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Dissertation

Kähler Differential Algebras for 0-Dimensional Schemes and Applications

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Kähler Differential Algebras for 0-Dimensional Schemes and Applications

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Dedicated to my parents, my husband and my daughter

Contents

Chapter 1

Introduction

"Kähler's concept of a differential module of a ring
had a great impact on commutative algebra and algebraic geometry"
(Rolf Berndt)
"like the differentials of analysis,
differential modules "linearize" problems,
i.e. reduce questions about algebras (non-linear problems)
to questions of linear algebra"
(Ernst Kunz)

1.1 Motivation and Overview

The description "Kähler differentials" was used in the mathematical literature for the first time in Zariski's note in 1966 [Za]. However, the concept of "the universal module of differentials" was introduced by Kähler, who used differentials to study inseparable field extensions [Ka1], [Ka2]. Some of the many applications of Kähler differentials in algebraic geometry and commutative algebra were contributed by E. Kunz, R. Waldi, L. G. Roberts, and J. Johnson (see [Kun], [KW1], [KW2], [Joh1], [Joh2], [Rob1], [Rob2] and [Rob3]). In [C], Cartier gave a different approach to the universal module of differentials: Let R_o be a ring, and let R/R_o be an algebra. By J we denote the kernel of the canonical multiplication map $\mu : R \otimes_{R_o} R \to R$ given by $r_1 \otimes r_2 \mapsto r_1 r_2$. Then the module of Kähler differential 1-forms of R/R_o is the R-module $\Omega^1_{R/R_o} = J/J^2$. The Kähler differential algebra Ω_{R/R_o} of R/R_o is the exterior algebra of Ω^1_{R/R_o}. Let K be a field of characteristic zero. When R_o is a standard graded K-algebra and R/R_o is a graded algebra, the Kähler differential algebra $\Omega_{R/R_o} = \bigoplus_{m\in\mathbb{N}} \bigwedge^m_R(\Omega^1_{R/R_o})$ is a bigraded R-algebra. The Hilbert function of Ω_{R/R_o}, defined by $\mathrm{HF}_{\Omega_{R/R_o}}(m,i) = \mathrm{HF}_{\Omega^m_{R/R_o}}(i) = \dim_K(\Omega^m_{R/R_o})_i$, is a basic and interesting invariant for many questions about the Kähler differential algebra.

For some special classes of graded algebras R/R_o, one can describe the Hilbert function of the graded R-modules Ω^m_{R/R_o} explicitly. For instance, let F be a product of s linear factors in the standard graded polynomial rings of two variables $S = K[X,Y]$, and let $R = S/\langle F\rangle_S$. By taking the derivative of F, the Hilbert functions of $\Omega^1_{R/K}$ and $\Omega^2_{R/K}$ are all found (see [Rob1, Section 4]). Moreover, in this case we see that the ideal $\langle F\rangle_S$ is the homogeneous vanishing ideal of a set of s distinct K-rational points in the projective line $\mathbb{P}^1$.

In 1999, G. Dominicis and M. Kreuzer generalized this result by giving a concrete formula for the Hilbert function of the module of Kähler differential 1-forms for a reduced 0-dimensional complete intersection $\mathbb{X}$ in the projective n-space $\mathbb{P}^n$. More precisely, they showed that if the homogeneous vanishing ideal $\mathcal{I}_\mathbb{X}$ of $\mathbb{X}$ is generated by n homogeneous polynomials in $S = K[X_0,\dots,X_n]$ of degrees $d_1,\dots,d_n$ and if $R_\mathbb{X} = S/\mathcal{I}_\mathbb{X}$ is the homogeneous coordinate ring of $\mathbb{X}$ then the Hilbert function of $\Omega^1_{R_\mathbb{X}/K}$ satisfies $\mathrm{HF}_{\Omega^1_{R_\mathbb{X}/K}}(i) = (n+1)\,\mathrm{HF}_\mathbb{X}(i-1) - \sum_{j=1}^n \mathrm{HF}_\mathbb{X}(i-d_j)$ for all $i \in \mathbb{Z}$ (see [DK, Proposition 4.3]). The proof of this formula is based on the construction of an exact sequence of graded $R_\mathbb{X}$-modules

$$0 \longrightarrow \mathcal{I}_{2\mathbb{X}}/\mathcal{I}_\mathbb{X}^2 \longrightarrow \mathcal{I}_\mathbb{X}/\mathcal{I}_\mathbb{X}^2 \longrightarrow R_\mathbb{X}^{n+1}(-1) \longrightarrow \Omega^1_{R_\mathbb{X}/K} \longrightarrow 0 \qquad (1.1)$$

(see [DK, Proposition 3.9]). This exact sequence establishes a connection between the module of Kähler differential 1-forms $\Omega^1_{R_\mathbb{X}/K}$ for $\mathbb{X}$ and the Hilbert function of the double point scheme $2\mathbb{X}$ in $\mathbb{P}^n$.

Double point schemes is a particular class of fat point schemes: Let $\mathbb{X} = \{P_1,...,P_s\}$, and let $\wp_i$ be the associate prime ideal of P_i in S. For a sequence of positive integers $m_1,...,m_s$, the scheme $\mathbb{W}$, defined by the saturated ideal $I_\mathbb{W} = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$, is called a fat point scheme in $\mathbb{P}^n$. If $m_1 = \cdots = m_s = \nu$ then $\mathbb{W}$ is called an equimultiple fat point scheme and we denote $\mathbb{W}$ by $\nu\mathbb{X}$. The structure of the Kähler differential algebras for fat point schemes, and in general for 0-dimensional schemes in $\mathbb{P}^n$ has received little attention so far. The aim of this thesis is to investigate Kähler differential algebras and their Hilbert functions for 0-dimensional schemes in $\mathbb{P}^n$.

There are many reasons to study this topic. First, although the Hilbert functions of the homogeneous coordinate rings of 0-dimensional schemes in $\mathbb{P}^n$ provide us information about the geometry of those schemes, we believe that the Hilbert functions of their Kähler differential algebras contain even more information about the geometry of these schemes. Second, the exact sequence (1.1) mentioned above inspires us to find a connection between the Kähler differential algebra of a fat point scheme and other fat point schemes in $\mathbb{P}^n$. This gives us a tool to study fat point schemes via their Kähler differential algebras. Third, the study of the Kähler differential algebras for 0-dimensional schemes in $\mathbb{P}^n$ provides a number of concrete examples of Kähler differential algebras to which computer algebra methods can be applied.

Now we give an overview of the thesis and mention our main contributions. The thesis is divided into five chapters and one appendix. The first chapter is this introduction.

In Chapter 2 we define some basic concepts, introduce notation and recall results that will be used in the rest of the thesis. Most of these results are well known. An original contribution is the following upper bound for the regularity index of a submodule of a free $R_{\mathbb{X}}$-module, where $R_{\mathbb{X}}$ is the homogeneous coordinate ring of a 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n$.

Proposition 2.4.10. *Let $\mathbb{X}$ be a 0-dimensional scheme in $\mathbb{P}^n$, and let $r_{\mathbb{X}}$ be the regularity index of $\mathrm{HF}_{\mathbb{X}}$. Let V be a graded $R_{\mathbb{X}}$-module generated by the set of homogeneous elements $\{v_1, \dots, v_d\}$ for some $d \geq 1$, let $\delta_j = \deg(v_j)$ for $j = 1, \dots, d$, and let $m \geq 1$. Assume that $\delta_1 \leq \cdots \leq \delta_d$. Then the regularity index of $\bigwedge^m_{R_{\mathbb{X}}}(V)$ satisfies: $\mathrm{ri}(\bigwedge^m_{R_{\mathbb{X}}}(V)) = -\infty$ for $m > d$, and for $1 \leq m \leq d$ we have*

$$\mathrm{ri}(\textstyle\bigwedge^m_{R_{\mathbb{X}}}(V)) \leq \max\left\{ r_{\mathbb{X}} + \delta + \delta_d - \delta_{d-m+1},\, \mathrm{ri}(V) + \delta - \delta_{d-m+1} \right\},$$

where $\delta = \delta_{d-m+1} + \cdots + \delta_d$. In particular, if $1 \leq m \leq d$ and $\delta_1 = \cdots = \delta_d = t$ then we have $\mathrm{ri}(\bigwedge^m_{R_{\mathbb{X}}}(V)) \leq \max\{ r_{\mathbb{X}} + mt, \mathrm{ri}(V) + (m-1)t \}$.

This proposition can be applied to get upper bounds for the regularity indices of the module of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/R_o}$, where R_o is either K or $K[x_0]$, which will be presented in the later chapters.

In Chapter 3 we study the Kähler differential algebras of finitely generated graded algebras R/R_o and apply them to investigate 0-dimensional schemes in $\mathbb{P}^n$. One of the main tools for studying the Kähler differential algebra of R/R_o is its Hilbert function, which plays a fundamental role throughout this chapter. Let R_o be a $\mathbb{N}$-graded ring, let S be a standard graded polynomial ring over R_o, let I be a homogeneous ideal of S, and let $R = S/I$. We denote by d_{S/R_o} the universal derivation of the graded algebra S/R_o. The universal property of the module of Kähler differential 1-forms Ω^1_{R/R_o} implies the presentation $\Omega^1_{R/R_o} = \Omega^1_{S/R_o}/\langle d_{S/R_o}I + I\Omega^1_{S/R_o}\rangle_S$. Based on this presentation, we establish an algorithm for computing Ω^1_{R/R_o} and its Hilbert function (see Proposition 3.1.9). Moreover, for $m \geq 1$, we use the universal property of the m-th exterior power to get a presentation of the module of Kähler differential m-forms $\Omega^m_{R/R_o} = \Omega^m_{S/R_o}/(\langle d_{S/R_o}I\rangle_S \wedge_S \Omega^{m-1}_{S/R_o} + I\Omega^m_{S/R_o})$ (see Proposition 3.2.11). Using this presentation, we also write a procedure for the computation of a presentation of Ω^m_{R/R_o} and its Hilbert function (see Proposition 3.2.14).

Next, we look at the Kähler differential algebra of a 0-dimensional scheme $\mathbb{X}$ in $\mathbb{P}^n$. We always assume that $\mathrm{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ and we let x_0 be the image of X_0 in the homogeneous coordinate ring $R_\mathbb{X} = S/\mathcal{I}_\mathbb{X}$ of $\mathbb{X}$. We see that x_0 is a non-zero divisor of $R_\mathbb{X}$ and $R_\mathbb{X}$ is a graded free $K[x_0]$-algebra. Let $\overline{R}_\mathbb{X} = R_\mathbb{X}/\langle x_0 \rangle$, and let $m \geq 1$. Then we have the following relations between the modules of Kähler differential m-forms $\Omega^m_{R_\mathbb{X}/K}, \Omega^m_{R_\mathbb{X}/K[x_0]}$, and $\Omega^m_{\overline{R}_\mathbb{X}/K}$.

Proposition 3.3.3. *Let $m \geq 1$. There is an exact sequence of graded $R_\mathbb{X}$-modules*

$$0 \longrightarrow R_\mathbb{X} dx_0 \wedge_{R_\mathbb{X}} \Omega^{m-1}_{R_\mathbb{X}/K} \longrightarrow \Omega^m_{R_\mathbb{X}/K} \longrightarrow \Omega^m_{R_\mathbb{X}/K[x_0]} \longrightarrow 0.$$

Moreover, the module $\Omega^m_{\overline{R}_\mathbb{X}/K}$ has the presentation $\Omega^m_{\overline{R}_\mathbb{X}/K} = \Omega^m_{R_\mathbb{X}/K[x_0]}/\langle x_0 \rangle \Omega^m_{R_\mathbb{X}/K[x_0]}$.

From this proposition we derive some consequences for the Hilbert functions and the regularity indices of the modules of Kähler differential m-forms for the scheme $\mathbb{X}$. In particular, in some special degrees, we can predict the Hilbert functions of $\Omega^m_{R_\mathbb{X}/K}$ and $\Omega^m_{R_\mathbb{X}/K[x_0]}$.

Proposition 3.3.8. *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, and let $\alpha_\mathbb{X}$ be the initial degree of $\mathcal{I}_\mathbb{X}$, i.e. let $\alpha_\mathbb{X} = \min\{i \in \mathbb{N} \mid (\mathcal{I}_\mathbb{X})_i \neq 0\}$.*

- *(i) For $i < m$, we have $\mathrm{HF}_{\Omega^m_{R_\mathbb{X}/K}}(i) = \mathrm{HF}_{\Omega^m_{R_\mathbb{X}/K[x_0]}}(i) = 0$.*
- *(ii) For $m \leq i < \alpha_\mathbb{X} + m - 1$, we have $\mathrm{HF}_{\Omega^m_{R_\mathbb{X}/K}}(i) = \binom{n+1}{m} \cdot \binom{n+i-m}{n}$ and $\mathrm{HF}_{\Omega^m_{R_\mathbb{X}/K[x_0]}}(i) = \binom{n}{m} \cdot \binom{n+i-m}{n}$.*
- *(iii) The Hilbert polynomials of $\Omega^m_{R_\mathbb{X}/K}$ and $\Omega^m_{R_\mathbb{X}/K[x_0]}$ are constant polynomials.*
- *(iv) Let R_o denote either K or $K[x_0]$. We have $\mathrm{HF}_{\Omega^m_{R_\mathbb{X}/R_o}}(r_\mathbb{X} + m) \geq \mathrm{HF}_{\Omega^m_{R_\mathbb{X}/R_o}}(r_\mathbb{X} + m + 1) \geq \cdots$, and if $\mathrm{ri}(\Omega^m_{R_\mathbb{X}/R_o}) \geq r_\mathbb{X} + m$ then*

$$\mathrm{HF}_{\Omega^m_{R_\mathbb{X}/R_o}}(r_\mathbb{X} + m) > \mathrm{HF}_{\Omega^m_{R_\mathbb{X}/R_o}}(r_\mathbb{X} + m + 1) > \cdots > \mathrm{HF}_{\Omega^m_{R_\mathbb{X}/R_o}}(\mathrm{ri}(\Omega^m_{R_\mathbb{X}/R_o})).$$

Furthermore, by applying the inequaltity $\mathrm{ri}(\Omega^1_{R_\mathbb{X}/K[x_0]}) \leq \max\{\mathrm{ri}(\Omega^1_{R_\mathbb{X}/K}), r_\mathbb{X} + 1\}$ and Proposition 2.4.10, we get the following sharp upper bound for the regularity index of the module of Kähler differential m-forms $\Omega^m_{R_\mathbb{X}/R_o}$ (see Proposition 3.3.11):

$$\mathrm{ri}(\Omega^m_{R_\mathbb{X}/R_o}) \leq \max\{\, r_\mathbb{X} + m, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K}) + m - 1 \,\}.$$

Next we consider the application of the Kähler differential algebras to prove geometric results for a special class of 0-dimensional schemes in $\mathbb{P}^n$. Let $\mathbb{X} = \{P_1, ..., P_s\} \subseteq \mathbb{P}^n$

be a set of s distinct K-rational points in $\mathbb{P}^n$, and let $\varrho_{\mathbb{X}}$ denote the dimension of the linear span of $\mathbb{X}$ plus 1 (see Section 3.4). We show that $\varrho_{\mathbb{X}} = m$ if and only if $\Omega^m_{R_{\mathbb{X}}/K} \neq \langle 0 \rangle$ and $\Omega^{m+1}_{R_{\mathbb{X}}/K} = \langle 0 \rangle$ (see Proposition 3.4.7). Moreover, if $n = 2$ then the Hilbert function of the module of Kähler differential 3-forms reflects some geometrical properties of $\mathbb{X}$ as the next result shows.

Corollary 3.4.20. *Let $s \geq 5$, and let $\mathbb{X} = \{P_1, ..., P_s\} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points.*

(i) If $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 1$, then $\mathbb{X}$ lies on two different lines and no $s-1$ points of $\mathbb{X}$ lie on a line.

(ii) Suppose that $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) = 0$ for $i \neq 3$. If $\Delta\,\mathrm{HF}_{\mathbb{X}}(2) = 1$, then $\mathbb{X}$ contains $s-1$ points on a line. Else, $\mathbb{X}$ lies on a non-singular conic.

In Chapter 4 we examine Kähler differential algebras for fat point schemes in $\mathbb{P}^n$. In the last fifty years, fat point schemes in $\mathbb{P}^n$ have been extensively studied by many authors (see for instance [BGT], [BFL], [Ca], [CTV], [DG], [DSG], [GMT], [GT], [Th1], [Th2], [TT]). However, as far as we know, these papers do not use the Kähler differential algebra to study fat point schemes in $\mathbb{P}^n$. This motivates us to use Kähler differential algebras as a new tool to study fat point schemes. To start the chapter, we collect some facts about a fat point scheme $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$ in Section 4.1. The first main result of this chapter is a generalization of the exact sequence (1.1) which has the following form.

Theorem 4.2.1. *Consider the two fat point schemes $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ and $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ in $\mathbb{P}^n$. Then the sequence of graded $R_{\mathbb{W}}$-modules*

$$0 \longrightarrow \mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{V}} \longrightarrow R_{\mathbb{W}}^{n+1}(-1) \longrightarrow \Omega^1_{R_{\mathbb{W}}/K} \longrightarrow 0$$

is exact.

This theorem shows that one can compute the Hilbert functions of the modules of Kähler differential 1-forms of $R_{\mathbb{W}}/K$ from the Hilbert functions of $\mathbb{W}$ and $\mathbb{V}$. Furthermore, this result can be applied to determine the Hilbert polynomials of these modules and to give a bound for the regularity indices of $\Omega^1_{R_{\mathbb{W}}/K}$ (see Corollary 4.2.3). By the short exact sequence of graded $R_{\mathbb{W}}$-modules

$$0 \longrightarrow R_{\mathbb{W}}dx_0 \hookrightarrow \Omega^1_{R_{\mathbb{W}}/K} \xrightarrow{\beta} \Omega^1_{R_{\mathbb{W}}/K[x_0]} \longrightarrow 0$$

(see Proposition 3.3.1) we get similar properties of the Hilbert function and the Hilbert polynomial of $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$. The Hilbert polynomial of $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ provides a condition for $\mathbb{W}$ to be a reduced scheme (see Corollary 4.2.4). Using [CTV, Theorem 6], we also bound the regularity indices of $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ if the support of $\mathbb{W}$ is in general position (see Corollary 4.2.10).

Next we investigate the modules of Kähler differential m-forms of fat point schemes. It is not easy to determine the Hilbert polynomials of these modules in general. Fortunately, we can give bounds for these invariants (see Proposition 4.3.1). Also, we prove the following sharp upper bounds for the regularity indices of $\Omega^m_{R_{\mathbb{W}}/K}$ and $\Omega^m_{R_{\mathbb{W}}/K[x_0]}$.

Proposition 4.3.4. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points, and let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$, and let $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$.*

(i) For all $1 \le m \le n$ we have

$$\max\{\mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K}), \mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K[x_0]})\} \le \max\{r_{\mathbb{W}} + m, r_{\mathbb{V}} + m - 1\}.$$

(ii) We have $\mathrm{ri}(\Omega^{n+1}_{R_{\mathbb{W}}/K}) \le \max\{r_{\mathbb{W}} + n, r_{\mathbb{V}} + n - 1\}$.

In particular, if $m_1 \le \cdots \le m_s$ and if $\mathbb{X}$ is in general position, then for $1 \le m \le n$ we have

$$\max\{\mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K}), \mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K[x_0]})\} \le \max\{m_s + m_{s-1} + m, \lfloor \tfrac{\sum_{j=1}^s m_j + s + n - 2}{n} \rfloor + m - 1\}$$

and $\mathrm{ri}(\Omega^{n+1}_{R_{\mathbb{W}}/K}) \le \max\{m_s + m_{s-1} + n, \lfloor \tfrac{\sum_{j=1}^s m_j + s + n - 2}{n} \rfloor + n - 1\}$.

When $\mathbb{W} = \nu\mathbb{X}$ is an equimultiple fat point scheme, we get further properties and insights. First of all, we show that the Hilbert polynomial of $\Omega^{n+1}_{R_{\mathbb{W}}/K}$ is determined by $\mathrm{HP}_{\Omega^{n+1}_{R_{\nu\mathbb{X}}/K}}(z) = \mathrm{HP}_{(\nu-1)\mathbb{X}}(z) = s\binom{\nu+n-2}{n}$, where $s = \#\mathbb{X}$ (see Proposition 4.3.11). Second, we establish relations between the module of Kähler differential 2-forms $\Omega^2_{R_{\nu\mathbb{X}}/K}$ and other fat point schemes via the complex of K-vector spaces

$$0 \longrightarrow (\mathcal{I}_{(\nu+1)\mathbb{X}}/\mathcal{I}_{(\nu+2)\mathbb{X}})_i \xrightarrow{\alpha} (\mathcal{I}_{\nu\mathbb{X}}\Omega^1_{S/K}/\mathcal{I}_{(\nu+1)\mathbb{X}}\Omega^1_{S/K})_i$$
$$\xrightarrow{\beta} (\Omega^2_{S/K}/\mathcal{I}_{\nu\mathbb{X}}\Omega^2_{S/K})_i \xrightarrow{\gamma} (\Omega^2_{R_{\nu\mathbb{X}}/K})_i \longrightarrow 0$$

(see Proposition 4.3.14). When $\mathbb{X}$ is a set of s distinct K-rational points in $\mathbb{P}^2$, this complex is exact for $i \gg 0$ (see Corollary 4.3.16). Using these relations, the Hilbert polynomial of $\Omega^2_{R_{\nu\mathbb{X}}/K}$ is $\mathrm{HP}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(z) = \frac{1}{2}(3\nu^2 - \nu - 2)s$ (see Corollary 4.3.17).

If the support $\mathbb{X} \subseteq \mathbb{P}^n$ is a complete intersection, then we have $\mathcal{I}_{\nu\mathbb{X}} = \mathcal{I}^{\nu}_{\mathbb{X}}$ [ZS, Appendix 6, Lemma 5]. Hence, using some results of [BGT], we can explicitly described the Hilbert function and the regularity index of $\Omega^1_{R_{\mathbb{W}}/K}$ (see Proposition 4.4.2 and Corollary 4.4.4). In this case we use techniques similar to the ones introduced in the papers [GMT] and [GT] to prove that the Hilbert function of the module of Kähler differential 1-forms of $\mathbb{Y}_j = \sum_{i\neq j} \nu P_i + (\nu-1)P_j$ is independent of j if $\nu \geq 2$ (see Proposition 4.4.9). This result seems to hold in the reduced case $\nu = 1$, too, but we can only offer a proof for the special case of a complete intersection $\mathbb{X}$ of type $(d,\dots,d)$. By applying the above results, we also provide bounds for the Hilbert function and regularity index of $\Omega^1_{R_{\mathbb{W}}/K}$ when $\mathbb{W}$ is a non-equimultiple fat point scheme in $\mathbb{P}^n$ supported at a complete intersection (see Proposition 4.4.6 and Corollary 4.4.8).

In the final chapter, Chapter 5, we look more closely at the Kähler differential algebras for some special fat point schemes in $\mathbb{P}^n$ where $n = 2$ or $n = 4$. Based on some results of M.V. Catalisano [Ca], we give a concrete description of the Hilbert function of the module of Kähler differential 1-forms of a fat point scheme $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^2$ whose support lies on a non-singular conic (see Proposition 5.1.3). If, in addition, $m_1 = \cdots = m_s = \nu$ then we obtain the following presentation of $\Omega^3_{R_{\mathbb{W}}/K}$.

Proposition 5.1.7. *Let $s \geq 4$, and let $\mathbb{X} = \{P_1, \cdots, P_s\} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points which lie on a non-singular conic $\mathcal{C} = \mathcal{Z}^+(C)$, and let $\nu \geq 1$. Then we have $\Omega^3_{R_{\nu\mathbb{X}}/K} \cong (S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}})(-3)$. In particular, for all $i \in \mathbb{Z}$, we have*

$$\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = \mathrm{HF}_{S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}}(i-3).$$

Moreover, this result can be applied to exhibit the Hilbert functions of $\Omega^3_{R_{\nu\mathbb{X}}/K}$ and $\Omega^2_{R_{\nu\mathbb{X}}/K}$ in terms of degrees of generators of $\mathcal{I}_{\mathbb{X}}$ (or of $\mathcal{I}_{(\nu-1)\mathbb{X}}$) (see Corollary 5.1.8 and Corollary 5.1.9).

In $\mathbb{P}^4$, we prove the so-called Segre bound for the regularity index of a set of s distinct K-rational points by using the method of proof of [Th2] (see Theorem 5.2.8). Furthermore, we show that this bound holds for equimultiple fat point schemes in $\mathbb{P}^4$ under an additional hypothesis as follows.

Theorem 5.2.12. *Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of $s \geq 5$ distinct K-rational points in $\mathbb{P}^4$, let $T_{\mathbb{X},j} = \max\big\{\big\lfloor \frac{1}{j}(\sum_{l=1}^{q} m_{i_l} + j - 2)\big\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a } j\text{-plane}\big\}$ for $j = 1, \dots, 4$, and let $\nu \geq 2$. If $\max\{T_{\mathbb{X},j} \mid 1 \leq j \leq 4\} = T_{\mathbb{X},1}$, then the equimultiple fat point scheme $\nu\mathbb{X} = \nu P_1 + \cdots + \nu P_s$ satisfies*

$$r_{\nu\mathbb{X}} = \max\big\{\nu q - 1 \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a line}\big\}.$$

This theorem also allows us to determine the regularity index of the module of Kähler differential 1-forms and bound the regularity index of the module of Kähler differential m-forms for equimultiple fat point schemes in $\mathbb{P}^4$ under the same additional hypothesis (see Proposition 5.2.14 and Corollary 5.2.15).

Many results in this thesis are illustrated by concrete examples. These examples have been computed by using the computer algebra system ApCoCoA [ApC]. In the appendix we provide the functions which implement the algorithms and procedures for the computation of the modules of Kähler differential m-forms and their Hilbert functions for 0-dimensional schemes in $\mathbb{P}^n$.

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Chapter 2

Preliminaries

In this chapter we collect the definitions, results and techniques that we require for the later chapters. As a consequence, most of the material in this chapter is well known.

The chapter is divided into four sections. The main task of Section 1 is to introduce graded rings, graded modules and exact sequences of graded modules. Also in this section we review some of the standard facts on resolutions of graded modules which we use for computing the degrees of minimal separators of fat point schemes in Chapter 3. In the second section we discuss homogeneous Gröbner Bases and present the Homogeneous Buchberger Algorithm for computing a homogeneous Gröbner basis of a given graded module. In Section 3 we introduce the definition of the exterior algebra and concentrate on some remarkable properties of homomorphisms of exterior algebras which we use later in Chapter 3 and Chapter 4. The last section is about 0-dimensional schemes, a main subject of study in this thesis. Some required results about 0-dimensional schemes are mentioned in this section. We refer to [KR1] and [KR2] as standard text books, in particular for the discussion of graded rings, graded modules and Gröbner bases. We refer to [SS] for studying exterior algebras and [DK] for the notions introduced in the last section.

Throughout this chapter we let K be a field of characteristic zero and R a commutative ring with 1 unless stated otherwise.

2.1 Resolutions of Graded Modules

Definition 2.1.1. Let R be a ring containing K.

(i) The ring R is called a $\mathbb{Z}$**-graded ring** if there exists a family of additive subgroups $\{R_i\}_{i\in\mathbb{Z}}$ such that $R = \oplus_i R_i, R_0 = K$ and $R_i \cdot R_j \subseteq R_{i+j}$. For brevity we call R a **graded ring**.

(ii) A module M over a graded ring R is said to be **graded** if there exists a family of subgroups $\{M_i\}_{i\in\mathbb{Z}}$ of M such that $M = \oplus_i M_i$ and $R_i \cdot M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$.

(iii) An element $u \in M$ is called **homogeneous of degree** i if $u \in M_i$ for some $i \in \mathbb{Z}$. In this case we write $\deg(u) = i$.

Remark 2.1.2. Let R be a graded ring, let M be a graded R-module, and let N be a graded submodule of M, i.e. let N be a submodule of M which is a graded R-module. Then $N_i = M_i \cap N$ for all $i \in \mathbb{Z}$. Furthermore, M/N is a graded R-module where $(M/N)_i = M_i/N_i$ for $i \in \mathbb{Z}$.

Definition 2.1.3. The **Hilbert function** of a finitely generated graded R-module M is defined by $\mathrm{HF}_M(i) = \dim_K M_i$ for $i \in \mathbb{Z}$. Its first difference function $\Delta\,\mathrm{HF}_M : \mathbb{Z} \to \mathbb{Z}$ given by $\Delta\,\mathrm{HF}_M(i) = \mathrm{HF}_M(i) - \mathrm{HF}_M(i-1)$ is called the **Castelnuovo function** of M.

Theorem 2.1.4. *The Hilbert function of a finitely generated graded R-module M of dimension d is of polynomial type of degree $d-1$, i.e. there exists a number $i_0 \in \mathbb{Z}$ and an integer valued polynomial $Q \in \mathbb{Q}[z]$ of degree $d-1$ such that $\mathrm{HF}_M(i) = Q(i)$ for all $i \geq i_0$.*

Proof. See Theorems 5.1.21 and 5.4.15 of [KR2] or Theorem 4.1.3 of [BH]. □

Definition 2.1.5. The unique polynomial in $\mathbb{Q}[z]$, denoted by $\mathrm{HP}_M(z)$, for which $\mathrm{HF}_M(i) = \mathrm{HP}_M(i)$ for $i \gg 0$ is called the **Hilbert polynomial** of M. The minimal number, denoted by $\mathrm{ri}(M)$, such that $\mathrm{HF}_M(i) = \mathrm{HP}_M(i)$ for all $i \geq \mathrm{ri}(M)$ is called the **regularity index** of M. Whenever $\mathrm{HF}_M(i) = \mathrm{HP}_M(i)$ for all $i \in \mathbb{Z}$, we let $\mathrm{ri}(M) = -\infty$.

From now on we denote the polynomial ring $K[X_0, \ldots, X_n]$ by S and we equip S with the standard grading, i.e. let $\deg(X_i) = 1$ for $i = 0, \ldots, n$.. The following example gives us formulas for the Hilbert function, Hilbert polynomial and regularity index of the simplest case $M = S$.

Example 2.1.6. For every $t \in \mathbb{N}$ we have $\mathrm{HF}_S(t) = \binom{n+t}{n}$. The Hilbert polynomial of S is $\mathrm{HP}_S(z) = \binom{z+n}{n}$, and the regularity index is $\mathrm{ri}(S) = -n$.

The next proposition provides useful rules for computing the Hilbert polynomial under ideal-theoretic operations.

Proposition 2.1.7. *Let I, J be proper homogeneous ideal of S.*

(i) We have $\mathrm{HP}_{S/(I\cap J)}(z) = \mathrm{HP}_{S/I}(z) + \mathrm{HP}_{S/J}(z) - \mathrm{HP}_{I+J}(z)$.

(ii) If $\sqrt{J} = (X_0, \ldots, X_n)$ *then* $\mathrm{HP}_{S/(I\cap J)}(z) = \mathrm{HP}_I(z)$.

Proof. See [KR2, Proposition 5.4.16]. □

The remaining part of this section is devoted to providing the reader with the definitions of minimal graded free resolution of a module, the mapping cone of a homomorphism of complexes and the subsequent propositions.

Definition 2.1.8. Let R be a graded ring.

(i) Let $i \in \mathbb{Z}$. A homomorphism of graded R-modules $\alpha : N \to M$ is called a **homogeneous homomorphism of degree i** if $\alpha(N_j) \subseteq M_{j+i}$ for all $j \in \mathbb{Z}$.

(ii) A sequence of graded R-modules

$$\mathcal{F} : \cdots \xrightarrow{\alpha_{t+1}} F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} \ldots,$$

denoted by $(F_\bullet, \alpha_\bullet)$, is called a **complex** if $\mathrm{Im}(\alpha_t) \subseteq \mathrm{Ker}(\alpha_{t-1})$ for all t. If $\mathrm{Im}(\alpha_t) = \mathrm{Ker}(\alpha_{t-1})$ for all t, then the sequence $(F_\bullet, \alpha_\bullet)$ is called an **exact sequence** of graded R-modules.

(iii) A **graded free resolution** of an R-module M is an exact sequence of free R-modules

$$\mathcal{F} : \cdots \xrightarrow{\alpha_{t+1}} F_t \xrightarrow{\alpha_t} F_{t-1} \xrightarrow{\alpha_{t-1}} \cdots \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

such that α_i are homogeneous homomorphisms of degree zero for all $i \geq 0$ and $\mathrm{Coker}(\alpha_1) = M$. If there exists a positive integer $n \in \mathbb{N}$ such that $F_{n+1} = \langle 0 \rangle = F_j$ for $j \leq -2$ but $F_i \neq \langle 0 \rangle$ for $0 \leq i \leq n$, we say that $(F_\bullet, \alpha_\bullet)$ is a **finite graded free resolution of length** n.

(iv) Let M be a finitely generated graded R-module. A graded free resolution

$$\mathcal{F} : \cdots \xrightarrow{\alpha_3} F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

of M is called a **minimal graded free resolution** of M if the images of the canonical basis of vectors of F_i are a minimal system of generators of $\mathrm{Ker}(\alpha_{i-1})$ for every $i \geq 1$.

The following proposition says that the minimal graded free resolution of a given graded S-module M indeed exists and is unique.

Proposition 2.1.9. *Let M be a finitely generated graded S-module. Then the following claims hold true:*

(i) The module M has a minimal graded free resolution of length at most $n+1$.

(ii) Let

$$0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$
$$\cdots \longrightarrow F'_\ell \xrightarrow{\varphi_\ell} \cdots \xrightarrow{\varphi_2} F'_1 \xrightarrow{\varphi_1} F'_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

be two minimal graded free resolutions of M. Then for every $0 \le i \le \ell$, we have $F_i = F'_i$ and $F'_j = 0$ for $j > \ell$.

Proof. See Corollary 4.8.7 and Theorem 4.8.9 of [KR2]. □

The length of the minimal graded free resolution of an S-module M exists and is unique. This constant is called the **projective dimension** of M, and is denoted by $\mathrm{pd}_S(M)$.

Definition 2.1.10. Let R be a graded ring and let M be a graded R-module.

(i) A sequence of homogeneous elements $F_1, \ldots, F_t \in R$ is called a **regular sequence** for M if $\langle F_1, \ldots, F_t \rangle M \neq M$ and for $i = 1, \ldots, t$ we have F_i is a non-zerodivisor for $M/\langle F_1, \ldots, F_{i-1} \rangle M$.

(ii) A graded ring R is a **complete intersection** if there is a graded regular ring $\widetilde{R}$ and a regular sequence of homogeneous elements $F_1, \ldots, F_t \in \widetilde{R}$ such that $R \cong \widetilde{R}/\langle F_1, \ldots, F_t \rangle$. The sequence of degrees $(\deg(F_1), \ldots, \deg(F_t))$ is called the type of R.

We define the depth of a graded S-module M, written by $\mathbf{depth}(M)$, to be the maximal length of regular sequences for M. In order to compute the projective dimension of a graded S-module M, we use the following connection between projective dimension and depth discovered by Auslander and Buchsbaum.

Proposition 2.1.11. (The Auslander-Buchsbaum Formula) *Let M be a graded finitely generated S-module. Its projective dimension is*

$$\mathrm{pd}_S(M) = n + 1 - \mathrm{depth}(M).$$

Proof. See [Pe, 15.3]. □

Definition 2.1.12. Let R be a graded ring, let M be a graded R-module, and let $h \in \mathbb{Z}$. We define $M(h)$ to be graded R-module with

$$M(h)_i = M_{h+i} \text{ for all } i \in \mathbb{Z}.$$

We call $M(h)$ the **h-th twist** of M.

Example 2.1.13. (Koszul Complex) Let $S = K[X_0, \dots, X_n]$ be the polynomial ring and $I = \langle X_1, \dots, X_n \rangle$. Let $\{e_{k_1} \wedge \cdots \wedge e_{k_j}\}_{1 \le k_1 < \cdots < k_j \le n}$ be a basis of the free S-modules $S^{\binom{n}{j}}$. The sequence $X_1, \dots, X_n$ is a regular sequence on S, and the minimal graded free resolution of the residue ring $R = S/I$ is

$$0 \longrightarrow S(-n) \xrightarrow{\varphi_n} S^{\binom{n}{n-1}}(-n+1) \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_2} S^{\binom{n}{1}}(-1) \xrightarrow{\varphi_1} S \xrightarrow{\varphi_0} R \longrightarrow 0,$$

where the module homomorphism $\varphi_j : S^{\binom{n}{j}}(-j) \to S^{\binom{n}{j-1}}(-j+1)$ given by $e_{k_1} \wedge \cdots \wedge e_{k_j} \mapsto \sum_{i=1}^{j}(-1)^{i+1} x_{k_i} e_{k_1} \wedge \cdots \wedge \widehat{e_{k_i}} \wedge \cdots \wedge e_{k_j}$. Thus the projective dimension of the residue ring R is $\mathrm{pd}_S(R) = n$. Therefore the Hilbert function of R in degree i is $\mathrm{HF}_R(i) = \sum_{j=0}^{n}(-1)^j \cdot \binom{n}{j} \cdot \mathrm{HF}_S(i-j) = \sum_{j=0}^{n}(-1)^j \cdot \binom{n}{j} \cdot \binom{n+i-j}{n}$. Moreover, by Example 2.1.6, the regularity index of R is $\mathrm{ri}(R) = 0$.

Example 2.1.13 is an easy case of the following result.

Theorem 2.1.14. (Koszul Resolution) *Let $R = S/I$ be a complete intersection of type $(d_1, \dots, d_n)$. Then the minimal graded free resolution of R has the form*

$$0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_2 \longrightarrow H_1 \longrightarrow S \longrightarrow R \longrightarrow 0,$$

where $H_j = \oplus_{1 \le i_1 < i_2 < \cdots < i_j \le n} S(-d_{i_1} - \cdots - d_{i_j})$ for $j = 1, \dots, n$.

Proof. See [GT, Theorem 1.1]. □

Corollary 2.1.15. *Let S/I and S/J be a complete intersections of types $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$, respectively. Assume that $\mathrm{HF}_{S/I}(t) = \mathrm{HF}_{S/J}(t)$ for all $t \in \mathbb{N}$. Then the residue rings S/I and S/J have the same type, i.e. if $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_n$ then $a_j = b_j$ for $j = 1, \dots, n$.*

Proof. Let

$$0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_2 \longrightarrow H_1 \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

and

$$0 \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow S \longrightarrow S/J \to 0$$

be the Koszul resolutions of S/I and S/J as in Theorem 2.1.14, respectively. We have $\mathrm{HF}_I(k) - \mathrm{HF}_J(k) = \sum_{i=1}^{n}(-1)^{i+1}\,\mathrm{HF}_{H_i}(k) - \sum_{i=1}^{n}(-1)^{i+1}\,\mathrm{HF}_{L_i}(k) = 0$ for all $k \in \mathbb{N}$. By the definition of H_i, L_i for $i = 0, \ldots, n-1$ and the assumption that $a_1 \leq \cdots \leq a_n$ and that $b_1 \leq \cdots \leq b_n$, we get $a_j = b_j$ for $j = 1, \ldots, n$. □

In what follows, we mention a simple, yet powerful method for constructing free resolutions of rings.

Definition 2.1.16. (i) Let $\mathcal{F} = (F_\bullet, \alpha_\bullet)$ and $\mathcal{H} = (H_\bullet, \beta_\bullet)$ be complexes of graded R-modules. The sequence of homomorphisms of degree zero $\varphi_t : F_t \to H_t$ such that the following diagram

$$\begin{array}{ccc} F_t & \xrightarrow{\alpha_t} & F_{t-1} \\ \big\downarrow{\scriptstyle \varphi_t} & & \big\downarrow{\scriptstyle \varphi_{t-1}} \\ H_t & \xrightarrow{\beta_t} & H_{t-1} \end{array}$$

is commutative for $t \in \mathbb{Z}$, is called **homomorphism of complexes**, and is denoted by $\varphi : \mathcal{F} \to \mathcal{H}$.

(ii) Let $\varphi : \mathcal{F} \to \mathcal{H}$ be a homomorphism of complexes. It easy to check that the sequence $\mathcal{L} = (L_\bullet, \gamma_\bullet)$, where $L_i = F_{i-1} \oplus H_i$, where the homomorphism is given by $\gamma_i : L_i \to L_{i-1}, \gamma_i(f,h) = (-\alpha_{i-1}(f), \beta_i(h) + \varphi_{i-1}(f))$ for all $f \in F_{i-1}$, $h \in H_i$ and for $i \in \mathbb{Z}$, is a complex. The complex $\mathcal{L}$ is called the **mapping cone** of φ.

Remark 2.1.17. Let $\gamma : M \to N$ be a homomorphism of graded S-modules. Let $\mathcal{F} : \cdots \overset{\alpha_2}{\to} F_1 \overset{\alpha_1}{\to} F_0 \overset{\alpha_0}{\to} M \to 0$ and $\mathcal{H} : \cdots \overset{\beta_2}{\to} H_1 \overset{\beta_1}{\to} H_0 \overset{\beta_0}{\to} N \to 0$ be free resolutions of graded S-modules M and N, respectively. Since H_0 maps onto N, the composite map $\gamma \circ \alpha_0$ may be lifted to a map $\varphi_0 : F_0 \to H_0$. We have $\beta_0 \circ \varphi_0 \circ \alpha_1(F_1) = \gamma \circ \alpha_0 \circ \alpha_1(F_1) = 0$, therefore $\varphi_0 \circ \alpha_1(F_1) \subseteq \mathrm{Ker}(\beta_0) = \mathrm{Im}(\beta_1)$. So, the map $\varphi_0 \circ \alpha_1$ has a lifting $\varphi_1 : F_1 \to H_1$. Continuing in this way we get the map of complexes $\varphi : \mathcal{F} \to \mathcal{H}$. The homomorphism φ is called a **complex homomorphism lifting** γ.

Proposition 2.1.18. *Let I be a homogeneous ideal of S and let H be a homogeneous polynomial in S. Assume that $I :_S H = \langle H_1, \ldots, H_t \rangle_S$, where the sequence of homogenous polynomials $H_1, \ldots, H_t$ is a regular sequence for S. Let $\mathcal{F}$ be a graded free resolution of S/I, let $\mathcal{H}$ be the minimal graded free resolution of the ideal $\langle H_1, \ldots, H_t \rangle$, and let $\psi : \mathcal{H} \to \mathcal{F}$ be a complex homomorphism lifting $S/(I :_S H) \to S/I$. Then the mapping cone $\mathcal{L}$ of ψ yields a graded free resolution of $S/\langle I, H \rangle$.*

Proof. See [HT, p. 280]. □

We illustrate this proposition by the following example.

Example 2.1.19. In the polynomial ring $S = \mathbb{Q}[X_0, X_1, X_2]$, let I be the ideal generated by the regular sequence (F_1, F_2) where $F_1 = X_1(X_1 - X_0)(X_1 - 2X_0)(X_1 - 3X_0)$ and $F_2 = X_2(X_2 - X_0)(X_2 - 2X_0))$. Let $F = F_1F_2/X_1X_2$. We have $I :_S \langle F \rangle = \langle X_1, X_2 \rangle$. According to Example 2.1.13, the Koszul resolutions of the ideals $J = \langle X_1, X_2 \rangle$ and I are

$$0 \to S(-2) \to S^2(-1) \to J \to 0,$$

$$0 \to S(-7) \to S(-3) \oplus S(-4) \to I \to 0,$$

respectively. Using Proposition 2.1.18 and the fact that $\deg(F) = 5$, we have a graded free resolution of the residue class ring $S/\langle I, F \rangle$ of the form

$$0 \to S(-7) \to S(-6)^2 \oplus S(-7) \to S(-5) \oplus S(-3) \oplus S(-4) \to S \to S/\langle I, F \rangle \to 0.$$

2.2 Introduction to Homogeneous Gröbner Bases

As in the previous section, we let K be a field of characteristic zero. Let $n \geq 0$, and let $S = K[X_0, \ldots, X_n]$ be the polynomial ring in $n + 1$ indeterminates, graded by $\deg(X_i) = 1$ for $i = 0, \ldots, n$ unless stated otherwise.

A polynomial $F \in S$ of the form $X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ such that $(\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ is called a **term**. The set of all terms of S is denoted by $\mathbb{T}^{n+1}$. Then $(\mathbb{T}^{n+1}, \cdot)$ is a monoid. Let $r \geq 1$ and let $\{e_1, \ldots, e_r\}$ be the canonical basis of the free S-module S^r. A **term** of S^r is an element of the form te_i such that $t \in \mathbb{T}^{n+1}$ and $1 \leq i \leq r$. The set of all terms of S^r is denoted by $\mathbb{T}^{n+1}\langle e_1, \ldots, e_r \rangle$. Recall that $\mathbb{T}^{n+1}\langle e_1, \ldots, e_r \rangle$, together with the operation $* : \mathbb{T}^{n+1} \times \mathbb{T}^{n+1}\langle e_1, \ldots, e_r \rangle \to \mathbb{T}^{n+1}\langle e_1, \ldots, e_r \rangle$ given by $(t, v) \mapsto t * v$, is a $\mathbb{T}^{n+1}$-**monomodule** (see [KR1, Definition 1.3.1.c]). Now we recall the notions of term ordering on $\mathbb{T}^{n+1}$ and module term ordering on $\mathbb{T}^{n+1}\langle e_1, \ldots, e_r \rangle$. We refer to [KR1] for more details about these notions.

Definition 2.2.1. A complete relation σ on $\mathbb{T}^{n+1}$ is called a **term ordering** on $\mathbb{T}^{n+1}$ if the following conditions are satisfied for all $t_1, t_2, t_3 \in \mathbb{T}^{n+1}$:

(i) $t_1 \geq_\sigma 1$

(ii) $t_1 \geq_\sigma t_1$

(iii) $t_1 \geq_\sigma t_2$ and $t_2 \geq_\sigma t_1$ imply $t_1 = t_2$

(iv) $t_1 \geq_\sigma t_2$ and $t_2 \geq_\sigma t_3$ imply $t_1 \geq_\sigma t_3$

(v) $t_1 \geq_\sigma t_2$ implies $t_1 t_3 \geq_\sigma t_2 t_3$.

Example 2.2.2. (i) Let $t = X_0^{\alpha_0} \cdots X_n^{\alpha_n} \in \mathbb{T}^{n+1}$ with $(\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$. We write $\log(t) = (\alpha_0, \ldots, \alpha_n)$. The following definition yields a term ordering on $\mathbb{T}^{n+1}$. We call it the **lexicographic term ordering** and is denoted by `Lex`. For two terms $t_1, t_2 \in \mathbb{T}^{n+1}$ we say $t_1 \geq_{\texttt{Lex}} t_2$ if and only if the first non-zero component of $\log(t_1) - \log(t_2)$ is positive or $t_1 = t_2$.

(ii) The following definition yields a term ordering on $\mathbb{T}^{n+1}$. It is called the **degree-lexicographic term ordering** and is denoted by `DegLex`. For $t_1, t_2 \in \mathbb{T}^{n+1}$ we let $t_1 \geq_{\texttt{DegLex}} t_2$ if $\deg(t_1) > \deg(t_2)$, or if $\deg(t_1) = \deg(t_2)$ and $t_1 \geq_{\texttt{Lex}} t_2$.

Definition 2.2.3. A complete relation σ on $\mathbb{T}^{n+1}\langle e_1, \ldots, e_r\rangle$ is called a **module term ordering** if for all $s_1, s_2, s_3 \in \mathbb{T}^{n+1}\langle e_1, \ldots, e_r\rangle$ and all $t \in \mathbb{T}^{n+1}$ we have

(i) $s_1 \geq_\sigma s_1$

(ii) $s_1 \geq_\sigma s_2$ and $s_2 \geq_\sigma s_1$ imply $s_1 = s_2$

(iii) $s_1 \geq_\sigma s_2$ and $s_2 \geq_\sigma s_3$ imply $s_1 \geq_\sigma s_3$

(iv) $s_1 \geq_\sigma s_2$ implies $t * s_1 \geq_\sigma t * s_2$

(v) $t * s_1 \geq_\sigma s_1$.

The existence of module term ordering is showed by the next example.

Example 2.2.4. Given two elements $t_1 e_i, t_2 e_j \in \mathbb{T}^{n+1}\langle e_1, \ldots, e_r\rangle$, where $t_1, t_2 \in \mathbb{T}^{n+1}$, where $i, j \in \{1, \ldots, r\}$. The following definition yields a module term ordering on $\mathbb{T}^{n+1}\langle e_1, \ldots, e_r\rangle$. It is denoted by `DegLexPos`. We let $t_1 e_i \geq_{\texttt{DegLexPos}} t_2 e_j$ if and only if $t_1 >_{\texttt{DegLex}} t_2$ or ($t_1 = t_2$ and $i \leq j$).

Let $r \geq 1$, and let σ be a module term ordering on $\mathbb{T}^{n+1}\langle e_1, \ldots, e_r\rangle$. We see that every element $H \in S^r \setminus \{0\}$ has a unique representation as a linear combination of terms $H = \sum_{i=1}^{s} c_i t_i e_{\gamma_i}$, where $c_1, \ldots, c_s \in K \setminus \{0\}$, $t_1, \ldots, t_s \in \mathbb{T}^n$, $\gamma_1, \ldots, \gamma_s \in \{1, \ldots, r\}$, and where $t_1 e_{\gamma_1} \geq_\sigma t_2 e_{\gamma_2} \geq_\sigma \cdots \geq_\sigma t_s e_{\gamma_s}$.

Definition 2.2.5. Let $H = \sum_{i=1}^{s} c_i t_i e_{\gamma_i} \in S^r \setminus \{0\}$ as above, and let M be an S-submodule of S^r.

(i) The term $\mathrm{LT}_\sigma(H) = t_1 e_{\gamma_1} \in \mathbb{T}^{n+1}\langle e_1, \dots, e_r\rangle$ is called the **leading term** of H with respect to σ.

(ii) The element $\mathrm{LC}_\sigma(H) = c_1 \in K \setminus \{0\}$ is called the **leading coefficient** of H with respect to σ.

(iii) We let $\mathrm{LM}_\sigma(H) = \mathrm{LC}_\sigma(H) \cdot \mathrm{LT}_\sigma(H) = c_1 t_1 e_{\gamma_1}$ and call it the **leading monomial** of H with respect to σ.

(iv) The module $\mathrm{LT}_\sigma(M) = \langle \mathrm{LT}_\sigma(H) \mid H \in M \setminus \{0\}\rangle$ is called the **leading term module** of M with respect to σ.

(v) In particular, if $M \subseteq S$ is an ideal of S, then $\mathrm{LT}_\sigma(M) \subseteq S$ is also called the **leading term ideal** of M with respect to σ.

In order to perform the division on the free S-module S^r, we present the following algorithm.

Theorem 2.2.6. (The Division Algorithm) *Let $t \geq 1$, and let $F, H_1, \dots, H_t \in S^r \setminus \{0\}$. Consider the following sequence of instructions.*

1) Let $Q_1 = \cdots = Q_t = 0, P = 0, L = F$.

2) Find the smallest $i \in \{1, \dots, t\}$ such that $\mathrm{LT}_\sigma(L)$ is a multiple of $\mathrm{LT}_\sigma(H_i)$. If such an i exists, replace Q_i by $Q_i + \frac{\mathrm{LM}_\sigma(L)}{\mathrm{LM}_\sigma(H_i)}$ and L by $L - \frac{\mathrm{LM}_\sigma(L)}{\mathrm{LM}_\sigma(H_i)} H_i$.

3) Repeat step 2) until there is no more $i \in \{1, \dots, s\}$ such that $\mathrm{LT}_\sigma(L)$ is a multiple of $\mathrm{LT}_\sigma(H_i)$. Then replace P by $P + \mathrm{LM}_\sigma(L)$ and L by $\mathrm{LM}_\sigma(L)$.

4) If $L \neq 0$ then start again with the step 2). If $L = 0$ then return the tuple $(Q_1, \dots, Q_t) \subseteq S^t$ and the vector $P \in S^r$.

This is an algorithm which returns vectors $(Q_1, \dots, Q_t) \in S^t$ and $P \in S^r$ such that $F = \sum_{i=1}^t Q_i H_i + P$ and either $P = 0$ or P is a K-linear combination of monomials none of which is divisible by any of $\mathrm{LT}(H_1), \dots, \mathrm{LT}(H_t)$. Furthermore, if $Q_i \neq 0$ for some $i = 1, \dots, t$ then we have $\mathrm{LT}_\sigma(Q_i H_i) \leq_\sigma \mathrm{LT}_\sigma(F)$.

Proof. See [KR1, Theorem 1.6.4]. □

Let $F, H_1, \dots, H_t \in S^r \setminus \{0\}$ and let $\mathcal{H}$ be the tuple $(H_1, \dots, H_t)$. The vector P given in Theorem 2.2.6 is called the **normal remainder** of F with respect to $\mathcal{H}$ and is denoted by $\mathrm{NR}_{\sigma,\mathcal{H}}(F)$. For $F = 0$, we let $\mathrm{NR}_{\sigma,\mathcal{H}}(F) = 0$.

Let us look at the next example to clarify the Division Algorithm.

Example 2.2.7. Let $S = \mathbb{Q}[X_0, \ldots, X_3]$.

(i) Let $F = X_0^2X_1^4X_2 + X_0^2X_3^5$, and let $\mathcal{G}$ be the tuple $\mathcal{G} = (G_1, G_2) \in S^2$, where $G_1 = X_1^4X_2 + X_1^2X_3^3$ and $G_2 = X_0^2X_1 + X_0^2X_3$. We follow the Division Algorithm 2.2.6 to eliminate $\mathrm{LT}_{\mathtt{Lex}}(F)$ step by step and get

$$F = (X_1^3X_2 - X_1^2X_2X_3 + X_1X_2X_3^2 - X_2X_3^3)G_2 + X_0^2X_2X_3^4 + X_0^2X_3^5.$$

Thus $\mathrm{NR}_{\mathtt{Lex},\mathcal{G}}(F) = X_0^2X_2 + X_0^2$. However, by using the term ordering `DegLex`, Theorem 2.2.6 yields that $F = X_0^2G_1 - (X_1X_3^3 - X_3^4)G_2$, and therefore we get $\mathrm{NR}_{\mathtt{DegLex},\mathcal{G}}(F) = 0$.

(ii) Let $\{e_1, e_2, e_3\}$ be the canonical basis of the free S-module S^3, let $G_1 = X_0^2e_1 + X_0X_1e_2 + X_2^2e_3$, $G_2 = X_0X_2^2e_1 + X_0X_2^2e_3$, $G_3 = X_2^5e_3$, $G_4 = X_0X_1X_2^5e_2$, and let $F = X_2^5G_1 + X_0^4G_2 + X_0^2G_3 + 0G_4$. We let $\mathcal{G}_1, \mathcal{G}_2$ be the tuples $\mathcal{G}_1 = (G_1, G_2, G_3, G_4)$ and $\mathcal{G}_2 = (G_3, G_2, G_4, G_1)$, respectively. Let $\sigma =$ `DegLexPos`. Then an application of Theorem 2.2.6 gives us $\mathrm{NR}_{\sigma,\mathcal{G}_1}(F) = -X_0^4X_1X_2^2e_2 + (X_0^5X_2^2 - X_0^3X_2^4)e_3$ and $\mathrm{NR}_{\sigma,\mathcal{G}_2}(F) = 0$.

In the view of Example 2.2.7, the normal remainder of a non-zero vector F with respect to a set of generators $\mathcal{G} = \{G_1, \ldots, G_t\}$ of an S-module $M \subseteq S^r$ depends not only on the choice of a term ordering but also on the order of elements in $\mathcal{G}$. One question is: can one choose a set of generators $\mathcal{G}'$ of M such that the normal remainder of every vector H with respect to $\mathcal{G}'$ is the same. A positive answer is given using the concept of Gröbner basis. In the following we fix a module term ordering σ on $\mathbb{T}^{n+1}\langle e_1, \ldots, e_r\rangle$.

Definition 2.2.8. A system of generators $\mathcal{H} = \{H_1, .., H_t\}$ of an S-module $M \subseteq S^r$ is called a **σ-Gröbner basis** of M if $\langle \mathrm{LT}_\sigma(H_1), \ldots, \mathrm{LT}_\sigma(H_t)\rangle = \mathrm{LT}_\sigma(M)$. In the case that M is a graded S-submodule of $\bigoplus_{i=1}^r S(-\delta_i)$ where $\delta_1, \ldots, \delta_r \in \mathbb{Z}$, a σ-Gröbner basis of M is called a **homogeneous σ-Gröbner basis** of M if it consists of only homogeneous elements.

Proposition 2.2.9. (Existence of a Homogeneous σ-Gröbner Basis) *Let M be a non-zero graded S-submodule of $\bigoplus_{i=1}^r S(-\delta_i)$.*

(i) Given a sequence of homogeneous elements $H_1, \ldots, H_t \in M \setminus \{0\}$ such that $\mathrm{LT}_\sigma(M) = \langle \mathrm{LT}_\sigma(H_1), \ldots, \mathrm{LT}_\sigma(H_t)\rangle$, we have $M = \langle H_1, \ldots, H_t\rangle$, and the set $\mathcal{H} = \{H_1, \ldots, H_t\}$ is a homogeneous σ-Gröbner basis of M.

(ii) The module M has a homogeneous σ-Gröbner basis $\mathcal{H} = \{H_1, \ldots, H_t\} \subseteq M \setminus \{0\}$.

Proof. See [KR1, Proposition 2.4.3] and [KR2, Proposition 4.5.1]. □

In order to compute a homogeneous σ-Göbner basis of a graded S-module $M \subseteq \bigoplus_{i=1}^r S(-\delta_i)$ from a given system of homogeneous generators, we can use the homogeneous version of Buchberger's algorithm which is stated below. To ease the notation, we shall use the following convention. Given a tuple $\mathcal{H} = (H_1, \ldots, H_t)$ of non-zero homogeneous vectors in $\bigoplus_{i=1}^r S(-\delta_i)$, we write $\mathrm{LM}_\sigma(H_i) = c_i t_i e_{\gamma_i}$ where $t_i \in \mathbb{T}^{n+1}$ and $c_i \in K$ and $\gamma_i \in \{1, \ldots, r\}$. For all pairs (i, j) such that $1 \leq i < j \leq t$ and $\gamma_i = \gamma_j$, the S-vector of H_i and H_j is $S_{ij} = \frac{\mathrm{lcm}(t_i,t_j)}{c_i t_i} H_i - \frac{\mathrm{lcm}(t_i,t_j)}{c_j t_j} H_j$. By $\mathbb{S}$ we denote the set of all S-vectors of $\mathcal{H}$.

Theorem 2.2.10. (The Homogeneous Buchberger Algorithm) *Let M be a graded S-submodule of the (standard) graded free S-module $\bigoplus_{i=1}^r S(-\delta_i)$, and let $\mathcal{H} = (H_1, \ldots, H_s)$ be a tuple of non-zero homogeneous vectors which generate M. Suppose that $\deg(H_1) \leq \cdots \leq \deg(H_s)$. Consider the following sequence of instructions.*

1) *Let $\mathbb{S} = \emptyset$, $\mathcal{V} = \mathcal{H}$, $\mathcal{G} = \emptyset$, and $s' = 0$.*

2) *Let d be the smallest degree of an element in $\mathbb{S}$ or in $\mathcal{V}$. Form the subset $\mathbb{S}_d = \{S_{ij} \in \mathbb{S} \mid \deg(S_{ij}) = d\}$ of $\mathbb{S}$, form the subtuple $\mathcal{V}_d = (H \subseteq \mathcal{V} \mid \deg(H) = d)$ of $\mathcal{V}$, and delete their entries from $\mathbb{S}$ and $\mathcal{V}$, respectively.*

3) *If $\mathbb{S}_d = \emptyset$ then continue with step 6). Otherwise, choose an element $F \in \mathbb{S}_d$ and remove it from $\mathbb{S}_d$.*

4) *Compute $F' = \mathrm{NR}_{\sigma,\mathcal{G}}(F)$. If $F' = 0$ continue with step 3).*

5) *Increase s' by one, append $G_{s'} = F'$ to the tuple $\mathcal{G}$, and append the set $\{S_{is'} \mid 1 \leq i < s', \gamma_i = \gamma_{s'}\}$ to the set $\mathbb{S}$. Continue with the step 3).*

6) *If $\mathcal{V}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $V \in \mathcal{V}_d$ and remove it from $\mathcal{V}_d$.*

7) *Compute $V' = \mathrm{NR}_{\sigma,\mathcal{G}}(V)$. If $V' = 0$, continue with step 6).*

8) *Increase s' by one, append $G_{s'} = V'$ to the tuple $\mathcal{G}$ and append the set $\{S_{is'} \mid 1 \leq i < s', \gamma_i = \gamma_{s'}\}$ to the set $\mathbb{S}$. Continue with step 6).*

9) If $\mathbb{S} = \emptyset$ and $\mathcal{V} = \emptyset$, return the tuple $\mathcal{G}$ and stop. Otherwise, continue with step 2).

This is an algorithm which returns a tuple $\mathcal{G} = (G_1, \ldots, G_{s'})$ such that $\deg(G_1) \leq \cdots \leq \deg(G_{s'})$. The elements of $\mathcal{G}$ are a homogeneous σ-Gröbner basis of M.

Proof. See [KR2, Theorem 4.5.5]. $\square$

Example 2.2.11. Let us go back to Example 2.2.7(ii). We want to compute a homogeneous σ-Gröbner basis of the graded S-module M generated by $\{G_1, \ldots, G_4\}$. We let $\sigma = \texttt{DegLexPos}$. By applying Theorem 2.2.10, a homogeneous σ-Gröbner basis of M is given by $\mathcal{G} = (G_1, G_2, \widetilde{G}, G_3, G_4)$, where $\widetilde{G} = X_0 X_1 X_2^2 e_2 + (X_2^4 - X_0^2 X_2^2) e_3$.

2.3 Exterior Algebras

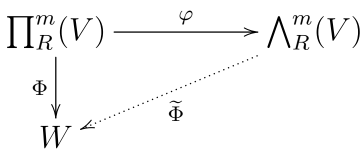
Let R be a ring, and let V be an R-module. For every integer $m \geq 1$, the **m-th tensor power** of V over R is the R-module $\bigotimes_R^m(V) = V \otimes_R V \otimes_R \cdots \otimes_R V$, where there are m factors. Note that $\bigotimes_R^1(V) = V$ and $\bigotimes_R^0(V) = R$. Furthermore, if V is a free R-module with a basis B, then $\bigotimes_R^m(V)$ is also free and it has the elements $b_1 \otimes b_2 \otimes \cdots \otimes b_m$, where $b_i \in B$, as a basis. For more information about tensor powers of a module, we refer to [SS, Section 80-81] and [Nor, Chapter 1].

Now we let $I_R^m(V)$ denote the submodule of $\bigotimes_R^m(V)$ which is generated by all elements of the form $x_1 \otimes \cdots \otimes x_m$ with $x_i = x_j$ for some $i, j \in \{1, \ldots, m\}$ and $i \neq j$, and let $\bigwedge_R^m(V) = \bigotimes_R^m(V)/I_R^m(V)$. The canonical map $\varphi\colon \prod_R^m(V) \to \bigotimes_R^m(V) \twoheadrightarrow \bigwedge_R^m(V)$ is an alternating multilinear mapping, i.e. whenever $(v_1, \ldots, v_m) \in \prod_R^m(V)$ contains a repetition we have $\varphi(v_1, \ldots, v_m) = 0$. For all $v_1, \ldots, v_m \in V$, we denote by $v_1 \wedge \cdots \wedge v_m$ the element $\varphi(v_1, \ldots, v_m)$ and call it the **exterior product** of $v_1, \ldots, v_m$. Then $\bigwedge_R^m(V)$ is an R-module and its elements are finite sums of elements of the form $v_1 \wedge \cdots \wedge v_m$ with $v_1, \ldots, v_m \in V$.

Definition 2.3.1. The R-module $\bigwedge_R^m(V)$ is called the **m-th exterior power** of V over R.

We remark that $I_R^0(V) = I_R^1(V) = \langle 0 \rangle$, and so $\bigwedge_R^0(V) = R$ and $\bigwedge_R^1(V) = V$. If R is a graded ring and V is a graded R-module, then $\bigwedge_R^m(V)$ is also a graded R-module for all $m \geq 0$, where $(\bigwedge_R^m(V))_i = \{\sum v_1 \wedge \cdots \wedge v_m \mid v_k \in V, k = 1, \ldots, m \text{ and } \sum_{j=1}^m \deg(v_j) = i\}$. Moreover, the m-th exterior power of V over R has the following universal property (see [SS, 83.1]).

Proposition 2.3.2. (Universal Property of the m-th Exterior Power) *Let V be an R-module, let $m \in \mathbb{N}$, and let $\varphi : \prod_R^m(V) \to \bigwedge_R^m(V)$ be the canonical alternating multilinear map given by $(v_1, \dots, v_m) \mapsto v_1 \wedge \cdots \wedge v_m$. For any alternating multilinear map $\Phi : \prod_R^m(V) \to W$ of $\prod_R^m(V)$ to an R-module W, there is a unique R-linear map $\widetilde{\Phi} : \bigwedge_R^m(V) \to W$ such that the diagram*



is commutative (i.e. $\widetilde{\Phi} \circ \varphi = \Phi$).

A set of generators of the m-th exterior power of a finitely generated R-module V can be described as follows.

Lemma 2.3.3. *Let V be an R-module generated by $\{v_1, \dots, v_n\}$ for some $n \geq 1$. Then we have $\bigwedge_R^m(V) = \langle 0 \rangle$ for $m > n$. For $1 \leq m \leq n$, the R-module $\bigwedge_R^m(V)$ is generated by the elements $v_{i_1} \wedge \cdots \wedge v_{i_m}$, where $1 \leq i_1 < \cdots < i_m \leq n$.*

Proof. See [Nor, Chapter 5, Section 5.2, Theorem 5]. □

If V is a free R-module, then $\bigwedge_R^m(V)$ is also a free R-module, as the following proposition shows.

Proposition 2.3.4. *Let V be a free R-module with basis $\{e_1, \dots, e_n\}$, and let $1 \leq m \leq n$. Then $\bigwedge_R^m(V)$ is a free R-module with the basis*

$$\{\, e_{i_1} \wedge \cdots \wedge e_{i_m} \mid 1 \leq i_1 < \cdots < i_m \leq n \,\}.$$

In particular, the rank of $\bigwedge_R^m(V)$ is $\binom{n}{m}$.

Proof. See [SS, 83.4]. □

Combine this proposition and [KR2, Proposition 5.1.14], we get the value of the Hilbert function of a free S-module as follows.

Corollary 2.3.5. *Let $S = K[X_0, \dots, X_n]$ be the standard graded polynomial ring over a field K, let $V = \bigoplus_{i=1}^r S(-d_j)$ be a graded free S-module, and let $1 \leq m \leq r$. Then the Hilbert function of $\bigwedge_R^m(V)$ is given by*

$$\mathrm{HF}_{\bigwedge_R^m(V)}(i) = \sum_{1 \leq j_1 < \cdots < j_m \leq r} \mathrm{HF}_S(i - \sum_{k=1}^m d_{j_k}) = \sum_{1 \leq j_1 < \cdots < j_m \leq r} \binom{i - \sum_{k=1}^m d_{j_k} + n}{n}$$

for all $i \in \mathbb{Z}$.

Let $\alpha : V \to W$ be a homomorphism of R-modules, and let $m \geq 0$. We let $\widetilde{\alpha} : \prod_R^m(V) \to \prod_R^m(W)$ be the map given by $\widetilde{\alpha}(v_1, \dots, v_m) = (\alpha(v_1), \dots, \alpha(v_m))$ for all $(v_1, \dots, v_m) \in \prod_R^m(V)$. Then the composition map of $\widetilde{\alpha}$ and the canonical alternating multilinear map $\varphi_W : \prod_R^m(W) \to \bigwedge_R^m(W)$ is an alternating multilinear map. By Proposition 2.3.2, we get a unique R-linear map

$$\bigwedge\nolimits^m(\alpha) : \bigwedge\nolimits_R^m(V) \to \bigwedge\nolimits_R^m(W), \quad v_1 \wedge \cdots \wedge v_m \mapsto \alpha(v_1) \wedge \cdots \wedge \alpha(v_m).$$

Furthermore, this map makes the following diagram commutative

$$\begin{array}{ccc} \prod_R^m(V) & \xrightarrow{\ \widetilde{\alpha}\ } & \prod_R^m(W) \\ \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_W} \\ \bigwedge_R^m(V) & \xrightarrow{\ \bigwedge^m(\alpha)\ } & \bigwedge_R^m(V) \end{array}$$

The map $\bigwedge^m(\alpha)$ is called the **m-th exterior power** of α.

Now we present some connections between the homomorphism $\alpha : V \to W$ and its m-th exterior power.

Lemma 2.3.6. *Using the notation as above, the following statements hold true.*

(i) If α is an isomorphism, then $\bigwedge^m(\alpha)$ is also an isomorphism.

(ii) If α is surjective, then $\bigwedge^m(\alpha)$ is surjective as well.

(iii) If R is a field and α is injective, then $\bigwedge^m(\alpha)$ is injective.

