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## FREE VIBRATION OF PLATE ASSEMBLIES USING SPECTRAL ELEMENT METHOD

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**Abstract** –In this paper, the Spectral Element Method (SEM) is applied to analyze free vibration of two plates perpendicularly assembled, so-called L plate. In order to obtain necessary results, the transformation matrices have been developed for two positions of rectangular plates. Numerical example is conducted for L plate assemblies consisting of rectangular plates with same mechanical and geometrical properties. The accuracy of the results obtained by the SEM is verified by comparing them with the solutions obtained by conventional Finite Element Method (FEM).

### 1. INTRODUCTION

Plates form many structural components ranging from walls and floors of high-rise buildings, panels in ship hull and aircraft, to printed circuit boards and silicon chips. It is rare that plates are independent in the structures, but assembled. The most common solution method for analysing these types of structures is the Finite Element Method (FEM) [2]. The size of the finite element depends on the highest frequency in the analysis. Consequently, to gain accurate results for large structures with high eigenfrequencies it is required a lot of finite elements and the increase the number of finite elements takes greater computer time and effort to solve the problem. As an alternative to the FEM in dynamic analysis the Spectral Element Method can be used. The SEM is based on the spectral representation of the displacement field and on the exact solution of the governing equations of motion defined in the frequency domain. Consequently, the dynamic stiffness matrix is frequency dependent, i.e. the analysis is performed in the frequency domain. The SEM is especially useful for one-dimensional elements where the accurate solutions for governing equations of motions are obtained. However, for two-dimensional elements, it is not possible to obtain exact solutions of partial differential equations that satisfy arbitrary boundary conditions. In order to find a solution of a problem, plate displacements are presented as infinite Fourier type series. For practical purposes, the series have to be truncated, which introduces an error. Consequently, the solutions are approximate and satisfy the prescribed degree of accuracy.

The procedure for the development of the dynamic stiffness matrix for rectangular plate undergoing in-plane and transverse vibration can be found in the literature, [4], [7]. Earlier studies of modal characteristics of plate assemblies

were conducted for plates with specific boundary conditions. Bercin [2] analyzed plate assemblies that were simply-supported along longitudinal edges.

The main objective of this paper is to present the development of the transformation matrix necessary for obtaining the dynamic stiffness matrix of L plate assembly for general case of boundary conditions. The obtained results were compared with the results obtained by the FEM.

### 2. DYNAMIC STIFFNESS MATRIX OF PLATES

General form of equation of motion of the plates in the frequency domain without presence of external load can be given as:

$$L(\mathbf{u}) + \rho h \omega^2 \mathbf{u} = 0 \quad (1)$$

where  $\mathbf{u} = \mathbf{u}(x, y)$  is displacement vector,  $\rho$  is the mass density,  $h$  is the plate thickness,  $\omega$  is the circular frequency and  $L$  is the differential operator.

In order to find a solution of equation of motion, plate displacements are presented as series:

$$u(x, y) \approx \sum_{m=1}^M C_m f_m(x, y) \quad (2)$$

where  $C_m$  are integration constants and  $f_m(x, y)$  are base functions that satisfy Eq. (1).

Relation between the displacements  $\hat{q}(s)$  and forces  $\hat{Q}(s)$  along the boundaries is obtained by performing so-called projection method [1]. This method is based on projections of the displacements and forces on the boundaries onto a set of functions  $h(s)$ :

$$\begin{aligned} \hat{q}(s) &\approx \sum_{n=1}^M \langle \hat{q}, h_n \rangle h_n(s) \\ \hat{Q}(s) &\approx \sum_{n=1}^M \langle \hat{Q}, h_n \rangle h_n(s) \end{aligned} \quad (3)$$

where  $\tilde{q}_n = \langle \hat{q}, h_n \rangle$  is projection of displacements and  $\tilde{Q}_n = \langle \hat{Q}, h_n \rangle$  is projection of forces along boundaries.

