Optimal quantization of probabilities concentrated on small balls

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Abstract

We consider probability distributions which are uniformly distributed on a disjoint union of balls with equal radius. For small enough radius the optimal quantization error is calculated explicitly in terms of the ball centroids. We apply the results to special self-similar measures.

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1 Introduction

Approximating a probability distribution by another discrete one with finite support, one can study the deviation in terms of the induced L_r-error and ask for an optimal approximation measure under the constraint of fixed (finite) support cardinality.

More exactly let $d \in \mathbb{N} = \{1, 2, ..\}$ and μ be a Borel probability distribution on $\mathbb{R}^d$. For $n \in \mathbb{N}$ and $r > 0$, we define the $n-$optimal quantization error

$$V_{n,r}(\mu) = \inf\{\int \min_{b \in \beta} \| x - b \|^r \, d\mu(x) : \beta \subset \mathbb{R}^d, \mathrm{card}(\beta) \leq n\},$$

where $\| \cdot \|$ is the Euclidean norm and card is the cardinality. A set $\alpha \subset \mathbb{R}^d$ consisting of at most n-points is called $n-$optimal (of order r) for the probability μ, if

$$V_{n,r}(\mu) = \int \min_{a \in \alpha} \| x - a \|^r \, d\mu(x).$$

The problem of optimal quantization is to determine for every $n \in \mathbb{N}$ all $n-$optimal sets, which are also called $n-$optimal codebooks, and to calculate the optimal quantization error $V_{n,r}(\mu)$.

Historically the problem of optimal quantization is mainly motivated from electrical engineering and information theory in connection with signal processing and data compression. It's history goes back to the 1940's. A good survey is

*this work is a part of the author's doctoral thesis (cf. [9])

the article of Gray and Neuhoff [6]. A comprehensive mathematical treatment of this problem was given by Graf and Luschgy [3],[4].

Despite of the difficulties in determing an explicit solution of the quantization problem, the asymptotic behaviour of $(V_{n,r}(\mu))_{n\in\mathbb{N}}$ could be described for large classes of probability distributions μ. Mainly the existence of the so-called quantization dimension and quantization coefficient was investigated by several authors (cf. [3],[5],[8],[11],[12],[14],[15]).

Only for a very few non-singular distributions, the optimal quantization error and the optimal codebooks can be determined exactly (cf. [3], Sections 4.4 and 5.2). A few years ago, progress was made for one-dimensional singular distributions. The quantization problem was solved for the classical self-similar Cantor distribution (cf. [2]) and later on for more generalized Cantor distributions, which are not necessarily self-similar (cf. [8],[10]). For $r = 2$ and probabilities which are uniformly distributed on a finite support of cardinality N, the problem of optimal quantization reduces to the calculation of the centroids of all appearing partitions of the support (cf. [3], p.35).

The main objective of this paper is to generalize this centroidal representation of the optimal quantization error for distributions with finite support to distributions which are uniformly supported on a collection of N disjoint balls. If the balls are small enough, this generalization is possible for singular and non-singular distributions in arbitrary finite dimension. If, additionally, the distribution concerned consists of identical parts on each ball (modulo translation), we can calculate the quantization error explicitly for $n = 1, .., N$. This is the main vantage of our approach and we will utilise it for special self-similar measures. Somewhat more precisely we investigate the quantization problem for measures, which are concentrated on a disjoint union of N closed balls $(B(x_i, l))_{i\in\{1,..,N\}}$ on $\mathbb{R}^d$ with equal radius $l > 0$ and midpoints $x_i \in \mathbb{R}^d$. We assume that μ is equidistributed on the balls, i.e.

$$\mu(B(x_i, l)) = \frac{1}{N} \text{ for every } i \in \{1, .., N\}. \tag{1}$$

For small enough radius $l > 0$ we derive a formula for the optimal quantization error $V_{n,2}(\mu)$ for all $n \in \{1, .., N\}$ in terms of the $\mu-$centroids on the balls and we give a characterization of the optimal codebooks (cf. Theorem 4.4). The main idea in our proofs is an approximation argument between μ and the equidistribution Q_ω on the finite set $\omega = \{x_1, .., x_N\}$.

The results will then be applied to self-similar measures, which are satisfying condition (1). If, additionally, the iterated function system, which generates the self-similar measure, does not contain any rotation part, we can calculate the ball centroids explicitly. Hence, we will get a formula for the optimal quantization error of these special singular distributions, which does not contain any $\mu-$integrals. (cf. Theorem 5.4). As special examples we briefly discuss the uniform distributions on modified versions of the Cantor sets, the Sierpinski gasket and the Cantor dust.

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2 Basic notions and results about optimal quantization

For the reader's convenience we briefly present in this chapter some well-known general facts about optimal quantization, which will be frequently used in the sequel. Let μ be a Borel probability distribution and B be a Borel-measurable subset of $\mathbb{R}^d$ with $\mu(B) > 0$. We define

$$s_\mu(B) = (\mu(B))^{-1} \int_B x d\mu(x) \tag{2}$$

as the $\mu-$centroid of B. Let $n \leq \text{card}(\text{supp}(\mu))$, where $\text{supp}(\mu)$ is the support of μ. For each $r \geq 1$, we denote by $C_{n,r}(\mu)$ the set of all $n-$optimal sets for μ of order r. For any finite nonempty set $\alpha \subset \mathbb{R}^d$ and $a \in \alpha$ let

$$W(a \mid \alpha) := \{x \in \mathbb{R}^d : \parallel x - a \parallel \; = \min_{b \in \alpha} \parallel x - b \parallel\} \tag{3}$$

be the Voronoi cell of a with respect to α. A bijective mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ is called similarity transformation if there exists $c \in]0, \infty[$, the scaling number, such that $\parallel Tx - Ty \parallel = c \parallel x - y \parallel$ for every $x, y \in \mathbb{R}^d$.

Theorem 2.1.
(1) There always exists an $n-$optimal set for μ of order r.
(2) Only one $1-$optimal set for μ of order 2 exists. It equals $\{s_\mu(\mathbb{R}^d)\}$.
(3) For any $\alpha \in C_{n,r}(\mu)$ and $\emptyset \neq \beta \subset \alpha$ with $\text{card}(\beta) = m \leq n$ we have

(a) $\text{card}(\alpha) = n$,

(b) if $r > 1$, then $\mu(W(a \mid \alpha) \cap W(b \mid \alpha)) = 0$ for every $a, b \in \alpha$ with $a \neq b$,

(c) $\mu(W(a \mid \alpha)) > 0$ *for every* $a \in \alpha$,

(d) $\beta \in C_{m,r}(\mu(\cdot \mid \bigcup_{b \in \beta} W(b \mid \alpha)))$.

(4) For a similarity transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ with scaling number $c > 0$ we have $C_{n,r}(\mu \circ T^{-1}) = TC_{n,r}(\mu)$ resp. $V_{n,r}(\mu \circ T^{-1}) = c^r V_{n,r}(\mu)$.

Proof. For a proof of (1) see [3], Theorem 4.12. A proof of (3a),(3c) and (3d) can be found in [3], Theorem 4.1. The assertion (2) follows from [3], Theorem 2.4 (i) and [3], Example 2.3 (b). From [3], Theorem 4.2 we deduce (3b). The assertion (4) is an easy consequence of the definition, but also stated as Lemma 3.2 in [3]. □

Denote $< \cdot, \cdot >$ as the inner product on $\mathbb{R}^d$. The following two results are quite simple but useful in later chapters.

Lemma 2.2. *Let $a = s_\mu(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$. Then*

$$\int \| x - z \|^2 \, d\mu(x) = \int \| x - a \|^2 \, d\mu(x) + \| a - z \|^2 \, .$$

Proof. Obviously we have

$$\| x - z \|^2 \; = \; \| x - a \|^2 + \| a - z \|^2 + 2 < x - a, a - z > .$$

Integration with μ and the linearity of $< \cdot, \cdot >$ are yielding the assertion. □

Corollary 2.3. *Let B be a Borel measurable subset of $\mathbb{R}^d$ with $\mu(B) > 0$. If $a = s_\mu(B)$, then*

$$\int_B \| x - z \|^2 \, d\mu(x) = \int_B \| x - a \|^2 \, d\mu(x) + \mu(B) \| a - z \|^2 \, .$$

If we approximate μ by another probability distribution, the optimal quantization problem will also be approximated. To state the exact result for the optimal codebooks, we first have to define the distance between two probability measures in terms of the so called Wasserstein-Kantorovich-distance. Let $\mathfrak{M}_r = \mathfrak{M}_r(\mathbb{R}^d)$ be the space of all Borel probability measures ν on $\mathbb{R}^d$ with $\int \| x \|^r \, d\nu(x) < \infty$ and $\nu_1, \nu_2 \in \mathfrak{M}_r$. Then

$$\rho_r(\nu_1, \nu_2) = \inf_\lambda \left(\int \| x - y \|^r \, d\lambda(x, y) \right)^{1/r}$$

is called the Wasserstein-Kantorovich (or $\mathbf{L_r}-$minimal) distance between ν_1 and ν_2. The infimum is taken over all Borel probability measures λ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginal measures ν_1 and ν_2. Let $U, V \subset \mathbb{R}^d$ be arbitrary sets. We denote

$$d_H(U, V) = \max \left\{ \max_{u \in U} \min_{v \in V} \| u - v \|, \max_{v \in V} \min_{u \in U} \| u - v \| \right\}$$

as the Hausdorff distance between U and V.

