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FORMING THE CUPOLAE WITH CONCAVE POLYHEDRAL SURFACES BY CORRUGATING A FOURFOLD STRIP OF EQUILATERAL TRIANGLES

Slobodan Mišić[74]
Marija Obradović[75]

RESUME

The cupolae with concave polyhedral surfaces consist of: two regular polygons, n-gone and 2n-gone in parallel planes, interconnected by an envelope constituted of series of equilateral triangles. The paper describes cupolae which originate by corrugating of a fourfold strip of equilateral triangles, forming thereby the envelope of a cupola. In this manner, a non-convex polyhedron is emerged. Such a method of corrugating the envelope, allows the solutions for generating cupolae with base polygon which number of sides exceedes n=10, which was the maximal number of sides for cupole with the envelope consisted of twofold strip of equilateral triangles. By analyzing the elements of these polyhedra and by help of their paper models, we find geometric constructions and projection procedures by which it is possible to graphically display the cupolae.

Key words: cupola, concave polyhedral surface, envelope, model

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1. INTRODUCTION

Convex polyhedron is a polyhedron such that meets the condition that each segment that connects any two points on the inner surfaces of the polyhedron, itself fully belongs to its interior. Otherwise polyhedron is non-convex or concave. Therefore, the convex polyhedra are those whose sides belong to planes that do not intersect the interior of the solid, so that they form a solid whose every vertex is protruding outwards. At the contrary, concave polyhedra include sides that belong to the planes that intersect the interior of the solid, ie, there are at least two adjacent sides of the solid between which the dihedral angle is greater than 180^o. The envelope of the concave polyhedron is a concave polyhedral surface.

Polyhedral cupola is a polyhedron composed of two regular polygons: n-and 2n-gons in parallel planes, associated with alternating series of regular polygons: triangles and squares. The cupolae formed by corrugating the envelope consisting of two series of equilateral triangles have been already described [1], and are therefore called concave cupolae of the second sort, whereat it is accepted that the sort of the cupola is dictated by the number of triangles' stripes in the envelope's net. If the envelope's net consists of three rows of equilateral triangles, the obtained concave cupola will be of the Third Sort [4].

A cupola with the concave polyhedral surface, obtained by corrugating a fourfold strip of equilateral triangles are actually concave cupola of the Fourth Sort, and so far they have not been studied in constructively geometrical manner.

The research began with making a paper model of pentadecagon based cupola, and that base was retained throughout the further research. The paper model was a practical experiment, and at the same time, it was the evidence that it is possible to construct a cupola of the Fourth Sort.

2. STARTING ASSUMPTIONS

In **Figure 1**, it is given the layout of the envelope's net, which consists of the fourfold strip of equilateral triangles, and the basic polygons: pentadecagon and its parallel triacontagon (2x15-sided polygon). By folding and joining the corresponding edges of the net, we obtain a closed envelope, a ring fragment of concave polyhedral surface.

