

PAPER • OPEN ACCESS

On the uniform summability of the Fourier-Laplace series on the sphere

To cite this article: Ahmad Fadly Nurullah bin Rasedee *et al* 2020 *J. Phys.: Conf. Ser.* **1489** 012024

View the [article online](#) for updates and enhancements.



IOP | ebooks™

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection—download the first chapter of every title for free.

On the uniform summability of the Fourier-Laplace series on the sphere

Ahmad Fadly Nurullah bin Rasedee¹, Abdumalik Rakhimov²,
Mohamad Hasan Abdul Sathar³

¹Faculty of Economics and Muamalat, Universiti Sains Islam Malaysia, 71800 Bandar Baru Nilai, Negeri Sembilan, Malaysia.

²Kuliyah of Engineering, International Islamic University Malaysia, 50728 Kuala Lumpur, Malaysia.

³Centre for Agricultural Science Foundation Studies, Universiti Putra Malaysia

E-mail: fadlynurullah@usim.edu.my (corresponding author), abdulalik@iiu.edu.my

Abstract. Convergence problems has been the focus of interest for researchers that are working in the fields of spectral theory. In the current research we investigate issues relating to the summability of the Fourier-Laplace series on the unit sphere. The necessary conditions which are required to obtain good estimation for summability of the Fourier-Laplace series investigated. This research will also provide new and sufficient conditions in the form of theorems and lemmas which will validate the uniform summability of the Fourier-Laplace series on the sphere.

1. Introduction

Let S^N is N dimensional sphere in R^{N+1} :

$$S^N = \{x = (x_1, x_2, \dots, x_{N+1}) \in R^{N+1} : \sum_{n=1}^{N+1} x_n^2 = 1\}$$

For any two point x and y in S^N by $\gamma = \gamma(x, y)$ denote spherical distance between these two points which is radial value of an angle between vectors x and y . It is clear that $\gamma \leq \pi$.

Denote by Δ_s be Laplace-Beltrami operator on S^N which has the following expression in the spherical coordinates $x = (\xi_1, \xi_2, \dots, \xi_{N-1}, \zeta)$ as:

$$\Delta_s = \frac{1}{\sin^{N-1} \xi_1} \frac{\partial}{\partial \xi_1} \left(\sin^{N-1} \xi_1 \frac{\partial}{\partial \xi_1} \right) + \frac{1}{\sin^2 \xi_1 \sin^{N-2} \xi_2} \frac{\partial}{\partial \xi_2} \left(\sin^{N-2} \xi_2 \frac{\partial}{\partial \xi_2} \right) + \dots + \frac{1}{\sin^2 \xi_1 \sin^2 \xi_2 \dots \sin^2 \xi_{N-1}} \frac{\partial^2}{\partial \zeta^2}.$$

Consider this operator as a formal differential operator with domain of definition $C^\infty(S^N)$. It is a symmetric, non negative and essentially selfadjoint. Thus its closure $-\overline{\Delta_s}$ is a selfadjoint operator in $L_2(S^N)$. Its eigenfunctions Y^k are known as spherical harmonics and they are complete and orthogonal system in $L_2(S^N)$.



For the any function $f \in L_2(S^N)$ its Fourier series by spherical harmonics $\{Y_j^k\}_{j=1}^{a_k}$ is called Fourier-Laplace series on sphere:

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \quad (1)$$

where $f_{k,j} = \int_{S^N} f(y) Y_j^k(y) d\sigma(y)$, and $a_k = \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{N!(k-2)!}$ is a frequency of the corresponding eigenvalues $\lambda_k = k(k+N-1)$. Equality (1) should be understood in the sense $L_2(S^N)$.

A partial sum of the series (1) can be written as follows

$$E_n f(x) = \int_{S^N} f(y) \Theta(x, y, n) d\sigma(y),$$

where $\Theta(x, y, n)$ is called a spectral kernel and has a form:

$$\Theta(x, y, n) = \sum_{k=0}^n \sum_{j=1}^{a_k} Y_j^k(x) Y_j^k(y). \quad (2)$$

Equation (1) can be understood in the sense other than L_2 topology. But in this case application of the summation methods of the partial sum (regularization of the partial sum) is required depending on the smoothness of the function. Traditionally summation of the Fourier-Laplace series carried out by the Cesaro means [4]. In [4] Kogbetliantz obtained asymptotic representations of the Cesaro means of the spectral kernel of the Fourier-Laplace series.