Proposition 2.4. *Let $\mu \in \mathfrak{M}_r$ and $n \leq \mathrm{card}(\mathrm{supp}(\mu))$. For every $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: for every $\nu \in \mathfrak{M}_r$ with $\rho_r(\mu, \nu) < \delta$ and every $\alpha \in C_{n,r}(\mu)$, there exists $\beta \in C_{n,r}(\nu)$ such that $d_H(\alpha, \beta) < \varepsilon$.*

Proof. See [3], Theorem 4.21 (b). □

Essential is the following characterization.

Proposition 2.5. *Let $\mu \in \mathfrak{M}_r$ and $\mu_k \in \mathfrak{M}_r$ for every $k \in \mathbb{N}$. Then*

$$\lim_{k \to \infty} \rho_r(\mu_k, \mu) = 0,$$

if and only if μ_k converges weakly to μ and

$$\int \| x \|^r \, d\mu_k(x) \to \int \| x \|^r \, d\mu(x).$$

Proof. See Theorem 2.6.4 in [13]. □

3 Optimal quantization of distributions with finite support

At first in this section we briefly discuss the optimal quantization problem of distributions with finite support. In this situation, optimal quantization is reduced to an optimal partitioning problem for the support (Remark 3.2). If $r = 2$, the optimal quantization error can be calculated in terms of the centroids of an optimal partition (Proposition 3.1). Moreover we will prove, that an optimal partition of the finite support, which is induced by an appropriate optimal set, will also be generated by another set with the same cardinality, if the Hausdorff distance between the two sets is small enough (Lemma 3.5).

Let $N \geq 2$ and $\omega = \{x_1, .., x_N\} \subset \mathbb{R}^d$ be a set consisting of N different points. We denote Q_ω as the equidistribution on ω, i.e.

$$Q_\omega = \frac{1}{N} \sum_{x \in \omega} \delta_x,$$

if δ_x denotes the Dirac measure on x. For $n \in \{1, .., N\}$ let $\mathcal{Z}_n$ be the set of all partitions of ω consisting of $n-$elements. For any non-empty finite set $\alpha \subset \mathbb{R}^d$ and $a \in \alpha$ we write $\omega(a \mid \alpha) = \omega \cap W(a \mid \alpha)$.

Proposition 3.1. *Let* $n \in \{1, .., N\}$. *Then*

$$V_{n,2}(Q_\omega) = \frac{1}{N} \min_{Z \in \mathcal{Z}_n} \sum_{\gamma \in Z} \sum_{x \in \gamma} \| x - s_{Q_\omega}(\gamma) \|^2 . \tag{4}$$

Proof. From [3], Example 3.5 we obtain

$$\begin{aligned} V_{n,2}(Q_\omega) &= \frac{1}{N} \min_{Z \in \bigcup_{k=1}^{n} \mathcal{Z}_k} \sum_{\gamma \in Z} \sum_{x \in \gamma} \| x - s_{Q_\omega}(\gamma) \|^2 \\ &\leq \frac{1}{N} \min_{Z \in \mathcal{Z}_n} \sum_{\gamma \in Z} \sum_{x \in \gamma} \| x - s_{Q_\omega}(\gamma) \|^2 . \end{aligned} \tag{5}$$

On the other hand let α be an $n-$optimal set for Q_ω, i.e. $\alpha \in C_{n,2}(Q_\omega)$. From Theorem 2.1 (3) we deduce, that $\{\omega(a \mid \alpha) : a \in \alpha\}$ is a $n-$partition of ω. Moreover due to Theorem 2.1 (2) and (3) we have $a = s_{Q_\omega}(\omega(a \mid \alpha))$ for any a in α. Hence,

$$\begin{aligned} V_{n,2}(Q_\omega) &= \int \min_{a \in \alpha} \| x - a \|^2 \, dQ_\omega(x) \\ &= \frac{1}{N} \sum_{a \in \alpha} \sum_{x \in \omega(a|\alpha)} \| x - s_{Q_\omega}(\omega(a \mid \alpha)) \|^2 \\ &\geq \frac{1}{N} \min_{Z \in \mathcal{Z}_n} \sum_{\gamma \in Z} \sum_{x \in \gamma} \| x - s_{Q_\omega}(\gamma) \|^2 . \end{aligned} \tag{6}$$

The combination of (5) and (6) yields the assertion. □

Let $n \in \{1, .., N\}$ and define

$$\mathcal{Z}_n^* = \left\{ Z \in \mathcal{Z}_n : V_{n,2}(Q_\omega) = \frac{1}{N} \sum_{\gamma \in Z} \sum_{x \in \gamma} \| x - s_{Q_\omega}(\gamma) \|^2 \right\}$$

as the set of all $n-$optimal partitions of ω. For any $Z \in \mathcal{Z}_n$ denote $s(Z) = \{s_{Q_\omega}(\gamma) : \gamma \in Z\}$ as the set of all Q_ω-centroids induced by the partition Z.

Remark 3.2. *For any $n \in \{1, .., N\}$ Proposition 3.1 reduces the calculation of $V_{n,2}(Q_\omega)$ to the analysis of all possible $n-$partitions of ω. The proof of Proposition 3.1 also shows, that every n-optimal set of Q_ω is generated by the Q_ω-centroids of an appropriate $n-$optimal partition, i.e.*

$$C_{n,2}(Q_\omega) = \{s(Z) : Z \in \mathcal{Z}_n^*\}.$$

On the other hand , every $n-$optimal partition is induced by an $n-$optimal set, i.e.

$$\mathcal{Z}_n^* = \{\{\omega(a \mid \alpha) : a \in \alpha\} : \alpha \in C_{n,2}(Q_\omega)\}.$$

Even stronger, if $\alpha \in C_{n,2}(Q_\omega)$ and $Z = \{\omega(a \mid \alpha) : a \in \alpha\}$ we have $\alpha = s(Z)$. If $Z \in \mathcal{Z}_n^*$, then $Z = \{\omega(a \mid s(Z)) : a \in s(Z)\}$.

From Theorem 2.1 (4) we know, how the quantization error scales under a similarity transformation. The following lemma does preserve this result for the discrete distribution Q_ω, if the transformation is defined only on ω instead of $\mathbb{R}^d$. We will need this result in Section 5.

Lemma 3.3. *Let $c > 0$ and $f : \omega \to \mathbb{R}^d$ with*

$$\| f(x) - f(y) \| = c \, \| x - y \|$$

for every $x, y \in \omega$. If $n \in \{1, .., N\}$, then

$$V_{n,r}(Q_{f(\omega)}) = c^2 V_{n,r}(Q_\omega).$$

Proof. Immediate consequence of Proposition 3.1. □

Remark 3.4. *Obviously the set $f(\mathcal{Z}_n^*)$ is identical with the set of all optimal $n-$partitions of $f(\omega)$.*

For a set $B \subset \mathbb{R}^d$ let $\overset{\circ}{B}$ be the interior of B. Let $\beta \subset \mathbb{R}^d$ be finite and consisting of more than one point. Let $\alpha \subset \mathbb{R}^d$ with $\text{card}(\alpha) = \text{card}(\beta)$. We define

$$d_{\min}(\beta) = \min\{\| x - y \| : x, y \in \beta; x \neq y\}$$

as the minimal appearing distance in β and $\rho = \min(d_{\min}(\alpha), d_{\min}(\beta))/2$. If

$$\alpha \cap \overset{\circ}{B}(b, \rho) \neq \emptyset \text{ for all } b \in \beta \tag{7}$$

and

$$\beta \cap \overset{\circ}{B}(a,\rho) \neq \emptyset \text{ for all } a \in \alpha, \tag{8}$$

we can define a bijection $G_{\alpha,\beta}$ from β to α by taking $G_{\alpha,\beta}(b)$ as a unique element of $\alpha \cap \overset{\circ}{B}(b,\rho)$ for every $b \in \beta$.

The conditions (7) and (8) are satisified, if

$$d_H(\alpha,\beta) < \rho = \frac{1}{2}\min(d_{\min}(\alpha), d_{\min}(\beta)). \tag{9}$$

If $d_H(\alpha,\beta) < d_{\min}(\beta)/2$, it is easy to see, that

$$\begin{aligned}\frac{1}{2}\min(d_{\min}(\alpha), d_{\min}(\beta)) &\geq \frac{1}{2}\min(d_{\min}(\beta) - 2d_H(\alpha,\beta), d_{\min}(\beta)) \\ &= \frac{1}{2}d_{\min}(\beta) - d_H(\alpha,\beta).\end{aligned}$$

As a consequence, if $d_H(\alpha,\beta) < d_{\min}(\beta)/4$ holds, then (9) is satisified and therefore (7) and (8) are also satisfied.

