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Binding energy of the $X(3872)$ at unphysical pion masses

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The chiral extrapolation of the $X(3872)$ binding energy is investigated using the modified Weinberg formulation of chiral effective field theory for the $D\bar{D}^*$ scattering. Given its explicit renormalizability, this approach is particularly useful to explore the interplay of the long- and short-range $D\bar{D}^*$ forces in the $X(3872)$ from studying the light-quark (pion) mass dependence of its binding energy. In particular, the parameter-free leading-order calculation shows that the X pole disappears for unphysically large pion masses. On the other hand, without contradicting the naive dimensional analysis, the higher-order pion-mass-dependent contact interaction can change the slope of the binding energy at the physical point, yielding the opposite scenario of a more strongly bound X at pion masses larger than its physical value. An important role of the pion dynamics and of the three-body $D\bar{D}\pi$ effects for chiral extrapolations of the X pole is emphasized. The results of the present study should be of practical value for lattice simulations since they provide a nontrivial connection between lattice points at unphysical pion masses and the physical world.

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I. INTRODUCTION

After more than a decade since the discovery by the Belle Collaboration of the $X(3872)$ charmonium-like state [1], its nature still remains an open question (see Ref. [2] for a review). According to the Particle Data Group, this state has the mass $M_X = (3871.68 \pm 0.17)$ MeV [3] and thus resides very close to the neutral $D\bar{D}^*$ threshold with

$$E_B = M_{D^0} + M_{\bar{D}^{*0}} - M_X = (0.12 \pm 0.26)\ \text{MeV}. \quad (1)$$

It is therefore natural to assume that its wave function has a large molecular admixture. [See the vast literature on hadronic molecules—for example, Refs. [4–15]—also in the context of the $X(3872)$.]

The 1^{++} quantum numbers of the X determined recently by the LHCb Collaboration [16,17] are consistent with its interpretation as an S-wave $D^0\bar{D}^{*0}$ bound state[1] (see, for example, Refs. [8,13,15]). The small binding energy relative to the $D^0\bar{D}^{*0}$ threshold allows for an effective field theory (EFT) formulation of the problem in analogy to the deuteron. The pionless EFT based on pure contact $D\bar{D}^*$ interactions was first applied to the $X(3872)$ in Ref. [18], while implications of the heavy quark and heavy flavor symmetries were utilized in Refs. [19,20] to predict partner states of the $X(3872)$. However, in the presence of other relevant dynamical scales such a treatment is expected to be valid only in a very narrow energy region around the threshold. In particular, the three-body neutral channel $D^0\bar{D}^0\pi^0$ opens at approximately 7 MeV below the $D^0\bar{D}^{*0}$ threshold, while the charged three-body thresholds $D^\pm D^\mp\pi^0$ and $D^\pm\bar{D}^0\pi^\mp$ reside about 2 MeV above it. In addition, the charged two-body threshold $D^\pm\bar{D}^{*\mp}$ is located around 8 MeV above the neutral one. The mass difference between the charged and neutral $D\bar{D}^*$ thresholds was shown in Ref. [21] to play a crucial role in the understanding of isospin violation in the decays of the X into $\pi^+\pi^- J/\psi$ [22] and $\pi^+\pi^-\pi^0 J/\psi$ [23], for which approximately equal branching fractions were observed. To incorporate the long-range pion physics the so-called X-EFT approach was developed in Ref. [24] based on the assumption that pions can be treated perturbatively. Recently, this framework was extended to include higher-order corrections and then used to predict the pion-mass dependence of the X pole [25] and the finite-volume corrections to the X binding energy [26]. On the other hand, the perturbative treatment of pions has a smaller range of validity compared to nonperturbative approaches and it has to be used with caution; for example, the perturbative framework is known to be not applicable in the deuteron channel [27] which demonstrates certain similarities with the $X(3872)$.

[1]A proper C-parity eigenstate is always meant by this (and similar) shorthand notation.

The frameworks with nonperturbative pions were employed in many phenomenological studies (see, for example, Refs. [28–30]); however, all of them include one-pion exchange (OPE) in the static limit, that is, under the assumption that the D mesons are infinitely heavy. Meanwhile, the close proximity of the $D^{\pm}D^{\mp}\pi^0$, $D^{\pm}\bar{D}^0\pi^{\mp}$, and $D^0\bar{D}^0\pi^0$ thresholds to the $X(3872)$ pole suggests that three-body scales can play an important role in this state, so that by neglecting the three-body dynamics one distorts the analytical structure of the amplitude in the kinematical region of interest. It has to be noticed, however, that the proper inclusion of the three-body dynamics requires special care. For example, it was shown in Ref. [31] that the three-body unitary cuts play a very important role in the $D_\alpha\bar{D}_\beta$ system if the D_β width is dominated by the S-wave $D_\beta \to D_\alpha\pi$ decay. In particular, it was demonstrated that if the $D_\beta \to D_\alpha\pi$ coupling is sufficiently strong to produce a bound state [32,33] it is, at the same time, necessarily sufficiently strong to provide the state with such a large width that it becomes unobservable. In turn, in the case of P-wave vertices, the system at hand demonstrates additional difficulties since the self-energy loops diverge, so that the system requires a proper treatment to avoid false conclusions (see an example of such conclusions in Ref. [34] and its detailed discussion in Ref. [35]). In particular, contrary to the claims of Ref. [34], it was shown in Ref. [35] that the OPE potential in the $\bar{D}D^*$ system is well defined in the sense of an effective field theory only in connection with a contact operator. This contact operator absorbs all the details of the short-range dynamics present in the system and it is taken in the form of a polynomial function in the pion mass and in the small momentum.

In Refs. [36,37], the properties of the $X(3872)$ molecular state were studied in a heavy-meson EFT framework with nonperturbative pions including all relevant three-body scales. It was understood that the dynamical treatment of pions had a big impact on the X line shape and, in particular, on the partial decay width $X \to D^0\bar{D}^0\pi^0$. Furthermore, it was shown in Ref. [36] that the static OPE approximation was not adequate to analyze the role of the long-range pion dynamics in the $X(3872)$, since it corresponded to an uncontrolled modification of the proper dynamical scales related to the $D\bar{D}\pi$ cuts and to the neglect of the imaginary part of the $D\bar{D}^*$ potential. Meanwhile, the role of nonperturbative effects for these observables appeared to be quite moderate, as follows from the agreement between the results of the nonperturbative calculations [36] and those in the X-EFT [8,24].

In Ref. [37], the nonperturbative framework developed in Ref. [36] was generalized to study the dependence of the X binding energy on the light-quark mass or, equivalently, on the pion mass. The use of nonperturbative one-pion exchange for chiral extrapolations allows one to extend the region of applicability of the approach to larger pion masses which is important for analyzing the results of lattice QCD calculations.

In this work we address another important issue which is related to nonperturbative renormalization of the three-body Lippmann-Schwinger or Faddeev-type equations to describe the interaction between heavy mesons in the X. The standard nonrelativistic approach to heavy mesons leads to coupled-channel integral equations for the scattering amplitudes which, at leading order in the EFT expansion, are linearly divergent. As a consequence, iterations of the truncated potential within the dynamical equation generate an infinite series of ultraviolet (UV) divergent higher-order contributions to the amplitude which cannot be absorbed into a finite number of counterterms (contact interactions) included in the potential. In other words, the coefficients in front of the logarithmic and power-law divergences appearing in the iterations of the equation involve powers of external momenta which can only be removed if an infinite number of higher-order (derivative) contact interactions is included. The standard way to deal with this problem is to employ a finite UV cutoff of the order of a natural hard scale in the problem which would suppress the unwanted higher-order contributions, as advocated in Ref. [38]. This strategy was followed, in particular, in Refs. [19,20,36,37,39]. Exactly the same problem with renormalization also emerges in the context of nuclear chiral EFT (see, for example, Refs. [40,41] and references therein). In particular, a finite cutoff was employed for the construction of the NN potential and the few-body nuclear forces within chiral EFT (see Ref. [42] for a review). This procedure induces cutoff artifacts which might become a nontrivial issue for (in particular) chiral extrapolations since it might be difficult to control the pion-mass dependence of short-range interactions in a systematic way. Note also that in Ref. [37] the m_π dependence of the contact interaction was promoted to the leading order to maintain the renormalizability of the scattering amplitude at unphysical pion masses.