In the present paper we consider the Riesz method of summation [1]. The Riesz means of the partial sums of the series (1) also has an integral form:

$$E_n^\alpha f(x) = \int_{S^N} f(y) \Theta^\alpha(x, y, n) dy,$$

where the kernel $\Theta^\alpha(x, y, n)$, is the Riesz means of the spectral kernel (2) and has the following asymptotic formula (see [10]):

if $|\frac{\pi}{2} - \gamma(x, y)| < \frac{\pi n}{2(n+1)}, n \rightarrow \infty$, then

$$\begin{aligned} \Theta^\alpha(x, y, n) = & n^{\frac{N-1}{2}-\alpha} (N-1) \frac{\sin \left[\left(n + \frac{N}{2} + \frac{\alpha}{2} \right) \gamma - \pi \frac{(N-1+2\alpha)}{4} \right]}{(2 \sin \gamma)^{\frac{N-1}{2}} (2 \sin \frac{\gamma}{2})^{1+\alpha}} \\ & + n^{\frac{N-3}{2}-\alpha} \frac{\eta_n(\gamma)}{(\sin \gamma)^{\frac{N+1}{2}} (\sin \frac{\gamma}{2})^{1+\alpha}} + \frac{\varepsilon_n(\gamma)}{(n+1) (\sin \frac{\gamma}{2})^{1+N}} \end{aligned} \quad (3)$$

where $|\eta_n(\gamma)| < C$, $|\varepsilon_n(\gamma)| < C$;

if $0 < \gamma_0 \leq \gamma \leq \pi$

$$|\Theta^\alpha(x, y, n)| \leq C_4 n^{N-1-\alpha} \quad (4)$$

if $0 \leq \gamma \leq \pi$

$$|\Theta^\alpha(x, y, n)| \leq C_5 n^N. \quad (5)$$

In the present paper we will study the problems of the uniform summability by the Riesz means of the Fourier-Laplace series in the Nikolskii spaces $H_p^\alpha(S^N)$ [13]. These questions for the Cesaro means studied by A.K.Pulatov [5].

2. Main theorem

For any domain $\Sigma \subset S^N$ by Σ^* denote its diametrically opposite: $\Sigma^* = \{x^* \in S^N : \text{there exist } x \in \Sigma \text{ such that } \gamma(x, x^*) = \pi. \}$

Theorem 2.1 Let $f \in H_p^a(S^N)$, $p \geq 1$, $a > 0$.

(i) If $pa > N$ and $a + s > \frac{N-1}{2}$, then

$$\lim_{n \rightarrow \infty} E_n^s f(x) = f(x),$$

uniformly on S^N .

(ii) If $pa = N$ and $a + s > \frac{N-1}{2}$, and in addition a function f above is continuous in the domain $\Sigma \subset S^N$, then

$$\lim_{n \rightarrow \infty} E_n^s f(x) = f(x),$$

uniformly on any compact $K \subset \Sigma$.

(iii) If $f(x)$ is vanishing in some domain $\Sigma \subset S^N$ and $a + s > \max\{\frac{N}{p} - 1, \frac{N-1}{2}\}$, then

$$\lim_{n \rightarrow \infty} E_n^s f(x) = 0,$$

uniformly on any compact $K \subset \Sigma$.

(iv) If $f(x)$ is vanishing in some domain $\Sigma \subset S^N$ and also in its diametrically opposite Σ^* and $a + s > \frac{N-1}{2}$, then

$$\lim_{n \rightarrow \infty} E_n^s f(x) = 0,$$

uniformly on any compact $K \subset \Sigma$.

Theorem 2.1 provides sufficient conditions for the uniform convergence of $E_n^s f(x)$ functions from the class $H_p^a(S^N)$. A condition $ap > N$ is precise because in other case there exists an unbounded function in $H_p^a(S^N)$ whose Fourier-Laplace series trivially is divergent and cannot be uniformly summable on S^N . Item (ii) in the theorem shows that in critical case when $ap = N$ additional conditions gives positive answer for the uniform summability problem. This clarification methods developed in [1] and [6]. Inequality $a + s > \max\{\frac{N}{p} - 1, \frac{N-1}{2}\}$, of the item (iii) of the theorem above makes corrections of the corresponding condition in theorem 2 of the paper [11]. Necessity of the conditions in theorem 2.1 is discussed in the the theorem below.