For an arbitrary set $U \subset \mathbb{R}^d$ we denote its boundary by ∂U. For any $x \in \mathbb{R}^d$ and $l > 0$ let $B(x,l) := \{z \in \mathbb{R}^d : \parallel z \parallel \leq l\}$ be the closed ball with radius l and midpoint x.

Lemma 3.5. *There exists a $\delta \in\ ]0, d_{\min}(\omega)/2]$, such that for every $n \in \{1,..,N\}$, every $\beta \in C_{n,2}(Q_\omega)$ and every $\alpha \subset \mathbb{R}^d$ with* $\mathrm{card}(\alpha) = n$ *and $d_H(\alpha,\beta) < \delta$ the relation*

$$\begin{aligned}\forall\, b \in \beta : \omega(b \mid \beta) &= \{x \in \omega : B(x,\delta) \subset W(b \mid \beta)\} \\ &= \{x \in \omega : B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\} = \omega(G_{\alpha,\beta}(b) \mid \alpha)\end{aligned}$$

hold.

Proof.
1. We determine $\delta \in]0, d_{\min}(\omega)/2]$.

By taking the minimum over $\{1,..,N\}$ it suffices to prove the assertion for one arbitrary $n \in \{1,..,N\}$. Let $n \in \{1,..,N\}$ and $\beta \in C_{n,2}(Q_\omega)$. From Theorem 2.1 (3b) we obtain $Q_\omega(\bigcup_{b\in\beta} \partial W(b \mid \beta)) = 0$, which yields

$$\omega \cap \bigcup_{b\in\beta} \partial W(b \mid \beta) = \emptyset. \tag{10}$$

According to the definition (3) of a Voronoi cell and the identity (10) we obtain

$$\parallel x - t' \parallel - \parallel x - t \parallel > 0$$

for every $t, t' \in \beta$ with $t \neq t'$ and $x \in \omega(t \mid \beta)$. Because β and ω are finite, we get

$$H(\omega,\beta) := \min_{t\in\beta} \min_{t'\in\beta\setminus\{t\}} \min_{x\in\omega(t|\beta)} (\parallel x - t' \parallel - \parallel x - t \parallel) > 0. \tag{11}$$

Let $\gamma \subset \mathbb{R}^d$ with $\mathrm{card}(\gamma) = n$ and $d_H(\gamma, \beta) < \min(H(\omega, \beta)/2, d_{\min}(\beta)/4)$. Next we will show, that for every $t \in \beta$ the relation

$$\omega(t \mid \beta) = \omega(G_{\gamma,\beta}(t) \mid \gamma) \tag{12}$$

holds. Let $t \in \beta$ and $x \in \omega(t \mid \beta)$. Let $t' \in \beta \backslash \{t\}$. Then we have

$$\begin{aligned}
& \| x - G_{\gamma,\beta}(t') \| - \| x - G_{\gamma,\beta}(t) \| \\
\geq \; & \| x - t' \| - \| t' - G_{\gamma,\beta}(t') \| - (\| x - t \| + \| t - G_{\gamma,\beta}(t) \|) \\
\geq \; & \| x - t' \| - \| x - t \| - 2d_H(\gamma, \beta) \\
\geq \; & H(\omega, \beta) - 2d_H(\gamma, \beta) > 0.
\end{aligned}$$

The bijectivity of $G_{\gamma,\beta}$ ensures $G_{\gamma,\beta}(\beta \backslash \{t\}) = \gamma \backslash \{G_{\gamma,\beta}(t)\}$. Hence, we obtain for all $a' \in \gamma \backslash \{G_{\gamma,\beta}(t)\}$, that

$$\| x - a' \| - \| x - G_{\gamma,\beta}(t) \| > 0,$$

which yields $x \in \omega(G_{\gamma,\beta}(t) \mid \gamma)$.

On the other hand, let $x \in \omega(G_{\gamma,\beta}(t) \mid \gamma)$. By similar arguments one gets $x \in \omega(t \mid \beta)$. Thus, the equality (12) is proved.

Now we will show, that for every $\varepsilon > 0$ and every $\gamma \subset \mathbb{R}^d$ with $\mathrm{card}(\gamma) = n$ under the condition

$$d_H(\gamma, \beta) < \min\left(\frac{1}{2} H(\omega, \beta), \frac{1}{4} d_{\min}(\beta), \frac{\varepsilon}{2}\right)$$

the relation

$$| H(\omega, \beta) - H(\omega, \gamma) | \leq \varepsilon \tag{13}$$

holds. The definition (11) and the identity (12) lead to

$$\begin{aligned}
H(\omega, \beta) - H(\omega, \gamma) \; = \; & \min_{t \in \beta} \min_{t' \in \beta \backslash \{t\}} \min_{x \in \omega(t|\beta)} (\| x - t' \| - \| x - t \|) \\
- \; & \min_{a \in \gamma} \min_{a' \in \gamma \backslash \{a\}} \min_{x \in \omega(a|\gamma)} (\| x - a' \| - \| x - a \|) \\
= \; & \min_{t \in \beta} \min_{t' \in \beta \backslash \{t\}} \min_{x \in \omega(G_{\gamma,\beta}(t)|\gamma)} (\| x - t' \| - \| x - t \|) \\
- \; & \min_{a \in \gamma} \min_{a' \in \gamma \backslash \{a\}} \min_{x \in \omega(a|\gamma)} (\| x - a' \| - \| x - a \|).
\end{aligned}$$

Because $G_{\gamma,\beta}$ is bijective, we obtain

$$\begin{aligned}
H(\omega, \beta) - H(\omega, \gamma) \; = \; & \min_{b \in \gamma} \min_{b' \in \gamma \backslash \{b\}} \min_{x \in \omega(b|\gamma)} (\| x - G_{\beta,\gamma}(b') \| - \| x - G_{\beta,\gamma}(b) \|) \\
- \; & \min_{a \in \gamma} \min_{a' \in \gamma \backslash \{a\}} \min_{x \in \omega(a|\gamma)} (\| x - a' \| - \| x - a \|) \\
\geq \; & \min_{b \in \gamma} \min_{b' \in \gamma \backslash \{b\}} \min_{x \in \omega(b|\gamma)} (\| x - b' \| - \| x - b \| - 2d_H(\gamma, \beta)) \\
- \; & \min_{a \in \gamma} \min_{a' \in \gamma \backslash \{a\}} \min_{x \in \omega(a|\gamma)} (\| x - a' \| - \| x - a \|) \\
= \; & -2d_H(\gamma, \beta).
\end{aligned}$$

In the same way one can show, that $H(\omega,\beta) - H(\omega,\gamma) \leq 2d_H(\gamma,\beta)$, which implies (13). As an immediate consequence of (13) we derive the d_H-continuity of the mapping $H(\omega,\cdot)$ in β. Therefore a $\delta_1(\beta) > 0$ exists such that for all $\delta \leq \delta_1(\beta)$ and for all $\gamma \subset \mathbb{R}^d$ with $\mathrm{card}(\gamma) = n$ and $d_H(\beta,\gamma) < \delta$ the relation

$$H(\omega,\gamma) > 4\delta_1(\beta) \geq 4\delta > 0 \tag{14}$$

hold.

Note, that $C_{n,2}(Q_\omega)$ consists of finitely many partitions of ω (cf. Remark 3.2). Hence the inequality (14) still holds if we exchange $\delta_1(\beta)$ by the positive value

$$\delta_1' = \min\{\delta_1(\beta') : \beta' \in C_{n,2}(Q_\omega)\}.$$

Let $\delta_2 = \min\{d_{\min}(\beta')/4 : \beta' \in C_{n,2}(Q_\omega)\}$ and

$$\delta \in \left]0, \min\left(\frac{d_{\min}(\omega)}{2}, \delta_2, \delta_1'\right)\right]$$

be chosen independently of β. Let $\alpha \subset \mathbb{R}^d$ such that $\mathrm{card}(\alpha) = n$ and $d_H(\alpha,\beta) < \delta$. Let $b \in \beta$. To show the assertion of Lemma 3.5, we divide the rest of the proof into several steps.

2. We will show that $\omega(b \mid \beta) \subset \{x \in \omega : B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\}$.