Projections of the displacements and forces along boundaries are collected into vector:

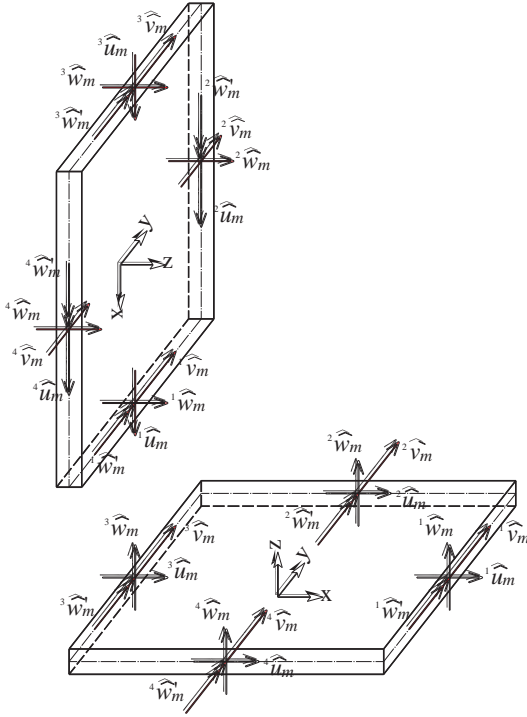
$$\tilde{\mathbf{q}} = [\langle \hat{q}, h_n \rangle] \quad \tilde{\mathbf{Q}} = [\langle \hat{Q}, h_n \rangle] \quad (4)$$

Now, it is possible to define a relation between displacements and forces for general case by using a diagonal dynamic stiffness matrix:

$$\tilde{\mathbf{K}}_D = \begin{bmatrix} \tilde{\mathbf{K}}_{D_t} & 0 \\ 0 & \tilde{\mathbf{K}}_{D_i} \end{bmatrix} \quad (5)$$

where  $\mathbf{K}_{D_t}$  is dynamic stiffness matrix for out-of-plane vibration and  $\mathbf{K}_{D_i}$  is dynamic stiffness matrix for in-plane vibration. Presented dynamic stiffness matrix (5) gives the relation between the displacement vector  $\tilde{\mathbf{q}}$  and the force vector  $\tilde{\mathbf{Q}}$ :

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{K}}_D \tilde{\mathbf{q}} \quad (6)$$



**Figure 1.** Edge displacements for L plate in local coordinate systems

The force vector  $\tilde{\mathbf{Q}}$  is defined as:

$$\begin{aligned} \tilde{\mathbf{Q}}^T &= [\tilde{\mathbf{Q}}_t \quad \tilde{\mathbf{Q}}_i] \\ \tilde{\mathbf{Q}}_t^T &= [\tilde{\mathbf{Q}}_{0,t} \quad \tilde{\mathbf{Q}}_{1,t} \quad \dots \quad \tilde{\mathbf{Q}}_{m,t} \quad \dots \quad \tilde{\mathbf{Q}}_{M,t}] \\ \tilde{\mathbf{Q}}_{m,t}^T &= [{}^1\tilde{\mathbf{Q}}_{m,t}^T \quad {}^2\tilde{\mathbf{Q}}_{m,t}^T \quad {}^3\tilde{\mathbf{Q}}_{m,t}^T \quad {}^4\tilde{\mathbf{Q}}_{m,t}^T] \\ {}^j\tilde{\mathbf{Q}}_{0,t}^T &= [{}^j\bar{T}_{x_{S_0}} \quad {}^jM_{x_{S_0}}] \quad {}^k\tilde{\mathbf{Q}}_{0,t}^T = [{}^j\bar{T}_{y_{S_0}} \quad {}^jM_{y_{S_0}}] \\ {}^j\tilde{\mathbf{Q}}_{m,t}^T &= [{}^j\bar{T}_{x_{S_m}} \quad {}^j\bar{T}_{x_{A_m}} \quad {}^jM_{x_{S_m}} \quad {}^jM_{x_{A_m}}] \\ {}^k\tilde{\mathbf{Q}}_{m,t}^T &= [{}^k\bar{T}_{y_{S_m}} \quad {}^k\bar{T}_{y_{A_m}} \quad {}^kM_{y_{S_m}} \quad {}^kM_{y_{A_m}}] \end{aligned} \quad (7)$$