Recently a novel, renormalizable (in the EFT sense) approach to nucleon-nucleon scattering with nonperturbative pions was proposed in Ref. [43]. Starting from the Lorentz-invariant form of the effective Lagrangian, the authors of Ref. [43] derived a three-dimensional dynamical equation which complies with the relativistic elastic unitarity and which is renormalizable at the leading order of EFT. Indeed, in the suggested approach, all logarithmically divergent contributions generated by iterations of the potential can be fully absorbed into the redefinition of the leading-order contact terms. Then higher-order contributions are subject to a perturbative treatment in this approach. It should be stressed that the central point of the approach is noncommutativity of the nonrelativistic expansion and the renormalization procedure, while after renormalization relativistic effects as such provide only a

minor impact on the low-energy observables, as it should be in EFT. Apart from its transparency with regard to renormalization, this approach allows one to remove finite cutoff artifacts and it is very well suited for carrying out chiral extrapolations and studying correlations between the effective range parameters induced by the analytic structure of the long-range forces (see Refs. [44,45] for the corresponding results in the NN sector). Given the same UV behavior of the dynamical equations for NN and heavy meson-antimeson scattering, we demonstrate that the method of Ref. [43] can be used to reformulate the nonrelativistic three-body approach of Refs. [36,37] in terms of renormalizable integral equations. We apply the resulting theoretical framework to study the quark-mass-dependence of the $X(3872)$ binding energy. In contrast to the finite cutoff formulation, the m_π dependence of the binding energy is predicted at leading order in a renormalizable approach.

The paper is organized as follows. In Sec. II we give a brief introduction to the method suggested in Ref. [43]. In Sec. III the problem of the $D\bar{D}^*$ interaction is formulated in a closed self-consistent form which makes it possible to appeal to the approach discussed in Sec. II. In Sec. IV we present and discuss the results of our calculations. We summarize our findings in Sec. V. All necessary technical details are collected in the Appendix.

II. NONRELATIVISTIC (LIPPMANN-SCHWINGER) EQUATIONS VERSUS EQUATIONS WITH RELATIVISTIC UNITARITY

A proper nonrelativistic expansion of low-energy physical quantities can be done by calculating these quantities in a Lorenz-invariant theory and expanding the final result in the powers of the velocity v (we work in the natural system of units, setting the speed of light $c = 1$); see, e.g., Ref. [46] where this issue is discussed in the one-nucleon sector. On the other hand, one can perform the nonrelativistic expansion at the level of the Lagrangian of the theory. However, this expansion does not commute with the loop integration. This can be exemplified by a simple calculation adapted from Ref. [47]. Consider a scalar two-point loop function which is logarithmically divergent and therefore should be regularized. With the simplest regularization prescription given by a sharp cutoff in the three-dimensional momentum, it reads

$$I = \frac{4i}{(2\pi)^4}\int \frac{d^4k\theta(\Lambda - |\boldsymbol{k}|)}{[k^2 - m^2 + i0][(P-k)^2 - m^2 + i0]}, \quad (2)$$

where $P = (2\sqrt{m^2 + \boldsymbol{p}^2}, \boldsymbol{0})$. The integral can be evaluated analytically for $\Lambda > |\boldsymbol{p}|$ with the result

$$I = -\frac{i|\boldsymbol{p}|}{2\pi\sqrt{m^2+\boldsymbol{p}^2}} + \frac{|\boldsymbol{p}|}{\pi^2\sqrt{m^2+\boldsymbol{p}^2}}\ln\frac{\Lambda\sqrt{m^2+\boldsymbol{p}^2} + |\boldsymbol{p}|\sqrt{\Lambda^2+m^2}}{m\sqrt{\Lambda^2-\boldsymbol{p}^2}} - \frac{1}{\pi^2}\ln\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}. \quad (3)$$

A nonrelativistic expansion of the integrand in Eq. (2) implies the strong inequality $|\boldsymbol{p}| \ll \Lambda \ll m$ that is equivalent to the $1/m$ expansion made prior to the $1/\Lambda$ expansion in the exact result (3), which yields

$$I = -\frac{i|\boldsymbol{p}|}{2\pi m} - \frac{\Lambda}{\pi^2 m} + \ldots, \quad (4)$$

where the ellipsis denotes suppressed terms. The divergence in Eq. (4) is linear, that is, it is stronger than that in the original integral (2) which is a consequence of the nonrelativistic expansion of the integrand.

On the contrary, keeping the integrand relativistic and performing the nonrelativistic expansion after integration is equivalent to imposing a different (and more natural) strong inequality $|\boldsymbol{p}| \ll m \ll \Lambda$ and, therefore, the $1/\Lambda$ expansion is to be performed in Eq. (3) before the $1/m$ expansion. This leads to a different result for the real part of the integral,

$$I = -\frac{i|\boldsymbol{p}|}{2\pi m} - \frac{1}{\pi^2}\ln\frac{2\Lambda}{m} + \cdots, \quad (5)$$

which reveals the logarithmic divergence, in agreement with the UV behavior of the original integral.

Thus the nonrelativistic expansion of the integrand changes its ultraviolet behavior and the final result differs from the relativistic expansion of the exact expression for the integral. This difference is caused by the noncommutativity of the nonrelativistic expansion and the loop integration. Because of this noncommutativity, in order to reproduce the results of the Lorentz-invariant theory, one needs to add compensating terms to the nonrelativistic effective Lagrangian. Therefore, more singular behavior of the nonrelativistic equation leads to perturbative nonrenormalizability already for the leading-order (LO) potential. In particular, iterations in the Lippmann-Schwinger equation generate power-law divergences with coefficients of progressively increasing powers of the momentum/energy. As the LO potential does not contain momentum-dependent contact interactions, one cannot get rid of these

divergences by absorbing them into a redefinition of the parameters of the LO potential. Adding any finite number of momentum/energy-dependent terms does not resolve the issue. While this is not a problem when calculating a finite number of diagrams, it is rather disturbing when solving integral equations. Except for some trivial cases, it is not possible to take into account contributions of an infinite number of compensating terms required for "correcting" an infinite number of iterations. One is, therefore, forced either to keep the ultraviolet cutoff finite ($\Lambda \sim m$) or to resort back to the original Lorentz-invariant formulation of the theory, although the effect of relativistic corrections at low energies is, of course, small after renormalization.

On the other hand, iterations of the Lippmann-Schwinger equation without nonrelativistic expansion generate only logarithmic divergences. This guarantees a perturbative renormalizability of the theory at LO, that is, all divergences can be removed by renormalizing the coupling constant of the LO contact interaction.