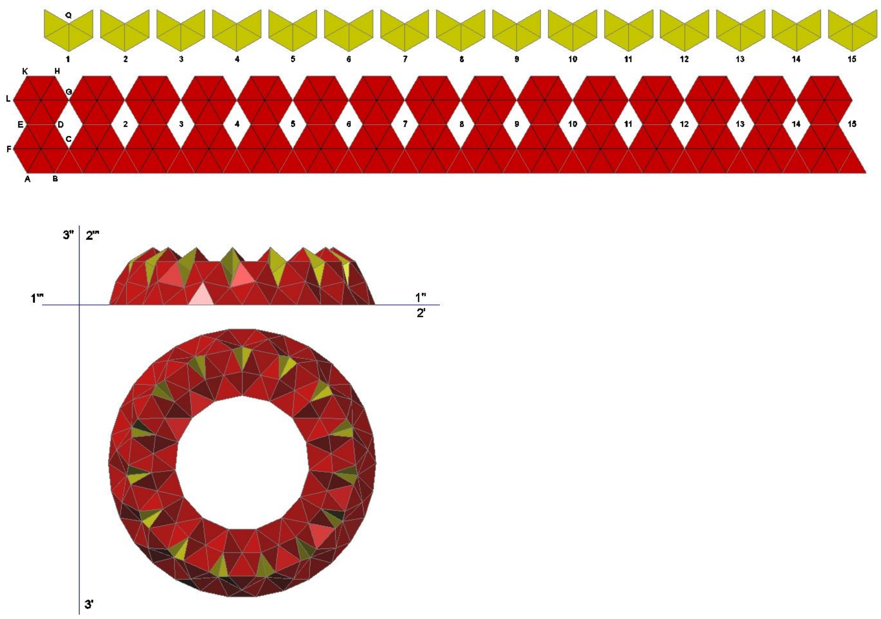

Figure 1: *The Net of the Envelope of the Cupola, Top View and Front View of the Cupola's Envelope*

The initially chosen pentadecagon is taken as a base for further construction. Let us adopt that it lies in the horizontal plane (**x-y**), so to see its original size in the first orthogonal projection. Further on, let us suppose that the adopted base polygon lies above the parallel plane of the double sided brim polygon. The basic and the brim polygons, as seen in the first orthogonal projection, are concentric, i.e. the centers of the circumscribed (and inscribed) circles of these polygons will be seen as merged because they will be situated on the vertical line - the axis of the polyhedral solid.

The construction of ***2n***-sided regular polygon, for the assigned ***n-***sided regular polygon, is well known (Euclid 300.B.C.). It is enough just for the radius of the circle circumscribed about an n-sided polygon to be transferred to perpendicular on a side of the polygon, from one of its vertices. The resulting point is the newly created vertex of the ***2n***-sided polygon, while the center of the circumscribed circles is a common for both of the starting ***n*-**sided polygon, and the brim ***2n***-sided polygon.

The **Figure 2** shows the unit cell ***ABCDEF*** participating in the formatting of cupola's lower belt, closer to the brim of the ***2n***-gon. The unit cell ***ABCDEF*** consists of **6** identical equilateral triangles formed around a common vertex, labeled as ***O_1***. By radial array of these identical cells around the axis (***k)*** and by their linking with additional equilateral triangles in the bottom zone, we get a closed geometric entity. On the unit cell of the lower belt, there are attached ***EDGHKL*** cells***,*** also made of ***6*** equilateral triangles arranged around common vertex, marked as ***O_2***.

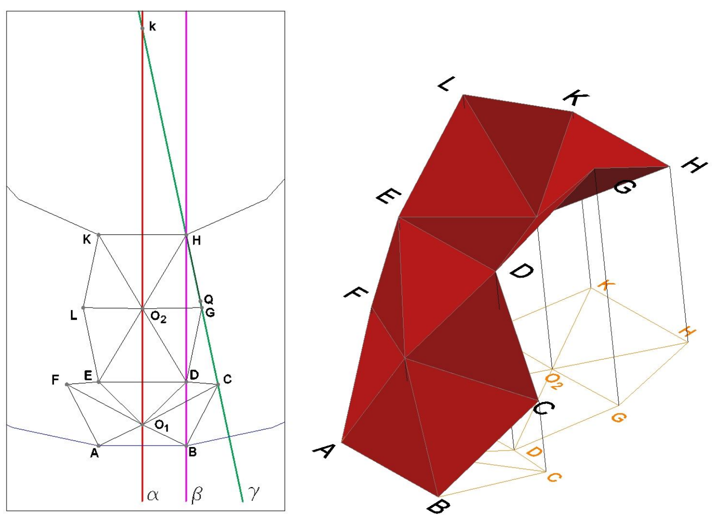

Figure 2: *An Unit Cell of the Cupola – the Base and 3D model of Spatial Hexahedrons ABCDEF and EDGHKL*

In order to define the desired parameters of the surface, it is necessary to set the initial conditions that such a spatial structure must fulfill:

1. The edge **AB** belongs to the horizontal plane (**x-y**) of the triacontagon and this is the plane we adopt for the basic horizontal plane. 2. The edge ***ED*** is parallel to the edge ***AB*** and belongs to the unit cells of lower and upper belt of the concave cupola.