Proof. See [SS, 83.6]. □

In the case that the homomorphism α is surjective, we can describe the kernel of its m-th exterior power explicitly. This result is well known (see Exercise 26 of Chapter X of [SS]). However, for convenience of the reader, we include its proof.

Proposition 2.3.7. *Let $\alpha : V \longrightarrow W$ be an epimorphism of R-modules. We set $\mathcal{G} := \mathrm{Ker}(\alpha)$ and $\mathcal{G} \wedge_R \bigwedge_R^{m-1}(V) := \langle f \wedge g \,|\, f \in \mathcal{G}, g \in \bigwedge_R^{m-1}(V) \rangle_R$. Then, for all $m \in \mathbb{N}$, we have the exact sequence of R-modules*

$$0 \longrightarrow \mathcal{G} \wedge_R \bigwedge\nolimits_R^{m-1}(V) \longrightarrow \bigwedge\nolimits_R^m(V) \overset{\bigwedge^m(\alpha)}{\longrightarrow} \bigwedge\nolimits_R^m(W) \longrightarrow 0.$$

Proof. Notice that $\bigwedge^m(\alpha)$ is surjective, since α is surjective (see Lemma 2.3.6). Now we let $\iota : \mathcal{G} \wedge_R \bigwedge_R^{n-1}(V) \longrightarrow \bigwedge_R^n(V)$ be the inclusion map. Then we have the short exact sequence

$$0 \longrightarrow \mathcal{G} \wedge_R \bigwedge\nolimits_R^{m-1}(V) \longrightarrow \bigwedge\nolimits_R^m(V) \overset{\psi}{\longrightarrow} \mathrm{Coker}(\iota) \longrightarrow 0.$$

Since α is surjective, for every element $(w_1, \ldots, w_m) \in W^m$ there exists an element $(v_1, \ldots, v_m) \in V^m$ such that $\alpha(v_i) = w_i$ for all $i = 1, \ldots, m$. We define a map $\phi : W^m \to \mathrm{Coker}(\iota)$ given by $\phi((w_1, \ldots, w_m)) = v_1 \wedge v_2 \wedge \cdots \wedge v_m + \mathcal{G} \wedge_R \bigwedge_R^{m-1}(V)$.

We first need to prove that the map ϕ is well-defined and is an R-multilinear, alternating homomorphism. Assume that there are elements $(v_1, \ldots, v_m), (u_1, \ldots, u_m) \in V^m$ such that $\alpha(u_i) = \alpha(v_i) = w_i$ for $i = 1, \ldots, m$. This implies that $u_i - v_i \in \mathrm{Ker}(\alpha)$ and $v_1 \wedge v_2 \wedge \cdots \wedge v_m - u_1 \wedge u_2 \wedge \cdots \wedge u_m = (v_1 - u_1) \wedge v_2 \wedge \cdots \wedge v_m + u_1 \wedge (v_2 - u_2) \wedge v_3 \wedge \cdots \wedge v_m + \cdots + u_1 \wedge u_2 \wedge \cdots \wedge u_{m-1} \wedge (v_m - u_m) \in \mathrm{Ker}(\alpha) \wedge \bigwedge_R^{m-1}(V)$. Thus the map ϕ is well-defined. By using the multilinear and alternating properties of $\bigwedge_R^m(V)$, it is not difficult to verify that the map ϕ is an R-multilinear, alternating homomorphism.

Secondly, we show that the sequence

$$0 \longrightarrow \mathcal{G} \wedge_R \textstyle\bigwedge_R^{m-1}(V) \longrightarrow \bigwedge_R^m(V) \longrightarrow \bigwedge_R^m(W) \longrightarrow 0$$

is exact. By the universal property of m-th exterior power, there is a unique homomorphism of R-modules $\varphi : \bigwedge_R^m(W) \to \mathrm{Coker}(\iota)$ such that $\varphi \circ \gamma = \phi$, where the map $\gamma : W^m \to \bigwedge_R^m(W)$ is given by $\gamma((w_1, \ldots, w_m)) = w_1 \wedge \cdots \wedge w_m$. Let $\sum_i r_i v_{1i} \wedge \cdots \wedge v_{mi}$ be an element of $\bigwedge_R^m(V)$. Then we have

$$\begin{aligned}(\varphi \circ \textstyle\bigwedge^m(\alpha))(\sum_i r_i v_{1i} \wedge \cdots \wedge v_{mi}) &= \sum_i \varphi(r_i \alpha(v_{1i}) \wedge \cdots \wedge \alpha(v_{mi})) \\ &= \sum_i r_i v_{1i} \wedge \cdots \wedge v_{mi} + \mathcal{G} \wedge_R \textstyle\bigwedge_R^{m-1}(V) \\ &= \psi(\sum_i r_i v_{1i} \wedge \cdots \wedge v_{mi}).\end{aligned}$$

It follows that $\varphi \circ \bigwedge^m(\alpha) = \psi$. Consequently, we get relations $\mathcal{G} \wedge_R \bigwedge_R^{m-1}(V) = \mathrm{Ker}(\psi) = \mathrm{Ker}(\varphi \circ \bigwedge^m(\alpha)) \supseteq \mathrm{Ker}(\bigwedge^m(\alpha))$. Moreover, we have $\bigwedge^m(\alpha)\big(\mathcal{G} \wedge_R \bigwedge_R^{m-1}(V)\big) = \langle 0 \rangle$ and so $\mathcal{G} \wedge_R \bigwedge_R^{m-1}(V) \subseteq \mathrm{Ker}(\bigwedge^m(\alpha))$. Therefore we obtain $\mathcal{G} \wedge_R \bigwedge_R^{m-1}(V) = \mathrm{Ker}(\bigwedge^m(\alpha))$, and this finishes the proof. □

Remark 2.3.8. In the setting of Proposition 2.3.7, assume that the R-module $\mathcal{G}$ is generated by $H_1, \ldots, H_p$ and the $(m-1)$-th exterior power of V over R is generated by $L_1, \ldots, L_q$. Then the R-module $\mathcal{G} \wedge_R \bigwedge_R^{m-1}(V)$ is generated by the set

$$\{\, H_i \wedge L_j \mid 1 \le i \le p,\ 1 \le j \le q \,\}.$$

Now we denote the direct sum $\bigoplus_{m \in \mathbb{N}} \bigwedge_R^m(V)$ by $\bigwedge_R(V)$ and equip it with the multiplication $\wedge : (\bigoplus_{m \in \mathbb{N}} \bigwedge_R^m(V), \bigoplus_{m \in \mathbb{N}} \bigwedge_R^m(V)) \to \bigoplus_{m \in \mathbb{N}} \bigwedge_R^m(V), (\omega, \nu) \mapsto \omega \wedge \nu$. Then $\bigwedge_R(V)$ is an R-algebra. Also, V is a submodule of $\bigwedge_R(V)$ and it generates $\bigwedge_R(V)$ as an R-algebra.

Definition 2.3.9. Let V be an R-module. The algebra $\bigwedge_R(V)$ is called the **exterior algebra** of V over R.

Note that $v \wedge v = 0$ for all $v \in \bigwedge_R(V)$. The universal property of the exterior algebra $\bigwedge_R(V)$ is given by the following proposition.

Proposition 2.3.10. (Universal Property of the Exterior Algebras) *Let $\bigwedge_R(V)$ be the exterior algebra of an R-module V and let $\phi : V \to W$ be an R-linear mapping of V into an R-algebra W, such that $(\phi(v))^2 = 0$ for all $v \in V$. Then there exists a unique R-algebra homomorphism $\psi : \bigwedge_R(V) \to W$ such that $\phi = \psi \circ \phi_V$, where $\phi_V : V \to \bigwedge_R(V)$ is the canonical injection.*

Proof. See [SS, 85.2] or [Nor, Chapter 5, Section 5.1, Theorem 2]. □

Similar to Lemma 2.3.6, we have basic properties of R-algebra homomorphisms of exterior algebras.

Proposition 2.3.11. *Let $\alpha : V \to W$ be a homomorphism of R-modules. Then there exists a unique R-algebra homomorphism $\bigwedge(\alpha) : \bigwedge_R(V) \to \bigwedge_R(W)$. Moreover, the following claims hold true:*

(i) The homomorphism $\bigwedge(\alpha)$ is the direct sum of the maps $\bigwedge^m(\alpha) : \bigwedge_R^m(V) \to \bigwedge_R^m(W)$.

(ii) If α is an isomorphism then $\bigwedge(\alpha)$ is also an isomorphism.

(iii) If α is surjective then $\bigwedge(\alpha)$ is surjective as well.

(iv) If R is a field and α is injective then so is $\bigwedge(\alpha)$.

Proof. See [SS, 85.5 and 85.7]. □

We end this section with an immediate consequence of Proposition 2.3.7.

Corollary 2.3.12. *Let $\alpha : V \to W$ be an epimorphism of R-modules, and let $\mathcal{G}$ be the kernel of α. Then the sequence*

$$0 \longrightarrow \mathcal{G} \wedge \bigwedge_R(V) \longrightarrow \bigwedge_R(V) \longrightarrow \bigwedge_R(W) \longrightarrow 0$$

is exact.

2.4 Some Properties of 0-Dimensional Schemes

First of all we fix the notation that will be used throughout the section. We work over a field K of characteristic zero. By $\mathbb{P}^n$ we denote the projective n-space over K, where $n \geq 1$. The homogeneous coordinate ring of $\mathbb{P}^n$ is the polynomial ring $S = K[X_0, ..., X_n]$ equipped with its standard grading $\deg(X_0) = \cdots = \deg(X_n) = 1$. If $I \subseteq S$ is a homogeneous ideal defining the scheme $\mathbb{X}$ (i.e. $\mathbb{X} = \mathrm{Proj}(S/I)$) and $\widetilde{I} = \oplus_{i \geq i_0} I_i$ for some $i_0 \geq 0$, then $\widetilde{I}$ also defines $\mathbb{X}$. We refer to [Har, Chapter II] for more details about the theory of schemes.

Definition 2.4.1. Let I be a homogeneous ideal of S. The set

$$I^{\mathrm{sat}} := \{ F \in S \mid \langle X_0, \ldots, X_n \rangle^i F \subseteq I \text{ for some } i \in \mathbb{N} \}$$

is a homogeneous ideal of S and is called the **saturation** of I. The ideal I is called a **saturated ideal** if $I = I^{\mathrm{sat}}$.

Example 2.4.2. Let s be a positive integer, and let $m_1, \ldots, m_s \in \mathbb{N}$. For $j = 1, \ldots, s$, we let $I_j = \langle X_1 - a_{j1}X_0, \ldots, X_n - a_{jn}X_0 \rangle^{m_j} \subseteq S$ for some $a_{j1}, \ldots, a_{jn} \in K$, and let $I = \cap_{j=1}^s I_j$. Then the ideal I is a saturated ideal of S. Indeed, it is clear that $I \subseteq I^{\mathrm{sat}}$. Moreover, for a homogeneous element $F \in I^{\mathrm{sat}}$, there is $i \in \mathbb{N}$ such that $X_0^i F \in I$. Note that $X_0 \notin I_j$ for all $j = 1, \ldots, s$. So, the image x_0 of X_0 in S/I is a non-zerodivisor. This implies $F \in I$. Therefore we get $I = I^{\mathrm{sat}}$, as desired.

Notice that the saturation of a given homogeneous ideal $I \subseteq S$ is finitely generated, since S is Noetherian. It follows from the definition of I^{sat} that $\mathrm{HF}_{I^{\mathrm{sat}}}(i) = \mathrm{HF}_I(i)$ for $i \gg 0$, and hence I and I^{sat} defines the same subscheme of $\mathbb{P}^n$. Furthermore, two homogeneous ideals I and J of S define the same subscheme of $\mathbb{P}^n$ if and only if $I^{\mathrm{sat}} = J^{\mathrm{sat}}$ (see [Per, Proposition 1.3]). Thus if $\mathbb{X} \subseteq \mathbb{P}^n$ is a scheme defined by a homogeneous ideal I of S, the saturation I^{sat} is the largest homogeneous ideal of S which defines $\mathbb{X}$.

Definition 2.4.3. Let $\mathbb{X}$ be a scheme of $\mathbb{P}^n$.

(i) The homogeneous saturated ideal of S defining $\mathbb{X}$ is called the **homogeneous vanishing ideal** of $\mathbb{X}$ and is denoted by $\mathcal{I}_{\mathbb{X}}$.

(ii) The residue class ring $R_{\mathbb{X}} = S/\mathcal{I}_{\mathbb{X}}$ is called the **homogeneous coordinate ring** of $\mathbb{X}$. Its homogeneous maximal ideal is denoted by $\mathfrak{m}_{\mathbb{X}}$.

(iii) The coefficient $\dim(S/I_{\mathbb{X}}) - 1$, denoted by $\dim(\mathbb{X})$, is called the **dimension** of $\mathbb{X}$.

Now we turn our attention to 0-dimensional scheme $\mathbb{X}$ of $\mathbb{P}^n$. Given a 0-dimensional scheme $\mathbb{X}$, it is known that the ideal $\mathcal{I}_{\mathbb{X}}$ can be decomposed as $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^{s} \mathfrak{q}_j$, where each $\mathfrak{q}_j$ is a homogeneous primary ideal (see for instance [KR2, Proposition 5.6.21]). Let $\mathfrak{P}_j = \sqrt{\mathfrak{q}_j}$ be the corresponding radical ideal of $\mathfrak{q}_j$. Then $\mathfrak{P}_i$ is a homogeneous prime ideal and the only homogeneous prime ideal which contains $\mathfrak{P}_j$ is $\mathfrak{M} = \bigoplus_{i\geq 1} S_i$. Notice that $\mathfrak{P}_j$ is the homogeneous ideal of the standard graded ring S so, the number of elements of minimal sets of generators of $\mathfrak{P}_j$ is unique. Let $\{L_{j1}, \dots, L_{jt_j}, H_{j1}, \dots, H_{jm_j}\}$ be a set of minimal generators of the ideal $\mathfrak{P}_j$, where $\deg(L_{jk}) = 1$ for $k = 1, \dots, t_j$, where $\deg(H_{jl}) \geq 2$ for $l = 1, \dots, m_j$. Since $\mathfrak{P}_j \subsetneq \mathfrak{M}$ and the sequence of linear forms $L_{j1}, \dots, L_{jt_j}$ is a regular sequence for S, we can find linear forms $L_{j\,t_j+1}, \dots, L_{jn}$ such that $L_{j1}, \dots, L_{jn}$ is a regular sequence for S. By Q_i we denote the point corresponding to the homogeneous prime ideal $\wp_j = \langle L_{j1}, \dots, L_{jn}\rangle_S$. Due to [KR2, Lemma 6.3.20] and the assumption that the field K is infinite, there exist a linear form L such that $L(Q_j) \neq 0$ for all $j = 1, \dots, s$. This means that $L \notin \wp_j$ for all $j = 1, \dots, s$. By the Prime Avoidance Theorem (cf. [KR2, Proposition 5.6.22]), we have $L \notin \bigcup_{j=1}^{s} \wp_j$. This implies $L \notin \bigcup_{j=1}^{s} \langle L_{j1}, \dots, L_{jt_j}\rangle_S$, and consequently $L \notin \bigcup_{j=1}^{s} \mathfrak{P}_j$. Thus [KR2, Proposition 5.6.17] yields that L is a non-zerodivisor for $R_{\mathbb{X}}$.

Let $\operatorname{Supp}(\mathbb{X}) = \{P_1, \dots, P_s\}$ be the set of closed points of $\mathbb{P}^n$ in $\mathbb{X}$. Note that $\mathfrak{P}_j$ is the homogeneous prime ideal corresponding to P_j for $j = 1, \dots, s$. The local ring $\mathcal{O}_{\mathbb{X},P_j}$ is then the homogeneous localization of R at the image of $\mathfrak{P}_j$ in R. The **degree** of $\mathbb{X}$ is given by $\deg(\mathbb{X}) = \sum_{j=1}^{s} \dim_K \mathcal{O}_{\mathbb{X},P_j}$. The above argument allows us to make the following assumption.

Assumption 2.4.4. From now on, the coordinates $\{X_0, \dots, X_n\}$ of $\mathbb{P}^n$ are always chosen such that no point of $\operatorname{Supp}(\mathbb{X})$ lies on the hyperplane $\mathcal{Z}^+(X_0)$.

The image of X_i in $R_{\mathbb{X}}$ is denoted by x_i for $i = 0, \dots, n$. By the choice of the coordinates, x_0 is a non-zerodivisor for $R_{\mathbb{X}}$ and $K[x_0] \cong K[X_0]$ is a subring of $R_{\mathbb{X}}$. Thus $R_{\mathbb{X}}$ is a 1-dimensional Cohen-Macaulay ring and $\overline{R}_{\mathbb{X}} = R_{\mathbb{X}}/\langle x_0\rangle$ is a 0-dimensional local ring. Moreover, it follows from Proposition 2.1.11 that $\operatorname{pd}_S(R_{\mathbb{X}}) = n$.

The Hilbert function of $R_{\mathbb{X}}$ is denoted by $\mathrm{HF}_{\mathbb{X}} : \mathbb{Z} \to \mathbb{N}$ $(i \mapsto \dim_K(R_{\mathbb{X}})_i)$. The Hilbert polynomial of $R_{\mathbb{X}}$ is $\mathrm{HP}_{\mathbb{X}}(z) = \deg(\mathbb{X})$. By $r_{\mathbb{X}}$ we denote the regularity index of $\mathrm{HF}_{\mathbb{X}}$. The following proposition collects some elementary properties of the Hilbert function of 0-dimensional schemes which we use in the next chapters.

Proposition 2.4.5. *Let $\mathbb{X}$ be a 0-dimensional scheme.*

(i) We have $\mathrm{HF}_{\mathbb{X}}(i) = 0$ for $i < 0$ and $\mathrm{HF}_{\mathbb{X}}(0) = 1$.

(ii) *There is an integer* $\alpha_{\mathbb{X}} \geq 1$ *such that* $\mathrm{HF}_{\mathbb{X}}(i) = \binom{n+i}{n}$ *if and only if* $i < \alpha_{\mathbb{X}}$.

(iii) *We have* $1 = \mathrm{HF}_{\mathbb{X}}(0) < \mathrm{HF}_{\mathbb{X}}(1) < \cdots < \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) < \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = \deg(\mathbb{X})$ *and* $\mathrm{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X})$ *for all* $i \geq r_{\mathbb{X}}$.

Proof. See [Kre2, Lemma 1.3] and [DK, p. 155]. □

The number $\alpha_{\mathbb{X}}$ given in Proposition 2.4.5 is called the **initial degree** of $\mathcal{I}_{\mathbb{X}}$. This can be described as $\alpha_{\mathbb{X}} = \min\{i \in \mathbb{N} \mid (\mathcal{I}_{\mathbb{X}})_i \neq 0\}$. The degree of an element F of a minimal homogeneous system of generators of $\mathcal{I}_{\mathbb{X}}$ is bounded by $\alpha_{\mathbb{X}} \leq \deg(F) \leq r_{\mathbb{X}}+1$, as the following proposition shows (cf. [GM, Proposition 1.1]).

Proposition 2.4.6. *Let* $\mathbb{X}$ *be a* 0*-dimensional subscheme of* $\mathbb{P}^n$. *Then the homogeneous vanishing ideal of* $\mathbb{X}$ *satisfies* $\mathcal{I}_{\mathbb{X}} = \langle (\mathcal{I}_{\mathbb{X}})_{\alpha_{\mathbb{X}}}, (\mathcal{I}_{\mathbb{X}})_{\alpha_{\mathbb{X}}+1}, \dots, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}+1} \rangle_S$.

Below we proceed to give a bound for the regularity index of the m-th exterior power of a finitely generated $R_{\mathbb{X}}$-module V, where $m \geq 1$. For this, we need the following result.

Lemma 2.4.7. *Let* $d \geq 1$, *let* $\delta_1, \dots, \delta_d \in \mathbb{Z}$, *and let* V *be a non-trivial graded submodule of the graded free* $R_{\mathbb{X}}$*-module* $\bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j)$. *Then* x_0 *is not a zerodivisor for* V, *i.e. if* $x_0 \cdot v = 0$ *for some* $v \in V$ *then* $v = 0$.

Proof. Let $\{e_1, \dots, e_d\}$ be the canonical $R_{\mathbb{X}}$-basis of $\bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j)$, and let $i \in \mathbb{Z}$. Then every homogeneous element $v \in V_i$ has a representation $v = g_1e_1 + \cdots + g_de_d$ for some homogeneous elements $g_1, \dots, g_d \in R_{\mathbb{X}}$, where $\deg(g_j) = \deg(v) - \delta_j$ for $j = 1, \dots, d$. Suppose that $x_0 \cdot v = 0$. This implies that $x_0g_1e_1 + \cdots + x_0g_de_d = 0$, and so $x_0g_1 = \cdots = x_0g_d = 0$ in $R_{\mathbb{X}}$. Since x_0 is a non-zerodivisor for $R_{\mathbb{X}}$, we have $g_1 = \cdots = g_d = 0$, and hence $v = 0$. Thus the claim follows. □

Proposition 2.4.8. *Let* $d \geq 1$, *let* $\delta_1, \dots, \delta_d \in \mathbb{Z}$ *such that* $\delta_1 \leq \cdots \leq \delta_d$, *let* $W = \bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j)$ *be the graded free* $R_{\mathbb{X}}$*-module, and let* V *be a non-trivial graded submodule of* W. *Then, for* $1 \leq m \leq d$, *we have*

$$\mathrm{ri}(V \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^m_{R_{\mathbb{X}}}(W)) \leq \mathrm{ri}(V) + \delta_{d-m+1} + \cdots + \delta_d.$$

Proof. First we note that the Hilbert polynomial of W is $\mathrm{HP}_W(z) = d \cdot \deg(\mathbb{X})$ and that $\mathrm{ri}(W) = r_{\mathbb{X}} + \delta_d$. This shows that the Hilbert polynomial of V is a constant polynomial $\mathrm{HP}_V(z) = u \leq d \cdot \deg(\mathbb{X})$. Let $r = \mathrm{ri}(V)$, and let $v_1, \dots, v_u$ be a K-basis of V_r. By Lemma 2.4.7, the elements $\{x_0^iv_1, \dots, x_0^iv_u\}$ form a K-basis of the K-vector space V_{r+i} for all $i \in \mathbb{N}$. We let $\{e_1, \dots, e_d\}$ be the canonical $R_{\mathbb{X}}$-basis of W, we let $t = \binom{d}{m}$, and we let $\{\varepsilon_1, \dots, \varepsilon_t\}$ be a basis of the graded free $R_{\mathbb{X}}$-module $\bigwedge^m_{R_{\mathbb{X}}}(W)$ w.r.t. $\{e_1, \dots, e_d\}$ (see Proposition 2.3.4). We set $\delta = \delta_{d-m+1} + \cdots + \delta_d$, and let

$$N = \langle x_0^{\delta - \deg(\varepsilon_k)} v_j \wedge \varepsilon_k \in V \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^m_{R_{\mathbb{X}}}(W) \mid 1 \le j \le u, 1 \le k \le t \rangle_K.$$

Let $\varrho = \dim_K N$, and let $w_1, \dots, w_\varrho$ be a K-basis of N. It is not difficult to verify that $N = \langle w_1, \dots, w_\varrho \rangle_K = (V \wedge_{R_{\mathbb{X}}} \bigwedge^m_{R_{\mathbb{X}}}(W))_{\delta+r}$. Moreover, for any $i \ge 0$, the set $\{x_0^i w_1, \dots, x_0^i w_\varrho\}$ is K-linearly independent. Indeed, assume that there are elements $a_1, \dots, a_\varrho \in K$ such that $\sum_{j=1}^{\varrho} x_0^i a_j w_j = 0$. Since x_0 is a non-zerodivisor for $V \wedge_{R_{\mathbb{X}}} \bigwedge^m_{R_{\mathbb{X}}}(W)$ by Lemma 2.4.7, we get $\sum_{j=1}^{\varrho} a_j w_j = 0$, and hence $a_1 = \cdots = a_\varrho = 0$.

Now it is sufficient to prove that the set $\{x_0^i w_1, \dots, x_0^i w_\varrho\}$ generates the K-vector space $(V \wedge_{R_{\mathbb{X}}} \bigwedge^m_{R_{\mathbb{X}}}(W))_{\delta+r+i}$ for all $i \ge 0$. Let $w \in (V \wedge_{R_{\mathbb{X}}} \bigwedge^m_{R_{\mathbb{X}}}(W))_{\delta+r+i}$ be a non-zero homogeneous element. Then there are homogeneous elements $\widetilde{v}_j \in V$, $h_k \in R_{\mathbb{X}}$ such that $w = \sum_{j,k} \widetilde{v}_j \wedge h_k \varepsilon_k = \sum_{j,k} h_k \widetilde{v}_j \wedge \varepsilon_k$, where $\deg(\widetilde{v}_j) + \deg(h_k) = \delta + r + i - \deg(\varepsilon_k)$ for all j,k. Note that $\deg(h_k \widetilde{v}_j) = \delta + r + i - \deg(\varepsilon_k) \ge r + i$ for all $i \ge 0$. Also, we have

$$h_k \widetilde{v}_j \in V_{\delta+r+i-\deg(\varepsilon_k)} = \langle x_0^{\delta+i-\deg(\varepsilon_k)} v_1, \dots, x_0^{\delta+i-\deg(\varepsilon_k)} v_u \rangle_K.$$

Thus there are $b_{jk1}, \dots, b_{jku} \in K$ such that $h_k \widetilde{v}_j = \sum_{l=1}^{u} b_{jkl} x_0^{\delta+i-\deg(\varepsilon_k)} v_l$. Hence we have

$$\begin{aligned} w &= \sum_{j,k} h_k \widetilde{v}_j \wedge \varepsilon_k = \sum_{j,k} \sum_{l=1}^{u} b_{jkl} x_0^{\delta+i-\deg(\varepsilon_k)} v_l \wedge \varepsilon_k \\ &= x_0^i \sum_{j,k} \sum_{l=1}^{u} b_{jkl} x_0^{\delta-\deg(\varepsilon_k)} v_l \wedge \varepsilon_k \\ &= \sum_{j,k} \sum_{l=1}^{u} \sum_{q=1}^{\varrho} b_{jkl} c_{jklq} x_0^i w_q \end{aligned}$$

with $c_{jklq} \in K$. The last equality follows from the fact that $\{x_0^i w_1, \dots, x_0^i w_\varrho\}$ is a K-basis of $x_0^i N$. Thus we get $w \in \langle x_0^i w_1, \dots, x_0^i w_\varrho \rangle_K$, and $\mathrm{HF}_{V \wedge_{R_{\mathbb{X}}} \bigwedge^m_{R_{\mathbb{X}}}(W)}(i) = \varrho$ for all $i \ge \delta + r$. Therefore we obtain $\mathrm{ri}(V \wedge_{R_{\mathbb{X}}} \bigwedge^m_{R_{\mathbb{X}}}(W)) \le \mathrm{ri}(V) + \delta$, as we wanted to show. □

Using Proposition 2.4.8, we get immediately the following corollary.

Corollary 2.4.9. *In the setting of Proposition 2.4.8, assume that* $\delta_1 = \cdots = \delta_d = 0$. *Then we have* $\mathrm{ri}(V \wedge_{R_{\mathbb{X}}} \bigwedge^m_{R_{\mathbb{X}}}(W)) \le \mathrm{ri}(V)$ *for* $1 \le m \le d$.

At this point we can bound the regularity index of the m-th exterior power of a finitely generated graded $R_{\mathbb{X}}$-module V as follows.

Proposition 2.4.10. *Let V be a graded $R_{\mathbb{X}}$-module generated by the set of homogeneous elements $\{v_1,\dots,v_d\}$ for some $d\geq 1$, let $\delta_j=\deg(v_j)$ for $j=1,\dots,d$, and let $m\geq 1$. Assume that $\delta_1\leq\cdots\leq\delta_d$, and set $\delta=\delta_{d-m+1}+\cdots+\delta_d$ if $m\leq d$. Then the regularity index of $\bigwedge^m_{R_{\mathbb{X}}}(V)$ satisfies $\mathrm{ri}(\bigwedge^m_{R_{\mathbb{X}}}(V))=-\infty$ if $m>d$ and*

$$\mathrm{ri}(\textstyle\bigwedge^m_{R_{\mathbb{X}}}(V))\leq\max\left\{\, r_{\mathbb{X}}+\delta+\delta_d-\delta_{d-m+1},\ \mathrm{ri}(V)+\delta-\delta_{d-m+1}\,\right\}$$

if $1\leq m\leq d$. In particular, if $1\leq m\leq d$ and $\delta_1=\cdots=\delta_d=t$ then we have $\mathrm{ri}(\bigwedge^m_{R_{\mathbb{X}}}(V))\leq\max\{\,r_{\mathbb{X}}+mt,\mathrm{ri}(V)+(m-1)t\,\}$.

Proof. According to Lemma 2.3.3, we have $\bigwedge^m_{R_{\mathbb{X}}}(V)=\langle 0\rangle$ if $m>d$. Hence we see that $\mathrm{ri}(\bigwedge^m_{R_{\mathbb{X}}}(V))=-\infty$ if $m>d$. Now we assume that $1\leq m\leq d$. It is easy to see that the $R_{\mathbb{X}}$-linear map $\alpha: W=\bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j)\to V$ given by $e_j\mapsto v_j$ is a homogeneous $R_{\mathbb{X}}$-epimorphism of degree zero. We let $\mathcal{G}=\mathrm{Ker}(\alpha)$. By Proposition 2.3.7, we get the short exact sequence of graded $R_{\mathbb{X}}$-modules

$$0\longrightarrow\mathcal{G}\wedge_{R_{\mathbb{X}}}\textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(W)\longrightarrow\bigwedge^m_{R_{\mathbb{X}}}(W)\overset{\bigwedge^m(\alpha)}{\longrightarrow}\bigwedge^m_{R_{\mathbb{X}}}(V)\longrightarrow 0.$$

Thus an application of Proposition 2.4.8 implies that

$$\begin{aligned}\mathrm{ri}(\textstyle\bigwedge^m_{R_{\mathbb{X}}}(V)) &\leq\max\left\{\,\mathrm{ri}(\textstyle\bigwedge^m_{R_{\mathbb{X}}}(W)),\ \mathrm{ri}(\mathcal{G}\wedge_{R_{\mathbb{X}}}\textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(W))\,\right\}\\ &\leq\max\left\{\,r_{\mathbb{X}}+\delta,\ \mathrm{ri}(\mathcal{G})+\delta-\delta_{d-m+1}\,\right\}\\ &\leq\max\left\{\,r_{\mathbb{X}}+\delta+\delta_d-\delta_{d-m+1},\ \mathrm{ri}(V)+\delta-\delta_{d-m+1}\,\right\}.\end{aligned}$$

Here the last inequality follows from the fact that $\mathrm{ri}(\mathcal{G})\leq\max\{r_{\mathbb{X}}+\delta_d,\mathrm{ri}(V)\}$. □

In the remainder of this section, we consider a special class of 0-dimensional subschemes of $\mathbb{P}^n$, namely finite sets of K-rational points in $\mathbb{P}^n$. Let $\mathbb{X}=\{P_1,\dots,P_s\}$ be a set of s-distinct K-rational points in $\mathbb{P}^n$. Note that we always assume $P_j\notin\mathcal{Z}^+(X_0)$ for all $j=1,\dots,s$. This allows us to write $P_j=(1:p_{j1}:\cdots:p_{jn})$ with $p_{j1},\dots,p_{jn}\in K$ for $j=1,\dots,s$. We let $\wp_j$ be the associated homogeneous prime ideal of the point P_j. Then the homogeneous vanishing ideal of $\mathbb{X}$ is given by $\mathcal{I}_{\mathbb{X}}=\wp_1\cap\cdots\cap\wp_s\subseteq S$. For $f\in R_{\mathbb{X}}$ and $j\in\{1,\dots,s\}$, we write $f(P_j):=F(1,p_{j1},\dots,p_{jn})$, where F is any representative of f in S. The element $f\in(R_{\mathbb{X}})_{r_{\mathbb{X}}}$ is called a **separator** of $\mathbb{X}\setminus\{P_j\}$ in $\mathbb{X}$ if $f(P_j)\neq 0$ and $f(P_k)=0$ for all $k\neq j$. A separator $f_j\in(R_{\mathbb{X}})_{r_{\mathbb{X}}}$ of $\mathbb{X}\setminus\{P_j\}$ in $\mathbb{X}$ is called the **normal separator** if $f_j(P_k)=\delta_{jk}$ for all $1\leq k\leq s$.

In the following proposition we collect some properties of separators of a finite set of reduced K-rational points in $\mathbb{P}^n$ (cf. [GKR, Propositions 1.13 and 1.14].).

Proposition 2.4.11. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points, and let f_j be a normal separator of $\mathbb{X} \setminus \{P_j\}$ in $\mathbb{X}$ for all $j = 1, \dots, s$.*

(i) There is an index $j \in \{1, \dots, s\}$ such that $x_0 \nmid f_j$.

(ii) For every $i \geq r_{\mathbb{X}}$, the set $\{x_0^{i-r_{\mathbb{X}}} f_1, \dots, x_0^{i-r_{\mathbb{X}}} f_s\}$ is a K-basis of $(R_{\mathbb{X}})_i$.

(iii) If $g \in (R_{\mathbb{X}})_i$ for some $i \geq 0$ and $c_1 x_0^k f_1 + \cdots + c_s x_0^k f_s \in (R_{\mathbb{X}})_{k+r_{\mathbb{X}}}$ for some $k \geq 0$ and $c_1, \dots, c_s \in K$, we have

$$g \cdot (c_1 x_0^k f_1 + \cdots + c_s x_0^k f_s) = c_1 g(P_1) x_0^{i+k} f_1 + \cdots + c_s g(P_s) x_0^{i+k} f_s.$$

In particular, we have $f_j f_k = \delta_{jk} x_0^{r_{\mathbb{X}}} f_j$ for all $j, k \in \{1, \dots, s\}$.

Chapter 3

Kähler Differential Algebras

In this chapter we investigate Kähler differential algebras and their Hilbert functions for finitely generated graded algebras. In particular, we look more closely at them for 0-dimensional schemes in the projective n-space $\mathbb{P}^n$.

The chapter contains four sections. In Section 3.1 we study the module of Kähler differential 1-forms Ω^1_{R/R_o} of an algebra R/R_o. We first recall the definition of Ω^1_{R/R_o} and present some basic results on it (see Proposition 3.1.5 and Corollary 3.1.6). When R_o is a standard graded algebra over a field K and $R = S/I$, where $S = R_o[X_0, \dots, X_n]$ and I is a homogeneous ideal of S, we give a short exact sequence of graded R-module Ω^1_{R/R_o} (see Corollary 3.1.7). Moreover, we also show how to compute the Hilbert function of Ω^1_{R/R_o} in this case (see Proposition 3.1.9).

The Kähler differential algebra of an algebra R/R_o is introduced in Section 3.2. We use the exterior algebra to define the Kähler differential algebra Ω_{R/R_o} of R/R_o. If R/R_o is an $\mathbb{N}$-graded algebra, we indicate that Ω_{R/R_o} has a natural structure of a bi-graded algebra (see Proposition 3.2.4). Also, we give a presentation of the module of Kähler differential m-forms of a graded algebra R/R_o, where $R = S/I$, where S/R_o is a graded algebra, and where I is a homogeneous ideal of S (see Proposition 3.2.11). Furthermore, we examine algorithms for the computation of Ω^m_{R/R_o} and its Hilbert function (see Proposition 3.2.14 and Corollary 3.2.17).

In next sections, we restrict our attention to Kähler differential algebras for 0-dimensional schemes $\mathbb{X} \subseteq \mathbb{P}^n$. Section 3.3 is concerned with relations between the Kähler differential algebras of $R_{\mathbb{X}}/K$ and of $R_{\mathbb{X}}/K[x_0]$, where $R_{\mathbb{X}}$ is the homogeneous coordinate ring of $\mathbb{X}$. We first present connections between $\Omega^1_{R_{\mathbb{X}}/K}$ and $\Omega^1_{R_{\mathbb{X}}/K[x_0]}$ and their Hilbert functions (see Proposition 3.3.1 and Corollary 3.3.2). Then we establish a connection between $\Omega^m_{R_{\mathbb{X}}/K}$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ for $m \geq 1$ (see Proposition 3.3.3) and describe their Hilbert functions (see Lemma 3.3.7 and Proposition 3.3.8). In the case

$n = 1$ it is easy to determine the regularity indices of $\Omega^m_{R_\mathbb{X}/K}$ and $\Omega^m_{R_\mathbb{X}/K[x_0]}$. However, it is hard to determine them in the general case, so we try to give a sharp bound (see Proposition 3.3.11).

In the last section 3.4 we consider the Kähler differential algebras for a special class of 0-dimensional schemes in $\mathbb{P}^n$, namely finite sets of K-rational points of $\mathbb{P}^n$. Given a finite set of distinct K-rational points $\mathbb{X} \subseteq \mathbb{P}^n$, we describe the Hilbert polynomials of $\Omega^m_{R_\mathbb{X}/K}$ and $\Omega^m_{R_\mathbb{X}/K[x_0]}$ and bound their regularity indices (see Proposition 3.4.1 and Corollary 3.4.3). Also, we show that $\mathcal{A}_\mathbb{X}$ has rank $\varrho_\mathbb{X} = m$ if and only if $\Omega^m_{R_\mathbb{X}/K} \neq \langle 0 \rangle$ and $\Omega^{m+1}_{R_\mathbb{X}/K} = \langle 0 \rangle$ (see Proposition 3.4.7). Finally, we apply the Hilbert function of Kähler differential algebra $\Omega_{R_\mathbb{X}/K}$ to characterize some configurations of a set of distinct K-rational points $\mathbb{X} \subseteq \mathbb{P}^n$ (see Proposition 3.4.7 and Corollary 3.4.20).

The techniques we use in the first and second section are mainly inspired by the work on Kähler differentials of E. Kunz [Kun] and the results of [AKR]. The material of the last two sections is based on the work of G. Dominicis and M. Kreuzer [DK].

3.1 Modules of Kähler Differential 1-Forms of Algebras

Let R_o be a ring and let R/R_o be an algebra. The ring $R \otimes_{R_o} R$ can be considered as an R-module by defining the product $r \cdot (\sum_i a_i \otimes b_i) = (1 \otimes r) \cdot (\sum_i a_i \otimes b_i)$ for $r \in R$ and $\sum_i a_i \otimes b_i \in R \otimes_{R_o} R$. Let J denote the kernel of the multiplication map $\mu : R \otimes_{R_o} R \to R$ given by $\mu(r_1 \otimes r_2) = r_1 r_2$ for $r_1, r_2 \in R$. For an element $\sum_i r_i \otimes t_i \in J$, we see that $\sum_i r_i t_i = 0$ and $\sum_i r_i \otimes t_i = \sum_i (1 \otimes t_i)(r_i \otimes 1 - 1 \otimes r_i) - \sum_i 1 \otimes (t_i r_i) = \sum_i (1 \otimes t_i)(r_i \otimes 1 - 1 \otimes r_i)$. This implies that J is the R-module generated by the set $\{r \otimes 1 - 1 \otimes r \mid r \in R\}$.

The mapping d_{R/R_o} which is defined by $d_{R/R_o} : R \to J/J^2, r \mapsto r \otimes 1 - 1 \otimes r + J^2$, is R_o-linear. Moreover, d_{R/R_o} satisfies the Leibniz rule, i.e. for every elements $r_1, r_2 \in R$, we have

$$\begin{aligned}
d_{R/R_o}(r_1 r_2) &= r_1 r_2 \otimes 1 - 1 \otimes r_1 r_2 + J^2 \\
&= (r_2 \otimes 1 - 1 \otimes r_2)(r_1 \otimes 1 - 1 \otimes r_1) + \\
&\quad (1 \otimes r_2)(r_1 \otimes 1 - 1 \otimes r_1) + (1 \otimes r_1)(r_2 \otimes 1 - 1 \otimes r_2) + J^2 \\
&= (1 \otimes r_2)(r_1 \otimes 1 - 1 \otimes r_1) + (1 \otimes r_1)(r_2 \otimes 1 - 1 \otimes r_2) + J^2 \\
&= r_2 d_{R/R_o} r_1 + r_1 d_{R/R_o} r_2.
\end{aligned}$$

Thus the mapping d_{R/R_o} is an R_o-derivation of R into J/J^2.

Definition 3.1.1. Let M be an R-module. An R_o-derivation $d : R \to M$ is called **universal** if for every derivation $\delta : R \to N$ of R into an R-module N, there is one and only one linear mapping $\ell : M \to N$ with $\delta = \ell \circ d$.

If $d_1 : R \to M_1$ and $d_2 : R \to M_2$ are universal derivations of the algebra R/R_o then by the universal property, there is a unique R-linear map $\ell : M_1 \to M_2$ such that $d_2 = \ell \circ d_1$, and ℓ is an isomorphism. So, we can say that there is exactly one universal derivation of R/R_o up to isomorphism. The existence of a universal derivation of R/R_o is provided by the following proposition (cf. [Kun, Section 1]).

Proposition 3.1.2. *The derivation* $d_{R/R_o} : R \to J/J^2, r \mapsto r \otimes 1 - 1 \otimes r + J^2$, *is universal.*

Definition 3.1.3. If $d\colon R \to M$ is a universal derivation of R/R_o then the R-module M which is unique up to isomorphism is denoted by Ω^1_{R/R_o} and is called the **module of Kähler differential 1-forms** of R/R_o.

When R/R_o is a graded algebra, the ring $R \otimes_{R_o} R$ is a graded ring by the grading $(R \otimes_{R_o} R)_i = \oplus_{j=0}^{i}(R_j \otimes R_{i-j})$. The kernel J of the canonical multiplication map $\mu : R \otimes_{R_o} R \to R$ given by $r_1 \otimes r_2 \mapsto r_1 r_2$ is therefore a homogeneous ideal of $R \otimes_{R_o} R$. Thus the module of Kähler differential 1-forms Ω^1_{R/R_o} is a graded R-module. If $r \in R_i$ is a homogeneous element of degree i in R, then $d_{R/R_o} r$ is a homogeneous element of degree i in Ω^1_{R/R_o} as well.

Let us illustrate the concept of module of Kähler differential 1-forms with the simplest case $R = R_o[X_0, \dots, X_n]$.

Example 3.1.4. Let R_o be a ring, and let $S = R_o[X_0, \dots, X_n]$ be the polynomial ring in $n+1$ indeterminates over R_o. Let J be the kernel of the multiplication map $\mu : S \otimes_{R_o} S \to S$ defined by $F \otimes G \mapsto FG$. The derivation $d_{S/R_o} : S \to \Omega^1_{S/R_o} = J/J^2$ given by $d_{S/R_o} F = F \otimes 1 - 1 \otimes F + J^2$ for $F \in S$ is a universal derivation of S/R_o. Let us show that Ω^1_{S/R_o} is a free S-module with a basis $\{d_{S/R_o} X_0, \dots, d_{S/R_o} X_n\}$. Therefore we deduce immediately that if $R_o = K$ is a field then the Hilbert function of Ω^1_{S/R_o} is

$$\mathrm{HF}_{\Omega^1_{S/R_o}}(i) = (n+1)\binom{n+i-1}{n}$$

for $i \in \mathbb{Z}$.

First we check that the set $\{d_{S/R_o} X_0, \dots, d_{S/R_o} X_n\}$ generates the S-module Ω^1_{S/R_o}. For two elements $F, G \in S$, we have

$$FG \otimes 1 - 1 \otimes FG + J^2 = F(G \otimes 1 - 1 \otimes G) + G(F \otimes 1 - 1 \otimes F) + J^2.$$

By repeating this process, we see that the set $\{d_{S/R_o}X_0, \ldots, d_{S/R_o}X_n\}$ is a generating set of the module of Kähler differential 1-forms Ω^1_{S/R_o}.

Second we use the universal property of the derivation d_{S/R_o} to indicate that $\Omega^1_{S/R_o} \cong S^{n+1}$. Let $v_0, \ldots, v_n$ be an S-basis of a free S-module $M = Sv_0 \oplus \cdots \oplus Sv_n$. The mapping

$$d : S \longrightarrow M,\ F \longmapsto \sum_{i=0}^{n} \tfrac{\partial F}{\partial X_i} v_i,$$

where $\partial F/\partial X_i$ is the formal derivative of a polynomial F at the indeterminate X_i, is an R_o-derivation of S into M. Also, for an S-module N and an R_o-derivation $\delta : S \to N$, there is one and only one map $\widetilde{\delta} : M \to N, v_i \mapsto \delta(X_i)$ such that $\delta = \widetilde{\delta} \circ d$. So, we have d is a universal derivation of S/R_o. Hence, by the universal property of the module of Kähler differential 1-forms Ω^1_{S/R_o} we have $\Omega^1_{S/R_o} \cong S^{n+1}$, and a basis of Ω^1_{S/R_o} is $\{d_{S/R_o}X_0, \ldots, d_{S/R_o}X_n\}$.

Let S/R_o be an arbitrary algebra, and let I be an ideal of S. The residual class ring $R = S/I$ is an R_o-algebra. In general, the module of Kähler differential 1-forms of R/R_o is not a free R-module (see Example 3.1.10). However, we can study Ω^1_{R/R_o} via its presentation. Let d_{S/R_o} and d_{R/R_o} be universal derivations of the algebras S/R_o and R/R_o, respectively. We let π denote the canonical projection from S to R. Then the map π induces the mapping $\theta : \Omega^1_{S/R_o} \to \Omega^1_{R/R_o}$ given by $\theta(\sum_i F_i d_{S/R_o} G_i) = \sum_i (F_i + I) d_{R/R_o}(G_i + I)$. We denote by $\langle d_{S/R_o} I\rangle_S$ the S-module generated by $\{d_{S/R_o}F \mid F \in I\}$. It is clear that $\theta(\langle d_{S/R_o} I\rangle_S) = 0$. Additionally, $\langle d_{S/R_o} I\rangle_S$ is an S-submodule of Ω^1_{S/R_o} containing $I\Omega^1_{S/R_o}$, since for $F \in S$ and $G \in I$ we have $G d_{S/R_o} F = F d_{S/R_o} G - d_{S/R_o}(FG) \in \langle d_{S/R_o} I\rangle_S$ and $\Omega^1_{S/R_o} = \langle d_{S/R_o} F \mid F \in S\rangle_S$.

Proposition 3.1.5. *Using the notation as above, we have* $\mathrm{Ker}(\theta) = \langle d_{S/R_o} I\rangle_S$.

Proof. Fist we note that $\Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$ is an R-module via the multiplication

$$(F + I) \cdot (d_{S/R_o} G + \langle d_{S/R_o} I\rangle_S) = F d_{S/R_o} G + \langle d_{S/R_o} I\rangle_S$$

for all $F, G \in S$. Let $d_1 : R \to \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$ be the map defined by $d_1(F + I) = d_{S/R_o} F + \langle d_{S/R_o} I\rangle_S$. Let us show that d_1 is a derivation of R into $\Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$ makes the diagram

$$\begin{array}{ccc} S & \longrightarrow & S/I = R \\ \big\downarrow{\scriptstyle d_{S/R_o}} & & \big\downarrow{\scriptstyle d_1} \\ \Omega^1_{S/R_o} & \longrightarrow & \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S \end{array}$$

commutative. It is clear that d_1 is well-defined. For $F, G \in S$, we have $d_1(FG+I) = d_{S/R_o}(FG) + \langle d_{S/R_o} I\rangle_S = F d_{S/R_o} G + G d_{S/R_o} F + \langle d_{S/R_o} I\rangle_S = F d_1(G+I) + G d_1(F+I)$. Since d_{S/R_o} is R_o-linear, so is d_1. Hence d_1 is a derivation of R into $\Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$. Consequently, the diagram is commutative.

Now we prove that the derivation $d_1 : R \to \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$ is universal. For this, let N be an R-module such that there is a derivation $d_2 : R \to N$. Then we can check that $d_2 \circ \pi : S \to N$ is a derivation of S/R_o. By the universal property of Ω^1_{S/R_o}, there is a unique S-linear map $\widetilde{\phi} : \Omega^1_{S/R_o} \to N$ such that $d_2 \circ \pi = \widetilde{\phi} \circ d_{S/R_o}$. For $F \in I$, we have $\widetilde{\phi}(d_{S/R_o} F) = \widetilde{\phi} \circ d_{S/R_o}(F) = d_2 \circ \pi(F) = 0$. It follows that $\widetilde{\phi}$ factors through $\Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$, i.e. there is an S-linear map $\phi : \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S \to N$ such that $\widetilde{\phi} = \phi \circ \widetilde{\pi}$, where $\widetilde{\pi}$ is the canonical projection from Ω^1_{S/R_o} to $\Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$. Moreover, the map ϕ is an R-linear map which satisfies $(\phi \circ d_1)(F+I) = \phi(d_{S/R_o} F + \langle d_{S/R_o} I\rangle_S) = d_2(F+I)$ for all $F \in S$. In other words, we have $\phi \circ d_1 = d_2$.

Suppose that there is another R-linear map $\phi' : \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S \to N$ such that $\phi' \circ d_1 = d_2$. Then we have the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\ \pi\ } & S/I = R & & \\ \big\downarrow{\scriptstyle d_{S/R_o}} & & \big\downarrow{\scriptstyle d_1} & \searrow^{d_2} & \\ \Omega^1_{S/R_o} & \xrightarrow{\ \widetilde{\pi}\ } & \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S & \overset{\phi}{\underset{\phi'}{\dashrightarrow}} & N \end{array}$$

From this we deduce $\phi' \circ \widetilde{\pi} = \phi \circ \widetilde{\pi}$, and so $\phi' = \phi$ (as $\widetilde{\pi}$ is surjective). Thus d_1 is a universal derivation of R/R_o. Consequently, we have $\Omega^1_{R/R_o} = \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$, and hence $\mathrm{Ker}(\theta) = \mathrm{Ker}(\widetilde{\pi}) = \langle d_{S/R_o} I\rangle_S$, as we wanted to show. □

If $R = S/I$, where I is an ideal of $S = R_o[X_0, \dots, X_n]$ and $d_{S/R_o} : S \to \Omega^1_{S/R_o}$ is the universal derivation of S/R_o, then by Example 3.1.4 and Proposition 3.1.5, we have $\Omega^1_{S/R_o} = S d_{S/R_o} X_0 \oplus \cdots \oplus S d_{S/R_o} X_n$, and the presentation of the module of Kähler differential Ω^1_{R/R_o} is $\Omega^1_{R/R_o} = S d_{S/R_o} X_0 \oplus \cdots \oplus S d_{S/R_o} X_n / \langle d_{S/R_o} I\rangle_S$. In this case, we can explicitly describe a system of generators of $\langle d_{S/R_o} I\rangle_S$ as follows.

Corollary 3.1.6. *Let S be the polynomial ring $R_o[X_0, \dots, X_n]$ and let $\{F_1, \dots, F_t\}$ be a system of generators of a given ideal I of S. Let d_{S/R_o} be the universal derivation of S/R_o. Then the set*

$$\{d_{S/R_o} F_1, \dots, d_{S/R_o} F_t, F_1 d_{S/R_o} X_0, \dots, F_1 d_{S/R_o} X_n, \dots, F_t d_{S/R_o} X_0, \dots, F_t d_{S/R_o} X_n\}$$

is a system of generators of the S-module $\langle d_{S/R_o} I\rangle_S$.

Proof. It is clear that $d_{S/R_o}F_i \in \langle d_{S/R_o}I\rangle_S$ and $F_i d_{S/R_o}X_j = d_{S/R_o}(F_iX_j) - X_j d_{S/R_o}F_i \in \langle d_{S/R_o}I\rangle_S$ for $i = 1,\dots,t$ and $j = 0,\dots,n$. Let $F \in I = \langle F_1,\dots,F_t\rangle_S$. Then there are $G_1,\dots,G_t \in S$ such that $F = \sum_{i=1}^t F_iG_i$. By the Leibniz rule applied to the derivation d_{S/R_o}, we get $d_{S/R_o}F = d_{S/R_o}(\sum_{i=1}^t F_iG_i) = \sum_{i=1}^t F_i d_{S/R_o}G_i + \sum_{i=1}^t G_i d_{S/R_o}F_i$. Moreover, we have $d_{S/R_o}G_i = \sum_{k=0}^n \frac{\partial G_i}{\partial X_k} d_{S/R_o}X_k$. Thus we get

$$d_{S/R_o}F = \sum_{i=1}^t \sum_{k=0}^n \frac{\partial G_i}{\partial X_k} F_i d_{S/R_o}X_k + \sum_{i=1}^t G_i d_{S/R_o}F_i$$

Hence the conclusion follows. □

Now we turn our attention to graded algebras. In what follows R_o denotes a standard graded algebra over a field K. Let $S = R_o[X_0,\dots,X_n]$ be the graded polynomial ring over R_o with $\deg(X_0) = \cdots = \deg(X_n) = 1$, and let I be a homogenous ideal of S. By R we denote the residual class ring S/I. We set $\mathcal{G} = \langle (\partial F/\partial x_0,\dots,\partial F/\partial x_n) \in R^{n+1} \mid F \in I\rangle_R$.

Corollary 3.1.7. *The sequence of graded R-modules*

$$0 \longrightarrow \mathcal{G}(-1) \longrightarrow R^{n+1}(-1) \longrightarrow \Omega^1_{R/R_o} \longrightarrow 0$$

is exact.

Proof. It follows immediately from Proposition 3.1.5 that

$$\begin{aligned}\Omega^1_{R/R_o} &= \Omega^1_{S/R_o}/\langle d_{S/R_o}I\rangle_S = \Omega^1_{S/R_o}/(\langle d_{S/R_o}I\rangle_S + I\Omega^1_{S/R_o}) \\ &= (\Omega^1_{S/R_o}/I\Omega^1_{S/R_o})/((\langle d_{S/R_o}I\rangle_S + I\Omega^1_{S/R_o})/I\Omega^1_{S/R_o})\end{aligned} \qquad (*)$$

Since $\Omega^1_{S/R_o} = Sd_{S/R_o}X_0 \oplus \cdots \oplus Sd_{S/R_o}X_n$, we get the presentation $\Omega^1_{S/R_o}/I\Omega^1_{S/R_o} = Re_0 \oplus \cdots \oplus Re_n$, where e_i is of degree 1. Thus we conclude from Corollary 3.1.6 and $(*)$ that the sequence of graded R-modules

$$0 \longrightarrow \mathcal{G}(-1) \longrightarrow R^{n+1}(-1) \longrightarrow \Omega^1_{R/R_o} \longrightarrow 0$$

is exact. □

Similar to Proposition 1.8 of [DK] we get the similar properties of the structure of the module $\mathcal{G}$.

Proposition 3.1.8. *(i) For every $i \in \mathbb{N}$, we have*

$$\mathcal{G}_i = \{\, (\partial F/\partial x_0,\dots,\partial F/\partial x_n) \in R^{n+1} \;\mid\; F \in I_{i+1} \,\}.$$

(ii) *If* $\{F_1, \dots, F_t\}$ *is a minimal homogeneous system of generators of* I*, then the set* $\{(\partial F_j/\partial x_0, \dots, \partial F_j/\partial x_n) \in R^{n+1} \mid 1 \leq j \leq t\}$ *is a minimal homogeneous system of generators of the* R*-module* $\mathcal{G}$.

At this point we can write an algorithm for computing the Hilbert function of the graded R-module Ω^1_{R/R_o}. This algorithm is based on the presentation $\Omega^1_{R/R_o} = \Omega^1_{S/R_o}/\langle d_{S/R_o} I\rangle_S$.

Proposition 3.1.9. (Computation of the Hilbert function of Ω^1_{R/R_o})
Let K *be a field, let* $R_o = K[Y_0, \dots, Y_m]$*, and let* $S = R_o[X_0, \dots, X_n]$ *be the polynomial ring over* R_o *with* $\deg(Y_j) = \deg(X_i) = 1$ *for* $j = 0, ..., m$*, for* $i = 0, \dots, n$*. Let* I *be a homogeneous ideal of* S *given by a set of homogeneous generators* $\{F_1, \dots, F_t\}$*, and let* $R = S/I$*. Consider the following sequence of instructions.*

1) *Compute a minimal homogeneous system of generators* L *of* I *by Buchberger's Algorithm with Minimalization (see [KR2, Theorem 4.6.3]).*

2) *Form* $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in S^{n+1}$ *for* $i = 0, \dots, n+1$*, and compute the set* $L' = \{\sum_{i=0}^n \partial F/\partial X_i e_i \in S^{n+1} \mid F \in L\}$.

3) *Form the graded* S*-submodule* N *of the graded-free* S*-module* S^{n+1} *which is generated by* $L' \cup \{Ge_j \mid G \in L, 0 \leq j \leq n\}$*, and compute the quotient module* $M = S^{n+1}/N$.

4) *Compute the Hilbert function and the regularity index of the module* M*, and return the Hilbert function of* Ω^1_{R/R_o}.

This algorithm computes the Hilbert function of the module of Kähler differential 1-forms Ω^1_{R/R_o}.

Proof. The correctness of this algorithm follows from Proposition 3.1.5 and Corollary 3.1.6. Note that $\Omega^1_{S/R_o} \cong S^{n+1}(-1)$, and $\langle d_{S/R_o} I\rangle_S \cong N(-1)$ hence we get $\mathrm{HF}_{\Omega^1_{R/R_o}}(i) = 0$ for $i \leq 0$ and $\mathrm{HF}_{\Omega^1_{R/R_o}}(i) = \mathrm{HF}_{S^{n+1}/N}(i-1)$ for $1 \leq i \leq ri(S^{n+1}/N)$, and $\mathrm{HF}_{\Omega^1_{R/R_o}}(i) = \mathrm{HP}_{S^{n+1}/N}(i-1)$ for all $i \geq ri(S^{n+1}/N) + 1$. The finiteness of this algorithm is clear. □

Let us apply Proposition 3.1.9 to some concrete cases.

Example 3.1.10. (i) Consider the homogeneous ideal $I = \langle X_0^3 - X_1^2 X_2\rangle$ of the polynomial ring $S = \mathbb{Q}[X_0, X_1, X_2]$. Let $R = S/I$. Then the Hilbert function of $\Omega^1_{R/\mathbb{Q}}$ is

$$\mathrm{HF}_{\Omega^1_{R/\mathbb{Q}}} : 0\ 3\ 9\ 17\ 6t+6 \text{ for } t \geq 3.$$

(ii) Consider the ideal $I = \bigcap_{i=1}^{6} \wp_i$ in $S = \mathbb{Q}[X_0, \dots, X_3]$, where $\wp_1 = \langle X_1, X_2, X_3\rangle^2$, $\wp_2 = \langle X_1 - X_0, X_2, X_3\rangle$, $\wp_3 = \langle X_1, X_2 - X_0, X_3\rangle$, $\wp_4 = \langle X_1, X_2, X_3 - X_0\rangle$, $\wp_5 = \langle X_1 - 2X_0, X_2 - X_0, X_3 - 10X_0\rangle$, and where $\wp_6 = \langle X_1 - X_0, X_2 - X_0, X_3 - X_0\rangle$. Note that the ideal I is the homogeneous vanishing ideal of a 0-dimensional scheme whose support is in general position. The residual class ring $R = S/I$ is not only a $\mathbb{Q}$-algebra but also a $\mathbb{Q}[x_0]$-algebra, where x_0 is the image of X_0 in R. We have $\mathrm{HF}_R : 1\ 4\ 9\ 9\ \dots$. By applying Proposition 3.1.9, the Hilbert functions of $\Omega^1_{R/\mathbb{Q}}$ and $\Omega^1_{R/\mathbb{Q}[x_0]}$ are computed as follows:

$$\begin{aligned} \mathrm{HF}_{\Omega^1_{R/\mathbb{Q}}} &: 0\ 4\ 15\ 25\ 15\ 15\ \dots \\ \mathrm{HF}_{\Omega^1_{R/\mathbb{Q}[x_0]}} &: 0\ 3\ 11\ 16\ 6\ 6\ \dots \end{aligned}$$

Since the Hilbert polynomials of $\Omega^1_{R/\mathbb{Q}}$ and of $\Omega^1_{R/\mathbb{Q}[x_0]}$ are not multiples of the Hilbert polynomial of R, we have neither the R-module $\Omega^1_{R/\mathbb{Q}}$ nor the R-module $\Omega^1_{R/\mathbb{Q}[x_0]}$ is a free R-module.

We end this section with a relation between modules of Kähler differential 1-forms. Let $I \subseteq J$ be homogeneous ideals of S. We denote the residual class rings S/I and S/J by R_I and R_J, respectively. Let $\varrho : R_I \to R_J$ be the canonical surjection given by $\varrho(F + I) = F + J$ for all $F \in S$. We observe that the graded R_J- module $\Omega^1_{R_J/K}$ can be considered as an R_I-module via ϱ, i.e. the multiplication defined by $f(g_1 d_{R_J/K} g_2) := \varrho(f) g_1 d_{R_J/K} g_2$ for $f \in R_I$ and $g_1, g_2 \in R_J$. Then the canonical projection ϱ induces an R_I-homomorphism of graded modules of Kähler differential 1-forms

$$\gamma : \Omega^1_{R_I/K} \longrightarrow \Omega^1_{R_J/K}, f d_{R_I/K} g \mapsto \varrho(f) d_{R_J/K} \varrho(g).$$

Lemma 3.1.11. *The homomorphism γ is an R_I-module epimorphism.*

Proof. The surjective property of γ follows from that one of ϱ since every element of $\Omega^1_{R_J/K}$ is of the form $d_{R_J/K} h$ for $h \in R_J$. □

The preceding proposition induces immediately the following inequality, which will be used in Section 3.4.

Corollary 3.1.12. *For all $i \in \mathbb{N}$ we have* $\mathrm{HF}_{\Omega^1_{R_I/K}}(i) \geq \mathrm{HF}_{\Omega^1_{R_J/K}}(i)$.

3.2 Kähler Differential Algebras

Let R/R_o be an algebra, and let Ω^1_{R/R_o} be the module of Kähler differential 1-forms with the universal derivation d.

Definition 3.2.1. The exterior algebra $\bigwedge_R(\Omega^1_{R/R_o})$ of the R-module of Kähler differential 1-forms over R is called the **Kähler differential algebra** of R/R_o and is denoted by Ω_{R/R_o}. The m-th exterior power of Ω^1_{R/R_o} over R is called the module of **Kähler differential m-forms** of R/R_o and is denoted by Ω^m_{R/R_o}.

From the definition of the exterior power we see that $\Omega_{R/R_o} = \bigoplus_{m\in\mathbb{N}} \Omega^m_{R/R_o}$. Let us collect some basic properties of Kähler differential algebras (cf. [Kun, 2.2]).

Proposition 3.2.2. *(i) The restriction $d : R \to \Omega^1_{R/R_o}$ of the differentiation d to elements of degree zero is a derivation of R/R_o.*

(ii) For all $f, f' \in R$, we have $df \wedge df' + df' \wedge df = 0$.

(iii) If $\omega = \sum f_0 df_1 \wedge \cdots \wedge df_m \in \Omega_{R/R_o}$, then $d\omega = \sum df_0 \wedge df_1 \wedge \cdots \wedge df_m$.

(iv) The Kähler differential algebra Ω_{R/R_o} is anti-commutative, i.e. for $\omega_m \in \Omega^m_{R/R_o}$ and $\omega_n \in \Omega^n_{R/R_o}$ we have $\omega_m \wedge \omega_n = (-1)^{mn}\omega_n \wedge \omega_m \in \Omega^{m+n}_{R/R_o}$.

(v) The map d is an anti-derivation, i.e. for $\omega_m \in \Omega^m_{R/R_o}$ and $\omega \in \Omega_{R/R_o}$ we have $d(\omega_m \wedge \omega) = d\omega_m \wedge \omega + (-1)^m \omega_m d\omega$.

Recall that the module of Kähler differential 1-forms Ω^1_{R/R_o} of a graded algebra R/R_o is a graded R-module. Hence for $m \in \mathbb{N}$, the Kähler differential m-forms Ω^m_{R/R_o} is also a graded R-module. In the analogy with the definition of grading on Ω^m_{R/R_o}, we can now define the grading on the Kähler algebra Ω_{R/R_o}.

Definition 3.2.3. Let R be a graded ring. An R-algebra Ω is called **bi-graded** if there exists a family of R-modules Ω^m and a family of subgroups $\Omega^{m,p} \subseteq \Omega^m$ for $m, p \in \mathbb{N}$ such that the following conditions are satisfied:

(i) $\Omega^m = \bigoplus_{p\in\mathbb{N}} \Omega^{m,p}$.

(ii) $\Omega^{m,p} \cdot \Omega^{m',p'} \subseteq \Omega^{m+m',p+p'}$ for all $m, m', p, p' \in \mathbb{N}$.

(iii) $\Omega^{0,p} = R_p$ for all $p \in \mathbb{N}$.

Proposition 3.2.4. *Let R_o be an $\mathbb{N}$-graded ring and let R/R_o be an $\mathbb{N}$-graded algebra. The Kähler differential algebra Ω_{R/R_o} has a natural structure of a bi-graded R-algebra.*

Proof. The Kähler differential algebra Ω_{R/R_o} has the presentation

$$\Omega_{R/R_o} = \bigoplus_{m\in\mathbb{N}} \Omega^m_{R/R_o} = \bigoplus_{m\in\mathbb{N}} R\,\underbrace{dR\wedge dR\wedge\cdots\wedge dR}_{m-\text{times}}.$$

For every $m, p \in \mathbb{N}$, we set

$$\Omega^{m,p}_{R/R_o} = \sum_{p_0+\cdots+p_m=p} R_{p_0} dR_{p_1}\wedge\cdots\wedge dR_{p_m}.$$

Since the derivation d is an R_o-linear map, this implies that $\Omega^{m,p}_{R/R_o}$ is a subgroup of the module of Kähler differential m-forms Ω^m_{R/R_o}. Additionally, we have $\Omega^{0,p}_{R/R_o} = R_p$.

Suppose that there are natural numbers m, p, m', p' such that $\Omega^{m,p}_{R/R_o} \cap \Omega^{m',p'}_{R/R_o} \neq \{0\}$. Let ω be a non-zero element of $\Omega^{m,p}_{R/R_o} \cap \Omega^{m',p'}_{R/R_o}$. Clearly, we have $\omega \in \Omega^m_{R/R_o} \cap \Omega^{m'}_{R/R_o}$, and therefore we get $m = m'$. Moreover, the module of Kähler differential m-forms Ω^m_{R/R_o} is a residue class module of the graded R-module $\bigotimes_{i=1}^m \Omega^1_{R/R_o}$. So we get $p = p'$ by the definition of $\Omega^{m,p}_{R/R_o}$. Hence we have $\Omega^m_{R/R_o} = \bigoplus_{p\in\mathbb{N}} \Omega^{m,p}_{R/R_o}$. Furthermore, using the grading induced by Ω^1_{R/R_o} on the tensor product and the fact that

$$\Omega^m_{R/R_o} \cdot \Omega^{m'}_{R/R_o} = \Omega^m_{R/R_o} \wedge_R \Omega^{m'}_{R/R_o} \subseteq \Omega^{m+m'}_{R/R_o},$$

we obtain $\Omega^{m,p}_{R/R_o} \cdot \Omega^{m',p'}_{R/R_o} \subseteq \Omega^{m+m',p+p'}_{R/R_o}$, and the conclusion follows. □

For $m, p \in \mathbb{N}$, each element of the group $\Omega^{m,p}_{R/R_o}$ is a finite sum of elements of the form $f_o df_1\wedge\cdots\wedge df_m$, where $f_o, \dots, f_m \in R$ are homogeneous such that $\sum_{i=0}^m \deg(f_i) = p$. In particular, for a graded K-algebra R_o and a graded algebra R/R_o, the group $\Omega^{m,p}_{R/R_o}$ is a finite dimensional K-vector space. The **Hilbert function** of the bi-graded R-algebra Ω_{R/R_o} is defined by

$$\mathrm{HF}_{\Omega_{R/R_o}}(m,p) = \dim_K(\Omega^{m,p}_{R/R_o}) = \mathrm{HF}_{\Omega^m_{R/R_o}}(p)$$

for all $(m, p) \in \mathbb{N}^2$.

Let us search for the Hilbert function of Kähler differential algebra in an easy case.

Proposition 3.2.5. *Let $S = R_o[X_0, \dots, X_n]$ be a standard graded polynomial ring over R_o. Then for $1 \le m \le n+1$ and $p \in \mathbb{Z}$ we have*

$$\mathrm{HF}_{\Omega_{S/R_o}}(m,p) = \tbinom{n+1}{m}\,\mathrm{HF}_S(p-m).$$

In particular, if $R_o = K$ then $\mathrm{HF}_{\Omega_{S/K}}(m,p) = \binom{n+1}{m}\binom{n+p-m}{n}$.

To prove this proposition, we use the following property.