Theorem 2.2 Let $p \geq 1$, $a > 0$.

(i) if $s + \alpha = \frac{N}{p} - 1$, then for any $x_0 \in S^N$ there is a function $f \in H_p^a(S^N)$, that is equal zero in some neighborhood of this point and satisfies the following inequality

$$\overline{\lim}_{n \rightarrow \infty} E_n^s f(x_0) > 0.$$

(ii) if $s + \alpha = \frac{N-1}{2}$, then for any $x_0 \in S^N$ there is a function $f \in C^a(S^N)$, that is equal zero in some neighborhood of this point as well as in some neighborhood of the diametrically opposite point and satisfies the following inequality

$$\overline{\lim}_{n \rightarrow \infty} E_n^s f(x_0) > 0.$$

A condition $s + \alpha = \frac{N}{p} - 1$ in (i) of theorem 2.2 proves preciseness of the condition in item (iii) of the theorem 2.1. A condition $s + \alpha = \frac{N-1}{2}$ proves preciseness of the corresponding conditions in (i),(ii),(iii) and (iv)in the theorem 2.1. Moreover theorem 2.2 also corrects theorem 1 in [12].

3. Preliminaries and proof of the theorem.

The proof of the main theorem based on number of supplementary statements. From the estimations (3), (4) and (5) we obtain

Lemma 1 *Let $n > 1$, $\alpha > -1$ and $1 \leq q \leq \infty$ ($1/q + 1/p = 1$). Uniformly with the respect variable x we have*

$$\|\Theta^\alpha(x, y, n)\|_{L_q(S^N)} \leq C(n^{N/p} + n^{(N-1)/2-\alpha}), \quad q \neq \frac{2N}{N+1+2\alpha} \quad (6)$$

$$\|\Theta^\alpha(x, y, n)\|_{L_q(S^N \cap \{\gamma(x, y) > \gamma_0 > 0\})} \leq C(n^{N/p-1-\alpha} + n^{(N-1)/2-\alpha}), \quad q \neq \frac{2N}{N-1} \quad (7)$$

$$\|\Theta^\alpha(x, y, n)\|_{L_q(S^N \cap \{\pi - \gamma_0 > \gamma(x, y) > \gamma_0 > 0\})} \leq Cn, \quad (8)$$

where the norm in L_q is taken with the respect to the variable y .

Let τ is a positive number. Using equality (1) we can define powers of the selfadjoint operator $1 + \overline{-\Delta_s}$

$$(1 + \overline{-\Delta_s})^\tau f(x) = \sum_{k=0}^{\infty} (1 + \lambda_k)^\tau \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x). \quad (9)$$

Note that operator $(1 + \overline{-\Delta_s})^\tau$ implements isomorphic mapping between spaces $H_p^a(S^N)$ and $H_p^{a+2\tau}(S^N)$ by the modulo $C^\infty(S^N)$ [2].

Let $g(x) = (1 + \overline{-\Delta_s})^\tau f(x)$. Then we can obtain the new representation for $E_n^\alpha f(x)$

$$E_n^\alpha f(x) = \int_{S^N} \Theta_\tau^\alpha(x, y, n) g(y) d\sigma(y), \quad (10)$$

where $\Theta_\tau^\alpha(x, y, n)$ defined as follows

$$\begin{aligned} \Theta_\tau^\alpha(x, y, n) &= (1 + \overline{-\Delta_s})^{-\tau} \Theta^\alpha(x, y, n) \\ &= \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha (1 + \lambda_k)^{-\tau} \sum_{j=1}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y). \end{aligned} \quad (11)$$

Using the Holder inequality we can estimate (10)

$$\begin{aligned} |E_n^\alpha f(x)| &\leq \int_{S^N} |\Theta_\tau^\alpha(x, y, n)| |g(y)| d\sigma(y) \\ &\leq \left(\int_{S^N} |\Theta_\tau^\alpha(x, y, n)|^q d\sigma(y) \right)^{\frac{1}{q}} \|g\|_{L_p}, \quad \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \quad (12)$$