Let $x \in \omega(b \mid \beta)$ and $z \in B(x,\delta)$. It holds that

$$\begin{aligned}
&\min_{a\in\alpha\setminus\{G_{\alpha,\beta}(b)\}} (\| z-a \| - \| z - G_{\alpha,\beta}(b) \|) \\
= &\min_{a\in\alpha\setminus\{G_{\alpha,\beta}(b)\}} (\| z-x+x-G_{\beta,\alpha}(a)+G_{\beta,\alpha}(a)-a \| - \| z-x+x-b+b-G_{\alpha,\beta}(b) \|) \\
\geq &\min_{a\in\alpha\setminus\{G_{\alpha,\beta}(b)\}} (\| x - G_{\beta,\alpha}(a) \| - \| z-x \| - \| G_{\beta,\alpha}(a) - a \| \\
&\quad - \| x-b \| - \| z-x \| - \| b - G_{\alpha,\beta}(b) \|) \\
\geq &\min_{a\in\alpha\setminus\{G_{\alpha,\beta}(b)\}} (\| x - G_{\beta,\alpha}(a) \| - \| x-b \|) - 2\delta - 2d_H(\alpha,\beta) \\
> &\min_{a\in\alpha\setminus\{G_{\alpha,\beta}(b)\}} (\| x - G_{\beta,\alpha}(a) \| - \| x-b \|) - 4\delta.
\end{aligned}$$

Due to $G_{\beta,\alpha} = G_{\alpha,\beta}^{-1}$ and (14) we have

$$\begin{aligned}
&\min_{a\in\alpha\setminus\{G_{\alpha,\beta}(b)\}} (\| x - G_{\beta,\alpha}(a) \| - \| x-b \|) - 4\delta \\
= &\min_{b'\in\beta\setminus\{b\}} (\| x-b' \| - \| x-b \|) - 4\delta \\
\geq &\; H(\omega,\beta) - 4\delta > 0,
\end{aligned}$$

which yields $z \in W(G_{\alpha,\beta}(b) \mid \alpha)$, resp. $B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)$.

3. We will show that $\omega(b \mid \beta) \supset \{x \in \omega : B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\}$.
Let $x \in \omega$ and assume that $B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)$.
Hence $x \in W(G_{\alpha,\beta}(b) \mid \alpha)$ and we get

$$\min_{b' \in \beta \setminus \{b\}} (\| x - b' \| - \| x - b \|)$$

$$= \min_{b' \in \beta \setminus \{b\}} (\| x - G_{\alpha,\beta}(b') + G_{\alpha,\beta}(b') - b' \| - \| x - G_{\alpha,\beta}(b) + G_{\alpha,\beta}(b) - b \|)$$

$$\begin{aligned} &> \min_{b' \in \beta \setminus \{b\}} (\| x - G_{\alpha,\beta}(b') \| - \| x - G_{\alpha,\beta}(b) \|) - 2\delta \\ &= \min_{a \in \alpha \setminus \{G_{\alpha,\beta}(b)\}} (\| x - a \| - \| x - G_{\alpha,\beta}(b) \|) - 2\delta \\ &\geq H(\omega,\alpha) - 2\delta > 4\delta - 2\delta > 0, \end{aligned}$$

which implies $x \in W(b \mid \beta)$.

4. We will verify the identity $\omega(b \mid \beta) = \{x \in \omega : B(x,\delta) \subset W(b \mid \beta)\}$.
Note, that $d_H(\beta,\beta) = 0 < \delta$. Hence, the identity of Step 4 is an immediate consequence of Step 1 to Step 3.

5. We will prove, that $\{x \in \omega : B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\} = \omega(G_{\alpha,\beta}(b) \mid \alpha)$.
It is obvious, that

$$\{x \in \omega : B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\} \subset \omega(G_{\alpha,\beta}(b) \mid \alpha). \tag{15}$$

Let $x \in \omega(G_{\alpha,\beta}(b) \mid \alpha)$ and $z \in B(x,\delta)$. We deduce

$$\begin{aligned} &\min_{a \in \alpha \setminus \{G_{\alpha,\beta}(b)\}} (\| z - a \| - \| z - G_{\alpha,\beta}(b) \|) \\ = &\min_{a \in \alpha \setminus \{G_{\alpha,\beta}(b)\}} (\| z - x + x - a \| - \| z - x + x - G_{\alpha,\beta}(b) \|) \\ \geq &\min_{a \in \alpha \setminus \{G_{\alpha,\beta}(b)\}} (\| x - a \| - \| x - G_{\alpha,\beta}(b) \|) - 2\delta \\ \geq &\; H(\omega,\alpha) - 2\delta \geq 4\delta - 2\delta > 0, \end{aligned}$$

which implies

$$\omega(G_{\alpha,\beta}(b) \mid \alpha) \subset \{x \in \omega \mid B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\}. \tag{16}$$

The combination of (15) and (16) shows the assertion of Step 5.
Hence Step 1 to Step 5 are proving the assertion of Lemma 3.5. □

4 The quantization error for ball-separated measures

Based on the results from the previous section we intend in this section to derive a formula similar to (4), which decomposes the quantization error by an optimal partition and the according centroidal sums (see Theorem 4.4). The main idea is an approximation argument between the discrete set ω and the ball collection $(B(x_i,l))_{i\in\{1,..,N\}}$ whose union contains the support of the Borel distribution μ.

Recall $\omega=\{x_1,..,x_N\}\subset\mathbb{R}^d$ and consider the collection $(B(x_i,l))_{i\in\{1,..,N\}}$ of N pairwise disjoint closed balls on $\mathbb{R}^d$ with equal radius $l>0$. The distribution μ is called $(l,\omega)-$separated, if relation (1) holds. Obviously Q_ω is $(l,\omega)-$separated for every $l\in]0,d_{\min}(\omega)/2[$. Let $Z\in\mathcal{Z}_n$ and recall $s(Z)=\{s_{Q_\omega}(\gamma):\gamma\in Z\}$ as the set of all Q_ω-centroids induced by the partition Z.

Lemma 4.1. *There exists a $l_0\in]0,d_{\min}(\omega)/2]$, such that for every $n\in\{1,..,N\}$, every $l<l_0$ and every (l,ω)-separated probability measure ν the following hold:*

(a) for every $Z\in\mathcal{Z}_n^*$ and $\gamma\in Z$ we have

$$\gamma=\{x\in\omega:B(x,l)\subset W(s_{Q_\omega}(\gamma)\mid s(Z))\}=\omega(s_{Q_\omega}(\gamma)\mid s(Z))$$

(b) for every $n-$optimal set $\alpha\in C_{n,2}(\nu)$ there exists an $n-$optimal partition $Z\in\mathcal{Z}_n^*$ coinciding with the partition induced by α, i.e.

$$Z=\{\omega(s_{Q_\omega}(\gamma)\mid s(Z)):\gamma\in Z\}=\{\omega(a\mid\alpha):a\in\alpha\}.$$

Moreover

$$\begin{aligned}Z&=\{\{x\in\omega:B(x,l)\subset W(a\mid\alpha)\}:a\in\alpha\}\\&=\{\{x\in\omega:B(x,l)\subset W(s_{Q_\omega}(\gamma)\mid s(Z))\}:\gamma\in Z\}.\end{aligned}$$

Proof. As a consequence of Proposition 2.5 one derives that for all $\varepsilon>0$ an $l_1>0$ exists such that for all $l\le l_1$ and for all $(l,\omega)-$separated measures ν the inequality

$$\rho_2(\nu,Q_\omega)<\varepsilon \tag{17}$$

holds. From (17) and Proposition 2.4 we get for every $\varepsilon>0$ the existence of $l_1>0$ satisfying the following: for any $l\le l_1$, $n\in\{1,..,N\}$ and $\alpha\in C_{n,2}(\nu)$, there exists $\beta\in C_{n,2}(Q_\omega)$ such that

$$d_H(\alpha,\beta)<\varepsilon. \tag{18}$$

Now fix $\delta\in]0,d_{\min}(\omega)/2]$ according to Lemma 3.5 and let $n\in\{1,..,N\}$. We take l_1 so that (18) holds with $\varepsilon=\delta$. We define $l_0:=\min(l_1,\delta)$ and choose $l<l_0$. Let ν be a $(l,\omega)-$separated measure.

(a) Let $Z\in\mathcal{Z}_n^*$ and $\gamma\in Z$. Remark 3.2 and Lemma 3.5 are yielding

$$\begin{aligned}\gamma&=\omega(s_{Q_\omega}(\gamma)\mid s(Z))\\&=\{x\in\omega:B(x,\delta)\subset W(s_{Q_\omega}(\gamma)\mid s(Z))\}.\end{aligned}$$

If we substitute $B(x,l)$ for $B(x,\delta)$, the part (a) of the assertion follows.

