{\mathcal{D}}$, we have

$$F_{\widetilde{f}\otimes\widehat{f}}(\widetilde{D}\times\widehat{D}) = -\frac{\widetilde{\mu}(\widetilde{D})\widehat{\mu}(\widehat{D})}{\underline{\widetilde{\mu}}(\partial\widetilde{D})\underline{\widehat{\mu}}(\partial\widehat{D})}\mathfrak{R}_{\widetilde{f}\otimes\widehat{f}}(\widetilde{D}\times\widehat{D})$$

The sets $\widetilde{D}$ and $\widehat{D}$ being fixed, the mapping $\widetilde{f}\otimes\widehat{f}\mapsto\mathfrak{R}_{\widetilde{f}\otimes\widehat{f}}(\widetilde{D}\times\widehat{D})$ corresponds to an integration of $\widetilde{f}\otimes\widehat{f}$ on the boundary of $\widetilde{D}\otimes\widehat{D}$, while $\widetilde{f}\otimes\widehat{f}\mapsto F_{\widetilde{f}\otimes\widehat{f}}(\widetilde{D}\times\widehat{D})$ correspond to an integration of $\widetilde{f}\otimes\widehat{f}$ on the interior of $\widetilde{D}\otimes\widehat{D}$. Thus for any function $\widetilde{f}$ (respectively $\widehat{f}$) whose support is included in the interior of $\widetilde{D}$ (resp. $\widehat{D}$), we get $F_{\widetilde{f}\otimes\widehat{f}}(\widetilde{D}\times\widehat{D})=0$, i.e. $\widetilde{\mu}\otimes\widehat{\mu}$ vanishes on the interior of $\widetilde{D}\otimes\widehat{D}$. Since this is true for any $\widetilde{D}\in\widetilde{\mathcal{D}}$ and $\widehat{D}\in\widehat{\mathcal{D}}$, we would conclude that $\widetilde{\mu}=0$ and $\widehat{\mu}=0$, a contradiction.

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