$$\begin{aligned} \tilde{\mathbf{Q}}_i^T &= [\tilde{\mathbf{Q}}_{0,i} \quad \tilde{\mathbf{Q}}_{1,i} \quad \dots \quad \tilde{\mathbf{Q}}_{m,i} \quad \dots \quad \tilde{\mathbf{Q}}_{M,i}] \\ \tilde{\mathbf{Q}}_{0,i}^T &= [{}^1N_{x_{S_0}} \quad {}^2N_{y_{S_0}} \quad {}^3N_{x_{S_0}} \quad {}^4N_{y_{S_0}}] \\ \tilde{\mathbf{Q}}_{m,i}^T &= [{}^1\tilde{\mathbf{Q}}_{m,i}^T \quad {}^2\tilde{\mathbf{Q}}_{m,i}^T \quad {}^3\tilde{\mathbf{Q}}_{m,i}^T \quad {}^4\tilde{\mathbf{Q}}_{m,i}^T] \\ {}^j\tilde{\mathbf{Q}}_{m,i}^T &= [{}^jN_{x_{S_m}} \quad {}^jN_{x_{A_m}} \quad {}^jN_{xy_{S_m}} \quad {}^jN_{xy_{A_m}}] \\ {}^k\tilde{\mathbf{Q}}_{m,i}^T &= [{}^kN_{y_{S_m}} \quad {}^kN_{y_{A_m}} \quad {}^kN_{xy_{S_m}} \quad {}^kN_{xy_{A_m}}] \\ j &= 1, 3; \quad k = 2, 4 \end{aligned}$$

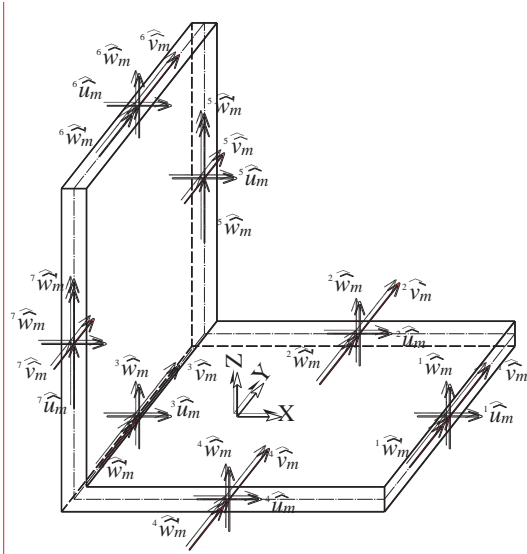
The displacement vector  $\tilde{\mathbf{q}}$  is defined as:

$$\begin{aligned} \tilde{\mathbf{q}}^T &= [\tilde{\mathbf{q}}_t \quad \tilde{\mathbf{q}}_i] \\ \tilde{\mathbf{q}}_t^T &= [\tilde{\mathbf{q}}_{0,t} \quad \tilde{\mathbf{q}}_{1,t} \quad \dots \quad \tilde{\mathbf{q}}_{m,t} \quad \dots \quad \tilde{\mathbf{q}}_{M,t}] \\ \tilde{\mathbf{q}}_{m,t}^T &= [{}^1\tilde{\mathbf{q}}_{m,t} \quad {}^2\tilde{\mathbf{q}}_{m,t} \quad {}^3\tilde{\mathbf{q}}_{m,t} \quad {}^4\tilde{\mathbf{q}}_{m,t}] \\ {}^j\tilde{\mathbf{q}}_{0,t}^T &= [{}^jW_{S_0} \quad {}^jW'_{S_0}] \\ {}^j\tilde{\mathbf{q}}_{m,t}^T &= [{}^jW_{S_m} \quad {}^jW_{A_m} \quad {}^jW'_{S_m} \quad {}^jW'_{A_m}] \\ \tilde{\mathbf{q}}_i^T &= [\tilde{\mathbf{q}}_{0,i} \quad \tilde{\mathbf{q}}_{1,i} \quad \dots \quad \tilde{\mathbf{q}}_{m,i} \quad \dots \quad \tilde{\mathbf{q}}_{M,i}] \\ \tilde{\mathbf{q}}_{0,i} &= [{}^1u_{S_0} \quad {}^2v_{S_0} \quad {}^3u_{S_0} \quad {}^4v_{S_0}] \\ \tilde{\mathbf{q}}_{m,i}^T &= [{}^1\tilde{\mathbf{q}}_{m,i} \quad {}^2\tilde{\mathbf{q}}_{m,i} \quad {}^3\tilde{\mathbf{q}}_{m,i} \quad {}^4\tilde{\mathbf{q}}_{m,i}] \\ {}^j\tilde{\mathbf{q}}_{m,i}^T &= [{}^ju_{S_m} \quad {}^ju_{A_m} \quad {}^jv_{S_m} \quad {}^jv_{A_m}] \\ j &= 1, 2, 3, 4 \end{aligned} \quad (8)$$

Represented vectors are given in the local coordinate system. The plate's displacements in the local coordinate system are given in Fig. 1.