(b) Let $\alpha \in C_{n,2}(\nu)$. We take β so that (18) holds with $\varepsilon = \delta$. Lemma 3.5 guarantees for all $b \in \beta$, that

$$\begin{aligned} \omega(G_{\alpha,\beta}(b) \mid \alpha) &= \{x \in \omega : B(x,\delta) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\} \\ &\subset \{x \in \omega : B(x,l) \subset W(G_{\alpha,\beta}(b) \mid \alpha)\} \\ &\subset \omega(G_{\alpha,\beta}(b) \mid \alpha). \end{aligned} \tag{19}$$

Moreover Lemma 3.5 implies that

$$\begin{aligned} \omega(b \mid \beta) &= \{x \in \omega : B(x,\delta) \subset W(b \mid \beta)\} \\ &\subset \{x \in \omega : B(x,l) \subset W(b \mid \beta)\} \subset \omega(b \mid \beta) \end{aligned} \tag{20}$$

and

$$\omega(G_{\alpha,\beta}(b) \mid \alpha) = \omega(b \mid \beta). \tag{21}$$

Now we define the partition $Z = \{\omega(b \mid \beta) : b \in \beta\}$. From Remark 3.2 we deduce $Z \in \mathcal{Z}_n^*$. Applying Remark 3.2 and Theorem 2.1 (2) we obtain

$$\beta = s(Z). \tag{22}$$

From the definition of Z and (22), it follows that

$$Z = \{\omega(b \mid \beta) : b \in \beta\} = \{\omega(s_{Q_\omega}(\gamma) \mid s(Z)) : \gamma \in Z\}.$$

The identity (21) and the bijectivity of $G_{\alpha,\beta}$ yield

$$Z = \{\omega(b \mid \beta) : b \in \beta\} = \{\omega(G_{\alpha,\beta}(b) \mid \alpha) : b \in \beta\} = \{\omega(a \mid \alpha) : a \in \alpha\}.$$

Hence we get from (19) and (20) the relation

$$\begin{aligned} Z &= \{\{x \in \omega : B(x,l) \subset W(a \mid \alpha)\} : a \in \alpha\} \\ &= \{\{x \in \omega : B(x,l) \subset W(s_{Q_\omega}(\gamma) \mid s(Z))\} : \gamma \in Z\}, \end{aligned}$$

which proves the part (b) of the assertion. □

One could expect in general the existence of a lower bound for l_0 in Lemma 4.1. E.g. it could be conjectured the existence of $\xi > 0$ independent of ω such that $l_0 > \xi \cdot d_{\min}(\omega)$. The following example shows that this is unfortunately not true.

Example 4.2. *Let $q \in \,]0, 1/4[$ and*

$$\omega = \{(0,0); (1,0); (1-q,1); (q,1)\}.$$

Applying Proposition 3.1 and Remark 3.2 one recognizes that

$$C_{3,2}(Q_\omega) = \{\{(0,0); (1,0); (\frac{1}{2}, 1)\}\}.$$

Now let $l \in [0, 2q]$ and

$$\kappa = \{(0,0); (1,0); (1-q+l,1); (q-l,1)\}.$$

The equidistribution Q_κ is a $(l,\omega)-$separated measure.
If $l < q$, then $C_{3,2}(Q_\kappa) = C_{3,2}(Q_\omega)$. If $l > q$, then

$$C_{3,2}(Q_\kappa) = \{\{(q-l,1);(1-q+l,1);(\frac{1}{2},0)\}\}.$$

In case of $l = q$ we have

$$C_{3,2}(Q_\kappa) = \left\{\left\{\frac{x+y}{2}\right\} \cup \kappa - \{x,y\} : x,y \in \kappa, \| x-y \| = 1\right\}.$$

Obviously the statements (a) and (b) in Lemma 4.1 are becoming wrong, if we set $\nu = Q_\kappa$ and $l \geq q$. They are true if $l < q$. Because $d_{\min}(\omega) = 1-2q > 1/4 > q$, and q could be chosen arbitrary small, it is not possible to fix a $\xi > 0$, which is independent of ω, such that the relation $l_0 > \xi \cdot d_{\min}(\omega)$ would hold.

As a last auxiliary result in this section we need a partition formula for the quantization error in case of $n = 1$.

Lemma 4.3. *Let $l \in]0, d_{\min}(\omega)/2[$ and $\emptyset \neq \kappa \subset \omega$. Let ν be a (l,ω)-separated measure and $n = \operatorname{card}(\kappa)$. Then*

$$V_{1,2}\left(\nu\left(\cdot \mid \bigcup_{x\in\kappa} B(x,l)\right)\right)$$

$$= \frac{1}{n}\sum_{x\in\kappa}\left(\| s_\nu(B(x,l)) - \frac{1}{n}\sum_{y\in\kappa} s_\nu(B(y,l)) \|^2 + V_{1,2}(\nu(\cdot \mid B(x,l)))\right).$$

Proof. Using Theorem 2.1 (2) one gets

$$\{s_\nu(\bigcup_{x\in\kappa} B(x,l))\} \in C_{1,2}(\nu(\cdot \mid \bigcup_{x\in\kappa} B(x,l))),$$

which yields

$$V_{1,2}(\nu(\cdot \mid \bigcup_{x\in\kappa} B(x,l))) = \frac{N}{n}\sum_{x\in\kappa}\int_{B(x,l)} \| z - s_\nu(\bigcup_{y\in\kappa} B(y,l)) \|^2 \, d\nu(z).$$

The application of Corollary 2.3 implies

$$\begin{aligned} & V_{1,2}(\nu(\cdot \mid \bigcup_{x\in\kappa} B(x,l))) \\ = \; & \frac{N}{n}\sum_{x\in\kappa}\left(\frac{1}{N} \| s_\nu(\bigcup_{y\in\kappa} B(y,l)) - s_\nu(B(x,l)) \|^2 \right. \\ & \left. + \int_{B(x,l)} \| z - s_\nu(B(x,l)) \|^2 \, d\nu(z)\right) \\ = \; & \frac{1}{n}\sum_{x\in\kappa}\left(\| s_\nu(\bigcup_{y\in\kappa} B(y,l)) - s_\nu(B(x,l)) \|^2 + V_{1,2}(\nu(\cdot \mid B(x,l)))\right). \end{aligned}$$

From the definition of s_ν we obtain

$$\begin{aligned} s_\nu(\bigcup_{y\in\kappa} B(y,l)) &= \left(\nu(\bigcup_{y\in\kappa} B(y,l))\right)^{-1} \int_{\bigcup_{y\in\kappa} B(y,l)} z d\nu(z) \\ &= \frac{N}{n}\sum_{y\in\kappa}\int_{B(y,l)} z d\nu(z) \\ &= \frac{1}{n}\sum_{y\in\kappa} s_\nu(B(y,l)). \end{aligned}$$

As a consequence we deduce

$$V_{1,2}(\nu(\cdot \mid \bigcup_{x\in\kappa} B(x,l)))$$

$$= \frac{1}{n}\sum_{x\in\kappa}\left(\| \frac{1}{n}\sum_{y\in\kappa} s_\nu(B(y,l)) - s_\nu(B(x,l)) \|^2 + V_{1,2}(\nu(\cdot \mid B(x,l)))\right).$$

□

Now we can state and prove the main result in this section.

Theorem 4.4. *There exists a $l_0 \in]0, d_{\min}(\omega)/2]$, such that for every $n \in \{1,..,N\}$, every $l < l_0$ and $(l,\omega)-$separated probability distribution ν the identity*

$$V_{n,2}(\nu) = \frac{1}{N}\sum_{x\in\omega} V_{1,2}(\nu(\cdot \mid B(x,l))) +$$

$$\min_{Z\in\mathcal{Z}_n^*} \frac{1}{N}\sum_{\gamma\in Z}\sum_{x\in\gamma} \| s_\nu(B(x,l)) - \left(\frac{1}{\operatorname{card}(\gamma)}\sum_{y\in\gamma} s_\nu(B(y,l))\right) \|^2 \tag{23}$$

hold. Moreover for every $n-$optimal set $\alpha \in C_{n,2}(\nu)$ an $n-$optimal partition $Z \in \mathcal{Z}_n^*$ of ω exists, which induces α, i.e.

$$C_{n,2}(\nu) \subset \{\{s_\nu(\bigcup_{x\in\gamma} B(x,l)) : \gamma \in Z\} : Z \in \mathcal{Z}_n^*\}. \tag{24}$$

Proof. We choose l_0 according to Lemma 4.1. Let $n \in \{1,..,N\}$. Let $l < l_0$ and ν a (l,ω)-separated probability measure. Let $\alpha \in C_{n,2}(\nu)$. We subdivide the remaining proof into several steps.

1. We show relation (24).

According to Lemma 4.1 (b) the set α induces an $n-$optimal partition Z of ω, i.e.

$$Z = \{\omega(a \mid \alpha) : a \in \alpha\} \in \mathcal{Z}_n^*. \tag{25}$$

Again by Lemma 4.1 (b) we have for every $a \in \alpha$, that

$$\{x \in \omega : B(x,l) \subset W(a \mid \alpha)\} = \omega(a \mid \alpha). \tag{26}$$

In the same way as for Q_ω in Remark 3.2 we deduce from Theorem 2.1 (2) and (3) that α consists of the ν-centroids of its Voronoi cells, i.e.

$$\alpha = \{s_\nu(W(a \mid \alpha)) : a \in \alpha\}. \tag{27}$$

Because the support of ν is a subset of $\cup_{x\in\omega} B(x,l)$ we get from (26) and (27), that

$$\begin{aligned}\alpha &= \{s_\nu(\bigcup_{x\in\omega(a|\alpha)} B(x,l)) : a \in \alpha\} \\ &= \{s_\nu(\bigcup_{x\in\gamma} B(x,l)) : \gamma \in Z\}.\end{aligned}$$

Because α was chosen arbitrarily, we have proven (24).