Proposition 3.2.6. *Let $S = R_o[X_0, \dots, X_n]$ be a standard graded polynomial ring over R_o. Then the module of Kähler differential m-forms Ω^m_{S/R_o} is a S-free module of rank $\binom{n+1}{m}$.*

Proof. As we computed in Example 3.1.4, the module Ω^1_{S/R_o} is a S-free module with basis $\{dX_0, \dots, dX_n\}$. According to Satz 83.4 of [SS], for every $m \in \mathbb{N}$, the module of Kähler differential m-forms Ω^m_{S/R_o} is a S-free module with basis $\{dX_{i_1} \wedge \cdots \wedge dX_{i_m} \mid 0 \le i_1 < \cdots < i_m \le n\}$. Hence, Ω^m_{S/R_o} is a free S-module of rank $\binom{n+1}{m}$ as we wished. □

Proof of Proposition 3.2.5. By Proposition 3.2.6, the module of Kähler differential m-forms Ω^m_{S/R_o} is a S-free module with basis $\{dX_{i_1} \wedge \cdots \wedge dX_{i_m} \mid 0 \le i_1 < \cdots < i_m \le n\}$. By Lemma 2.3.3, we have $\Omega^k_{S/R_o} = 0$ for all $k > n+1$. Let $0 \le m \le n+1$ and $p \in \mathbb{N}$. Thus the Hilbert function of Kähler differential algebra Ω_{S/R_o} at degree (m,p) is

$$\mathrm{HF}_{\Omega_{S/R_o}}(m,p) = \dim_K \Omega^{m,p}_{S/R_o} = \tbinom{n+1}{m} \mathrm{HF}_S(p-m).$$

In particular, if $R_o = K$ then $\mathrm{HF}_{\Omega_{S/K}}(m,p) = \binom{n+1}{m} \mathrm{HF}_S(p-m) = \binom{n+1}{m}\binom{n+p-m}{n}$. □

Notice that, given a graded residue ring $R = S/I$, the module of Kähler differential m-forms $\Omega^m_{R/K} = 0$ for $m > n+1$. Thus we have $\Omega_{R/K} = \oplus_{i=0}^{n+1} \Omega^i_{R/K}$. A relation of the modules $\Omega^m_{R/K}$ for $0 \le m \le n+1$ is provided by our next proposition.

Proposition 3.2.7. *Let $S = K[X_0, \dots, X_n]$ be a standard graded polynomial ring over K, let I be a homogeneous ideal of S, and let $R = S/I$. We let $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$ be the homogeneous maximal ideal of R, where x_i is the image of X_i in R for $i = 0, \dots, n$. Then we have a homogeneous exact sequence of graded R-modules*

$$0 \longrightarrow \Omega^{n+1}_{R/K} \xrightarrow{\gamma} \Omega^n_{R/K} \xrightarrow{\gamma} \cdots \xrightarrow{\gamma} \Omega^1_{R/K} \xrightarrow{\gamma} \mathfrak{m} \longrightarrow 0 \tag{3.1}$$

where $\gamma(dx_{i_1} \wedge \cdots \wedge dx_{i_m}) = \sum_{j=1}^m (-1)^{j+1} x_{i_j} dx_{i_1} \wedge \cdots \wedge dx_{i_{j-1}} \wedge dx_{i_{j+1}} \wedge \cdots \wedge dx_{i_m}$ for $0 \le i_1 < \cdots < i_m \le n$.

Proof. First we notice that $\Omega^m_{R/K}$ is a graded R-module generated by the homogeneous system $\{dx_{i_1} \wedge \cdots \wedge dx_{i_m} \mid 0 \le i_0 < \cdots < i_m \le n\}$ and that $\Omega^k_{R/K} = 0$ for all $k > n+1$. Also, it is not hard to verify that the mapping $\gamma : \Omega^m_{R/K} \to \Omega^{m-1}_{R/K}$ is a homogeneous homomorphism of graded R-modules of degree zero.

Now we show that $\gamma \circ \gamma = 0$, i.e. that the sequence (3.1) is a complex. Let $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$ be an element of $\Omega^m_{R/K}$. Then we have

$$(\gamma \circ \gamma)(\omega) = \gamma(\sum_{j=1}^{m}(-1)^{j+1}x_{i_j}dx_{i_1} \wedge \cdots \wedge dx_{i_{j-1}} \wedge dx_{i_{j+1}} \wedge \cdots \wedge dx_{i_m})$$
$$= \sum_{j=1}^{m}\sum_{k<j}(-1)^{j+k+2}x_{i_k}x_{i_j}dx_{i_1}\wedge\cdots\wedge dx_{i_{k-1}}\wedge dx_{i_{k+1}}\wedge\cdots\wedge dx_{i_{j-1}}\wedge dx_{i_{j+1}}\wedge\cdots\wedge dx_{i_m} +$$
$$\sum_{j=1}^{m}\sum_{k>j}(-1)^{j+k+1}x_{i_j}x_{i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_{j-1}}\wedge dx_{i_{j+1}}\wedge\cdots\wedge dx_{i_{k-1}}\wedge dx_{i_{k+1}}\wedge \cdots \wedge dx_{i_m}$$
$$= 0.$$

It remains to show that for each homogeneous element $\omega = \sum_i f_i dx_{i_1} \wedge \cdots \wedge dx_{i_m}$ in $(\Omega^m_{R/K})_{t+m}$ with $\gamma(\omega) = 0$ there is a homogeneous element $\widetilde{\omega} \in (\Omega^{m+1}_{R/K})_{t+m}$ such that $\gamma(\widetilde{\omega}) = \omega$. For such an ω, we have

$$\gamma(d\omega) = \gamma(\sum_i df_i dx_{i_1} \wedge \cdots \wedge dx_{i_m})$$
$$= \gamma(\sum_i \sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j} dx_j dx_{i_1} \wedge \cdots \wedge dx_{i_m})$$
$$= \sum_i \sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j} x_j dx_{i_1}\wedge\cdots\wedge dx_{i_m} + \sum_i \sum_{j=0}^{n}\sum_{k=1}^{m} \frac{\partial f_i}{\partial x_j} x_{i_k}(-1)^k dx_j \wedge dx_{i_1}\wedge\cdots\wedge\widehat{dx_{i_k}}\wedge\cdots\wedge dx_{i_m}$$
$$= \sum_i t f_i dx_{i_1} \wedge \cdots \wedge dx_{i_m} + \sum_i \sum_{k=1}^{m} x_{i_k}(-1)^k df_i \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_k}} \wedge \cdots \wedge dx_{i_m}$$
$$= t\omega + d(\gamma(\omega)) = t\omega.$$

Thus, by letting $\widetilde{\omega} = \frac{1}{t}d(\omega)$, we obtain $\widetilde{\omega} \in \Omega^{m+1}_{R/K}$ and $\gamma(\widetilde{\omega}) = \omega$, as desired. □

Remark 3.2.8. Observe that the exact sequence (3.1) can be applied to compute the Hilbert function of the bi-graded algebra $\Omega_{R/K}$. In fact, we will use it to compute $\mathrm{HF}_{\Omega_{R/K}}$ when R is the homogeneous coordinate ring of a 0-dimensional scheme in the projective plane (see later).

The remaining part of this section is devoted to providing the reader with some useful properties of Kähler differential algebras. Let R and S be $\mathbb{N}$-graded R_o-algebras, and let $\pi : S \to R$ be a homogeneous ring epimorphism of degree zero. Furthermore, let $(\Omega_{R/R_o}, d)$ and $(\Omega_{S/R_o}, d)$ be the corresponding Kähler differential algebras of R/R_o and S/R_o, respectively. Then π induces a homogeneous epimorphism of graded Kähler differential algebras

$$\theta : \Omega_{S/R_o} \to \Omega_{R/R_o},\ \sum F_0 dF_1 \wedge \cdots \wedge dF_m \mapsto \sum \pi(F_0)d(\pi(F_1)) \wedge \cdots \wedge d(\pi(F_m)).$$

In this situation, in order to explicitly describe the kernel $\mathrm{Ker}(\theta)$ of θ we need only find the kernel of the restriction of θ on the m-th direct summand of Ω_{S/R_o} for $m \in \mathbb{N}$. The following lemma, which follows immediately from Proposition 2.3.7, gives us an explicit description of $\mathrm{Ker}(\theta|_{\Omega^m_{S/R_o}})$, where $m \in \mathbb{N}$.

Lemma 3.2.9. *Using the notation introduced above, let $\alpha = \theta|_{\Omega^1_{S/R_o}} : \Omega^1_{S/R_o} \to \Omega^1_{R/R_o}$ be the canonical homogeneous epimorphism of graded S-modules of degree zero. We set $\mathcal{H} = \mathrm{Ker}(\alpha)$ and $\mathcal{H} \wedge_S \bigwedge_S^{m-1}(\Omega^1_{S/R_o}) = \langle v \wedge \omega \mid v \in \mathcal{H}, \omega \in \bigwedge_S^{m-1}(\Omega^1_{S/R_o})\rangle_S$. Then, for all $m \in \mathbb{N}$, we have a homogeneous exact sequence of graded S-modules*

$$0 \longrightarrow \mathcal{H} \wedge_S \textstyle\bigwedge_S^{m-1}(\Omega^1_{S/R_o}) \longrightarrow \bigwedge_S^{m}(\Omega^1_{S/R_o}) \stackrel{\bigwedge^m(\alpha)}{\longrightarrow} \bigwedge_S^{m}(\Omega^1_{R/R_o}) \longrightarrow 0$$

where $\wedge^m(\alpha) : \bigwedge_S^m(\Omega^1_{S/R_o}) \to \bigwedge_S^m(\Omega^1_{R/R_o})$ is given by $\wedge^m(\alpha)(v_1 \wedge \cdots \wedge v_m) = \alpha(v_1) \wedge \cdots \wedge \alpha(v_m)$.

Remark 3.2.10. Let us make some observations about Lemma 3.2.9.

(a) If the graded S-module $\mathcal{H}$ is generated by homogeneous elements $v_1, \dots, v_p$ and the $(m-1)$-th exterior power of the graded S-module Ω^1_{S/R_o} over S is generated by $\omega_1, \dots, \omega_q$, then $\mathcal{H} \wedge_S \bigwedge_S^{m-1}(\Omega^1_{S/R_o})$ is generated by $\{v_i \wedge \omega_j \mid 1 \le i \le p; 1 \le j \le q\}$ as an S-module.

(b) When Ω^1_{S/R_o} is a graded-free S-module of rank $n+1$ with a basis $\{e_0, \dots, e_n\}$ and $\mathcal{H} = \langle v_1, \dots, v_p\rangle_S$, where $v_i = H_{i0}e_0 + \cdots + H_{in}e_n$ for $i = 1, \dots, p$, the S-submodule $\mathcal{H} \wedge_S \bigwedge_S^n(\Omega^1_{S/R_o})$ of $\bigwedge_S^{n+1}(\Omega^1_{S/R_o})$ is generated by the set

$$\{H_{ij}e_0 \wedge e_1 \wedge \cdots \wedge e_n \mid 1 \le i \le p; 0 \le j \le n\}.$$

Indeed, we set $M = \langle H_{ij}e_0 \wedge \cdots \wedge e_n \mid 1 \le i \le p; 0 \le j \le n\rangle_S$. It follows from [SS, Satz 83.4] that the n-th exterior power $\bigwedge_S^n(\Omega^1_{S/R_o})$ is a graded-free module of rank $n+1$ with a basis $\{\epsilon_0, \dots, \epsilon_n\}$ where $\epsilon_j = e_0 \wedge \cdots \wedge e_{j-1} \wedge \widehat{e_j} \wedge e_{j+1} \wedge \cdots \wedge e_n$ for $j = 0, \dots, n$. We also see that $v_i \wedge \epsilon_j = H_{ij}e_0 \wedge e_1 \wedge \cdots \wedge e_n$. So, we get the inclusion $\mathcal{H} \wedge_S \bigwedge_S^{n-1}(\Omega^1_{S/R_o}) \supseteq M$. For the other inclusion, we let $\sum_k \nu_k \wedge \omega_k \in \mathcal{H} \wedge_S \bigwedge_S^{n-1}(\Omega^1_{S/R_o})$. By a suitable arrangement, we may assume that $\nu_k \in \mathcal{H}$ and $\omega_k = \epsilon_k \in \bigwedge_S^{n-1}(\Omega^1_{S/R_o})$. Since the S-module $\mathcal{H}$ is generated by $\{v_1, \dots, v_p\}$, for each ν_k there are $G_{ik} \in S$ such that $\nu_k = \sum_{i=1}^p G_{ik}v_i$. Thus we have $\sum_k \nu_k \wedge \epsilon_k = \sum_k (\sum_{i=1}^p G_{ik}v_i) \wedge \epsilon_k = \sum_k (\sum_{i=1}^p G_{ik}(\sum_{j=1}^n H_{ij}e_j)) \wedge \epsilon_k = \sum_{i,k} G_{ik}H_{ik}e_1 \wedge \cdots \wedge e_n \in M$.

Proposition 3.2.11. *Let S be an $\mathbb{N}$-graded R_o-algebra, let I be a homogeneous ideal of S, and let R be the residual class ring $R = S/I$. Then the module of Kähler differential m-forms Ω^m_{R/R_o} has the presentation*

$$\Omega^m_{R/R_o} = \Omega^m_{S/R_o} / (\langle dI\rangle_S \wedge_S \Omega^{m-1}_{S/R_o} + I\Omega^m_{S/R_o}).$$

To prove Proposition 3.2.11, we require the following lemma.

Lemma 3.2.12. *Let S be an $\mathbb{N}$-graded R_o-algebra, let R/S be an algebra, and let V be a graded S-module. For all $m \in \mathbb{N}$, the universal canonical R-homomorphism $\beta : \bigwedge_S^m(V) \otimes_S R \to \bigwedge_R^m(V \otimes_S R)$ given by $(v_1 \wedge \cdots \wedge v_m) \otimes 1 \mapsto (v_1 \otimes 1) \wedge \cdots \wedge (v_m \otimes 1)$, is a homogeneous isomorphism of R-modules of degree zero.*

Proof. We see that the map $\gamma : (\bigwedge_S^m(V)) \times R \to \bigwedge_R^m(V \otimes_S R)$ given by $\gamma(v_1 \wedge \cdots \wedge v_m, 1) = (v_1 \otimes 1) \wedge \cdots \wedge (v_m \otimes 1)$ is an S-multilinear map. By the universal property of tensor product, there is an S-linear map $\beta : \bigwedge_S^m(V) \otimes_S R \to \bigwedge_R^m(V \otimes_S R)$ such that $\beta \circ \alpha = \gamma$, where $\alpha : (\bigwedge_S^m(V)) \times R \to \bigwedge_S^m(V) \otimes_S R$ is the canonical S-multilinear map. Then we deduce $\beta((v_1 \wedge \cdots \wedge v_m) \otimes 1) = (v_1 \otimes 1) \wedge \cdots \wedge (v_m \otimes 1)$. Clearly, the map β is also an R-module homomorphism and it is homogeneous of degree zero. Now we check that β is an isomorphism. Since the map $\theta : \prod_R^m(V \otimes_S R) \to \bigwedge_S^m V \otimes_S R$ defined by $(v_1 \otimes 1), \cdots, (v_m \otimes 1) \mapsto (v_1 \wedge \cdots \wedge v_m) \otimes 1$ for all $v_1, \ldots, v_m \in V$, is an R-multilinear map, it follows from the universal property of exterior power that there is an R-multilinear map $\psi : \bigwedge_R^m(V \otimes_S R) \to \bigwedge_S^m V \otimes_S R$ such that

$$\psi((v_1 \otimes 1) \wedge \cdots \wedge (v_m \otimes 1)) = (v_1 \wedge \cdots \wedge v_m) \otimes 1.$$

Observe that $\alpha \circ \psi = \mathrm{id}_{\bigwedge_R^m(V \otimes_S R)}$ and $\psi \circ \alpha = \mathrm{id}_{\bigwedge_S^m V \otimes_S R}$, therefore the claim follows. □

Proof of Proposition 3.2.11. By Proposition 3.1.5, we have $\langle dI \rangle_S = \langle I\Omega^1_{S/R_o}, dI \rangle_S$ and the homogeneous exact sequence of graded S-modules

$$0 \longrightarrow \langle I\Omega^1_{S/R_o}, dI \rangle_S \longrightarrow \Omega^1_{S/R_o} \longrightarrow \Omega^1_{R/R_o} \longrightarrow 0.$$

So, an application of Lemma 3.2.9 yields the following homogeneous exact sequence of graded S-modules

$$0 \longrightarrow \langle I\Omega^1_{S/R_o}, dI \rangle_S \wedge_S \Omega^{m-1}_{S/R_o} \longrightarrow \Omega^m_{S/R_o} \longrightarrow \textstyle\bigwedge_S^m \Omega^1_{R/R_o} \longrightarrow 0.$$

Thus we have

$$\begin{aligned}(\textstyle\bigwedge_S^m \Omega^1_{R/R_o}) \otimes_S R &= (\Omega^m_{S/R_o} / \langle I\Omega^1_{S/R_o}, dI \rangle_S \wedge_S \Omega^{m-1}_{S/R_o}) \otimes_S R \\ &= \Omega^m_{S/R_o} / (\langle dI \rangle_S \wedge_S \Omega^{m-1}_{S/R_o} + I\Omega^m_{S/R_o}).\end{aligned}$$

We can consider R as an S-algebra. By applying Lemma 3.2.12, we then get

$$(\textstyle\bigwedge_S^m \Omega^1_{R/R_o}) \otimes_S R = \bigwedge_R^m \Omega^1_{R/R_o} = \Omega^m_{R/R_o}.$$

Hence a presentation of the module of Kähler differential m-forms Ω^m_{R/R_o} is given by $\Omega^m_{R/R_o} = \Omega^m_{S/R_o} / (\langle dI \rangle_S \wedge_S \Omega^{m-1}_{S/R_o} + I\Omega^m_{S/R_o})$. □

Corollary 3.2.13. *Let I be a homogeneous ideal of $S = R_o[X_1, \dots, X_n]$ generated by the set $\{F_1, \dots, F_t\}$, and let $R = S/I$. Let $\{\epsilon_j \mid 1 \le j \le \binom{n+1}{m-1}\}$ and $\{\widetilde{\epsilon}_k \mid 1 \le k \le \binom{n+1}{m}\}$ be the canonical bases of the modules of Kähler differential Ω^{m-1}_{S/R_o} and Ω^m_{S/R_o}, respectively. Then we have*

$$\langle dI + I\Omega^1_{S/R_o}\rangle_S \wedge_S \Omega^{m-1}_{S/R_o} = \langle F_j\widetilde{\epsilon}_k, dF_j \wedge \epsilon_l \mid 1 \le j \le t; 1 \le l \le \tbinom{n+1}{m-1}; 1 \le k \le \tbinom{n+1}{m}\rangle_S.$$

Proof. According to Corollary 3.1.6, we have the S-module $\langle dI\rangle_S$ is generated by the set $\{F_j dX_i, dF_l \mid 1 \le j, l \le t; 0 \le i \le n\}$. Notice that $I\Omega^1_{S/R_o} \subseteq \langle dI\rangle_S$. Moreover, an element of the S-module $\langle dI + I\Omega^1_{S/R_o}\rangle_S \wedge_S \Omega^{m-1}_{S/R_o}$ is of the form $(\sum_{i=0}^n \sum_{j=1}^t G_{ij}F_j dX_i + \sum_{j=1}^t H_j dF_j) \wedge \epsilon_k = \sum_{i=0}^n \sum_{j=1}^j G_{ij}F_j dX_i \wedge \epsilon_k + \sum_{j=1}^t H_j dF_j \wedge \epsilon_k$, where $G_{ij}, H_j \in S$. It follows that $\langle dI + I\Omega^1_{S/R_o}\rangle_S \wedge_S \Omega^{m-1}_{S/R_o} = \langle F_j\widetilde{\epsilon}_k, dF_j \wedge \epsilon_l \mid 1 \le j \le t; 1 \le l \le \binom{n+1}{m-1}; 1 \le k \le \binom{n+1}{m}\rangle_S$. □

Based on the presentation of the module of Kähler differential m-forms Ω^m_{R/R_o} given in Proposition 3.2.11, we can describe how to compute Ω^m_{R/R_o} as well as its Hilbert function as follows.

Proposition 3.2.14. (Computation of Ω^m_{R/R_o} and its Hilbert function)
Let R_o be a standard graded K-algebra, let $S = R_o[X_0, \dots, X_n]$ be the graded polynomial ring over R_o with $\deg(X_i) = 1$, let I be a homogeneous ideal of S generated by $\{F_1, \dots, F_t\}$, let $R = S/I$, and let $1 \le m \le n+1$. Consider the following sequence of instructions.

1) *Compute a minimal homogeneous system of generators L of I by Buchberger's Algorithm with Minimalization (see [KR2, Theorem 4.6.3]).*

2) *Form the polynomial ring $T = S[e_0, \dots, e_n] = R_o[X_0, \dots, X_n, e_0, \dots, e_n]$ and compute the set $L' = \{\sum_{i=0}^n \partial F/\partial X_i e_i \mid F \in L\}$.*

3) *Compute the canonical basis $V_{m-1} = \{e_{j_1} \wedge \cdots \wedge e_{j_{m-1}} \mid 0 \le j_1 < \cdots < j_{m-1} \le n\}$ of the free S-module Ω^{m-1}_{S/R_o} and the set*

$$W_1 = \{\textstyle\sum_{i=0}^n \partial F/\partial X_i e_i \wedge e_{j_1} \wedge \cdots \wedge e_{j_{m-1}} \mid F \in L, 0 \le j_1 < \cdots < j_{m-1} \le n\}.$$

4) *Compute the canonical basis $V_m = \{e_{j_1} \wedge \cdots \wedge e_{j_m} \mid 0 \le j_1 < \cdots < j_m \le n\}$ of the free S-module Ω^m_{S/R_o} and the set*

$$W_2 = \{F \wedge e_{j_1} \wedge \cdots \wedge e_{j_m} \mid F \in L, 0 \le j_1 < \cdots < j_m \le n\}.$$

5) Form the canonical basis $\{\varepsilon_i = (0,\dots,0,1,0,\dots,0) \in S^{\binom{n+1}{m}} \mid 1 \le i \le \binom{n+1}{m}\}$ *of the graded-free* S*-module* $S^{\binom{n+1}{m}}$*, and take the image* W *of* $W_1 \cup W_2$ *in* $S^{\binom{n+1}{m}}$ *via the isomorphism* $\Omega^m_{S/R_o} \cong S^{\binom{n+1}{m}}$.

5) Compute the graded S*-submodule* $N = \langle W \rangle_S$ *of* $S^{\binom{n+1}{m}}$ *and the residue class module* $\widetilde{N} = S^{\binom{n+1}{m}}/N$.

6) Compute the regularity index of the module $\widetilde{N}$ *and return* $\widetilde{N}$ *and its Hilbert function, stop.*

This algorithm computes the module of Kähler differential m*-forms* Ω^m_{R/R_o} *and its Hilbert function.*

Proof. The correctness of this algorithm follows immediately from Corollary 3.2.13 and Proposition 3.2.11. The finiteness is clear. □

Let us compute the Hilbert function of the Kähler differential algebra using this algorithm.

Example 3.2.15. Let $I = \langle (X_0X_1^2 - X_2^3)(X_1 - X_2)^3, (X_0X_1^2 - X_2^3)(X_1 - X_0)^2 \rangle$ be the ideal of $S = \mathbb{Q}[X_0, X_1, X_2]$. We let $R = S/I$. The Hilbert function of the Kähler differential algebra $\Omega_{R/\mathbb{Q}}$ is

$$\begin{array}{ll} \mathrm{HF}_R & : \ 1\ 3\ 6\ 10\ 15\ 20\ 3t+6 \text{ for } t \ge 6 \\ \mathrm{HF}_{\Omega^1_{R/\mathbb{Q}}} & : \ 0\ 3\ 9\ 18\ 30\ 44\ 56\ 63\ 66\ 6t+22 \text{ for } t \ge 9 \\ \mathrm{HF}_{\Omega^2_{R/\mathbb{Q}}} & : \ 0\ 0\ 3\ 9\ 18\ 30\ 42\ 48\ 45\ 3t+22 \text{ for } t \ge 9 \\ \mathrm{HF}_{\Omega^3_{R/\mathbb{Q}}} & : \ 0\ 0\ 0\ 1\ 3\ 6\ 10\ 12\ 9\ 6\ \dots \text{for } t \ge 9. \end{array}$$

Note that for $S = R_o[X_0,\dots,X_n]$, the module of Kähler differential $(n+1)$-forms Ω^{n+1}_{S/R_o} is a graded free S-module generated by $dX_0 \wedge \cdots \wedge dX_n$. Let I be a homogeneous ideal of S and let $R = S/I$. According to Proposition 3.2.11, we have

$$\Omega^{n+1}_{R/R_o} = \Omega^{n+1}_{S/R_o}/(\langle dI \rangle_S \wedge_S \Omega^n_{S/R_o} + I dX_0 \wedge \cdots \wedge dX_n) \cong (S/J)(-n-1)$$

where $J(-n-1)$ is the image in $S(-n-1)$ of $\langle dI \rangle_S \wedge_S \Omega^n_{S/R_o} + I dX_0 \wedge \cdots \wedge dX_n$ under the isomorphism of S-graded modules $\Omega^{n+1}_{S/R_o} \cong S(-n-1)$. The homogeneous ideal $J \subseteq S$ is explicitly described by our next corollary.

Corollary 3.2.16. *Let $S = R_o[X_0, \dots, X_n]$ be the standard graded polynomial ring over R_o, let $I = \langle F_1, \dots, F_t \rangle$ be a homogeneous ideal of S, and let $R = S/I$. Then we have*

$$\Omega^{n+1}_{R/R_o} = (R/\widetilde{I})(-n-1) = (S/J)(-n-1)$$

where $\widetilde{I}$ is the ideal of R generated by $\{\partial F_i/\partial X_j + I \mid 1 \le i \le t; 0 \le j \le n\}$, and J is the ideal of S generated by $\{\partial F_i/\partial X_j \mid 1 \le i \le t; 0 \le j \le n\}$.

Proof. It is well-known (see Proposition 3.2.6) that the module of Kähler differential n-forms $\Omega^n_{S/R_o} = \bigwedge^n_S(\Omega^1_{S/R_o})$ is a graded-free S-module of rank $n+1$ with the basis $\{dX_0 \wedge \cdots \wedge dX_{j-1} \wedge \widehat{dX_j} \wedge dX_{j+1} \wedge \cdots \wedge dX_n \mid 0 \le j \le n\}$. Let $\mathcal{H} = \langle dF_1, \dots, dF_t \rangle_S$. Then Corollary 3.1.6 yields that

$$\mathcal{H} \wedge_S \Omega^n_{S/R_o} + I dX_0 \wedge \cdots \wedge dX_n = \langle dI \rangle_S \wedge_S \Omega^n_{S/R_o} + I dX_0 \wedge \cdots \wedge dX_n.$$

We write $dF_i = \partial F_i/\partial X_0 dX_0 + \cdots + \partial F_i/\partial X_n dX_n$ for $i = 1, \dots, t$. Then an application of Remark 3.2.10(b) implies

$$\mathcal{H} \wedge_S \Omega^n_{S/R_o} = \langle \partial F_i/\partial X_j dX_0 \wedge \cdots \wedge dX_n \mid 1 \le i \le t; 0 \le j \le n \rangle_S.$$

Hence the image of $(\mathcal{H} \wedge_S \Omega^n_{S/R_o} + I dX_0 \wedge \cdots \wedge dX_n)(n+1)$ in S under the isomorphism $\Omega^{n+1}_{S/R_o}(n+1) \cong S$ is $J = \langle \{\partial F_i/\partial X_j \mid 1 \le i \le t; 0 \le j \le n\} \cup \{F_i \mid 1 \le i \le t\} \rangle_S$. Moreover, the field K is of characteristic zero, and so Euler's formula yields that $\deg(F_i)F_i = \sum_{j=0}^n X_j \partial F_i/\partial X_j$ for all $1 \le i \le t$. Thus $J = \langle \partial F_i/\partial X_j \mid 1 \le i \le t; 0 \le j \le n \rangle_S$, and the conclusion follows. $\square$

Based on Corollary 3.2.16, we can improve the algorithm given in Proposition 3.2.14 for computing the Hilbert function of the module of Kähler differential $(n+1)$-forms Ω^{n+1}_{R/R_o} as follows.

Corollary 3.2.17. *In the setting of Corollary 3.2.16, we consider the following sequence of instructions.*

1) *Compute a minimal homogeneous system of generators L of I by Buchberger's Algorithm with Minimalization (see [KR2, Theorem 4.6.3]).*

2) *Compute the set $L' = \{\partial F/\partial X_j \mid F \in L, 0 \le j \le n\}$ and form the homogeneous ideal $J = \langle L' \rangle_S + I$ in S.*

3) *Compute the regularity index of the residue class ring S/I, return the module Ω^{n+1}_{R/R_o} and its Hilbert function by using equality $\Omega^{n+1}_{R/R_o} = (S/J)(-n-1)$, stop.*

This is an algorithm which computes the module Ω^{n+1}_{R/R_o} and its Hilbert function.

To wrap up this section we apply the algorithm of the corollary to the following example.

Example 3.2.18. Let I be the ideal $I = \cap_{i=1}^4 \wp_i$ of the ring $S = \mathbb{Q}[X_0, \dots, X_3]$, where $\wp_1 = \langle X_1, X_2, X_3\rangle_S^5$, $\wp_2 = \langle X_1 - X_0, X_2, X_3\rangle_S$, $\wp_3 = \langle X_1, X_2 - X_0, X_3\rangle_S^4$, and where $\wp_4 = \langle X_1, X_2, X_3 - X_0\rangle_S^4$. Note that the ideal I is the homogeneous vanishing ideal of a 0-dimensional scheme whose support is in general position. We let $R = S/I$. Then the Hilbert function of $\Omega^4_{R/\mathbb{Q}}$ is

$$\mathrm{HF}_{\Omega^4_{R/\mathbb{Q}}} : 0\ 0\ 0\ 0\ 1\ 4\ 10\ 20\ 31\ 38\ 40\ 40 \dots.$$

3.3 Kähler Differential Algebras for 0-Dimensional Schemes

Throughout this section we let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{P}^n$ such that $\mathrm{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$. By $\mathcal{I}_\mathbb{X}$ we denote the homogeneous vanishing ideal of $\mathbb{X}$ in $S = K[X_0, \dots, X_n]$. Then the homogeneous coordinate ring of $\mathbb{X}$ is $R_\mathbb{X} = S/\mathcal{I}_\mathbb{X}$. Let x_i be the image of X_i in $R_\mathbb{X}$ for $i = 0, \dots, n$. We know that x_0 is a non-zerodivisor for $R_\mathbb{X}$ and that $R_\mathbb{X}$ is a graded $K[x_0]$-algebra which is free of rank $\deg(\mathbb{X})$. In this section we are interested in the study of the Kähler differential algebras of the algebras $R_\mathbb{X}/K$ and $R_\mathbb{X}/K[x_0]$. In particular, we look more closely at their relations and their Hilbert functions.

Let R_o be either K or $K[x_0]$. We recall that the module of Kähler differential 1-forms of $R_\mathbb{X}/R_o$ is given by $\Omega^1_{R_\mathbb{X}/R_o} = J/J^2$, where J is the homogeneous ideal of the graded ring $R_\mathbb{X} \otimes_{R_o} R_\mathbb{X} = \bigoplus_{i\in\mathbb{Z}} \bigoplus_{j=0}^{i} (R_\mathbb{X})_j \otimes (R_\mathbb{X})_{i-j}$ which is generated by the set $\{x_i \otimes 1 - 1 \otimes x_i \mid i = 0, \dots, n\}$. The universal derivation $d : R_\mathbb{X} \to \Omega^1_{R_\mathbb{X}/R_o}$ is given by $r \mapsto r \otimes 1 - 1 \otimes r + J^2$. Then the Kähler differential algebra of $R_\mathbb{X}/R_o$ is $\Omega_{R_\mathbb{X}/R_o} = \bigwedge_{R_\mathbb{X}}(\Omega^1_{R_\mathbb{X}/R_o}) = \bigoplus_{m\in\mathbb{N}} \Omega^m_{R_\mathbb{X}/R_o}$.

First we have the following connection between $\Omega^1_{R_\mathbb{X}/K}$ and $\Omega^1_{R_\mathbb{X}/K[x_0]}$ which is given as Proposition 3.24 in [Kun].

Proposition 3.3.1. *There is an exact sequence of graded $R_\mathbb{X}$-modules*

$$R_\mathbb{X} \otimes_{K[x_0]} \Omega^1_{K[x_0]/K} \xrightarrow{\alpha} \Omega^1_{R_\mathbb{X}/K} \xrightarrow{\beta} \Omega^1_{R_\mathbb{X}/K[x_0]} \longrightarrow 0$$

where $\Omega^1_{K[x_0]/K} \cong K[x_0]dx_0$, where α is given by $\alpha(f_1 \otimes f_2 dx_0) = f_1 f_2 dx_0$, and where β is given by $\beta(f_1 d_{R_\mathbb{X}/K} f_2) = f_1 d_{R_\mathbb{X}/K[x_0]} f_2$.

We see that $\alpha(R_{\mathbb{X}} \otimes_{K[x_0]} \Omega^1_{K[x_0]/K}) = R_{\mathbb{X}}dx_0 \subset \Omega^1_{R_{\mathbb{X}}/K}$. Thus we get the exact sequence of graded $R_{\mathbb{X}}$-modules

$$0 \longrightarrow R_{\mathbb{X}}dx_0 \hookrightarrow \Omega^1_{R_{\mathbb{X}}/K} \xrightarrow{\beta} \Omega^1_{R_{\mathbb{X}}/K[x_0]} \longrightarrow 0 \tag{3.2}$$

By exact sequence (3.1), the map $\gamma : \Omega^1_{R_{\mathbb{X}}/K} \to R_{\mathbb{X}}$ given by $\gamma(FdG) = FG$ for all $F, G \in R_{\mathbb{X}}$ is a homomorphism of graded $R_{\mathbb{X}}$-modules. So, $\mathrm{Ann}_{R_{\mathbb{X}}}(dx_0) = \langle 0 \rangle$, indeed if there is $F \in R_{\mathbb{X}} \setminus \{0\}$ such that $Fdx_0 = 0$ then $\gamma(Fdx_0) = 0 = Fx_0$, contradict with the assumption that x_0 is a non-zero divisor of $R_{\mathbb{X}}$. From sequence (3.2) we obtain the following equality for Hilbert functions of modules of Kähler differentials 1-forms.

Corollary 3.3.2. *We have* $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K[x_0]}}(i) = \mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}}(i) - \mathrm{HF}_{\mathbb{X}}(i-1)$ *for all* $i \in \mathbb{Z}$. *In particular,* $\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K[x_0]}) \le \max\{\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K}), r_{\mathbb{X}} + 1\}$.

Let $\overline{R}_{\mathbb{X}} = R_{\mathbb{X}}/\langle x_0 \rangle$. An application of [DK, Proposition 1.6] yields the exact sequence of graded $R_{\mathbb{X}}$-modules

$$0 \longrightarrow R_{\mathbb{X}}dx_0 + \langle x_0 \rangle \Omega^1_{R_{\mathbb{X}}/K} \hookrightarrow \Omega^1_{R_{\mathbb{X}}/K} \longrightarrow \Omega^1_{\overline{R}_{\mathbb{X}}/K} \longrightarrow 0 \tag{3.3}$$

Our next proposition gives us connections between the modules of Kähler differential m-forms of $R_{\mathbb{X}}/K, \overline{R}_{\mathbb{X}}/K$ and $R_{\mathbb{X}}/K[x_0]$.

Proposition 3.3.3. *Let* $m \in \mathbb{N}$. *Then there is an exact sequence of graded* $R_{\mathbb{X}}$*-modules*

$$0 \longrightarrow R_{\mathbb{X}}dx_0 \wedge_{R_{\mathbb{X}}} \Omega^{m-1}_{R_{\mathbb{X}}/K} \longrightarrow \Omega^m_{R_{\mathbb{X}}/K} \longrightarrow \Omega^m_{R_{\mathbb{X}}/K[x_0]} \longrightarrow 0.$$

Moreover, $\Omega^m_{\overline{R}_{\mathbb{X}}/K}$ *has a presentation* $\Omega^m_{\overline{R}_{\mathbb{X}}/K} = \Omega^m_{R_{\mathbb{X}}/K[x_0]}/\langle x_0 \rangle \Omega^m_{R_{\mathbb{X}}/K[x_0]}$.

Proof. The exact sequence of Proposition 3.3.3 follows from Proposition 2.3.7 and sequence (3.2). We only need to prove the additional part of this proposition. Due to the short exact sequence (3.3) and Proposition 2.3.7, we get a short exact sequence of graded $R_{\mathbb{X}}$-modules

$$0 \longrightarrow (R_{\mathbb{X}}dx_0 + \langle x_0 \rangle \Omega^1_{R_{\mathbb{X}}/K}) \wedge_{R_{\mathbb{X}}} \Omega^{m-1}_{R_{\mathbb{X}}/K} \longrightarrow \Omega^m_{R_{\mathbb{X}}/K} \longrightarrow \textstyle\bigwedge^m_{R_{\mathbb{X}}}(\Omega^1_{\overline{R}_{\mathbb{X}}/K}) \longrightarrow 0.$$

We set $V = (R_{\mathbb{X}}dx_0 + \langle x_0 \rangle \Omega^1_{R_{\mathbb{X}}/K}) \wedge_{R_{\mathbb{X}}} \Omega^{m-1}_{R_{\mathbb{X}}/K}$. It follows from the exact sequence given in Proposition 3.3.3 that $\Omega^m_{R_{\mathbb{X}}/K[x_0]} = \Omega^m_{R_{\mathbb{X}}/K}/R_{\mathbb{X}}dx_0 \wedge_{R_{\mathbb{X}}} \Omega^{m-1}_{R_{\mathbb{X}}/K}$. Then we have

$$\begin{aligned}\textstyle\bigwedge^m_{R_{\mathbb{X}}}(\Omega^1_{\overline{R}_{\mathbb{X}}/K}) &= \Omega^m_{R_{\mathbb{X}}/K}/V = (\Omega^m_{R_{\mathbb{X}}/K}/R_{\mathbb{X}}dx_0 \wedge_{R_{\mathbb{X}}} \Omega^{m-1}_{R_{\mathbb{X}}/K})/(V/R_{\mathbb{X}}dx_0 \wedge_{R_{\mathbb{X}}} \Omega^{m-1}_{R_{\mathbb{X}}/K}) \\ &= \Omega^m_{R_{\mathbb{X}}/K[x_0]}/\langle x_0 \rangle \Omega^m_{R_{\mathbb{X}}/K[x_0]}.\end{aligned}$$

Thus Lemma 3.2.12 yields

$$\begin{aligned}\Omega^m_{\overline{R}_{\mathbb{X}}/K} &= \textstyle\bigwedge^m_{R_{\mathbb{X}}} \Omega^1_{\overline{R}_{\mathbb{X}}/K} \otimes_{R_{\mathbb{X}}} \overline{R}_{\mathbb{X}} = \Omega^m_{R_{\mathbb{X}}/K[x_0]}/\langle x_0\rangle \Omega^m_{R_{\mathbb{X}}/K[x_0]} \otimes_{R_{\mathbb{X}}} \overline{R}_{\mathbb{X}}\\ &= \Omega^m_{R_{\mathbb{X}}/K[x_0]}/\langle x_0\rangle \Omega^m_{R_{\mathbb{X}}/K[x_0]},\end{aligned}$$

as we wanted to show. □

As an immediate consequence of Proposition 3.2.11, we have the following corollary.

Corollary 3.3.4. *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, and let $1 \le m \le n+1$.*

(i) The module of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/K}$ has a presentation:

$$\Omega^m_{R_{\mathbb{X}}/K} = \Omega^m_{S/K}/(\langle d\mathcal{I}_{\mathbb{X}}\rangle_S \wedge_S \Omega^{m-1}_{S/K} + \mathcal{I}_{\mathbb{X}}\Omega^m_{S/K}).$$

(ii) The module of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ has a presentation:

$$\Omega^m_{R_{\mathbb{X}}/K[x_0]} = \Omega^m_{S/K[x_0]}/(\langle d\mathcal{I}_{\mathbb{X}}\rangle_S \wedge_S \Omega^{m-1}_{S/K[x_0]} + \mathcal{I}_{\mathbb{X}}\Omega^m_{S/K[x_0]}).$$

Remark 3.3.5. Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a 0-dimensional scheme. We let $\mathrm{Supp}(\mathbb{X}) = \{P_1,\dots,P_s\}$ for some $s \ge 1$, let $\wp_j$ be the homogeneous vanishing ideal of $\mathbb{X}$ at P_j for $j = 1,\dots,s$. In order to compute the homogeneous vanishing ideal $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^s \wp_j$ of $\mathbb{X}$ we use the results in the paper of J. Abbott, M. Kreuzer and L. Robbiano [AKR]. More precisely, we can either proceed degree by degree to compute a homogeneous σ-Gröbner basis of $\mathcal{I}_{\mathbb{X}}$ by using the GPBM-Algorithm (cf. [AKR, Theorem 4.6]), or we can apply the GBM-Algorithm (cf. [AKR, Theorem 3.1]).

When a homogeneous Gröbner basis $\{F_1,\dots,F_t\}$ of $\mathcal{I}_{\mathbb{X}}$ has been computed, we can apply Proposition 3.2.14 to compute presentations of the modules of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/K}$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ and their Hilbert functions. Note that $\Omega^m_{R_{\mathbb{X}}/K} = \langle 0\rangle$ if $m > n+1$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]} = \langle 0\rangle$ if $m > n$. In the case $m = n+1$ we can also use Corollary 3.2.17 to compute the Hilbert function of $\Omega^{n+1}_{R_{\mathbb{X}}/K}$.

Moreover, we have $\Omega^m_{\overline{R}_{\mathbb{X}}/K} = \Omega^m_{R_{\mathbb{X}}/K[x_0]}/\langle x_0\rangle \Omega^m_{R_{\mathbb{X}}/K[x_0]}$ (cf. Proposition 3.3.3). This enables us to compute $\Omega^m_{\overline{R}_{\mathbb{X}}/K}$ and its Hilbert function, too.

Example 3.3.6. Let $\mathbb{X}$ be the 0-dimensional scheme in $\mathbb{P}^3_{\mathbb{Q}}$ with the homogeneous vanishing ideal $I_{\mathbb{X}} = I_1 \cap I_2 \cap I_3$, where $I_1 = \langle X_1 - 9X_0, X_2, X_3\rangle^3$, $I_2 = \langle X_1 - 6X_0, X_2, X_3 - X_0\rangle^5$, and $I_3 = \langle X_1 - 2X_0, X_2 - 3X_0, X_3 - 3X_0\rangle^3$. The Hilbert function of $\Omega_{\overline{R}_{\mathbb{X}}/\mathbb{Q}}$ is

$$\begin{aligned}\Omega^1_{\overline{R}_{\mathbb{X}}/\mathbb{Q}} &: 0\ 3\ 9\ 18\ 30\ 36\ 23\ 8\ 0\ 0 \dots\\ \Omega^2_{\overline{R}_{\mathbb{X}}/\mathbb{Q}} &: 0\ 0\ 3\ 9\ 18\ 30\ 27\ 12\ 2\ 0\ 0 \dots\\ \Omega^3_{\overline{R}_{\mathbb{X}}/\mathbb{Q}} &: 0\ 0\ 0\ 1\ 3\ 6\ 10\ 6\ 2\ 0\ 0 \dots\end{aligned}$$

Now we describe the Hilbert functions of the module of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/K}$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$, where $1 \le m \le n+1$. We first consider the case $n = 1$.

Lemma 3.3.7. *Let $\mathbb{X} \subseteq \mathbb{P}^1$ be a 0-dimensional scheme with the homogeneous vanishing ideal $\mathcal{I}_{\mathbb{X}} = \langle F \rangle$, where $F = \prod_{i=1}^s (X_1 - a_i X_0)^{m_i}$ for some $m_i \ge 1$ and $a_i \in K$ such that $a_i \ne a_j$ if $i \ne j$. Let $\mu = \sum_{i=1}^s m_i$. Then the Hilbert functions of the modules of Kähler differential m-forms are given by*

$$\begin{aligned}
\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}} &: 0\ 2\ 4\ 6\ \cdots\ 2(\mu-2)\ 2(\mu-1)\ 2\mu-1\ 2\mu-2\ \cdots\ 2\mu-s\ 2\mu-s\cdots \\
\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}} &: 0\ 0\ 1\ 2\ \cdots\ \mu-2\ \mu-1\ \mu-2\ \mu-3\ \cdots\ \mu-s\ \mu-s\cdots \\
\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K[x_0]}} &: 0\ 1\ 2\ 3\ \ldots\ \mu-2\ \mu-1\ \mu-1\ \mu-2\ \ldots\ \mu-s\ \mu-s\cdots
\end{aligned}$$

and $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K[x_0]}}(i) = 0$ *for all* $i \in \mathbb{Z}$.

Proof. It is clear that the Hilbert function of $R_{\mathbb{X}}$ is $\mathrm{HF}_{R_{\mathbb{X}}}:\ 1\ 2\ 3\ 4\ \cdots\ \mu-1\ \mu\ \mu\cdots$. We let $G = \prod_{i=1}^s (X_1 - a_i X_0)^{m_i - 1}$, $H_1 = \sum_{i=1}^s m_i a_i \prod_{j \ne i}(X_1 - a_j X_0)$, and let $H_2 = \sum_{i=1}^s m_i \prod_{j\ne i}(X_1 - a_j X_0)$. Note that $\deg(G) = \sum_{i=1}^s (m_i - 1)$ and $\deg(H_1) = \deg(H_2) = s-1$. Then it is not hard to verify that the sequence H_1, H_2 is an S-regular sequence, and hence this is a regular sequence for the principle ideal $\langle G \rangle_S$. Thus it follows from Corollary 3.2.16 that

$$\begin{aligned}
\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}(i) &= \mathrm{HF}_{S/\langle \partial F/\partial X_0, \partial F/\partial X_1\rangle_S}(i-2) = \mathrm{HF}_{S/\langle GH_1, GH_2\rangle_S}(i-2) \\
&= \mathrm{HF}_{S/\langle G\rangle_S}(i-2) + \mathrm{HF}_{\langle G\rangle_S/\langle GH_1, GH_2\rangle_S}(i-2) \\
&= \mathrm{HF}_{S/\langle G\rangle_S}(i-2) + \mathrm{HF}_{\langle G\rangle_S}(i-2) - 2\,\mathrm{HF}_{\langle G\rangle_S}(i-1-s) + \mathrm{HF}_{\langle G\rangle_S}(i-2s) \\
&= \mathrm{HF}_S(i-2) - 2\,\mathrm{HF}_S(i-1-\mu) + \mathrm{HF}_S(i-s-\mu) \\
&= \binom{i-1}{1} - 2\binom{i-\mu}{1} + \binom{i-s-\mu+1}{1}
\end{aligned}$$

Therefore we have $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}:\ 0\ 0\ 1\ 2\ \cdots\ \mu-2\ \mu-1\ \mu-2\ \mu-3\ \cdots\ \mu-s\ \mu-s\cdots$. By the homogeneous exact sequence given in Proposition 3.2.7, the module of Kähler differentials 1-forms $\Omega^1_{R_{\mathbb{X}}/K}$ has the Hilbert function:

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}} : 0\ 2\ 4\ 6\ \cdots\ 2(\mu-2)\ 2(\mu-1)\ 2\mu-1\ 2\mu-2\ \cdots\ 2\mu-s\ 2\mu-s\cdots.$$

By Corollary 3.3.2, the Hilbert function of $\Omega^1_{R_{\mathbb{X}}/K[x_0]}$ is

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K[x_0]}} : 0\ 1\ 2\ \ldots\ \mu-2\ \mu-1\ \mu-1\ \mu-2\ \ldots\ \mu-s\ \mu-s\cdots.$$

Finally, we have $\Omega^1_{R_{\mathbb{X}}/K[x_0]} = R_{\mathbb{X}} dx_1$, and therefore $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K[x_0]}}(i) = 0$ for all $i \in \mathbb{Z}$. □

In some special degrees, we can predict the Hilbert function of $\Omega^m_{R_{\mathbb{X}}/K}$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$.

Proposition 3.3.8. *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, and let $\alpha_{\mathbb{X}}$ be the initial degree of $\mathcal{I}_{\mathbb{X}}$, i.e. let $\alpha_{\mathbb{X}} = \min\{i \in \mathbb{N} \mid (\mathcal{I}_{\mathbb{X}})_i \neq 0\}$.*

(i) For $i < m$, we have $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i) = \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(i) = 0$.

(ii) For $m \le i < \alpha_{\mathbb{X}}+m-1$, we have $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i) = \binom{n+1}{m}\cdot\binom{n+i-m}{n}$ and $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(i) = \binom{n}{m}\cdot\binom{n+i-m}{n}$.

(iii) The Hilbert polynomials of $\Omega^m_{R_{\mathbb{X}}/K}$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ are constant polynomials.

(iv) Let R_o denote either K or $K[x_0]$. We have $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(r_{\mathbb{X}}+m) \ge \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(r_{\mathbb{X}}+m+1) \ge \cdots$, and if $\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/R_o}) \ge r_{\mathbb{X}}+m$ then

$$\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(r_{\mathbb{X}}+m) > \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(r_{\mathbb{X}}+m+1) > \cdots > \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/R_o})).$$

Proof. (i) Let ω be a non-zero homogeneous element of the S-graded module Ω^m_{S/R_o}. Since ω can be written as a sum of elements of the form $F dX_{i_1}\cdots dX_{i_m}$ for some $F \in S$. We get $\deg(\omega) \ge m$. By Corollary 3.3.4, an element in $\Omega^m_{R_{\mathbb{X}}/R_o}$ is the residue class of an element in Ω^m_{S/R_o}, and therefore $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(i) = 0$ for all $i < m$.

(ii) Let R_o denote either K or $K[x_0]$. Let $m \le i < \alpha_{\mathbb{X}}+m-1$. Suppose that ω is a homogeneous element of degree i in the graded S-module $\mathcal{I}_{\mathbb{X}}\Omega^m_{S/R_o} + \langle d\mathcal{I}_{\mathbb{X}}\rangle_S\Omega^{m-1}_{S/R_o}$. We proceed to show that $\omega = 0$. Let us write $\omega = \sum_j F_j\omega_j + \sum_k dF_k\widetilde{\omega}_k$, where $F_j, F_k \in \mathcal{I}_{\mathbb{X}}$, $\omega_j \in \Omega^m_{S/R_o}$, and $\widetilde{\omega}_k \in \Omega^{m-1}_{S/R_o}$. Here $F_j, \omega_j, F_k, \widetilde{\omega}_k$ can be chosen homogeneous such that $\deg(F_j)+\deg(\omega_j) = \deg(F_k)+\deg(\widetilde{\omega}_k) = i$ for all j,k. Obviously, we have $\deg(F_j) \ge \alpha_{\mathbb{X}}$, $\deg(F_k) \ge \alpha_{\mathbb{X}}$, $\deg(\omega_j) \ge m$, and $\deg(\widetilde{\omega}_k) \ge m-1$. Therefore we get $\deg(\omega) \ge \alpha_{\mathbb{X}}+m-1$. This implies that $\omega = 0$, as wanted. Consequently, we have $(I_{\mathbb{X}}\Omega^m_{S/R_o} + \langle d\mathcal{I}_{\mathbb{X}}\rangle_S\Omega^{m-1}_{S/R_o})_i = \langle 0\rangle$ for all $i < \alpha_{\mathbb{X}}+m-1$. Thus for all $i < \alpha_{\mathbb{X}}+m-1$, we obtain

$$\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(i) = \mathrm{HF}_{\Omega^m_{S/R_o}}(i) = \begin{cases} \binom{n+1}{m}\cdot\binom{n+i-m}{n} & \text{if } R_o = K \\ \binom{n}{m}\cdot\binom{n+i-m}{n} & \text{if } R_o = K[x_0]. \end{cases}$$

(iii) Since the module of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/R_o}$ is a finitely generated graded $R_{\mathbb{X}}$-module and $R_{\mathbb{X}}$ is a Noetherian ring, by Theorem 2.1.4 we know that the Hilbert polynomial of $\Omega^m_{R_{\mathbb{X}}/R_o}$ exists. Corollary 3.3.4 implies $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/R_o}}(i) \le \mathrm{HF}_{\Omega^m_{S/R_o}/I_{\mathbb{X}}\Omega^m_{S/R_o}}(i)$ for $i \in \mathbb{N}$. Hence the Hilbert polynomial of $\Omega^m_{R_{\mathbb{X}}/R_o}$ is a constant polynomial.

(iv) The $R_{\mathbb{X}}$-module $\Omega^m_{R_{\mathbb{X}}/K}$ has the following presentation:

$$(\Omega^m_{R_{\mathbb{X}}/K})_{i+m} = (R_{\mathbb{X}})_i dx_0 \wedge \cdots \wedge dx_{m-1} + \cdots + (R_{\mathbb{X}})_i dx_{n-m+1} \wedge \cdots \wedge dx_n.$$

We remark that if $i \geq r_{\mathbb{X}}+1$ then $(R_{\mathbb{X}})_i = x_0(R_{\mathbb{X}})_{i-1}$. Therefore we have $(\Omega^m_{R_{\mathbb{X}}/K})_{i+m} = x_0(\Omega^m_{R_{\mathbb{X}}/K})_{i+m-1}$ for all $i \geq r_{\mathbb{X}}+1$. Thus we get the inequality

$$\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i+m) \leq \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i+m-1)$$

for all $i \geq r_{\mathbb{X}}+1$. If there is a degree $i \geq r_{\mathbb{X}}$ such that $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i+m) = \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i+m+1)$ then it follows from Corollary 3.1.7 and Proposition 2.3.7 that

$$\mathrm{HF}_{\bigwedge^m_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})}(i) - \mathrm{HF}_{\mathcal{G}\wedge_{R_{\mathbb{X}}}\bigwedge^{m-1}_{R_{\mathbb{X}}}(R_{\mathbb{X}}{}^{n+1})}(i) = \mathrm{HF}_{\bigwedge^m_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})}(i+1) - \mathrm{HF}_{\mathcal{G}\wedge_{R_{\mathbb{X}}}\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})}(i+1).$$

Here we have $\mathcal{G} = \langle (\partial F/\partial x_0, \dots, \partial F/\partial x_n) \mid F \in \mathcal{I}_{\mathbb{X}} \rangle_{R_{\mathbb{X}}}$. Since $i \geq r_{\mathbb{X}}$, we have $\mathrm{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X})$, and so $\mathrm{HF}_{\bigwedge^m_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})}(i) = \mathrm{HF}_{\bigwedge^m_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})}(i+1)$. The above equation yields

$$\mathrm{HF}_{\mathcal{G}\wedge_{R_{\mathbb{X}}}\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})}(i) = \mathrm{HF}_{\mathcal{G}\wedge_{R_{\mathbb{X}}}\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})}(i+1).$$

Notice that x_0 is a non-zerodivisor for $R_{\mathbb{X}}$, and hence by Lemma 2.4.7 this is also a non-zerodivisor for the graded $R_{\mathbb{X}}$-submodule $\mathcal{G} \wedge_{R_{\mathbb{X}}} \bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$ of the graded-free $R_{\mathbb{X}}$-module $\bigwedge^m_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$. This implies

$$(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_{i+1} = x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_i.$$

In the view of Proposition 2.4.6, the ideal $\mathcal{I}_{\mathbb{X}}$ can be generated by polynomials of degree less than or equal to $r_{\mathbb{X}}+1$. So, the graded $R_{\mathbb{X}}$-module $\mathcal{G} \wedge_{R_{\mathbb{X}}} \bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$ is generated in degree less than or equal to $r_{\mathbb{X}}$. Thus we obtain

$$\begin{aligned}(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_{i+2} &= x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_{i+1} + \cdots + x_n(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_{i+1}\\ &= x_0(x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_i + \cdots + x_n(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_i)\\ &= x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}))_{i+1}.\end{aligned}$$

Altogether, we have $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i+m+1) = \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i+m+2)$, and the claim follows by induction.

If $R_o = K[x_0]$, we have

$$(\Omega^m_{R_{\mathbb{X}}/K[x_0]})_{i+m} = (R_{\mathbb{X}})_i dx_1 \wedge \cdots \wedge dx_m + \cdots + (R_{\mathbb{X}})_i dx_{n-m+1} \wedge \cdots \wedge dx_n.$$

For $i \geq r_{\mathbb{X}}+1$ then $(R_{\mathbb{X}})_i = x_0(R_{\mathbb{X}})_{i-1}$. Thus we get $(\Omega^m_{R_{\mathbb{X}}/K[x_0]})_{i+m} = x_0(\Omega^m_{R_{\mathbb{X}}/K[x_0]})_{i+m-1}$ for all $i \geq r_{\mathbb{X}}+1$. Therefore the inequality $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(i+m) \leq \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(i+m-1)$ holds for all $i \geq r_{\mathbb{X}}+1$. If there is $i \geq r_{\mathbb{X}}$ such that $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(i+m) = \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(i+m+1)$ then we argue as above to get $\mathrm{HF}_{\bigwedge^m_{R_{\mathbb{X}}}(R^n_{\mathbb{X}})}(j) = \mathrm{HF}_{\bigwedge^m_{R_{\mathbb{X}}}(R^n_{\mathbb{X}})}(j+1)$ and

$$\mathrm{HF}_{(\mathcal{H}\wedge_{R_{\mathbb{X}}}\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^n_{\mathbb{X}}))}(j) = \mathrm{HF}_{x_0(\mathcal{H}\wedge_{R_{\mathbb{X}}}\bigwedge^{m-1}_{R_{\mathbb{X}}}(R^n_{\mathbb{X}}))}(j+1)$$

for $j \geq i$, where $\mathcal{H} = \langle(\partial F/\partial x_1, \ldots, \partial F/\partial x_n) | F \in I_{\mathbb{X}}\rangle_{R_{\mathbb{X}}}$. Altogether, we have $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(j+m+1) = \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(j+m+2)$, and the claims follows. $\square$

The following example shows that $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i+m)$ and $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(i+m)$ may or may not be monotonic in the range of $\alpha_{\mathbb{X}} + m \leq i \leq r_{\mathbb{X}} + m$.

Example 3.3.9. Let $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ be the set of nine points $\mathbb{X} = \{(1:0:0), (1:0:1), (1:0:2), (1:0:3), (1:0:4), (1:0:5), (1:1:0), (1:2:0), (1:1:1)\}$, which contains six points on a line and three points off that line. It is clear that $\mathrm{HF}_{\mathbb{X}}:\ 1\ 3\ 6\ 7\ 8\ 9\ 9\cdots$, $\alpha_{\mathbb{X}} = 3$, and $r_{\mathbb{X}} = 5$. The Hilbert functions of the modules of Kähler differential 1-forms, 2-forms, and 3-forms are computed as follows:

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}} : 0\ 3\ 9\ 15\ 14\ 13\ 14\ 13\ 12\ 11\ 10\ 9\ 9\cdots$$
$$\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/\mathbb{Q}}} : 0\ 0\ 3\ 9\ 9\ 4\ 5\ 4\ 3\ 2\ 1\ 0\ 0\cdots$$
$$\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/\mathbb{Q}}} : 0\ 0\ 0\ 1\ 3\ 0\ 0\cdots.$$

We see that $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}}(\alpha_{\mathbb{X}}+1) = 14 > 13 = \mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}}(\alpha_{\mathbb{X}}+2)$ and $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}}(\alpha_{\mathbb{X}}+2) = 13 < 14 = \mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}}(r_{\mathbb{X}}+1)$. So, $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}}$ is not monotonic in the range of $\alpha_{\mathbb{X}}+1 \leq i \leq r_{\mathbb{X}}+1$. But $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}}$ is monotonic in the range of $\alpha_{\mathbb{X}} + 3 \leq i \leq r_{\mathbb{X}} + 3$. Furthermore, we have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}[x_0]}}:\ 0\ 2\ 6\ 9\ 7\ 5\ 5\ 4\ 3\ 2\ 1\ 0\ 0\ldots$ and $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/\mathbb{Q}[x_0]}}:\ 0\ 0\ 1\ 3\ 1\ 0\ 0\ldots.$

Remark 3.3.10. By Propositions 3.3.3 and 3.3.8, the Hilbert polynomial of $\Omega^m_{R_{\mathbb{X}}/K}$ is constant for all $m = 1, \ldots, n$.

The regularity index of the modules $\Omega^m_{R_{\mathbb{X}}/K}$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ can be bounded as follows.

Proposition 3.3.11. *Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a 0-dimensional scheme, let $m \geq 1$. The regularity indices of the module of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/K}$ and $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ satisfies*

$$\max\{\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/K}), \mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/K[x_0]})\} \leq \max\{r_{\mathbb{X}} + m, \mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K}) + m - 1\}.$$

Proof. We set $\mathcal{G} = \langle(\partial F/\partial x_0, \ldots, \partial F/\partial x_n) \in R^{n+1} \mid F \in I\rangle_R$. Corollary 3.1.7 yields the short exact sequence of graded $R_{\mathbb{X}}$-modules

$$0 \longrightarrow \mathcal{G} \longrightarrow \Omega^1_{S/R_o}/I_{\mathbb{X}}\Omega^1_{S/R_o} \longrightarrow \Omega^1_{R_{\mathbb{X}}/R_o} \longrightarrow 0.$$

Applying Proposition 2.4.10 to the $R_{\mathbb{X}}$-module $\Omega^1_{R_{\mathbb{X}}/R_o}$ which has a set of generators of degree 1, we get $\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/R_o}) \leq \max\{r_{\mathbb{X}}+m, \mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/R_o})+m-1\}$. By Corollary 3.3.2, we get $\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K[x_0]}) \leq \max\{\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K}), r_{\mathbb{X}}+1\}$. Thus the claim follows. $\square$

Remark 3.3.12. If $R_o = K$ then $\mathrm{ri}(\Omega^{n+1}_{R_\mathbb{X}/K}) \le \max\{r_\mathbb{X}+n, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+n-1\}$. Indeed, the exact sequence of graded $R_\mathbb{X}$-modules

$$0 \longrightarrow \Omega^{n+1}_{R_\mathbb{X}/K} \longrightarrow \Omega^{n}_{R_\mathbb{X}/K} \longrightarrow \cdots \longrightarrow \Omega^{1}_{R_\mathbb{X}/K} \longrightarrow \mathfrak{m} \longrightarrow 0$$

deduces $\mathrm{ri}(\Omega^{n+1}_{R_\mathbb{X}/K}) \le \max\{\mathrm{ri}(\Omega^{i}_{R_\mathbb{X}/K}) | i = 0, \dots, n\} \le \max\{r_\mathbb{X}+n, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+n-1\}$. In conclusion, we obtain the following upper bounds for the regularity indices of $\Omega^m_{R_\mathbb{X}/K}$:

$$\mathrm{ri}(\Omega^m_{R_\mathbb{X}/K}) \le \min\{\max\{r_\mathbb{X}+n, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+n-1\}, \max\{r_\mathbb{X}+m, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+m-1\}\}.$$

If $R_o = K$ then the upper bound for the regularity index of the module of Kähler differential m-forms $\Omega^m_{R_\mathbb{X}/K}$ which is given in the above remark is sharp. Moreover, if $R_o = K[x_0]$ and $m < n$, the upper bound for the regularity index of the module of Kähler differential m-forms $\Omega^m_{R_\mathbb{X}/K[x_0]}$ which is given in Proposition 3.3.11 is sharp, as the following example shows.

Example 3.3.13. Let $\mathbb{X}$ be the 0-dimensional scheme in $\mathbb{P}^3_\mathbb{Q}$ with the homogeneous vanishing ideal $I_\mathbb{X} = \cap_{i=1}^6 \wp_i$, where $\wp_1 = \langle X_1, X_2, X_3\rangle^2, \wp_2 = \langle X_1 - X_0, X_2, X_3\rangle)$, $\wp_3 = \langle X_1, X_2 - X_0, X_3\rangle$, $\wp_4 = \langle X_1, X_2, X_3 - X_0\rangle$, $\wp_5 = \langle X_1 - 2X_0, X_2 - X_0, X_3\rangle$, and $\wp_6 = \langle X_1 - X_0, X_2 - X_0, X_3 - X_0\rangle$. We see that $r_\mathbb{X} = 2$ and $\mathrm{ri}(\Omega^1_{R_\mathbb{X}/\mathbb{Q}}) = 4$. Also, we have $\mathrm{ri}(\Omega^2_{R_\mathbb{X}/\mathbb{Q}}) = \min\{\max\{r_\mathbb{X}+n, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+n-1\}, \max\{r_\mathbb{X}+m, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+m-1\}\} = 5$ and

$$\begin{aligned}\mathrm{ri}(\Omega^3_{R_\mathbb{X}/\mathbb{Q}}) &= \mathrm{ri}(\Omega^4_{R_\mathbb{X}/\mathbb{Q}}) \\ &= \min\{\max\{r_\mathbb{X}+n, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+n-1\}, \max\{r_\mathbb{X}+m, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/K})+m-1\}\} = 6\end{aligned}$$

for $m = 3, 4$. Thus the upper bound for the regularity index of the module of Kähler differential m-forms $\Omega^m_{R_\mathbb{X}/\mathbb{Q}}$ given in Remark 3.3.12 is sharp. Furthermore, we have $\mathrm{ri}(\Omega^1_{R_\mathbb{X}/\mathbb{Q}[x_0]}) = 4$ and $\mathrm{ri}(\Omega^2_{R_\mathbb{X}/\mathbb{Q}[x_0]}) = 5 = \max\{r_\mathbb{X}+m, \mathrm{ri}(\Omega^1_{R_\mathbb{X}/\mathbb{Q}}+m-1)\}$ for $m = 2$. Therefore, for $m < n$, the upper bound for the regularity index of $\Omega^m_{R_\mathbb{X}/\mathbb{Q}[x_0]}$ given in Proposition 3.3.11 is sharp.

3.4 Kähler Differential Algebras for Finite Sets of K-Rational Points

In this section we restrict ourselves to investigating Kähler differential algebras for a special class of 0-dimensional schemes in $\mathbb{P}^n$, namely finite sets of distinct K-rational points in $\mathbb{P}^n$.

In the following we let $s \geq 1$, and we let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points in $\mathbb{P}^n$. Furthermore, we always assume that no point of $\mathbb{X}$ lies on the hyperplane at infinity $\mathcal{Z}^+(X_0)$. Let us write $P_j = (p_{j0} : p_{j1} : \cdots : p_{jn})$ with $p_{j1}, \dots, p_{jn} \in K$ and $p_{j0} = 1$ for $j = 1, \dots, s$. Then the vanishing ideal of P_j is $\wp_j = \langle p_{j1}X_0 - X_1, \dots, p_{jn}X_0 - X_n \rangle_S$, where $S = K[X_0, \dots, X_n]$. We have the image x_0 of X_0 in $R_{\mathbb{X}}$ is a non-zerodivisor for $R_{\mathbb{X}}$ and that $R_{\mathbb{X}}$ is a graded $K[x_0]$-algebra which is free of rank s. In [DK], Proposition 3.5 shows that the Hilbert polynomial of $\Omega^1_{R_{\mathbb{X}}/K}$ is $\mathrm{HP}_{\Omega^1_{R_{\mathbb{X}}/K}}(z) = \deg(\mathbb{X}) = s$ and the regularity index of $\Omega^1_{R_{\mathbb{X}}/K}$ satisfies $\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K}) \leq 2r_{\mathbb{X}} + 1$, where $r_{\mathbb{X}}$ is the regularity index of $\mathrm{HF}_{\mathbb{X}}$. This result was proved by using the theory of separators in 1999 by M. Kreuzer and G. Dominicis [DK]. For $j = 1, \dots, s$, let $f_j \in (R_{\mathbb{X}})_{r_{\mathbb{X}}}$ be the normal separator of $\mathbb{X} \setminus \{P_j\}$ in $\mathbb{X}$, i.e. $f_j(P_k) = \delta_{jk}$ for $j, k = 1, \dots, s$. We recall that the set $\{x_0^{i-r_{\mathbb{X}}} f_1, \dots, x_0^{i-r_{\mathbb{X}}} f_s\}$ is a K-basis of R_i for $i \geq r_{\mathbb{X}}$, and for $f \in R_j$ and $c_1 x_0^k f_1 + \cdots + c_s x_0^k f_s \in R_{k+r_{\mathbb{X}}}$ with $c_1, \dots, c_s \in K$ we have

$$f \cdot (c_1 x_0^k f_1 + \cdots + c_s x_0^k f_s) = c_1 f(P_1) x_0^{j+k} f_1 + \cdots + c_s f(P_s) x_0^{j+k} f_s.$$

Using the above tools, we can describe the Hilbert polynomial of the module of Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/K}$ for every $1 \leq m \leq n+1$ as follows.

Proposition 3.4.1. *Let $1 \leq m \leq n+1$. We have*

$$\mathrm{HP}_{\Omega^m_{R_{\mathbb{X}}/K}}(z) = \begin{cases} \deg(\mathbb{X}) & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases}$$

In particular, the regularity index of $\Omega^m_{R_{\mathbb{X}}/K}$ satisfies $\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/K}) \leq 2r_{\mathbb{X}} + m$.