In Ref. [37], the nonrelativistic Lippmann-Schwinger equation was solved that corresponds to the nonrelativistic expansion of the integrand, as was explained above. In the present paper we deal with the relativized Lippmann-Schwinger equation and, therefore, the LO amplitude is obtained by solving a renormalizable integral equation. Analogously to the nucleon-nucleon scattering in the modified Weinberg approach of Ref. [43], the relativized integral equation for the $D\bar{D}^*$ system has a milder ultraviolet behavior if compared to the nonrelativistic Lippmann-Schwinger equation. It has to be noticed, however, that the integral equation becomes nonrenormalizable if higher-order corrections to the leading-order potential are also treated nonperturbatively. In particular, by iterating higher-order contributions in the potential one generates divergences with such structures of momentum- and/or energy-dependent coefficients which are not present in the iterated potential, that is, these divergent contributions cannot be absorbed into the redefinition of the contact terms included in the potential at the given order. On the other hand, renormalizability is retained by treating corrections perturbatively. In particular, if we denote the LO amplitude as T_0 and the NLO corrections to it as T_1, we have the following perturbative expansion of the full amplitude:

$$T = T_0 + \varepsilon T_1 + O(\varepsilon^2). \tag{6}$$

Then, the inverse amplitude takes the form

$$T^{-1} = T_0^{-1}(T_0 - \varepsilon T_1)T_0^{-1} + O(\varepsilon^2 T_0^{-1}), \tag{7}$$

where ε stands either for the expansion parameter of chiral EFT, $\varepsilon \sim \{m_\pi/\Lambda_\chi, p/\Lambda_\chi\}$ with Λ_χ being the chiral symmetry-breaking scale, or it corresponds to the expansion around the physical pion mass $m_\pi^{\rm ph}$, that is, $\varepsilon \sim (m_\pi^2 - m_\pi^{\rm ph2})/m_\pi^{\rm ph2}$. In what follows, while we stick to the leading-order chiral potential, we investigate the pion-mass dependence of the $X(3872)$ binding energy including corrections at NLO which appear as one goes away from the physical point.

Note that the expression of Eq. (7) gives an explicitly unitary amplitude; however, it also includes selectively resummed higher-order contributions which do not affect the renormalizability of the scattering amplitude. A bound state corresponds to the zero of the inverse amplitude (7). Finally, we use the superscript "ph" to label quantities taken at the physical point, that is, for $m_\pi = m_\pi^{\rm ph}$.

III. SYSTEM OF COUPLED-CHANNEL INTEGRAL EQUATIONS FOR THE $D\bar{D}^*$ PROBLEM

In this section we briefly outline our theoretical formulation of the problem. We follow the lines of Ref. [37], adapting the approach according to Ref. [43]. In particular, we work in terms of the effective $D\bar{D}^*$ potential which, in addition to the long-range physics related to OPE, also includes the contact term to account for our ignorance of the short-range forces present in the system, such as heavy-meson exchanges between the D and D^*, other Fock components of the X wave function, and so on (see, for instance, examples of the reformulation of the problem through the effective $D\bar{D}^*$ potential in Refs. [48–50]).

The lowest-order $D^*D\pi$ interaction Lagrangian is taken in the form [24]

$$\mathcal{L} = \frac{g_c}{2f_\pi}(\boldsymbol{D}^{*\dagger}\cdot\boldsymbol{\nabla}\pi^a\tau^a D + D^\dagger\tau^a\boldsymbol{\nabla}\pi^a\cdot\boldsymbol{D}^*). \tag{8}$$

The dimensionless coupling constant g_c is related to the $D^{*0} \to D^0\pi^0$ decay width as

$$\Gamma(D^{*0} \to D^0\pi^0) = \frac{g_c^2 m_0 q^3}{24\pi f_\pi^2 m_{*0}}, \tag{9}$$

where $q = \lambda^{1/2}(m_{*0}^2, m_0^2, m_{\pi^0}^2)/(2m_{*0})$ is the center-of-mass 3-momentum of the outgoing particles and $\lambda(x,y,z)$ is the standard triangle function [see the definition in Eq. (A6)]. Here and in what follows, m_*, m, and m_π denote the masses of the D^* meson, D meson, and pion, respectively. Charged and neutral states are distinguished by an additional index, for example m_{*c} versus m_{*0}.

The $D\bar{D}^*$ potential at LO in chiral effective field theory consists of the OPE and the S-wave derivativeless contact interaction C_0,

$$V_{ij}^{nn'}(\boldsymbol{p},\boldsymbol{p}') = (\boldsymbol{p}+\boldsymbol{p}')^n(\boldsymbol{p}+\boldsymbol{p}')^{n'}F_{ij}(\boldsymbol{p},\boldsymbol{p}') + C_0\delta^{nn'}, \tag{10}$$

where the indices n and n' are contracted with the corresponding indices of the D^* polarization vectors. Here

$$F_{ij}(\boldsymbol{p},\boldsymbol{p}') = -\frac{g_c^2}{(4\pi f_\pi)^2}\left(\frac{1}{D_{3ij}^{(1)}(\boldsymbol{p},\boldsymbol{p}')} + \frac{1}{D_{3ij}^{(2)}(\boldsymbol{p},\boldsymbol{p}')}\right), \tag{11}$$

and $D_{3ij}^{(1)}$ and $D_{3ij}^{(2)}$ ($i, j = 0, c$) stand for the $D\bar{D}\pi$ and $D^*\bar{D}^*\pi$ propagators written in the framework of the time-ordered perturbation theory (see Fig. 1),

$$D_{3ij}^{(1)}(\boldsymbol{p},\boldsymbol{p}') = \begin{cases} E_{\pi^0}(\boldsymbol{p}+\boldsymbol{p}')(E_{D_i}(p)+E_{D_i}(p')+E_{\pi^0}(\boldsymbol{p}+\boldsymbol{p}')-M), & i=j, \\ E_{\pi^c}(\boldsymbol{p}+\boldsymbol{p}')(E_{D_i}(p)+E_{D_j}(p')+E_{\pi^c}(\boldsymbol{p}+\boldsymbol{p}')-M), & i\neq j, \end{cases} \tag{12}$$

$$D_{3ij}^{(2)}(\boldsymbol{p},\boldsymbol{p}') = \begin{cases} E_{\pi^0}(\boldsymbol{p}+\boldsymbol{p}')(E_{D_i^*}(p)+E_{D_i^*}(p')+E_{\pi^0}(\boldsymbol{p}+\boldsymbol{p}')-M), & i=j, \\ E_{\pi^c}(\boldsymbol{p}+\boldsymbol{p}')(E_{D_i^*}(p)+E_{D_j^*}(p')+E_{\pi^c}(\boldsymbol{p}+\boldsymbol{p}')-M), & i\neq j. \end{cases} \tag{13}$$

For convenience, the energy E is counted relative to the neutral two-body threshold,

$$M = m_{*0} + m_0 + E, \tag{14}$$

while the energies of the individual particles are

$$E_{\pi_i}(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m_{\pi_i}^2}, \qquad E_{D_i}(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m_i^2}, \qquad E_{D_i^*}(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m_{*i}^2}. \tag{15}$$

The OPE potential (10) interrelates the four D-meson channels defined as

$$|0\rangle = D^0\bar{D}^{*0}, \qquad |\bar{0}\rangle = \bar{D}^0 D^{*0}, \qquad |c\rangle = D^+D^{*-}, \qquad |\bar{c}\rangle = D^-D^{*+}. \tag{16}$$