3. The edge ***HK*** belongs to the horizontal plane of pentadecagon which is located above the basic plane.

4. The plane ***α*** is the symmetral plane of the edge ***AB*** and passes tgrough the axis ***k***, which connects the centroids of the bases of the pentadecagon and triacontagon.

5. The plane ***β*** is parallel to the plane ***α*** and passes through the vertices ***B, D*** and ***H***.

6. The plane ***γ*** is the symmetral plane of the contiguous edge ***BX*** of the brim ***30***-gon, and also passes through the axis ***k***.

7. In the plane ***γ,*** we find vertices ***C, G, H*** and an isolated vertex ***Q*** which do not belong to the basic cells, the spatial hexagons ***ABCDEF*** and ***EDGHKL***.

8. Spatial hexagons ***EDGHKL*** and ***ABCDEF*** are planary symmetric with respect to the plane ***α***.

9. In order to determine the solid's height, we set the spatial hexagons ***EDGHKL*** and ***ABCDEF*** to move around the edge ***AB***, which would be fixed axis of rotation.

3. THE CONSTRUCTIVE-GEOMETRICAL GENERATING OF THE CUPOLA

In the **Figure 3** we adopt the edge ***AB*** and the initial position of the first projection of the vertex O_1, the central vertex of the spatial hexagon ***ABCDEF***. The vertex O_1 lies in the plane ***α***. In the vertex O_1 it is located the center of the sphere M_1, which radius equals the edge ***AB*** (***AB=a***) of an equilateral triangle. The vertices of the spatial hexagon ***ABCDEF*** will all lie on the sphere M_1.

The vertex O_1 is of the height h_1 (in relation to the plane ***x-y*** of the ***30-***gon). The height h_1 is determined by the vertex's O_1 location on the plane ***α***. The triangle ABO_1 rotates around the edge ***AB*** by the circle k_1 whose radius equals the altitude of the equilateral triangle of the edge ***a***: $r= a\sqrt{3}/2$. This cycle of movement of the point O_1 is in the extra projection plane (α^{α}) seen in the real size.

The plane ***γ*** intersects the sphere M_1 by the circle c_1. Height of the center of the circle c_1 is equal to the height of the adopted vertex O_1 – the center of the sphere. In the extra projection plane (γ^{γ}) we see the real size and the position of the circle c_1. On the circle c_1, there lies the vertex ***C***, which lies simultaneously on the sphere M_1 and on the plane ***γ***.

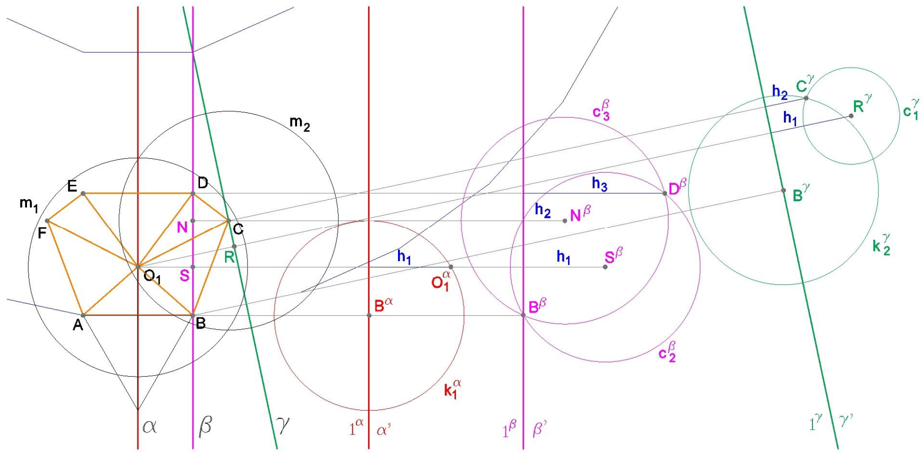

Figure 3: The C*onstruction of the Vertices' Heights of the Spatial Hexahedron ABCDEF for the Adopted Position of the Vertex* $\boldsymbol{O_1}$