Proof. For $m = 1$ we have $\mathrm{HP}_{\Omega^1_{R_{\mathbb{X}}/K}} = \deg(\mathbb{X})$ and $\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K}) \leq 2r_{\mathbb{X}} + 1$ (see [DK, Proposition 3.5]). Assume that $m \geq 2$. We see that $\Omega^m_{R_{\mathbb{X}}/K}$ is a graded $R_{\mathbb{X}}$-module generated by the set of $\binom{n+1}{m}$ elements $\{\, dx_{i_1} \wedge \cdots \wedge dx_{i_m} \mid 0 \leq i_1 < \cdots < i_m \leq n+1 \,\}$. For $j \in \{1, \dots, s\}$, let f_j be the normal separator of $\mathbb{X} \setminus \{P_j\}$ in $\mathbb{X}$. Since the set $\{x_0^{i-r_{\mathbb{X}}} f_1, \dots, x_0^{i-r_{\mathbb{X}}} f_s\}$ is a K-basis of the K-vector space $(R_{\mathbb{X}})_i$ for $i \geq r_{\mathbb{X}}$, the set

$$\{\, x_0^{k-r_{\mathbb{X}}-m} f_j dx_{i_1} \wedge \cdots \wedge dx_{i_m} \mid 0 \leq i_1 < \cdots < i_m \leq n+1; 1 \leq j \leq s \,\}$$

is a system of generators of the K-vector space $(\Omega^m_{R_{\mathbb{X}}/K})_k$ for all $k \geq r_{\mathbb{X}} + m$. Note that $f_j^2 = f_j(P_j) x_0^{r_{\mathbb{X}}} f_i = x_0^{r_{\mathbb{X}}} f_i$ and $x_i f_j = p_{ji} x_0 f_j$, where we write $P_j = (1 : p_{j1} : \cdots : p_{jn})$

with $p_{j1}, \ldots, p_{jn} \in K$ for $j = 1, \ldots, s$. Therefore we get

$$\begin{aligned} x_0^{r_{\mathbb{X}}} f_j dx_{i_1} \cdots dx_{i_m} &= f_j^2 dx_{i_1} \cdots dx_{i_m} = (d(f_j^2 x_{i_1}) - x_{i_1} df_i^2) dx_{i_2} \cdots dx_{i_m} \\ &= (d(p_{ji_1} x_0 f_j^2) - x_{i_1} df_j^2) dx_{i_2} \cdots dx_{i_m} \\ &= ((p_{ji_1} x_0 - x_{i_1}) df_j^2 + p_{ji_1} f_j^2 dx_0) dx_{i_2} \cdots dx_{i_m} \\ &= (2(p_{ji_1} x_0 - x_{i_1}) f_j df_j + p_{ji_1} f_j^2 dx_0) dx_{i_2} \cdots dx_{i_m} \\ &= p_{ji_1} f_j^2 dx_0 dx_{i_2} \cdots dx_{i_m} = p_{ji_1} x_0^{r_{\mathbb{X}}} f_j dx_0 dx_{i_2} \cdots dx_{i_m} \\ &= \cdots = p_{ji_1} p_{ji_2} x_0^{r_{\mathbb{X}}} f_j dx_0 dx_0 dx_{i_3} \cdots dx_{i_m}. \end{aligned}$$

Since we have $m \geq 2$, this implies that $x_0^{r_{\mathbb{X}}} f_j dx_{i_1} \cdots dx_{i_m} = 0$ for all $j = 1, \ldots, s$ and $\{i_1, \ldots, i_m\} \subseteq \{0, \ldots, n+1\}$. Therefore we obtain $(\Omega^m_{R_{\mathbb{X}}/K})_k = \langle 0 \rangle$ for all $k \geq 2r_{\mathbb{X}} + m$, and the conclusion follows. □

Since $\mathrm{ri}(\Omega^{n+1}_{R_{\mathbb{X}}/K}) \leq \max\{r_{\mathbb{X}}, \mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/K}) \mid 1 \leq m \leq n\}$, we get $\mathrm{ri}(\Omega^{n+1}_{R_{\mathbb{X}}/K}) \leq 2r_{\mathbb{X}} + n$. Hence we have $\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/K}) \leq \min\{2r_{\mathbb{X}} + m, 2r_{\mathbb{X}} + n\}$. These upper bounds for the regularity indices of the Kähler differential m-forms $\Omega^m_{R_{\mathbb{X}}/K}$ are sharp, as our next example shows.

Example 3.4.2. Let $\mathbb{X} = \{P_1, P_2, P_3, P_4\}$ be the set of four $\mathbb{Q}$-rational points in $\mathbb{P}^3_{\mathbb{Q}}$, where $P_1 = (1:9:0:0), P_2 = (1:6:0:1), P_3 = (1:2:3:3)$, and $P_4 = (1:9:3:5)$. It is clear that $\mathrm{HF}_{\mathbb{X}} : 1\ 4\ 4 \ldots$ and $r_{\mathbb{X}} = 1$. Moreover, we have

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}} : 0\ 4\ 10\ 4\ 4 \cdots, \qquad \mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/\mathbb{Q}}} : 0\ 0\ 6\ 4\ 0\ 0 \cdots,$$
$$\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/\mathbb{Q}}} : 0\ 0\ 0\ 4\ 1\ 0\ 0 \cdots, \quad \mathrm{HF}_{\Omega^4_{R_{\mathbb{X}}/\mathbb{Q}}} : 0\ 0\ 0\ 0\ 1\ 0\ 0 \cdots.$$

Thus we get $\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/\mathbb{Q}}) = 2r_{\mathbb{X}} + m = 2r_{\mathbb{X}} + 1 = 3$, $\mathrm{ri}(\Omega^2_{R_{\mathbb{X}}/\mathbb{Q}}) = 2r_{\mathbb{X}} + m = 2r_{\mathbb{X}} + 2 = 4$, and $\mathrm{ri}(\Omega^3_{R_{\mathbb{X}}/\mathbb{Q}}) = \mathrm{ri}(\Omega^4_{R_{\mathbb{X}}/\mathbb{Q}}) = \min\{2r_{\mathbb{X}} + m, 2r_{\mathbb{X}} + n\} = 2r_{\mathbb{X}} + n = 5$. Hence we obtain the equality $\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/\mathbb{Q}}) = \min\{2r_{\mathbb{X}} + m, 2r_{\mathbb{X}} + n\}$ for $m = 1, \ldots, 4$.

Follow from Corollary 3.3.2 and Propositions 3.3.3 and 3.4.1, we get an upper bound for the regularity index of $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$:

Corollary 3.4.3. *Let* $1 \leq m \leq n+1$. *Then the Hilbert polynomial of* $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ *is* $\mathrm{HP}_{\Omega^m_{R_{\mathbb{X}}/K[x_0]}}(z) = 0$ *and the regularity index of* $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ *satisfies* $\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/K[x_0]}) \leq 2r_{\mathbb{X}} + m$.

For $1 \leq m \leq n$, our next example shows that the upper bound for the regularity index of $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ given in Corollary 3.4.3 is sharp.

Example 3.4.4. Let $\mathbb{X} = \{P_1, \ldots, P_5\}$ be the set of five $\mathbb{Q}$-rational points in $\mathbb{P}^4_{\mathbb{Q}}$ where $P_1 = (1:9:0:0:1), P_2 = (1:1:6:0:1), P_3 = (1:0:2:3:3), P_4 = (1:9:3:0:5)$, and $P_5 = (1:3:0:4:1)$. Then we have $\mathrm{HF}_{\mathbb{X}}: 1\ 5\ 5 \ldots$ and $r_{\mathbb{X}} = 1$. Moreover, we have

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K[x_0]}}: 0\ 4\ 10\ 0\ 0 \cdots, \qquad \mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K[x_0]}}: 0\ 0\ 6\ 4\ 0\ 0 \cdots,$$
$$\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K[x_0]}}: 0\ 0\ 0\ 4\ 1\ 0\ 0 \cdots, \quad \mathrm{HF}_{\Omega^4_{R_{\mathbb{X}}/K[x_0]}}: 0\ 0\ 0\ 0\ 1\ 0\ 0 \cdots.$$

So, we get $\mathrm{ri}(\Omega^m_{R_{\mathbb{X}}/\mathbb{Q}}) = \min\{2r_{\mathbb{X}} + m, 2r_{\mathbb{X}} + n\} = 2 + m$ for $m = 1,2,3$. Hence the upper bound for the regularity index of $\Omega^m_{R_{\mathbb{X}}/K[x_0]}$ given in Corollary 3.4.3 is sharp.

We form the matrix $\mathcal{A}_{\mathbb{X}} = (p_{ij})_{\substack{i=1,\ldots,s\\ j=0,\ldots,n}} \in \mathrm{Mat}_{s\times(n+1)}(K)$, and let $\varrho_{\mathbb{X}}$ denote the rank of the matrix $\mathcal{A}_{\mathbb{X}}$.

Proposition 3.4.5. *Using the notation as above, we have* $\Omega^m_{R_{\mathbb{X}}/K} = \langle 0\rangle$ *for all* $m > \varrho_{\mathbb{X}}$.

Proof. Let $\mathcal{G} = \langle (\partial F/\partial x_0, \ldots, \partial F/\partial x_n) \in R_{\mathbb{X}}^{n+1} \mid F \in \mathcal{I}_{\mathbb{X}}\rangle_{R_{\mathbb{X}}}$. Then the sequence of graded $R_{\mathbb{X}}$-modules

$$0 \longrightarrow \mathcal{G} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R_{\mathbb{X}}^{n+1}) \longrightarrow \bigwedge^{\varrho_{\mathbb{X}}+1}_{R_{\mathbb{X}}}(R_{\mathbb{X}}^{n+1}) \longrightarrow \Omega^{\varrho_{\mathbb{X}}+1}_{R_{\mathbb{X}}/K} \longrightarrow 0 \tag{3.4}$$

is exact by Proposition 2.3.7. Let $\mathcal{Y} = \begin{pmatrix} y_0 & \cdots & y_n\end{pmatrix}^{\mathrm{tr}}$, let $u = n+1-\varrho_{\mathbb{X}}$, and let V be the space of solutions of the system of linear equations $\mathcal{A}_{\mathbb{X}} \cdot \mathcal{Y} = \mathbf{0}$. Then $\dim_K V = u$. Without loss of generality, we may assume that the set $\{v_1, \ldots, v_u\}$, where $v_1 = (1,0,\ldots,0,a_{1\,u+1},\ldots,a_{1\,n+1})$, $\ldots$, $v_u = (0,0,\ldots,1,a_{u\,u+1},\ldots,a_{u\,n+1})$, is a K-basis of V. Then the linear forms $L_1 = X_0 + a_{1\,u+1}X_u + \cdots + a_{1\,n+1}X_n$, $\ldots$, $L_u = X_{u-1} + a_{u\,u+1}X_u + \cdots + a_{u\,n+1}X_n$ are contained in $\mathcal{I}_{\mathbb{X}}$. This implies that $v_1, \ldots, v_u \in \mathcal{G}$, and so $\langle v_1, \ldots, v_u\rangle_{R_{\mathbb{X}}} \subseteq \mathcal{G} \subseteq R_{\mathbb{X}}^{n+1}$.

In order to prove $\Omega^{\varrho_{\mathbb{X}}+1}_{R_{\mathbb{X}}/K} = \langle 0\rangle$, it suffices to prove that

$$\textstyle\bigwedge^{\varrho_{\mathbb{X}}+1}_{R_{\mathbb{X}}}(R_{\mathbb{X}}^{n+1}) = \langle v_1, \ldots, v_u\rangle_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}} \bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R_{\mathbb{X}}^{n+1}).$$

Let $\{e_1, \ldots, e_{n+1}\}$ be a basis of the graded-free $R_{\mathbb{X}}$-module $R_{\mathbb{X}}^{n+1}$, and let $i \in \{1,\ldots,u\}$. Then we see that $v_i = e_i + a_{i\,u+1}e_{u+1} + \cdots + a_{in}e_{n+1}$ and

$$e_i \wedge e_{u+1} \wedge \cdots \wedge e_{u+\varrho_{\mathbb{X}}} = v_i \wedge e_{u+1} \wedge \cdots \wedge e_{u+\varrho_{\mathbb{X}}} \in \langle v_1, \ldots, v_u\rangle_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R_{\mathbb{X}}^{n+1}).$$

Let $1 \le k \le \varrho_{\mathbb{X}}$. We want to show that if

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_{\varrho_{\mathbb{X}}+1}} \in \langle v_1, \ldots, v_u\rangle_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R_{\mathbb{X}}^{n+1})$$

for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, u\}$ and $\{i_{k+1}, \ldots, i_{\varrho_{\mathbb{X}}+1}\} \subseteq \{u+1, \ldots, u+\varrho_{\mathbb{X}}\}$ then

$$e_{j_1} \wedge \cdots \wedge e_{j_{k+1}} \wedge e_{j_{k+2}} \wedge \cdots \wedge e_{j_{\varrho_{\mathbb{X}}+1}} \in \langle v_1, \ldots, v_u\rangle_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}} R_{\mathbb{X}}^{n+1}$$

for all $\{j_1,\dots,j_{k+1}\}\subseteq\{1,\dots,u\}$ and $\{j_{k+2},\dots,j_{\varrho_{\mathbb{X}}+1}\}\subseteq\{u+1,\dots,u+\varrho_{\mathbb{X}}\}$. We see that

$$\begin{aligned}
&e_{j_1}\wedge\cdots\wedge e_{j_{k+1}}\wedge e_{j_{k+2}}\wedge\cdots\wedge e_{j_{\varrho_{\mathbb{X}}+1}}\\
&\quad=(e_{j_1}+a_{j_1\,u+1}e_{u+1}+\cdots+a_{j_1\,u+\varrho_{\mathbb{X}}}e_{u+\varrho_{\mathbb{X}}})\wedge\cdots\wedge e_{j_{k+1}}\wedge e_{j_{k+2}}\wedge\cdots\wedge e_{j_{\varrho_{\mathbb{X}}+1}}\\
&\qquad-\textstyle\sum_{l=u+1}^{u+\varrho_{\mathbb{X}}}(-1)^{\varrho_{\mathbb{X}}}a_{j_1l}(e_{j_2}\wedge\cdots\wedge e_{j_{k+1}})\wedge(e_{j_{k+2}}\wedge\cdots\wedge e_{j_{\varrho_{\mathbb{X}}+1}}\wedge e_l)\\
&\quad=v_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_{k+1}}\wedge e_{j_{k+2}}\wedge\cdots\wedge e_{j_{\varrho_{\mathbb{X}}+1}}\\
&\qquad-\textstyle\sum_{l=u+1}^{u+\varrho_{\mathbb{X}}}(-1)^{\varrho_{\mathbb{X}}}a_{j_1l}(e_{j_2}\wedge\cdots\wedge e_{j_{k+1}})\wedge(e_{j_{k+2}}\wedge\cdots\wedge e_{j_{\varrho_{\mathbb{X}}+1}}\wedge e_l).
\end{aligned}$$

This implies that $e_{j_1}\wedge\cdots\wedge e_{j_{k+1}}\wedge e_{j_{k+2}}\wedge\cdots\wedge e_{j_{\varrho_{\mathbb{X}}+1}}\in\langle v_1,\dots,v_u\rangle_{R_{\mathbb{X}}}\wedge_{R_{\mathbb{X}}}\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$. Thus we have shown that the set $\{e_{i_1}\wedge\cdots\wedge e_{i_{\varrho_{\mathbb{X}}+1}}\mid\{i_1,\dots,i_{\varrho_{\mathbb{X}}+1}\}\subseteq\{1,\dots,n+1\},\exists i_l\in\{1,\dots,u\}\}$ is contained in $\langle v_1,\dots,v_u\rangle_{R_{\mathbb{X}}}\wedge_{R_{\mathbb{X}}}\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$. Since this set is also a system of generators of the $R_{\mathbb{X}}$-module $\bigwedge^{\varrho_{\mathbb{X}}+1}_{R_{\mathbb{X}}}R^{n+1}_{\mathbb{X}}$, we obtain the equality $\bigwedge^{\varrho_{\mathbb{X}}+1}_{R_{\mathbb{X}}}R^{n+1}_{\mathbb{X}}=\langle v_1,\dots,v_u\rangle_{R_{\mathbb{X}}}\wedge_{R_{\mathbb{X}}}\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$, as we wanted to show. □

Let us clarify this proposition using an example.

Example 3.4.6. Let $\mathbb{X}\subseteq\mathbb{P}^4$ be the set of seven $\mathbb{Q}$-rational points $P_1=(1:2:3:6:7)$, $P_2=(1:0:0:0:6)$, $P_3=(1:3:5:6:7)$, $P_4=(1:3/2:5/2:3:13/2)$, $P_5=(2:5:8:12:14)$, $P_6=(1:79:33:67:2)$, and $P_7=(1:1:3/2:3:13/2)$. Then we have $\varrho_{\mathbb{X}}=4$. A calculation gives $\Omega^4_{R_{\mathbb{X}}/K}\neq\langle 0\rangle$ and $\Omega^5_{R_{\mathbb{X}}/K}=\langle 0\rangle$.

This example is a particular case of the following proposition.

Proposition 3.4.7. *Let $\mathbb{X}$ be a set of s distinct K-rational points in $\mathbb{P}^n$. Then $\mathcal{A}_{\mathbb{X}}$ has rank $\varrho_{\mathbb{X}}=m$ if and only if $\Omega^m_{R_{\mathbb{X}}/K}\neq\langle 0\rangle$ and $\Omega^{m+1}_{R_{\mathbb{X}}/K}=\langle 0\rangle$.*

To prove Proposition 3.4.7 we use the following lemma, which has proved in [SS, Proposition 85.12].

Lemma 3.4.8. *Let R be a ring and V,W be R-modules. Then we have a canonical isomorphism*

$$\textstyle\bigwedge^n_R(V\oplus W)=\sum_{i=0}^n\bigwedge^{n-i}_R(V)\otimes_R\bigwedge^i_R(W).$$

Proof of Proposition 3.4.7.
In view of Proposition 3.4.5, it suffices to show that $\Omega^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}/K}\neq\langle 0\rangle$. We let $v_1,\dots,v_{n+1-\varrho_{\mathbb{X}}}$ and $L_1,\dots,L_{n+1-\varrho_{\mathbb{X}}}$ be defined as in the proof of Proposition 3.4.5. Then the vanishing ideal of $\mathbb{X}$ can be written as $\mathcal{I}_{\mathbb{X}}=\langle L_1,\dots,L_{n+1-\varrho_{\mathbb{X}}},F_1,\dots,F_r\rangle_S$ for some $F_j\in S$ with $\deg(F_j)\geq 2$. The $R_{\mathbb{X}}$-modules $M=\langle v_1,\dots,v_{n+1-\varrho_{\mathbb{X}}}\rangle_{R_{\mathbb{X}}}$ and $N=\langle e_{n+2-\varrho_{\mathbb{X}}},\dots,e_{n+1}\rangle_{R_{\mathbb{X}}}$

are graded-free $R_{\mathbb{X}}$-modules of rank $n+1-\varrho_{\mathbb{X}}$ and $\varrho_{\mathbb{X}}$ respectively. We proceed to show that $M \wedge_{R_{\mathbb{X}}} \bigwedge^{\varrho_{\mathbb{X}}-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}) \subsetneq \bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$. According to Lemma 3.4.8, we have

$$\begin{aligned} M \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{\varrho_{\mathbb{X}}-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}) &= M \wedge_{R_{\mathbb{X}}} \textstyle\bigwedge^{\varrho_{\mathbb{X}}-1}_{R_{\mathbb{X}}}(M \oplus N) \\ &= M \wedge_{R_{\mathbb{X}}} \Big(\textstyle\sum_{i=0}^{\varrho_{\mathbb{X}}-1} \bigwedge^{\varrho_{\mathbb{X}}-1-i}_{R_{\mathbb{X}}}(M) \otimes_{R_{\mathbb{X}}} \bigwedge^{i}_{R_{\mathbb{X}}}(N) \Big) \\ &= \textstyle\sum_{i=0}^{\varrho_{\mathbb{X}}-1} \bigwedge^{\varrho_{\mathbb{X}}-i}_{R_{\mathbb{X}}}(M) \otimes_{R_{\mathbb{X}}} \bigwedge^{i}_{R_{\mathbb{X}}}(N). \end{aligned}$$

Clearly, we have $M \cap N = \langle 0 \rangle$. Thus we get $\operatorname{rank}(\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})) = \sum_{i=0}^{\varrho_{\mathbb{X}}} \binom{n+1-\varrho_{\mathbb{X}}}{\varrho_{\mathbb{X}}-i} \cdot \binom{\varrho_{\mathbb{X}}}{i}$ and $\operatorname{rank}(M \wedge_{R_{\mathbb{X}}} \bigwedge^{\varrho_{\mathbb{X}}-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})) = \sum_{i=0}^{\varrho_{\mathbb{X}}-1} \binom{n+1-\varrho_{\mathbb{X}}}{\varrho_{\mathbb{X}}-i} \cdot \binom{\varrho_{\mathbb{X}}}{i}$, and hence

$$\operatorname{rank}(\textstyle\bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})) - \operatorname{rank}(M \wedge_{R_{\mathbb{X}}} \bigwedge^{\varrho_{\mathbb{X}}-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})) = 1.$$

Consequently, we have $M \wedge_{R_{\mathbb{X}}} \bigwedge^{\varrho_{\mathbb{X}}-1}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}}) \subsetneq \bigwedge^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}}(R^{n+1}_{\mathbb{X}})$. Based on the exact sequence (3.4) and the fact that $M \subseteq \mathcal{G}$ and $M_0 = \mathcal{G}_0$, we conclude that $\Omega^{\varrho_{\mathbb{X}}}_{R_{\mathbb{X}}/K} \neq \langle 0 \rangle$. $\square$

Let us illustrate this proposition with a concrete example.

Example 3.4.9. Let $\mathbb{X} = \{P_1, \dots, P_s\} \subset \mathbb{P}^n$ be a set of s distinct K-rational points lies on a line, i.e. $P_j = (1 : \lambda_j p_1 : \cdots : \lambda_j p_n)$ with $p_1, \dots, p_n \in K$, $p_1 \neq 0$, $\lambda_1 = 1$, and $\lambda_j \neq \lambda_k$ if $j \neq k$. Then we have

$$\begin{aligned} &\operatorname{HF}_{\Omega^1_{R_{\mathbb{X}}/K}} : 0\ 2\ 4\ 6 \cdots\ (2s-2)\ (2s-1)\ (2s-2)\ (2s-3) \cdots\ (s+1)\ s\ s \cdots \\ &\operatorname{HF}_{\Omega^2_{R_{\mathbb{X}}/K}} : 0\ 1\ 2\ \cdots\ (s-2)\ (s-1)\ (s-2)\ \cdots 2\ 1\ 0\ 0 \cdots \end{aligned}$$

and for $m \geq 3$ we have $\operatorname{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i) = 0$ for all $i \in \mathbb{N}$.

In the remainder of this section, we discuss some geometrical configurations of a finite set of s distinct K-rational points $\mathbb{X}$ in the projective plane $\mathbb{P}^2$ which are reflected in terms of Hilbert functions of the modules of Kähler differentials 3-forms. We begin with the following criterion for $\mathbb{X}$ to lie on a conic.

Proposition 3.4.10. *Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points in $\mathbb{P}^2$. Then $\mathbb{X}$ lies on a conic which is not a double line if and only if $\operatorname{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) \leq 1$ for all $i \in \mathbb{N}$ and $\operatorname{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$.*

Proof. Suppose that $\mathbb{X} \subseteq \mathcal{C} = \mathcal{Z}^+(C)$, where $C = a_{00}X_0^2 + 2a_{01}X_0X_1 + 2a_{02}X_0X_2 + a_{11}X_1^2 + 2a_{12}X_1X_2 + a_{22}X_2^2$ and $C \neq aL^2$ for any linear form $L \in S$ and $a \in K$. W.l.o.g. we may assume that $a_{00} \neq 0$. Since C is contained in the vanishing ideal $\mathcal{I}_{\mathbb{X}}$, the ideal $\langle \partial C/\partial X_0, \partial C/\partial X_1, \partial C/\partial X_2 \rangle_S$ is a subideal of $\langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\mathbb{X}}, 0 \leq i \leq 2 \rangle_S$. It is clear that we have $\langle \partial C/\partial X_0, \partial C/\partial X_1, \partial C/\partial X_2\rangle_S = \langle a_{00}X_0 + a_{01}X_1 + a_{02}X_2, a_{01}X_0 + a_{11}X_1 + a_{12}X_2, a_{02}X_0 + a_{12}X_1 + a_{22}X_2\rangle_S$. We see that the matrix $\mathcal{A} = (a_{ij})_{i,j=0,1,2}$ is of rank 1 if and only if it is of the form

$$\mathcal{A} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & \frac{a_{01}^2}{a_{00}} & \frac{a_{01}a_{02}}{a_{00}} \\ a_{02} & \frac{a_{01}a_{02}}{a_{00}} & \frac{a_{02}^2}{a_{00}} \end{pmatrix}.$$

If $\mathrm{rank}(\mathcal{A}) = 1$, then $a_{00}C = (a_{00}X_0 + a_{01}X_1 + a_{02}X_2)^2$, a contradiction. Thus we must have $\mathrm{rank}(\mathcal{A}) \geq 2$. We may assume that the two vectors (a_{00}, a_{01}, a_{02}) and (a_{01}, a_{11}, a_{12}) are linearly independent. Then the sequence of linear forms $a_{00}X_0 + a_{01}X_1 + a_{02}X_2, a_{01}X_0 + a_{11}X_1 + a_{12}X_2$ is a regular sequence for S. By Corollary 3.2.16, we have

$$\begin{aligned} \mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) &= \mathrm{HF}_{S/\langle \partial F/\partial X_i \mid F\in\mathcal{I}_{\mathbb{X}}, 0\leq i\leq 2\rangle_S}(i-3) \\ &\leq \mathrm{HF}_{S/\langle \partial C/\partial X_0, \partial C/\partial X_1, \partial C/\partial X_2\rangle_S}(i-3) \\ &\leq \mathrm{HF}_{S/\langle a_{00}X_0+a_{01}X_1+a_{02}X_2, a_{01}X_0+a_{11}X_1+a_{12}X_2\rangle_S}(i-3) \leq 1. \end{aligned}$$

Since $\mathbb{X}$ does not lie on any lines, we have $\langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\mathbb{X}}, 0 \leq i \leq 2\rangle_S \subseteq \langle X_0, X_1, X_2\rangle_S$, and therefore $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$.

Conversely, suppose $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) \leq 1$ for all $i \in \mathbb{N}$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$. If $\mathbb{X}$ lies on a line then Proposition 3.4.7 shows that $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) = 0$ for all $i \in \mathbb{N}$, a contradiction. Note that if $\mathbb{X}$ is contained in a double line then it lies on a line. Now we assume that $\mathbb{X}$ does not lie on any conic. Then $\mathcal{I}_{\mathbb{X}}$ is generated in degrees greater than 2. It follows that $\mathrm{HF}_{\langle \partial F/\partial X_i \mid F\in\mathcal{I}_{\mathbb{X}}, 0\leq i\leq 2\rangle_S}(1) = 0$. Thus the Hilbert function of $\Omega^3_{R_{\mathbb{X}}/K}$ satisfies $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 3$ by Corollary 3.2.16. This is a contradiction. □

Remark 3.4.11. It follows from the proof of Proposition 3.4.10 that a set of s distinct K-rational points $\mathbb{X}$ in $\mathbb{P}^2$ lies on a line if and only if $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) = 0$ for all $i \in \mathbb{N}$.

Lemma 3.4.12. *Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a 0-dimensional scheme, and let $\mathbb{Y}$ be a subscheme of $\mathbb{X}$. Then for all $m \in \mathbb{N}$ and for all $i \in \mathbb{Z}$ we have $\mathrm{HF}_{\Omega^m_{R_{\mathbb{Y}}/K}}(i) \leq \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i)$.*

Proof. Let $\mathcal{I}_{\mathbb{X}}$ and $\mathcal{I}_{\mathbb{Y}}$ be the homogeneous vanishing ideals of $\mathbb{X}$ and $\mathbb{Y}$, respectively. By Lemma 3.1.11, the canonical epimorphism $\pi : R_{\mathbb{X}} = S/\mathcal{I}_{\mathbb{X}} \to R_{\mathbb{Y}} = S/\mathcal{I}_{\mathbb{Y}}$ induces an epimorphism of graded $R_{\mathbb{X}}$-modules $\gamma : \Omega^1_{R_{\mathbb{X}}/K} \to \Omega^1_{R_{\mathbb{Y}}/K}$. By Proposition 2.3.7, for any $m \in \mathbb{N}$, the map γ induces an epimorphism of graded $R_{\mathbb{X}}$-modules $\phi : \Omega^m_{R_{\mathbb{X}}/K} \to \Omega^m_{R_{\mathbb{Y}}/K}$. Hence we obtain $\mathrm{HF}_{\Omega^m_{R_{\mathbb{Y}}/K}}(i) \leq \mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i)$ for all $i \in \mathbb{Z}$. □

Lemma 3.4.13. *Let $\mathbb{Y} = \{P_1, \ldots, P_5\}$ be a set of five reduced K-rational points in $\mathbb{P}^2$ which lie on 2 different lines. Assume that none of four points of $\mathbb{Y}$ lie on a line. Then we have* $\mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}} : 0\ 0\ 0\ 1\ 1\ 0\ \cdots$.

Proof. W.l.o.g, we may assume that $L_1 = a_0X_0 + a_1X_1 + a_2X_2$ and $L_2 = b_0X_0 + b_1X_1 + b_2X_2$ are lines through P_1, P_2, P_3 and P_4, P_5, respectively. For $i = 1, 2, 3$, the homogeneous vanishing ideal of P_i is of the form $\wp_i = \langle L_i', L_1\rangle_S$ with $L_i' = c_{i0}X_0 + c_{i1}X_1 + c_{i2}X_2$ and $c_{i0}, c_{i1}, c_{i2} \in K$. So, the vanishing ideal of $\{P_1, P_2, P_3\}$ is $\langle L_1'L_2'L_3', L_1\rangle_S$. Similarly, the vanishing ideal of $\{P_4, P_5\}$ is $\langle L_4'L_5', L_2\rangle_S$ for some linear forms $L_i' = c_{i0}X_0 + c_{i1}X_1 + c_{i2}X_2$, where $i = 4, 5$. By the assumption, we have $\gcd(L_1, L_2) = 1$. Therefore, if $F \in I_{\mathbb{Y}}$, then $\deg(F) \geq 2$. Let $F \in (I_{\mathbb{Y}})_2 \setminus \{0\}$. Then we have $F = F_1L_1 = F_2L_2 + \lambda L_4'L_5'$ for some $\lambda \in K$ and $\deg(F_1) = \deg(F_2) = 1$. For $i = 4, 5$, we have $F(P_i) = F_1(P_i)L_1(P_i) = (\lambda L_4'L_5')(P_i) = 0$. This means $F_1(P_4) = F_1(P_5) = 0$, and hence $F_1 = kL_2$ for some $k \in K \setminus \{0\}$. As a consequence, we have $F \in (I_{\mathbb{Y}})_2 \setminus \{0\}$ if and only if $F = kL_1L_2$ for some $k \in K \setminus \{0\}$. Since the matrix

$$\begin{pmatrix} 2a_0b_0 & a_0b_1 + a_1b_0 & a_0b_2 + a_2b_0 \\ a_0b_2 + a_2b_0 & 2a_1b_1 & a_1b_2 + a_2b_1 \\ a_2b_0 + a_0b_2 & a_2b_1 + a_1b_2 & 2a_2b_2 \end{pmatrix} = \begin{pmatrix} a_0 & b_0 & 0 \\ a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 \\ a_0 & a_1 & a_2 \\ 0 & 0 & 0 \end{pmatrix}$$

has determinant 0, we get $\langle a_0L_2+b_0L_1, a_1L_2+b_1L_1, a_2L_2+b_2L_1\rangle_S = \langle a_0L_2+b_0L_1, a_1L_2+b_1L_1\rangle_S \subseteq J = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\mathbb{X}}, 0 \leq i \leq 2\rangle_S$. Note that we have $\operatorname{rank}\left(\begin{smallmatrix} b_0 & b_1 & b_2 \\ a_0 & a_1 & a_2 \end{smallmatrix}\right) = 2$. This implies $\mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}}(4) = 1$. Moreover, we have $L_1L_4'L_5' \in \mathcal{I}_{\mathbb{Y}}$, and so $L_4'L_5' \in J$. By the inclusion $\langle L_1, L_2, L_4'L_5'\rangle_S \subseteq J$ and

$$\operatorname{rank}\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_{40} & c_{41} & c_{41} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_{50} & c_{51} & c_{51} \end{pmatrix} = 3$$

for all $H \in S$ of degree bigger than or equal to 2, we have $H \in \langle L_1, L_2, L_4'L_5'\rangle_S \subseteq J$. Thus we get $\mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}}(i) = 0$ for all $i > 4$. □

Corollary 3.4.14. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of $s \geq 5$ distinct K-rational points which lie on two different lines. Suppose there exist five points such that no four of them lie on a line. Then we have* $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = \mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 1$.

Proof. Apply Proposition 3.4.10 to the scheme $\mathbb{X}$ which lies on two different lines. This yields $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) \leq 1$. By Lemma 3.4.12 and Lemma 3.4.13, we get $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 1$. □

Lemma 3.4.15. *In $\mathbb{P}^2$, let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points which lie on a line $\mathcal{Z}^+(L)$, let $Q \notin \mathcal{Z}^+(L)$, and let $\mathbb{Y} = \mathbb{X} \cup \{Q\}$. Then we have $\mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}}(i) = 0$ for all $i \neq 3$.*

Proof. Since $\mathbb{X} \subseteq \mathcal{Z}^+(L)$, we may write $\wp_j = \langle L_j, L \rangle_S$ for some linear form $L_j \in S$, where $j \in \{1, \dots, s\}$. It follows that $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^s \wp_j = \langle F, L \rangle_S$ with $F = \prod_{j=1}^s L_j$. Since $Q \notin \mathcal{Z}^+(L)$, we may choose suitable linear forms L'_1 and L'_2 such that $\wp_Q = \langle L'_1, L'_2 \rangle_S$ and $L'_1(P_1) = 0$ and $L'_2(P_2) = 0$. Then the homogeneous vanishing ideal of $\mathbb{Y}$ is

$$\mathcal{I}_{\mathbb{Y}} = \mathcal{I}_{\mathbb{X}} \cap \wp_Q = \langle F, L \rangle_S \cap \langle L'_1, L'_2 \rangle_S \supseteq \langle FL'_1, FL'_2, L'_1 L, L'_2 L \rangle_S.$$

We write $L'_1 = a_0X_0 + a_1X_1 + a_2X_2$, $L'_2 = b_0X_0 + b_1X_1 + b_2X_2$, and $L = c_0X_0 + c_1X_1 + c_2X_2$, where $a_i, b_i, c_i \in K$. In the following we show that the rank of the matrix

$$\mathcal{L} = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix}$$

is 3. Otherwise, there are elements $\lambda_1, \lambda_2 \in K$ such that $L = \lambda_1 L'_1 + \lambda_2 L'_2$. We have $0 = L(P_1) = \lambda_1 L'_1(P_1) + \lambda_2 L'_2(P_1) = \lambda_2 L'_2(P_1)$. Since $L'_2(P_1) \neq 0$, we deduce $\lambda_2 = 0$. Similarly, we can show that $\lambda_1 = 0$. Thus we get $L = 0$, a contradiction. Hence we have $\mathrm{rank}(\mathcal{L}) = 3$, as claimed.

Now we see that $\langle \partial LL'_k/\partial X_i \mid 0 \leq i \leq 2, k = 1,2 \rangle_S = \langle a_0L + c_0L'_1, a_1L + c_1L'_1, a_2L + c_2L'_1, b_0L + c_0L'_2, b_1L + c_1L'_2, b_2L + c_2L'_2 \rangle_S \subseteq J = \langle \partial G/\partial X_i \mid G \in \mathcal{I}_{\mathbb{Y}}, 0 \leq i \leq 2 \rangle_S \subseteq \langle X_0, X_1, X_2 \rangle_S$. Since $L \neq L'_1$, we may assume that $\det \begin{pmatrix} a_0 & c_0 \\ a_1 & c_1 \end{pmatrix} \neq 0$ and $c_0 \neq 0$. Then

$$\det(\mathcal{M}) = \det \left(\begin{pmatrix} a_0 & c_0 & 0 \\ a_1 & c_1 & 0 \\ b_0 & 0 & c_0 \end{pmatrix} \begin{pmatrix} c_0 & c_1 & c_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \right) \neq 0.$$

Moreover, we have $\mathcal{M} \cdot (X_0 \ X_1 \ X_2)^{\mathrm{tr}} = (a_0L + c_0L'_1 \quad a_1L + c_1L'_1 \quad b_0L + c_0L'_2)^{\mathrm{tr}}$. Hence we get $\langle a_0L + c_0L'_1, a_1L + c_1L'_1, b_0L + c_0L'_2 \rangle_S = \langle X_0, X_1, X_2 \rangle_S$, and consequently $J = \langle X_0, X_1, X_2 \rangle_S$. Thus, by Corollary 3.2.16, we obtain $\mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}}(3) = \mathrm{HF}_{S/J}(0) = 1$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}}(i) = \mathrm{HF}_{S/J}(i-3) = 0$ for all $i \neq 3$. □

Corollary 3.4.16. *In the setting of Lemma 3.4.15, the Castelnuovo function of $\mathbb{Y}$ $\Delta \mathrm{HF}_{\mathbb{Y}}(i) := \mathrm{HF}_{\mathbb{Y}}(i) - \mathrm{HF}_{\mathbb{Y}}(i-1)$ satisfies $\Delta \mathrm{HF}_{\mathbb{Y}}(i) \leq 1$ for all $i \geq 2$.*

Proof. Using the notation as in the proof of Lemma 3.4.15, we have

$$\mathcal{I}_{\mathbb{Y}} = \mathcal{I}_{\mathbb{X}} \cap \wp_Q = \langle F, L\rangle_S \cap \langle L_1', L_2'\rangle_S \supseteq \langle FL_1', FL_2', L_1'L, L_2'L\rangle_S.$$

This yields $\Delta\,\mathrm{HF}_{\mathbb{Y}}(2) = 1$, and therefore $\Delta\,\mathrm{HF}_{\mathbb{Y}}(i) \le 1$ for all $i \ge 2$ (see [KR2, Corollary 5.5.28]). □

Lemma 3.4.17. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points which lie on a non-singular conic $\mathcal{C} = \mathcal{Z}^+(C)$. Then we have $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) = 0$ for $i \neq 3$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$.*

Proof. Let us write $C = a_{00}X_0^2 + 2a_{01}X_0X_1 + 2a_{02}X_0X_2 + a_{11}X_1^2 + 2a_{12}X_1X_2 + a_{22}X_2^2$, where $a_{jk} \in K$, and let $\mathcal{A} = (a_{jk})_{j,k=0,1,2}$. We know that $\mathcal{C} = \mathcal{Z}^+(C)$ is a non-singular conic if and only if $\mathrm{rank}(\mathcal{A}) = 3$. This implies the equality $\langle \partial C/\partial X_i \mid 0 \le i \le 2\rangle_S = \langle a_{00}X_0 + a_{01}X_1 + a_{02}X_2, a_{10}X_0 + a_{11}X_1 + a_{12}X_2, a_{20}X_0 + a_{21}X_1 + a_{22}X_2\rangle_S = \langle X_0, X_1, X_2\rangle_S$. Since $C \in \mathcal{I}_{\mathbb{X}}$, we have $J = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\mathbb{X}}, 0 \le i \le 2\rangle_S = \langle X_0, X_1, X_2\rangle_S$. Therefore it follows from the isomorphism $\Omega^3_{R_{\mathbb{X}}/K} \cong (S/J)(-3)$ that $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) = 0$ for $i \neq 3$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$, as desired. □

Corollary 3.4.18. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points with $s \ge 5$. If $\mathbb{X}$ lies on a non-singular conic $\mathcal{C} = \mathcal{Z}^+(C)$, then there is an index $i \ge 2$ such that $\Delta\,\mathrm{HF}_{\mathbb{X}}(i) \ge 2$.*

To prove the corollary, we use the following lemma which is mentioned in [Uen, Theorem 1.32].

Lemma 3.4.19. *Let $\mathcal{C}_1 = \mathcal{Z}\langle F\rangle$ and $\mathcal{C}_2 = \mathcal{Z}\langle G\rangle$ be plane curves of degree m and n, respectively, in projective plane $\mathbb{P}^2$. If F and G do not possess a common divisor, then*

$$\mathcal{C}_1 \cdot \mathcal{C}_2 = mn.$$

Proof of Corollary 3.4.18. Since $\deg(\mathbb{X}) = s > 4$, it follows from Lemma 3.4.19 that the homogeneous vanishing ideal $\mathcal{I}_{\mathbb{X}}$ contains only one homogeneous polynomial of degree 2. Assume that $\mathcal{I}_{\mathbb{X}} = \langle C, F_1, \dots, F_t\rangle_S$, where $\deg(F_j) > 2$ for all $j = 1, \dots, t$. Then we have $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_{S/\langle C, F_1, \dots, F_t\rangle_S} :\ 1\ 3\ 5\ *\ * \cdots$. This implies that $\Delta\,\mathrm{HF}_{\mathbb{X}}(2) = 2$. □

Corollary 3.4.20. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points with $s \ge 5$.*

(i) If $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 1$, then $\mathbb{X}$ lies on two different lines and no $s-1$ points of $\mathbb{X}$ lie on a line.

(ii) Suppose that $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$ *and* $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(i) = 0$ *for* $i \neq 3$. *If* $\Delta\,\mathrm{HF}_{\mathbb{X}}(2) = 1$, *then* $\mathbb{X}$ *contains* $s-1$ *points on a line. Otherwise,* $\mathbb{X}$ *lies on a non-singular conic.*

Proof. By Propositions 3.4.5 and 3.4.10, we have $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) \leq 1$ and $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$ if and only if $\mathbb{X}$ lies on a conic. There are three possibilities for the position of the points of $\mathbb{X}$.

If $\mathbb{X}$ lies on two different lines, such that no $s-1$ points lie on a line then Corollary 3.4.14 yields $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = \mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 1$.

If $\mathbb{X}$ lies on a non-singular conic then Lemma 3.4.17 and Corollary 3.4.18 yield $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1, \mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 0$ and $\Delta\,\mathrm{HF}_{\mathbb{X}}(2) = 2$.

If $\mathbb{X}$ contains $s-1$ points on a line and the other points of $\mathbb{X}$ do not lie on this line then by Lemma 3.4.15 and Corollary 3.4.16, we get $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(3) = 1$, $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}(4) = 0$, and $\Delta\,\mathrm{HF}_{\mathbb{X}}(2) = 1$. □

Example 3.4.21. Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of six K-rational points on a non-singular conic, e.g.,

$$\mathbb{X} = \{(1:0:1), (1:0:-1), (3:4:5), (3:-4:5), (3:-4:-5), (3:4:-5)\},$$

and let $\mathbb{Y} \subseteq \mathbb{P}^2$ be a set of six K-rational points on a singular conic, e.g.,

$$\mathbb{Y} = \{(1:0:2), (1:1:0), (1:0:1), (1:2:0), (1:-1:0), (1:0:-1)\}.$$

Then we see that $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_{\mathbb{Y}} :\ 1\ 3\ 5\ 6\ 6 \cdots$ and

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}} = \mathrm{HF}_{\Omega^1_{R_{\mathbb{Y}}/K}} :\ 0\ 3\ 8\ 11\ 10\ 7\ 6\ 6 \cdots .$$

Moreover, the minimal graded free resolutions of $R_{\mathbb{Y}}$ and $R_{\mathbb{X}}$ are the same, i.e. we have

$$0 \longrightarrow S(-5) \longrightarrow S(-2) \oplus S(-3) \longrightarrow S \longrightarrow S/\mathcal{I} \longrightarrow 0$$

where $\mathcal{I}$ is either $\mathcal{I}_{\mathbb{X}}$ or $\mathcal{I}_{\mathbb{Y}}$ (see [TT, Example 4.1]). However, in this case we have

$$\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}} : 0\ 0\ 3\ 6\ 4\ 1\ 0 \cdots, \quad \mathrm{HF}_{\Omega^2_{R_{\mathbb{Y}}/K}} :\ 0\ 0\ 3\ 6\ 5\ 1\ 0\ 0 \cdots$$
$$\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}} : 0\ 0\ 0\ 1\ 0\ 0 \cdots, \quad \mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}} :\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \cdots .$$

This shows that we can distinguish two sets $\mathbb{X}$ and $\mathbb{Y}$ by looking at the Hilbert functions of the modules of their Kähler differential m-forms, where $m = 2, 3$. But we can not distinguish them by either the Hilbert functions of their homogeneous coordinate rings or their minimal free resolutions.

Chapter 4

Kähler Differential Algebras for Fat Point Schemes

Given a finite set of points $\mathbb{X} = \{P_1, ..., P_s\}$ in the projective n-space $\mathbb{P}^n$, a *fat point scheme* $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ is a 0-dimensional scheme whose homogeneous vanishing ideal is of the form $\mathcal{I}_{\mathbb{W}} = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ where $m_i \geq 1$ and $\wp_i$ is the homogeneous vanishing ideal of P_i. In the last twenty years many papers have investigated fat point schemes $\mathbb{W}$ in $\mathbb{P}^n$ by looking at their algebraic properties such as the Hilbert function, minimal graded free resolution, and graded Betti numbers (see for instance [DG], [DK], [Ca], [GT],[GMT]). In [DK], G. Dominicis and M. Kreuzer used the module of Kähler differential 1-forms as a new tool for studying sets of K-rational points $\mathbb{X}$ in $\mathbb{P}^n$. Inspired by their ideas, in this chapter we explore Kähler differential algebras for fat point schemes in $\mathbb{P}^n$.

In the first section we recall the definition and some properties of minimal separators of a fat point scheme $\mathbb{W}$ in $\mathbb{P}^n$ (see Proposition 4.1.7 and Proposition 4.1.11). From that, we define $\mathrm{Deg}_{\mathbb{W}}(m_jP_j)$, the degree of the fat point m_jP_j in $\mathbb{W}$. Also, we deduce relations between $\mathrm{Deg}_{\mathbb{W}}(m_jP_j)$ and the Hilbert function of the subscheme $\mathbb{W}_j = m_1P_1 + \cdots + m_{j-1}P_{j-1} + m_{j+1}P_{j+1} + \cdots + m_sP_s$ of $\mathbb{W}$ (see Propositions 4.1.12 and 4.1.17 and Corollary 4.1.13). In addition, when $\mathbb{W} = \nu\mathbb{X}$ is an equimultiple fat point scheme whose support $\mathbb{X}$ is a complete intersection, we collect some results on the minimal graded free resolutions of $\mathbb{W}$ and its subschemes (see Propositions 4.1.18 and 4.1.19).

In Section 4.2 we focus on the modules of Kähler differential 1-forms for fat point schemes. A remarkable result in this section is the short exact sequence of graded $R_{\mathbb{W}}$-modules $0 \to \mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{V}} \to R_{\mathbb{W}}^{n+1} \to \Omega^1_{R_{\mathbb{W}}/K} \to 0$ where $\mathcal{I}_{\mathbb{V}}$ is the vanishing ideal of the fat point scheme $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ (see Theorem 4.2.1). From this result, we deduce various formulas for Hilbert functions, Hilbert polynomials and regularity indices of the modules $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ (see Corollary 4.2.3). Using the Hilbert polynomial of $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$, we can characterize whether a fat point scheme $\mathbb{W}$ is reduced or not (see Corollary 4.2.4). Also, we give relations between modules of Kähler differential 1-forms and degree of a fat point scheme when the support $\mathbb{X} = \{P_1, ..., P_s\}$ is a Caylay-Bacharach scheme (see Propositions 4.2.5 and 4.2.6). Furthermore, if $\mathbb{X}$ is in general position, we get sharp upper bounds for the regularity indices of the modules $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ (see Corollary 4.2.10 and Example 4.2.11).

Continuing the studies of Section 4.2, in the third section, we are concerned with the modules of Kähler differential m-forms of fat point schemes. We first bound the Hilbert polynomials of $\Omega^m_{R_{\mathbb{W}}/K}$ and $\Omega^m_{R_{\mathbb{W}}/K[x_0]}$ (see Proposition 4.3.1). Then we point out that for $2 \le m \le n+1$, the Hilbert polynomial of $\Omega^m_{R_{\mathbb{W}}/K}$ is a non-zero constant if and only if $\mathbb{W}$ is a non-reduced scheme (see Corollary 4.3.2). Also, sharp upper bounds for the regularity indices of $\Omega^m_{R_{\mathbb{W}}/K}$ and $\Omega^m_{R_{\mathbb{W}}/K[x_0]}$ are given in Proposition 4.3.4. Moreover, we present a formula for the Hilbert function of $\Omega^{n+1}_{R_{\mathbb{W}}/K}$ when $\mathrm{Supp}(\mathbb{W})$ lies on a hyperplane (see Propositions 4.3.6 and 4.3.9). If $\mathbb{W} = \nu\mathbb{X}$ is an equimultiple fat point scheme then the Hilbert polynomial of $\Omega^{n+1}_{R_{\mathbb{W}}/K}$ is given by $\mathrm{HP}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(z) = s\binom{\nu+n-2}{n}$ (see Proposition 4.3.11 and Corollary 4.3.13). In the remainder of this section, we establish a relation between the module of Kähler differential 2-forms $\Omega^2_{R_{\mathbb{W}}/K}$ and other fat point schemes which are deduced from the equimultiple fat point scheme $\mathbb{W}$ in $\mathbb{P}^2$ (see Proposition 4.3.14 and Corollaries 4.3.15 and 4.3.17). From this relation, we also deduce a formula for the Hilbert polynomial of $\Omega^2_{R_{\mathbb{W}}/K}$.

In Section 4.4 we study the modules of Kähler differential m-forms, where $m = 1$ or $m = n+1$, of a fat point scheme $\mathbb{W}$ supported at a complete intersection $\mathbb{X}$. First we show that the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K}$ of an equimultiple fat point scheme $\mathbb{W} = \nu\mathbb{X}$ can be calculated from that of $\mathbb{X}$ (see Proposition 4.4.2). In this case, the regularity index of $\Omega^1_{R_{\mathbb{W}}/K}$ is given by $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) = \nu d_n + \sum_{j=1}^n d_j - n$, where $(d_1, ..., d_n)$ is the complete intersection type of $\mathbb{X}$ and $d_1 \le \cdots \le d_n$ (see Corollary 4.4.4). When $\mathbb{W}$ is not an equimultiple fat point scheme, Proposition 4.4.6 provides upper and lower bounds for the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K}$. We also derive from this proposition an upper bound for the regularity index of $\Omega^m_{R_{\mathbb{W}}/K}$ for $m = 1, ..., n+1$ (see Corollary 4.4.8). Furthermore, if $\nu \ge 2$ and if $\mathbb{W} = \nu\mathbb{X}$ is supported at a complete intersection $\mathbb{X}$, we show that the Hilbert function of $\Omega^1_{R_{\mathbb{W}_j}/K}$ is independent of j where $\mathbb{W}_j = \sum_{i\ne j} \nu P_i + (\nu-1)P_j$ (see Proposition 4.4.9). In the case $\nu = 1$, this result holds when the complete intersection $\mathbb{X}$ is of type $(d, \ldots, d)$ (see Proposition 4.4.11). We end this section with a result on the module of Kähler differential $n+1$-forms of an equimultiple fat point scheme $\mathbb{W}$ in some special cases of complete intersection types $(d_1, \dots, d_n)$ (see Proposition 4.4.12).

Throughout this chapter we work over a field K of characteristic zero. By $\mathbb{P}^n$ we denote the projective n-space over K. The homogeneous coordinate ring of $\mathbb{P}^n$ is $S = K[X_0, \dots, X_n]$. It is equipped with the standard grading $\deg(X_i) = 1$ for $i = 0, \dots, n$.

4.1 Fat Point Schemes

Let $s \geq 1$, and let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points in $\mathbb{P}^n$. For $i = 1, \dots, s$, we let $\wp_i$ be the associated prime ideal of P_i in S.

Definition 4.1.1. Given a sequence of positive integers $m_1, \dots, m_s$, the intersection $\mathcal{I}_{\mathbb{W}} := \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$ is a saturated homogeneous ideal in S and is therefore the vanishing ideal of a 0-dimensional subscheme $\mathbb{W}$ of $\mathbb{P}^n$.

(i) The scheme $\mathbb{W}$, denoted by $\mathbb{W} = m_1P_1 + \dots + m_sP_s$, is called a **fat point scheme** in $\mathbb{P}^n$. The homogeneous vanishing ideal of $\mathbb{W}$ is $\mathcal{I}_{\mathbb{W}}$. The number m_j is called the **multiplicity** of the point P_j for $j = 1, \dots, s$.

(ii) If $m_1 = \dots = m_s = \nu$, we denote $\mathbb{W}$ also by $\nu\mathbb{X}$ and call it an **equimultiple fat point scheme**.

The homogeneous coordinate ring of the scheme $\mathbb{W}$ is $R_{\mathbb{W}} = S/\mathcal{I}_{\mathbb{W}}$. The ring $R_{\mathbb{W}} = \bigoplus_{i \geq 0} (R_{\mathbb{W}})_i$ is a standard graded K-algebra and its homogeneous maximal ideal is $\mathfrak{m}_{\mathbb{W}} = \bigoplus_{i \geq 1} (R_{\mathbb{W}})_i$. Notice that the support of $\mathbb{W}$ is $\operatorname{Supp}(\mathbb{W}) = \mathbb{X} = \{P_1, \dots, P_s\}$. Since K is infinite, we can choose the coordinate system $\{X_0, \dots, X_n\}$ such that no point of $\mathbb{X}$ lies on the hyperplane at infinity $\mathcal{Z}^+(X_0)$. The image of X_i in $R_{\mathbb{W}}$ is denoted by x_i for $i = 0, \dots, n$. Then x_0 is a non-zerodivisor for $R_{\mathbb{W}}$.

As usual, we let $r_{\mathbb{W}}$ denote the regularity index of $\mathrm{HF}_{\mathbb{W}}$, i.e. $r_{\mathbb{W}} = \min\{i \in \mathbb{Z} \mid \mathrm{HF}_{\mathbb{W}}(j) = \mathrm{HP}_{\mathbb{W}}(j) \text{ for all } j \geq i\}$. According to Proposition 2.4.5, we have $\mathrm{HF}_{\mathbb{W}}(i) = 0$ for $i < 0$, $1 = \mathrm{HF}_{\mathbb{W}}(0) < \mathrm{HF}_{\mathbb{W}}(1) < \dots < \mathrm{HF}_{\mathbb{W}}(r_{\mathbb{W}} - 1) < \deg(\mathbb{W})$, and for $i \geq r_{\mathbb{W}}$, we have $\mathrm{HF}_{\mathbb{W}}(i) = \deg(\mathbb{W})$.

Let us illustrate the concepts of fat point schemes and their Hilbert functions with the following example.

Example 4.1.2. Let $\mathbb{W}$ be a fat point scheme $\mathbb{W} = m_1P_1 + \dots + m_sP_s$ in $\mathbb{P}^1$, where $P_j = (1 : p_j)$, $p_j \in K$ for $j = 1, \dots, s$. Then the vanishing ideal of $\mathbb{W}$ is $\mathcal{I}_{\mathbb{W}} = \langle F \rangle_S$ with $F = (X_1 - p_1X_0)^{m_1} \cdots (X_1 - p_sX_0)^{m_s}$. Thus we have $\alpha_{\mathbb{W}} = \sum_{j=1}^{s} m_j = \deg(\mathbb{W})$,

$r_{\mathbb{W}} = \sum_{j=1}^{s} m_j - 1$, and the Hilbert function of $\mathbb{W}$ depend only on the sum of the multiplicities of the points in $\mathbb{W}$, i.e.

$$\mathrm{HF}_{\mathbb{W}} : 1\ 2\ \cdots\ \sum_{j=1}^{s} m_j - 1\ \sum_{j=1}^{s} m_j\ \sum_{j=1}^{s} m_j \cdots .$$

In many cases, the Hilbert function of a fat point scheme $\mathbb{W}$ depends not only on the positions of the points in $\mathbb{W}$ but also their multiplicities (see for instance Example 4.1.4). However, the following proposition show that the degree $\deg(\mathbb{W})$ is independent of the position of the points in $\mathbb{W}$. This result is well known. However, for convenience of the reader, we include its proof.

Proposition 4.1.3. *Let* $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ *be a fat point scheme in* $\mathbb{P}^n$. *Then the Hilbert polynomial of* $\mathbb{W}$ *is* $\mathrm{HP}_{\mathbb{W}}(z) = \deg(\mathbb{W}) = \sum_{j=1}^{s} \binom{m_j+n-1}{n}$. *In particular, if* $\mathbb{W} = \nu\mathbb{X}$ *is an equimultiple fat point scheme then* $\mathrm{HP}_{\mathbb{W}}(z) = \deg(\mathbb{W}) = s\binom{\nu+n-1}{n}$.

Proof. Let $1 \le t \le s-1$. We claim that $\mathrm{HP}_{S/(\cap_{j=1}^{t}\wp_j^{m_j}+\wp_{t+1}^{m_{t+1}})}(z) = 0$. Indeed, let F be a homogeneous polynomial $F \in \cap_{j=1}^{t}\wp_j^{m_j} \setminus \wp_{t+1}$. Then there are $k_t \in \mathbb{N}$ and $G \in \wp_{t+1}$ such that $X_0^{k_t} = F+G$. We set $n_{t+1} = k_t + m_{t+1}$. Then $(X_0^{k_t} - F)^{m_{t+1}} = G^{m_{t+1}} \in \wp_{t+1}^{m_{t+1}}$, and so $X_0^{n_{t+1}} \in \cap_{j=1}^{t}\wp_j^{m_j} + \wp_{t+1}^{m_{t+1}}$. We denote the maximal ideal $\langle X_0, ..., X_n\rangle \subseteq S$ by $\mathfrak{M}$. For any homogeneous polynomial $H \in \mathfrak{M}^{n_{t+1}+m_{t+1}}$, by the Dirichlet's box principle, there exist polynomials $H_1 \in \wp_{t+1}^{m_{t+1}}$ and $H_2 \in S$ such that $H = H_1 + X_0^{n_{t+1}}H_2 \in \cap_{j=1}^{t}\wp_j^{m_j} + \wp_{t+1}^{m_{t+1}}$. Thus we get $\mathfrak{M}^{n_{t+1}+m_{t+1}} \subseteq \cap_{j=1}^{t}\wp_j^{m_j} + \wp_{t+1}^{m_{t+1}} \subseteq \mathfrak{M}$, and the claim follows.

Given homogeneous ideals I and J in S, we have the exact sequence of graded rings:

$$0 \longrightarrow S/(I \cap J) \xrightarrow{\alpha} S/I \oplus S/J \xrightarrow{\beta} S/(I+J) \longrightarrow 0,$$

where α is given by $\alpha(a+(I\cap J)) = (a+I, a+J)$ and where β is given by $\beta(a+I, b+J) = (a-b) + (I+J)$. We deduce from this exact sequence that

$$\mathrm{HP}_{\mathbb{W}}(z) = \mathrm{HP}_{S/\cap_{j=1}^{s}\wp_j^{m_j}}(z) = \mathrm{HP}_{S/\cap_{j=1}^{s-1}\wp_j^{m_j}}(z) + \mathrm{HP}_{S/\wp_s^{m_s}}(z) = \sum_{j=1}^{s} \mathrm{HP}_{S/\wp_j^{m_j}}(z).$$

For $t \in \mathbb{N}$, the Hilbert polynomial of $\wp_j^{t-1}/\wp_j^{t}$ is $\binom{n-1+t-1}{n-1}$, the Hilbert polynomial of $S/\wp_j$ is 1, and hence

$$\begin{aligned}\mathrm{HP}_{P/\wp_j^{m_j}}(z) &= \mathrm{HP}_{\wp_j^{m_j-1}/\wp_j^{m_j}}(z) + \mathrm{HP}_{P/\wp_j^{m_j-1}}(z)\\ &= \binom{n-1+m_j-1}{n-1} + \cdots + \binom{n-1}{n-1} = \binom{m_j+n-1}{n}.\end{aligned}$$

Therefore the conclusion follows. □

The following example show that the Hilbert functions of fat point schemes depend on the position of their points.

Example 4.1.4. Let $\mathbb{X}$ be a set of six $\mathbb{Q}$-rational points on a conic and two $\mathbb{Q}$-rational points off that conic $\mathbb{X} = \{P_1, \dots, P_8\}$ in $\mathbb{P}^2$, where $P_1 = (1:1:1)$, $P_2 = (1:-1:-1)$, $P_3 = (1:1:-1)$, $P_4 = (1:-1:1)$, $P_5 = (1:3/4:-3/5)$, $P_6 = (1:-3/4:3/5)$, and $P_7 = (1:3:3)$, and $P_8 = (1:-3:-3)$. We consider two fat point schemes $\mathbb{V} = P_1+P_2+P_3+P_4+P_5+2P_6+P_7+P_8$ and $\mathbb{W} = P_1+P_2+P_3+P_4+P_5+P_6+2P_7+P_8$ in $\mathbb{P}^2$. Then we have $\mathrm{HF}_{\mathbb{V}}:\ 1\ 3\ 6\ 10\ 10 \cdots$ and $\mathrm{HF}_{\mathbb{W}}:\ 1\ 3\ 6\ 9\ 10\ 10 \cdots$. Clearly, the Hilbert polynomials of $\mathbb{V}$ and $\mathbb{W}$ are both given by

$$\mathrm{HP}_{\mathbb{V}}(z) = \mathrm{HP}_{\mathbb{W}}(z) = 7\binom{1+2-1}{2} + \binom{2+2-1}{2} = 10.$$

In Section 3.4 we use separators to relate a set of point to its subsets. Similar to the definition of separators of a finite set of reduced K-rational points in $\mathbb{P}^n$, we now introduce the concept of separators of fat point schemes, which shall be useful in later sections.

Definition 4.1.5. Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, and let $j \in \{1, \dots, s\}$. By $\mathbb{W}_j$ we denote the subscheme $\mathbb{W}_j = m_1P_1 + \cdots + m_{j-1}P_{j-1} + (m_j - 1)P_j + m_{j+1}P_{j+1} + \cdots + m_sP_s$ of $\mathbb{W}$. Let $\mathcal{I}_{\mathbb{W}_j/\mathbb{W}}$ be the ideal of $\mathbb{W}_j$ in $R_{\mathbb{W}}$, i.e. the residue class ideal of $\mathcal{I}_{\mathbb{W}_j}$ in $R_{\mathbb{W}}$. Then any non-zero homogeneous element of $\mathcal{I}_{\mathbb{W}_j/\mathbb{W}}$ is called a **separator** of $\mathbb{W}_j$ in $\mathbb{W}$ (or of P_j of multiplicity m_j).

Remark 4.1.6. If $m_1 = \cdots = m_s = 1$, i.e. if $\mathbb{W} = \mathbb{X} = \{P_1, \dots, P_s\}$ is a reduced scheme, and if $j \in \{1, \dots, s\}$ then a separator $f \in R_{\mathbb{W}}$ of $\mathbb{W} \setminus \{P_j\}$ in $\mathbb{W}$ satisfies $f(P_j) \neq 0$ and $f(P_k) = 0$ for all $k \neq j$. Thus f is a separator of $\mathbb{W} \setminus \{P_j\}$ in $\mathbb{W}$ in the usual sense (see Section 2.4).

It is clear that separators are not unique. Two separators of $\mathbb{W}_j$ in $\mathbb{W}$ differ by an element of $I_{\mathbb{W}}$. The existence of separators is given by the following proposition (cf. [GMT, Theorem 3.3]).

Proposition 4.1.7. *Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, and let $j \in \{1, \dots, s\}$. By $\mathbb{W}_j$ we denote the subscheme $\mathbb{W}_j = m_1P_1 + \cdots + m_{j-1}P_{j-1} + (m_j - 1)P_j + m_{j+1}P_{j+1} + \cdots + m_sP_s$ of $\mathbb{W}$. Then there exist $\nu_j = \deg(\mathbb{W}) - \deg(\mathbb{W}_j)$ homogenous elements $f_{j1}^*, \dots, f_{j\nu_j}^* \in R_{\mathbb{W}}$ such that $\mathcal{I}_{\mathbb{W}_j/\mathbb{W}} = \langle f_{j1}^*, \dots, f_{j\nu_j}^* \rangle_{R_{\mathbb{W}}}$ and these elements form a minimal system of generators of $\mathcal{I}_{\mathbb{W}_j/\mathbb{W}}$.*

Definition 4.1.8. Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, and let $j \in \{1,\dots,s\}$. For $k \in \{0,\dots,m_j\}$, we denote by $\mathbb{W}_{jk}$ the subscheme $\mathbb{W}_{jk} = m_1P_1 + \cdots + m_{j-1}P_{j-1} + (m_j - k)P_j + m_{j+1}P_{j+1} + \cdots + m_sP_s$ of $\mathbb{W}$. Let $\nu_j = \deg(\mathbb{W}) - \deg(\mathbb{W}_{jm_j})$.

a) The set $\{f^*_{j1},\dots,f^*_{j\nu_j}\}$ given in Proposition 4.1.7 is called a **minimal set of separators** of $\mathbb{W}_j$ in $\mathbb{W}$ (or of P_j of multiplicity m_j).

b) Let $\{f^*_{j1},\dots,f^*_{j\nu_j}\}$ be a minimal set of separators of $\mathbb{W}_{j1}$ in $\mathbb{W}$. Suppose that $\deg(f^*_{j1}) \le \cdots \le \deg(f^*_{j\nu_j})$. We define the **degree of** P_j **in** $\mathbb{W}$, denote by $\operatorname{sepdeg}_{\mathbb{W}}(P_j)$, as $\operatorname{sepdeg}_{\mathbb{W}}(P_j) = (\deg(f^*_{j1}),\dots,\deg(f^*_{j\nu_j}))$.

c) Let S be the set of all entries of the tuples $\operatorname{sepdeg}_{\mathbb{W}_{j0}}(P_j),\dots,\operatorname{sepdeg}_{\mathbb{W}_{j\,m_j-1}}(P_j)$. The tuple of elements of S in increasing order is called the **degree of the fat point** m_jP_j in $\mathbb{W}$ and is denoted by $\operatorname{Sepdeg}_{\mathbb{W}}(m_jP_j)$.

Remark 4.1.9. We make some observations about Definition 4.1.8.

(i) If $m_1 = \cdots = m_s = 1$, i.e. if $\mathbb{W}$ is a set of s distinct K-rational points in $\mathbb{P}^n$, then $\operatorname{Sepdeg}_{\mathbb{W}}(P_j) = \operatorname{sepdeg}_{\mathbb{W}}(P_j)$ for all $j = 1,\dots,s$. In this case we also write $\operatorname{sepdeg}_{\mathbb{W}}(P_j) = \deg(f^*_{j1})$ for all $j = 1,\dots,s$.

(ii) When $k = 0$, we have $\operatorname{sepdeg}_{\mathbb{W}_{jk}}(P_j) = \operatorname{sepdeg}_{\mathbb{W}}(P_j)$.

(iii) If $\operatorname{Sepdeg}_{\mathbb{W}}(m_jP_j) = \operatorname{Sepdeg}_{\mathbb{W}}(m_kP_k)$, then we have $m_j = m_k$.

The following example shows how to compute the degree of the fat point m_jP_j in $\mathbb{W}$, where $m_j \ge 2$. Moreover, this shows that the converse of Remark 4.1.9(iii) is not true in general.

Example 4.1.10. Let $\mathbb{X} = \{P_1,\dots,P_6\} \subseteq \mathbb{P}^3$ be the set of six K-rational points on a plane given by $P_1 = (1:0:0:0)$, $P_2 = (1:0:1:0)$, $P_3 = (1:0:2:0)$, $P_4 = (1:2:1:0)$, $P_5 = (1:1:1:0)$, and $P_6 = (1:2:3:0)$. Let $\mathbb{W}$ be the fat point scheme $\mathbb{W} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_5 + 3P_6$ in $\mathbb{P}^3$. We see that $\mathrm{HF}_{\mathbb{W}}:\ 1\ 4\ 10\ 18\ 25\ 30\ 30\cdots$ and $r_{\mathbb{W}} = 5$. Let $j = 6$. The subscheme $\mathbb{W}_{61} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_5 + 2P_6$ has $\mathrm{HF}_{\mathbb{W}_{61}}:\ 1\ 4\ 10\ 17\ 22\ 24\ 24\dots$ and $r_{\mathbb{W}_{61}} = 5$. By using Proposition 4.1.11, we get $\operatorname{sepdeg}_{\mathbb{W}_{60}}(P_6) = \operatorname{sepdeg}_{\mathbb{W}}(P_6) = (3,4,4,5,5,5)$. Similarly, the subscheme $\mathbb{W}_{62} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_5 + P_6$ has $\mathrm{HF}_{\mathbb{W}_{62}}:\ 1\ 4\ 10\ 16\ 20\ 21\ 21\dots$ and $r_{\mathbb{W}_{62}} = 5$, and so $\operatorname{sepdeg}_{\mathbb{W}_{61}}(P_6) = (3,4,5)$. The subscheme $\mathbb{W}_{63} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_5$ has $\mathrm{HF}_{\mathbb{W}_{63}}:\ 1\ 4\ 10\ 16\ 19\ 20\ 20\dots$ and $r_{\mathbb{W}_{63}} = 5$, and thus $\operatorname{sepdeg}_{\mathbb{W}_{62}}(P_6) = (4)$. Altogether, we obtain

$$\operatorname{Sepdeg}_{\mathbb{W}}(3P_6) = (3,3,4,4,4,4,5,5,5,5).$$

Moreover, we have

$$\begin{aligned}\mathrm{Sepdeg}_{\mathbb{W}}(3P_5) &= (3,3,4,4,4,5,5,5,5,5),\\ \mathrm{Sepdeg}_{\mathbb{W}}(2P_3) = \mathrm{Sepdeg}_{\mathbb{W}}(2P_4) &= (4,5,5,5),\\ \mathrm{Sepdeg}_{\mathbb{W}}(P_1) = \mathrm{Sepdeg}_{\mathbb{W}}(P_2) &= (5).\end{aligned}$$

In this case, we also see that $m_5 = 3 = m_6$, but $\mathrm{Sepdeg}_{\mathbb{W}}(3P_5) \neq \mathrm{Sepdeg}_{\mathbb{W}}(3P_6)$.