Then the system of coupled-channel Lippmann-Schwinger equations for the $D\bar{D}^*$ t-matrix elements $a_{00}^{nn'}(\boldsymbol{p},\boldsymbol{p}')$ and $a_{c0}^{nn'}(\boldsymbol{p},\boldsymbol{p}')$ in the C-even channel has the form [37]

$$\begin{cases} a_{00}^{nn'}(\boldsymbol{p},\boldsymbol{p}') = \lambda_0 V_{00}^{nn'}(\boldsymbol{p},\boldsymbol{p}') - \sum\limits_{i=0,c}\lambda_i \int d^3k V_{0i}^{nm}(\boldsymbol{p},\boldsymbol{k})\frac{1}{\Delta_i(\boldsymbol{k})}a_{i0}^{mn'}(\boldsymbol{k},\boldsymbol{p}'), \\ a_{c0}^{nn'}(\boldsymbol{p},\boldsymbol{p}') = \lambda_c V_{c0}^{nn'}(\boldsymbol{p},\boldsymbol{p}') - \sum\limits_{i=0,c}\lambda_i \int d^3k V_{ci}^{nm}(\boldsymbol{p},\boldsymbol{k})\frac{1}{\Delta_i(\boldsymbol{k})}a_{i0}^{mn'}(\boldsymbol{k},\boldsymbol{p}'), \end{cases} \tag{17}$$

where $\lambda_0 = \langle 0|\vec{\tau}_1\cdot\vec{\tau}_2|\bar{0}\rangle = \langle c|\vec{\tau}_1\cdot\vec{\tau}_2|\bar{c}\rangle = 1$ and $\lambda_c = \langle 0|\vec{\tau}_1\cdot\vec{\tau}_2|\bar{c}\rangle = \langle c|\vec{\tau}_1\cdot\vec{\tau}_2|\bar{0}\rangle = 2$ are the isospin factors for the π^0 and $\pi^\pm$ exchange, respectively.

The partial-wave projections of the potential (10) on the relevant ${}^3S_1$ and ${}^3D_1$ partial waves read ($x = \cos\theta$ where θ is the angle between the momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$)

$$\begin{aligned}
V_{ij}^{SS}(p,p') &= C_0 + \frac{1}{6}\int_{-1}^{1} F_{ij}(p,p',x)(p^2+p'^2+2pp'x)dx, \\
V_{ij}^{SD}(p,p') &= -\frac{\sqrt{2}}{6}\int_{-1}^{1} F_{ij}(p,p',x)\left[p'^2 + p^2\left(\frac{3}{2}x^2-\frac{1}{2}\right)+2pp'x\right]dx, \\
V_{ij}^{DS}(p,p') &= -\frac{\sqrt{2}}{6}\int_{-1}^{1} F_{ij}(p,p',x)\left[p^2 + p'^2\left(\frac{3}{2}x^2-\frac{1}{2}\right)+2pp'x\right]dx, \\
V_{ij}^{DD}(p,p') &= \frac{1}{3}\int_{-1}^{1} F_{ij}(p,p',x)\left[(p^2+p'^2)\left(\frac{3}{2}x^2-\frac{1}{2}\right)+\frac{11}{10}pp'x+\frac{9}{10}pp'\left(\frac{5}{2}x^3-\frac{3}{2}x\right)\right]dx.
\end{aligned}$$

Because of the P-wave nature of the $D^* \to D\pi$ vertex, the $D\pi$ loop operator $\Sigma(s, m_*, m, m_\pi)$ diverges and it is subject to renormalization. The necessary details of the renormalization procedure are given in the Appendix, while here we only quote the final result for the inverse two-body propagators Δ_0 and Δ_c entering the system of equations (17):

$$\Delta_0(\boldsymbol{p}) = \frac{E_{D^0}(\boldsymbol{p})E_{D^{*0}}(\boldsymbol{p})}{m_0 m_{*0}}\left[\frac{E_{D^0}(\boldsymbol{p}) + E_{D^{*0}}(\boldsymbol{p}) - M}{\zeta} - \frac{\tilde{\Sigma}_R(s, m_{*0}, m_{\pi^0}, m_0) + 2\tilde{\Sigma}_R(s, m_{*0}, m_{\pi_c}, m_c) + i m_{*0}\Gamma(D^{*0} \to D^0\gamma)}{2E_{D^{*0}}(\boldsymbol{p})}\right],$$

$$\Delta_c(\boldsymbol{p}) = \frac{E_{D^c}(\boldsymbol{p})E_{D^{*c}}(\boldsymbol{p})}{m_c m_{*c}}\left[\frac{E_{D^c}(\boldsymbol{p}) + E_{D^{*c}}(\boldsymbol{p}) - M}{\zeta} - \frac{\tilde{\Sigma}_R(s, m_{*c}, m_{\pi^0}, m_c) + 2\tilde{\Sigma}_R(s, m_{*c}, m_{\pi_c}, m_0)}{2E_{D^{*c}}(\boldsymbol{p})}\right], \tag{18}$$

where

$$s = m_*^2 + 2E_{D^*}(\boldsymbol{p})(M - E_D(\boldsymbol{p}) - E_{D^*}(\boldsymbol{p})) \tag{19}$$

for the off-shell D^* resonance and $\tilde{\Sigma}_R(s, m_*, m, m_\pi)$ is the renormalized loop operator defined at the "running" pion mass that, in particular, brings about the quantity ζ,

$$\zeta^{-1} \equiv 1 - \frac{g_R^2}{384\pi^2}\ln\frac{m_\pi^2}{(m_\pi^{\rm ph})^2}, \tag{20}$$

with the renormalized coupling constant defined as (see the Appendix)

$$g_R = g_c\frac{\sqrt{m_0 m_{*0}}}{f_\pi}. \tag{21}$$

For future discussion of the static approximation, we also consider the simplified case of a constant width which corresponds to the substitution of the constant $s = m_*^2$ in Eq. (18) instead of the "running" s, as given in Eq. (19).

We are now in a position to introduce the power counting in the parameter $\xi = m_\pi/m_\pi^{\rm ph}$. The m_π dependence of the coupling constant g_c is extracted from Ref. [51] and is discussed in Ref. [37]. In particular, at LO g_c remains constant while at NLO it acquires corrections of the order of m_π^2. Similarly, f_π, the masses of the D and D^* mesons, and the decay width $\Gamma(D^{*0} \to D^0\gamma)$ take their respective physical values in the LO calculation. The central issue of this work is related to the m_π dependence of the contact interaction C_0 [see Eq. (10)]. Since the nature of this interaction is obscure, the dependence $C_0(m_\pi)$ can only be guessed using the principle of naturalness. Below, we discuss in detail the generalization of the corresponding approach developed in Ref. [37]. Meanwhile, regardless of the particular source of the dependence $C_0(m_\pi)$ it only appears at NLO, so that the contact interaction remains constant at LO and, therefore, the problem is fully fixed to provide a prediction for the behavior of the X pole as the pion mass leaves the physical point. Furthermore, the pion-mass dependence at LO occurs only due to the pion energies in the $D\bar{D}\pi$ propagator and pion-mass effects in the renormalized loop—both are equally important parts of the three-body $D\bar{D}\pi$ dynamics. To finalize the setup of the problem, we quote the values of various parameters at the physical pion mass used in the calculations. In particular, in the physical limit of $m_\pi = m_\pi^{\rm ph}$ one has $f_\pi^{\rm ph} = 92.4$ MeV; then,

$$m_{\pi^0}^{\rm ph} = 134.98\ \text{MeV}, \quad m_{\pi^c}^{\rm ph} = 139.57\ \text{MeV},$$
$$m_0^{\rm ph} = 1864.84\ \text{MeV}, \quad m_c^{\rm ph} = 1869.62\ \text{MeV},$$
$$m_{*0}^{\rm ph} = 2006.97\ \text{MeV}, \quad m_{*c}^{\rm ph} = 2010.27\ \text{MeV},$$

and the values

$$\Gamma^{\rm ph}(D^{*0} \to D^0\pi^0) = 42\,\text{keV}, \quad \Gamma^{\rm ph}(D^{*0} \to D^0\gamma) = 21\,\text{keV} \tag{22}$$

can be deduced from the data for the charged D^* decay modes [3]. The physical values of the couplings introduced above are [37]

$$g_c^{\rm ph} = 0.61, \qquad g_R^{\rm ph} = 12.7. \tag{23}$$

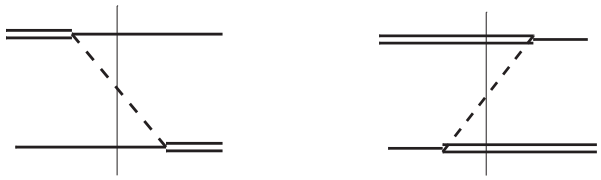

FIG. 1. Diagrams in time-ordered perturbation theory corresponding to the (inversed) three-body propagators $D_{3ij}^{(1)}$ (left plot) and $D_{3ij}^{(2)}$ (right plot) $(i, j = 0, c)$. The double and single solid lines refer to the D^* and D, respectively, while the dashed lines refer to pions. The thin vertical line pinpoints the intermediate state.