2. We prove a lower bound for $V_{n,2}(\nu)$.

Following the comments in [3], p.9 one recognizes that a Borel measurable partition $\{A_a : a \in \alpha\}$ exists with

$$W_0(a \mid \alpha) \subset A_\alpha \subset W(a \mid \alpha),$$

if we denote $W_0(a \mid \alpha) = \{x \in \mathbb{R}^d : \parallel x - a \parallel \ < \min_{b\in\alpha\setminus\{a\}} \parallel x - b \parallel\}$. From [3], p. 31/32 we obtain

$$V_{n,2}(\nu) = \sum_{a\in\alpha} \nu(A_a) \cdot V_{1,2}(\nu(\cdot \mid A_a)).$$

Applying Theorem 2.1 (3)(b) we get

$$V_{n,2}(\nu) = \sum_{a\in\alpha} \nu(W(a \mid \alpha)) \cdot V_{1,2}(\nu(\cdot \mid W(a \mid \alpha))).$$

Because ν is $(l,\omega)-$separated we have $\nu(B(x,l)) = 1/N$. The application of (25) and (26) yields

$$V_{n,2}(\nu) = \sum_{\gamma\in Z} \frac{\text{card}(\gamma)}{N} \cdot V_{1,2}(\nu(\cdot \mid \bigcup_{x\in\gamma} B(x,l))).$$

Therefore, Lemma 4.3 implies

$$\begin{aligned}V_{n,2}(\nu) \quad &= \quad \frac{1}{N}\sum_{\gamma\in Z}\sum_{x\in\gamma}\Big(V_{1,2}(\nu(\cdot \mid B(x,l))) \\ &\quad + \parallel s_\nu(B(x,l)) - \frac{1}{\text{card}(\gamma)}\sum_{y\in\gamma} s_\nu(B(y,l)) \parallel^2\Big),\end{aligned}$$

and

$$\begin{aligned}V_{n,2}(\nu) \quad &\geq \quad \min_{Z\in\mathcal{Z}_n^*}\frac{1}{N}\sum_{\gamma\in Z}\sum_{x\in\gamma}\Big(V_{1,2}(\nu(\cdot \mid B(x,l))) \\ &\quad + \parallel s_\nu(B(x,l)) - \frac{1}{\text{card}(\gamma)}\sum_{y\in\gamma} s_\nu(B(y,l)) \parallel^2\Big).\end{aligned} \tag{28}$$

3. For the upper bound we proceed indirectly.
Assume the existence of $Y \in \mathcal{Z}_n^* \backslash \{Z\}$ satisfying

$$\begin{aligned} V_{n,2}(\nu) \quad > \quad & \frac{1}{N} \sum_{\eta \in Y} \sum_{x \in \eta} \Big(V_{1,2}(\nu(\cdot \mid B(x,l))) \\ & + \parallel s_\nu(B(x,l)) - \frac{1}{\operatorname{card}(\eta)} \sum_{y \in \eta} s_\nu(B(y,l)) \parallel^2 \Big). \end{aligned}$$

Using Lemma 4.3 we recognize that the right hand side of this inequality is identical to

$$\sum_{\eta \in Y} \frac{\operatorname{card}(\eta)}{N} \cdot V_{1,2}\Big(\nu(\cdot \mid \bigcup_{x \in \eta} B(x,l))\Big).$$

We define $\beta = \{s_{Q_\omega}(\eta) : \eta \in Y\}$. From Lemma 4.1 (a) we deduce

$$\sum_{\eta \in Y} \frac{\operatorname{card}(\eta)}{N} \cdot V_{1,2}\Big(\nu(\cdot \mid \bigcup_{x \in \eta} B(x,l))\Big) = \sum_{b \in \beta} \nu(W(b \mid \beta)) V_{1,2}(\nu(\cdot \mid W(b \mid \beta))).$$

According to [3], Lemma 3.3 the right hand side of this inequality is greater than or equal to $V_{n,2}(\nu)$. Hence, we end into a contradiction. Therefore (28) turns into an equation and the identity (23) is proved. □

5 Application to self-similar measures

Let $d, N \in \mathbb{N}$, $N \geq 2$ and consider N contractions $S_1, .., S_N$ with identical contraction factor $c \in]0,1[$, which is the number satisfying

$$\parallel S_i(x) - S_i(y) \parallel = c \parallel x - y \parallel \tag{29}$$

for every $i \in \{1, .., N\}$ and $x, y \in \mathbb{R}^d$. We call these N contractions a uniform iterated function system ($UIFS$). Every $UIFS$ has a unique nonempty compact set $A \subset \mathbb{R}^d$ with the characteristic property

$$A = S_1(A) \cup ... \cup S_N(A). \tag{30}$$

For a proof of this fact the reader is referred to [7], Theorem 3.1 (3) (i). In the literature on fractal geometry (see e.g. [1], p.31) the set A is often called invariant attractor or invariant set.

Moreover a unique Borel probability distribution μ on $\mathbb{R}^d$ exists, which is characterized by

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \mu \circ S_i^{-1}. \tag{31}$$

We call μ the uniform distribution (UD) of the $UIFS$. The support of μ coincides with the invariant set A. A proof of these facts can also be found in [7].

Every contraction $S : \mathbb{R}^d \to \mathbb{R}^d$ with contraction factor $c \in]0,1[$ has a unique fixed point x. Moreover an orthonormal mapping $O : \mathbb{R}^d \to \mathbb{R}^d$ exists such that

$$S(z) = c \cdot O(z - x) + x$$

for every $z \in \mathbb{R}^d$. For a proof see [7], Proposition 2.3 (1)).

We denote by $\omega = \{x_1, .., x_N\}$ the set of fixed points of the $UIFS$. We will assume that the fixed points are pairwise different. It holds that $\omega \subset A$.

Now we intend to apply the results of Section 4 to self-similar measures. To this end, we need further restrictions to the $UIFS$. For the rest of this paper let us assume, that all contractions of the $UIFS$ do not contain a rotation part, i.e. for every $x \in \mathbb{R}^d$ and $i \in \{1, .., N\}$ we have

$$S_i(x) = c \cdot (x - x_i) + x_i. \tag{32}$$

For any nonempty set $B \subset \mathbb{R}^d$ we define

$$\operatorname{diam}(B) = \sup\{\| x - y \| : x, y \in \mathbb{R}^d; x \neq y\}.$$

We assume that

$$c < \frac{1}{2} \cdot \frac{d_{\min}(\omega)}{\operatorname{diam}(A)}. \tag{33}$$

Remark 5.1. *Because the set A depends also on the contraction factor c the existence of c_0 satisfying (33) for every $c \in]0, c_0[$ is not obvious. Due to $A = \bigcup_{i=1}^N S_i(A)$ there exists $i, j \in \{1, .., N\}$ and $x \in S_i(A)$, $y \in S_j(A)$, such that $\operatorname{diam}(A) = \| x - y \|$. Hence we obtain*

$$\begin{aligned} \operatorname{diam}(A) &\leq \| x - x_i \| + \| x_i - x_j \| + \| x_j - y \| \\ &\leq 2c \cdot \operatorname{diam}(A) + \operatorname{diam}(\omega), \end{aligned}$$

which yields

$$\operatorname{diam}(A) \leq \frac{1}{1 - 2c} \cdot \operatorname{diam}(\omega), \tag{34}$$

if we assume that $c \in]0, 1/2[$. A simple calculation using (34) shows that

$$c < \frac{1}{2} \cdot \frac{d_{\min}(\omega)}{\operatorname{diam}(A)}$$

is satisfied for every $c \in]0, c_0[$, if we set $c_0 = d_{\min}(\omega)/(2(d_{\min}(\omega) + \operatorname{diam}(\omega)))$.

Remark 5.2. *Due to (33) the balls $(B(x_i, c \cdot \operatorname{diam}(A)))_{i \in \{1,..,N\}}$ are pairwise disjoint. Using (30) and (31) we deduce that*

$$\mu(B(x_i, c \cdot \operatorname{diam}(A))) = \mu(S_i(A)) = \frac{1}{N}$$

for every $i \in \{1, .., N\}$. Hence, the probability distribution μ as UD of the $UIFS$ is a $(c \cdot \operatorname{diam}(A), \omega)-$separated measure.

Lemma 5.3. *Let* $\emptyset \neq I \subset \{1,..,N\}$ *and*

$$\widetilde{\omega} = \{S_i(s_\mu(A)) : i \in \{1,..,N\}\}.$$

Then

(a) $s_\mu(A) = \sum_{i=1}^N x_i/N$,

(b) $s_\mu\left(\bigcup_{i\in I} B(x_i, c\cdot \mathrm{diam}(A))\right) = \sum_{i\in I} S_i(s_\mu(A))/\,\mathrm{card}(I)$,

(c) $V_{n,2}(Q_{\widetilde{\omega}}) = (1-c)^2 V_{n,2}(Q_\omega)$ *for every* $n \in \{1,..,N\}$,

(d) $V_{1,2}(\mu) = ((1-c)/(1+c)) \cdot V_{1,2}(Q_\omega)$.

Moreover we obtain for every $i \in \{1,..,N\}$, *that*

(e) $V_{1,2}(\mu(\cdot \mid B(x_i, c\cdot\mathrm{diam}(A)))) = c^2 \cdot ((1-c)/(1+c)) \cdot V_{1,2}(Q_\omega)$.