Observe that if $\mathbb{W} = \mathbb{X}$ is a set of s distinct K-rational points in $\mathbb{P}^n$ then we have $\mathrm{sepdeg}_{\mathbb{W}}(P_j) = \alpha_{\mathbb{W}\setminus\{P_j\}/\mathbb{W}} = \min\{i \in \mathbb{N} \mid (\mathcal{I}_{\mathbb{W}\setminus\{P_j\}/\mathbb{W}})_i \neq 0\}$ for every $j \in \{1,\dots,s\}$. The Hilbert function of $\mathbb{W} \setminus \{P_j\}$ satisfies $\mathrm{HF}_{\mathbb{W}\setminus\{P_j\}}(i) = \mathrm{HF}_{\mathbb{W}}(i)$ for $i < \mathrm{sepdeg}_{\mathbb{W}}(P_j)$ and $\mathrm{HF}_{\mathbb{W}\setminus\{P_j\}}(i) = \mathrm{HF}_{\mathbb{W}}(i) - 1$ for $i \geq \mathrm{sepdeg}_{\mathbb{W}}(P_j)$ (cf. [GKR, Proposition 1.3]). In analogy with this observation, we can now use the minimal set of separators to describe the Hilbert functions of fat point schemes in $\mathbb{P}^n$.

Proposition 4.1.11. *Let $\mathbb{W}$ be the fat point scheme $\mathbb{W} = m_1P_1+\cdots+m_sP_s$ in $\mathbb{P}^n$. For $j \in \{1,\dots,s\}$, let $\mathbb{W}_j = m_1P_1+\cdots+m_{j-1}P_{j-1}+(m_j-1)P_j+m_{j+1}P_{j+1}+\cdots+m_sP_s$, let $n_j = \deg(\mathbb{W}) - \deg(\mathbb{W}_j)$, and let $f^*_{j1},\dots,f^*_{jn_j} \in R_{\mathbb{W}}$ be a minimal set of separators of $\mathbb{W}_j$ in $\mathbb{W}$. Then for all $i \in \mathbb{Z}$ we have*

$$\mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{W}_j}(i) = \#\{f^*_{jk} |\ \deg(f^*_{jk}) \leq i, 1 \leq k \leq n_j\}.$$

Proof. By Lemma 3.6 of [GMT], we have

$$\begin{aligned}\#\{f^*_{jk} |\ \deg(f^*_{jk}) \leq i, 1 \leq k \leq n_j\} &= \mathrm{HF}_{\mathcal{I}_{\mathbb{W}_j/\mathbb{W}}}(i)\\ &= \mathrm{HF}_S(i) - \mathrm{HF}_{\mathcal{I}_{\mathbb{W}}}(i) - \mathrm{HF}_S(i) - \mathrm{HF}_{\mathcal{I}_{\mathbb{W}_j}}(i)\\ &= \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{W}_j}(i)\end{aligned}$$

as we wished. □

Similar to Proposition 4.1.11, the difference between the Hilbert functions of two fat point schemes can be calculated by the degrees of fat points.

Proposition 4.1.12. *Let $\mathbb{W} = m_1P_1+\cdots+m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, let $j \in \{1,\dots,s\}$, and let $\mathbb{W}_{jm_j}$ be the subscheme of $\mathbb{W}$ where $\mathbb{W}_{jm_j} = m_1P_1+\cdots+m_{j-1}P_{j-1}+m_{j+1}P_{j+1}+\cdots+m_sP_s$. Then for all $i \in \mathbb{Z}$ we have*

$$\mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{W}_{jm_j}}(i) = \#\{d \in \mathrm{Sepdeg}_{\mathbb{W}}(m_jP_j) \mid d \leq i\}.$$

Proof. Note that if $m_j = 1$, then the claim follows immediately from Proposition 4.1.11. So we may assume that $m_j \geq 2$. In this case, we let $\mathbb{W}_{jk}$ be the subscheme of $\mathbb{W}$ given in Definition 4.1.8 for $k = 0, \dots, m_j$. Using Proposition 4.1.11, for all $i \in \mathbb{Z}$ we have

$$\begin{aligned}
\mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{W}_{jm_j}}(i) &= (\mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{W}_{j1}}(i)) + (\mathrm{HF}_{\mathbb{W}_{j1}}(i) - \mathrm{HF}_{\mathbb{W}_{j2}}(i)) \\
&\quad + \cdots + (\mathrm{HF}_{\mathbb{W}_{j\,m_j-1}}(i) - \mathrm{HF}_{\mathbb{W}_{jm_j}}(i)) \\
&= \#\{d \in \mathrm{sepdeg}_{\mathbb{W}_{j0}}(P_j) \mid d \leq i\} + \#\{d \in \mathrm{sepdeg}_{\mathbb{W}_{j1}}(P_j) \mid d \leq i\} \\
&\quad + \cdots + \#\{d \in \mathrm{sepdeg}_{\mathbb{W}_{j\,m_j-1}}(P_j) \mid d \leq i\} \\
&= \#\{d \in \mathrm{Sepdeg}_{\mathbb{W}}(m_jP_j) \mid d \leq i\}.
\end{aligned}$$

Thus the claim is proved. □

The following corollary follows immediately from Proposition 4.1.12.

Corollary 4.1.13. *In the setting of Proposition 4.1.12, let $j, k \in \{1, \dots, s\}$ be such that $j \neq k$. Then $\mathrm{HF}_{\mathbb{W}_{jm_j}}(i) = \mathrm{HF}_{\mathbb{W}_{km_k}}(i)$ for all $i \in \mathbb{Z}$ if and only if $\mathrm{Sepdeg}_{\mathbb{W}}(m_jP_j) = \mathrm{Sepdeg}_{\mathbb{W}}(m_kP_k)$.*

Example 4.1.14. Let us go back to Example 4.1.10. We see that the fat point scheme $\mathbb{W} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_5 + 3P_6$ has $\mathrm{Sepdeg}_{\mathbb{W}}(2P_3) = \mathrm{Sepdeg}_{\mathbb{W}}(2P_4) = (4, 5, 5, 5)$. Therefore the two subschemes $\mathbb{W}_{32} = P_1 + P_2 + 2P_4 + 3P_5 + 3P_6$ and $\mathbb{W}_{42} = P_1 + P_2 + 2P_3 + 3P_5 + 3P_6$ of $\mathbb{W}$ have the same Hilbert function which is given by

$$\mathrm{HF}_{\mathbb{W}_{32}} = \mathrm{HF}_{\mathbb{W}_{42}} : \ 1\ 4\ 10\ 18\ 24\ 26\ 26 \cdots .$$

Now we recall the following notion of a Cayley-Bacharach scheme which was carefully studied in [GKR] and [Kre1].

Definition 4.1.15. Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points. We say that $\mathbb{X}$ is a **Cayley-Bacharach scheme** (in short, **CB-scheme**) if every hypersurface of degree $r_{\mathbb{X}} - 1$ which contains all but one point of $\mathbb{X}$ must contain all points of $\mathbb{X}$.

Note that $\mathbb{X}$ is a CB-scheme if and only if $\mathrm{sepdeg}_{\mathbb{X}}(P_j) = r_{\mathbb{X}}$ for all $j = 1, \dots, s$. Equivalently, the Hilbert function of $\mathbb{X} \setminus \{P_j\}$ does not depend on the choice of j. Furthermore, it follows from the well-known Cayley-Bacharach Theorem that every reduced 0-dimensional complete intersection $\mathbb{X}$ of type $(d_1, \dots, d_n)$ is a CB-scheme. In this special case, for all $j \in \{1, \dots, s\}$ we have $\mathrm{sepdeg}_{\mathbb{X}}(P_j) = \sum_{i=1}^{n} d_i - n$ and the

Hilbert function of $\mathbb{X}_j = \mathbb{X} \setminus \{P_j\}$ satisfies

$$\mathrm{HF}_{\mathbb{X}_j}(i) = \begin{cases} \mathrm{HF}_{\mathbb{X}}(i) & \text{if } i < \mathrm{sepdeg}_{\mathbb{X}}(P_j), \\ \mathrm{HF}_{\mathbb{X}}(i) - 1 & \text{if } i \geq \mathrm{sepdeg}_{\mathbb{X}}(P_j). \end{cases}$$

If $\mathbb{W}$ is an equimultiple fat point scheme whose support is a complete intersection, we have a concrete formula for the Hilbert function of $\mathbb{W}$ as in the following proposition (cf. [GT, Corollary 2.3]).

Proposition 4.1.16. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a complete intersection of type $(d_1, \dots, d_n)$. Let $L_j = \{(a_1, \dots, a_n) \in \mathbb{N}^n | a_1 + \cdots + a_n = j\}$. For every $\nu \in \mathbb{N}$, the Hilbert function of the equimultiple fat point scheme $\nu\mathbb{X}$ satisfies*

$$\mathrm{HF}_{\nu\mathbb{X}}(i) = \sum_{j=0}^{\nu-1} \sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_{\mathbb{X}}(i - a_1 d_1 - \cdots - a_n d_n)$$

for all $i \in \mathbb{Z}$. In particular, the Hilbert function of $\nu\mathbb{X}$ depend only on the type of the scheme $\mathbb{X}$ and the multiplicity ν.

Using this proposition and Corollary 2.1.15, we now show that the degrees of fat points of a given scheme can be the same.

Proposition 4.1.17. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points. Suppose that $\mathbb{X}$ is a Cayley-Bacharach scheme, and that the subschemes $\mathbb{X}_i = \mathbb{X} \setminus \{P_i\}$, $\mathbb{X}_j = \mathbb{X} \setminus \{P_j\}$ are complete intersections, for some $i, j \in \{1, \dots, s\}$. Then the subschemes $\mathbb{X}_i$ and $\mathbb{X}_j$ have the same complete intersection types, and for all $\nu \geq 1$ and we have* $\mathrm{Sepdeg}_{\nu\mathbb{X}}(\nu P_i) = \mathrm{Sepdeg}_{\nu\mathbb{X}}(\nu P_j)$.

Proof. Since $\mathbb{X}$ is a CB-scheme, this implies $\mathrm{HF}_{\mathbb{X}_i}(\ell) = \mathrm{HF}_{\mathbb{X}_j}(\ell)$ for all $\ell \in \mathbb{Z}$. By Corollary 2.1.15, the complete intersections $\mathbb{X}_i$ and $\mathbb{X}_j$ have the same type. From this, Proposition 4.1.16 shows that $\mathrm{HF}_{\nu\mathbb{X}_i}(\ell) = \mathrm{HF}_{\nu\mathbb{X}_j}(\ell)$ for all $\ell \in \mathbb{Z}$. Thus for $\ell \in \mathbb{Z}$, we get

$$\begin{aligned} \mathrm{HF}_{\nu\mathbb{X}}(\ell) - \#\{d \in \mathrm{Sepdeg}_{\nu\mathbb{X}}(\nu P_i) \mid d \leq \ell\} &= \mathrm{HF}_{\nu\mathbb{X}_i}(\ell) = \mathrm{HF}_{\nu\mathbb{X}_j}(\ell) \\ &= \mathrm{HF}_{\nu\mathbb{X}}(\ell) - \#\{d \in \mathrm{Sepdeg}_{\nu\mathbb{X}}(\nu P_j) \mid d \leq \ell\} \end{aligned}$$

Hence we have $\mathrm{Sepdeg}_{\nu\mathbb{X}}(\nu P_i) = \mathrm{Sepdeg}_{\nu\mathbb{X}}(\nu P_j)$. □

The next proposition describes explicitly the minimal graded free resolution of an equimultiple fat point scheme $\mathbb{W} = \nu\mathbb{X}$ in $\mathbb{P}^n$ supported at a complete intersection $\mathbb{X}$.

Proposition 4.1.18. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which is a complete intersection of type $(d_1, \dots, d_n)$. The minimal graded free resolution of $\mathcal{I}_{\nu\mathbb{X}}$ has the form*

$$0 \longrightarrow \mathcal{F}_{n-1} \longrightarrow \mathcal{F}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{I}_{\nu\mathbb{X}} \longrightarrow 0,$$

where $\mathcal{F}_0 = \bigoplus_{(a_1,\dots,a_n)\in\mathcal{M}_{n,\nu,1}} S(-a_1d_1 - \cdots - a_nd_n)$, and for $i = 1, \dots, n$, we have

$$\mathcal{F}_i = \bigoplus_{l_1=i+1}^{n} \left[\bigoplus_{l_2=l_1}^{n} \left[\cdots \left[\bigoplus_{l_i=l_{i-1}}^{n} \left[\bigoplus_{(a_1,\dots,a_n)\in\mathcal{M}_{n,\nu+i,l_i}} S(-a_1d_1 - \cdots - a_nd_n) \right] \right] \cdots \right] \right]$$

where $\mathcal{M}_{n,s,t} := \{(a_1, \dots, a_n) \in \mathbb{N}^n \mid \sum_{i=1}^n a_i = s$ and at least t of a_i's are non-zero$\}$.

Proof. Since the homogeneous vanishing ideal of $\mathbb{X}$ is generated by a regular sequence, we can use [ZS, Appendix 6, Lemma 5] and get $\mathcal{I}_{\nu\mathbb{X}} = \mathcal{I}_{\mathbb{X}}^{(\nu)} = \mathcal{I}_{\mathbb{X}}^{\nu}$ for all $\nu \in \mathbb{N}$. Thus the claim follows from [GT, Theorem 2.1]. $\square$

We end this section with the following proposition which shows that the degree of any point P_j in a given equimultiple fat point scheme $\mathbb{W}$ and the Hilbert function of the fat point subscheme obtained from $\mathbb{W}$ by reducing the multiplicity of P_j by one do not depend on the position of P_j in the complete intersection $\mathbb{X}$.

Proposition 4.1.19. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which is a complete intersection of type $(d_1, \dots, d_n)$, let $\nu \geq 1$ and let $\mathcal{F}_{n-1}$ be given in Proposition 4.1.18. For every $j \in \{1, \dots, s\}$, let $\mathbb{W}_{j1}$ be the subscheme $\mathbb{W}_{j1} = \nu P_1 + \cdots + \nu P_{j-1} + (\nu-1)P_j + \nu P_{j+1} + \cdots + \nu P_s$ of $\nu\mathbb{X}$.*

(i) *We have* $\operatorname{rank}(\mathcal{F}_{n-1}) = \deg(\mathbb{W}) - \deg(\mathbb{W}_{j1}) = \binom{\nu+n-2}{n-1}$.

(ii) *Let we denote the tuple of all elements of the set $W = \{a_1d_1 + \cdots + a_nd_n - n \mid a_i \neq 0, \sum_{i=1}^n a_i = \nu + n - 1\}$ in increasing order by $(b_1, \dots, b_{\binom{\nu+n-2}{n-1}})$. Then we have*

$$\operatorname{sepdeg}_{\nu\mathbb{X}}(P_1) = \cdots = \operatorname{sepdeg}_{\nu\mathbb{X}}(P_s) = (b_1, \dots, b_{\binom{\nu+n-2}{n-1}}).$$

(iii) *For every $j \in \{1, \dots, s\}$, the schemes $\mathbb{W}_{j1}$ all have the same Hilbert function.*

(iv) *A graded free resolution of $R_{\mathbb{W}_{j1}}$ is of the form*

$$0 \to \bigoplus_{\prod_{i=1}^n a_i \neq 0, \sum_{i=1}^n a_i = \nu+n-1} S^{\binom{n}{n-1}}(-a_1d_1 - \cdots - a_nd_n + 1) \to \cdots \to S \to R_{\mathbb{W}_{j1}} \to 0.$$

Proof. To prove (i), let us compute the free S-module $\mathcal{F}_{n-1}$ first. Using Proposition 4.1.18, we have $\mathcal{F}_{n-1} = \bigoplus_{(a_1,\dots,a_n)} S(-a_1d_1 - \cdots - a_nd_n)$, where $a_i \neq 0$ for all $i = 1,\dots,n$ and $\sum_{i=1}^n a_i = \nu + n - 1$. Thus $\operatorname{rank}(\mathcal{F}_{n-1}) = \binom{\nu+n-2}{n-1}$. Moreover, Proposition 4.1.3 implies $\deg(\mathbb{W}) - \deg(\mathbb{W}_{j1}) = s\binom{\nu+n-1}{n} - (s-1)\binom{\nu+n-1}{n} - \binom{\nu-1+n-1}{n} = \binom{\nu+n-2}{n-1}$. Hence the first claim follows.

Now we prove the second claim. By (i), there exists a set V of $\binom{\nu+n-2}{n-1}$ polynomials such that V is a minimal set of separator of $\mathbb{W}_{j1}$ in $\mathbb{W}$. Let V' be the sequence of degrees of all entries of V in increasing order $V' = (c_1,\dots,c_{\binom{\nu+n-2}{n-1}})$. It follows from (i) and [GMT, Theorem 5.4] that there exists a free S-module $\mathcal{F}'_{n-1}$ such that

$$\begin{aligned}\mathcal{F}_{n-1} &= \bigoplus_{\prod_{i=1}^n a_i\neq 0, \sum_{i=1}^n a_i = \nu+n-1} S(-a_1d_1 - \cdots - a_nd_n)\\ &= \mathcal{F}'_{n-1} \oplus S(-c_1 - n) \oplus S(-c_2 - n) \oplus \cdots \oplus S(-c_{\binom{\nu+n-2}{n-1}} - n).\end{aligned}$$

By comparing coefficients, we get $\mathcal{F}'_{n-1} = 0$ and $c_i = b_i$ for $i = 1,\dots,\binom{\nu+n-2}{n-1}$, where $(b_1,\dots,b_{\binom{\nu+n-2}{n-1}})$ is the tuple of all elements of the set $W = \{a_1d_1 + \cdots + a_nd_n - n \mid a_i \neq 0, \sum_{i=1}^n a_i = \nu + n - 1\}$ in increasing order. Therefore the claim follows from the fact that $\operatorname{sepdeg}_{\nu\mathbb{X}}(P_j) = (c_1,\dots,c_{\binom{\nu+n-2}{n-1}})$.

Let us proceed to prove (iii). We let V be a minimal set of separator of $\mathbb{W}_{j1}$ in $\mathbb{W}$. Then Corollary 4.1.11 yields

$$\begin{aligned}\mathrm{HF}_{\mathbb{W}_{j1}}(i) &= \mathrm{HF}_{\mathbb{W}}(i) - \#\{f_j^* \in V \,|\, \deg(f_j^*) \leq i\}\\ &= \mathrm{HF}_{\mathbb{W}}(i) - \#\{b \in \{b_1,\dots,b_{\binom{\nu+n-2}{n-1}}\} \mid b \leq i\}.\end{aligned}$$

So, the claim (iii) follows from (ii).

Finally, we show (iv). From the last exact sequence in the proof of [GMT, Theorem 5.4], the resolution of the ring $R_{\mathbb{W}_{j1}}$ is

$$0 \to \bigoplus_{\prod_{i=1}^n a_i\neq 0, \sum_{i=1}^n a_i = \nu+n-1} S^{\binom{n}{n-1}}(-a_1d_1 - \cdots - a_nd_n + 1) \oplus \mathcal{H}_{n-1} \to \cdots \to S \to R_{\mathbb{W}_{j1}} \to 0$$

where the free S-module $\mathcal{H}_{n-1}$ satisfies the equality

$$\mathcal{F}_{n-1} = \bigoplus_{\prod_{i=1}^n a_i\neq 0, \sum_{i=1}^n a_i = \nu+n-1} S^{\binom{n}{n-1}}(-a_1d_1 - \cdots - a_nd_n) \oplus \mathcal{H}_{n-1}.$$

In the proof of (i) we have $\mathcal{F}_{n-1} = \bigoplus_{\prod_{i=1}^n a_i\neq 0, \sum_{i=1}^n a_i = \nu+n-1} S^{\binom{n}{n-1}}(-a_1d_1 - \cdots - a_nd_n)$. Consequently, it follows from the proof of (i) that $\mathcal{H}_{n-1} = 0$. $\square$

4.2 Modules of Kähler Differential 1-Forms for Fat Point Schemes

In this section we study the modules of Kähler differential 1-forms $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ for fat point schemes $\mathbb{W}$ in $\mathbb{P}^n$. More precisely, we work out their Hilbert functions and give bounds for their regularity indices.

Let $\mathbb{W} = \mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points in $\mathbb{P}^n$. The following presentation for the module of Kähler differentials $\Omega^1_{R_{\mathbb{X}}/K}$ was given in [DK]:

$$0 \longrightarrow \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{2\mathbb{X}} \longrightarrow R^{n+1}_{\mathbb{X}}(-1) \longrightarrow \Omega^1_{R_{\mathbb{X}}/K} \longrightarrow 0$$

where $2\mathbb{X} = 2P_1 + \cdots + 2P_s$ is the corresponding scheme of double points in $\mathbb{P}^n$. This presentation can be generalized to the case of an arbitrary fat point scheme in $\mathbb{P}^n$ as follows.

Theorem 4.2.1. *Consider the two fat point schemes $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ and $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ in $\mathbb{P}^n$. Then the sequence of graded $R_{\mathbb{W}}$-modules*

$$0 \longrightarrow \mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{V}} \longrightarrow R^{n+1}_{\mathbb{W}}(-1) \longrightarrow \Omega^1_{R_{\mathbb{W}}/K} \longrightarrow 0$$

is exact.

In the proof of this theorem we use the following lemma which follows for instance from [Mat, Chapter 3, §7, Theorem 7.4 (i)].

Lemma 4.2.2. *Let M be a free S-module of rank m and let I, J be ideals of S. Then we have $IM \cap JM = (I \cap J)M$.*

Proof of Theorem 4.2.1. Since $R^{n+1}_{\mathbb{W}}(-1) \cong \Omega^1_{S/K}/\mathcal{I}_{\mathbb{W}}\Omega^1_{S/K}$, it suffices to prove that the sequence of graded $R_{\mathbb{W}}$-modules

$$0 \longrightarrow \mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{V}} \stackrel{\alpha}{\longrightarrow} \Omega^1_{S/K}/\mathcal{I}_{\mathbb{W}}\Omega^1_{S/K} \longrightarrow \Omega^1_{R_{\mathbb{W}}/K} \longrightarrow 0$$

is exact, where $\alpha : \mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{V}} \to \Omega^1_{S/K}/\mathcal{I}_{\mathbb{W}}\Omega^1_{S/K}$ is given by $\alpha(F + \mathcal{I}_{\mathbb{V}}) = dF + \mathcal{I}_{\mathbb{W}}\Omega^1_{S/K}$ for all $F \in \mathcal{I}_{\mathbb{W}}$.

First we check that the map α is well defined: Let $F_1, F_2 \in \mathcal{I}_{\mathbb{W}}$ be such that $F_1 - F_2 \in \mathcal{I}_{\mathbb{V}}$. Then we have $F_1 - F_2 \in \wp_j^{m_j+1}$ for $j \in \{1, \dots, s\}$, and hence $dF_1 - dF_2 \in d(\wp_j^{m_j+1}) \subset \wp_j^{m_j}\Omega^1_{S/K}$. Therefore we see that $dF_1 - dF_2 \in \bigcap_{j=1}^s \wp_j^{m_j}\Omega^1_{S/K}$. Notice that $\Omega^1_{S/K}$ is a free S-module of rank $n+1$. By Lemma 4.2.2, we get $dF_1 - dF_2 \in \bigcap_{j=1}^s \wp_j^{m_j}\Omega^1_{S/K} = \mathcal{I}_{\mathbb{W}}\Omega^1_{S/K}$.

Next we show that the map α is $R_{\mathbb{W}}$-linear: For $F_1, F_2 \in \mathcal{I}_{\mathbb{W}}$ and $G_1, G_2 \in S$, we have

$$\begin{aligned}\alpha(G_1F_1 + G_2F_2 + \mathcal{I}_{\mathbb{V}}) &= d(G_1F_1 + G_2F_2) + \mathcal{I}_{\mathbb{W}}\Omega^1_{S/K}\\ &= G_1dF_1 + G_2dF_2 + \mathcal{I}_{\mathbb{W}}\Omega^1_{S/K}\\ &= (G_1 + \mathcal{I}_{\mathbb{W}})\cdot\alpha(F_1 + \mathcal{I}_{\mathbb{V}}) + (G_2 + \mathcal{I}_{\mathbb{W}})\cdot\alpha(F_2 + \mathcal{I}_{\mathbb{V}}).\end{aligned}$$

Now we prove that the map α is injective. For a contradiction, suppose that there is a homogeneous polynomial $F \in \mathcal{I}_{\mathbb{W}} \setminus \mathcal{I}_{\mathbb{V}}$ such that $\alpha(F + \mathcal{I}_{\mathbb{V}}) = dF + \mathcal{I}_{\mathbb{W}}\Omega^1_{S/K} = 0$. Since $F \notin \mathcal{I}_{\mathbb{V}} = \bigcap_{j=1}^s \wp_j^{m_j+1}$, there is an index $j_0 \in \{1,\dots,s\}$ such that $F \notin \wp_{j_0}^{m_{j_0}+1}$. Setting

$$\Sigma = \{G \in \wp_{j_0}^{m_{j_0}} \mid G \notin \wp_{j_0}^{m_{j_0}+1}, dG \in \wp_{j_0}^{m_{j_0}}\Omega^1_{S/K}\},$$

we see that $F \in \Sigma \neq \emptyset$. Let G be a homogeneous polynomial of minimal degree in Σ. By Euler's rule, we have

$$\deg(G)\cdot G = \sum_{i=0}^n \frac{\partial G}{\partial X_i}X_i = \sum_{i=1}^n \frac{\partial G}{\partial X_i}(X_i - p_{j_0i}X_0) + X_0\sum_{i=0}^n p_{j_0i}\frac{\partial G}{\partial X_i}$$

where we write $P_{j_0} = (1 : p_{j_01} : \cdots : p_{j_0n})$ with $p_{ji} \in K$. Let $\widetilde{G} = \sum_{i=0}^n p_{j_0i}\frac{\partial G}{\partial X_i}$. It follows from $dG = \sum_{i=0}^n \frac{\partial G}{\partial X_i}dX_i \in \wp_{j_0}^{m_{j_0}}\Omega^1_{S/K}$ that $\frac{\partial G}{\partial X_i} \in \wp_{j_0}^{m_{j_0}}$ for all $i = 0,\dots,n$, and hence $\widetilde{G} \in \wp_{j_0}^{m_{j_0}}$. On the other hand, we have $\sum_{i=1}^n \frac{\partial G}{\partial X_i}(X_i - p_{j_0i}X_0) \in \wp_{j_0}^{m_{j_0}+1}$, and thus, since $G \notin \wp_{j_0}^{m_{j_0}+1}$, we get $X_0\widetilde{G} \notin \wp_{j_0}^{m_{j_0}+1}$. This implies that $\widetilde{G} \notin \wp_{j_0}^{m_{j_0}+1}$, and in particular $\widetilde{G} \neq 0$. Moreover, both dG and $d\big(\sum_{i=1}^n \frac{\partial G}{\partial X_i}(X_i - p_{j_0i}X_0)\big)$ are contained in $\wp_{j_0}^{m_{j_0}}\Omega^1_{S/K}$. This implies $d(X_0\widetilde{G}) = X_0d\widetilde{G} + \widetilde{G}dX_0 \in \wp_{j_0}^{m_{j_0}}\Omega^1_{S/K}$. Clearly, $\widetilde{G}dX_0$ is an element of $\wp_{j_0}^{m_{j_0}}\Omega^1_{S/K}$, and so is $X_0d\widetilde{G}$. Since the image of X_0 in $S/\wp_{j_0}^{m_{j_0}}$ is not a zerodivisor for $S/\wp_{j_0}^{m_{j_0}}$, we have $d\widetilde{G} \in \wp_{j_0}^{m_{j_0}}\Omega^1_{S/K}$. Altogether, we find $\widetilde{G} \in \Sigma \setminus \{0\}$ and $\deg(\widetilde{G}) < \deg(G)$, in contradiction to the choice of G. Consequently, the map α is injective.

Now it is straightforward to see that the sequence of graded $R_{\mathbb{W}}$-modules

$$0 \longrightarrow \mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{V}} \stackrel{\alpha}{\longrightarrow} \Omega^1_{S/K}/\mathcal{I}_{\mathbb{W}}\Omega^1_{S/K} \longrightarrow (\Omega^1_{S/K}/\mathcal{I}_{\mathbb{W}}\Omega^1_{S/K})/\operatorname{Im}(\alpha) \longrightarrow 0$$

is exact. Furthermore, we have $\operatorname{Im}(\alpha) = \alpha(\mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{V}}) = (\langle d\mathcal{I}_{\mathbb{W}}\rangle_S + \mathcal{I}_{\mathbb{W}}\Omega^1_{S/K})/\mathcal{I}_{\mathbb{W}}\Omega^1_{S/K}$ and $\Omega^1_{R_{\mathbb{W}}/K} \cong \Omega^1_{S/K}/(\langle d\mathcal{I}_{\mathbb{W}}\rangle_S + \mathcal{I}_{\mathbb{W}}\Omega^1_{S/K})$. Therefore we obtain the desired exact sequence. $\square$

In view of Proposition 4.1.3, we see that the Hilbert polynomial of $\mathbb{W}$ depends only on the number of points in its support and their multiplicities. Moreover, the following corollary shows that the Hilbert polynomials of $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ also have the same property. Moreover, this corollary gives us a relation between the Hilbert functions of $\Omega^1_{R_{\mathbb{W}}/K}$ (as well as of $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$) and of $\mathbb{W}$ and $\mathbb{V}$.

Corollary 4.2.3. *Consider the two fat point schemes* $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ *and* $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ *in* $\mathbb{P}^n$*.*

(i) We have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i)$ *for all* $i \in \mathbb{Z}$*.*

(ii) The Hilbert polynomial of $\Omega^1_{R_{\mathbb{W}}/K}$ *is the constant polynomial*

$$\mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K}}(z) = (n+2)\sum_{j=1}^{s}\binom{m_j+n-1}{n} - \sum_{j=1}^{s}\binom{m_j+n}{n}.$$

(iii) We have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(i) = n\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i)$*.*

(iv) The Hilbert polynomial of $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ *is*

$$\mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(z) = (n+1)\sum_{j=1}^{s}\binom{m_j+n-1}{n} - \sum_{j=1}^{s}\binom{m_j+n}{n}.$$

(v) The regularity indices of the modules of Kähler differentials 1-forms $\Omega^1_{R_{\mathbb{W}}/K}$ *and* $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ *are bounded by*

$$\max\{\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}), \mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K[x_0]})\} \le \max\{r_{\mathbb{W}}+1, r_{\mathbb{V}}\}.$$

Proof. Claim (i) follows immediately from the exact sequence of the module of Kähler differential 1-forms $\Omega^1_{R_{\mathbb{W}}/K}$ given in Theorem 4.2.1. Claim (ii) follows from Proposition 4.1.3 and the fact that $\mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K}}(z) = (n+2)\,\mathrm{HP}_{\mathbb{W}}(z) - \mathrm{HP}_{\mathbb{V}}(z)$ which is induced by (i).

Now we prove (iii). By Corollary 3.3.2 and (i), we have

$$\begin{aligned}\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(i) &= \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) - \mathrm{HF}_{\mathbb{W}}(i-1)\\ &= (n+1)\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i) - \mathrm{HF}_{\mathbb{W}}(i-1)\\ &= n\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i)\end{aligned}$$

for all $i \in \mathbb{Z}$. Claim (iv) follows from (iii) and Proposition 4.1.3. Finally, claim (v) is an immediate consequence of (i) and (iii). □

Our next corollary indicates that the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ knows whether the scheme is reduced.

Corollary 4.2.4. *Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$. Then $m_1 = \cdots = m_s = 1$ if and only if $\mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(z) = 0$.*

Proof. By Corollary 4.2.3(v), we have the equalities

$$\begin{aligned}
\mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(z) &= \mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K}}(z) - \mathrm{HP}_{\mathbb{W}}(z)\\
&= (n+1)\sum_{j=1}^{s}\binom{m_j+n-1}{n} - \sum_{j=1}^{s}\binom{m_j+n}{n}\\
&= \sum_{j=1}^{s}\frac{(m_j+n-1)!}{n!\,m_j!}\big((n+1)m_j - m_j - n\big)\\
&= \sum_{j=1}^{s}\binom{m_j+n-1}{n-1}\big(m_j-1\big).
\end{aligned}$$

Hence we have $\mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(z) = 0$ if and only if $m_j = 1$ for $j = 1, \dots, s$. $\square$

When $\mathbb{W} = \mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ is a CB-scheme, we have the following property.

Proposition 4.2.5. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which is a CB-scheme, and let $\mathbb{X}_j = \mathbb{X} \setminus \{P_j\}$ for $j = 1, \dots, s$. Then we have $\mathrm{Sepdeg}_{2\mathbb{X}}(2P_j) = \mathrm{Sepdeg}_{2\mathbb{X}}(2P_k)$ if and only if $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}_j}/K}}(i) = \mathrm{HF}_{\Omega^1_{R_{\mathbb{X}_k}/K}}(i)$ for all $i \in \mathbb{Z}$.*

Proof. Let $j, k \in \{1, \dots, s\}$ with $j \neq k$. Since $\mathbb{X}$ is a CB-scheme, we have $\mathrm{HF}_{\mathbb{X}_j}(i) = \mathrm{HF}_{\mathbb{X}_k}(i)$ for all $i \in \mathbb{Z}$. Thus due to Theorem 4.2.1, we get

$$\begin{aligned}
\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}_j}/K}}(i) - \mathrm{HF}_{\Omega^1_{R_{\mathbb{X}_k}/K}}(i) &= (n+1)\,\mathrm{HF}_{\mathbb{X}_j}(i-1) + \mathrm{HF}_{\mathbb{X}_j}(i) - \mathrm{HF}_{2\mathbb{X}_j}(i)\\
&\quad - ((n+1)\,\mathrm{HF}_{\mathbb{X}_k}(i-1) + \mathrm{HF}_{\mathbb{X}_k}(i) - \mathrm{HF}_{2\mathbb{X}_k}(i))\\
&= \mathrm{HF}_{2\mathbb{X}_k}(i) - \mathrm{HF}_{2\mathbb{X}_j}(i)
\end{aligned}$$

for all $i \in \mathbb{Z}$. Hence $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}_j}/K}}(i) = \mathrm{HF}_{\Omega^1_{R_{\mathbb{X}_k}/K}}(i)$ for all $i \in \mathbb{Z}$ if and only if $\mathrm{HF}_{2\mathbb{X}_k}(i) = \mathrm{HF}_{2\mathbb{X}_j}(i)$ for all $i \in \mathbb{Z}$. According to Corollary 4.1.13, this is equivalent to the fact that $\mathrm{Sepdeg}_{2\mathbb{X}}(2P_j) = \mathrm{Sepdeg}_{2\mathbb{X}}(2P_k)$, and the claim is proved. $\square$

We can generalize the above property as follows.

Proposition 4.2.6. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which is a CB-scheme, let $\mathbb{X}_j = \mathbb{X} \setminus \{P_j\}$ for $j = 1, \dots, s$, and let $\nu \geq 1$. Then we have $\mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_j}/K}}(i) = \mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_k}/K}}(i)$ for all $i \in \mathbb{Z}$ and $1 \leq l \leq \nu$ if and only if $(\mathrm{Sepdeg}_{2\mathbb{X}}(2P_j), \dots, \mathrm{Sepdeg}_{(\nu+1)\mathbb{X}}((\nu+1)P_j)) = (\mathrm{Sepdeg}_{2\mathbb{X}}(2P_k), \dots, \mathrm{Sepdeg}_{(\nu+1)\mathbb{X}}((\nu+1)P_k))$.*

Proof. In the case $\nu = 1$, the claim follows from Proposition 4.2.5. Now we assume that $\nu \geq 2$. Notice that $\mathrm{HF}_{\mathbb{X}_j}(i) = \mathrm{HF}_{\mathbb{X}_k}(i)$ for all $i \in \mathbb{Z}$, since $\mathbb{X}$ is a CB-scheme. By Theorem 4.2.1, we have

$$\begin{aligned}\mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_j}/K}}(i) - \mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_k}/K}}(i) &= (n+1)\,\mathrm{HF}_{l\mathbb{X}_j}(i-1) + \mathrm{HF}_{l\mathbb{X}_j}(i) - \mathrm{HF}_{(l+1)\mathbb{X}_j}(i) \\ &\quad - ((n+1)\,\mathrm{HF}_{l\mathbb{X}_k}(i-1) + \mathrm{HF}_{l\mathbb{X}_k}(i) - \mathrm{HF}_{(l+1)\mathbb{X}_k}(i)) \\ &= (n+1)(\mathrm{HF}_{l\mathbb{X}_j}(i-1) - \mathrm{HF}_{l\mathbb{X}_k}(i-1)) \\ &\quad + (\mathrm{HF}_{l\mathbb{X}_j}(i) - \mathrm{HF}_{l\mathbb{X}_k}(i)) + (\mathrm{HF}_{(l+1)\mathbb{X}_k}(i) - \mathrm{HF}_{(l+1)\mathbb{X}_j}(i))\end{aligned}$$

for all $i \in \mathbb{Z}$. Thus, we get $\mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_j}/K}}(i) = \mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_k}/K}}(i)$ for all $i \in \mathbb{Z}$ and $1 \leq l \leq \nu$ if and only if $\mathrm{HF}_{(l+1)\mathbb{X}_k}(i) = \mathrm{HF}_{(l+1)\mathbb{X}_j}(i)$ for all $i \in \mathbb{Z}$ and $1 \leq l \leq \nu$, and an application of Corollary 4.1.13 finishes the proof. □

Remark 4.2.7. Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which satisfies the hypotheses of Proposition 4.1.17, and let $\nu \geq 1$. Then we have $\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}_i}/K}}(\ell) = \mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}_j}/K}}(\ell)$ for all $\ell \in \mathbb{Z}$.

Let us clarify the details of Proposition 4.2.6 using an example.

Example 4.2.8. Let $\mathbb{X} = \{P_1, \dots, P_5\} \subseteq \mathbb{P}^4$ given by $P_1 = (1:0:1:1:0)$, $P_2 = (1:1:0:1:0)$, $P_3 = (1:2:1:1:1)$, $P_4 = (1:2:2:0:1)$, and $P_5 = (1:0:2:1:1)$. Then $\mathbb{X}$ has the Hilbert function $\mathrm{HF}_{\mathbb{X}}:\ 1\ 5\ 5 \dots$, and so it is a CB-scheme. Notice that $\mathbb{X} \setminus \{P_1\}$ is not a complete intersection. Furthermore, for $j \in \{1, \dots, 5\}$, we can check that $\mathrm{Sepdeg}_{2\mathbb{X}}(2P_j) = (2,3,3,3,3)$ and

$$\mathrm{Sepdeg}_{3\mathbb{X}}(3P_j) = (3,4,4,4,4,4,4,4,4,4,4,5,5,5,5).$$

So, we have $\big(\mathrm{Sepdeg}_{2\mathbb{X}}(2P_j), \mathrm{Sepdeg}_{3\mathbb{X}}(3P_j)\big) = \big(\mathrm{Sepdeg}_{2\mathbb{X}}(2P_k), \mathrm{Sepdeg}_{3\mathbb{X}}(3P_k)\big)$ for all $j, k \in \{1, \dots, 5\}$. Hence Proposition 4.2.6 yields that $\mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_j}/K}}(i) = \mathrm{HF}_{\Omega^1_{R_{l\mathbb{X}_k}/K}}(i)$ for all $i \in \mathbb{Z}$ and $l = 1, 2$ and $j, k \in \{1, \dots, 5\}$.

A set of s distinct K-rational points in $\mathbb{P}^n$ is said to be in **general position** if no $h+2$ points of them are on an h-plane for $h < n$. For a fat point scheme $\mathbb{W}$ in $\mathbb{P}^n$ whose support is in general position, we recall the following bound for its regularity index which has been proved by M.V. Catalisano *et al* (cf. [CTV, Theorem 6]).

Proposition 4.2.9. *Let $s \geq 2$, let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points of $\mathbb{P}^n$ in general position, and let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme* *in $\mathbb{P}^n$ with support $\mathbb{X}$. Suppose that $m_1 \leq \cdots \leq m_s$. Then we have*

$$r_{\mathbb{W}} \leq \max\left\{ m_s + m_{s-1} - 1, \left\lfloor \frac{\sum_{j=1}^{s} m_j + n - 2}{n} \right\rfloor \right\}$$

where $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{Q}$.

Based on this result, a bound for the regularity index of the module of Kähler differentials for a fat point scheme in $\mathbb{P}^n$ whose support is in general position can be given as follows.

Corollary 4.2.10. *Let $s \geq 2$, let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points of $\mathbb{P}^n$ in general position, and let $\mathbb{W} = m_1 P_1 + \cdots + m_s P_s$ be a fat point scheme in $\mathbb{P}^n$ with $m_1 \leq \cdots \leq m_s$. Then we have*

$$\max\{\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}), \mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K[x_0]})\} \leq \max\left\{ m_s + m_{s-1} + 1, \left\lfloor \tfrac{1}{n} \sum_{j=1}^{s} m_j + s + n - 2 \right\rfloor \right\}.$$

Proof. Let ζ denote $\max\{\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}), \mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K[x_0]})\}$. Using Corollary 4.2.3 and Proposition 4.2.9, we have

$$\zeta \leq \max\left\{ m_s + m_{s-1} + 1, \left\lfloor \tfrac{1}{n}(\sum_{j=1}^{s} m_j + s + n - 2) \right\rfloor, \left\lfloor \tfrac{1}{n} \sum_{j=1}^{s} m_j + n - 2 \right\rfloor + 1 \right\}.$$

If $s \geq n$ then $\lfloor (\sum_{j=1}^{s} m_j + s + n - 2)/n \rfloor \geq \lfloor (\sum_{j=1}^{s} m_j + n - 2)/n \rfloor + 1$. If $s < n$, we use $s \geq 2$ and $1 \leq m_1 \leq \cdots \leq m_{s-1} \leq m_s$ and get $nm_s \geq \sum_{j=1}^{s} m_j$ as well as $nm_{s-1} \geq n$. This shows $m_s + m_{s-1} + 1 \geq \lfloor (\sum_{j=1}^{s} m_j + n - 2)/n \rfloor + 1$. Therefore we obtain $\zeta \leq \max\left\{ m_s + m_{s-1} + 1, \lfloor \frac{1}{n} \sum_{j=1}^{s} m_j + s + n - 2 \rfloor \right\}$. □

The upper bound for the regularity indices of $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ which is given in Corollary 4.2.10 is sharp, as the following examples show.

Example 4.2.11. (a) Let $\mathbb{W}$ be the fat point scheme $\mathbb{W} = P_1 + 4P_2 + 4P_3$ in $\mathbb{P}^4$, where $P_1 = (1 : 2 : 4 : 8 : 16)$, $P_2 = (1 : 0 : 0 : 0 : 0)$, and where $P_3 = (1 : 1 : 1 : 1 : 1)$. Clearly, the set $\mathbb{X} = \{P_1, P_2, P_3\}$ is in general position in $\mathbb{P}^4$. Now we calculate

$$\begin{aligned} \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}} &: 0\ 5\ 25\ 75\ 161\ 238\ 270\ 281\ 282\ 281\ 281 \cdots, \\ \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}} &: 0\ 4\ 20\ 60\ 126\ 182\ 204\ 211\ 211\ 210\ 210 \cdots. \end{aligned}$$

Then $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) = \mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K[x_0]}) = 9 = 4 + 4 + 1$.

(b) Let $\mathbb{W}$ be the fat point scheme $\mathbb{W} = P_1 + P_2 + P_3 + P_4 + 2P_5 + 2P_6$ in $\mathbb{P}^2$ where $P_1 = (1 : 2 : 4)$, $P_2 = (1 : 3 : 9)$, $P_3 = (1 : 4 : 16)$, $P_4 = (1 : 5 : 25)$, $P_5 = (1 : 0 : 0)$, and where $P_6 = (1:1:1)$. It is easy to check that $\mathbb{X} = \{P_1, \dots, P_6\}$ is in general position in $\mathbb{P}^2$. Then we calculate

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}} : 0\ 3\ 9\ 17\ 22\ 20\ 17\ 16\ 16 \cdots \text{ and } \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}} : 0\ 2\ 6\ 11\ 13\ 10\ 7\ 6\ 6 \cdots .$$

Therefore we have $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) = \mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K[x_0]}) = 7 = (8+6+2-2)/2$.

We conclude this section with the following corollary which summarizes some basic properties of the Hilbert functions of $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ for a fat point scheme $\mathbb{W}$ in $\mathbb{P}^n$.

Corollary 4.2.12. *Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, and let $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$.*

(i) *For $i \le 0$ we have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(i) = 0$.*

(ii) *For $1 \le i < \alpha_{\mathbb{W}}$ we have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = (n+1)\binom{n+i-1}{n}$.*

(iii) *For $1 \le i < \alpha_{\mathbb{W}}$ we have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(i) = n\binom{n+i-1}{n}$.*

(iv) *We have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(\alpha_{\mathbb{W}}) = \mathrm{HF}_{\mathbb{W}}(\alpha_{\mathbb{W}}) + (\alpha_{\mathbb{W}}-1)\binom{n+\alpha_{\mathbb{W}}-1}{n-1}$.*

(v) *We have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(\alpha_{\mathbb{W}}) = \mathrm{HF}_{\mathbb{W}}(\alpha_{\mathbb{W}}) + n\binom{n+\alpha_{\mathbb{W}}-1}{n} - \binom{n+\alpha_{\mathbb{W}}}{n}$.*

(vi) *For $\alpha_{\mathbb{W}} < i < \alpha_{\mathbb{V}}$, we have*

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \binom{i+n}{n}.$$

(vii) *For $\alpha_{\mathbb{W}} < i < \alpha_{\mathbb{V}}$, we have*

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(i) = n\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \binom{i+n}{n}.$$

(viii) *We have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(r_{\mathbb{W}}+1) \ge \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(r_{\mathbb{W}}+2) \ge \cdots$.*
If $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) \ge r_{\mathbb{W}}+1$, then $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(r_{\mathbb{W}}+1) > \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(r_{\mathbb{W}}+2) > \cdots \cdots > \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K})) = (n+2)\left(\sum_{j=1}^{s}\binom{m_j+n-1}{n}\right) - \sum_{j=1}^{s}\binom{m_j+n}{n}$.

Proof. First we note that, given a homogeneous polynomial $F \in (\mathcal{I}_{\mathbb{V}})_{\alpha_{\mathbb{V}}}$, Euler's rule shows that there is an index $i \in \{0, \dots, n\}$ such that $\partial F/\partial X_i \in (\mathcal{I}_{\mathbb{W}})_{\alpha_{\mathbb{V}}-1} \setminus \{0\}$. This yields $\alpha_{\mathbb{W}} < \alpha_{\mathbb{V}}$. Thus the claims of the corollary follow immediately from Proposition 3.3.8 and Corollary 4.2.3(i) and (iii) . □

4.3 Modules of Kähler Differential m-Forms for Fat Point Schemes

Throughout this section we let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points, and we let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$ supported at $\mathbb{X}$. In Section 4.2 we have taken a first step towards studying the modules of Kähler differential 1-forms for the fat point scheme $\mathbb{W}$. In the process we investigated many interesting properties of $\Omega^1_{R_\mathbb{W}/K}$ and $\Omega^1_{R_\mathbb{W}/K[x_0]}$. In this section we look more closely at the modules of Kähler differential m-forms for the fat point scheme $\mathbb{W}$, where $1 \le m \le n+1$.

First we bound the Hilbert polynomials of $\Omega^m_{R_\mathbb{W}/K}$ and $\Omega^m_{R_\mathbb{W}/K[x_0]}$ as follows.

Proposition 4.3.1. *Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$ such that $m_i \ge 2$ for some $i \in \{1, \dots, s\}$, and let $1 \le m \le n+1$.*

(i) The Hilbert polynomial of $\Omega^m_{R_\mathbb{W}/K}$ is a constant polynomial which is bounded by

$$\sum_{i=1}^{s} \binom{n+1}{m}\binom{m_i+n-2}{n} \le \mathrm{HP}_{\Omega^m_{R_\mathbb{W}/K}}(z) \le \sum_{i=1}^{s} \binom{n+1}{m}\binom{m_i+n-1}{n}.$$

(ii) If $1 \le m \le n$, then the Hilbert polynomial of $\Omega^m_{R_\mathbb{W}/K[x_0]}$ is a constant polynomial which is bounded by

$$\sum_{i=1}^{s} \binom{n}{m}\binom{m_i+n-2}{n} \le \mathrm{HP}_{\Omega^m_{R_\mathbb{W}/K[x_0]}}(z) \le \sum_{i=1}^{s} \binom{n}{m}\binom{m_i+n-1}{n}.$$

Proof. Let $\mathbb{Y}$ be the subscheme $\mathbb{Y} = (m_1-1)P_1 + \cdots + (m_s-1)P_s$ of $\mathbb{W}$. Since we have $\langle d\mathcal{I}_\mathbb{W}\rangle_S \subseteq \mathcal{I}_\mathbb{Y}\Omega^1_{S/K}$, this implies $\langle d\mathcal{I}_\mathbb{W}\rangle_S \wedge \Omega^{m-1}_{S/K} \subseteq \mathcal{I}_\mathbb{Y}\Omega^m_{S/K}$. Obviously, we have the inclusion $\mathcal{I}_\mathbb{W} \subseteq \mathcal{I}_\mathbb{Y}$. It follows that $\mathcal{I}_\mathbb{W}\Omega^m_{S/K} \subseteq \mathcal{I}_\mathbb{Y}\Omega^m_{S/K}$. From this we deduce $\mathcal{I}_\mathbb{W}\Omega^m_{S/K} + \langle d\mathcal{I}_\mathbb{W}\rangle_S\Omega^{m-1}_{S/K} \subseteq \mathcal{I}_\mathbb{Y}\Omega^m_{S/K}$. By Corollary 3.3.4, the Hilbert function of $\Omega^m_{R_\mathbb{W}/K}$ satisfies $\mathrm{HF}_{\Omega^m_{R_\mathbb{W}/K}}(i) = \mathrm{HF}_{\Omega^m_{S/K}/(\mathcal{I}_\mathbb{W}\Omega^m_{S/K}+\langle d\mathcal{I}_\mathbb{W}\rangle_S\Omega^{m-1}_{S/K})}(i) \ge \mathrm{HF}_{\Omega^m_{S/K}/\mathcal{I}_\mathbb{Y}\Omega^m_{S/K}}(i)$ for all $i \in \mathbb{Z}$. We see that $\mathrm{HP}_{\Omega^m_{S/K}/\mathcal{I}_\mathbb{Y}\Omega^m_{S/K}}(z) = \sum_{i=1}^{s}\binom{n+1}{m}\binom{m_i+n-2}{n} > 0$ since $m_i \ge 2$ for some $i \in \{1, \dots, s\}$. Hence we get the stated lower bound for Hilbert polynomial of $\Omega^m_{R_\mathbb{W}/K}$. In particular, $\mathrm{HP}_{\Omega^m_{R_\mathbb{W}/K}}(z) > 0$.

Furthermore, Proposition 3.3.8 shows that $\mathrm{HP}_{\Omega^m_{R_\mathbb{W}/K}}(z)$ is a constant polynomial. Now we find an upper bound for $\mathrm{HP}_{\Omega^m_{R_\mathbb{W}/K}}(z)$. Clearly, the $R_\mathbb{W}$-module $\Omega^m_{R_\mathbb{W}/K}$ is generated by the set $\{dx_{i_1} \wedge \cdots \wedge dx_{i_m} \mid \{i_1, \dots, i_m\} \subseteq \{0, \dots, n\}\}$ consisting of $\binom{n+1}{m}$ elements. This implies $\mathrm{HF}_{\Omega^m_{R_\mathbb{W}/K}}(i) \le \binom{n+1}{m}\mathrm{HF}_\mathbb{W}(i-m)$ for all $i \ge 0$. Hence we get

$\mathrm{HP}_{\Omega^m_{R_{\mathbb{W}}/K}}(z) \le \binom{n+1}{m}\sum_{i=1}^{s}\binom{m_i+n-1}{n}$, which completes the proof of claim (i). Claim (ii) follows from Corollary 3.3.4 with a similar argument as above. □

The following corollary in an immediate consequence of this proposition and Proposition 3.4.1.

Corollary 4.3.2. *Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, and let $m_{\max} = \max\{m_1, \dots, m_s\}$. The following conditions are equivalent.*

(i) The scheme $\mathbb{W}$ is not reduced i.e. $m_{\max} > 1$.

(ii) There exists $m \in \{2, \dots, n+1\}$ such that $\mathrm{HP}_{\Omega^m_{R_{\mathbb{W}}/K}}(z) > 0$.

(iii) $\mathrm{HP}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(z) > 0$.

(iv) There exists $m \in \{2, \dots, n\}$ such that $\mathrm{HP}_{\Omega^m_{R_{\mathbb{W}}/K[x_0]}}(z) > 0$.

Notice that the corollary does not always hold true for an arbitrary 0-dimensional subscheme of $\mathbb{P}^n$, as the following example shows.

Example 4.3.3. Let K be an algebraically closed field, and let $\mathbb{W}$ be the 0-dimensional complete intersection by two hypersurfaces $Z_1 = \mathcal{Z}^+(X_0^2 + X_1^2 - X_2^2)$ and $Z_2 = \mathcal{Z}^+(5X_0^2+5X_1^2+6X_0X_1+6X_1X_2-5X_2^2)$. By using Bézout's theorem, the scheme $\mathbb{W}$ contains the point $(1:0:-1)$ with multiplicity 3 and the point $(1:0:1)$ with multiplicity 1. A simple computation gives us $\mathrm{HP}_{\Omega^3_{R_{\mathbb{W}}/K}}(z) = 0$ and $\mathrm{HP}_{\Omega^1_{R_{\mathbb{W}}/K}}(z) = 6 \neq 4$. Therefore the scheme $\mathbb{W}$ is neither a set of distinct K-rational points nor a fat point scheme. In this case, we also have $\mathrm{HP}_{\Omega^2_{R_{\mathbb{W}}/K}}(z) = 2 \neq 0$.

Similar to Corollaries 4.2.3 and 4.2.10, we bound the regularity indices of $\Omega^m_{R_{\mathbb{W}}/K}$ and $\Omega^m_{R_{\mathbb{W}}/K[x_0]}$, for $m = 1, \dots, n+1$, as follows.

Proposition 4.3.4. *Let $\mathbb{W} = m_1P_1+\cdots+m_sP_s$ and $\mathbb{V} = (m_1+1)P_1+\cdots+(m_s+1)P_s$ be fat point schemes in $\mathbb{P}^n$.*

(i) For $m = 1, \dots, n$ we have

$$\max\{\mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K}), \mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K[x_0]})\} \le \max\{r_{\mathbb{W}} + m, r_{\mathbb{V}} + m - 1\}.$$

(ii) We have $\mathrm{ri}(\Omega^{n+1}_{R_{\mathbb{W}}/K}) \le \max\{r_{\mathbb{W}} + n, r_{\mathbb{V}} + n - 1\}$.

In particular, if $m_1 \le \cdots \le m_s$ and $\mathrm{Supp}(W) = \{P_1, \dots, P_s\}$ is in general position then for $1 \le m \le n$ we have

$$\max\{\mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K}), \mathrm{ri}(\Omega^m_{R_{\mathbb{W}}/K[x_0]})\} \le \max\{m_s+m_{s-1}+m, \lfloor\frac{\sum_{j=1}^{s} m_j + s + n - 2}{n}\rfloor+m-1\}$$

and

$$\mathrm{ri}(\Omega^{n+1}_{R_{\mathbb{W}}/K}) \le \max\{m_s + m_{s-1} + n, \lfloor \frac{\sum_{j=1}^{s} m_j + s + n - 2}{n} \rfloor + n - 1\}.$$

Proof. Claim (i) follows directly from Proposition 3.3.11 and Corollary 4.2.3. By (i) and the exact sequence of modules of Kähler differentials given in Proposition 3.2.7, we have $\mathrm{ri}(\Omega^{n+1}_{R_{\mathbb{W}}/K}) \le \max\left\{ r_{\mathbb{W}}, \mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}), \dots, \mathrm{ri}(\Omega^n_{R_{\mathbb{W}}/K}) \right\} \le \max\{r_{\mathbb{W}}+n, r_{\mathbb{V}}+n-1\}$, and hence claim (ii) follows.

Moreover, if Supp($\mathbb{W}$) is in general position, then Proposition 4.2.9 implies that $\max\{r_{\mathbb{W}}+m, r_{\mathbb{V}}+m-1\} \le \max\{m_s+m_{s-1}+m-1, \lfloor \frac{\sum_{j=1}^{s} m_j+n-2}{n} \rfloor + m, m_s+m_{s-1}+m+1, \lfloor \frac{\sum_{j=1}^{s} m_j+s+n-2}{n} \rfloor + m - 1\} \le \max\{m_s+m_{s-1}+m, \lfloor \frac{\sum_{j=1}^{s} m_j+s+n-2}{n} \rfloor + m - 1\}$. Thus the additional claim follows from (i) and (ii). □

The following example indicates that the upper bound for the regularity index of $\Omega^m_{R_{\mathbb{W}}/K}$ given in Proposition 4.3.4 is sharp.

Example 4.3.5. Let $\mathbb{W}$ be the fat point scheme $\mathbb{W} = P_1 + 2P_2 + P_3 + P_4 + 2P_5 + 2P_6 + 2P_7 + P_8$ in $\mathbb{P}^3$, where $P_1 = (1:9:0:0)$, $P_2 = (1:6:0:1)$, $P_3 = (1:2:3:3)$, $P_4 = (1:9:3:5)$, $P_5 = (1:3:0:4)$, $P_6 = (1:0:1:3)$, $P_7 = (1:0:2:0)$, and $P_8 = (1:3:0:10)$. Let $\mathbb{V}$ be the fat point scheme $\mathbb{V} = 2P_1 + 3P_2 + 2P_3 + 2P_4 + 3P_5 + 3P_6 + 3P_7 + 2P_8$ containing the scheme $\mathbb{W}$. We have $r_{\mathbb{W}} = 3$ and $r_{\mathbb{V}} = 5$ so, $\max\{r_{\mathbb{W}}+m, r_{\mathbb{V}}+m-1\} = m+4$ for $m = 1, ..., 4$. The regularity index of $\Omega^m_{R_{\mathbb{W}}/K}$ is $m+4$ for $m = 1, ..., 3$, and $\mathrm{ri}(\Omega^4_{R_{\mathbb{W}}/K}) = 7$. Furthermore, the regularity index of $\Omega^m_{R_{\mathbb{W}}/K[x_0]}$ is $m+4$ for $m = 1, 2$. Thus the bounds for regular indices in claims i) and ii) are sharp bounds.

Let $\mathbb{X}$ be the scheme $\mathbb{X} = \{P_4, P_5, P_6, P_7, P_8\}$ in $\mathbb{P}^3$. Then $\mathbb{X}$ is in general position. For $m = 1, \dots, 3$, the regularity index of $\Omega^m_{R_{2\mathbb{X}}/K}$ is $4+m$. Also we have $\mathrm{ri}(\Omega^m_{R_{2\mathbb{X}}/K[x_0]}) = 4+m$ for $m = 1, 2$ and $\mathrm{ri}(\Omega^3_{R_{2\mathbb{X}}/K[x_0]}) = 6$. Thus $\max\{\mathrm{ri}(\Omega^m_{R_{2\mathbb{X}}/K}), \mathrm{ri}(\Omega^m_{R_{2\mathbb{X}}/K[x_0]})\} = 4+m = \max\{2+2+m, \lfloor (\sum_{i=1}^{5} 2+5+3-2)/3 \rfloor + m - 1\}$ for $m = 1, 2, 3$. In addition, for $m = 4$, we have $\mathrm{ri}(\Omega^4_{R_{2\mathbb{X}}/K}) = 7 = \max\{2+2+3, \lfloor (\sum_{i=1}^{5} 2+5+3-2)/3 \rfloor + 3 - 1$, and so the bounds in the additional claim are sharp bounds.

Given a fat point scheme $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$, the Hilbert function of the $R_{\mathbb{W}}$-module $\Omega^1_{R_{\mathbb{W}}/K}$ is $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i)$ for all $i \in \mathbb{Z}$, where $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ (see Corollary 4.2.3). Naturally, we still want to give the formula for the Hilbert function of the module $\Omega^m_{R_{\mathbb{W}}/K}$ for $m \ge 2$. In fact, we can formulate the Hilbert function of $\Omega^{n+1}_{R_{\mathbb{W}}/K}$ under the following certain case.

Proposition 4.3.6. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points. For $m_1, \dots, m_s \in \mathbb{N}$, let $\mathbb{W}$ be the fat point scheme $\mathbb{W} = m_1P_1 + \cdots + m_sP_s \subseteq \mathbb{P}^n$, and let $\mathbb{Y}$ be the subscheme $\mathbb{Y} = (m_1-1)P_1 + \cdots + (m_s-1)P_s$ of $\mathbb{W}$. Suppose that the scheme $\mathbb{X}$ is contained in a hyperplane. Then we have $\Omega^{n+1}_{R_\mathbb{W}/K} \cong S/\mathcal{I}_\mathbb{Y}(-n-1)$. In particular, $\mathrm{HF}_{\Omega^{n+1}_{R_\mathbb{W}/K}}(i) = \mathrm{HF}_\mathbb{Y}(i-n-1)$ for all $i \in \mathbb{Z}$.*

Proof. Assume that $\mathbb{X} \subseteq \mathcal{Z}^+(H)$, where $0 \neq H = \sum_{i=0}^n a_iX_i \in S$, $a_0, \dots, a_n \in K$. By letting $J = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_\mathbb{W}, 0 \le i \le n\rangle_S + \mathcal{I}_\mathbb{W}$, we have $\Omega^{n+1}_{R_\mathbb{W}/K} \cong (S/J)(-n-1)$ (see Corollary 3.2.16). Let $F \in \mathcal{I}_\mathbb{W} = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ be a non-zero homogeneous polynomial. For any $i = 0, \dots, n$, we see that $\partial F/\partial X_i \in \wp_j^{m_j-1}$, and so $\partial F/\partial X_i \in \mathcal{I}_\mathbb{Y}$. This implies $J \subseteq \mathcal{I}_\mathbb{Y}$, and thus $\mathrm{HF}_{\Omega^{n+1}_{R_\mathbb{W}/K}}(i) \ge \mathrm{HF}_\mathbb{Y}(i-n-1)$ for all $i \in \mathbb{N}$.

Now we prove $\mathcal{I}_\mathbb{Y} = J$. Suppose for a contradiction that there exists a non-zero homogeneous polynomial G such that $G \in \mathcal{I}_\mathbb{Y} \setminus J$. Then $HG \in \mathcal{I}_\mathbb{W}$. Since $H \neq 0$, we may assume $a_i \neq 0$ for some $i \in \{0, \dots, n\}$. By taking the partial derivative of HG of X_i, we have $\partial(HG)/\partial X_i = a_iG + H\partial G/\partial X_i \in J$. Since $G \notin J$, we deduce $H\partial G/\partial X_i \in \mathcal{I}_\mathbb{Y} \setminus J$. We set $G_1 := H\partial G/\partial X_i \neq 0$. Then we continue to have $HG_1 \in \mathcal{I}_\mathbb{W}$, and so $\partial(H\partial G_1/\partial X_i)/\partial X_i = H^2\partial^2G/\partial X_i^2 + 2a_iH\partial G/\partial X_i \in J$. This implies that $0 \neq H^2\partial^2 G/\partial X_i^2 \in \mathcal{I}_\mathbb{Y} \setminus J$. Repeating this process, we eventually get $H^{\deg(G)} \in \mathcal{I}_\mathbb{Y} \setminus J$. On the other hand, since $G \in \mathcal{I}_\mathbb{Y}$, we have $\deg(G) \ge m := \max\{m_1-1, \dots, m_s-1\}$, and therefore $H^{m+1} \in \mathcal{I}_\mathbb{W}$. Thus $H^{\deg(G)} = \frac{1}{a_i}\partial H^{\deg(G)+1}/\partial X_i \in J$, a contradiction. Consequently, we get $\mathcal{I}_\mathbb{Y} = J$, and hence $\Omega^{n+1}_{R_\mathbb{W}/K} \cong R_\mathbb{Y}(-n-1)$, as desired. □

Example 4.3.7. Let $\mathbb{W}$ and $\mathbb{Y}$ be fat point schemes $\mathbb{W} = 2P_1 + 3P_2 + 4P_3 + 2P_4 + P_5 + 7P_6 + 5P_7$ and $\mathbb{Y} = P_1 + 2P_2 + 3P_3 + P_4 + 6P_6 + 4P_7$ in $\mathbb{P}^5_\mathbb{Q}$, respectively, where $P_1 = ((1:1:1:1:1:15/6)$, $P_2 = (1:2:1:1:1:17/6)$, $P_3 = (1:1:2:1:1:18/6)$, $P_4 = (1:2:3:4:5:55/6)$, $P_5 = (1:2:2:1:1:20/6)$, $P_6 = (1:3:2:1:1:22/6)$, and where $P_7 = (1:0:0:1:1:10/6)$. Then $\mathbb{X} = \{P_1, \dots, P_7\}$ is contained in the hyperplane $\mathcal{Z}^+(H)$, where $H = X_0 - 4X_3 + 3X_4$. Thus Proposition 4.3.6 yields that $\Omega^6_{R_\mathbb{W}/K} \cong R_\mathbb{Y}(-6)$ and the Hilbert function of $\Omega^6_{R_\mathbb{W}/K}$ is $\mathrm{HF}_{\Omega^6_{R_\mathbb{X}/K}}$: 0 0 0 0 0 0 1 6 21 56 126 252 306 329 336 337 337 $\cdots$.

Lemma 4.3.8. *In the setting of Proposition 4.3.6, if $\alpha_\mathbb{Y} + 2 \le \alpha_\mathbb{W}$ then*

$$\mathrm{HF}_{\Omega^{n+1}_{R_\mathbb{W}/K}}(\alpha_\mathbb{W} + n - 1) > \mathrm{HF}_\mathbb{Y}(\alpha_\mathbb{W} - 2).$$

Proof. Let $J = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_\mathbb{W}, 0 \le i \le n\rangle_S + \mathcal{I}_\mathbb{W}$. The inclusions $\mathcal{I}_\mathbb{W} \subseteq J \subseteq \mathcal{I}_\mathbb{Y}$ yield that $\alpha_\mathbb{W} = \alpha_J + 1 \ge \alpha_\mathbb{Y}$, where $\alpha_J = \min\{i \in \mathbb{N} \mid J_i \neq 0\}$, and that

$$\mathrm{HF}_{\Omega^{n+1}_{R_\mathbb{W}/K}}(\alpha_\mathbb{W}+n-1) = \mathrm{HF}_{S/J}(\alpha_\mathbb{W}-2) = \mathrm{HF}_{S/J}(\alpha_J-1) = \mathrm{HF}_S(\alpha_J-1) = \mathrm{HF}_S(\alpha_\mathbb{W}-2).$$

Also, it follows from the inequality $\alpha_{\mathbb{Y}} + 2 \leq \alpha_{\mathbb{W}}$ that

$$\begin{aligned}\mathrm{HF}_{\mathbb{Y}}(\alpha_{\mathbb{W}} - 2) &= \mathrm{HF}_{S}(\alpha_{\mathbb{W}} - 2) - \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}}}(\alpha_{\mathbb{W}} - 2)\\ &\leq \mathrm{HF}_{S}(\alpha_{\mathbb{W}} - 2) - \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}}}(\alpha_{\mathbb{Y}})\\ &< \mathrm{HF}_{S}(\alpha_{\mathbb{W}} - 2).\end{aligned}$$

Thus we get $\mathrm{HF}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(\alpha_{\mathbb{W}} + n - 1) = \mathrm{HF}_{S}(\alpha_{\mathbb{W}} - 2) > \mathrm{HF}_{\mathbb{Y}}(\alpha_{\mathbb{W}} - 2)$. □

Now we present a criterion for the support of a fat point scheme to lie on a hyperplane.