IV. RESULTS AND DISCUSSIONS

We are now in a position to discuss the results for the pion-mass (or, equivalently, light-quark mass) dependence of the $X(3872)$ binding energy $E_B(m_\pi)$. We start from the

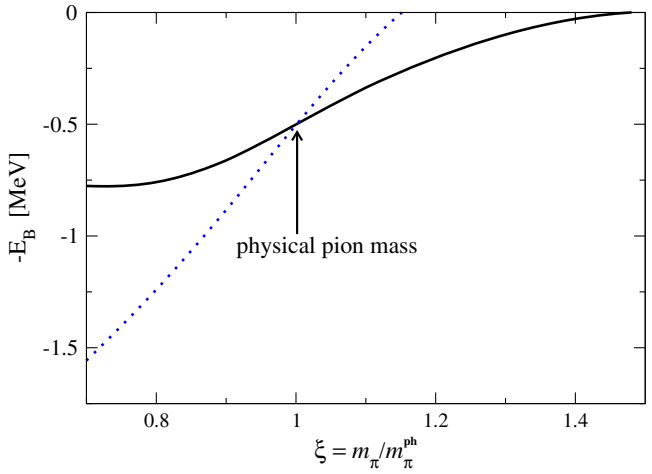

FIG. 2 (color online). Pion-mass dependence of the $X(3872)$ binding energy at LO. The results of the full dynamical theory with three-body effects included (black solid curve) are confronted with the simplified formulation with static OPE (blue dotted curve).

discussion of the LO results. As was explained above, the contact term C_0 is m_π independent at this order, so once it is adjusted to reproduce the binding energy at the physical pion mass [for definiteness we set $E_B(m_\pi^{\rm ph}) = 0.5$ MeV], the scattering amplitude can be calculated for unphysical pion masses without loss of renormalizability of the LO equations. Therefore, at LO of our EFT, the dependence $E_B(m_\pi)$ can be predicted in a parameter-free way (see Fig. 2). At this order, the pion-mass dependence of E_B originates only from the pionic effects in the OPE potential and from those in the renormalized self-energy loops $\tilde{\Sigma}_R$ [see Eq. (18)]. The binding energy at LO demonstrates a clear tendency to decrease with the growth of m_π. Note that a similar behavior of the binding energy was observed in Ref. [44] for the deuteron. Furthermore, the slope of the binding energy in m_π at the physical point, $(\partial E_B/\partial m_\pi)|_{m_\pi = m_\pi^{\rm ph}}$, exhibits a strong sensitivity to the three-body $D\bar{D}\pi$ effects. In particular, neglecting the three-body dynamics (the so-called static OPE) results in a much steeper decrease of the binding energy [compare the dotted (blue) line versus the solid (black) in Fig. 2].

Since no real experiment is possible for unphysical pion masses, the only source of information on the X pole fate for the m_π's exceeding the physical pion mass is provided by lattice simulations. Such calculations are indeed being performed and most of them predict an increase of the binding energy with the m_π growth. For example, different lattice collaborations observe this type of behavior for the deuteron (see, for example, Refs. [52–54] and references therein).[2] Also, the first lattice calculations for the $X(3872)$ indicate the existence of a more strongly bound X for $m_\pi > m_\pi^{\rm ph}$ [56–58]. Although these results still suffer from potentially large finite-range corrections, as pointed out in Ref. [26] within X-EFT, they raise an important question of whether such a behavior of the binding energy can be understood theoretically. To this end, we go beyond LO and proceed to NLO, thus including corrections quadratic in m_π. In particular, we allow for an m_π dependence of the short-range interaction which therefore goes away from its physical value. Thus, we consider (for simplicity all indices are omitted)

$$V_{\rm NLO} = V_{\rm OPE}(\boldsymbol{p},\boldsymbol{p}',\xi) + C_0 + D(\xi^2 - 1), \qquad \xi = m_\pi/m_\pi^{\rm ph}, \tag{24}$$

where the first two terms on the right-hand side stand for the LO potential (10) while the last term accounts for our ignorance of other dynamical scales except those related to the OPE. As was discussed in the previous section, the renormalizability of the theory requires all operators beyond the LO to be included perturbatively. Following Ref. [37], we fix the unknown coefficient D to the slope of the binding energy at the physical pion mass, which is therefore considered as an additional input quantity. For example, in Fig. 3 we illustrate the behavior of the binding energy for the slope $(\partial E_B/\partial m_\pi^2)|_{m_\pi = m_\pi^{\rm ph}} \approx E_B^{\rm ph}/m_\pi^{\rm ph2}$ (see the dashed curve in the left panel). While the sign of the slope was fixed to provide a growth of the binding energy with the pion mass, its magnitude was chosen to comply with naturalness. Specifically, we assume that the shift of the binding energy $\delta E_B \sim E_B^{\rm ph}$ for $\delta m_\pi \sim m_\pi^{\rm ph}$ can be interpreted as natural. Indeed, the slope predicted at LO due to OPE fulfills this criterion: $(\partial E_B/\partial m_\pi^2)|_{m_\pi = m_\pi^{\rm ph}} \approx -1.5 E_B^{\rm ph}/m_\pi^{\rm ph2}$. Therefore, to study the case of a more strongly bound X, we fix the slope to be $(\partial E_B/\partial m_\pi^2)|_{m_\pi = m_\pi^{\rm ph}} \approx E_B^{\rm ph}/m_\pi^{\rm ph2}$. Interestingly, in a theory with the same polynomial behavior of the contact operator but without pions, one would observe a much flatter behavior $E_B(m_\pi)$ for the same slope $(\partial E_B/\partial m_\pi^2)|_{m_\pi = m_\pi^{\rm ph}}$, as shown by the dash-dotted curve. The difference between the two curves demonstrates the role of dynamical pions as an explicit long-range degree of freedom. As seen from Fig. 3, the contact interaction provides a smooth background for a rapidly varying pion-mass dependence stemming from OPE. Therefore, by integrating out pions and the corresponding three-body soft scales while still trying to (at least partially) compensate for neglecting these long-range effects, one would inevitably arrive at unnaturally large m_π-dependent coefficients accompanying short-range operators.

On the other hand, one may question a justification of the perturbative inclusion of the m_π-dependent short-range interaction in Eq. (24). Given the shallowness of the

[2]On the other hand, the HAL QCD Collaboration found no bound state in the NN $^3S_1$-$^3D_1$ channel [55].