In the plane γ^{γ} we see the circle $\boldsymbol{k_2}$, of the radius ***r=a√3/2***, which will be the circular trajectory of the vertex ***C*** rotating around the edge ***BX.*** In the intersection of the circle $\boldsymbol{k_2}$ and the circle $\boldsymbol{c_1}$ we obtain two possible solutions for the position of the vertex **C**. We take the radius of the circle $\boldsymbol{c_1}$ from the first projection (top view), where the circle $\boldsymbol{c_1}$ is seen as a segment. We adopt just one of the solutions, depending on the way we corrugate the envelope of the cupola. There would exist two ways of folding the net, so there are the two solutions for the position of the vertex ***C***.

In the vertex ***C,*** we set the center of the new sphere $\boldsymbol{M_2}$ of radius ***r=a***. The plane ***β*** intersects the sphere $\boldsymbol{M_2}$ by the circle $\boldsymbol{c_3}$**,** and the sphere $\boldsymbol{M_1}$ by the circle $\boldsymbol{c_2}$. In the additional projection plane (β^{β}**)** we see the real size of the circles $\boldsymbol{c_3}$ and $\boldsymbol{c_2}$. The height of the center ***S*** of the circle $\boldsymbol{c_2}$ equals the height of the vertex $\boldsymbol{O_1}$. The height of the center ***N*** of the circle $\boldsymbol{c_3}$ is equal to the height of the vertex ***C***. The intersection of circles $\boldsymbol{c_2}$ and $\boldsymbol{c_3}$ obtains the position of the vertex ***D***. The second section point of the circles $\boldsymbol{c_2}$ and $\boldsymbol{c_3}$ is the point ***B***, which is the graphic control of the whole process of determining vertices ***C*** and ***D***.

In order to determine the position of the vertices of the spatial hexagon ***EDGHKL*** (**Figure 4**) we set the sphere $\boldsymbol{M_3}$ in the vertex ***D***. the sphere radius is always ***r=a***. On the sphere $\boldsymbol{M_3}$ lie the vertices ***G,*** $\boldsymbol{O_2}$***, E,*** $\boldsymbol{O_1}$***, C,*** and concave-periphery vertex ***Q***. The plane ***γ*** intersects the sphere $\boldsymbol{M_3}$ by the circle $\boldsymbol{c_4}$ centered at point ***P***. In the additional projection plane ($\boldsymbol{\gamma^{\gamma}}$) we see the real size of circle $\boldsymbol{c_4}$. The height of the central point ***P*** is equal to the height of the vertex ***D***.

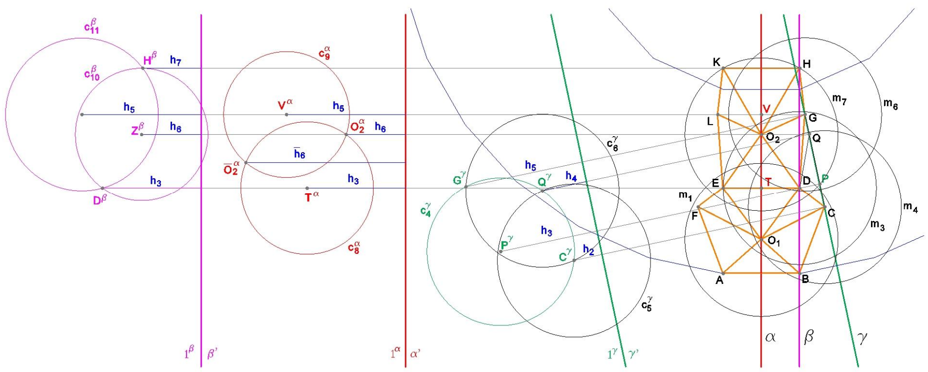

Figure 4: *The Construction of the Heights of Spatial Hexahedron EDGHKL Vertices, and of Peripheral Vertex* ***Q***