### 3. TRANSFORMATION OF PLATE DYNAMIC STIFFNESS MATRIX

As shown in the previous section, in the SEM the force-displacement relation of plate is defined by the dynamic stiffness matrix. Therefore, the same assemblage procedure is used as in the FEM. First, it is necessary to transform displacements and forces along the edges from local coordinate system to global coordinate system. Fig. 1. shows the system consisting of two plates, so-called L plate, with edge displacements in the local coordinate systems. It is necessary to transform the displacements in the global coordinate system, Figure 2.



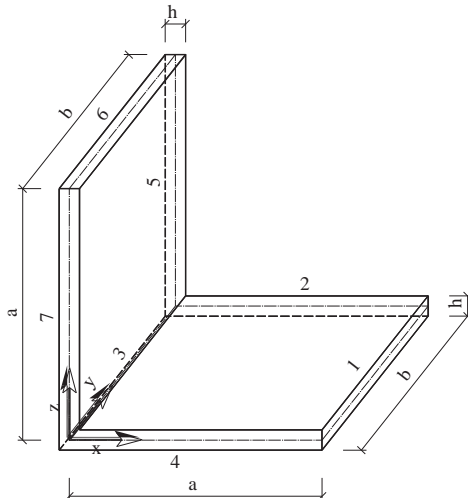
**Figure 2.** Edge displacement of L plate in global coordinate system

Using the transformation matrix the displacements have been transformed:

$$\tilde{\mathbf{q}} = \mathbf{T}_T \mathbf{q}_T \quad (9)$$

where vector  $\tilde{\mathbf{q}}$  is given in Eq. (8), and displacement vector in global coordinate system is defined as:

$$\begin{aligned} \tilde{\mathbf{q}}_T^T &= [\tilde{\mathbf{q}}_0 \quad \tilde{\mathbf{q}}_1 \quad \dots \quad \tilde{\mathbf{q}}_m \quad \dots \quad \tilde{\mathbf{q}}_M] \\ \tilde{\mathbf{q}}_0^T &= [{}^1\tilde{\mathbf{q}}_0 \quad {}^2\tilde{\mathbf{q}}_0 \quad {}^3\tilde{\mathbf{q}}_0 \quad {}^4\tilde{\mathbf{q}}_0] \\ {}^j\tilde{\mathbf{q}}_0^T &= [{}^j u_{S_0} \quad {}^j v_{S_0} \quad {}^j w'_{S_0}] \\ {}^k\tilde{\mathbf{q}}_0^T &= [{}^k v_{S_0} \quad {}^k w_{S_0} \quad {}^k w'_{S_0}] \\ \tilde{\mathbf{q}}_m^T &= [{}^1\tilde{\mathbf{q}}_m \quad {}^2\tilde{\mathbf{q}}_m \quad {}^3\tilde{\mathbf{q}}_m \quad {}^4\tilde{\mathbf{q}}_m] \\ {}^i\tilde{\mathbf{q}}_m^T &= [{}^i u_m \quad {}^i v_m \quad {}^i w_m \quad {}^i w'_m] \\ {}^i \mathbf{r}_m &= [{}^i r_{S_m} \quad {}^i r_{A_m}] \quad i = 1, 2, 3, 4; \quad r = u, v, w, w' \end{aligned} \quad (10)$$