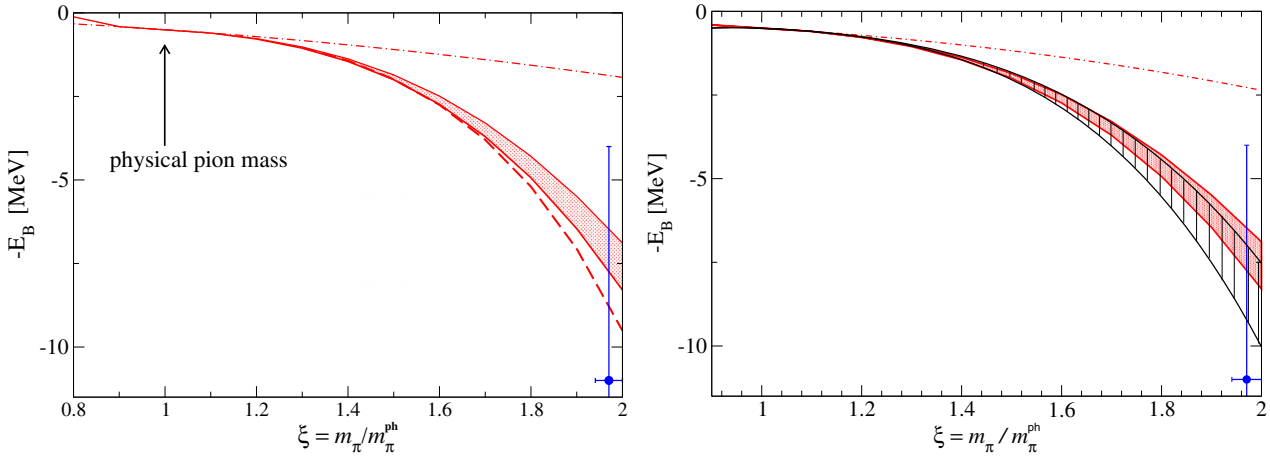

FIG. 3 (color online). Pion-mass dependence of the $X(3872)$ binding energy at NLO. Left panel: The dashed line is for the perturbative treatment of the m_π-dependent contact operator at NLO, while the red dotted band represents the nonperturbative results employing resonance saturation. Right panel: Comparison of the results obtained in the heavy-meson formulation of Ref. [37] with the finite cutoff $\Lambda \in [500\ \text{MeV}, 700\ \text{MeV}]$ (black hatched band) with the nonperturbative results of the current study employing resonance saturation and the cutoff $\Lambda \to \infty$ (red dotted band). The dash-dotted line in both panels corresponds to the calculation without pions. The (blue) dot with the error bars shows the result of the lattice calculation of Ref. [58].

physical X state, even a small variation of the slope within its natural range (as discussed above) has a sizable impact on the m_π dependence of the binding energy. In order to verify the validity of the perturbative approach, we employ resonance saturation to model higher-order contact interactions by means of a heavy-meson exchange. In particular, we consider the NLO potential in the form

$$V_{\rm NLO} = V_{\rm OPE}(\boldsymbol{p},\boldsymbol{p}',\xi) + C_0 + \beta\frac{g_c^2}{(4\pi f_\pi)^2}\frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{q})(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{q})}{\boldsymbol{q}^2 + M^2 + \delta M^2(\xi^2-1)}, \qquad (25)$$

where $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{p}'$ and the scale M is varied in the range $M = 600$–800 MeV that corresponds to a typical heavy-meson mass. The parameter β accounts for the difference in the strength of the heavy-meson exchange potential relative to that of OPE. It is expected to take values around unity and it could be, in principle, adjusted to the $D\bar{D}^*$ effective range. However, given that the latter is unknown, β is varied within a suitable range of values from 1 to 2 which we treat as natural. The term $\delta M^2(\xi^2-1)$ in the denominator accounts for the pion-mass dependence of the heavy-meson mass[3] with δM adjusted to the slope $(\partial E_B/\partial m_\pi^2)|_{m_\pi = m_\pi^{\rm ph}}$. We have verified that for $\delta M = 0$, the dependence $E_B(m_\pi)$ is basically indistinguishable from the LO one that confirms the results to be insensitive to the details of the short-range interaction, as expected. The form of the NLO potential (25) ensures that the corresponding scattering amplitude is renormalizable so that the NLO calculations can be carried out in the same way as the LO ones. Then, fixing the slope as before, $(\partial E_B/\partial m_\pi^2)|_{m_\pi = m_\pi^{\rm ph}} \approx E_B^{\rm ph}/m_\pi^{\rm ph2}$, we obtain the dotted (red) band in Fig. 3. A very good agreement between this band and the dashed curve in the considered range of pion masses confirms that higher-order effects originating from the nonperturbative resummation of pion-mass-dependent short-range interactions are minor. Therefore, the perturbative treatment of the pion-mass-dependent short-range interaction, as given by Eq. (24), is indeed justified even though it brings about a new effect: the slope of the binding energy may change its sign compared to the LO result depicted in Fig. 2.

It is also instructive to compare the results of the present study with those obtained in Ref. [37] in the heavy-meson formulation with a finite cutoff (see the right panel in Fig. 3). At NLO, both approaches are consistent with each other, as may be expected since both formulations are justified in general. Meanwhile, as was already discussed above, the LO equation is explicitly renormalizable and predictive in the present formulation in contrast to the approach of Ref. [37], where the m_π-dependent contact interaction requires an additional input to be included to maintain the renormalizability of the scattering amplitude. In addition, the requirements of naturalness are much easier to formulate and to apply in the current approach since the dependence of the results on the cutoff is eliminated. Thus, we emphasize that the role of pion dynamics can be understood in a much more transparent way using the explicitly renormalizable theory which is free of finite cutoff artifacts.

To further clarify the role of the dynamical pions at NLO, we assume that there exist gedanken lattice data at

[3]Notice that a similar EFT approach based on the resonance saturation hypothesis was used in Ref. [59] to constrain the pion-mass dependence of the short-range NN forces.

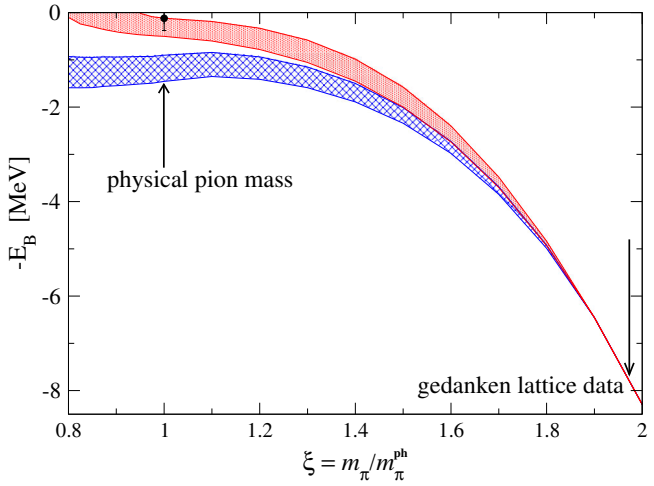

FIG. 4 (color online). Pion-mass dependence of the $X(3872)$ binding energy. The red dotted band is for the full calculation with dynamical pions at NLO, while the blue crossed band is for the static OPE.

unphysical pion masses. These data could be used to adjust the parameters of the short-range potential. Then, once the short-range physics is fixed, the theory can be extrapolated to the physical point in m_π and confronted with the experimental data. In particular, if the lattice calculations provide two measurements of the binding energy of the X made for two unphysically large pion masses then the suggested approach allows us to establish the correct extrapolating formula to the physical point and thus to predict the corresponding value $E_B(m_\pi^{\rm ph})$. In addition, information on the behavior of the short-range interactions in the X—which can, in this way, be extracted from the lattice data—may shed light on the nature of the binding mechanisms in the X. This establishes an important link between the EFT approach and lattice simulations for hadronic molecule states.