In the vertex ***C*** we set the sphere $\boldsymbol{M_4}$ of radius ***R=a***. The plane ***γ*** intersects the sphere $\boldsymbol{M_4}$ by the circle $\boldsymbol{c_5}$. The center of the circle $\boldsymbol{c_5}$ is the point ***C*** which height we have already defined. The intersection of circles $\boldsymbol{c_4}$ and $\boldsymbol{c_5}$, in the projection plane ($\boldsymbol{\gamma^{\gamma}}$) will provide the solution to the position of the point ***Q***. Again, the two solutions will occur, one of which will be adopted, according to pre-set conditions.

In the vertex ***Q*** we set the sphere $\boldsymbol{M_5}$, which intersects the plane ***γ*** by the circle $\boldsymbol{c_6}$. On the sphere $\boldsymbol{M_5}$ there are situated vertices ***G, D,*** and ***C***. The vertices ***G*** and ***C*** belong to both the spheres $\boldsymbol{M_3}$ and $\boldsymbol{M_5}$. In the projection plane ($\boldsymbol{\gamma^{\gamma}}$), at the intersection points of the circles $\boldsymbol{c_4}$ and $\boldsymbol{c_6}$ we obtain the position of the vertex ***G***. The second intersection point of the circles $\boldsymbol{c_4}$ and $\boldsymbol{c_6}$ is the point ***C***, which is the graphic control of the whole process of determining the vertices ***Q*** and ***G***.

The vertex $\boldsymbol{O_2}$ is located both on the sphere $\boldsymbol{M_3}$ (centered in the vertex ***D***) and on the sphere $\boldsymbol{M_6}$ (centered in the vertex ***G***). The plane ***α*** intersects the sphere $\boldsymbol{M_3}$ by the circle $\boldsymbol{c_8}$***,*** and the sphere $\boldsymbol{M_6}$ by the circle $\boldsymbol{c_9}$. The center of the circle $\boldsymbol{c_9}$ is the point ***V***. The height of the point ***V*** is equal to the height of the vertex ***G***. The center of the circle $\boldsymbol{c_8}$ is the point ***T***, which is located on ***DE*** and its height is equal to the height of vertex ***D***. In the projection plane ($\boldsymbol{a^{a}}$) we see the real size of the circles $\boldsymbol{c_8}$ and $\boldsymbol{c_9}$ in which intersections we find the vertex $\boldsymbol{O_2}$. We obtain the two solutions, and choose the one that meets the initial requirements.

The vertex ***H*** is obtained in the intersection point of the sphere $\boldsymbol{M_7}$ (centered in the vertex $\boldsymbol{O_2}$) and the sphere $\boldsymbol{M_6}$ (centered in the vertex ***G***), and the intersection of circles $\boldsymbol{c_{10}}$ and $\boldsymbol{c_9}$. The second intersection point of the circles $\boldsymbol{c_{10}}$ and $\boldsymbol{c_9}$ is the point ***D***. In this way we found all the vertices of the spatial patterns, but without meeting the conditions for the vertices of ***H*** and ***K*** to belong to the pentadecagon, whose center of the circumscribed circle is merged with the center of the circle circumscribed around the basic triacontagon of the edge ***AB***.

In **Figure 5** and ***6*** we repeat the procedure several times in order to get an approximation of the trajectory of the vertex ***H,*** depending on the position of the originally adopted vertex $\boldsymbol{A_1}$. Respecting the initial assumptions, the vertex $\boldsymbol{O_1}$ is always chosen to belong to the plane ***a*** and above the horizontal plane of starting triacontagon.