Proposition 4.3.9. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points, let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, and let $\mathbb{Y} = (m_1 - 1)P_1 + \cdots + (m_s - 1)P_s$ be a subscheme of $\mathbb{W}$. Suppose that $\alpha_{\mathbb{W}} = \alpha_{\mathbb{Y}} + \alpha_{\mathbb{X}}$. Then the scheme $\mathbb{X}$ is contained in a hyperplane if and only if $\mathrm{HF}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(\alpha_{\mathbb{W}} + n - 1) = \mathrm{HF}_{\mathbb{Y}}(\alpha_{\mathbb{W}} - 2)$.*

Proof. If $\mathbb{X}$ is contained in a hyperplane, then Proposition 4.3.6 implies $\mathrm{HF}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(i) = \mathrm{HF}_{\mathbb{Y}}(i-n-1)$ for all $i \in \mathbb{Z}$. In particular, we have $\mathrm{HF}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(\alpha_{\mathbb{W}}+n-1) = \mathrm{HF}_{\mathbb{Y}}(\alpha_{\mathbb{W}}-2)$. Conversely, suppose that $\mathrm{HF}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(\alpha_{\mathbb{W}} + n - 1) = \mathrm{HF}_{\mathbb{Y}}(\alpha_{\mathbb{W}} - 2)$ and that $\mathbb{X}$ does not lie on any hyperplane. It is clear that $\alpha_{\mathbb{X}} \geq 2$. By assumption, we have $\alpha_{\mathbb{Y}} + 2 \leq \alpha_{\mathbb{Y}} + \alpha_{\mathbb{X}} = \alpha_{\mathbb{W}}$. So, Lemma 4.3.8 implies $\mathrm{HF}_{\Omega^{n+1}_{R_{\mathbb{W}}/K}}(\alpha_{\mathbb{W}} + n - 1) > \mathrm{HF}_{\mathbb{Y}}(\alpha_{\mathbb{W}} - 2)$, a contradiction. □

Let us apply the proposition to the following concrete examples.

Example 4.3.10. (a) Let $\mathbb{W}$ be the fat point scheme $\mathbb{W} = 3P_1 + 2P_2 + 8P_3 + 5P_4 + 4P_5 + 2P_6 + 5P_7 + P_8 + 2P_9$ in $\mathbb{P}^3$, where $P_1 = (1 : 3/4 : 0 : 0)$, $P_2 = (1 : 2 : 0 : 0)$, $P_3 = (1 : 1/23 : 0 : 0)$, $P_4 = (1 : 1 : 1 : 0)$, $P_5 = (1 : 7 : 1 : 0)$, $P_6 = (1 : 2 : 1 : 0)$, $P_7 = (1 : 1/2 : 2 : 0)$, $P_8 = (1 : 3 : 1/6 : 0)$, and where $P_9 = (1 : 3/17 : 1/4 : 0)$. Let $\mathbb{Y}$ be the subscheme $\mathbb{Y} = 2P_1 + P_2 + 7P_3 + 4P_4 + 3P_5 + P_6 + 4P_7 + P_9$ of $\mathbb{W}$. Then $\mathbb{X} = \mathrm{Supp}(\mathbb{W})$ lies on the hyperplane $\mathcal{Z}^+(X_3)$ and $\alpha_{\mathbb{W}} = 13 = 3 + 10 = \alpha_{\mathbb{X}} + \alpha_{\mathbb{Y}}$. Thus Proposition 4.3.9 yields that $\mathrm{HF}_{\Omega^4_{R_{\mathbb{W}}/K}}(i) = \mathrm{HF}_{\mathbb{Y}}(i - n - 1)$ for all $i \in \mathbb{Z}$. Explicitly, we have $\mathrm{HF}_{\Omega^4_{R_{\mathbb{W}}/K}(n+1)} = \mathrm{HF}_{R_{\mathbb{Y}}} :$ 1 4 10 20 35 56 84 106 124 136 141 141 $\cdots$.
(b) Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a complete intersection of type $(d_1, \dots, d_n)$, and let $\nu \geq 2$. Clearly, we have

$$\begin{aligned}\alpha_{\nu\mathbb{X}} &= \nu \min\{d_1, \dots, d_n\}\\ &= (\nu - 1) \min\{d_1, \dots, d_n\} + \min\{d_1, \dots, d_n\}\\ &= \alpha_{(\nu-1)\mathbb{X}} + \alpha_{\mathbb{X}}.\end{aligned}$$

So, if $\alpha_{\mathbb{X}} = 1$ then $\mathbb{X}$ is contained in a hyperplane, and therefore, an application of Proposition 4.3.9 implies $\mathrm{HF}_{\Omega^{n+1}_{R_{\nu\mathbb{X}}/K}}(i) = \mathrm{HF}_{(\nu-1)\mathbb{X}}(i-n-1)$ for all $i \in \mathbb{Z}$.

(c) Let $\mathbb{X} = \{P_1, P_2, P_3\} \subseteq \mathbb{P}^n$ be a set of three points in general position, and let $\nu \geq 1$. It is well known (cf. [DSG, Example 3.6]) that

$$\alpha_{\nu\mathbb{X}} = \begin{cases} 3k-1 & \text{for } \nu = 2k-1, \\ 3k & \text{for } \nu = 2k. \end{cases}$$

From this we deduce $\alpha_{\mathbb{X}} = 2$, $\alpha_{2\nu\mathbb{X}} = 3\nu$, and $\alpha_{(2\nu+1)\mathbb{X}} = 3\nu + 2 = \alpha_{\mathbb{X}} + \alpha_{2\nu\mathbb{X}}$. If $n \geq 3$, then $\mathbb{X}$ is always contained in some hyperplane, and thus Proposition 4.3.9 shows that $\mathrm{HF}_{\Omega^{n+1}_{R_{(2\nu+1)\mathbb{X}}/K}}(i) = \mathrm{HF}_{2\nu\mathbb{X}}(i-n-1)$ for all $i \in \mathbb{Z}$.

Although no formula for the Hilbert function of the module of Kähler differential $(n+1)$-forms of an equimultiple fat point scheme is known, the following proposition provides a formula for its Hilbert polynomial.

Proposition 4.3.11. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points, and let $\nu \geq 1$. Then we have* $\mathrm{HP}_{\Omega^{n+1}_{R_{(\nu+1)\mathbb{X}}/K}}(z) = \mathrm{HP}_{\nu\mathbb{X}}(z)$.

To prove this proposition, we use the following remark which is mentioned in [HC, Remark 4.2].

Remark 4.3.12. Let $\mathcal{I}_{\mathbb{X}} = \cap_{i=1}^{s} \wp_i$ be the homogeneous vanishing ideal of the scheme $\mathbb{X} = \{P_1, \cdots, P_s\} \subseteq \mathbb{P}^n$. For $\nu \geq 1$, there is a $\mathfrak{M}$-primary ideal J such that the intersection $\mathcal{I}_{\nu\mathbb{X}} \cap J$ is a primary decomposition of the ideal $\mathcal{I}_{\mathbb{X}}^{\nu}$. Moreover, we have $(\mathcal{I}_{\mathbb{X}}^{(\nu)})_i = (\mathcal{I}_{\mathbb{X}}^{\nu})_i$ for $i \gg 0$.

Proof of Proposition 4.3.11. Let $I = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\mathbb{X}}, 0 \leq i \leq n \rangle_S + \mathcal{I}_{\mathbb{X}}$ be a homogeneous ideal of S. Then Corollary 3.2.16 and Proposition 3.4.1 yield that $\Omega^{n+1}_{R_{\mathbb{X}}/K} \cong (S/I)(-n-1)$ and $\mathrm{HF}_{\Omega^{n+1}_{R_{\mathbb{X}}/K}}(i) = \mathrm{HF}_{S/I}(i-n-1) = 0$ for $i \gg 0$. So, there exists $t_1 \in \mathbb{N}$ such that $I_{t_1+i} = \mathfrak{M}_{t_1+i}$ for all $i \in \mathbb{N}$. Moreover, it follows from Remark 4.3.12 that there is $t_2 \in \mathbb{N}$ such that $(\mathcal{I}_{\nu\mathbb{X}})_{t_2+i} = (\mathcal{I}_{\mathbb{X}}^{\nu})_{t_2+i}$ for all $i \in \mathbb{N}$. Let $t = \max\{t_1, t_2, r_{\nu\mathbb{X}}\}$, let $r = \binom{n+t}{n} - s$, let $\{F_1, \dots, F_r\}$ be a K-basis of the K-vector space $(\mathcal{I}_{\mathbb{X}})_t$, and let

$$J = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_{(\nu+1)\mathbb{X}}, 0 \leq i \leq n \rangle_S + \mathcal{I}_{(\nu+1)\mathbb{X}}.$$

Clearly, we have $\mathcal{I}_{(\nu+1)\mathbb{X}} \subseteq J \subseteq \mathcal{I}_{\nu\mathbb{X}}$. Since $\Omega^{n+1}_{R_{(\nu+1)\mathbb{X}}/K} \cong (S/J)(-n-1)$ (by Corollary 3.2.16) and since $\mathcal{I}_{\nu\mathbb{X}}$ is generated by elements of degree $\leq t+1$ (see Proposition 2.4.6), it suffices to show that $\mathrm{HF}_J(i) = \mathrm{HF}_{\mathcal{I}_{\nu\mathbb{X}}}(i)$ for some $i \geq t+1$.

We observe that

$$\begin{aligned}I_t &= (\langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\mathbb{X}}, 0 \le i \le n\rangle_S + \mathcal{I}_{\mathbb{X}})_t\\ &= \langle \{\partial F/\partial X_i \mid F \in (\mathcal{I}_{\mathbb{X}})_{t+1}, 0 \le i \le n\}\\ &\qquad \cup \{G\partial H/\partial X_i \mid G \in (\mathcal{I}_{\mathbb{X}})_k, H \in S_{t+1-k}, 0 \le i \le n\}\rangle_K\\ &= \langle \partial F/\partial X_i \mid F \in (\mathcal{I}_{\mathbb{X}})_{t+1}, 0 \le i \le n\rangle_K + (\mathcal{I}_{\mathbb{X}})_t.\end{aligned}$$

For $F \in (\mathcal{I}_{\mathbb{X}})_t$, since $\langle F_1, \dots, F_r\rangle_K = (\mathcal{I}_{\mathbb{X}})_t$, Euler's relation implies that there are elements $a_1, \dots, a_r \in K$ such that $F = \sum_{j=1}^r a_j F_j = \sum_{j=1}^r \sum_{i=0}^n \frac{1}{\deg(F_j)} a_j X_i \partial F_j/\partial X_i$. Thus F is contained in $\langle \partial F_j/\partial X_i \mid 0 \le i \le n, 1 \le j \le r\rangle_K \mathfrak{M}_1$. Moreover, we have

$$\begin{aligned}\langle \partial G/\partial X_i \mid G \in (\mathcal{I}_{\mathbb{X}})_{t+1}, 0 \le i \le n\rangle_K &= \langle \partial X_j H/\partial X_i \mid H \in (\mathcal{I}_{\mathbb{X}})_t, 0 \le i, j \le n\rangle_K\\ &= \langle \partial F_j/\partial X_i \mid 0 \le i \le n, 1 \le j \le r\rangle_K \mathfrak{M}_1.\end{aligned}$$

Thus $I_t = \langle \partial F_j/\partial X_i \mid 0 \le i \le n, 1 \le j \le r\rangle_K \mathfrak{M}_1 + (\mathcal{I}_{\mathbb{X}})_t$. So, we get equalities

$$\begin{aligned}(\mathcal{I}_{\nu\mathbb{X}})_{(\nu+1)(nr+1)t} &= (\mathcal{I}_{\nu\mathbb{X}})_{\nu t}\mathfrak{M}_{((\nu+1)nr+1)t} = (\mathcal{I}_{\nu\mathbb{X}})_{\nu t}(\mathfrak{M}^{(\nu+1)nr+1})_{((\nu+1)nr+1)t}\\ &= (\mathcal{I}_{\mathbb{X}}^{\nu})_{\nu t}(\mathfrak{M}^{(\nu+1)nr+1})_{((\nu+1)nr+1)t}\\ &= \underbrace{(\mathcal{I}_{\mathbb{X}})_t \cdots (\mathcal{I}_{\mathbb{X}})_t}_{\nu \text{ times}} \underbrace{\mathfrak{M}_t \cdots \mathfrak{M}_t}_{(\nu+1)nr+1 \text{ times}} = \underbrace{(\mathcal{I}_{\mathbb{X}})_t \cdots (\mathcal{I}_{\mathbb{X}})_t}_{\nu \text{ times}} \underbrace{I_t \cdots I_t}_{(\nu+1)nr+1 \text{ times}}\\ &= \langle F_1, \dots, F_r\rangle_K^{\nu} \cdot (\langle \partial F_j/\partial X_i \mid 0 \le i \le n, 1 \le j \le r\rangle_K \mathfrak{M}_1)^{(\nu+1)nr+1}\\ &\quad + (\mathcal{I}_{(\nu+1)\mathbb{X}})_{(\nu+1)(nr+1)t}.\end{aligned}$$

So, Proposition 4.3.11 holds if we have the following inclusion

$$\langle F_1, \dots, F_r\rangle_K^{\nu} \cdot (\langle \partial F_j/\partial X_i \mid 0 \le i \le n, 1 \le j \le r\rangle_K \mathfrak{M}_1)^{(\nu+1)nr+1} \subseteq J.$$

Indeed, for $0 \le i_1 \le n$ and $1 \le j_1 \le r$ we have $(\nu+1)F_{j_1}^{\nu}\frac{\partial F_{j_1}}{\partial X_{i_1}} = \frac{\partial F_{j_1}^{\nu+1}}{\partial X_{i_1}} \in J$. Also, for $i_1, i_2 \in \{0, \dots, n\}$ and $j_1, j_2 \in \{1, \dots, r\}$, we get

$$\nu F_{j_1}^{\nu-1}F_{j_2}\frac{\partial F_{j_1}}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} = \frac{\partial F_{j_1}^{\nu}F_{j_2}}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} - F_{j_1}^{\nu}\frac{\partial F_{j_1}}{\partial X_{i_2}} \cdot \frac{\partial F_{j_2}}{\partial X_{i_1}} \in J.$$

Now we assume that $F_{j_1}^{\nu-k}F_{j_2}\cdots F_{j_{k+1}}\frac{\partial F_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} \in J$ for all $i_1, \dots, i_{k+1} \in \{0, \dots, n\}$ and $j_1, \dots, j_{k+1} \in \{1, \dots, r\}$ and $1 \le k \le \nu$. We shall prove

$$F_{j_1}^{\nu-k-1}F_{j_2}\cdots F_{j_{k+2}}\frac{\partial F_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+2}}} \in J$$

for all $i_1, \dots, i_{k+2} \in \{0, \dots, n\}$ and $j_1, \dots, j_{k+2} \in \{1, \dots, r\}$. We have

$$\begin{aligned}&(\nu-k)F_{j_1}^{\nu-k-1}F_{j_2}\cdots F_{j_{k+2}}\frac{\partial F_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+2}}}\\ &= \frac{\partial F_{j_1}^{\nu-k}F_{j_2}\cdots F_{j_{k+2}}}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+2}}} - F_{j_1}^{\nu-k}\frac{\partial F_{j_2}\cdots F_{j_{k+2}}}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+2}}}\\ &= \frac{\partial F_{j_1}^{\nu-k}F_{j_2}\cdots F_{j_{k+2}}}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+2}}} - \sum_{l=2}^{k+2} F_{j_1}^{\nu-k}\frac{F_{j_2}\cdots F_{j_{k+2}}}{F_{j_l}}\frac{\partial F_{j_l}}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+2}}}.\end{aligned}$$

By the inductive hypothesis, we have $F_{j_1}^{\nu-k-1}F_{j_2}\cdots F_{j_{k+2}}\frac{\partial F_{j_1}}{\partial X_{i_1}}\cdots\frac{\partial F_{j_1}}{\partial X_{i_{k+2}}}\in J$. Thus we have shown that $F_{j_1}^{\nu-k}F_{j_2}\cdots F_{j_{k+1}}\frac{\partial F_{j_1}}{\partial X_{i_1}}\cdots\frac{\partial F_{j_1}}{\partial X_{i_{k+1}}}\in J$ for all $i_1,\dots,i_{k+1}\in\{0,\dots,n\}$ and $j_1,\dots,j_{k+1}\in\{1,\dots,r\}$ and $1\le k\le\nu$. In particular, if $k=\nu$, then we have $F_{j_1}F_{j_2}\cdots F_{j_\nu}\frac{\partial F_j}{\partial X_{i_1}}\cdots\frac{\partial F_j}{\partial X_{i_{\nu+1}}}\in J$ for all $i_1,\dots,i_{\nu+1}\in\{0,\dots,n\}$ and $j,j_1,\dots,j_\nu\in\{1,\dots,r\}$. On the other hand, due to Dirichlet's box principle, for any element F of the set $\langle\partial F_j/\partial X_i\mid 0\le i\le n,1\le j\le r\rangle_K^{\nu r+1}$, there is an integer $k\in\{1,...,r\}$ such that $F=\frac{\partial F_k}{\partial X_{i_1}}\cdots\frac{\partial F_k}{\partial X_{i_{\nu+1}}}$ for some $i_1,...,i_{\nu+1}\in\{0,...,n\}$. Since $(\nu+1)nr+1\ge(\nu r+1)$, we get any element of the K-vector space

$$\langle F_1,\dots,F_r\rangle_K^\nu\cdot\langle\partial F_j/\partial X_i\mid 0\le i\le n,1\le j\le r\rangle_K^{(\nu+1)nr+1}$$

is a sum of elements of the form $F_{j_1}\cdots F_{j_\nu}\frac{\partial F_j}{\partial X_{i_1}}\cdots\frac{\partial F_j}{\partial X_{i_{\nu+1}}}H$ for some homogeneous polynomial H in S. Therefore we get

$$\langle F_1,\dots,F_r\rangle_K^\nu\cdot(\langle\partial F_j/\partial X_i\mid 0\le i\le n,1\le j\le r\rangle_K\mathfrak{M}_1)^{(\nu+1)nr+1}\subseteq J,$$

and this completes the proof. □

The following corollary follows immediately from Propositions 4.1.3 and 4.3.11,

Corollary 4.3.13. *Let $\mathbb{X}=\{P_1,\dots,P_s\}\subseteq\mathbb{P}^n$ be a set of s distinct K-rational points, and let $\nu\ge1$. Then we have $\mathrm{HP}_{\Omega^{n+1}_{R_{(\nu+1)\mathbb{X}}/K}}(z)=s\binom{\nu+n-1}{n}$.*

In the last part of this section we study the Kähler differential 2-forms of fat point schemes $\mathbb{W}$. Let us begin with a short exact sequence of $R_\mathbb{W}$-modules.

Proposition 4.3.14. *Let $\mathbb{W}=m_1P_1+\cdots+m_sP_s$ be a fat point scheme in $\mathbb{P}^n$. For every $i\in\mathbb{N}$, we let $\mathbb{W}_i:=(m_1+i)P_1+\cdots+(m_s+i)P_s$. Then the sequence of graded $R_\mathbb{W}$-modules*

$$0\longrightarrow\mathcal{I}_{\mathbb{W}_1}/\mathcal{I}_{\mathbb{W}_2}\xrightarrow{\alpha}\mathcal{I}_\mathbb{W}\Omega^1_{S/K}/\mathcal{I}_{\mathbb{W}_1}\Omega^1_{S/K}\xrightarrow{\beta}\Omega^2_{S/K}/\mathcal{I}_\mathbb{W}\Omega^2_{S/K}\xrightarrow{\gamma}\Omega^2_{R_\mathbb{W}/K}\longrightarrow0\qquad(**)$$

is a complex, where $\alpha(F+\mathcal{I}_{\mathbb{W}_2})=dF+\mathcal{I}_{\mathbb{W}_1}\Omega^1_{S/K}$, where $\beta(GdX_i+\mathcal{I}_{\mathbb{W}_1}\Omega^1_{S/K})=d(GdX_i)+\mathcal{I}_\mathbb{W}\Omega^2_{S/K}$, and where $\gamma(H+\mathcal{I}_\mathbb{W}\Omega^2_{S/K})=H+(\mathcal{I}_\mathbb{W}\Omega^2_{S/K}+d\mathcal{I}_\mathbb{W}\Omega^1_{S/K})$. Moreover,

(i) the map α is injective,

(ii) the map γ is surjective, and

(iii) $\beta(\mathcal{I}_\mathbb{W}\Omega^1_{S/K}/\mathcal{I}_{\mathbb{W}_1}\Omega^1_{S/K})=\mathrm{Ker}(\gamma)$.

(iv) For all $i \geq 0$, we have

$$\begin{aligned}\mathrm{HF}_{\Omega^2_{R_{\mathbb{W}}/K}}(i+2) \geq \frac{n(n+1)}{2}\,\mathrm{HF}_{\mathbb{W}}(i) + \mathrm{HF}_{\mathbb{W}_2}(i+2)\\ - \mathrm{HF}_{\mathbb{W}_1}(i+2) - (n+1)(\mathrm{HF}_{\mathbb{W}_1}(i+1) - \mathrm{HF}_{\mathbb{W}}(i+1)).\end{aligned}$$

Proof. Similar to the proof of Theorem 4.2.1, it follows that the map α is an injective map. Claims (ii) and (iii) are a consequence of induced from the presentation

$$\Omega^2_{R_{\mathbb{W}}/K} = \Omega^2_{S/K}/(\mathcal{I}_{\mathbb{W}}\Omega^2_{S/K} + \langle d\mathcal{I}_{\mathbb{W}}\rangle_S \Omega^1_{S/K})$$

(see Proposition 3.2.11). The map d is an anti-derivation, hence $\beta \circ \alpha(\mathcal{I}_{\mathbb{W}_1}/\mathcal{I}_{\mathbb{W}_2}) = \langle 0 \rangle$. Therefore the sequence $(**)$ is a complex. Additionally, claim (iv) follows from the fact that $(**)$ is a complex and from claim (iii). □

The following corollary is an immediate consequence of Proposition 4.3.14.

Corollary 4.3.15. *In the setting of Proposition 4.3.14, we have*

$$\mathrm{HP}_{\Omega^2_{R_{\mathbb{W}}/K}}(z) \geq \tfrac{(n+2)(n+1)}{2}\,\mathrm{HP}_{\mathbb{W}}(z) + \mathrm{HP}_{\mathbb{W}_2}(z) - (n+2)\,\mathrm{HP}_{\mathbb{W}_1}(z).$$

Let us consider the special case that $\mathbb{W} = \nu\mathbb{X}$ is an equimultiple fat point scheme in $\mathbb{P}^2$. First, we show that the sequence of K-vector space $(**)$ is exact. Then we establish the formula for the Hilbert polynomial of $\Omega^2_{R_{\mathbb{W}}/K}$.

Corollary 4.3.16. *Let $\mathbb{X} = \{P_1, \cdots, P_s\} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points, and let $\nu \geq 1$. For $i \gg 0$, we have the following exact sequence of K-vector spaces:*

$$\begin{aligned}0 \longrightarrow (\mathcal{I}_{(\nu+1)\mathbb{X}}/\mathcal{I}_{(\nu+2)\mathbb{X}})_i \stackrel{\alpha}{\longrightarrow} (\mathcal{I}_{\nu\mathbb{X}}\Omega^1_{S/K}/\mathcal{I}_{(\nu+1)\mathbb{X}}\Omega^1_{S/K})_i\\ \stackrel{\beta}{\longrightarrow} (\Omega^2_{S/K}/\mathcal{I}_{\nu\mathbb{X}}\Omega^2_{S/K})_i \stackrel{\gamma}{\longrightarrow} (\Omega^2_{R_{\nu\mathbb{X}}/K})_i \longrightarrow 0.\end{aligned}$$

Here the maps α, β and γ are defined as in Proposition 4.3.14.

Proof. According to Proposition 4.3.14, it suffices to show that the Hilbert polynomial of $\Omega^2_{R_{\nu\mathbb{X}}/K}$ is

$$\mathrm{HP}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(z) = \tfrac{(n+2)(n+1)}{2}\,\mathrm{HP}_{\nu\mathbb{X}}(z) + \mathrm{HP}_{(\nu+2)\mathbb{X}}(z) - (n+2)\,\mathrm{HP}_{(\nu+1)\mathbb{X}}(z).$$

Due to Proposition 3.2.7, we have the exact sequence of graded $R_{\nu\mathbb{X}}$-modules

$$0 \longrightarrow \Omega^3_{R_{\nu\mathbb{X}}/K} \longrightarrow \Omega^2_{R_{\nu\mathbb{X}}/K} \longrightarrow \Omega^1_{R_{\nu\mathbb{X}}/K} \longrightarrow \mathfrak{m}_{\nu\mathbb{X}} \longrightarrow 0.$$

Thus, by Corollary 4.2.3 and Proposition 4.3.11, we get

$$\begin{aligned}
\mathrm{HP}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(z) &= \mathrm{HP}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(z) + \mathrm{HP}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(z) - \mathrm{HP}_{\nu\mathbb{X}}(z)\\
&= ((n+2)\,\mathrm{HP}_{\nu\mathbb{X}}(z) - \mathrm{HP}_{(\nu+1)\mathbb{X}}(z)) + \mathrm{HP}_{(\nu-1)\mathbb{X}}(z) - \mathrm{HP}_{\nu\mathbb{X}}(z)\\
&= 3s\tbinom{\nu+1}{2} - s\tbinom{\nu+2}{2} + s\tbinom{\nu}{2}\\
&= \tfrac{1}{2}(3\nu^2-\nu-2)s\\
&= 6s\tbinom{\nu+1}{2} + s\tbinom{\nu+3}{2} - 4s\tbinom{\nu+2}{2}\\
&= \tfrac{(n+2)(n+1)}{2}\,\mathrm{HP}_{\nu\mathbb{X}} + \mathrm{HP}_{(\nu+2)\mathbb{X}} - (n+2)\,\mathrm{HP}_{(\nu+1)\mathbb{X}},
\end{aligned}$$

and this finishes the proof. □

From the proof of Corollary 4.3.16 we get the formula for the Hilbert polynomial of $\Omega^2_{R_{\nu\mathbb{X}}/K}$ as follows.

Corollary 4.3.17. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points, and let $\nu \geq 1$. Then $\mathrm{HP}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(z) = \frac{1}{2}(3\nu^2-\nu-2)s$.*

4.4 Kähler Differential 1-Forms for Fat Point Schemes Supported at Complete Intersections

As in the previous sections, we let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct reduced K-rational points, and we let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme supported at $\mathbb{X}$. We have computed the Hilbert polynomial and the regularity index of the module of Kähler differential m-forms of $R_{\mathbb{X}}$, and we have extended this to arbitrary fat point schemes as far as we could. In some special cases, the modules $\Omega^1_{R_{\mathbb{W}}/K}$ and $\Omega^1_{R_{\mathbb{W}}/K[x_0]}$ have particular Hilbert functions. In this section we examine such special fat point schemes.

For reduced complete intersections $\mathbb{X}$, we can determine the Hilbert function of the module $\Omega^1_{R_{\mathbb{X}}/K}$ and its regularity index explicitly. This result has been shown in [DK, Proposition 4.3].

Proposition 4.4.1. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct reduced K-rational points which is a complete intersection of type $(d_1, \dots, d_n)$. Then we have*

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{X}}(i-1) - \sum_{j=1}^{n} \mathrm{HF}_{\mathbb{X}}(i-d_j)$$

for all $i \geq 0$ and $\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K}) = \sum_{i=1}^{n} d_i - n + \max\{d_j \mid 1 \leq j \leq n\}$.

Now we extend this proposition to equimultiple fat point schemes whose supports are complete intersections.

Proposition 4.4.2. *Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a set of s distinct reduced K-rational points which is a complete intersection of type $(d_1,\dots,d_n)$. Let $L_j = \{(a_1,\dots,a_n) \in \mathbb{N}^n \mid a_1+\cdots+a_n = j\}$ for $j \geq 1$ and let $\nu \geq 1$.*

(i) We have the following exact sequence of graded $R_{\nu\mathbb{X}}$-modules:

$$0 \longrightarrow \mathcal{I}_{\mathbb{X}}^{\nu}/\mathcal{I}_{\mathbb{X}}^{\nu+1} \longrightarrow (S/\mathcal{I}_{\mathbb{X}}^{\nu})^{n+1}(-1) \longrightarrow \Omega^1_{R_{\nu\mathbb{X}}/K} \longrightarrow 0$$

(ii) For all $i \in \mathbb{Z}$, we have

$$\begin{aligned}\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(i) =&(n+1)\Big(\sum_{j=0}^{\nu-1}\sum_{(a_1,\dots,a_n)\in L_j}\mathrm{HF}_{\mathbb{X}}(i-1-a_1d_1-\cdots-a_nd_n)\Big)\\ &-\sum_{(a_1,\dots,a_n)\in L_\nu}\mathrm{HF}_{\mathbb{X}}(i-a_1d_1-\cdots-a_nd_n).\end{aligned}$$

(iii) For all $i \in \mathbb{N}$, we have

$$\begin{aligned}\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}}(i) =&n\Big(\sum_{j=0}^{\nu-1}\sum_{(a_1,\dots,a_n)\in L_j}\mathrm{HF}_{\mathbb{X}}(i-1-a_1d_1-\cdots-a_nd_n)\Big)\\ &-\sum_{(a_1,\dots,a_n)\in L_\nu}\mathrm{HF}_{\mathbb{X}}(i-a_1d_1-\cdots-a_nd_n).\end{aligned}$$

Proof. (i) By Theorem 4.2.1, we have an exact sequence of graded $R_{\nu\mathbb{X}}$-modules

$$0 \longrightarrow \mathcal{I}_{\mathbb{X}}^{(\nu)}/\mathcal{I}_{\mathbb{X}}^{(\nu+1)} \longrightarrow R_{\nu\mathbb{X}}^{n+1}(-1) \longrightarrow \Omega^1_{R_{\nu\mathbb{X}}/K} \longrightarrow 0.$$

Since the homogeneous vanishing ideal of $\mathbb{X}$ is generated by a regular sequence, we can use [ZS, Appendix 6, Lemma 5] and get $\mathcal{I}_{\mathbb{X}}^{(\nu)} = \mathcal{I}_{\mathbb{X}}^{\nu}$ for all $\nu \in \mathbb{N}$. Then we have $R_{\nu\mathbb{X}} = S/\mathcal{I}_{\mathbb{X}}^{(\nu)} = S/\mathcal{I}_{\mathbb{X}}^{\nu}$. Hence we get the desired exact sequence.

(ii) By Proposition 4.1.16, the Hilbert function of $R_{\nu\mathbb{X}}$ is

$$\mathrm{HF}_{\nu\mathbb{X}}(i) = \sum_{j=0}^{\nu-1}\sum_{(a_1,\dots,a_n)\in L_j}\mathrm{HF}_{\mathbb{X}}(i-a_1d_1-\cdots-a_nd_n).$$

So, we get

$$\begin{aligned}\mathrm{HF}_{\mathcal{I}_{\mathbb{X}}^{\nu}/\mathcal{I}_{\mathbb{X}}^{\nu+1}}(i) &= \mathrm{HF}_{(\nu+1)\mathbb{X}}(i) - \mathrm{HF}_{\nu\mathbb{X}}(i)\\ &= \sum_{(a_1,\dots,a_n)\in L_\nu}\mathrm{HF}_{\mathbb{X}}(i-a_1d_1-\cdots-a_nd_n).\end{aligned}$$

By the first part of this proposition, we obtain

$$\begin{aligned}\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(i) &= (n+1)\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{\mathcal{I}^\nu_\mathbb{X}/\mathcal{I}^{\nu+1}_\mathbb{X}}(i)\\ &= (n+1)\Big(\sum_{j=0}^{\nu-1}\sum_{(a_1,\dots,a_n)\in L_j}\mathrm{HF}_\mathbb{X}(i-1-a_1d_1-\cdots-a_nd_n)\Big)\\ &\quad - \sum_{(a_1,\dots,a_n)\in L_\nu}\mathrm{HF}_\mathbb{X}(i-a_1d_1-\cdots-a_nd_n).\end{aligned}$$

(iii) By Corollary 3.3.2, we have $\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}}(i) = \mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(i) - \mathrm{HF}_{\nu\mathbb{X}}(i-1)$ for all $i\in\mathbb{Z}$. Thus, according to (ii), we get

$$\begin{aligned}\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}}(i) = n\Big(&\sum_{j=0}^{\nu-1}\sum_{(a_1,\dots,a_n)\in L_j}\mathrm{HF}_\mathbb{X}(i-1-a_1d_1-\cdots-a_nd_n)\Big)\\ &- \sum_{(a_1,\dots,a_n)\in L_\nu}\mathrm{HF}_\mathbb{X}(i-a_1d_1-\cdots-a_nd_n)\end{aligned}$$

as we wished. □

Remark 4.4.3. Let $\mathbb{X}=\{P_1,\dots,P_s\}\subseteq\mathbb{P}^n$ be a set of s distinct reduced K-rational points which is a complete intersection of type $(d_1,\dots,d_n)$ with $d_1\le\cdots\le d_n$.

(a) We have $r_\mathbb{X}=\sum_{j=1}^n d_j - n$.

(b) If $\nu=1$ then $\mathrm{ri}(\Omega^1_{R_\mathbb{X}/K}) = d_n+\sum_{j=1}^n d_j - n$ (see Proposition 4.4.1). Moreover, the Hilbert function of $\Omega^1_{R_\mathbb{X}/K[x_0]}$ satisfies

$$\mathrm{HF}_{\Omega^1_{R_\mathbb{X}/K[x_0]}}(i) = n\,\mathrm{HF}_\mathbb{X}(i-1) - \sum_{(a_1,\dots,a_n)\in L_1}\mathrm{HF}_\mathbb{X}(i-a_1d_1-\cdots-a_nd_n)$$

for all $i\in\mathbb{Z}$ (by Proposition 4.4.2(iii)). Thus $\mathrm{ri}(\Omega^1_{R_\mathbb{X}/K[x_0]}) = d_n+\sum_{j=1}^n d_j-n$ if $d_n\ge 2$ and $\mathrm{ri}(\Omega^1_{R_\mathbb{X}/K[x_0]}) = -\infty$ if $d_n=1$.

(c) In the case $n=1$, we see that $\mathrm{HF}_{\nu\mathbb{X}}(i)=\sum_{j=0}^{\nu-1}\mathrm{HF}_\mathbb{X}(i-jd_1)$ and $\mathrm{HF}_{\mathcal{I}^\nu_\mathbb{X}/\mathcal{I}^{\nu+1}_\mathbb{X}}(i) = \mathrm{HF}_\mathbb{X}(i-\nu d_1)$ for all $i\in\mathbb{Z}$. Thus Proposition 4.4.2(ii) implies that

$$\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(i) = 2\textstyle\sum_{j=0}^{\nu-1}\mathrm{HF}_\mathbb{X}(i-1-jd_1) - \mathrm{HF}_\mathbb{X}(i-\nu d_1)$$

for all $i\in\mathbb{Z}$, and the regularity index of $\Omega^1_{R_{\nu\mathbb{X}}/K}$ satisfies $\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K}) = \nu d_1+d_1-1$. Also, Proposition 4.4.2(iii) yields

$$\begin{aligned}\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}}(i) &= \sum_{j=0}^{\nu-1}\mathrm{HF}_\mathbb{X}(i-1-jd_1) - \mathrm{HF}_\mathbb{X}(i-\nu d_1)\\ &= \sum_{j=0}^{\nu-2}\mathrm{HF}_\mathbb{X}(i-1-jd_1) + (\mathrm{HF}_\mathbb{X}(i-1-(\nu-1)d_1) - \mathrm{HF}_\mathbb{X}(i-\nu d_1)\end{aligned}$$

for all $i \in \mathbb{Z}$. If $d_1 = 1$ then $\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}}(i) = \sum_{j=1}^{\nu-2} \mathrm{HF}_{\mathbb{X}}(i-1-j)$ for all $i \in \mathbb{Z}$, and so $\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}) = (\nu-1)d_1 + d_1 - 1$. In the case $d_1 \geq 2$, we see that for $j \leq \nu - 1$ we have $1 + jd_1 < \nu d_1$, and consequently $\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}) = \nu d_1 + d_1 - 1$.

Now we apply Proposition 4.4.2 and the above remark to precisely describe the regularity indices of $\Omega^1_{R_{\nu\mathbb{X}}/K}$ and of $\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}$ as follows.

Corollary 4.4.4. *Let $s \geq 2$, let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which is a complete intersection of type $(d_1, \dots, d_n)$ with $d_1 \leq \cdots \leq d_n$, and let $\nu \geq 1$. Then the $R_{\nu\mathbb{X}}$-modules $\Omega^1_{R_{\nu\mathbb{X}}/K}$ and $\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}$ have the same regularity index which is given by*

$$\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K}) = \mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}) = \nu d_n + \sum_{j=1}^{n} d_j - n.$$

Proof. First we remark that $s \geq 2$. This implies $d_n \geq 2$. In view of Remark 4.4.3, we may assume that $n \geq 2$ and $\nu \geq 2$. For $j \geq 1$ we let $L_j = \{(a_1, \dots, a_n) \in \mathbb{N}^n \mid a_1 + \cdots + a_n = j\}$, and we let

$$t_j := \max\{a_1 d_1 + \cdots + a_n d_n \mid (a_1, \dots, a_n) \in L_j\}.$$

Note that $d_1 \leq \cdots \leq d_n$ and $r_{\mathbb{X}} = \sum_{j=1}^{n} d_j - n$. So, if $j < k$ then $t_j = jd_n < kd_n = t_k$. This implies $\max\{1 + \sum_{i=1}^{n} a_i d_i \mid (a_1, \dots, a_n) \in L_j, 1 \leq j \leq \nu - 1\} = 1 + (\nu-1)d_n < \nu d_n = t_\nu$, since $d_n \geq 2$. Thus it follows from Proposition 4.4.2(ii) and (iii) that

$$\begin{aligned}
\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K}) &= \mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}) \\
&= \max\{r_{\mathbb{X}} + a_1 d_1 + \cdots + a_n d_n \mid (a_1, \dots, a_n) \in L_\nu\} \\
&= r_{\mathbb{X}} + \max\{a_1 d_1 + \cdots + a_n d_n \mid (a_1, \dots, a_n) \in L_\nu\} \\
&= r_{\mathbb{X}} + t_\nu = \sum_{j=1}^{n} d_j - n + \nu d_n.
\end{aligned}$$

Hence the conclusion follows. □

Let us look at an example to illustrate this corollary.

Example 4.4.5. Let $\mathbb{X} = \{P_1, \dots, P_8\} \subseteq \mathbb{P}^2$ be the set consisting of eight points $P_1 = (1:0:0)$, $P_2 = (1:0:1)$, $P_3 = (1:0:2)$, $P_4 = (1:0:3)$, $P_5 = (1:1:0)$, $P_6 = (1:1:1)$, $P_7 = (1:1:2)$, and $P_8 = (1:1:3)$. Then it is easy to see that $\mathbb{X}$ is a complete intersection of type $(2, 4)$. Let $\nu \geq 1$. An application of Corollary 4.4.4 implies

$$\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K}) = \mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}) = 4\nu + 2 + 4 - 2 = 4(\nu+1).$$

For instance, if $\nu = 5$ then we have

$$\mathrm{HF}_{\Omega^1_{R_{5\mathbb{X}}/K}} : 0\ 3\ 9\ 18\ 30\ 45\ 63\ 84\ 108\ 135\ 164\ 192\ 219\ 242\ 262\ 279\ 293\ 304\ 312\ 317\ 319\ 318\ 315\ 313\ 312\ 312 \dots$$

$$\mathrm{HF}_{\Omega^1_{R_{5\mathbb{X}}/K[x_0]}} : 0\ 2\ 6\ 12\ 20\ 30\ 42\ 56\ 72\ 90\ 109\ 127\ 144\ 158\ 170\ 180\ 188\ 194\ 198\ 200\ 200\ 198\ 195\ 193\ 192\ 192 \dots.$$

and $\mathrm{ri}(\Omega^1_{R_{5\mathbb{X}}/K}) = \mathrm{ri}(\Omega^1_{R_{5\mathbb{X}}/K[x_0]}) = 24 = 4(5+1)$.

Observe that Proposition 4.4.2 and Corollary 4.4.4 contain formulas for the Hilbert functions of $\Omega^1_{R_{\nu\mathbb{X}}/K}$ and $\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}$ and their regularity indices. For a non-equimultiple fat point scheme $\mathbb{W}$ whose support $\mathrm{Supp}(\mathbb{W}) = \mathbb{X}$ is a reduced complete intersection, these results can be applied to give bounds for the Hilbert functions of $\Omega^1_{R_\mathbb{W}/K}$ and $\Omega^1_{R_\mathbb{W}/K[x_0]}$ and their regularity indices.

Proposition 4.4.6. *Let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$ supported at a complete intersection $\mathbb{X} = \{P_1, \dots, P_s\}$ of type $(d_1, \dots, d_n)$. Suppose that $\mathbb{W}$ is not an equimultiple fat point scheme. We set $m_{\min} := \min\{m_1, \dots, m_s\}$ and $m_{\max} := \max\{m_1, \dots, m_s\}$, and we set*

$$\begin{aligned}\mathrm{HF}_1(i) = (n+1)\sum_{j=0}^{m_{\min}-1}\ &\sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_\mathbb{X}(i-1-a_1d_1-\cdots-a_nd_n)\\ &- \sum_{j=m_{\min}}^{m_{\max}} \sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_\mathbb{X}(i-a_1d_1-\cdots-a_nd_n)\end{aligned}$$

and

$$\begin{aligned}\mathrm{HF}_2(i) = (n+1)\sum_{j=0}^{m_{\max}-1}\ &\sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_\mathbb{X}(i-1-a_1d_1-\cdots-a_nd_n))\\ &+ \sum_{j=m_{\min}+1}^{m_{\max}-1} \sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_\mathbb{X}(i-a_1d_1-\cdots-a_nd_n)\end{aligned}$$

for all $i \in \mathbb{Z}$. Then, for all $i \in \mathbb{Z}$, we have

$$\mathrm{HF}_1(i) \le \mathrm{HF}_{\Omega^1_{R_\mathbb{W}/K}}(i) \le \mathrm{HF}_2(i).$$

Proof. For $t \ge 0$ and $i \in \mathbb{Z}$, we set

$$\mathrm{H}_1(t,i) = \sum_{j=0}^{t} \sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_\mathbb{X}(i-a_1d_1-\cdots-a_nd_n)$$

and

$$\mathrm{H}_2(t,i) = \sum_{j=0}^{t} \sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_\mathbb{X}(i-1-a_1d_1-\cdots-a_nd_n).$$

It follows from the inclusions $m_{\min}\mathbb{X} \subseteq \mathbb{W} \subseteq m_{\max}\mathbb{X}$ that for all $i \in \mathbb{Z}$, we have $\mathrm{HF}_{m_{\min}\mathbb{X}}(i) \le \mathrm{HF}_{\mathbb{W}}(i) \le \mathrm{HF}_{m_{\max}\mathbb{X}}(i)$. Thus Proposition 4.1.16 yields

$$\begin{aligned}\mathrm{H}_1(m_{\min}-1,i) &= \sum_{j=0}^{m_{\min}-1} \sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_{\mathbb{X}}(i-a_1d_1-\cdots-a_nd_n)\\ &\le \mathrm{HF}_{\mathbb{W}}(i)\\ &\le \sum_{j=0}^{m_{\max}-1} \sum_{(a_1,\dots,a_n)\in L_j} \mathrm{HF}_{\mathbb{X}}(i-a_1d_1-\cdots-a_nd_n) = \mathrm{H}_1(m_{\max}-1,i).\end{aligned}$$

Let $\mathbb{V}$ be the fat point scheme $\mathbb{V} = (m_1+1)P_1+\cdots+(m_s+1)P_s$ in $\mathbb{P}^n$. For the same reason as above, we get $\mathrm{H}_1(m_{\min},i) \le \mathrm{HF}_{\mathbb{V}}(i) \le \mathrm{H}_1(m_{\max},i)$ for all $i \in \mathbb{Z}$. According to Corollary 4.2.3, for every $i \in \mathbb{Z}$ the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K}$ satisfies

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i).$$

Hence we get

$$\begin{aligned}&(n+1)\mathrm{H}_2(m_{\min}-1,i) + \mathrm{H}_1(m_{\min}-1,i) - \mathrm{H}_1(m_{\max},i)\\ &\qquad \le \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) \le (n+1)\mathrm{H}_2(m_{\max}-1,i) + \mathrm{H}_1(m_{\max}-1,i) - \mathrm{H}_1(m_{\min},i).\end{aligned}$$

Note that $m_{\min} < m_{\max}$. So, we have

$$\mathrm{HF}_1(i) = (n+1)\mathrm{H}_2(m_{\min}-1,i) + \mathrm{H}_1(m_{\min}-1,i) - \mathrm{H}_1(m_{\max},i)$$

and

$$\mathrm{HF}_2(i) = (n+1)\mathrm{H}_2(m_{\max}-1,i) + \mathrm{H}_1(m_{\max}-1,i) - \mathrm{H}_1(m_{\min},i)$$

for all $i \in \mathbb{Z}$. Therefore the inequalities $\mathrm{HF}_1(i) \le \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) \le \mathrm{HF}_2(i)$ hold true for all $i \in \mathbb{Z}$, as we wanted to show. □

Remark 4.4.7. In the setting of Proposition 4.4.6, we can give bounds for the Hilbert function of $\Omega^1_{R_{\mathbb{X}}/K[x_0]}$ as follows. For every $i \in \mathbb{Z}$, we have $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(i) = \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) - \mathrm{HF}_{\mathbb{W}}(i-1)$. We use the same argument as in the proof of Proposition 4.4.6 to get

$$\begin{aligned}&n\mathrm{H}_2(m_{\min}-1,i) + \mathrm{H}_1(m_{\min}-1,i) - \mathrm{H}_1(m_{\max},i)\\ &\qquad \le \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K[x_0]}}(i) \le n\mathrm{H}_2(m_{\max}-1,i) + \mathrm{H}_1(m_{\max}-1,i) - \mathrm{H}_1(m_{\min},i)\end{aligned}$$

for all $i \in \mathbb{Z}$.

Corollary 4.4.8. *Using the notation introduced in Proposition 4.4.6, the regularity index of module of Kähler differential* 1*-forms* $\Omega^1_{R_{\mathbb{W}}/K}$ *is bounded by*

$$\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) \le (m_{\max}+1)d_n + d_1 + \cdots + d_{n-1} - n.$$

Proof. Let $\mathbb{V}$ be the fat point scheme $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ in $\mathbb{P}^n$. Since $\mathbb{W} \subseteq m_{\max}\mathbb{X}$, this implies that $r_{\mathbb{W}} \le r_{m_{\max}\mathbb{X}} = (m_{\max}-1)d_n + \sum_{j=1}^n d_j - n$ (see Corollary 4.4.4). Similarly, we have $r_{\mathbb{V}} \le r_{(m_{\max}+1)\mathbb{X}} = m_{\max}d_n + \sum_{j=1}^n d_j - n$. Corollary 4.2.3 implies that $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) \le \max\{r_{\mathbb{W}}+1, r_{\mathbb{V}}\}$. Thus an upper bound for the regularity index of $\Omega^1_{R_{\mathbb{W}}/K}$ is

$$\begin{aligned}\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) &\le \max\left\{(m_{\max}-1)d_n + \textstyle\sum_{j=1}^n d_j - n + 1, m_{\max}d_n + \sum_{j=1}^n d_j - n\right\}\\ &\le (m_{\max}+1)d_n + d_1 + \cdots + d_{n-1} - n,\end{aligned}$$

as we wished. □

For an equimultiple fat point scheme $\nu\mathbb{X}$ supported at a complete intersection, Proposition 4.4.2 shows that the Hilbert functions of the Kähler differential modules $\Omega^1_{R_{\nu\mathbb{X}}/K}$ and $\Omega^1_{R_{\nu\mathbb{X}}/K[x_0]}$ depend only on the type of $\mathbb{X}$. Our next proposition says that, if we reduce in $\nu\mathbb{X}$ the multiplicity of one point P_j by one, the Hilbert function of the module of Kähler differentials of the resulting scheme does not depend on the choice of j.

Proposition 4.4.9. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which is a complete intersection of type $(d_1, \dots, d_n)$, and let $\nu \ge 2$. For $j \in \{1, \dots, s\}$, let $\mathbb{Y}_j = \nu P_1 + \cdots + \nu P_{j-1} + (\nu-1)P_j + \nu P_{j+1} + \cdots + \nu P_s$. Then the Hilbert function of $\Omega^1_{R_{\mathbb{Y}_j}/K}$ does not depend on the choice of j.*

Proof. According to Corollary 4.1.19, the schemes $\mathbb{Y}_j$ all have the same Hilbert function. Similarly, the schemes $\mathbb{W}_j = (\nu+1)P_1 + \cdots + (\nu+1)P_{j-1} + \nu P_j + (\nu+1)P_{j+1} + \cdots + (\nu+1)P_s$ all have the same Hilbert function. By Theorem 4.2.1, the sequence of graded $R_{\mathbb{Y}_j}$-modules

$$0 \longrightarrow \mathcal{I}_{\mathbb{Y}_j}/\mathcal{I}_{\mathbb{W}_j} \longrightarrow R^{n+1}_{\mathbb{Y}_j}(-1) \longrightarrow \Omega^1_{R_{\mathbb{Y}_j}/K} \longrightarrow 0$$

is exact. Hence we have

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{Y}_j}/K}}(i) = (n+1)\,\mathrm{HF}_{\mathbb{Y}_j}(i-1) + \mathrm{HF}_{\mathbb{Y}_j}(i) - \mathrm{HF}_{\mathbb{W}_j}(i)$$

for all $i \in \mathbb{Z}$, and the conclusion follows. □

Surprisingly, in the case $\nu = 1$, i.e. in the case of a reduced 0-dimensional complete intersection, the analogue of the preceding proposition seems to be more difficult. We offer two partial results in this case. First of all, the claim holds for subschemes of $\mathbb{P}^2$, as the following example shows.

Example 4.4.10. Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points which is a complete intersection of type (a, b), and let $a \le b$. For every $j \in \{1, \dots, ab\}$, let $\mathbb{Y}_j$ denote the scheme $2(\mathbb{X} \setminus \{P_j\})$ of double points in $\mathbb{P}^2$. By Theorem 4.4 of [BGT], a minimal separator of $\mathbb{Y}_j$ in $\mathbb{W}_j = 2P_1 + \cdots + 2P_{j-1} + P_j + 2P_{j+1} + \cdots + 2P_{ab}$ is of degree $2b + a - 3$. Thus all schemes $\mathbb{Y}_j$ have the same Hilbert function. Now we can use Theorem 4.2.1 to conclude that the Hilbert function of $\Omega^1_{R_{\mathbb{X}\setminus\{P_j\}}/K}$ does not depend on the choice of j.

Next we prove the desired property for reduced 0-dimensional complete intersections of type $(d, \dots, d)$, where $d \ge 1$.

Proposition 4.4.11. *Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points which is a complete intersection of type $(d, \dots, d)$ for some $d \ge 1$, and let $\mathbb{X}_j = \mathbb{X} \setminus \{P_j\}$ for $j = 1, \dots, s$. Then the Hilbert function of $\Omega^1_{R_{\mathbb{X}_j}/K}$ does not depend on the choice of j.*

Proof. For $j = 1, \dots, s$, let $\mathbb{W}_j$ and $\mathbb{Y}_j$ be the subschemes $\mathbb{W}_j = 2P_1 + \cdots + 2P_{j-1} + P_j + 2P_{j+1} + \cdots + 2P_s$ and $\mathbb{Y}_j = 2P_1 + \cdots + 2P_{j-1} + 2P_{j+1} + \cdots + 2P_s$ of the scheme $2\mathbb{X}$. By Corollary 4.1.19(iii), the Hilbert function of $\mathbb{W}_j$ does not depend on j. For $j = 1, \dots, s$, let $f_j^* \in R_{\mathbb{W}_j}$ be a minimal separator of $\mathbb{Y}_j$ in $\mathbb{W}_j$, and let $F_j^* \in S$ be a representative of f_j^*. It is clear that $\deg(F_j^*) = \alpha_{\mathbb{Y}_j/\mathbb{W}_j}$, where $\alpha_{\mathbb{Y}_j/\mathbb{W}_j} = \min\{i \in \mathbb{N} \mid (\mathcal{I}_{\mathbb{Y}_j/\mathbb{W}_j})_i \ne 0\}$. Then we have a short exact sequence

$$0 \longrightarrow S/(\mathcal{I}_{\mathbb{W}_j} : F_j^*)(-\alpha_{\mathbb{Y}_j/\mathbb{W}_j}) \xrightarrow{\times F_j^*} S/\mathcal{I}_{\mathbb{W}_j} \longrightarrow S/\mathcal{I}_{\mathbb{Y}_j} \longrightarrow 0.$$

By [GMT, Lemmata 2.2 and 5.1], we have $\mathcal{I}_{\mathbb{W}_j} : F_j^* = \wp_j$, and $\wp_j$ has a minimal graded free resolution

$$0 \to S(-n) \to S^{\binom{n}{n-1}}(-n+1) \to \cdots \to S^{\binom{n}{1}}(-1) \to \wp_j \to 0.$$

Since $2\mathbb{X}$ is a scheme of double points in $\mathbb{P}^n$, Corollary 4.1.19 implies that all minimal separators of $\mathbb{W}_j$ in $2\mathbb{X}$ have the same degree, namely $(n+1)d - n$, and the graded free resolution of $R_{\mathbb{W}_j}$ is given by

$$0 \to S^{2n}(-(n+1)d + 1) \to \mathcal{G}_{n-2} \to \cdots \to \mathcal{G}_0 \to S \to S/\mathcal{I}_{\mathbb{W}_j} \to 0.$$

Now an application of the mapping cone construction (see Definition 2.1.16) yields the following graded free resolution of $R_{\mathbb{Y}_j}$

$$\begin{aligned} 0 \to S(-\alpha_{\mathbb{Y}_j/\mathbb{W}_j} - n) &\to S^{2n}(-(n+1)d + 1) \oplus S^n(-\alpha_{\mathbb{Y}_j/\mathbb{W}_j} - n + 1) \\ &\to \cdots \to S(-\alpha_{\mathbb{Y}_j/\mathbb{W}_j}) \oplus \mathcal{G}_0 \to S \to S/\mathcal{I}_{\mathbb{Y}_j} \to 0. \end{aligned}$$

Since the ideal $\mathcal{I}_{\mathbb{Y}_j}$ is saturated, its projective dimension is $n-1$. Hence the module $S(-\alpha_{\mathbb{Y}_j/\mathbb{W}_j}-n)$ must be a submodule of $S^{2n}(-(n+1)d+1)\oplus S^{\binom{n}{n-1}}(-\alpha_{\mathbb{Y}_j/\mathbb{W}_j}-n+1)$. By degree comparison, the term $S(-\alpha_{\mathbb{Y}_j/\mathbb{W}_j}-n)$ must cancel against something in the module $S^{2n}(-(n+1)d+1)$. This implies $\alpha_{\mathbb{Y}_j/\mathbb{W}_j} = (n+1)(d-1)$. Thus the Hilbert function of $\mathbb{Y}_j$ does not depend on the choice of P_j in $\mathbb{X}$. Furthermore, the Hilbert function of $\mathbb{X}\setminus\{P_j\}$ is independent of the choice of j, because $\mathbb{X}$ is a complete intersection. Therefore the claim follows from Theorem 4.2.1. □

Our last proposition of this section presents some results on the the module of Kähler differential $(n+1)$-forms of an equimultiple fat point scheme supported at a complete intersection.

Proposition 4.4.12. *Let $\mathbb{X}=\{P_1,\dots,P_s\}\subseteq\mathbb{P}^n$ be a set of s distinct K-rational points which is a complete intersection with $\mathcal{I}_{\mathbb{X}}=\langle F_1,\dots,F_n\rangle_S$, let $d_j=\deg(F_j)$ for $j=1,\dots,n$, let $\mathfrak{M}=\langle X_0,\dots,X_n\rangle_S$, and let $\nu\geq 1$. Suppose that $d_1\leq\cdots\leq d_n$.*

(i) If $d_1=1$, then we have $\Omega^{n+1}_{R_{\nu\mathbb{X}}/K}\cong R_{(\nu-1)\mathbb{X}}(-n-1)$.

(ii) If $2=d_1=\cdots=d_t<d_{t+1}\leq\cdots\leq d_n$, $1\leq t\leq n$, and there is $j\in\{1,\dots,t\}$ such that $\mathcal{Z}^+(F_j)$ is a non-singular conic, then we have

$$\Omega^{n+1}_{R_{\nu\mathbb{X}}/K}\cong(S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}})(-n-1).$$

(iii) Suppose that $2=d_1=\cdots=d_t<d_{t+1}\leq\cdots\leq d_n$ for some $t\in\{1,\dots,n\}$. We let $\mathcal{X}=(X_0\ \cdots\ X_n)$ and write $\partial F_j/\partial X_k=\mathcal{X}\cdot\mathcal{A}_{jk}$, where $\mathcal{A}_{jk}\in\mathrm{Mat}_{1\times(n+1)}(K)$, $1\leq j\leq t$, and $0\leq k\leq n$. If $\operatorname{rank}\big((\mathcal{A}_{10}\ \cdots\ \mathcal{A}_{tn})\big)<n+1$, then

$$\Omega^{n+1}_{R_{\nu\mathbb{X}}/K}\ncong(S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}})(-n-1).$$

(iv) If $1<d_1<d_2\leq\cdots\leq d_n$, then we have

$$\Omega^{n+1}_{R_{\nu\mathbb{X}}/K}\cong(S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}})(-n-1)$$

if and only if $\mathcal{Z}^+(F_1)$ is a non-singular conic.

Proof. Claim (i) follows directly from Proposition 4.3.6. Now we prove claim (ii). For this, we may assume without lost of generality that $\mathcal{Z}^+(F_1)$ is a non-singular conic. Then we write $F_1=\mathcal{X}\mathcal{A}\mathcal{X}^{\mathrm{tr}}$, where $\mathcal{A}=(a_{ij})_{i,j=0,\dots,n}$ with $a_{ij}=a_{ji}$ for all $i,j\in\{0,\dots,n\}$ and $i\neq j$. It follows from the assumption that $\det(\mathcal{A})\neq 0$. For every $i \in \{0, \dots, n\}$, we have $\partial F_1/\partial X_i = 2(a_{i0}X_0 + \cdots + a_{in}X_n)$. This implies that $\langle \partial F_1/\partial X_0, \dots, \partial F_1/\partial X_n\rangle_S = \mathfrak{M} = \langle X_0, \dots, X_n\rangle_S$.

Moreover, since the scheme $\mathbb{X}$ is a complete intersection, we deduce that

$$\mathcal{I}_{\nu\mathbb{X}} = \langle F_1, \dots, F_n\rangle_S^{\nu} = \langle F_1^{\nu}, F_1^{\nu-1}F_2, \dots, F_n^{\nu}\rangle_S.$$

Let $J = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\nu\mathbb{X}}, 0 \le i \le n\rangle_S + \mathcal{I}_{\nu\mathbb{X}}$. According to Corollary 3.2.16, it suffices to show that $J = \mathcal{I}_{(\nu-1)\mathbb{X}}\mathfrak{M}$. Since $d_1 > 1$, it follows that $J \subseteq \mathcal{I}_{(\nu-1)\mathbb{X}}\mathfrak{M}$. For other inclusion, it is enough to show that $F_{i_1}\cdots F_{i_{\nu-1}}\mathfrak{M} \subseteq J$ for all $i_1, \dots, i_{\nu-1} \in \{1, \dots, n\}$.

In the following, we use induction on k to prove $F_1^k F_{i_1}\cdots F_{i_{\nu-1-k}}\mathfrak{M} \subseteq J$ for all $k \in \mathbb{N}$ and $i_1, \dots, i_{\nu-1-k} \in \{1, \dots, n\}$. We see that the claim is clearly true for $k \ge \nu$ and $F_1^{\nu-1}\mathfrak{M} = \langle \partial F_1^{\nu}/\partial X_0, \dots, \partial F_1^{\nu}/\partial X_n\rangle_S \subseteq J$. Moreover, since $\partial F_j/\partial X_k \in \mathfrak{M}$, we see that

$$\begin{aligned} F_1^{\nu-2}F_{i_1}\mathfrak{M} &= \langle F_1^{\nu-2}F_{i_1}\partial F_1/\partial X_i \mid 0 \le i \le n\rangle_S \\ &= \langle \partial(F_1^{\nu-1}F_{i_1})/\partial X_i - F_1^{\nu-1}\partial F_{i_1}/\partial X_i \mid 0 \le i \le n\rangle_S \subseteq J. \end{aligned}$$

Now we assume that $F_1^k F_{i_1}\cdots F_{i_{\nu-1-k}}\mathfrak{M} \subseteq J$ for some $1 \le k \le \nu - 2$ and for any $i_1, \dots, i_{\nu-1-k} \in \{1, \dots, n\}$. We need to show $F_1^{k-1}F_{i_1}\cdots F_{i_{\nu-k}}\mathfrak{M} \subseteq J$ for any $i_1, \dots, i_{\nu-k} \in \{1, \dots, n\}$. It is clear that $F_1^k F_{i_1}\cdots F_{i_{\nu-k}} \in \mathcal{I}_{\nu\mathbb{X}}$. Therefore for every $i \in \{0, \dots, n\}$, we have $\partial(F_1^k F_{i_1}\cdots F_{i_{\nu-k}})/\partial X_i = F_1^{k-1}F_{i_1}\cdots F_{i_{\nu-k}}\partial F_1/\partial X_i + \sum_{j=1}^{\nu-k} F_1^k F_{i_1}\cdots \widehat{F_{i_j}}\cdots F_{i_{\nu-k}}\partial F_{i_j}/\partial X_i \in J$. By the inductive hypothesis, the elements $F_1^k F_{i_1}\cdots \widehat{F_{i_j}}\cdots F_{i_{\nu-k}}\partial F_{i_j}/\partial X_i$ are contained in J (as $\partial F_{i_j}/\partial X_i \in \mathfrak{M}$), and so we get $F_1^{k-1}F_{i_1}\cdots F_{i_{\nu-k}}\partial F_1/\partial X_i \in J$ for all $i = 0, \dots, n$. Consequently,

$$F_1^{k-1}F_{i_1}\cdots F_{i_{\nu-k}}\mathfrak{M} = \langle F_1^{k-1}F_{i_1}\cdots F_{i_{\nu-k}}\partial F_1/\partial X_i \mid 0 \le i \le n\rangle_S \subseteq J.$$

Hence $F_1^k F_{i_1}\cdots F_{i_{\nu-1-k}}\mathfrak{M} \subseteq J$ for all $k \in \mathbb{N}$ and $i_1, \dots, i_{\nu-1-k} \in \{1, \dots, n\}$. In other words, the equality $J = \mathcal{I}_{(\nu-1)\mathbb{X}}\mathfrak{M}$ is proved.

Next we prove (iii). We assume that $\operatorname{rank}\big((\mathcal{A}_{1\,0} \ \cdots \ \mathcal{A}_{tn})\big) < n + 1$. This implies that $\mathfrak{M} \not\subseteq \langle \partial F_j/\partial X_i \mid 0 \le i \le n, 1 \le j \le t\rangle_S$. Let $i \in \{0, \dots, n\}$ be an index such that $X_i \in \mathfrak{M} \setminus \langle \partial F_j/\partial X_i \mid 0 \le i \le n, 1 \le j \le t\rangle_S$. Then it is easy to see that $X_i F_1^{\nu-1} \in \mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}$. As above, let $J = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\nu\mathbb{X}}, 0 \le i \le n\rangle_S + \mathcal{I}_{\nu\mathbb{X}}$. Then $J_{2\nu-1} = \langle \partial F_{i_1}\cdots F_{i_\nu}/\partial X_i \mid 0 \le i \le n, 1 \le i_1, \dots, i_\nu \le t\rangle_K$ and $J_{2\nu-2} = \langle 0\rangle$. Now we distinguish two cases.

- If $\nu = 1$ then it is clear that $J \ne \mathfrak{M}$.
- If $\nu \ge 2$ then we have $X_i F_1^{\nu-1} \notin J$, and hence $J \ne \mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}$.

Consequently, an application of Corollary 3.2.16 yields that

$$\Omega^{n+1}_{R_\nu\mathbb{X}/K} \cong (S/J)(-n-1) \ncong S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}(-n-1),$$

and this finishes the proof of (iii).

Finally, claim (iv) follows from the claims (ii) and (iii). $\square$

Chapter 5

Some Special Cases and Applications

In this chapter we investigate the Hilbert functions and the regularity indices of the Kähler differential algebras for some special fat point schemes in $\mathbb{P}^n$, where $n = 2$ or $n = 4$.

In the first section we consider the case of fat point scheme $\mathbb{W}$ on a non-singular conic in $\mathbb{P}^2$. We first recall one of the results of Catalisano [Ca] which gives a description of the Hilbert function of $\mathbb{W}$ in terms of a certain subscheme. Then we use this description to compute the Hilbert function of the module of Kähler differential 1-forms of the scheme $\mathbb{W}$ (see Proposition 5.1.3). Next we show that if, in addition, $\mathbb{W} = \nu\mathbb{X}$ is an equimultiple fat point scheme, then $\Omega^3_{R_{\mathbb{W}}/K} \cong S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}(-3)$ (see Proposition 5.1.7), and apply this isomorphism to exhibit the Hilbert function of $\Omega^3_{R_{\nu\mathbb{X}}/K}$ in terms of degrees of generators of $\mathcal{I}_{\mathbb{X}}$ (or of $\mathcal{I}_{(\nu-1)\mathbb{X}}$) (see Corollary 5.1.8). Finally, we apply the exact sequence given in Proposition 3.2.7 to write down the Hilbert function of $\Omega^2_{R_{\nu\mathbb{X}}/K}$ (see Corollary 5.1.9).

In Section 5.2, we study the case of fat point schemes in $\mathbb{P}^4$. By following the method of proof of [Th2] (which is a rather extended case-by-case argument), we prove the Segre bound for the regularity index of a set of points in $\mathbb{P}^4$ (see Theorem 5.2.8). Then we prove that, under an additional hypothesis, this bound holds for equimultiple fat point schemes in $\mathbb{P}^4$ and is actually an equality (see Theorem 5.2.12). Finally, we use the latter result to compute the regularity index of the module of Kähler differential 1-forms and bound the regularity index of the module of Kähler differential m-forms under this additional hypothesis (see Proposition 5.2.14 and Corollary 5.2.15).

Throughout this chapter let K be a field of characteristic zero, and let $\mathbb{P}^n$ be the projective n-space over K. The homogeneous coordinate ring of $\mathbb{P}^n$ is $S = K[X_0, \dots, X_n]$.

It is equipped with the standard grading $\deg(X_i) = 1$ for $i = 0, \dots, n$. Furthermore, we let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ be a set of s distinct K-rational points, and we let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$ supported at $\mathbb{X}$, where $m_j \geq 1$ for all $j = 1, \dots, s$.

5.1 Kähler Differential Algebras for Fat Point Schemes on a Non-Singular Conic in $\mathbb{P}^2$

In Lemma 3.3.7, we described concretely the Hilbert functions of $\Omega^m_{R_{\mathbb{W}}/K}$ and $\Omega^m_{R_{\mathbb{W}}/K[x_0]}$ when $\mathbb{W}$ is a fat point scheme in $\mathbb{P}^1$. This result leads us to the following question:

Question 5.1.1. *Can we compute explicitly the Hilbert function of the bi-graded $R_{\mathbb{W}}$-algebra $\Omega_{R_{\mathbb{W}}/K}$ for a fat point scheme $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^2$?*

In this section we answer some parts of the above question. More precisely, we give concrete formulas for the Hilbert function of the bi-graded algebra $\Omega_{R_{\mathbb{W}}/K}$ if $\mathbb{W}$ is an equimultiple fat point scheme in $\mathbb{P}^2$ whose support $\mathbb{X}$ lies on a non-singular conic.

In what follows, we let $\mathcal{C} = \mathcal{Z}^+(C)$ be a non-singular conic defined by a quadratic $C \in S = K[X_0, X_1, X_2]$, we let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points in $\mathcal{C}$, and we let $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^2$ supported at $\mathbb{X}$. Suppose that $0 \leq m_1 \leq \cdots \leq m_s$ and $s \geq 4$. Then it is well-known (cf. [Ca, Proposition 2.2]) that the regularity index of $\mathbb{W}$ is $r_{\mathbb{W}} = \max\{m_s + m_{s-1} - 1, \lfloor \sum_{j=1}^{s} m_j/2 \rfloor\}$. Moreover, the Hilbert function of $\mathbb{W}$ can be effectively computed from the Hilbert function of a certain subscheme $\mathbb{Y}$ of $\mathbb{W}$, as the following proposition points out.

Proposition 5.1.2. *Using the notation introduced as above, we define a fat point subscheme $\mathbb{Y}$ of $\mathbb{W}$ as follows*
a) for $\sum_{j=1}^{s} m_j \geq 2m_{s-1} + 2m_s$, let $\mathbb{Y} = \max\{m_1 - 1, 0\}P_1 + \cdots + \max\{m_s - 1, 0\}P_s$
b) and for $\sum_{j=1}^{s} m_j \leq 2m_{s-1} + 2m_s - 1$, we let $\mathbb{Y} = m_1P_1 + m_2P_2 \cdots + m_{s-2}P_{s-2} + \max\{m_{s-1} - 1, 0\}P_{s-1} + \max\{m_s - 1, 0\}P_s$.
Then we have

$$\mathrm{HF}_{\mathbb{W}}(i) \begin{cases} = \sum_{j=1}^{s} \binom{m_j+1}{2} & \text{if } i \geq r_{\mathbb{W}}, \\ = 2i + 1 + \mathrm{HF}_{\mathbb{Y}}(i-2) & \text{if } 0 \leq i < r_{\mathbb{W}} \text{ and } \sum_{j=1}^{s} m_j \geq 2m_{s-1} + 2m_s, \\ = i + 1 + \mathrm{HF}_{\mathbb{Y}}(i-1) & \text{if } 0 \leq i < r_{\mathbb{W}} \text{ and } \sum_{j=1}^{s} m_j \leq 2m_{s-1} + 2m_s - 1, \\ = 0 & \text{if } i < 0. \end{cases}$$

Proof. See [Ca, Theorem 3.1]. □

Due to Proposition 5.1.2, the Hilbert function of the module of Kähler differential 1-forms $\Omega^1_{R_{\mathbb{W}}/K}$ satisfies the following conditions.