We choose $\tau = \frac{\alpha-\varepsilon}{2}$, $\varepsilon > 0$. The estimation of $(\int_{S^N} |\Theta_\tau^\alpha(x, y, n)|^q d\sigma(y))^{\frac{1}{q}}$ follows from the Lemma 1. Then for the estimation of $\|g\|_{L_p}$ we apply theorem on isomorphism of the Nikolskii spaces [2]

$$\|g\|_{L_p} = \|(1 + \overline{-\Delta_s})^{\frac{\alpha-\varepsilon}{2}} f\|_{L_p} \leq C \|f\|_{H_p^{\alpha-\varepsilon}} \leq C \|f\|_{H_p^\alpha}$$

This conclude the proof of the Theorem 1. Its statements follows from the statements of the Lemma 1 and estimation for $\|g\|_{L_p}$ above.

The proof of the theorem 2 is based on the analogue of the Lemma 3 in [5] for the Riesz means.

4. Discussions

The questions of the uniform summability of the spectral expansions associated with the selfadjoint extensions of elliptic differential operators of arbitrary order in N -dimensional domain studied in [1]. These problems for the Fourier-Laplace series for the Cesaro means studied by Pulatov in [5]. The Fourier-Laplace series in the spaces of singular distributions studied in [3] and [9]. More general expansions for the singular distributions studied in [6], [7] and [8].

5. Conclusion

From the main statements of these paper and paper [5] we observe similar behavior of the Reisz means of the Fourier-Laplace series on the sphere and the Chezaro means similar in terms of uniform convergence.

6. Acknowledgement.

This research has been supported by Ministry of Education (MoE) and Universiti Sains Islam Malaysia (USIM) under the Fundamental Research Grant Scheme (FRGS) project number USIM/FRGS/FEM/055002/51517.

References

- [1] Sh.A. Alimov 1976. On spectral decompositions of functions in H_p^α , Mat. Sb. (N.S.), v. 101(143):1(9), pp.3-20.
- [2] Sh.A. Alimov 1972. Fractional powers of the elliptic operators and isomorphism of the classes of the differential able functions, J. Differential Equations, 8:9, pp.1609-1626.
- [3] Ahmedov, A., Nurullah, A.F., Rakhimov, A. 2013 Localization of fourier-laplace series of distributions, Malaysian Journal of Mathematical Sciences 7(2), pp.315-326.
- [4] E. Kogbetliantz 1924 Recherches sur la sommabilité des series ultrasphériques par la méthode des moyennes arithmétiques, J. Math. Pures Appl., v.9, pp.107-187.
- [5] A.K. Pulatov 1981. On the uniform convergence and localization of arithmetic means of Fourier-Laplace series. J. Soviet Doclads, v. 258:3, pp.554-556.
- [6] A.A. Rakhimov 2000 On the summability of multiple trigonometric fourier series of distributions, Doklady Mathematics 62(2), pp.163-165.
- [7] A.A. Rakhimov 1996. Localization conditions for spectral decompositions related to elliptic operators from the class Ar , J. Mathematical Notes 59(3-4), pp.298-302.
- [8] A.A. Rakhimov 1996. Spectral decompositions of distributions from negative Sobolev classes, Differential Equations 32(7), pp.1011-1013.
- [9] A.A. Rakhimov 2016. On the Uniform Convergence of Fourier Series, Malaysian Journal of Mathematical Sciences, v. 10, pp.55-60.
- [10] A.F.N. Rasedee 2015. Spectral expansions of Laplace-Beltrami operator on unit sphere, University Putra of Malaysia.
- [11] A.F.N Rasedee, Abdumalik Rakhimov, and Anvarjon A. Ahmedov 2017. Uniform convergence of the Fourier-Laplace series, AIP Conference Proceedings 1830, 070006 (2017); doi: 10.1063/1.4980955
- [12] A.F.N Rasedee, Abdumalik Rakhimov, Anvarjon A. Ahmedov, Norizarina Ishak, and Siti Raihana Hamzah 2017 Absence of localization of Fourier-Laplace series, AIP Conference Proceedings 1830, 070034 (2017); doi: 10.1063/1.4980983
- [13] H. Triebel 1978. Interpolation Theory, Function Spaces, Differential Operators, Berlin.