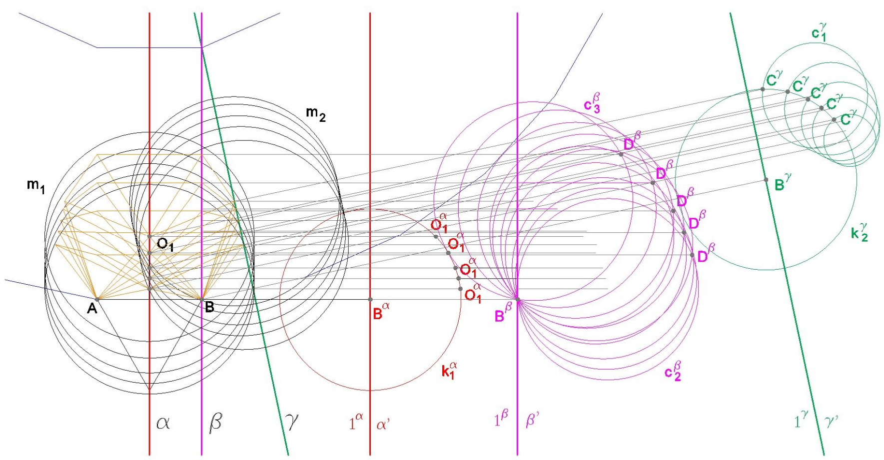

Figure 5: *Determination of the Trajectory of Vertex D by changing the position of the vertex O_1 – The Center of the Spatial Hexahedron ABCDEF*

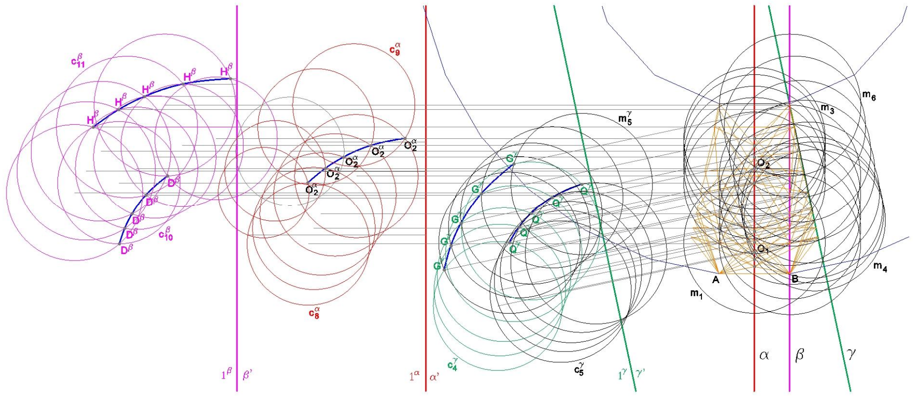

Figure 6: *Determination of the Trajectory of Vertex H by changing the position of the vertex O_1 – The Center of the Spatial Hexahedron ABCDEF*

By intersection of the trajectory of the vertex ***H*** and vertical plane (***v***), ***Figure 7***, where the required position of vertex ***H*** is expected, we get the final position of the vertices of ***H***. Vertical plane (***v***) is assigned by the position of the point ***H*** as a vertex of the initial pentadecagon.