As an illustration, the chiral extrapolations for the two theories—the one with dynamical pions and the one with the static OPE—are compared with each other in Fig. 4. For definiteness the gedanken lattice result is taken at $m_\pi = 2m_\pi^{\rm ph}$, as indicated by the arrow in Fig. 4. If the slope is chosen such that the theory with dynamical pions provides the correct extrapolation to the physical pion mass, the static theory with the same slope yields a significant overbinding of the $X(3872)$, by more than a factor of 3 in the binding energy. In addition, one can see from Fig. 4 that the extrapolation curve from an unphysically large pion mass (close to the values used on the lattices) to the physical point is nontrivial and the corresponding extrapolating behavior has a strong curvature. This illustration emphasizes the importance of the three-body effects for the chiral extrapolations for the X and puts in question the possibility of using any simple ansatz for the extrapolation formula.

V. SUMMARY AND CONCLUSIONS

In this work we developed an explicitly renormalizable framework to study chiral extrapolations of the binding energy of the $X(3872)$ beyond the physical pion mass. This approach is free of the finite cutoff artifacts which is a precondition for a systematic control over the pion-mass dependence from the short-range interactions. The pertinent results of our work can be summarized as follows. First, the interplay between the long- and short-range forces in the X appears to be quite nontrivial, as was already pointed out in Ref. [37]. If the X turns out to be less bound for the pion masses exceeding its physical value, the m_π dependence of the X binding energy is entirely governed by the explicit pion-mass dependence of the OPE potential. On the other hand, a more strongly bound X would signal the importance of the m_π-dependent short-range interactions in addition to pionic effects. Confronting our results with those of the lattice simulations could allow one to extract valuable information on such short-range interactions and, possibly, to disclose the nature of the binding forces in the X.

Second, our findings are of a practical value for the lattice simulations since they open the way to override the gap between the unphysically large pion masses used on the lattices and the physical limit. It follows from our results that the corresponding interpolating curve has a strong curvature and it is strongly affected by the three-body effects in the X.

Last but not least, the approach developed in this paper can also be adapted to other near-threshold states, the $X(3872)$ being just the most prominent and therefore the most extensively studied one.

ACKNOWLEDGMENTS

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APPENDIX: The $D\pi$ LOOP OPERATOR AND THE D^* PROPAGATOR

Consider an unstable vector-mesonic state which decays in the P wave into a pair of (pseudo)scalar mesons. We start from its inverse propagator

$$D(s) = s - m_0^2 + \Sigma(s), \tag{A1}$$

where s is the invariant energy ($s = p^2$), m_0 stands for the "bare" mass, and $\Sigma(s)$ denotes the self-energy loop operator. The one-loop contribution to the self-energy has the form[4]

$$\Sigma(s) = -g_0^2 I_{\mu\nu}\varepsilon^\mu \varepsilon^{*\nu}, \tag{A2}$$

where g_0 is a dimensionless bare coupling constant and the ε's stand for the polarization vectors of the unstable meson such that $(\varepsilon \cdot \varepsilon^*) = -1$ and $(\varepsilon \cdot p) = 0$.

The loop integral corresponding to the decay of the vector meson into two mesons of the masses m_1 and m_2 reads

$$\begin{aligned} I_{\mu\nu} &= i\int \frac{d^n k}{(2\pi)^n}\frac{k_\mu k_\nu}{(k^2-m_1^2)((k+p)^2-m_2^2)} \\ &= A g_{\mu\nu} + B\frac{p_\mu p_\nu}{p^2} \end{aligned} \tag{A3}$$

and, due to the property $(\varepsilon \cdot p) = 0$, only the function A is relevant for the loop operator (A2). Since the loop integral (A3) diverges quadratically in the limit of $n \to 4$, the coefficients A_0 and A_1 in the Taylor series of the function $A(s)$,

$$A(s) = A_0 + A_1 s + A_{\rm reg}(s), \tag{A4}$$

are singular in this limit, while the residual function $A_{\rm reg}(s)$ is regular. By a straightforward calculation in the dimensional regularization scheme, it is easy to find that

$$\begin{aligned} A_0 &= \frac{1}{192\pi^2}((2m_2^2+m_1^2)C(m_2)+(2m_1^2+m_2^2)C(m_1)), \\ A_1 &= -\frac{1}{384\pi^2}(C(m_1)+C(m_2)), \\ A_{\rm reg}(s) &= \frac{1}{12s}\left[\frac{m_1^4-m_2^4}{16\pi^2}\ln\frac{m_1}{m_2}+\lambda(s,m_1^2,m_2^2)I_2^{\rm reg}(m_1,m_2,s)\right], \end{aligned} \tag{A5}$$

where

$$I_2^{\rm reg}(s,m_1,m_2) = -\frac{1}{16\pi^2}\left(-1+\frac{m_1^2-m_2^2}{s}\ln\frac{m_1}{m_2}+\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{s}\ln\frac{m_1^2+m_2^2-s-\lambda^{1/2}(s,m_1^2,m_2^2)}{2m_1m_2}\right)$$

and

$$\begin{aligned} C(m) &= \left(\frac{1}{\epsilon}-\log(4\pi)+\gamma_E-1\right)+\ln\frac{m^2}{\mu^2}, \\ \epsilon &= \frac{1}{2}(4-n)\to 0, \qquad \gamma_E = -\Gamma'(1)\approx 0.5772, \end{aligned}$$

while μ is the scale in dimensional regularization and the triangle function is defined in the standard way,

$$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz. \tag{A6}$$

Then, the one-loop contribution to the self-energy in Eq. (A2) takes the form

$$\Sigma(s) = g_0^2 A(s) = g_0^2(A_0 + A_1 s + A_{\rm reg}(s)), \tag{A7}$$

and it is subject to renormalization which we perform by expanding $\Sigma(s)$ near the renormalized vector-meson mass m_R,[5]

$$\Sigma(s) = {\rm Re}\Sigma(m_R^2) + {\rm Re}\Sigma'(m_R^2)(s-m_R^2) + \Sigma_{\rm reg}(s), \tag{A8}$$

where

$$\begin{aligned} {\rm Re}\Sigma(m_R^2) &= g_0^2(A_0 + A_1 m_R^2 + {\rm Re}A_{\rm reg}(m_R^2)), \\ {\rm Re}\Sigma'(m_R^2) &= g_0^2(A_1 + {\rm Re}A'_{\rm reg}(m_R^2)), \end{aligned} \tag{A9}$$

while

$$\Sigma_{\rm reg}(s) = \Sigma(s) - {\rm Re}\Sigma(m_R^2) - {\rm Re}\Sigma'(m_R^2)(s-m_R^2) \equiv g_0^2 A_R(s), \tag{A10}$$

with

$$A_R(s) = A_{\rm reg}(s) - {\rm Re}A_{\rm reg}(m_R^2) - {\rm Re}A'_{\rm reg}(m_R^2)(s-m_R^2). \tag{A11}$$

Notice that $A_R(s)$ is finite and does not depend on the auxiliary regularization scale μ.

Defining the renormalized mass such that $m_R^2 = m_0^2 - {\rm Re}\Sigma(m_R^2)$, we have

$$\begin{aligned} D(s) &= (s - m_0^2 + {\rm Re}\Sigma(m_R^2)) + {\rm Re}\Sigma'(m_R^2)(s-m_R^2) \\ &\quad + \Sigma_R(s) = Z^{-1}(s-m_R^2) + g_0^2 A_R(s), \end{aligned}$$

with

[4]Vector-meson self-energies in chiral EFT were first discussed in Ref. [60].