Proof.
(a) Obviously

$$S_i(A) \subset B(x_i, c\cdot \mathrm{diam}(A)) \tag{35}$$

for every $i \in \{1,..,N\}$. Due to (33) we therefore get

$$S_i(A) \cap S_j(A) = \emptyset, \tag{36}$$

for all $i,j \in 1,..,N$ with $i \neq j$. From (30), (36) and the definition (2) of the centroid we deduce

$$s_\mu(A) = \int_A x d\mu(x) = \sum_{i=1}^N \int_{S_i(A)} x d\mu(x).$$

Using (31) and again (36) we obtain

$$s_\mu(A) = \sum_{i=1}^N \int_{S_i(A)} x d(\frac{1}{N}\sum_{j=1}^N \mu\circ S_j^{-1}(x)) = \sum_{i=1}^N \frac{1}{N}\int_{S_i(A)} x d\mu\circ S_i^{-1}(x).$$

From (32) we deduce

$$\begin{aligned} s_\mu(A) &= \sum_{i=1}^N \frac{1}{N}\int_A (c(x-x_i)+x_i)\, d\mu(x) \\ &= \sum_{i=1}^N \left(\frac{c}{N}\int_A x d\mu(x) + \frac{1-c}{N} x_i\right) \\ &= c\cdot s_\mu(A) + (1-c)\frac{1}{N}\sum_{i=1}^N x_i, \end{aligned}$$

which yields $s_\mu(A) = \sum_{i=1}^N x_i/N$.

(b) Because μ is a $(c \cdot \operatorname{diam}(A), \omega)-$separated measure (cf. Remark 5.2), we get by a simple calculation, that

$$s_\mu\left(\bigcup_{i\in I} B(x_i, c\cdot \operatorname{diam}(A))\right) = \frac{1}{\operatorname{card}(I)} \sum_{i\in I} s_\mu(B(x_i, c\cdot \operatorname{diam}(A))). \tag{37}$$

Due to $\operatorname{supp}(\mu) = A$ and (30) we have for every $k \in \{1, .., N\}$ that

$$\mu(\cdot \mid B(x_k, c\cdot \operatorname{diam}(A))) = \mu\left(\cdot \mid \bigcup_{i=1}^{N} S_i(A) \cap B(x_k, c\cdot \operatorname{diam}(A))\right).$$

The combination of (36), (35) and (33) implies

$$\mu(\cdot \mid B(x_k, c\cdot \operatorname{diam}(A))) = \mu\left(\cdot \mid S_k(A)\right). \tag{38}$$

From (37) and (38) we deduce

$$\begin{aligned} s_\mu\left(\bigcup_{i\in I} B(x_i, c\cdot \operatorname{diam}(A))\right) &= \frac{1}{\operatorname{card}(I)} \sum_{i\in I} s_\mu(S_i(A)) \\ &= \frac{1}{\operatorname{card}(I)} \sum_{i\in I} (\mu(S_i(A)))^{-1} \int_{S_i(A)} x d\mu(x). \end{aligned}$$

Using (31) and (36) we get together with (32), that

$$\begin{aligned} & s_\mu\left(\bigcup_{i\in I} B(x_i, c\cdot \operatorname{diam}(A))\right) \\ = \; & \frac{1}{\operatorname{card}(I)} \sum_{i\in I} N\left(\frac{1}{N}\int_A S_i(x) d\mu(x)\right) \\ = \; & \frac{1}{\operatorname{card}(I)} \sum_{i\in I} \int_A \left(c(x - x_i) + x_i\right) d\mu(x) \\ = \; & \frac{1}{\operatorname{card}(I)} \sum_{i\in I} (c\int_A x d\mu(x) + (1-c)x_i) = \frac{1}{\operatorname{card}(I)} \sum_{i\in I} S_i(s_\mu(A)). \end{aligned}$$

(c) We define a mapping T from ω to $\mathbb{R}^d$ by

$$T(x) = c\cdot\left(\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) - x\right) + x$$

for every $x \in \omega$. Applying (a) and (32) we get $T(\omega) = \widetilde{\omega}$. Now let $x, y \in \omega$. Obviously

$$\| T(x) - T(y) \| = (1-c) \| x - y \| .$$

The assertion (c) is now a direct consequence of Lemma 3.3.

(d) With (31) and (32) we derive

$$\int \|x\|^2 d\mu(x) = \frac{1}{N}\sum_{i=1}^{N}\int \|c(x-x_i)+x_i\|^2 d\mu(x).$$

Using $s_\mu(A) = \int x d\mu(x)$ and by some elementary calculations we deduce

$$\int \|x\|^2 d\mu(x) = \frac{1-c}{1+c}\cdot\left((\frac{1}{N}\sum_{i=1}^{N}\|x_i\|^2) - \|s_\mu(A)\|^2\right) + \|s_\mu(A)\|^2.$$

By (a) we have

$$V_{1,2}(Q_\omega) = \frac{1}{N}\sum_{i=1}^{N} \| x_i - s_\mu(A) \|^2 = \left(\frac{1}{N}\sum_{i=1}^{N}\|x_i\|^2\right) - \|s_\mu(A)\|^2.$$

Hence we obtain

$$\int \|x\|^2 d\mu(x) - \|s_\mu(A)\|^2 = \frac{1-c}{1+c}\cdot V_{1,2}(Q_\omega).$$

By Theorem 2.1 (2) we know, that $\{s_\mu(A)\}$ is a $1-$optimal set for μ. Thus we get from Lemma 2.2 that $V_{1,2}(\mu) = \int \|x\|^2 d\mu(x) - \|s_\mu(A)\|^2$, which proves the assertion (d).

(e) From identity (38) and (31) it follows for all $i \in \{1,..,N\}$, that

$$\mu(\cdot \mid B(x_i, c\cdot \operatorname{diam}(A))) = \mu\,(\cdot \mid S_i(A)) = \mu\circ S_i^{-1}.$$

Using (29) and Theorem 2.1 (4) we deduce

$$V_{1,2}(\mu(\cdot \mid B(x_i, c\cdot \operatorname{diam}(A)))) = V_{1,2}(\mu\circ S_i^{-1}) = c^2 V_{1,2}(\mu).$$

The assertion (e) then follows immediately from (d). □

In order to keep notation simple, we denote $\mathcal{Z}_n^*$ from now on as the set of all $n-$partitions of the set $\{1,..,N\}$ instead of the set ω. To stress the dependence of the UD of the $UIFS$ on the contraction factor $c \in]0,1[$ we denote μ_c instead of μ.

Theorem 5.4. *Let μ_c be the UD of a $UIFS$, which consists of the mappings*

$$S_i(x) = c\cdot(x-x_i)+x_i.$$

with contraction factor $c \in]0,1[$ and $i \in \{1,..,N\}$, $x \in \mathbb{R}^d$.
Then a $c_0 \in]0, d_{\min}(\omega)/(2\operatorname{diam}(A))]$ exists such that for every $n \in \{1,..,N\}$ and every $c < c_0$ the equation

$$V_{n,2}(\mu_c) = \frac{1-c}{1+c}\cdot c^2 V_{1,2}(Q_\omega) + (1-c)^2 V_{n,2}(Q_\omega) \tag{39}$$

holds. Additionally for every $n-$*optimal set* $\alpha \in C_{n,2}(\mu_c)$ *an* $n-$*optimal partition* Z *of* $\{1,..,N\}$ *exists such that*

$$\alpha = \left\{ \frac{1}{\text{card}(I)} \sum_{i \in I} \left(x_i + \frac{c}{N} \sum_{j=1}^{N} (x_j - x_i) \right) : I \in Z \right\}.$$

Proof. Fix $l_0 > 0$ according to Theorem 4.4. Let $c_0 := l_0/(2l_0 + \text{diam}(\omega))$ and choose $c < c_0$. As in Remark 5.1 one recognizes, that

$$c < \frac{l_0}{\text{diam}(A)} \leq \frac{d_{\min}(\omega)}{2\,\text{diam}(A)}.$$

Remark 5.2 ensures that μ_c is a $(c \cdot \text{diam}(A), \omega)-$separated probability distribution. Now let $n \in \{1,..,N\}$ and $l = c \cdot \text{diam}(A)$.