Proposition 5.1.3. *(i) If* $\sum_{j=1}^s m_j + s \geq 2m_s + 2m_{s-1} + 4$ *then*

$$\begin{aligned}
\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) &= \textstyle\sum_{j=1}^s \frac{(m_j+1)(3m_j-2)}{2} \quad \text{for } i \geq \lfloor (\sum_{j=1}^s m_j + s)/2 \rfloor, \\
&= 3\textstyle\sum_{j=1}^s \binom{m_j+1}{2} - 2i - 1 \quad \text{for } r_{\mathbb{W}} + 2 \leq i < \lfloor (\sum_{j=1}^s m_j + s)/2 \rfloor, \\
&= 4\textstyle\sum_{j=1}^s \binom{m_j+1}{2} - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2) \quad \text{for } i = r_{\mathbb{W}} + 1, \\
&= \textstyle\sum_{j=1}^s \binom{m_j+1}{2} + 3\,\mathrm{HF}_{\mathbb{W}}(i-1) - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2) \text{ for } i = r_{\mathbb{W}}, \\
&= 3\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2) \text{ for } 0 \leq i < r_{\mathbb{W}}.
\end{aligned}$$

(ii) If $\sum_{j=1}^s m_j + s \leq 2m_s + 2m_{s-1} + 3$ *and let* $\mathbb{Y} = (m_1+1)P_1 + \cdots + (m_{s-2}+1)P_{s-2} + m_{s-1}P_{s-1} + m_sP_s$*, then*

$$\begin{aligned}
\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) &= \textstyle\sum_{j=1}^s \frac{(m_j+1)(3m_j-2)}{2} \quad \text{for } i \geq m_s + m_{s-1} + 1, \\
&= 4\textstyle\sum_{j=1}^n \binom{m_j+1}{2} - i - 1 - \mathrm{HF}_{\mathbb{Y}}(i-1) \text{ for } r_{\mathbb{W}} + 1 \leq i \leq m_s + m_{s-1}, \\
&= \textstyle\sum_{j=1}^s \binom{m_j+1}{2} + 3\,\mathrm{HF}_{\mathbb{W}}(i-1) - i - 1 - \mathrm{HF}_{\mathbb{Y}}(i-1) \text{ for } i = r_{\mathbb{W}}, \\
&= 3\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - i - 1 - \mathrm{HF}_{\mathbb{Y}}(i-1) \quad \text{for } 0 \leq i < r_{\mathbb{W}}.
\end{aligned}$$

Proof. Let $\mathbb{V}$ be the fat point scheme $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$ containing $\mathbb{W}$. Then we have $r_{\mathbb{V}} = \max\{m_s + m_{s-1} + 1, \lfloor (\sum_{j=1}^s m_j + s)/2 \rfloor\}$. By Corollary 4.2.3(i), we have

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = 3\,\mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i)$$

for all $i \in \mathbb{Z}$. In the following we distinguish two cases.

First we consider the case $\sum_{j=1}^s m_j + s \geq 2m_s + 2m_{s-1} + 4$. In this case, we have $r_{\mathbb{V}} = \lfloor (\sum_{j=1}^s m_j + s)/2 \rfloor$. Since $s \geq 4$, we get the inequality

$$r_{\mathbb{W}} + 1 = \max\{m_s + m_{s-1} - 1, \lfloor \sum_{j=1}^s m_j/2 \rfloor\} + 1 < \lfloor (\sum_{j=1}^s m_j + s)/2 \rfloor = r_{\mathbb{V}}.$$

So, Corollary 4.2.3(iii) yields $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) \leq r_{\mathbb{V}} = \lfloor (\sum_{j=1}^s m_j + s)/2 \rfloor$. Also, we see that

$$\begin{aligned}
\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(r_{\mathbb{V}} - 1) &= 4\deg(\mathbb{W}) - \mathrm{HF}_{\mathbb{V}}(r_{\mathbb{V}} - 1) > 4\deg(\mathbb{W}) - \mathrm{HF}_{\mathbb{V}}(r_{\mathbb{V}} + i) \\
&= 4\deg(\mathbb{W}) - \deg(\mathbb{V}) = \mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(r_{\mathbb{V}} + i)
\end{aligned}$$

for all $i \geq 0$, and hence $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) = r_{\mathbb{V}}$. Consequently, we can apply Proposition 5.1.2 to work out the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K}$ with respect to the range of degree i as follows:

(a) For $i \geq r_{\mathbb{V}} = \lfloor (\sum_{j=1}^{s} m_j + s)/2 \rfloor$, the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K}$ satisfies

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = 4\sum_{j=1}^{s} \binom{m_j+1}{2} - \binom{m_j+2}{2} = \sum_{j=1}^{s} \tfrac{(m_j+1)(3m_j-2)}{2}.$$

(b) Let $r_{\mathbb{W}} + 2 \leq i < r_{\mathbb{V}} = \lfloor (\sum_{j=1}^{s} m_j + s)/2 \rfloor$. Then we have $\mathrm{HF}_{\mathbb{W}}(i-2) = \mathrm{HF}_{\mathbb{W}}(i-1) = \mathrm{HF}_{\mathbb{W}}(i) = \deg(\mathbb{W}) = \sum_{j=1}^{s} \binom{m_j+1}{2}$ and $\mathrm{HF}_{\mathbb{V}}(i) = 2i+1+\mathrm{HF}_{\mathbb{W}}(i-2)$. This follows that

$$\mathrm{HF}_{\Omega^1_{\mathbb{W}/K}}(i) = 4\sum_{j=1}^{s} \binom{m_j+1}{2} - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2) = 3\sum_{j=1}^{s} \binom{m_j+1}{2} - 2i - 1.$$

(c) Let $i = r_{\mathbb{W}} + 1$. Then we have $\mathrm{HF}_{\mathbb{W}}(i-1) = \mathrm{HF}_{\mathbb{W}}(i) = \sum_{j=1}^{s} \binom{m_j+1}{2}$ and $\mathrm{HF}_{\mathbb{V}}(i) = 2i + 1 + \mathrm{HF}_{\mathbb{W}}(i-2)$. So,

$$\mathrm{HF}_{\Omega^1_{\mathbb{W}/K}}(i) = 4\sum_{j=1}^{s} \binom{m_j+1}{2} - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2).$$

(d) Similarly, if $i = r_{\mathbb{W}}$ then $\mathrm{HF}_{\mathbb{W}}(i) = \sum_{j=1}^{m} \binom{m_j+1}{2}$ and $\mathrm{HF}_{\mathbb{V}}(i) = 2i+1+\mathrm{HF}_{\mathbb{W}}(i-2)$, and hence

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = \sum_{j=1}^{s} \binom{m_j+1}{2} + 3\,\mathrm{HF}_{\mathbb{W}}(i-1) - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2).$$

(e) In the case $i < r_{\mathbb{W}}$, we have

$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}(i) = \mathrm{HF}_{\mathbb{W}}(i) + 3\,\mathrm{HF}_{\mathbb{W}}(i-1) - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2).$$

Altogether, when $\sum_{j=1}^{s} m_j + s \geq 2m_s + 2m_{s-1} + 4$, we have proved the formula for the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K}$.

Next we consider the second case $\sum_{j=1}^{s} m_j + s \leq 2m_s + 2m_{s-1} + 3$. In this case, the relation between Hilbert functions of $\mathbb{V}$ and of $\mathbb{Y}$ follows from Proposition 5.1.2. Also, we have $r_{\mathbb{V}} = m_s + m_{s-1} + 1$ and $r_{\mathbb{W}} = m_s + m_{s-1} - 1 < r_{\mathbb{V}}$, and hence $\mathrm{ri}(\Omega^1_{R_{\mathbb{W}}/K}) = r_{\mathbb{V}}$. Therefore a similar argument as in the first case yields the desired formula for the Hilbert function of $\Omega^1_{R_{\mathbb{W}}/K}$. $\square$

It is worth noting that Propositions 5.1.2 and 5.1.3 give us a procedure for computing the Hilbert function of the module of Kähler differential 1-forms of $R_{\mathbb{W}}/K$ from some suitable fat point schemes. Moreover, $\mathrm{HF}_{\Omega^1_{R_{\mathbb{W}}/K}}$ is completely determined by s and the multiplicities $m_1, \dots, m_s$.

Using Proposition 5.1.3 we can extend Proposition 4.4.11 by the next example.

Example 5.1.4. On a non-singular conic $\mathcal{C}$, let $\mathbb{W}$ be a complete intersection of type $(2, n)$. Let $P \in \mathbb{W}$, and let $\mathbb{Y} = \mathbb{W} \setminus \{P\}$. The regularity index of the CB-scheme $\mathbb{Y}$ is $n-1$. Using Proposition 5.1.3 we see that the Hilbert function of $\Omega^1_{R_{\mathbb{Y}}/K}$ is independent of position of the point P.

The following lemma can be used to find out a connection between the Hilbert functions of $\Omega^3_{R_{\mathbb{W}}/K}$ (as well as of $\Omega^2_{R_{\mathbb{W}}/K}$) and of a suitable subscheme of $\mathbb{W}$ if $\mathbb{W}$ is an equimultiple fat point scheme, i.e. if $m_1 = \cdots = m_s = \nu$.

Lemma 5.1.5. *Let $\mathbb{X}$, $\mathbb{W}$ and $\mathbb{Y}$ be as in Proposition 5.1.2, and let*

$$q = \max\{m_s + m_{s-1}, \lfloor (\sum_{j=1}^{s} m_j + 1)/2 \rfloor\}.$$

Let $\{G_1, \dots, G_r\}$ be a minimal homogenous system of generator of $\mathcal{I}_{\mathbb{Y}}$, let L be the linear form such that $P_s, P_{s-1} \in \mathcal{Z}^+(L)$, and write $\mathcal{C} = \mathcal{Z}^+(C)$.

(i) If $\sum_{j=1}^{s} m_j \geq 2m_s + 2m_{s-1}$, and $\sum_{j=1}^{s} m_j$ is odd, then there exist $F_1, F_2 \in (\mathcal{I}_{\mathbb{W}})_q$ such that the set $\{CG_1, \cdots, CG_r, F_1, F_2\}$ is a minimal homogeneous system of generators of $\mathcal{I}_{\mathbb{W}}$.

(ii) If $\sum_{j=1}^{s} m_j \geq 2m_s + 2m_{s-1}$, and $\sum_{j=1}^{s} m_j$ is even, then there exists $F \in (\mathcal{I}_{\mathbb{W}})_q$ such that the set $\{CG_1, \cdots, CG_r, F\}$ is a minimal homogeneous system of generators of $\mathcal{I}_{\mathbb{W}}$.

(iii) If $\sum_{j=1}^{s} m_j \leq 2m_s + 2m_{s-1} - 1$, then there exists $G \in (\mathcal{I}_{\mathbb{W}})_q$ such that the set $\{LG_1, \cdots, LG_r, G\}$ is a minimal homogeneous system of generators of $\mathcal{I}_{\mathbb{W}}$.

Proof. This result follows from [Ca, Proposition 4.3]. □

In particular, if $\mathbb{X} = \{P_1, \dots, P_s\}$ is a set of s distinct K-rational points on a non-singular conic $\mathcal{C} = \mathcal{Z}^+(C)$ then, for every $k \in \mathbb{N}$, a minimal homogeneous system of generators of $\mathcal{I}_{k\mathbb{X}}$ has the following form.

Corollary 5.1.6. *Let $s \geq 4$ and let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points on a non-singular conic $\mathcal{C} = \mathcal{Z}^+(C)$. Let $\mathcal{I}_{\mathbb{X}} = \langle C, G_1, \dots, G_t \rangle_S$ and let $k \in \mathbb{N}$.*

(i) If $s = 2v$ for some $v \in \mathbb{N}$ then

$$\{C^k, C^{k-1}G_1, \dots, C^{k-1}G_t, C^{k-2}F_{2\,1}, C^{k-3}F_{3\,1}, \dots, CF_{(k-1)\,1}, F_{k\,1}\}$$

is a minimal homogeneous system of generators of $\mathcal{I}_{k\mathbb{X}}$, where $\deg(F_{j\,1}) = jv$ for every $j = 2, \dots, k$.

(ii) *If $s = 2v+1$ and $k = 2h$ for some $v, h \in \mathbb{N}$ then*

$$\{C^k, C^{k-1}G_1, \ldots, C^{k-1}G_t, C^{k-2}F_{2\,1}, C^{k-3}F_{3\,1}, C^{k-3}F_{3\,2}, \ldots, CF_{(k-1)\,1}, CF_{(k-1)\,2}, F_{k\,1}\}$$

is a minimal homogeneous system of generators of $\mathcal{I}_{k\mathbb{X}}$, where $\deg(F_{j\,l})$ satisfies $\deg(F_{j\,l}) = \lfloor (j(2v+1)+1)/2 \rfloor$ for every $j = 2, \ldots, k$ and $l = 1, 2$.

(iii) *If $s = 2v+1$ and $k = 2h+1$ for some $v, h \in \mathbb{N}$ then*

$$\{C^k, C^{k-1}G_1, \ldots, C^{k-1}G_t, C^{k-2}F_{2\,1}, C^{k-3}F_{3\,1}, C^{k-3}F_{3\,2}, \ldots, CF_{(k-1)\,1}, F_{k\,1}, F_{k\,2}\}$$

is a minimal homogeneous system of generators of $\mathcal{I}_{k\mathbb{X}}$, where $\deg(F_{j\,l})$ satisfies $\deg(F_{j\,l}) = \lfloor (j(2v+1)+1)/2 \rfloor$ for every $j = 2, \ldots, k$ and $l = 1, 2$.

Proof. Since $s \geq 4$, we have $q_t = \max\{t+t, \lfloor (st+1)/2 \rfloor\} = \lfloor (st+1)/2 \rfloor$ for every t. By applying Lemma 5.1.5 and by induction on k, we get the claimed minimal homogeneous system of generators of the ideal $\mathcal{I}_{k\mathbb{X}}$. □

Now we present a relation between the Hilbert functions of the module of Kähler differential 3-forms $\Omega^3_{R_{\nu\mathbb{X}}/K}$ and of $S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}$.

Proposition 5.1.7. *Let $s \geq 4$ and $\nu \geq 1$, and let $\mathbb{X} = \{P_1, \cdots, P_s\} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points which lie on a non-singular conic $\mathcal{C} = \mathcal{Z}^+(C)$. Then we have $\Omega^3_{R_{\nu\mathbb{X}}/K} \cong (S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}})(-3)$. In particular, for all $i \in \mathbb{N}$, we have*

$$\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = \mathrm{HF}_{S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}}(i-3).$$

Proof. Let $\mathcal{B}_1 = \{C, G_1, \ldots, G_t\}$ be a minimal homogeneous system of generators of $\mathcal{I}_{\mathbb{X}}$, and let $J^{(\nu)} = \langle \partial F/\partial X_i \mid F \in \mathcal{I}_{\nu\mathbb{X}}, 0 \leq i \leq n \rangle_S$. By Corollary 3.2.16, we have $\Omega^3_{R_{\nu\mathbb{X}}/K} \cong (S/J^{(\nu)})(-3)$. Thus it suffices to prove the equality $J^{(\nu)} = \mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}$ for all $\nu \geq 1$. We remark that $J^{(\nu)} \subseteq \mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}$ is always true for all $\nu \geq 1$, and that $\langle \frac{\partial C}{\partial X_i} \mid 0 \leq i \leq 2 \rangle_S = \mathfrak{M}$, since $\mathcal{C}$ is a non-singular conic.

We proceed to prove $\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}} = J^{(\nu)}$ and $C \cdot J^{(\nu)} \subseteq J^{(\nu+1)}$ by induction on ν. If $\nu = 1$, then we have $J^{(1)} = \mathfrak{M} = \mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}$, and

$$C \cdot J^{(1)} = C\mathfrak{M} = \langle \tfrac{\partial C^2}{\partial X_i} \mid 0 \leq i \leq 2 \rangle_S \subseteq J^{(2)},$$

and hence the claim is true for the case $\nu = 1$. Now we distinguish the following two cases.

Case (a): Suppose that s is even.

If $\nu = 2$ then we apply Corollary 5.1.6(i) and find a polynomial F_{21} of degree s such that $\mathcal{B}_2 = \{C^2, CG_1, \dots, CG_t, F_{21}\}$ is a minimal homogeneous system of generators of $\mathcal{I}_{2\mathbb{X}}$. For $1 \le j \le t$ and $0 \le i \le 2$, we see that $\frac{\partial CG_j}{\partial X_i} = G_j \frac{\partial C}{\partial X_i} + C\frac{\partial G_j}{\partial X_i} \in J^{(2)}$. Since $C\frac{\partial G_j}{\partial X_i} \in C\mathfrak{M} \subseteq J^{(2)}$, we get $G_j \frac{\partial C}{\partial X_i} \in J^{(2)}$, and hence $G_j\mathfrak{M} \subseteq J^{(2)}$. This implies

$$\mathcal{I}_{\mathbb{X}}\mathfrak{M} = \langle X_iG \mid 0 \le i \le 2, G \in \mathcal{B}_1\rangle_S \subseteq J^{(2)},$$

and hence $\mathcal{I}_{\mathbb{X}}\mathfrak{M} = J^{(2)}$. Moreover, we have $C^2\mathfrak{M} = C^2\langle \frac{\partial C}{\partial X_i} \mid 0 \le i \le 2\rangle_S \subseteq J^{(3)}$ and $CG_j\frac{\partial C}{\partial X_i} = \frac{1}{2}(\frac{\partial (C^2G_j)}{\partial X_i} + C^2\frac{\partial G}{\partial X_i}) \in J^{(3)}$. From this we deduce

$$C \cdot J^{(2)} = C \cdot \mathcal{I}_{\mathbb{X}}\mathfrak{M} = \langle C^2, CG_1, \dots, CG_t\rangle_S \cdot \langle \tfrac{\partial C}{\partial X_i} \mid 0 \le i \le 2\rangle_S \subseteq J^{(3)}.$$

Now we assume that for $2 \le k \le \nu-1$ we have $\mathcal{I}_{(k-1)\mathbb{X}}\mathfrak{M} = J^{(k)}$, $C\cdot J^{(k)} \subseteq J^{(k+1)}$, and $\mathcal{B}_k = \{C^k, C^{k-1}G_1, \dots, C^{k-1}G_t, C^{k-2}F_{21}, \dots, CF_{k-1\,1}, F_{k\,1}\}$ is a minimal homogeneous system of generators of $\mathcal{I}_{k\mathbb{X}}$, where $\deg(F_{k1}) = \frac{sk}{2}$ and

$$2 \le \max\{\deg(G_j) \mid 1 \le j \le t\} < s = \deg(F_{21}) < \cdots < \tfrac{sk}{2} = \deg(F_{k\,1}).$$

Again, Corollary 5.1.6(i) enables us to find a homogeneous polynomial $F_{\nu\,1} \in (\mathcal{I}_{\nu\mathbb{X}})_{s\nu/2}$ such that $\mathcal{B}_\nu = \{C^\nu, C^{\nu-1}G_1, \dots, C^{\nu-1}G_t, C^{\nu-2}F_{21}, \dots, CF_{\nu-1\,1}, F_{\nu\,1}\}$ is a minimal homogeneous system of generators of $\mathcal{I}_{\nu\mathbb{X}}$. Clearly, we have $C^{\nu-1}\mathfrak{M} \subseteq J^{(\nu)}$. Let $i \in \{0,1,2\}$, $j \in \{1,\dots,t\}$, and $k \in \{2,\dots,\nu-1\}$. We deduce

- $(\nu-1)C^{\nu-2}G_j\frac{\partial C}{\partial X_i} = \frac{\partial (C^{\nu-1}G_j)}{\partial X_i} - C^{\nu-1}\frac{\partial G_j}{\partial X_i} \in J^{(\nu)}$ (as $\frac{\partial G_j}{\partial X_i} \in \mathfrak{M}$).
- $(\nu-k)C^{\nu-k-1}F_{k\,1}\frac{\partial C}{\partial X_i} = \frac{\partial (C^{\nu-k}F_{k\,1})}{\partial X_i} - C^{\nu-k}\frac{\partial F_{k\,1}}{\partial X_i} \in J^{(\nu)}$ (as $\frac{\partial F_{k\,1}}{\partial X_i} \in \mathcal{I}_{(k-1)\mathbb{X}}\mathfrak{M}$).

Thus we get $\mathcal{I}_{(\nu-1)\mathbb{X}}\mathfrak{M} \subseteq J^{(\nu)}$, and so $\mathcal{I}_{(\nu-1)\mathbb{X}}\mathfrak{M} = J^{(\nu)}$. Furthermore, we see that

(a1) $C^\nu\langle \frac{\partial C}{\partial X_i} \mid 0 \le i \le 2\rangle_S = C^\nu\mathfrak{M} \subseteq J^{(\nu+1)}$.

(a2) $\nu C^{\nu-1}G_j\frac{\partial C}{\partial X_i} = \frac{\partial (C^\nu G_j)}{\partial X_i} - C^\nu\frac{\partial G_j}{\partial X_i} \in J^{(\nu+1)}$ (by (a1)).

(a3) $(\nu-1)C^{\nu-2}F_{21}\frac{\partial C}{\partial X_i} = \frac{\partial (C^{\nu-1}F_{21})}{\partial X_i} - C^{\nu-1}\frac{\partial F_{21}}{\partial X_i} \in J^{(\nu+1)}$, since $C^{\nu-1}\frac{\partial F_{21}}{\partial X_i} \in C^{\nu-1}\mathcal{I}_{\mathbb{X}}\langle \frac{\partial C}{\partial X_i} \mid 0 \le i \le 2\rangle_S \subseteq J^{(\nu+1)}$ (by (a1) and (a2)).

(a4) $(\nu-k+1)C^{\nu-k}F_{k\,1}\frac{\partial C}{\partial X_i} = \frac{\partial (C^{\nu-k+1}F_{k\,1})}{\partial X_i} - C^{\nu-k+1}\frac{\partial F_{k\,1}}{\partial X_i} \in J^{(\nu+1)}$, since we have $C^{\nu-k+1}\frac{\partial F_{k\,1}}{\partial X_i} \in C^{\nu-k+1}\mathcal{I}_{(k-1)\mathbb{X}}\langle \frac{\partial C}{\partial X_i} \mid 0 \le i \le 2\rangle_S \subseteq J^{(\nu+1)}$ (by (a1), (a2), (a3) and induction on k).

This shows that $C \cdot J^{(\nu)} \subseteq J^{(\nu+1)}$, as wanted.

Case (b): Suppose that s is odd.

If $\nu = 2$ we argue the same as in the case (a) and get $\mathfrak{M}\mathcal{I}_{\mathbb{X}} = J^{(2)}$ and $C \cdot J^{(2)} \subseteq J^{(3)}$. Suppose that $\nu = 3$. By Corollary 5.1.6(ii), there are polynomials $F_{3\,1}, F_{3\,2} \in \mathcal{I}_{3\mathbb{X}}$ of degree $\lfloor (3s+1)/2 \rfloor$ such that $\mathcal{B}_3 = \{C^3, C^2G_1, \dots, C^2G_t, CF_{2\,1}, F_{3\,1}, F_{3\,2}\}$ is a minimal homogeneous base of $\mathcal{I}_{3\mathbb{X}}$. Then

(b1) For $k = 2, 3$ we have $C^k \cdot \mathfrak{M} \subseteq J^{(k+1)}$.

(b2) For $k = 2, 3$ we have $kC^{k-1}G_j \frac{\partial C}{\partial X_i} = \frac{\partial (C^k G_j)}{\partial X_i} - C^k \frac{\partial G_j}{\partial X_i} \in J^{(k+1)}$ (as $\frac{\partial G_j}{\partial X_i} \in \mathfrak{M}$ and (b1)).

(b3) For $l = 0, 1$, we have $(l+1)C^l F_{2\,1} \frac{\partial C}{\partial X_i} = \frac{\partial (C^{l+1} F_{2\,1})}{\partial X_i} - C^{l+1} \frac{\partial F_{2\,1}}{\partial X_i} \in J^{(l+3)}$, since $C^{l+1} \frac{\partial F_{2\,1}}{\partial X_i} \in C^{l+1}\mathcal{I}_{\mathbb{X}} \langle \frac{\partial C}{\partial X_i} \mid 0 \le i \le 2 \rangle_S \subseteq J^{(l+3)}$ (by (b1) and (b2)).

(b4) For $l = 1, 2$ we have $F_{3\,l} \frac{\partial C}{\partial X_i} = \frac{\partial (CF_{3\,l})}{\partial X_i} - C \frac{\partial F_{3\,l}}{\partial X_i} \in J^{(4)}$, since we have $C \frac{\partial F_{k\,l}}{\partial X_i} \in C\mathcal{I}_{2\mathbb{X}} \langle \frac{\partial C}{\partial X_i} \mid 0 \le i \le 2 \rangle_S \subseteq J^{(4)}$ (by (b1), (b2), and (b3)).

Thus we get $\mathfrak{M}\mathcal{I}_{2\mathbb{X}} \subseteq J^{(3)}$ and $C \cdot J^{(3)} \subseteq J^{(4)}$.

Now we assume that $2 \le k \le \nu - 1$ we have $\mathcal{I}_{(k-1)\mathbb{X}}\mathfrak{M} = J^{(k)}$, $C \cdot J^{(k)} \subseteq J^{(k+1)}$, and the minimal homogeneous system of generators $\mathcal{B}_k$ of $\mathcal{I}_{k\mathbb{X}}$ is given by

$$\mathcal{B}_k = \{C^k, C^{k-1}G_1, \dots, C^{k-1}G_t, C^{k-2}F_{2\,1}, C^{k-3}F_{3\,1}, C^{k-3}F_{3\,1}, \dots, CF_{2l-1\,1}, CF_{2l-1\,2}, F_{2l\,1}\}$$

if $k = 2l$ and

$$\mathcal{B}_k = \{C^k, C^{k-1}G_1, \dots, C^{k-1}G_t, C^{k-2}F_{2\,1}, C^{k-3}F_{3\,1}, C^{k-3}F_{3\,1}, \dots, CF_{2l\,1}, F_{2l+1\,1}, F_{2l+1\,2}\}$$

if $k = 2l + 1$. Note that $2 \le \max\{\deg(G_j) \mid 1 \le j \le t\} < s = \deg(F_{2\,1}) < \cdots < \deg(F_{k\,1})$. If ν is even, then we use the same argument as in the case (a) and get $\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}} \subseteq J^{(\nu)}$ and $C \cdot J^{(\nu)} \subseteq J^{(\nu+1)}$. Otherwise, we can argue similarly as the subcase $\nu = 3$ and get the same result.

Altogether, we have shown that $\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}} = J^{(\nu)}$ and $C \cdot J^{(\nu)} \subseteq J^{(\nu+1)}$ for all $\nu \ge 1$, and this finishes the proof of the proposition. □

Corollary 5.1.8. *In the setting of Proposition 5.1.7, let $\mathcal{B}_1 = \{C, G_1, \dots, G_t\}$ be a minimal homogeneous system of generators of $\mathcal{I}_{\mathbb{X}}$, let $d_j = \deg(G_j)$ for $j = 1, \dots, t$,* *and assume that* $d_1 \le \cdots \le d_t$. *If* $\nu = 1$ *then* $\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = 1$ *for* $i = 3$ *and* $\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = 0$ *for* $i \ne 3$. *If* $\nu \ge 2$ *then*

$$\begin{aligned}\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) &= s\tbinom{\nu}{2} + h_i + \delta_i \quad \text{for } i \ge \lfloor s(\nu-1)/2 \rfloor + 3\\ &= \mathrm{HF}_{\nu\mathbb{X}}(i-1) - 2i + 1 + h_i + \delta_i \quad \text{for } 2 < i < \lfloor s(\nu-1)/2 \rfloor + 3\\ &= 0 \quad \text{for } i \le 2.\end{aligned}$$

Here $h_i = \#\{F \in \mathcal{B}_1 \mid \deg(F) = i - 1 - 2\nu\}$ *and* δ_i *is defined as follows.*

(i) *If* $s = 4$ *then* $\delta_i = \nu - 2$ *if* $i = 2\nu - 4$ *and* $\delta_i = 0$ *otherwise.*

(ii) *If* $s = 5$ *then:*

- *If* ν *is odd then* $\delta_i = 1$ *if* $i = 2(\nu - 3) + 5$, *and* $\delta_i = 3$ *if* $i = 2\nu + k - 1$ *for some* $k = 1, \dots, (\nu - 3)/2$ *and* $\delta_i = 0$ *otherwise.*
- *If* ν *is even then* $\delta_i = 1$ *if* $i = 2(\nu - 3) + 5$, *and* $\delta_i = 3$ *if* $i = 2\nu + k - 1$ *for some* $k = 1, \dots, (\nu-2)/2$ *and* $\delta_i = 2$ *of* $i = (5\nu-4)/2$ *and* $\delta_i = 0$ *otherwise.*

(iii) *If* $s \ge 6$ *then:*

- *If* s *is even then* $\delta_i = 1$ *if* $i = 2(\nu-k)+(k-1)s/2+3$ *for some* $k = 3, \dots, \nu$ *and* $\delta_i = 0$ *otherwise.*
- *If* s *is odd then we have* $\delta_i = 1$ *if* $i = 2(\nu - 2k - 1) + ks + 3$ *for some* $k = 1, \dots, \lfloor (\nu-1)/2 \rfloor$ *and* $\delta_i = 2$ *if* $i = 2(\nu-2k-2) + \lfloor (2sk+s+1)/2 \rfloor + 3$ *for some* $k = 1, \dots, \lfloor (\nu-2)/2 \rfloor$ *and* $\delta_i = 0$ *otherwise.*

Proof. By Proposition 5.1.7, we have $\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = \mathrm{HF}_{S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}}(i-3)$. In the case $\nu = 1$ we have $\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = \mathrm{HF}_{S/\mathfrak{M}}(i-3)$, and hence $\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = 0$ for $i \ne 3$ and $\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) = 1$ for $i = 3$. If $\nu \ge 2$ we have

$$\begin{aligned}\mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) &= \mathrm{HF}_{S/\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}}(i-3)\\ &= \mathrm{HF}_S(i-3) - \mathrm{HF}_{\mathfrak{M}\mathcal{I}_{(\nu-1)\mathbb{X}}}(i-3)\\ &= \mathrm{HF}_S(i-3) - \dim_K(\mathfrak{M}_1(\mathcal{I}_{(\nu-1)\mathbb{X}})_{i-4})\\ &= \mathrm{HF}_S(i-3) - (\dim_K(\mathcal{I}_{(\nu-1)\mathbb{X}})_{i-3} - \#(\mathcal{B}_{\nu-1})_{i-3})\\ &= \mathrm{HF}_{(\nu-1)\mathbb{X}}(i-3) + \#(\mathcal{B}_{\nu-1})_{i-3}\end{aligned}$$

The sequence of degrees of the elements in the sequence

$$\mathcal{A} = (C^{\nu-3}F_{2\,1}, C^{\nu-4}F_{3\,1}, \dots, CF_{(\nu-2)\,1}, F_{(\nu-1)\,1})$$

is $\mathcal{A}^* = (2(\nu-3)+\lfloor(2s+1)/2\rfloor, 2(\nu-4)+\lfloor(3s+1)/2\rfloor, \dots, 2+\lfloor((\nu-2)s+1)/2\rfloor, \lfloor((\nu-1)s+1)/2\rfloor$.

(i) If $s=4$ then every element of the sequence $\mathcal{A}^*$ equals $2\nu-4$ so, $\#(\mathcal{B}_{\nu-1})_{i-3} = \#\{F \in \mathcal{B}_1 \mid \deg(F) = i-1-2\nu\} + \delta_i$ where $\delta_i = \nu-2$ if $i = 2\nu-4$ and $\delta_i = 0$ otherwise.

(ii) If $s=5$ then we see that $2(\nu-2k-1)+\lfloor(2ks+1)/2\rfloor < 2(\nu-2k-2)+\lfloor((2k+1)s+1)/2\rfloor = 2(\nu-2k-3)+\lfloor((2k+2)s+1)/2\rfloor$ for all $k = 1,\dots,\lfloor(\nu-3)/2\rfloor$.

- If ν is odd then $\#(\mathcal{B}_{\nu-1})_{i-3} = \#\{F \in \mathcal{B}_1 \mid \deg(F) = i-1-2\nu\} + \delta_i$ where $\delta_i = 1$ if $i = 2(\nu-3)+5$, and $\delta_i = 3$ if $i = 2\nu+k-1$ for some $k = 1,\dots,(\nu-3)/2$ and $\delta_i = 0$ otherwise.
- If ν is even then $\#(\mathcal{B}_{\nu-1})_{i-3} = \#\{F \in \mathcal{B}_1 \mid \deg(F) = i-1-2\nu\} + \delta_i$ where $\delta_i = 1$ if $i = 2(\nu-3)+5$, and $\delta_i = 3$ if $i = 2\nu+t-1$ for some $t = 1,\dots,(\nu-2)/2$ and $\delta_i = 2$ if $i = (5\nu-4)/2$ and $\delta_i = 0$ otherwise.

(iii) If $s \geq 6$ then the sequence of elements in $\mathcal{A}^*$ is strictly increasing.

- If s is even then Corollary 5.1.6(i) implies that $\#(\mathcal{B}_{\nu-1})_{i-3} = \#\{F \in \mathcal{B}_1 \mid \deg(F) = i-1-2\nu\} + \delta_i$ where $\delta_i = 1$ if $i = 2(\nu-k)+(k-1)s/2+3$ for some $k = 3,\dots,\nu$ and $\delta_i = 0$ otherwise.
- If s is odd then Corollary 5.1.6(ii) and (iii) yields that $\#(\mathcal{B}_{\nu-1})_{i-3} = \#\{F \in \mathcal{B}_1 \mid \deg(F) = i-1-2\nu\}+\delta_i$, where $\delta_i = 1$ if $i = 2(\nu-2k-1)+ks+3$ for some $k = 1,\dots,\lfloor(\nu-1)/2\rfloor$ and $\delta_i = 2$ if $i = 2(\nu-2k-2)+\lfloor(2sk+s+1)/2\rfloor+3$ for some $k = 1,\dots,\lfloor(\nu-2)/2\rfloor$ and $\delta_i = 0$ otherwise.

If $i \geq \lfloor s(\nu-1)/2\rfloor+3$ then by [Ca, Proposition 2.2] we have $\mathrm{HF}_{(\nu-1)\mathbb{X}}(i-3) = s\binom{\nu}{2}$. Otherwise, if $i < \lfloor s(\nu-1)/2\rfloor+3$ then $\mathrm{HF}_{(\nu-1)\mathbb{X}}(i-3) = \mathrm{HF}_{\nu\mathbb{X}}(i-1)-2i+1$. Hence the claims follow. □

Corollary 5.1.9. *Using the notation given in Corollary 5.1.8, we have.*

(i)

$$\begin{aligned}\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}(i) &= 0 \quad \textit{for } i \geq s\\ &= 3\,\mathrm{HF}_{\mathbb{X}}(2)-9 \quad \textit{for } i=3\\ &= 3\,\mathrm{HF}_{\mathbb{X}}(i-1)-\mathrm{HF}_{\mathbb{X}}(i-2)-2i-1 \quad \textit{for } i<s \textit{ and } i\neq 3.\end{aligned}$$

(ii) For every $\nu \geq 2$, let $t = \lfloor s(\nu+1)/2 \rfloor$ and $u = \lfloor s(\nu-1)/2 \rfloor + 3$. Then we have

$$\begin{aligned}\mathrm{HF}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(i) &= s(3\nu+2)(\nu-1)/2 + h_i + \delta_i \quad \textit{for } i \geq t \\ &= 3\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{\nu\mathbb{X}}(i-2) + s\tbinom{\nu}{2} + h_i + \delta_i - 2i - 1 \ \textit{for } u \leq i < t \\ &= 4\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) + \mathrm{HF}_{\nu\mathbb{X}}(i-2) - 4i + h_i + \delta_i \quad \textit{for } i < u.\end{aligned}$$

Proof. Clearly, $\mathrm{HF}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(i) = 0$ for $i \leq 1$. By Proposition 3.2.7 we have a short exact sequence of graded $R_{\nu\mathbb{X}}$-modules

$$0 \longrightarrow \Omega^3_{R_{\nu\mathbb{X}}/K} \longrightarrow \Omega^2_{R_{\nu\mathbb{X}}/K} \longrightarrow \Omega^1_{R_{\nu\mathbb{X}}/K} \longrightarrow \mathfrak{m}_{\nu\mathbb{X}} \longrightarrow 0.$$

For $i \geq 2$ we get

$$\begin{aligned}\mathrm{HF}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(i) &= \mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(i) + \mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) - \mathrm{HF}_{\nu\mathbb{X}}(i) \\ &\overset{(*)}{=} (\mathrm{HF}_{\nu\mathbb{X}}(i) + 3\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{(\nu+1)\mathbb{X}}(i)) + \mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i) - \mathrm{HF}_{\nu\mathbb{X}}(i) \\ &= 3\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{(\nu+1)\mathbb{X}}(i) + \mathrm{HF}_{\Omega^3_{R_{\nu\mathbb{X}}/K}}(i)\end{aligned}$$

where $(*)$ follows from Corollary 4.2.3(i).

We consider the case $\nu = 1$. We see that $\mathrm{HF}_{2\mathbb{X}}(3) = 10$, so $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}(3) = 3\,\mathrm{HF}_{\mathbb{X}}(2) - 10 + 1 = 3\,\mathrm{HF}_{\mathbb{X}}(2) - 9$. For $i < s$ and $i \neq 3$ we have $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}(i) = 3\,\mathrm{HF}_{\mathbb{X}}(i-1) - (2i + 1 + \mathrm{HF}_{\mathbb{X}}(i-2))$. For $i \geq s$ we get $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}(i) = 3\,\mathrm{HF}_{\mathbb{X}}(i-1) - \mathrm{HF}_{2\mathbb{X}}(i) = 3s - 3s = 0$. Thus claim (i) follows.

Now we suppose $\nu \geq 2$. By Corollary 5.1.8 we have $\mathrm{HF}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(i) = 3s\binom{\nu+1}{2} - s\binom{\nu+2}{2} + s\binom{\nu}{2} + h_i + \delta_i = s(3\nu+2)(\nu-1)/2 + h_i + \delta_i$ for $i \geq \lfloor s(\nu+1)/2 \rfloor$. For $\lfloor s(\nu-1)/2 \rfloor + 3 \leq i < \lfloor s(\nu+1)/2 \rfloor$ we obtain $\mathrm{HF}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(i) = 3\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) - (2i + 1 + \mathrm{HF}_{\nu\mathbb{X}}(i-2)) + s\binom{\nu}{2} + h_i + \delta_i$. For $i < \lfloor s(\nu-1)/2 \rfloor + 3$, by Corollary 5.1.8 we get $\mathrm{HF}_{\Omega^2_{R_{\nu\mathbb{X}}/K}}(i) = 3\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) - (2i+1+\mathrm{HF}_{\nu\mathbb{X}}(i-2)) + (\mathrm{HF}_{\nu\mathbb{X}}(i-1) - 2i + 1 + h_i + \delta_i) = 4\,\mathrm{HF}_{\nu\mathbb{X}}(i-1) + \mathrm{HF}_{\nu\mathbb{X}}(i-2) - 4i + h_i + \delta_i$. Therefore claim (ii) is completely proved. □

5.2 Segre's Bound for Fat Point Schemes in $\mathbb{P}^4$

Given a fat point scheme $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$, we define

$$T_{\mathbb{W},j} := \max\left\{ \left\lfloor \frac{m_{i_1} + \cdots + m_{i_q} + j - 2}{j} \right\rfloor \mid P_{i_1}, \ldots, P_{i_q} \text{ lie on a } j\text{-plane} \right\}$$

for $j = 1, \ldots, n$ and set $T_{\mathbb{W}} := \max\{T_{\mathbb{W},j} \mid j = 1, \ldots, n\}$. In 1996, the following conjecture, called **Segre's bound**, was formulated by N.V. Trung (cf. [Th1]) and independently by G. Fatabbi and A. Lorenzini (cf. [FL]).

Conjecture 5.2.1. *The regularity index of a fat point scheme* $\mathbb{W}$ *satisfies* $r_{\mathbb{W}} \le T_{\mathbb{W}}$.

In this section, we show that $r_{\mathbb{W}} \le T_{\mathbb{W}}$ if $\mathbb{W}$ is a set of s distinct reduced K-rational points in $\mathbb{P}^4$. We also show that Segre's bound is attained by an equimultiple fat point scheme in $\mathbb{P}^4$ whose support $\mathbb{X} := \mathrm{Supp}(\mathbb{W})$ satisfies $T_{\mathbb{X}} = T_{\mathbb{X},1}$. Moreover, we prove that the regularity index of $\Omega^1_{R_{\nu\mathbb{X}}/K}$ is exactly $T_{(\nu+1)\mathbb{X}}$ in this case.

These proofs require a number of preparations. The following techniques were developed in [CTV, Lemmata 1 and 3].

Lemma 5.2.2. *Let* $\mathbb{X} = \{P_1, \dots, P_s, Q\}$ *be a set of* $s+1$ *distinct* K*-rational points in* $\mathbb{P}^n$*, let* $\wp_i$ *(resp.* $\mathfrak{q}$*) be the associated prime ideal of* P_i *(resp.* Q*), and let* $m_1, \dots, m_s$*,* a *be positive integers. Define two ideals* $J := \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ *and* $I = J \cap \mathfrak{q}^a$.

(i) We have $\mathrm{ri}(S/I) = \max\{a-1, \mathrm{ri}(S/J), \mathrm{ri}(S/(J+\mathfrak{q}^a))\}$.

(ii) In the case $Q = (1 : 0 : \cdots : 0)$*, i.e. in the case* $\mathfrak{q} = \langle X_1, \dots, X_n\rangle$*, we have* $\mathrm{ri}(S/(J+\mathfrak{q}^a)) \le T$ *if and only if* $X_0^{T-i}M \in J + \mathfrak{q}^{i+1}$ *for every monomial* $M \in \mathfrak{q}^i$ *and for* $i = 0, \dots, a-1$.

In many cases, the regularity index $\mathrm{ri}(S/J)$ can be estimated by induction on the number s of points in the support of the scheme. In order to use the lemma, one needs to find a good bound for $\mathrm{ri}(S/(J+\mathfrak{q}^a))$. This is equivalent to finding the minimal number T such that for any monomial $M \in \mathfrak{q}^i$ with $0 \le i \le a-1$ there exist $T-i$ linearly independent linear forms $L_1, \dots, L_{T-i}$ which do not vanish at Q and satisfy $L_1 \cdots L_{T-i} M \in J$. Since we may assume $\mathfrak{q} = \langle X_1, \dots, X_n\rangle$, we can write $L_j = X_0 + G_j$ with a linear form $G_j \in \mathfrak{q}$ for $j = 1, \dots, T-i$. Then the relation

$$L_1 \cdots L_{T-i} M = (X_0 + G_1) \cdots (X_0 + G_{T-i}) M \in J$$

implies $X_0^{T-i} M \in J + \mathfrak{q}^{i+1}$. This means that we have $\mathrm{ri}(S/(J+\mathfrak{q}^a)) \le T$.

The next two lemmata are useful tools to construct suitable linear forms L_i. They follow from [Th2, Lemmata 2.3 and 2.4].

Lemma 5.2.3. *Let* $\mathbb{X} = \{P_1, \dots, P_s\}$ *be a set of* s *distinct* K*-rational points in a* 2*-plane* $K_0 \cong \mathbb{P}^2$ *in* $\mathbb{P}^n$*, let* $Q \in K_0 \setminus \mathbb{X}$*, and let* $m_1, \dots, m_s$ *be positive integers. We define*

$$t_j := \max\Big\{ \Big\lfloor \frac{\sum_{k=1}^{q} m_{i_k} + j - 1}{j} \Big\rfloor \;\Big|\; P_{i_1}, \dots, P_{i_q}, Q \text{ lie on a } j\text{-plane} \Big\}$$

for $j = 1, 2$*, and let* $t := \max\{t_1, t_2\}$.

Then there exist t *lines contained in* K_0*, say* $L_1, \dots, L_t$*, which do not pass through* Q *and have the property that for every* $j \in \{1, \dots, s\}$ *there exist* m_j *lines* $L_{i_1}, \dots, L_{i_{m_j}}$ *with* $P_j \in L_{i_1} \cap \dots \cap L_{i_{m_j}}$.

Lemma 5.2.4. *Let* $\mathbb{X} = \{P_1, \dots, P_s\}$ *be a set of* s *distinct* K*-rational points in a* 3*-plane* $H \cong \mathbb{P}^3$ *in* $\mathbb{P}^n$*, let* $Q \in H \setminus \mathbb{X}$*, and let* $m_1, \dots, m_s$ *be positive integers. We define*

$$t_j := \max\Big\{\Big\lfloor \frac{\sum_{k=1}^{q} m_{i_k} + j - 1}{j} \Big\rfloor \mid P_{i_1}, \dots, P_{i_q}, Q \text{ lie on a } j\text{-plane}\Big\}$$

for $j = 1, 2, 3$ *and let* $t := \max\{t_1, t_2, t_3\}$.

Then we can find t 2*-planes contained in* H*, say* $K_1, \dots, K_t$*, which do not pass through* Q *and have the property that for every* $j \in \{1, \dots, s\}$ *there exist* m_j 2*-planes* $K_{i_1}, \dots, K_{i_{m_j}}$ *with* $P_j \in K_{i_1} \cap \dots \cap K_{i_{m_j}}$.

By using the above lemmata, P. V. Thien proved the following result (cf. [Th2, Theorem 1.1]).

Lemma 5.2.5. *Let* $\mathbb{X} = \{P_1, \dots, P_s\}$ *be a set of* s *distinct* K*-rational points in* $\mathbb{P}^4$*. Then we have* $r_{2\mathbb{X}} \le T_{2\mathbb{X}}$.

The next proposition tells us that the conjecture holds true for a fat point scheme in $\mathbb{P}^4$ whose support is contained in a hyperplane.

Proposition 5.2.6. *Let* $\mathbb{X} = \{P_1, \dots, P_s\}$ *be a set of* s *distinct* K*-rational points in* $\mathbb{P}^4$*, and let* $\mathbb{W} = m_1P_1 + \dots + m_sP_s$ *be a fat point scheme. If* $\mathbb{X}$ *is contained in a hyperplane* $H \cong \mathbb{P}^3$*, then we have* $r_{\mathbb{W}} \le T_{\mathbb{W}}$.

Proof. For $j = 1, \dots, s$, let P'_j denote the point of $H \cong \mathbb{P}^3$ corresponding to P_j, and let $\mathbb{W}' = m_1P'_1 + \dots + m_sP'_s$ be the fat point scheme of $\mathbb{P}^3$ corresponding to $\mathbb{W}$. By [Th1, Theorem 1.1], we know that $r_{\mathbb{W}'} \le T_{\mathbb{W}'}$ holds. Now it follows from Lemma 4.4 in [BFL] that $r_{\mathbb{W}} \le T_{\mathbb{W}'}$. In view of the inequality $T_{\mathbb{W}'} \le T_{\mathbb{W}}$ which follows immediately from the definition of $T_{\mathbb{W}}$, we get the desired result. □

Next we start to work towards the first main result of this section, namely the proof of Segre's bound in the case $m_1 = \dots = m_s = 1$, i.e. for a reduced 0-dimensional scheme in $\mathbb{P}^4$. We use the method of proof in [Th2] and estimate the regularity index $\mathrm{ri}(S/(J+\mathfrak{q}))$ as follows.

Proposition 5.2.7. *Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct K-rational points in $\mathbb{P}^4$ which is not contained in a hyperplane. Then there exists a point $P_{i_0} \in \mathbb{X}$ such that the ideal $J := \bigcap_{j \neq i_0} \wp_j$ satisfies the inequality*

$$\mathrm{ri}(S/(J + \wp_{i_0})) \leq T_{\mathbb{X}}.$$

Proof. By Lemma 5.2.2(ii), we only need to show that there exist a point $P_{i_0} \in \mathbb{X}$ and a number $t \in \{1, \dots, T_{\mathbb{X}}\}$ such that there are linear forms $L_1, \dots, L_t \in S$ with $L_1 \cdots L_t \in J \setminus \wp_{i_0}$.

To begin with, let $d \geq 1$ be the least integer such that there are hyperplanes $H_1, \dots, H_d$ in $\mathbb{P}^4$ for which the following conditions are satisfied:

(i) $\mathbb{X} \subseteq H_1 \cup \cdots \cup H_d$

(ii) For $i = 1, \dots, d-1$, we have

$$\big|H_i \cap (\mathbb{X} \setminus \bigcup_{j=1}^{i-1} H_j)\big| = \max\big\{|H \cap (\mathbb{X} \setminus \bigcup_{j=1}^{i-1} H_j)| \mid H \text{ is a hyperplane}\big\}.$$

Notice that we have $d \geq 2$ here, because $\mathbb{X}$ is not contained in a hyperplane. For convenience we put $\mathbb{X}_i := H_i \cap (\mathbb{X} \setminus \bigcup_{j=1}^{i-1} H_j)$ for $i = 1, \dots, d$. By the choice of d we have $\mathbb{X}_d \neq \emptyset$. We write $\mathbb{X}_d = \{P_1, \dots, P_r\}$ for some $r \geq 1$. The desired point P_{i_0} will be chosen in $\mathbb{X}_d$. Depending on the geometry of $\mathbb{X}_d$, we distinguish three cases.

Case (a) Suppose that $\mathbb{X}_d$ is not contained in a 2-plane.

Notice that this implies $r = |\mathbb{X}_d| \geq 4$. Let $u := |\mathbb{X}_{d-1}|$. Since $d \geq 2$ and $r \geq 4$, we have $(dr+2)/4 \geq d + \frac{r-1}{2} - 1$. This implies

$$T_{\mathbb{X}} \geq T_{\mathbb{X},4} \geq \lfloor \frac{(d-1)u + r + 2}{4} \rfloor \geq \lfloor \frac{dr+2}{4} \rfloor \geq d + \lfloor \frac{r-1}{2} \rfloor - 1.$$

Let ℓ be a line contained in H_d such that

$$|\ell \cap \mathbb{X}_d| = \max\{|\ell' \cap \mathbb{X}_d| \mid \ell' \text{ is a line contained in } H_d\}.$$

Note that, since $\mathbb{X}_d \nsubseteq \ell$, we can choose a point $P_{i_0} \in \mathbb{X}_d \setminus \ell$. Now we set

$$t_j := \max\big\{\lfloor \tfrac{1}{j}\left(|H \cap (\mathbb{X}_d \setminus \{P_{i_0}\})| + j - 1\right)\rfloor \mid H \text{ is a } j\text{-plane containing } P_{i_0}\big\}$$

for $j = 1, 2, 3$ and $t := \max\{t_1, t_2, t_3\}$. Let ℓ_1 be a line containing P_{i_0} such that $|\ell_1 \cap \mathbb{X}_d \setminus \{P_{i_0}\}| = t_1$. Since $\mathbb{X}_d$ does not lie on a 2-plane and by the choice of ℓ, we get

$$r = |\mathbb{X}_d| \geq |\ell \cap \mathbb{X}_d| + |\ell_1 \cap \mathbb{X}_d| \geq 2|\ell_1 \cap \mathbb{X}_d|.$$

This yields $t_1 \le \lfloor \frac{r-2}{2} \rfloor \le \lfloor \frac{r-1}{2} \rfloor$.

Next we let $E \subseteq H_d$ be a 2-plane containing P_{i_0} such that $\left\lfloor \frac{|E\cap(\mathbb{X}_d \setminus \{P_{i_0}\})|+1}{2} \right\rfloor = t_2$. Then we have $|E \cap \mathbb{X}_d| \le r-1$, and therefore $t_2 \le \lfloor \frac{r-1}{2} \rfloor$. Since $r \ge 4$, we have $t_3 \le \left\lfloor \frac{(r-1)+2}{3} \right\rfloor \le \lfloor \frac{r-1}{2} \rfloor$. Thus we obtain $t = \max\{t_1, t_2, t_3\} \le \lfloor \frac{r-1}{2} \rfloor$. By Lemma 5.2.4, we can find $\lfloor \frac{r-1}{2} \rfloor$ 2-planes $K_1, \dots, K_{\lfloor \frac{r-1}{2} \rfloor}$ contained in H_d avoiding P_{i_0} such that, for $i \ne i_0$, there is an index $j_i \in \{1, \dots, \lfloor \frac{r-1}{2} \rfloor\}$ with $P_i \in K_{j_i}$. If we choose hyperplanes $\mathcal{Z}^+(L_j)$ containing K_j and avoiding P_{i_0} for $j = 1, \dots, \lfloor \frac{r-1}{2} \rfloor$ and let $L_{\lfloor \frac{r-1}{2} \rfloor + j}$ be the linear form defining the hyperplane H_j for $j = 1, \dots, d-1$, we have $L_1 \cdots L_{d+\lfloor \frac{r-1}{2} \rfloor - 1} \in J \setminus \wp_{i_0}$, as we wanted.

Case (b) Suppose that $\mathbb{X}_d$ is contained in a 2-plane K_0, but not in a line.

W.l.o.g. we may assume that $\mathbb{X}_{d-1} = \{P_{r+1}, \dots, P_{r+u}\}$ for some $u \in \mathbb{N}$. Then we have $r \ge 3$ and $u \ge |K_0 \cap (\mathbb{X}_{d-1} \cup \mathbb{X}_d)| + 1 \ge |K_0 \cap \mathbb{X}_d| + 1 = r+1 \ge 4$ by the choice of H_{d-1}. Depending on the size of u, we distinguish the two subcases.

Subcase (b.1) Suppose that $u \ge r+2$.

In this case we see that $\lfloor \frac{d(r+2)}{4} \rfloor \ge d + \lfloor \frac{r}{2} \rfloor - 1$ iff $d(r+2) \ge 4d + 4\lfloor \frac{r}{2} \rfloor - 4$ iff $dr - 2d - 4\lfloor \frac{r}{2} \rfloor + 4 \ge 0$ iff $(2r - 4\lfloor \frac{r}{2} \rfloor) + (r-2)(d-2) \ge 0$. The last inequality holds for all $r \ge 3$ and $d \ge 2$. This implies

$$T_{\mathbb{X}} \ge T_{\mathbb{X},4} \ge \left\lfloor \frac{(d-1)u+r+2}{4} \right\rfloor \ge \left\lfloor \frac{d(r+2)}{4} \right\rfloor \ge d + \left\lfloor \frac{r}{2} \right\rfloor - 1.$$

Let ℓ be a line contained in K_0 with $|\ell \cap \mathbb{X}_d| = \max\{|\ell' \cap \mathbb{X}_d| \mid \ell' \text{ is a line contained in } K_0\}$. There is a point $P_{i_0} \in \mathbb{X}_d \setminus \ell$ because $\mathbb{X}_d$ is not contained in a line. For $j = 1, 2$, we set

$$t_j := \max\left\{ \left\lfloor \tfrac{1}{j}\left(|H \cap (\mathbb{X}_d \setminus \{P_{i_0}\})| + j - 1\right) \right\rfloor \mid H \text{ is a } j\text{-plane containing } P_{i_0} \right\}$$

and we let $t = \max\{t_1, t_2\}$. Furthermore, let $\ell_1 \subset K_0$ be a line containing P_{i_0} such that $|\ell_1 \cap \mathbb{X}_d \setminus \{P_{i_0}\}| = t_1$. By the choice of ℓ, we get

$$r + 1 = |\mathbb{X}_d| + 1 \ge |\ell \cap \mathbb{X}_d| + |\ell_1 \cap \mathbb{X}_d| \ge 2|\ell_1 \cap \mathbb{X}_d|.$$

This implies $t_1 \le \lfloor \frac{r-1}{2} \rfloor \le \lfloor \frac{r}{2} \rfloor$. Also, we observe that $t_2 \le \lfloor \frac{1}{2}(|\mathbb{X}_d \setminus \{P_{i_0}\}| + 1) \rfloor = \lfloor \frac{r}{2} \rfloor$. Hence we conclude that $t = \max\{t_1, t_2\} \le \lfloor \frac{r}{2} \rfloor$. By Lemma 5.2.3, we can find $\lfloor \frac{r}{2} \rfloor$ lines $\ell_1, \dots, \ell_{\lfloor \frac{r}{2} \rfloor}$ contained in K_0 and avoiding P_{i_0} such that for $i \ne i_0$ there exists an index $j_i \in \{1, \dots, \lfloor \frac{r}{2} \rfloor\}$ with $P_i \in \ell_{j_i}$. We choose hyperplanes $\mathcal{Z}^+(L_j)$ containing ℓ_j and avoiding P_{i_0} for $j = 1, \dots, \lfloor \frac{r}{2} \rfloor$, and we let $L_{\lfloor \frac{r}{2} \rfloor + j}$ be a linear form defining the hyperplane H_j for $j = 1, \dots, d-1$. Then we find $L_1 \cdots L_{d+\lfloor \frac{r}{2} \rfloor -1} \in J \setminus \wp_{i_0}$, as we wanted.

Subcase (b.2) Suppose that $u = r+1$.

In this case we have $r+1 = u \geq |K_0 \cap (\mathbb{X}_{d-1} \cup \mathbb{X}_d)| + 1 \geq |K_0 \cap \mathbb{X}_{d-1}| + r + 1$, and therefore $K_0 \cap \mathbb{X}_{d-1} = \emptyset$. Observe that we have $d(r+1)/4 \geq d + (r-3)/2$ for $d \geq 2$ and $r \geq 3$, and Thus

$$T_{\mathbb{X}} \geq T_{\mathbb{X},4} \geq \left\lfloor \tfrac{d(r+1)+1}{4} \right\rfloor \geq \left\lfloor d + \tfrac{2r-5}{4} \right\rfloor \geq d + \left\lfloor \tfrac{2r-5}{4} \right\rfloor.$$

If $r = 2k+1$ for some $k \geq 1$, then $\lfloor \frac{r}{2} \rfloor = k$ and $T_{\mathbb{X}} \geq d+k-1$. Thus if we choose P_{i_0} and $L_1, \dots, L_{d+k-1}$ as in Subcase (b.1), we obtain the desired relation $L_1 \cdots L_{d+k-1} \in J \setminus \wp_{i_0}$.

It remains to consider the case $r = 2k$ for some $k \geq 2$. In this case, we have $\lfloor \frac{r}{2} \rfloor = k$ and $T_{\mathbb{X}} \geq d + k - 2 \geq d$. Let $\mathbb{Y} = \{P_1, \dots, P_{r-1}\} \subseteq \mathbb{X}_d$, and let ℓ be a line contained in K_0 such that $|\ell \cap \mathbb{Y}| = \max\{|\ell' \cap \mathbb{Y}| \mid \ell' \text{ is a line contained in } K_0\}$.

Now there are two possibilities. If $\mathbb{Y} \subseteq \ell$, then we have $P_r \notin \ell$ since $\mathbb{X}_d$ does not lie on a line. Thus we may define $P_{i_0} := P_r$. We choose a linear form L such that $\mathcal{Z}^+(L)$ contains ℓ and avoids P_{i_0}, and we let $H_j = \mathcal{Z}^+(L_j)$ for $j = 1, \dots, d-1$. Then we have $L_1 \cdots L_{d-1} L \in J \setminus \wp_{i_0}$, as we wanted.

The second possibility is $\mathbb{Y} \nsubseteq \ell$. In this case we let $P_{i_0} \in \mathbb{Y} \setminus \ell$ and define

$$t_j := \max\left\{ \left\lfloor \tfrac{1}{j} \left(|H \cap (\mathbb{Y} \setminus \{P_{i_0}\})| + j - 1 \right) \right\rfloor \mid H \text{ is a } j\text{-plane containing } P_{i_0} \right\}$$

for $j = 1, 2$, and then $t := \max\{t_1, t_2\}$. By the choice of ℓ, we have $t_1 \leq \left\lfloor \frac{|\mathbb{Y}|-1}{2} \right\rfloor = \left\lfloor \frac{r-2}{2} \right\rfloor = k-1$, and $t_2 \leq \left\lfloor \frac{|\mathbb{Y} \setminus \{P_{i_0}\}|+1}{2} \right\rfloor = \left\lfloor \frac{r-1}{2} \right\rfloor = k-1$. Consequently, we have $t = \max\{t_1, t_2\} \leq k-1$. By Lemma 5.2.3, we can find $k-1$ lines $\ell_1, \dots, \ell_{k-1}$ contained in K_0 and avoiding P_{i_0} such that for every $i \neq i_0$ there is an index $j_i \in \{1, \dots, k-1\}$ with $P_i \in \ell_{j_i}$. Since $P_{i_0} \notin H_{d-1}$, there exist four points of $\mathbb{X}_{d-1}$, say P_{2r-2}, P_{2r-1}, P_{2r}, and P_{2r+1}, such that P_{i_0} and these four points do not lie on a hyperplane.

W.l.o.g. we may assume that $P_{i_0} = (1:0:0:0:0)$, $P_{2r-2} = (1:1:0:0:0)$, $P_{2r-1} = (1:0:1:0:0)$, $P_{2r} = (1:0:0:1:0)$, and $P_{2r+1} = (1:0:0:0:1)$. Since $P_r \neq P_{i_0}$, $P_r \notin H_{d-1} = \mathcal{Z}^+(X_0 - X_1 - X_2 - X_3 - X_4)$ and $\operatorname{span}(P_{i_0}, P_{2r-2}, P_{2r-1}, P_{2r}, P_{2r+1}) = \operatorname{span}(P_{i_0}, P_r, P_{2r-2}, P_{2r-1}, P_{2r}, P_{2r+1}) = \mathbb{P}^4$, there are $i_1, i_2, i_3 \in \{2r-2, \dots, 2r+1\}$ such that $\operatorname{span}(P_{i_0}, P_{i_1}, P_{i_2}, P_{i_3}, P_r) = \mathbb{P}^4$. W.l.o.g. we may therefore assume that $\operatorname{span}(P_{i_0}, P_{2r-1}, P_{2r}, P_{2r+1}, P_r) = \mathbb{P}^4$. Let $K_1 \subseteq \mathbb{P}^4$ be the 2-plane passing through the three points $P_{2r-1}, P_{2r}, P_{2r+1}$, and let ℓ' be the line passing through P_{i_0} and P_r. Then there is no hyperplane in $\mathbb{P}^4$ containing K_1 and ℓ'. Let L_k be a linear form defining the hyperplane of $\mathbb{P}^4$ containing both K_1 and P_r.

If $P_{i_0} \in \mathcal{Z}^+(L_k)$ then we have $K_1 \subseteq \mathcal{Z}^+(L_k)$ and $\ell' \subseteq \mathcal{Z}^+(L_k)$ which is impossible. Hence we have $P_{i_0} \notin \mathcal{Z}^+(L_k)$. For $j = 1, \dots, k-1$, let L_j be a linear form defining the hyperplane containing ℓ_j, P_{r+j} and P_{2r-j-1}. Then we also see that $P_{i_0} \notin \mathcal{Z}^+(L_j)$ for $j = 1, \dots, k-1$, since otherwise $\ell_j \cup P_{i_0} \subseteq K_0 \subseteq \mathcal{Z}^+(L_j)$ implies $|\mathcal{Z}^+(L_j) \cap (\mathbb{X}_{d-1} \cup \mathbb{X}_d)| \geq r+2 > u$, in contradiction to the choice of H_{d-1}. Thus we have

$$L_1 \cdots L_k \in \wp_1 \cap \cdots \cap \widehat{\wp}_{i_0} \cap \cdots \cap \wp_r \cap \cdots \cap \wp_{2r+1}.$$

Letting L_{k+j} be a linear form defining H_j for $j = 1, \dots, d-2$, we therefore obtain $L_1 \cdots L_{d+k-2} \in J \setminus \wp_{i_0}$, as we wanted.

Case (c) Suppose that $\mathbb{X}_d$ is contained in a line ℓ. In this case we distinguish four subcases depending on the size of $T_{\mathbb{X}}$.

Subcase (c.1) $T_{\mathbb{X}} \geq r+d-2$

Let $P_{i_0} = P_r$. We choose a linear from L_j defining a hyperplane passing through P_j and avoiding P_{i_0} for $j = 1, \dots, r-1$, and choose a linear form L_{r-1+k} defining H_k for $k = 1, \dots, d-1$. Then $L_1 \cdots L_{r+d-2} \in J \setminus \wp_{i_0}$.

Hence we assume from here on that $T_{\mathbb{X}} \leq r+d-3$. Before we move on to the other subcases, let us write down some general inequalities which will come in handy later. For $j = 1, \dots, d-1$, let $P'_j = \ell \cap H_j$. Clearly, the points $P_1, \dots, P_r, P'_1, \dots, P'_{d-1}$ all lie on ℓ. Since we have $T_{\mathbb{X}} \geq T_{\mathbb{X},1}$, there is an index $j \in \{1, \dots, d-1\}$ such that $P'_j \notin \mathbb{X}_j$. Let $e \in \{1, \dots, d-1\}$ be the largest integer such that $P'_e \notin \mathbb{X}_e$, and let $v = |\mathbb{X}_e|$. Then we have $|\mathbb{X}_j| \geq |\ell \cap (\mathbb{X}_j \cup \cdots \cup \mathbb{X}_d)| + 2$ for every $j = 1, \dots, d-1$, and $P'_j \in \mathbb{X}_j$ and $|\mathbb{X}_j| \geq r+2+d-j \geq 4$ for $j = e+1, \dots, d-1$. In particular, this shows

$$v \geq |\ell \cap (\mathbb{X}_e \cup \cdots \cup \mathbb{X}_d)| + 2 \geq |\ell \cap (\mathbb{X}_{e+1} \cup \cdots \cup \mathbb{X}_d)| + 2 \geq r+2+(d-e-1). \quad (*)$$

Consequently, we have two further inequalities

$$\begin{aligned} T_{\mathbb{X}} \geq T_{\mathbb{X},4} &= \left\lfloor \tfrac{1}{4}\Big(\sum_{i=1}^{e} |\mathbb{X}_i| + \sum_{i=e+1}^{d-1} |\mathbb{X}_i| + |\mathbb{X}_d| + 2\Big) \right\rfloor \\ &\geq \left\lfloor \tfrac{1}{4}\big(ev + (r+d+1-e) + \cdots + (r+3) + r + 2\big) \right\rfloor \\ &\geq \left\lfloor \tfrac{1}{4}\Big(ev + \tfrac{(2r+d-e+4)(d-e-1)}{2} + r + 2\Big) \right\rfloor \end{aligned} \quad (**)$$

and

$$T_{\mathbb{X},1} \geq |\ell \cap (\mathbb{X}_{e+1} \cup \cdots \cup \mathbb{X}_d)| - 1 = r+d-e-2. \quad (***)$$

Notice that, for $x, y \in \mathbb{N}$ and $z \in \mathbb{N}_+$, the inequality $x \geq \lfloor \frac{y}{z} \rfloor$ implies $x+1 \geq \frac{y+1}{z}$. From the first inequality in $(**)$ we deduce $r+d-3 \geq \lfloor \frac{1}{4}(ev + 4(d-e-1) + r + 2) \rfloor$. This yields $r+d-2 \geq \frac{1}{4}(e\,v+4(d-e-1)+r+3)$, and thus $e(v-4) \leq 3r-7$. Therefore we have $r \geq 3$ and $v \leq 3r-3$.

Now we turn our attention to the next subcase.

Subcase (c.2) $T_{\mathbb{X}} = r+d-3$

We put $t_j := \max\big\{\lfloor\frac{|H\cap\mathbb{X}_e|+j-1}{j}\rfloor \mid H$ is a j-plane containing $P'_e\big\}$ for $j=1,2,3$, and let $t=\max\{t_1,t_2,t_3\}$. We choose a line ℓ_1 containing P'_e such that $|\ell_1\cap\mathbb{X}_e| = t_1$, and we let K_1 be the 2-plane containing ℓ and ℓ_1. By the definition of H_e, we have

$$v = |\mathbb{X}_e| \geq |K_1 \cap (\mathbb{X}_e \cup \cdots \cup \mathbb{X}_d)| + 1 \geq |\ell_1 \cap \mathbb{X}_e| + |\ell \cap (\mathbb{X}_{e+1} \cup \cdots \cup \mathbb{X}_d)| + 1.$$

If $|\ell_1 \cap \mathbb{X}_e| \geq r$, then we have $v \geq 2r+d-e$. Hence it follows from $(**)$ that $3r-7 \geq e(v-4) + \frac{(2r+d-e-4)(d-e-1)}{2} \geq e(2r+(d-e-1)-3) + \frac{(2r+d-e-4)(d-e-1)}{2}$. Since $d-e-1 \geq 0$ and $2r+d-e-4 \geq 1$ and $r \geq 3$, we have $e \leq \frac{3r-7}{2r-3} \leq \frac{3}{2}$. This shows $e=1$. Therefore we find $3r-7 \geq 2r+d-5+\frac{(2r+d-5)(d-2)}{2} = 2r-3+\frac{(2r+d-3)(d-2)}{2}$. Hence we have $d=2$ and $T_{\mathbb{X},2} \geq \lfloor\frac{|K_1\cap\mathbb{X}|}{2}\rfloor \geq \lfloor\frac{|\ell_1\cap\mathbb{X}_e|+|\ell\cap\mathbb{X}_d|}{2}\rfloor > r-1 = T_{\mathbb{X}}$, a contradiction. Altogether, we conclude that we may assume $t_1 \leq r-1$.