[5]In general, it is preferable to relate the renormalized mass to the complex pole of the propagator [61–66]. However, at the one-loop level it is sufficient to use the real part of the inverse propagator since the difference between the two approaches occurs only starting from the two-loop order.

$$Z^{-1} \equiv 1 + \mathrm{Re}\Sigma'(m_R^2) = 1 + g_0^2(A_1 + \mathrm{Re}A'_{\mathrm{reg}}(m_R^2)), \quad \text{(A12)}$$

where it was used that A_1 is real.

Consider now the combination entering the system of equations for the scattering amplitudes (17),

$$\frac{g_0^2}{D(s)} = \frac{g_0^2}{Z^{-1}(s - m_R^2) + g_0^2 A_R(s)} = \frac{g_R^2}{s - m_R^2 + \Sigma_R(s)},$$
$$\Sigma_R(s) = g_R^2 A_R(s), \quad \text{(A13)}$$

where the renormalized coupling constant g_R is defined as

$$g_R^2 \equiv Z g_0^2. \quad \text{(A14)}$$

In particular, for the case of the D^{*0}, an obvious identification of the parameters is $m_R = m_{*0}$, $m_1 = m_0$, $m_2 = m_{\pi^0}$. Then, using Eq. (9) and the standard relation between the loop operator and the width,

$$\Gamma(D^{*0} \to D^0\pi^0) = \frac{1}{m_{*0}} \mathrm{Im}\Sigma_R(s = m_{*0}^2, m_R = m_{*0}, m_{\pi^0}, m_0)$$
$$= \frac{g_R^2}{m_{*0}} \mathrm{Im}A_{\mathrm{reg}}(s = m_{*0}^2, m_R = m_{*0}, m_{\pi^0}, m_0), \quad \text{(A15)}$$

where $\mathrm{Im}A_{\mathrm{reg}}$ can be found from Eq. (A5) to be

$$\mathrm{Im}A_{\mathrm{reg}}(s = m_{*0}^2, m_R = m_{*0}, m_{\pi^0}, m_0) = \frac{q^3}{24\pi m_{*0}},$$
$$q = \frac{1}{2m_{*0}} \lambda^{1/2}(m_{*0}^2, m_0^2, m_{\pi^0}^2), \quad \text{(A16)}$$

one arrives at the following relation between the couplings g_R and g_c:

$$g_R = g_c \frac{\sqrt{m_0 m_{*0}}}{f_\pi}, \quad \text{(A17)}$$

which completes the renormalization program at the physical point.

Away from the physical value of the pion mass one can write

$$\begin{aligned} D(s) &= s - m_0^2 + \Sigma(s) = s - m_R^2 + [\Sigma(s) - \mathrm{Re}\Sigma(m_R^2)] = s - m_R^2 + g_0^2[A(s) - \mathrm{Re}A(m_R^2)] \\ &= s - m_R^2 + g_0^2[A_1(s - m_R^2) + A_{\mathrm{reg}}(s) - \mathrm{Re}A_{\mathrm{reg}}(m_R^2)] \\ &= s - m_R^2 + g_0^2[A_1(s - m_R^2) + \mathrm{Re}A'^{\mathrm{ph}}_{\mathrm{reg}}(m_R^2)(s - m_R^2) + \tilde{A}_R(s)], \end{aligned} \quad \text{(A18)}$$

where

$$\tilde{A}_R(s) = A_{\mathrm{reg}}(s) - \mathrm{Re}A_{\mathrm{reg}}(m_R^2) - \mathrm{Re}A'^{\mathrm{ph}}_{\mathrm{reg}}(m_R^2)(s - m_R^2), \quad \text{(A19)}$$

that is, $\tilde{A}_R(s)$ is defined with the derivative in the subtracted term evaluated at the physical point. This allows one to avoid m_π dependence in the renormalization factor Z and to preserve its definition in the form of Eq. (A12). Then

$$\begin{aligned} D(s) &= (s - m_R^2)(1 + g_0^2(A_1 + \mathrm{Re}A'^{\mathrm{ph}}_{\mathrm{reg}}(m_R^2))) + g_0^2\tilde{A}_R(s) \\ &= (s - m_R^2)(1 + g_0^2(A_1 - A_1^{\mathrm{ph}}) + g_0^2(A_1^{\mathrm{ph}} + \mathrm{Re}A'^{\mathrm{ph}}_{\mathrm{reg}}(m_R^2))) + g_0^2\tilde{A}_R(s) \\ &= (s - m_R^2)(g_0^2(A_1 - A_1^{\mathrm{ph}}) + Z^{-1}) + g_0^2\tilde{A}_R(s) \\ &= Z^{-1}[(s - m_R^2)(1 + g_R^2(A_1 - A_1^{\mathrm{ph}})) + g_R^2\tilde{A}_R(s)] \\ &= Z^{-1}\left[(s - m_R^2)\left(1 - \frac{g_R^2}{384\pi^2}\ln\frac{m^2}{(m^{\mathrm{ph}})^2}\right) + g_R^2\tilde{A}_R(s)\right], \end{aligned} \quad \text{(A20)}$$

where the definition of the renormalized coupling g_R [Eq. (A14)] and the explicit form of A_1 [Eq. (A5)] were used. Therefore, for the "running" pion mass instead of Eq. (A13) one has

$$\frac{g_0^2}{D(s)} = \frac{g_0^2}{Z^{-1}[(s - m_R^2)\zeta^{-1} + g_R^2\tilde{A}_R(s)]} = \frac{g_R^2}{(s - m_R^2)\zeta^{-1} + \tilde{\Sigma}_R(s)} \equiv \frac{g_R^2}{D_R(s)}, \quad \text{(A21)}$$

where

$$\zeta^{-1} \equiv 1 - \frac{g_R^2}{384\pi^2} \ln \frac{m_\pi^2}{(m_\pi^{\rm ph})^2} \tag{A22}$$

and

$$\tilde{\Sigma}_R(s) = g_R^2 [A_{\rm reg}(s) - {\rm Re} A_{\rm reg}(m_R^2) - {\rm Re} A'^{\rm ph}_{\rm reg}(m_R^2)(s - m_R^2)]. \tag{A23}$$

After integrating over the zeroth component in the loop (one picks the D meson to be on energy shell), one finds

$$\begin{aligned} s - m_{*0}^2 &\approx 2E_{D^{*0}}(\boldsymbol{p})(p_0 - E_{D^{*0}}(\boldsymbol{p})) \\ &= 2E_{D^{*0}}(\boldsymbol{p})(M - E_{D^0}(\boldsymbol{p}) - E_{D^{*0}}(\boldsymbol{p})). \end{aligned} \tag{A24}$$

Therefore, one arrives at the formula

$$D_{D^{*0}}(\boldsymbol{p}) = \frac{2E_{D^{*0}}(\boldsymbol{p})}{\zeta} [M - E_{D^0}(\boldsymbol{p}) - E_{D^{*0}}(\boldsymbol{p})] + \tilde{\Sigma}_R(\boldsymbol{p}) \tag{A25}$$

for the renormalized inverse D^{*0} propagator valid for unphysical pion masses. Obviously, a similar formula holds for the inverse D^{*c} propagator for the charged particles as well.

The inverse two-body propagators Δ_0 and Δ_c entering the system of equations (17) can be written as

$$\Delta_0(\boldsymbol{p}) = -\frac{E_{D^0}(\boldsymbol{p})}{m_0} \frac{D_{D^{*0}}(\boldsymbol{p})}{2m_{*0}}, \qquad \Delta_c(\boldsymbol{p}) = -\frac{E_{D^c}(\boldsymbol{p})}{m_c} \frac{D_{D^{*c}}(\boldsymbol{p})}{2m_{*c}}. \tag{A26}$$

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