1. From Theorem 4.4 it follows that

$$V_{n,2}(\mu_c) = \frac{1}{N} \sum_{i=1}^{N} V_{1,2}(\mu_c(\cdot \mid B(x_i, l)))+$$

$$\min_{Z \in \mathcal{Z}_n^*} \frac{1}{N} \sum_{I \in Z} \sum_{i \in I} \parallel s_{\mu_c}(B(x_i, l)) - \left(\frac{1}{\text{card}(I)} \sum_{j \in I} s_{\mu_c}(B(x_j, l)) \right) \parallel^2 .$$

The application of Lemma 5.3 (b) and (e) yields

$$V_{n,2}(\mu_c) = c^2 \cdot \frac{1-c}{1+c} \cdot V_{1,2}(Q_\omega)+$$

$$\min_{Z \in \mathcal{Z}_n^*} \frac{1}{N} \sum_{I \in Z} \sum_{i \in I} \parallel S_i(s_{\mu_c}(A)) - \left(\frac{1}{\text{card}(I)} \sum_{j \in I} S_j(s_{\mu_c}(A)) \right) \parallel^2 .$$

According to Remark 3.4 and Proposition 3.1 we have

$$V_{n,2}(\mu_c) = c^2 \cdot \frac{1-c}{1+c} \cdot V_{1,2}(Q_\omega) + V_{n,2}(Q_{\tilde{\omega}}).$$

By Lemma 5.3 (c) we deduce

$$V_{n,2}(\mu_c) = c^2 \cdot \frac{1-c}{1+c} \cdot V_{1,2}(Q_\omega) + (1-c)^2 V_{n,2}(Q_\omega).$$

2. Theorem 4.4 implies

$$C_{n,2}(\mu_c) \subset \{\{s_{\mu_c}(\bigcup_{i \in I} B(x_i, c \cdot \text{diam}(A))) : I \in Z\} : Z \in \mathcal{Z}_n^*\}.$$

Lemma 5.3 (b) yields

$$
\begin{aligned}
& \{\{s_{\mu_c}(\bigcup_{i\in I} B(x_i, c\cdot \operatorname{diam}(A))) : I \in Z\} : Z \in \mathcal{Z}_n^*\} \\
= \; & \{\{\frac{1}{\operatorname{card}(I)} \sum_{i\in I} S_i(s_{\mu_c}(A)) : I \in Z\} : Z \in \mathcal{Z}_n^*\}.
\end{aligned}
$$

From Lemma 5.3 (a) and (32) we obtain

$$
\begin{aligned}
& \{\{\frac{1}{\operatorname{card}(I)} \sum_{i\in I} S_i(s_{\mu_c}(A)) : I \in Z\} : Z \in \mathcal{Z}_n^*\} \\
= \; & \{\{\frac{1}{\operatorname{card}(I)} \sum_{i\in I} \left(x_i + \frac{c}{N} \sum_{j=1}^{N} (x_j - x_i) \right) : I \in Z\} : Z \in \mathcal{Z}_n^*\},
\end{aligned}
$$

which finishes the proof. □

From the equation (39) we obtain that the quantization error differences of μ_c are scaled versions of the ones of Q_ω. To be exactly let ν be a Borel probability distribution on $\mathbb{R}^d$ with a support consisting of at least N points. If $N > 2$ we define

$$
\begin{aligned}
& D_{\min}(N, \nu) \\
= \; & \min\{V_{n,2}(\nu) - V_{n+1,2}(\nu) - (V_{n+1,2}(\nu) - V_{n+2,2}(\nu)) : n \in \{1, .., N-2\}\} \\
= \; & \min\{V_{n,2}(\nu) + V_{n+2,2}(\nu) - 2V_{n+1,2}(\nu) : n \in \{1, .., N-2\}\}
\end{aligned}
$$

as the smallest quantization error difference for $n = 1, .., N$.

Corollary 5.5. *There exists* $c_0 \in \,]0, d_{\min}(\omega)/(2\operatorname{diam}(A))]$ *such that for every* $c < c_0$

(a) $V_{N,2}(\mu_c) = c^2 V_{1,2}(\mu_c)$ *and*

(b) $D_{\min}(N, \mu_c) = (1-c)^2 D_{\min}(N, Q_\omega)$, *if* $N > 2$.

Proof. Immediate consequence of Theorem 5.4. □

To demonstrate the applicability of our results we discuss briefly three famous $UIFS$ and the optimal quantization of their related UD's.

Example 5.6 (one-dimensional Cantor set)**.** *Let* $d = 1$ *and* $\omega = \{x_1, x_2\}$ *with* $x_1 = 0$, $x_2 = 1$. *With a contracting factor* $c \in]0, 1/2]$ *we consider the* $UIFS$ *defined by (32). From Theorem 5.4 we obtain for small enough* $c \in]0, 1/2]$ *that*

$$
V_{1,2}(\mu) = \frac{1-c}{1+c} \cdot V_{1,2}(Q_\omega) = \frac{1-c}{4(1+c)} \tag{40}
$$

and

$$
V_{2,2}(\mu) = \frac{1-c}{1+c} \cdot c^2 \cdot V_{1,2}(Q_\omega) = c^2 \frac{1-c}{4(1+c)}.
$$

By elementary considerations using (31) and Theorem 2.1 (2), one can prove the equation (40) directly for every $c \in]0, 1/2]$. For the special case $c = 1/3$ the reader is also referred to [2], Lemma 3.4. and [2], Theorem 5.2. Insofar our results are substantially incorporated by the already known facts regarding this example.

Example 5.7 (Sierpinski gasket)**.** *Let $d = 2$ and $\omega = \{x_1, x_2, x_3\}$ with $x_1 = (0,0)$, $x_2 = (1,0)$ and $x_3 = (1/2, \sqrt{3}/2)$. With a contracting factor $c \in]0, 1/2]$ we consider the UIFS defined by (32). If $c = 1/2$, the invariant set A of the UIFS is the (classical) Sierpinski gasket. Using Proposition 3.1 we obtain*

$$V_{1,2}(Q_\omega) = \frac{1}{3}\sum_{i=1}^{3} \| x_i - s_{Q_\omega}(\omega) \|^2 = \frac{1}{3}.$$

Applying Theorem 5.4 and Proposition 3.1 we derive

$$V_{1,2}(\mu) = \frac{1-c}{1+c} \cdot V_{1,2}(Q_\omega) = \frac{1-c}{3(1+c)},$$

$$V_{2,2}(\mu) = \frac{c^2(1-c)}{3(1+c)} + \frac{1}{6}(1-c)^2, \tag{41}$$

and

$$V_{3,2}(\mu) = \frac{c^2(1-c)}{3(1+c)}$$

for small enough $c > 0$. By direct calculations or Corollary 5.5 (b) one gets

$$D_{\min}(3,\mu) = (1-c)^2 \left(\left(\frac{1}{\sqrt{3}}\right)^2 - \frac{4}{3}\left(\frac{1}{2}\right)^2 \right) = 0.$$

From Theorem 2.1 (2) we know that $s_\mu(A)$ is the only $1-$optimal set for μ. Using Lemma 5.3 (a) we obtain that $s_\mu(A) = (1/2, \sqrt{3}/6)$. For $i \in \{1,2,3\}$ we denote

$$x_{1,i} = S_i(s_\mu(A))$$

and

$$x_{2,i} = \frac{1}{2}\sum_{\substack{j=1 \\ j \neq i}}^{3} S_j(s_\mu(A)).$$

Applying Theorem 5.4 and Lemma 5.3 (a) we get

$$C_{2,2}(\mu) \subset \{\{x_{1,i}, x_{2,i}\} : i = 1,2,3\}.$$

According to symmetry arguments one easily recognizes that

$$C_{2,2}(\mu) = \{\{x_{1,i}, x_{2,i}\} : i = 1,2,3\}. \tag{42}$$

Again by Theorem 5.4 we obtain

$$C_{3,2}(\mu) = \{\{x_{1,1}, x_{1,2}, x_{1,3}\}\}.$$

The identities (42) and (41) are becoming wrong, if $c > 3/7$. *To see this, let* $\beta = \{b_1, b_2\}$ *with* $b_1 = \big(1/2, \sqrt{3}\,(1/2 - c/3)\big)$ *and* $b_2 = \big(1/2, c\sqrt{3}/6\big)$. *A direct calculation shows that*

$$\mu(W(b_1 \mid \beta) - S_3(A)) > 0,$$

if $c > 3/7$. *Hence, in this case*

$$b_1 \neq E(\mu(\cdot \mid W(b_1 \mid \beta))), \tag{43}$$

if $E(\cdot)$ *denotes the expected value. If we assume (42), then Theorem 2.1 (3)(d) implies* $C_{1,2}(\mu(\cdot \mid W(b_1 \mid \beta))) = \{\{b_1\}\}$. *Moreover, Theorem 2.1 (2) yields*

$$b_1 = E(\mu(\cdot \mid W(b_1 \mid \beta))),$$

which contradicts (43). Thus, $\beta = \{b_1, b_2\}$ *could not be a* $2-$*optimal set, and (41) and (42) do not hold in this case.*

Example 5.8 (Cantor dust)**.** *Let* $d = 2$ *and* $\omega = \{x_1, x_2, x_3, x_4\}$ *with* $x_1 = (0,0)$, $x_2 = (1,0)$, $x_3 = (1,1)$ *and* $x_4 = (0,1)$. *Consider the* $UIFS$ *defined by (32) with a contracting factor* $c \in]0, 1/2]$. *Similar to the other examples we get for small enough contracting factor* $c > 0$ *the following identities*

$$\begin{aligned} V_{1,2}(\mu) &= \frac{1-c}{2(1+c)}, \\ V_{2,2}(\mu) &= \frac{1-c}{4(1+c)}(1+c^2), \\ V_{3,2}(\mu) &= \frac{(1-c)(1+3c^2)}{8(1+c)}, \\ V_{4,2}(\mu) &= \frac{1-c}{2(1+c)} \cdot c^2. \end{aligned}$$

Here as well, the relation $D_{\min}(4, \mu) = 0$ *hold.*

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