**Figure 3.** L plate with dimensions and edge numeration

**Table 1.** Eigenfrequencies of L plate assembly

SEM			FEM			
			Mesh size			
M=1	M=3	M=10	5x10	10x20	20x40	40x80
15.0	15.1	15.1	14.8	15.1	15.1	15.1
19.3	19.3	19.3	18.7	19.1	19.2	19.3
52.7	53.0	53.1	49.7	52.2	52.9	53.0
65.7	65.9	65.9	60.8	64.5	65.5	65.8
66.9	67.2	67.3	62.7	66.1	67.0	67.2
70.4	70.6	70.3	66.0	69.4	70.3	70.6
117.3	116.0	115.8	106.5	115.3	115.7	115.8
118.4	119.5	119.6	112.7	116.2	118.8	119.4
130.8	131.6	131.7	116.6	127.5	130.6	131.4
133.8	135.1	135.2	126.0	133.1	134.7	135.1

It is necessary to transform edge forces too, using the same transformation matrix:

$$\tilde{\mathbf{Q}} = \mathbf{T}_T \mathbf{Q}_T \quad (11)$$

where vector  $\tilde{\mathbf{Q}}$  is given in Eq. (7), and force vector in global coordinate system is defined as:

$$\begin{aligned} \tilde{\mathbf{Q}}_T^T &= [\tilde{\mathbf{Q}}_0 \quad \tilde{\mathbf{Q}}_1 \quad \dots \quad \tilde{\mathbf{Q}}_m \quad \dots \quad \tilde{\mathbf{Q}}_M] \\ \tilde{\mathbf{Q}}_0^T &= [{}^1\tilde{\mathbf{Q}}_0 \quad {}^2\tilde{\mathbf{Q}}_0 \quad {}^3\tilde{\mathbf{Q}}_0 \quad {}^4\tilde{\mathbf{Q}}_0] \\ {}^j\tilde{\mathbf{Q}}_0^T &= [{}^j N_{x_{S_0}} \quad {}^j \bar{T}_{x_{S_0}} \quad {}^j M_{x_{S_0}}] \\ {}^k\tilde{\mathbf{Q}}_0^T &= [{}^k N_{y_{S_0}} \quad {}^k \bar{T}_{y_{S_0}} \quad {}^k M_{y_{S_0}}] \\ \tilde{\mathbf{Q}}_m^T &= [{}^1\tilde{\mathbf{Q}}_m \quad {}^2\tilde{\mathbf{Q}}_m \quad {}^3\tilde{\mathbf{Q}}_m \quad {}^4\tilde{\mathbf{Q}}_m] \\ {}^j\tilde{\mathbf{Q}}_m^T &= [{}^j N_{x_m} \quad {}^j N_{xy_m} \quad {}^j \bar{T}_{x_m} \quad {}^j M_{x_m}] \\ {}^k\tilde{\mathbf{Q}}_m^T &= [{}^k N_{y_m} \quad {}^k N_{xy_m} \quad {}^k \bar{T}_{y_m} \quad {}^k M_{y_m}] \\ {}^j \mathbf{r}_m &= [{}^j r_{S_m} \quad {}^j r_{A_m}] \quad {}^k \mathbf{r}_m = [{}^k r_{S_m} \quad {}^k r_{A_m}] \\ & \quad j = 1, 3; \quad k = 2, 4 \\ {}^j r_m &= [{}^j N_{x_m}, {}^j N_{xy_m}, {}^j \bar{T}_{x_m}, {}^j M_{x_m}] \\ {}^k r_m &= [{}^k N_{y_m}, {}^k N_{xy_m}, {}^k \bar{T}_{y_m}, {}^k M_{y_m}] \end{aligned} \quad (12)$$

Substituting Eq. (9) and Eq. (11) into Eq. (6) the relation between the force vector and displacement vector in global coordinate system is obtained:

$$\tilde{\mathbf{Q}}_T = \mathbf{T}_T^T \tilde{\mathbf{K}}_D \mathbf{T}_T \tilde{\mathbf{q}}_T = \tilde{\mathbf{K}}_{D_r} \tilde{\mathbf{q}}_T \quad (13)$$

where the dynamic stiffness matrix in the global coordinate system is given as:

$$\tilde{\mathbf{K}}_{D_r} = \mathbf{T}_T^T \tilde{\mathbf{K}}_D \mathbf{T}_T \quad (14)$$

In the following, it will be shown the transformation matrix for horizontal plate:

$$\mathbf{T}^T = [\mathbf{T}_1 \quad \mathbf{T}_2] \quad (15)$$