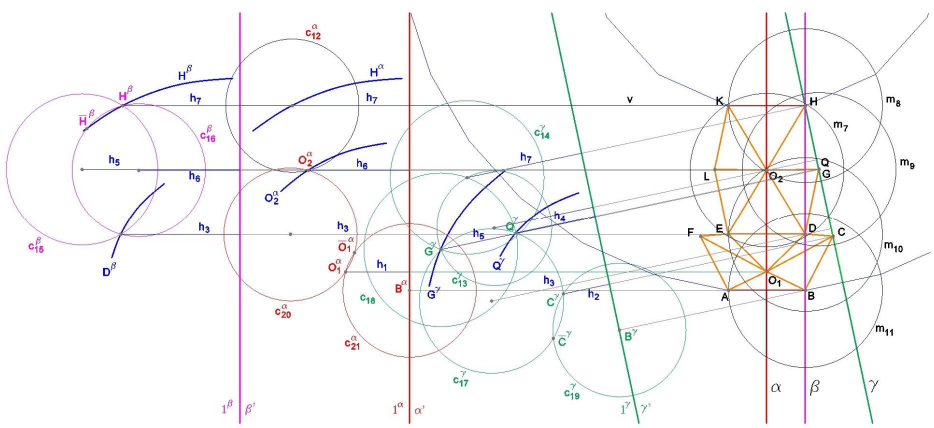

Figure 7: *Construction of the Position and Height of All the Vertices of the Pentadecagonal Based Cupola's Unit Cell*

With the known position of the vertex ***H***, by the reverse constructive steps, we obtain the remaining vertices of the spatial hexagonal unit assemblies.

1. $\alpha \cap M_8(H, a) = c_{12}$

   The intersection of the plane α and the sphere M_8 (the sphere's center point is ***H*** and its radius is equal to the edge (***a***) of an equilateral triangle) is the circle c_{12}. The intersection of the circle c_{12} and the trajectory of the vertex O_2 we obtain the height of the vertex O_2.

2. $\gamma \cap M_7(O_2; a) = c_{13}$, $\gamma \cap M_8(H; a) = c_{14}$

   $c_{13} \cap c_{14} = G$
3. $\beta \cap M_9(G; a) = c_{15}$, $\beta \cap M_7(O_2; a) = c_{16}$

   $c_{15} \cap c_{16} = D$
4. $\gamma \cap M_{10}(D; a) = c_{17}$, $\gamma \cap M_9(G; a) = c_{18}$

   $c_{17} \cap c_{18} = Q$
5. $\gamma \cap M_{11}(B; a) = c_{19}$, $c_{19} \cap c_{17} = C$
6. $\alpha \cap M_{10}(D; a) = c_{20}$, $\alpha \cap M_{11}(B; a) = c_{21}$

   $c_{20} \cap c_{21} = O_1$

In the presented procedure for graphically determining the vertices of the spatial hexagons ***EDGHKL*** and ***ABCDEF,*** we obtain two solutions for the position of the vertices ***H,*** O_2**,** ***C*** and O_1. We adopt one of the solutions, according to pre-set initial conditions. The spatial structure of two hexagons ***ABCDEF*** and ***EDGHKL***, with a common edge ***DE*** is plane symmetric with respect to the plane α. In this manner, we found the location and height of all the vertices of unit cells of the cupola with concave polyhedral surfaces formed by corrugating the fourfold strip of equilateral triangles. The whole cupola is presented in the **Figure 8**[76]**.**

[76] The 3D model of the cupola is provided by application of AutoCAD software package.

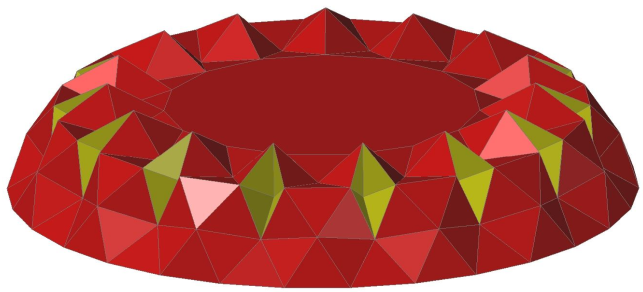

Figure 8: *3D Model of the Pentadecagonal Based Cupola*

4. CONCLUSIONS

In this paper we have shown the graphical-constructive method for finding the positions of the vertices of a complex, concave polyhedron – the concave cupola with pentadecagonal base. This method can be applied to other bases (n < 21), which makes it unique process for obtaining the concave cupola of the Fourth Sort. Since these polyhedra consist of regular polygons, and their envelopes exclusively from equilateral triangles, it makes these structures very inspiring to consider in the possible application in architecture and construction, and in other branches of engineering, as well.

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