Let K_2 be a 2-plane which contains P'_e and satisfies $\lfloor\frac{|K_2\cap\mathbb{X}_e|+1}{2}\rfloor = t_2$. Then the plane K_2 and the line ℓ lie on a hyperplane containing P'_e. By the choice of H_e, we have $v \geq |K_2\cap\mathbb{X}_e| + |\ell\cap\mathbb{X}_d| = |K_2\cap\mathbb{X}_e|+r$. This implies $|K_2\cap\mathbb{X}_e| \leq v-r \leq 2r-3$. Hence we have $t_2 = \lfloor\frac{|K_2\cap\mathbb{X}_e|+1}{2}\rfloor \leq \lfloor\frac{2r-3+1}{2}\rfloor = r-1$. We also see that $t_3 = \lfloor\frac{|\mathbb{X}_e|+2}{3}\rfloor = \lfloor\frac{v+2}{3}\rfloor \leq \lfloor\frac{3r-3+2}{3}\rfloor \leq r-1$. Combining these insights, we obtain $t = \max\{t_j \mid j=1,2,3\} \leq r-1$. By Lemma 5.2.4, we can now find 2-planes $K'_1,\dots,K'_{r-1}$ avoiding P'_e and contained in H_e such that, for every index i with $P_i \in \mathbb{X}_e$, there exists a 2-plane K'_{j_i} containing P_i.

Let $P_{i_0} := P_1$. For $j=1,\dots,r-1$, we take L_j to be a linear form defining the hyperplane containing K'_j and P_{j+1}. Since we have $P'_e \notin K'_j$, and since P_{j+1}, P_{i_0}, P'_e are collinear, we must have $P_{i_0} \notin \mathcal{Z}^+(L_j)$ for $j=1,\dots,r-1$. For $j=1,\dots,e-1,e+1,\dots,d-1$, we let L_{r+j-1} be a linear form defining H_j. Then we have $L_1\cdots L_{r+e-2}L_{r+e}\cdots L_{r+d-2} \in J \setminus \wp_{i_0}$, as we wanted.

Subcase (c.3) $T_{\mathbb{X}} = d+r-4$

From $(***)$ we deduce that $r+d-4 \geq r+d-e-2$, and hence $e \geq 2$. By $(**)$, we have $r+d-4 \geq \lfloor\frac{1}{4}(e\,v+\frac{(2r+d-e+4)(d-e-1)}{2}+r+2)\rfloor$. It follows that $4(r+d-3) \geq e\,v+\frac{(2r+d-e+4)(d-e-1)}{2}+r+3$, and thus $3r-11 \geq e(v-4)+\frac{(2r+d-e-4)(d-e-1)}{2}$. We also have $v \geq r+2+(d-e-1) \geq r+2$ by $(*)$. Putting these inequalities together, we find $3r-11 \geq e(r-2+(d-e-1)) + \frac{(2r+d-e-4)(d-e-1)}{2} \geq e(r-2)$, and therefore we have $2 \le e \le \frac{3r-11}{r-2} < 3$ or $e = 2$. If $d \ge 4$, then $3r-11 \ge 2(r-2)+r-1 = 3r-5$ is impossible. Consequently, we must have $d = e+1 = 3$.

Now we note that $2(r-2) \le 2(v-4) \le 3r-11$ implies $r \ge 7$ and $v \le \frac{3r-3}{2}$. Let $\mathbb{X}_e = \mathbb{X}_2 = \{P_{r+1}, \dots, P_{r+v}\}$, and let L be a linear form defining the hyperplane containing ℓ and the two points P_{r+v-1} and P_{r+v}. Since $\mathbb{X}_2$ and ℓ are not contained in a hyperplane, there is a point, say P_{r+v-2}, which does not belong to $\mathcal{Z}^+(L)$. We put $P_{i_0} = P_{r+v-2}$ and let $\mathbb{Y} = \{P_{r+1}, \dots, P_{r+v-2}\}$. Moreover, we let

$$t_j = \max\left\{ \left\lfloor \tfrac{1}{j}\left(|H \cap (\mathbb{Y} \setminus \{P_{i_0}\})| + j - 1\right)\right\rfloor \mid H \text{ is a } j\text{-plane containing } P_{i_0} \right\}$$

for $j = 1,2,3$ and define $t = \max\{t_1, t_2, t_3\}$.

Given a line ℓ_1 which contains P_{i_0} and satisfies $|\ell_1 \cap (\mathbb{Y} \setminus \{P_{i_0}\})| = t_1$, we see that $v = |\mathbb{X}_2| \ge |\ell_1 \cap \mathbb{Y}| + |\ell \cap \mathbb{X}_3| = |\ell_1 \cap \mathbb{Y}| + r$. Hence we have $|\ell_1 \cap \mathbb{Y}| \le v - r \le \frac{3r-3}{2} - r = \frac{r-3}{2}$, and therefore $t_1 = |\ell_1 \cap (\mathbb{Y} \setminus \{P_{i_0}\})| \le \frac{r-3}{2} - 1 = \frac{r-5}{2} \le r-3$.

Given a 2-plane K_1 which contains P_{i_0} and satisfies $\lfloor \frac{|K_1 \cap (\mathbb{Y} \setminus \{P_{i_0}\})|+1}{2} \rfloor = t_2$, we have $|K_1 \cap \mathbb{Y}| \le v-2 \le \frac{3r-3}{2} - 2 = \frac{3r-7}{2}$. This yields $|K_1 \cap (\mathbb{Y} \setminus \{P_{i_0}\})| \le \frac{3r-9}{2}$. Using $r \ge 7$, we obtain the bound $t_2 = \lfloor \frac{|K_1 \cap (\mathbb{Y} \setminus \{P_{i_0}\})|+1}{2} \rfloor \le \lfloor \frac{3r-7}{4} \rfloor \le r-3$.

Since $r \ge 7$, we also have $t_3 = \lfloor \frac{|\mathbb{Y} \setminus \{P_{i_0}\}|+2}{3} \rfloor = \lfloor \frac{v-1}{3} \rfloor \le \lfloor \frac{3r-5}{6} \rfloor \le r-3$. Now we combine these inequalities and get $t = \max\{t_j \mid j = 1,2,3\} \le r-3$. By Lemma 5.2.4, we can find 2-planes $K'_1, \dots, K'_{r-3}$ avoiding P_{i_0} and contained in H_2 such that for every index i with $P_i \in \mathbb{Y} \setminus \{P_{i_0}\}$ there exists an index j_i for which K'_{j_i} contains P_i. Since $P_{i_0} \notin K'_j$, we can choose a linear from L_j defining a hyperplane containing K'_j and avoiding P_{i_0} for $j = 1, \dots, r-3$. Letting L_{r-2} be a linear form defining H_1 and $L_{r-1} = L$, we get $L_1 \cdots L_{r-1} \in J \setminus \wp_{i_0}$, as we wanted. Notice that in this subcase we have $T_{\mathbb{X}} = r+d-4 = r-1$.

Subcase (c.4) $\quad T_{\mathbb{X}} \le r+d-5$

From $(***)$ and $r+d-5 \ge T_{\mathbb{X}} \ge T_{\mathbb{X},1}$ it follows that $e \ge 3$. The inequality $(**)$ yields $3r-15 \ge e(v-4) + \frac{(2r+d-e-4)(d-e-1)}{2} \ge e(v-4)$. Moreover, we deduce from $(*)$ that $v \ge r+2$. Combining this with $e \ge 3$ and $r \ge 3$, we get $3r-15 \ge e(r-2) \ge 3(r-2)$, a contradiction. Therefore the subcase $T_{\mathbb{X}} \le r+d-5$ cannot occur.

Altogether, cases (a), (b), and (c) cover all possibilites for the geometry of $\mathbb{X}_d$, and the proposition is completely proved. □

Now we are already to prove the Segre bound in the case $m_1 = \cdots = m_s = 1$, i.e. for sets of points in $\mathbb{P}^4$.

Theorem 5.2.8. *Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct reduced K-rational points in $\mathbb{P}^4$. Then we have $r_{\mathbb{X}} \le T_{\mathbb{X}}$.*

Proof. To prove $r_{\mathbb{X}} \leq T_{\mathbb{X}}$ we proceed by induction on s. The case $s = 1$ is trivial. For the induction step, we have $s \geq 2$. If $\mathbb{X}$ is contained in a hyperplane $H \cong \mathbb{P}^3_K$, the desired inequality follows from Proposition 5.2.6. Hence we may assume that $\mathbb{X}$ is not contained in a hyperplane. By Proposition 5.2.7, there is a point $P_{i_0} \in \mathbb{X}$ such that $\mathrm{ri}(S/(J + \wp_{i_0})) \leq T_{\mathbb{X}}$ where $J = \bigcap_{j \neq i_0} \wp_j$. Moreover, it follows from Lemma 5.2.2(i) that $r_{\mathbb{X}} = \max\{0, \mathrm{ri}(S/J), \mathrm{ri}(S/(J + \wp_{i_0}))\}$. By induction, we may assume that

$$\mathrm{ri}(S/J) \leq \max\{T'_j \mid j = 1, 2, 3, 4\} \leq T_{\mathbb{X}}$$

where $T'_j = \max\left\{\lfloor \frac{q+j-2}{j} \rfloor \mid P_{i_1}, \ldots, P_{i_q} \text{ lie on a } j\text{-plane},\ i_k \neq i_0,\ k = 1, \ldots, q\right\}$. Therefore we obtain $r_{\mathbb{X}} \leq T_{\mathbb{X}}$, as claimed. $\square$

Our next goal is to prove the Segre bound for equimultiple fat point schemes in $\mathbb{P}^4$ under the additional hypothesis $T_{\mathbb{X}} = T_{\mathbb{X},1}$. We need several preparations.

Lemma 5.2.9. *Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of $s \geq 2$ distinct K-rational points in $\mathbb{P}^4$, let $i \geq 1$, let $J^{(i)} = \wp_1^i \cap \cdots \cap \wp_s^i$, and let $\mathfrak{q} = \langle X_1, \ldots, X_4 \rangle$. Assume that there are positive integers r_1, r_a such that the bounds $\mathrm{ri}(S/(J^{(1)} + \mathfrak{q})) \leq r_1$ and $\mathrm{ri}(S/(J^{(a)} + \mathfrak{q}^a)) \leq r_a$ hold for some $a \geq 2$.*

Then we have $\mathrm{ri}(S/(J^{(a+1)} + \mathfrak{q}^{a+1})) \leq r_1 + r_a + 1$.

Proof. In view of Lemma 5.2.2(ii), it suffices to show $X_0^{r_1+r_a+1-i} M \in J^{(a+1)} + \mathfrak{q}^{i+1}$ for all $M \in \mathfrak{q}^i$ and $i = 0, \ldots, a$. Since $\mathrm{ri}(S/(J^{(1)} + \mathfrak{q})) \leq r_1$, we have $X_0^{r_1} \in J^{(1)} + \mathfrak{q}$. Furthermore, the hypothesis $\mathrm{ri}(S/J^{(a)} + \mathfrak{q}^a) \leq r_a$ implies $X_0^{r_a - j} M' \in J^{(a)} + \mathfrak{q}^{j+1}$ for all $M' \in \mathfrak{q}^j$ and $j = 0, \ldots, a-1$. Consequently, we get

$$\begin{aligned} X_0^{r_1+r_a-i} M &= X_0^{r_1}(X_0^{r_a-i} M) \in (J^{(1)} + \mathfrak{q})((J^{(a)} + \mathfrak{q}^{i+1}) \cap \mathfrak{q}^i) \\ &\subseteq J^{(1)} \cdot J^{(a)} + \mathfrak{q}^{i+1} \subseteq J^{(a+1)} + \mathfrak{q}^{i+1} \end{aligned}$$

for $i = 0, \ldots, a-1$ and $M \in \mathfrak{q}^i$.

Now we consider the case $i = a$ and $M = X_1^{\alpha_1} \cdots X_4^{\alpha_4} \in (\mathfrak{q}^a)_a$. W.l.o.g. we may assume that $\alpha_1 \geq 1$ and write $M = X_1 \cdot M'$ with $M' \in S$. Then we have $X_0^{r_1+r_a+1-a} M = X_0^{r_1} X_1 (X_0^{r_a-(a-1)} M') \in (J^{(1)}\mathfrak{q} + \mathfrak{q}^2)((J^{(a)} + \mathfrak{q}^a) \cap \mathfrak{q}^{a-1}) \subseteq J^{(1)} \cdot J^{(a)} + \mathfrak{q}^{a+1} \subseteq J^{(a+1)} + \mathfrak{q}^{a+1}$. Altogether, we have verified the hypothesis of Lemma 5.2.2 and the conclusion follows. $\square$

Let us give an example for the application of the preceding lemma.

Example 5.2.10. Let $\mathbb{X} = \{P_1, \dots, P_4\} \subseteq \mathbb{P}^4$ be the set of reduced points given by $P_1 = (1 : 0 : -1 : 1 : 1)$, $P_2 = (1 : 0 : 2 : 0 : 2)$, $P_3 = (1 : 1 : 1 : 0 : 1)$, and $P_4 = (1 : 2 : 2 : 2 : 1)$. Then we can check that $\operatorname{ri}(S/(J^{(1)} + \mathfrak{q})) = 1$ and $\operatorname{ri}(S/(J^{(2)} + \mathfrak{q}^2)) = 3$. The preceding lemma yields the bound $\operatorname{ri}(S/(J^{(3)} + \mathfrak{q}^3)) \le 5$ which turns out to be an equality in this case.

The hypothesis $T_{\mathbb{X}} = T_{\mathbb{X},1}$ in the theorem below can be exploited as follows.

Lemma 5.2.11. *Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of $s \ge 2$ distinct reduced K-rational points in $\mathbb{P}^4$. If we have $T_{\mathbb{X}} = T_{\mathbb{X},1}$, then $T_{\nu\mathbb{X}} \ge T_{(\nu-1)\mathbb{X}} + T_{\mathbb{X}} + 1$ for all $\nu \ge 2$.*

Proof. Let $q = T_{\mathbb{X}} + 1$. First we show that $T_{\nu\mathbb{X}} = T_{\nu\mathbb{X},1} = \nu q - 1$ for every $\nu \ge 2$. Let $j_0 \in \{1, 2, 3, 4\}$ and $q_0 \in \mathbb{N}$ be chosen such that $T_{\nu\mathbb{X}} = T_{\nu\mathbb{X},j_0} = \lfloor \frac{\nu q_0 + j_0 - 2}{j_0} \rfloor$.

If $j_0 = 1$, the claim is clearly true. Hence it suffices to consider $j_0 \in \{2, 3, 4\}$. By the definition of $T_{\mathbb{X}}$ we have $q - 1 \ge \lfloor \frac{q_0 + j_0 - 2}{j_0} \rfloor$. This implies $\nu q - 1 \ge \nu \lfloor \frac{q_0 + j_0 - 2}{j_0} \rfloor + \nu - 1$. Now we let $A_{j_0} = \nu \lfloor \frac{q_0 + j_0 - 2}{j_0} \rfloor + \nu - 1$ and $B_{j_0} = \lfloor \frac{\nu q_0 + j_0 - 2}{j_0} \rfloor$, and we write $q_0 = k j_0 + l$ with $k \in \mathbb{N}$ and $0 \le l \le j_0 - 1$. Then we have $A_{j_0} - B_{j_0} = \nu - 1 + \nu \lfloor \frac{l + j_0 - 2}{j_0} \rfloor - \lfloor \frac{\nu l + j_0 - 2}{j_0} \rfloor$. Using the fact that $(\nu - 2)(j_0 - 2) \ge 0$, this implies that we have $\nu j_0 - j_0 \ge j_0 + \nu + (\nu - 4) \ge j_0 + \nu - 2$ or $\nu - 1 \ge \frac{\nu + j_0 - 2}{j_0}$. Thus if $l < 2$, then we have $A_{j_0} - B_{j_0} \ge \nu - 1 - \lfloor \frac{\nu + j_0 - 2}{j_0} \rfloor \ge 0$. In the case $l \ge 2$, we see that $A_{j_0} - B_{j_0} \ge 2\nu - 1 - \lfloor \frac{\nu(j_0 - 1) + j_0 - 2}{j_0} \rfloor \ge 0$, since $\lfloor \frac{\nu(j_0 - 1) + j_0 - 2}{j_0} \rfloor \le \nu$. Altogether, we obtain $\nu q - 1 \ge A_{j_0} \ge B_{j_0}$. In other words, we have $T_{\nu\mathbb{X}} = \nu q - 1 = T_{\nu\mathbb{X},1}$.

Now it is clear that if $T_{\mathbb{X}} = q - 1$, then $T_{(\nu-1)\mathbb{X}} = (\nu - 1)q - 1$ and $T_{\nu\mathbb{X}} = \nu q - 1$. It therefore follows that $T_{\nu\mathbb{X}} \ge T_{(\nu-1)\mathbb{X}} + T_{\mathbb{X}} + 1$, as claimed. □

At this point we have assembled all tools that we need to prove the Segre bound for equimultiple fat point schemes satisfying $T_{\mathbb{X}} = T_{\mathbb{X},1}$.

Theorem 5.2.12. *Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of $s \ge 5$ distinct reduced K-rational points in $\mathbb{P}^4$ such that $T_{\mathbb{X}} = T_{\mathbb{X},1} = \max\{q - 1 \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a line}\}$, and let $\nu \ge 3$. Then the equimultiple fat point scheme $\nu\mathbb{X} = \nu P_1 + \cdots + \nu P_s$ satisfies*

$$r_{\nu\mathbb{X}} = T_{\nu\mathbb{X}} = \max\{\nu q - 1 \mid P_{i_1}, \dots, P_{i_q} \textit{ lie on a line}\}.$$

Proof. First we fix an index $i_0 \in \{1, \dots, s\}$. For very $a \ge 1$, we let $J^{(a)} = \bigcap_{j \ne i_0} \wp_j^a$. By induction on a, we prove $\operatorname{ri}(S/(J^{(a)} + \wp_j^a)) \le T_{a\mathbb{X}}$. The cases $a = 1, 2$ are true by Theorem 5.2.8 and Lemma 5.2.5. Now we assume that we have $\operatorname{ri}(S/(J^{(a-1)} + \wp_{i_0}^{a-1})) \le T_{(a-1)\mathbb{X}}$ for some $a \ge 3$. From Lemmata 5.2.9 and 5.2.11 we obtain $\operatorname{ri}(S/(J^{(a)} + \wp_{i_0}^a)) \le T_{(a-1)\mathbb{X}} + T_{\mathbb{X}} + 1 \le T_{a\mathbb{X}}$, and this finishes the induction.

Next we prove the inequality $r_{\nu\mathbb{X}} \le T_{\nu\mathbb{X}}$ by induction on s. In the case $s = 5$ we let $i_0 = 1$ and $\mathbb{Y} = \mathbb{X}\setminus\{P_{i_0}\}$. Then we have $\mathrm{ri}(S/(J^{(\nu)} + \wp_1^\nu)) \le T_{\nu\mathbb{X}}$, where $J^{(\nu)} = \bigcap_{j=2}^{s} \wp_j^\nu$. Since $\mathbb{Y}$ is contained in a hyperplane $H \cong \mathbb{P}^3$, it follows from Proposition 5.2.6 that

$$\mathrm{ri}(S/J^{(\nu)}) \le \max\{T_{\nu\mathbb{Y},j} \mid j = 1,2,3,4\} \le T_{\nu\mathbb{X}}$$

where $T_{\nu\mathbb{Y},j} = \max\big\{\lfloor \frac{\nu q + j - 2}{j} \rfloor \mid P_{i_1},\dots,P_{i_q} \in \mathbb{Y} \text{ lie on a } j\text{-plane}\big\}$ for $j = 1,2,3,4$. By Lemma 5.2.2(i), we conclude that

$$r_{\nu\mathbb{X}} = \max\{\nu - 1, \mathrm{ri}(S/J^{(\nu)}), \mathrm{ri}(S/(J^{(\nu)} + \wp_1^\nu))\} \le T_{\nu\mathbb{X}}.$$

Now let $s > 5$. For every $i \in \{5,\dots,s\}$, we can find a subset $\mathbb{Y}' \subseteq \mathbb{X}$ of degree i such that

$$T_{\mathbb{Y}'} = T_{\mathbb{Y}',1} = \max\big\{q - 1 \mid P_{i_1},\dots,P_{i_q} \in \mathbb{Y}' \text{ lie on a line}\big\}.$$

Thus by induction, we can assume that there is an index $i_0 \in \{1,\dots,s\}$ such that $\mathbb{Y}'' = \mathbb{X} \setminus \{P_{i_0}\}$ satisfies $r_{\nu\mathbb{Y}''} = \mathrm{ri}(S/\bigcap_{j\ne i_0} \wp_j^\nu) \le T_{\nu\mathbb{Y}''}$. As above, we also have $\mathrm{ri}(S/(\bigcap_{j\ne i_0} \wp_j^\nu + \wp_{i_0}^\nu)) \le T_{\nu\mathbb{X}}$. Thus we see that $r_{\nu\mathbb{X}} \le T_{\nu\mathbb{X}}$ by using Lemma 5.2.2(i) again.

In order to prove the equality $r_{\nu\mathbb{X}} = T_{\nu\mathbb{X}}$, we let $\mathbb{X}' = \{P_{i_1},\dots,P_{i_q}\}$ be a subset of $\mathbb{X}$ which lies on a line and satisfies $T_{\mathbb{X}'} = q - 1$. The homogeneous saturated ideal of $\nu\mathbb{X}'$ is $J' = \bigcap_{k=1}^{q} \wp_{i_k}^\nu$. It is well known (cf. [DG]) that $r_{\nu\mathbb{X}'} = \mathrm{ri}(S/J') = \nu q - 1$. Since $\nu\mathbb{X}'$ is a subscheme of $\nu\mathbb{X}$, this implies $r_{\nu\mathbb{X}} \ge r_{\nu\mathbb{X}'} = \nu q - 1$. On the other hand, we have $T_{\nu\mathbb{X}} = \nu q - 1$ as in the proof of Lemma 5.2.11. Therefore we obtain $r_{\nu\mathbb{X}} = T_{\nu\mathbb{X}}$, as we wanted to show. □

Let us apply the statement of this theorem to a family of examples.

Example 5.2.13. Let $\ell_1,\dots,\ell_5$ be five distinct lines in $\mathbb{P}^4$, and let $q_1 \ge 5$. We take q_1 points on ℓ_1, say $P_{11},\dots,P_{1q_1}$. Then, for $i = 2,\dots,5$, we choose numbers $q_i \le q_1/5$ and points $P_{i1},\dots,P_{iq_i}$ in $\ell_i \setminus \{P_{11},\dots,P_{i-1\,q_{i-1}}\}$. Now we form the set $\mathbb{X} = \{P_{11},\dots,P_{5q_5}\} \subseteq \mathbb{P}^4$. By the choice of the points P_{ij}, we have $T_{\mathbb{X}} = q_1 - 1$. Hence it follows from Theorem 5.2.12 that $r_{\nu\mathbb{X}} = \nu q_1 - 1$ for every $\nu \ge 1$.

The final result of this section applies the previous bounds to derive a bound for the regularity index of the module of Kähler differentials of an equimultiple fat point scheme $\nu\mathbb{X}$ in $\mathbb{P}^4$ whose support satisfies $T_{\mathbb{X}} = T_{\mathbb{X},1}$.

Proposition 5.2.14. *Let $\mathbb{X} = \{P_1,\dots,P_s\}$ be a set of $s \ge 2$ distinct reduced K-rational points of $\mathbb{P}^4$ such that $T_{\mathbb{X}} = T_{\mathbb{X},1}$. Then we have $\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K}) = T_{(\nu+1)\mathbb{X}}$ for every $\nu \ge 1$.*

Proof. By Theorem 5.2.12, we have $r_{\nu\mathbb{X}} = T_{\nu\mathbb{X}}$ for all $\nu \geq 1$. Notice that this clearly holds also for the cases $s = 2, 3, 4$. From Lemma 5.2.11 we get $r_{(\nu+1)\mathbb{X}} = T_{(\nu+1)\mathbb{X}} \geq T_{\nu\mathbb{X}} + T_{\mathbb{X}} \geq r_{\nu\mathbb{X}} + 2 > r_{\nu\mathbb{X}} + 1$. Hence Corollary 4.2.3(i) implies $\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(i) = (n+2)\deg(\nu\mathbb{X}) - \deg((\nu+1)\mathbb{X})$ for $i \geq T_{(\nu+1)\mathbb{X}}$ and $\mathrm{HF}_{\Omega^1_{R_{\nu\mathbb{X}}/K}}(T_{(\nu+1)\mathbb{X}} - 1) = (n+2)\deg(\nu\mathbb{X}) - \mathrm{HF}_{(\nu+1)\mathbb{X}}(T_{(\nu+1)\mathbb{X}} - 1) > (n+2)\deg(\nu\mathbb{X}) - \deg((\nu+1)\mathbb{X})$. Altogether we obtain $\mathrm{ri}(\Omega^1_{R_{\nu\mathbb{X}}/K}) = T_{(\nu+1)\mathbb{X}}$, as claimed. □

The following corollary is directly induced from Propositions 2.4.10 and 5.2.14.

Corollary 5.2.15. *In the setting of Proposition 5.2.14, let* $1 \leq m \leq 5$. *Then we have* $\mathrm{ri}(\Omega^m_{R_{\nu\mathbb{X}}/K}) \leq \min\{T_{(\nu+1)\mathbb{X}} + 3, T_{(\nu+1)\mathbb{X}} + m - 1\}$ *for every* $\nu \geq 1$.

Example 5.2.16. Let $\mathbb{X} = \{(1:0:0:0:0), (1:1:0:0:0), (1:0:1:0:0), (1:0:0:1:0), (1:0:0:0:1)\}$ be a set of 5 points in $\mathbb{P}^4$. We have $\mathrm{HF}_{\mathbb{X}}:\ 1\ 5\ 5 \ldots$, $\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}}:\ 0\ 5\ 15\ 5\ 5 \ldots$, $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}:\ 0\ 0\ 10\ 10\ 0\ 0 \ldots$, $\mathrm{HF}_{\Omega^3_{R_{\mathbb{X}}/K}}:\ 0\ 0\ 0\ 10\ 5\ 0\ 0 \ldots$, $\mathrm{HF}_{\Omega^4_{R_{\mathbb{X}}/K}}:\ 0\ 0\ 0\ 0\ 5\ 1\ 0\ 0 \ldots$ and $\mathrm{HF}_{\Omega^5_{R_{\mathbb{X}}/K}}:\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0 \ldots$. Then $q_1 = 2$ and the regularity indices are $r_{\mathbb{X}} = 2 - 1 = 1$ and $\mathrm{ri}(\Omega^1_{R_{\mathbb{X}}/K}) = 3 = 4 - 1 = \mathrm{ri}(\Omega^2_{R_{\mathbb{X}}/K}) - 1 = \mathrm{ri}(\Omega^3_{R_{\mathbb{X}}/K}) - 2 = \mathrm{ri}(\Omega^4_{R_{\mathbb{X}}/K}) - 3 = \mathrm{ri}(\Omega^5_{R_{\mathbb{X}}/K}) - 3$ respectively. Thus the bounds which given in Corollary 5.2.15 are sharp bounds. For $m = 4$ we have $r_{4\mathbb{X}} = 4 \cdot 2 - 1 = 7$ and $\mathrm{ri}(\Omega^i_{R_{4\mathbb{X}}/K}) = 9 \leq 7 + 2 + i - 1$ for $i = 1, 2, 3$ and $\mathrm{ri}(\Omega^i_{R_{4\mathbb{X}}/K}) = 10 \leq 7 + 2 + i - 1$ for $i = 4, 5$.

Appendix

In this appendix we provide the functions which implement the algorithms for computing the Kähler differential modules and their Hilbert functions in ApCoCoA and describe their usage with some examples. The ApCoCoA computer algebra system is primarily designed for working with applied problems by using the symbolic computation methods of CoCoA [Co] and by developing new libraries for the necessary computations. It can be obtained for free from the ApCoCoA home page:

http://www.apcocoa.org

There is also a comprehensive manual and a series of tutorials available at this web address.

A.1 Computing the Kähler Differential Module $\Omega^m_{R/K}$ and Its Hilbert Function

```
--------------------------------------------------------------------------------
-- KaehlerDifferentialModuleAndHF(IP,M): Computes the Kaehler diffenrential Module
--         of the algebra R/K and its Hilbert function, where R=S/I is the residual
--         class ring, S=K[X_0,..,X_n], I is a homogeneous ideal of S.
-- Input:  IP = A non-zero homogeneous ideal of S or a list of points of the form
--         (1:a_1:...:a_n)
--         M = the number of form of the Kaehler diffenrential module \Omega^M(R/K)
-- Output: A presentation and the Hilbert function of \Omega^M(R/K)
--------------------------------------------------------------------------------
Define KaehlerDifferentialModuleAndHF(IP,M);
    If Type(IP) = IDEAL Then I := GBasis(IP);
    Else I := GBasis(IdealOfProjectivePoints(IP));
    EndIf;
    N := Len(Indets());
    S ::= CoeffRing[x[1..N],y[1..N]], DegRevLex;
```

```
Using S Do
    -- Form the canonical bases V1, V2 of \Omega^(M-1)(S/K), \Omega^M(S/K)
    S1 := Subsets(1..N,M-1); V1 := NewList(Len(S1),0);
    For I1 := 1 To Len(S1) Do
        V1[I1] := Product([y[S1[I1][J]] | J In 1..(M-1)]);
    EndFor;
    S2 := Subsets(1..N,M); V2 := NewList(Len(S2),0);
    For I2 := 1 To Len(S2) Do
        V2[I2] := Product([y[S2[I2][J]] | J In 1..M ]);
    EndFor;
    -- Compute the submodule B2 = dI.\Omega^(M-1)(S/K) of \Omega^M(S/K)
    H := NewList(Len(V1)*Len(I),0); Q := 1;
    LI1 := Indets(); LI2 := First(LI1, N); F1 := RMap(LI2);
    NewI1 := Image(Ideal(I),F1);
    NewI := GBasis(NewI1);
    For W := 1 To Len(V1) Do
        C := Log(V1[W]);
        For K := 1 To Len(I) Do
            For J := 1 To N Do
                If Der(NewI[K],x[J]) <> 0 Then
                    If GCD(V1[W],y[J]) <> y[J] Then
                        D := N; D1 := 0;
                        Repeat D := D + 1;
                            If C[D] <> 0 Then D1 := D1 + 1; EndIf;
                        Until D = N + J;
                        H[Q] := H[Q] + (-1)^(D1)*Der(NewI[K],x[J])*y[J]*V1[W];
                    EndIf;
                EndIf;
            EndFor;
            Q := Q+1;
        EndFor;
    EndFor;
    B2 := [];
    For J := 1 To Len(H) Do
        If H[J] <> 0 Then B2 := Concat(B2,[H[J]]); EndIf;
    EndFor;
    -- Compute the submodule B3 = I.\Omega^M(S/K) of \Omega^M(S/K)
    B3 := NewList(Len(V2)*Len(NewI));
    Q := 1;
    For W := 1 To Len(V2) Do
        For J := 1 To Len(NewI) Do
            B3[Q] := NewI[J]V2[W];
            Q := Q+1;
```

```
        EndFor;
    EndFor;
    -- Compute the submodule U = B2+B3 of \Omega^M(S/K)\cong S^{C^M_N}
    D := List(Identity(Len(V2)));
    G := NewList(Len(B2),0D[1]);
    H := NewList(Len(B3),0D[1]);
    For J := 1 To Len(B3) Do
        X := Monomials(B3[J]);
        For Y := 1 To Len(X) Do
            For K := 1 To Len(V2) Do
                If GCD(X[Y],V2[K]) = V2[K] Then
                    H[J] := H[J] + (X[Y]/V2[K])*D[K];
                EndIf;
            EndFor;
        EndFor;
    EndFor;
    For J := 1 To Len(B2) Do
        X := Monomials(B2[J]);
        For Y := 1 To Len(X) Do
            For K := 1 To Len(V2) Do
                If GCD(X[Y],V2[K]) = V2[K] Then
                    G[J] := G[J] + (X[Y]/V2[K])*D[K];
                EndIf;
            EndFor;
        EndFor;
    EndFor;
    U := Concat(G,H);
EndUsing;
-- Return the image of U in S^{C^M_N}
LI3 := Indets();
LI4 := ConcatLists([LI3, [1 | J In 1..N]]);
F := RMap(LI4);
NewU := Image(U,F);
K1 := Len(NewU[1]);
LO1 := CurrentRing()^K1/Module(NewU);
LO2 := Hilbert(LO1);
LO3 := RegularityIndex(LO2);
LO4 := NewList(LO3+M+1,0);
LO5 := HilbertPoly(LO1);
PrintLn "A presentation of Omega^",M, "_(R/K):";
PrintLn "Omega^",M, "_(R/K)(-",M,") = ",LO1;
For J := M+1 To Len(LO4) Do
    LO4[J] := LO4[J]+ EvalHilbertFn(LO2,J-M-1);
```

```
    EndFor;
    PrintLn "The Hilbert function of Omega^",M, "_(R/K):";
    For J:=1 To Len(L04)-1 Do
        PrintLn "H(",J-1,") = ",L04[J]
    EndFor;
    Using QQt Do
        PrintLn "H(t) = ", L05 , "   for t >= ", Len(L04)-1
    EndUsing;
EndDefine;
```

Example A.1.1. Let us compute a presentation and the Hilbert function of the Kähler differential module $\Omega^m_{R/\mathbb{Q}}$ of the algebra $R/\mathbb{Q}$ given in Example 3.2.15. We run the following commands in ApCoCoA:

```
Use QQ[x[0..2]];
IP := Ideal((x[0]x[1]^2-x[2]^3)(x[1]-x[2])^3, (x[0]x[1]^2-x[2]^3)(x[1]-x[0])^2);
KaehlerDifferentialModuleAndHF(IP,1);
KaehlerDifferentialModuleAndHF(IP,2);
KaehlerDifferentialModuleAndHF(IP,3);
```

The results of these commands are the following presentations and Hilbert functions of $\Omega^m_{R/\mathbb{Q}}$, where $m = 1, 2, 3$:

```
A presentation of Omega^1_(R/K):
Omega^1_(R/K)(-1) = CurrentRingEnv^3/Module([[3x[0]^2x[1]^2 - 4x[0]x[1]^3 + x[1]^4 -
2x[0]x[2]^3 + 2x[1]x[2]^3, 2x[0]^3x[1] - 6x[0]^2x[1]^2 + 4x[0]x[1]^3 + 2x[0]x[2]^3 -
2x[1]x[2]^3, -3x[0]^2x[2]^2 + 6x[0]x[1]x[2]^2 -3x[1]^2x[2]^2], [x[1]^5 -3x[1]^4x[2]+
3x[1]^3x[2]^2 - x[1]^2x[2]^3,  5x[0]x[1]^4 -  12x[0]x[1]^3x[2] + 9x[0]x[1]^2x[2]^2 -
2x[0]x[1]x[2]^3- 3x[1]^2x[2]^3+ 6x[1]x[2]^4- 3x[2]^5, -3x[0]x[1]^4+ 6x[0]x[1]^3x[2]-
3x[0]x[1]^2x[2]^2  -  3x[1]^3x[2]^2  +  12x[1]^2x[2]^3 -  15x[1]x[2]^4  +  6x[2]^5],
[x[0]^3x[1]^2  -  2x[0]^2x[1]^3 +  x[0]x[1]^4  -  x[0]^2x[2]^3  +  2x[0]x[1]x[2]^3 -
x[1]^2x[2]^3, 0, 0], [x[0]x[1]^5- 3x[0]x[1]^4x[2]- x[1]^3x[2]^3+ 3x[0]x[1]^3x[2]^2 -
x[0]x[1]^2x[2]^3 + 3x[1]^2x[2]^4 - 3x[1]x[2]^5 + x[2]^6, 0, 0],   [0, x[0]^3x[1]^2 -
2x[0]^2x[1]^3 +  x[0]x[1]^4 -  x[0]^2x[2]^3 +  2x[0]x[1]x[2]^3 -  x[1]^2x[2]^3,  0],
[0, x[0]x[1]^5 - 3x[0]x[1]^4x[2]+ 3x[0]x[1]^3x[2]^2- x[0]x[1]^2x[2]^3- x[1]^3x[2]^3+
3x[1]^2x[2]^4 - 3x[1]x[2]^5 + x[2]^6,  0],   [0,  0,  x[0]^3x[1]^2 - 2x[0]^2x[1]^3 +
x[0]x[1]^4 - x[0]^2x[2]^3 + 2x[0]x[1]x[2]^3 - x[1]^2x[2]^3],   [0,  0,  x[0]x[1]^5 -
3x[0]x[1]^4x[2]+ 3x[0]x[1]^3x[2]^2- x[0]x[1]^2x[2]^3 - x[1]^3x[2]^3+ 3x[1]^2x[2]^4 -
3x[1]x[2]^5 + x[2]^6]])
The Hilbert function of Omega^1_(R/K):
H(0) = 0
H(1) = 3
H(2) = 9
H(3) = 18
```

```
H(4) = 30
H(5) = 44
H(6) = 56
H(7) = 63
H(8) = 66
H(t) = 6t + 22   for t >= 9
-------------------------------
A presentation of Omega^2_(R/K):
Omega^2_(R/K)(-2) = CurrentRingEnv^3/Module([[-2x[0]^3x[1]     +    6x[0]^2x[1]^2 -
4x[0]x[1]^3  -  2x[0]x[2]^3  +  2x[1]x[2]^3,   3x[0]^2x[2]^2  -  6x[0]x[1]x[2]^2  +
3x[1]^2x[2]^2, 0], [-5x[0]x[1]^4 + 12x[0]x[1]^3x[2] - 9x[0]x[1]^2x[2]^2 + 3x[2]^5 +
2x[0]x[1]x[2]^3 +  3x[1]^2x[2]^3 -  6x[1]x[2]^4,   3x[0]x[1]^4 -  6x[0]x[1]^3x[2] +
3x[0]x[1]^2x[2]^2  + 3x[1]^3x[2]^2 -  12x[1]^2x[2]^3  + 15x[1]x[2]^4 - 6x[2]^5, 0],
[3x[0]^2x[1]^2 - 4x[0]x[1]^3 + x[1]^4 -2x[0]x[2]^3 + 2x[1]x[2]^3, 0, 3x[0]^2x[2]^2-
6x[0]x[1]x[2]^2+3x[1]^2x[2]^2], [x[1]^5 - 3x[1]^4x[2] + 3x[1]^3x[2]^2-x[1]^2x[2]^3,
0, 3x[0]x[1]^4 - 6x[0]x[1]^3x[2] +3x[0]x[1]^2x[2]^2+3x[1]^3x[2]^2 -12x[1]^2x[2]^3 +
15x[1]x[2]^4 - 6x[2]^5], [0, 3x[0]^2x[1]^2 - 4x[0]x[1]^3 +  x[1]^4 -  2x[0]x[2]^3 +
2x[1]x[2]^3, 2x[0]^3x[1] - 6x[0]^2x[1]^2 + 4x[0]x[1]^3 + 2x[0]x[2]^3 -2x[1]x[2]^3],
[0, x[1]^5 -3x[1]^4x[2]+3x[1]^3x[2]^2 -x[1]^2x[2]^3, 5x[0]x[1]^4- 12x[0]x[1]^3x[2]+
9x[0]x[1]^2x[2]^2 -  2x[0]x[1]x[2]^3  -  3x[1]^2x[2]^3  +  6x[1]x[2]^4 -  3x[2]^5],
[x[0]^3x[1]^2 -  2x[0]^2x[1]^3 +  x[0]x[1]^4  -  x[0]^2x[2]^3  +  2x[0]x[1]x[2]^3 -
x[1]^2x[2]^3,  0,  0],   [x[0]x[1]^5  -  3x[0]x[1]^4x[2]  +   3x[0]x[1]^3x[2]^2   -
x[0]x[1]^2x[2]^3 - x[1]^3x[2]^3+ 3x[1]^2x[2]^4 -  3x[1]x[2]^5 +  x[2]^6,   0,   0],
[0, x[0]^3x[1]^2 - 2x[0]^2x[1]^3 +  x[0]x[1]^4 -  x[0]^2x[2]^3 +  2x[0]x[1]x[2]^3 -
x[1]^2x[2]^3,   0], [0,  x[0]x[1]^5  -  3x[0]x[1]^4x[2]   +   3x[0]x[1]^3x[2]^2   -
x[0]x[1]^2x[2]^3 - x[1]^3x[2]^3 + 3x[1]^2x[2]^4 - 3x[1]x[2]^5 + x[2]^6, 0], [0,  0,
x[0]^3x[1]^2- 2x[0]^2x[1]^3+x[0]x[1]^4-x[0]^2x[2]^3+ 2x[0]x[1]x[2]^3-x[1]^2x[2]^3],
[0, 0, x[0]x[1]^5-3x[0]x[1]^4x[2]+3x[0]x[1]^3x[2]^2-x[0]x[1]^2x[2]^3-x[1]^3x[2]^3 +
3x[1]^2x[2]^4 - 3x[1]x[2]^5 + x[2]^6]])
The Hilbert function of Omega^2_(R/K):
H(0) = 0
H(1) = 0
H(2) = 3
H(3) = 9
H(4) = 18
H(5) = 30
H(6) = 42
H(7) = 48
H(8) = 45
H(t) = 3t + 22   for t >= 9
-------------------------------
A presentation of Omega^3_(R/K):
Omega^3_(R/K)(-3) = CurrentRingEnv^1/Module([[-3x[0]^2x[2]^2  +  6x[0]x[1]x[2]^2  -
```

```
3x[1]^2x[2]^2], [-3x[0]x[1]^4 + 6x[0]x[1]^3x[2] - 3x[0]x[1]^2x[2]^2 -3x[1]^3x[2]^2+
12x[1]^2x[2]^3 - 15x[1]x[2]^4 + 6x[2]^5], [-2x[0]^3x[1] +6x[0]^2x[1]^2-4x[0]x[1]^3-
2x[0]x[2]^3 + 2x[1]x[2]^3], [-5x[0]x[1]^4 + 12x[0]x[1]^3x[2] -  9x[0]x[1]^2x[2]^2 +
2x[0]x[1]x[2]^3 + 3x[1]^2x[2]^3 - 6x[1]x[2]^4+3x[2]^5], [3x[0]^2x[1]^2-4x[0]x[1]^3+
x[1]^4  -  2x[0]x[2]^3  +  2x[1]x[2]^3],  [x[1]^5 -  3x[1]^4x[2] +  3x[1]^3x[2]^2 -
x[1]^2x[2]^3],  [x[0]^3x[1]^2  -  2x[0]^2x[1]^3  +  x[0]x[1]^4  -   x[0]^2x[2]^3  +
2x[0]x[1]x[2]^3 - x[1]^2x[2]^3], [x[0]x[1]^5 - 3x[0]x[1]^4x[2] + 3x[0]x[1]^3x[2]^2-
x[1]^3x[2]^3- x[0]x[1]^2x[2]^3 + 3x[1]^2x[2]^4 - 3x[1]x[2]^5 + x[2]^6]])
The Hilbert function of Omega^3_(R/K):
H(0) = 0
H(1) = 0
H(2) = 0
H(3) = 1
H(4) = 3
H(5) = 6
H(6) = 10
H(7) = 12
H(8) = 9
H(t) = 6   for t >= 9
-------------------------------
```

Example A.1.2. Let $\mathbb{X}$ be the set of $\mathbb{Q}$-rational points $\mathbb{X} = \{P_1, \ldots, P_9\} \subseteq \mathbb{P}^2$ given by $P_1 = (1:0:0)$, $P_2 = (1:0:1)$, $P_3 = (1:0:2)$, $P_4 = (1:0:3)$, $P_5 = (1:0:4)$, $P_6 = (1:0:5)$, $P_7 = (1:1:0)$, $P_8 = (1:2:0)$, and $P_9 = (1:1:1)$ (see also Example 3.3.9). To compute a presentation and the Hilbert function of $\Omega^2_{R/\mathbb{Q}}$, we run the following commands in ApCoCoA:

```
Use QQ[x[0..2]];
PP := [[1,0,0],[1,0,1],[1,0,2],[1,0,3],[1,0,4],[1,0,5],[1,1,0],[1,2,0],[1,1,1]];
KaehlerDifferentialModuleAndHF(PP,2);
```

The results of the above commands are the following presentation and the Hilbert function of $\Omega^2_{R/\mathbb{Q}}$:

```
A presentation of Omega^2_(R/K):
Omega^2_(R/K)(-2) = CurrentRingEnv^3/Module([[-x[0]^2 + 3x[0]x[1] -3/2x[1]^2, 0, 0],
[-x[0]x[2]+ x[2]^2, -x[0]x[1]+2x[1]x[2], 0], [-2x[1]x[2] +x[2]^2, -x[1]^2+2x[1]x[2],
0], [0, -x[0]^5+ 137/30x[0]^4x[2]-45/8x[0]^3x[2]^2+ 17/6x[0]^2x[2]^3- 5/8x[0]x[2]^4+
1/20x[2]^5, 0], [2x[0]x[1] - 3/2x[1]^2, 0, 0], [x[1]x[2], 0, -x[0]x[1] + 2x[1]x[2]],
[0, 0, -x[1]^2 + 2x[1]x[2]], [5x[0]^4x[2] - 137/15x[0]^3x[2]^2 + 45/8x[0]^2x[2]^3 -
17/12x[0]x[2]^4 + 1/8x[2]^5,  0,  -x[0]^5 +  137/30x[0]^4x[2] -  45/8x[0]^3x[2]^2 +
17/6x[0]^2x[2]^3 - 5/8x[0]x[2]^4 + 1/20x[2]^5], [0, 2x[0]x[1] - 3/2x[1]^2, x[0]^2 -
3x[0]x[1] + 3/2x[1]^2], [0, x[1]x[2], x[0]x[2] - x[2]^2], [0, 0, 2x[1]x[2] -x[2]^2],
```

```
[0, 5x[0]^4x[2]-137/15x[0]^3x[2]^2 + 45/8x[0]^2x[2]^3-17/12x[0]x[2]^4+1/8x[2]^5, 0],
[x[0]^2x[1] - 3/2x[0]x[1]^2 + 1/2x[1]^3, 0, 0], [x[0]x[1]x[2] - x[1]x[2]^2,  0,  0],
[x[1]^2x[2] - x[1]x[2]^2, 0, 0], [x[0]^5x[2] - 137/60x[0]^4x[2]^2+15/8x[0]^3x[2]^3-
17/24x[0]^2x[2]^4 + 1/8x[0]x[2]^5-1/120x[2]^6, 0, 0], [0, x[0]^2x[1]-3/2x[0]x[1]^2+
1/2x[1]^3, 0], [0, x[0]x[1]x[2] - x[1]x[2]^2, 0], [0, x[1]^2x[2] -  x[1]x[2]^2,  0],
[0, x[0]^5x[2]-137/60x[0]^4x[2]^2+15/8x[0]^3x[2]^3-17/24x[0]^2x[2]^4+1/8x[0]x[2]^5-
1/120x[2]^6, 0], [0, 0, x[0]^2x[1] - 3/2x[0]x[1]^2+1/2x[1]^3], [0, 0, x[0]x[1]x[2]-
x[1]x[2]^2], [0, 0, x[1]^2x[2] - x[1]x[2]^2], [0, 0, x[0]^5x[2]-137/60x[0]^4x[2]^2+
15/8x[0]^3x[2]^3 - 17/24x[0]^2x[2]^4 + 1/8x[0]x[2]^5 - 1/120x[2]^6]])
The Hilbert function of Omega^2_(R/K):
H(0) = 0
H(1) = 0
H(2) = 3
H(3) = 9
H(4) = 9
H(5) = 4
H(6) = 5
H(7) = 4
H(8) = 3
H(9) = 2
H(10) = 1
H(t) = 0   for t >= 11
-------------------------------
```

A.2 Computing the Kähler Differential Module $\Omega^{n+1}_{R/K}$ and Its Hilbert Function

```
-------------------------------------------------------------------------------
-- KaehlerDifferentialModuleN1AndHF(IP): Compute the module \Omega^{n+1}(R/K) of
--          R/K and its Hilbert function
-- Input:  IP = A non-zero homogeneous ideal of S or a list of points of the form
--          (1:a_1:...:a_n)
-- Output: A presentation and the Hilbert function of \Omega^{n+1}(R/K)
-------------------------------------------------------------------------------
Define KaehlerDifferentialModuleN1AndHF(IP);
    If Type(IP) = IDEAL Then I := GBasis(IP);
    Else I := GBasis(IdealOfProjectivePoints(IP));
    EndIf;
    N := Len(Indets());
    S ::= CoeffRing[x[1..N]], DegRevLex;
    Using S Do
        M := Len(I)*N; A := NewList(M,0); H := 0;
```

```
        For K := 1 To Len(I) Do
            For J := 1 To N Do
                A[H+J] := Der(I[K],x[J]);
            EndFor;
            H := N*K;
        EndFor;
        D := Concat(I,A);
    EndUsing;
    LI1 := Indets(); F := RMap(LI1);
    NewD := Image(D,F);
    LO1:= CurrentRing()/Minimalized(Ideal(NewD));
    LO2 := Hilbert(LO1);
    LO3 := RegularityIndex(LO2);
    LO4 := HilbertPoly(LO1);
    LO5 := NewList(LO3+N+1,0);
    PrintLn "A presentation of Omega^",N, "_(R/K):";
    PrintLn "Omega^",N, "_(R/K)(-",N,") = ",LO1;
    For J := N+1 To Len(LO5) Do
        LO5[J] := LO5[J]+ EvalHilbertFn(LO2,J-N-1);
    EndFor;
    PrintLn "The Hilbert function of Omega^",N, "_(R/K):";
    For J := 1 To Len(LO5)-1 Do
        PrintLn "H(",J-1,") = ",LO5[J]
    EndFor;
    Using QQt Do
        Print "H(t) = ", LO4, "   for t >= ", Len(LO5)-1
    EndUsing;
EndDefine;
```

Example A.2.1. Let $R/\mathbb{Q}$ be the algebra given in Example 3.2.18. We compute a presentation and the Hilbert function of $\Omega^{n+1}_{R/\mathbb{Q}}$ by running the following commands in ApCoCoA:

```
Use QQ[x[0..3]];
I1 := Ideal(x[1], x[2], x[3])^5;
I2 := Ideal(x[1]-x[0], x[2], x[3]);
I3 := Ideal(x[1], x[2]-x[0], x[3])^4;
I4 := Ideal(x[1],x[2],x[3]-x[0])^4;
IP := Intersection(I1,I2,I3,I4);
KaehlerDifferentialModuleN1AndHF(IP);
```

The results of these commands are the following presentation and Hilbert function of $\Omega^4_{R/\mathbb{Q}}$:

```
A presentation of Omega^4_(R/K):
Omega^4_(R/K)(-4) = CurrentRingEnv/Ideal(x[1]^3x[2],  x[1]^3x[3],   3x[1]^2x[2]x[3],
x[1]^4,  x[0]x[1]^2x[3]^2 -  x[1]^2x[3]^3,  2x[0]x[1]x[2]x[3]^2  -  2x[1]x[2]x[3]^3,
x[0]x[1]^2x[2]^2    -   x[1]^2x[2]^3,     2x[0]x[1]x[2]^2x[3]   -   2x[1]x[2]^3x[3],
2x[1]x[2]^2x[3]^2,  x[0]x[2]^2x[3]^2 - x[2]^3x[3]^2 - x[2]^2x[3]^3, - 2x[2]^3x[3]^3,
2x[0]^2x[2]x[3]^3 - 4x[0]x[2]x[3]^4 + 2x[2]x[3]^5,  x[0]^2x[1]x[3]^3 + x[1]x[3]^5 -
2x[0]x[1]x[3]^4, 2x[0]^2x[2]^3x[3]- 4x[0]x[2]^4x[3] +2x[2]^5x[3], x[0]^2x[1]x[2]^3-
2x[0]x[1]x[2]^4 + x[1]x[2]^5,  x[0]^3x[2]^4 -  3x[0]^2x[2]^5 + 3x[0]x[2]^6 - x[2]^7,
x[0]^3x[3]^4 - 3x[0]^2x[3]^5 + 3x[0]x[3]^6 - x[3]^7)
The Hilbert function of Omega^4_(R/K):
H(0) = 0
H(1) = 0
H(2) = 0
H(3) = 0
H(4) = 1
H(5) = 4
H(6) = 10
H(7) = 20
H(8) = 31
H(9) = 38
H(t) = 40   for t >= 10
-------------------------------
```

A.3 Computing the Kähler Differential Module $\Omega^m_{R/K[x_0]}$ and Its Hilbert Function

```
--------------------------------------------------------------------------------
-- KaehlerDiffModuleRelAndHF(IP,M): Compute the Kaehler differential module of
--      the algebra R/K[x_0] and its Hilbert function
-- Input: IP = A non-zero homogeneous ideal of S or a list of points of the form
--      (1:a_1:...:a_n)
--      M = the number of form of Kaehler differential module \Omega^M(R/K[x_0])
-- Output: A presentation and the Hilbert function of \Omega^M(R/K[x_0])
--------------------------------------------------------------------------------
Define KaehlerDiffModuleRelAndHF(IP,M);
    If Type(IP) = IDEAL Then I := GBasis(IP);
    Else I := GBasis(IdealOfProjectivePoints(IP));
    EndIf;
    N := Len(Indets());
    S ::= CoeffRing[x[1..N],y[1..(N-1)]], DegRevLex;
    Using S Do
        -- Form bases V1, V2 of \Omega^(M-1)(P/K[x_0]), \Omega^M(P/K[x_0])
```

```
S1 := Subsets(1..(N-1),M-1); V1 := NewList(Len(S1),0);
For I1 := 1 To Len(S1) Do
    V1[I1] := Product([y[S1[I1][J]] | J In 1..(M-1)]);
EndFor;
S2 := Subsets(1..(N-1),M); V2 := NewList(Len(S2),0);
For I2 := 1 To Len(S2) Do
    V2[I2] := Product([y[S2[I2][J]] | J In 1..M ]);
EndFor;
-- Compute B2 = dI.\Omega^(M-1)(P/K[x_0])
H:=NewList(Len(V1)*Len(I),0); Q := 1;
LI1 := Indets(); LI2 := First(LI1, N); F1 := RMap(LI2);
NewI1 := Image(Ideal(I),F1);
NewI := GBasis(NewI1);
For W := 1 To Len(V1) Do
    C := Log(V1[W]);
    For K := 1 To Len(I) Do
        For J:=2 To N Do
            If Der(NewI[K],x[J]) <> 0 Then
                If GCD(V1[W],y[J-1]) <> y[J-1] Then
                    D := N; D1 := 0;
                    Repeat D := D+1;
                        If C[D] <> 0 Then D1 := D1+1; EndIf;
                    Until D = N+J-1;
                    H[Q] := H[Q]+(-1)^(D1)*Der(NewI[K],x[J])*y[J-1]*V1[W];
                EndIf;
            EndIf;
        EndFor;
        Q := Q+1;
    EndFor;
EndFor;
B2 := [];
For J := 1 To Len(H) Do
    If H[J] <> 0 Then B2 := Concat(B2,[H[J]]);EndIf;
EndFor;
-- Compute B3 = I.\Omega^M(P/K[x_0])
B3 := NewList(Len(V2)*Len(NewI));
Q := 1;
For W := 1 To Len(V2) Do
    For J := 1 To Len(NewI) Do
        B3[Q] := NewI[J]V2[W];
        Q := Q+1;
    EndFor;
EndFor;
```

```
    -- Compute MM = B2+B3 in P^(C^M_{N-1}) and return the result
    D := List(Identity(Len(V2)));
    H := NewList(Len(B3),0D[1]);
    G := NewList(Len(B2),0D[1]);
    For J := 1 To Len(B3) Do
        X := Monomials(B3[J]);
        For Y := 1 To Len(X) Do
            For K := 1 To Len(V2) Do
                If GCD(X[Y],V2[K]) = V2[K] Then H[J] := H[J]+(X[Y]/V2[K])*D[K];
                EndIf;
            EndFor;
        EndFor;
    EndFor;
    For J := 1 To Len(B2) Do
        X := Monomials(B2[J]);
        For Y := 1 To Len(X) Do
            For K := 1 To Len(V2) Do
                If GCD(X[Y],V2[K]) = V2[K] Then G[J] := G[J]+(X[Y]/V2[K])*D[K];
                EndIf;
            EndFor;
        EndFor;
    EndFor;
    MM:=Concat(G,H);
EndUsing;
L5 := Indets();
L6 := ConcatLists([L5, [1 | J In 1..(N-1)]]);
F := RMap(L6);
NewM := Image(MM,F); -- Image of MM in R=K[x_1,...,X-n]
K1 := Len(NewM[1]);
L01 := CurrentRing()^K1/Module(NewM);
L02 := Hilbert(L01);
L03 := RegularityIndex(L02);
L04 := HilbertPoly(L01);
L05 := NewList(L03+M+1,0);
PrintLn "A presentation of Omega^",M, "_(R/K[x_0]):";
PrintLn "Omega^",M, "_(R/K[x_0])(-",M,") = ",L01;
For J :=M+1 To Len(L05) Do
    L05[J] := L05[J]+ EvalHilbertFn(L02,J-M-1);
EndFor;
PrintLn "The Hilbert function of Omega^",M, "_(R/K[x_0]):";
For J:=1 To Len(L05)-1 Do
    PrintLn "H(",J-1,") = ",L05[J]
EndFor;
```

```
    Using QQt Do
        Print "H(t) = ", L04, "   for t >= ", Len(L05)-1
    EndUsing;
EndDefine;
```

Example A.3.1. Let us go back to Example A.1.2 and compute a presentation and the Hilbert function of the Kähler differential module $\Omega^m_{R/\mathbb{Q}[x_0]}$ for the set $\mathbb{X}$, where $m = 1, 2$. We run the following commands in ApCoCoA:

```
Use QQ[x[0..2]];
PP:=[[1,0,0],[1,0,1],[1,0,2],[1,0,3],[1,0,4],[1,0,5],[1,1,0],[1,2,0],[1,1,1]];
KaehlerDiffModuleRelAndHF(PP,1);
KaehlerDiffModuleRelAndHF(PP,2);
```

The output of these commands is as follows:

```
A presentation of Omega^1_(R/K[x_0]):
Omega^1_(R/K[x_0])(-1) = CurrentRingEnv^2/Module([[x[0]^2 - 3x[0]x[1]+3/2x[1]^2, 0],
[x[0]x[2] - x[2]^2, x[0]x[1] - 2x[1]x[2]], [2x[1]x[2] - x[2]^2, x[1]^2 - 2x[1]x[2]],
[0, x[0]^5 - 137/30x[0]^4x[2] + 45/8x[0]^3x[2]^2 - 17/6x[0]^2x[2]^3 + 5/8x[0]x[2]^4-
1/20x[2]^5], [x[0]^2x[1] - 3/2x[0]x[1]^2 + 1/2x[1]^3, 0], [x[0]x[1]x[2] -x[1]x[2]^2,
0], [x[1]^2x[2] - x[1]x[2]^2, 0], [x[0]^5x[2] - 137/60x[0]^4x[2]^2+15/8x[0]^3x[2]^3-
17/24x[0]^2x[2]^4 + 1/8x[0]x[2]^5 - 1/120x[2]^6, 0], [0, x[0]^2x[1] - 3/2x[0]x[1]^2+
1/2x[1]^3], [0, x[0]x[1]x[2]-x[1]x[2]^2], [0, x[1]^2x[2]-x[1]x[2]^2], [0,x[0]^5x[2]-
137/60x[0]^4x[2]^2 + 15/8x[0]^3x[2]^3-17/24x[0]^2x[2]^4+1/8x[0]x[2]^5-1/120x[2]^6]])
The Hilbert function of Omega^1_(R/K[x_0]):
H(0) = 0
H(1) = 2
H(2) = 6
H(3) = 9
H(4) = 7
H(5) = 5
H(6) = 5
H(7) = 4
H(8) = 3
H(9) = 2
H(10) = 1
H(t) = 0   for t >= 11
-------------------------------
A presentation of Omega^2_(R/K[x_0]):
Omega^2_(R/K[x_0])(-2)=CurrentRingEnv^1/Module([[-x[0]x[1] + 2x[1]x[2]],  [-x[1]^2 +
2x[1]x[2]], [-x[0]^5 + 137/30x[0]^4x[2] - 45/8x[0]^3x[2]^2-5/8x[0]x[2]^4+ 1/20x[2]^5
+ 17/6x[0]^2x[2]^3], [x[0]^2 - 3x[0]x[1]+3/2x[1]^2], [x[0]x[2]-x[2]^2], [2x[1]x[2] -
x[2]^2],   [x[0]^2x[1] -  3/2x[0]x[1]^2 + 1/2x[1]^3],  [x[0]x[1]x[2]  - x[1]x[2]^2],
```

```
[x[1]^2x[2] - x[1]x[2]^2], [x[0]^5x[2] - 137/60x[0]^4x[2]^2 + 15/8x[0]^3x[2]^3 -
17/24x[0]^2x[2]^4 + 1/8x[0]x[2]^5 - 1/120x[2]^6]])
The Hilbert function of Omega^2_(R/K[x_0]):
H(0) = 0
H(1) = 0
H(2) = 1
H(3) = 3
H(4) = 1
H(t) = 0   for t >= 5
-------------------------------
```

A.4 Computing the Kähler differential Module $\Omega^m_{(R/(x_0))/K}$ and Its Hilbert Function

```
--------------------------------------------------------------------------------
-- KaehlerDiffModuleBarAndHF(IP,M): Compute the Kaehler differential
--          module of the algebra (R/(x_0))/K and its Hilbert function
-- Input:  IP = A non-zero homogeneous ideal of S or a list of points of the form
--          (1:a_1:...:a_n)
--          M = the number of form of Kaehler differential module
--          \Omega^M((R/(x_0))/K)
-- Output: A presentation and the Hilbert function of \Omega^M((R/(x_0))/K)
--------------------------------------------------------------------------------
Define KaehlerDiffModuleBarAndHF(IP,M);
    If Type(IP) = IDEAL Then I := GBasis(IP);
    Else I := GBasis(IdealOfProjectivePoints(IP));
    EndIf;
    N := Len(Indets());
    S ::= CoeffRing[x[1..N],y[1..(N-1)]], DegRevLex;
    Using S Do
        -- Form bases V1, V2 of \Omega^(M-1)(P/K[x_0]), \Omega^M(P/K[x_0])
        S1 := Subsets(1..(N-1),M-1); V1 := NewList(Len(S1),0);
        For I1 := 1 To Len(S1) Do
            V1[I1] := Product([y[S1[I1][J]] | J In 1..(M-1)]);
        EndFor;
        S2 := Subsets(1..(N-1),M); V2 := NewList(Len(S2),0);
        For I2 := 1 To Len(S2) Do
            V2[I2] := Product([y[S2[I2][J]] | J In 1..M ]);
        EndFor;
        -- Compute B2 = dI.\Omega^(M-1)(P/K[x_0]);
        H := NewList(Len(V1)*Len(I),0); Q := 1;
        LI1 := Indets(); LI2 := First(LI1, N); F1 := RMap(LI2);
```

```
NewI1 := Image(Ideal(I),F1);
NewI := GBasis(NewI1);
For W := 1 To Len(V1) Do
    C := Log(V1[W]);
    For K := 1 To Len(I) Do
        For J := 2 To N Do
            If Der(NewI[K],x[J]) <> 0 Then
                If GCD(V1[W],y[J-1]) <> y[J-1] Then
                    D := N; D1 := 0;
                    Repeat D := D+1;
                        If C[D] <> 0 Then D1 := D1+1; EndIf;
                    Until D = N+J-1;
                    H[Q] := H[Q]+(-1)^(D1)*Der(NewI[K],x[J])*y[J-1]*V1[W];
                EndIf;
            EndIf;
        EndFor;
        Q:=Q+1;
    EndFor;
EndFor;
B2 := [];
For J := 1 To Len(H) Do
    If H[J] <> 0 Then B2 := Concat(B2,[H[J]]); EndIf;
EndFor;
-- Compute B3 = I.\Omega^M(P/K[x_0]);
B3 := NewList(Len(V2)*Len(NewI));
Q := 1;
For W := 1 To Len(V2) Do
    For J := 1 To Len(NewI) Do
        B3[Q] := NewI[J]V2[W];
        Q := Q+1;
    EndFor;
EndFor;
-- Compute MM = B2+B3 in P^(C^M_{N-1}) and return the result
D := List(Identity(Len(V2)));
H := NewList(Len(B3),0D[1]);
G := NewList(Len(B2),0D[1]);
For J := 1 To Len(B3) Do
    X := Monomials(B3[J]);
    For Y := 1 To Len(X) Do
        For K := 1 To Len(V2) Do
            If GCD(X[Y],V2[K]) = V2[K] Then H[J] := H[J]+(X[Y]/V2[K])*D[K];
            EndIf;
        EndFor;
```

```
            EndFor;
        EndFor;
        For J := 1 To Len(B2) Do
            X := Monomials(B2[J]);
            For Y := 1 To Len(X) Do
                For K := 1 To Len(V2) Do
                    If GCD(X[Y],V2[K]) = V2[K] Then G[J] := G[J]+(X[Y]/V2[K])*D[K];
                    EndIf;
                EndFor;
            EndFor;
        EndFor;
        MM := Concat(G,H);
    EndUsing;
    L5 := Indets();
    L6 := ConcatLists([L5, [1 | J In 1..(N-1)]]);
    F := RMap(L6);
    NewM := Image(MM,F);  -- Image of MM in R=K[x_1,...,X-n]
    K1 := Len(NewM[1]); X0 := Ideal(x[0]);
    MT1 := X0*CurrentRing()^K1 + Module(NewM);
    LO1 := CurrentRing()^K1/Minimalized(MT1);
    LO2 := Hilbert(LO1);
    LO3 := RegularityIndex(LO2);
    LO4 := HilbertPoly(LO1);
    LO5 := NewList(LO3+M+1,0);
    PrintLn "A presentation of Omega^",M, "_((R/(x_0)/K):";
    PrintLn "Omega^",M, "_((R/(x_0)/K)(-",M,") = ",LO1;
    For J := M+1 To Len(LO5) Do
        LO5[J] := LO5[J]+ EvalHilbertFn(LO2,J-M-1);
    EndFor;
    PrintLn "The Hilbert function of Omega^",M, "_((R/(x_0))/K):";
    For J := 1 To Len(LO5)-1 Do
        PrintLn "H(",J-1,") = ",LO5[J]
    EndFor;
    Using QQt Do
        Print "H(t) = ", LO4, "   for t >= ", Len(LO5)-1
    EndUsing;
EndDefine;
```

Example A.4.1. Let $\mathbb{X}$ be the set of $\mathbb{Q}$-rational points $\mathbb{X} = \{P_1, \dots, P_6\} \subseteq \mathbb{P}^2$ given by $P_1 = (1 : 0 : 1)$, $P_2 = (1 : 0 : -1)$, $P_3 = (3 : 4 : 5)$, $P_4 = (3 : -4 : 5)$, $P_5 = (3 : -4 : -5)$, and $P_6 = (3 : 4 : -5)$, and let $m = 1, 2$. We apply `KaehlerDiffModuleBarAndHF(...)` to compute a presentation and the Hilbert function of $\Omega^m_{(R/(x_0))/\mathbb{Q}}$ of $(R/(x_0))/\mathbb{Q}$ as follows. We run the following commands in ApCoCoA:

```
Use QQ[x[0..2]];
PP :=[[1,0,1],[1,0,-1],[3,4,5],[3,-4,5],[3,-4,-5],[3,4,-5]];
KaehlerDiffModuleBarAndHF(PP,1);
KaehlerDiffModuleBarAndHF(PP,2);
```

The output of these commands is as follows:

```
A presentation of Omega^1_((R/(x_0)/K):
Omega^1_((R/(x_0)/K)(-1) = CurrentRingEnv^2/Module([[2x[1], -2x[2]], [0, x[0]],
[x[0], 0], [0, x[1]^2 - x[2]^2], [-x[2]^2, x[1]x[2]], [27/25x[2]^2, 0]])
The Hilbert function of Omega^1_((R/(x_0))/K):
H(0) = 0
H(1) = 2
H(2) = 3
H(3) = 1
H(t) = 0   for t >= 4
-------------------------------
A presentation of Omega^2_((R/(x_0)/K):
Omega^2_((R/(x_0)/K)(-2) = CurrentRingEnv^1/Module([[2x[1]], [2x[2]], [x[0]]])
The Hilbert function of Omega^2_((R/(x_0))/K):
H(0) = 0
H(1) = 0
H(2) = 1
H(t) = 0   for t >= 3
-------------------------------
```

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