NORTH-HOLLAND

A Remark on Equidistance in Hilbert Spaces

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ABSTRACT

It is shown that in a Hilbert space the only infinite equidistant systems of points are those obtained by translation and scaling from (in the complex case: almost) orthonormal systems.

1. INTRODUCTION

A trivial geometric argument leads to the conclusion that a maximal system of equidistant points in the three-dimensional Euclidean space consists of the four corners of a tetrahedron.

In this note we want to discuss the structure of equidistant sets in arbitrary Hilbert spaces. Our results imply a characterization of infinite orthonormal sets in Hilbert spaces which, though completely elementary, seems to have remained unnoticed in the literature.

2. FINITE EQUIDISTANT SETS IN REAL HILBERT SPACES

THEOREM 1.

(a) *Let H be a real Hilbert space, and let $x_1, \ldots, x_m \in H$ be given such that*

$$|x_j - x_k| = c > 0 \qquad (1 \le j < k \le m) \tag{1}$$

and

$$x_1 + \cdots + x_m = 0. \tag{2}$$

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Then

$$|x_1| = \cdots = |x_m| = \frac{c}{\sqrt{2}}\sqrt{1 - \frac{1}{m}} \tag{3}$$

follows, and any $m-1$ of the x_j are linearly independent.

(b) *If for two systems of m points in H (1) holds for the same constant c, then the systems are congruent.*

REMARK 1. If we do not refer to Euclidean norms, the assertions become false in general; e.g., in $\mathbf{R}^n$ with the Chebyshev norm $\|\cdot\|_\infty$ all 2^n points having components 0 and/or 1 form an equidistant system, whereas in the Euclidean case according to the theorem a maximum equidistant set in $\mathbf{R}^n$ consists of $n+1$ points.

Proof. In order to prove (a) we first derive (3) by induction on m.

The cases $m = 1$ and $m = 2$ being trivial, we assume that (3) holds for some $m \geq 2$. Now let $x_1, \ldots, x_m, x_{m+1} \in H$ be given such that $x_1 + \cdots + x_{m+1} = 0$, $|x_j - x_k| = c > 0$ for $1 \leq j < k \leq m+1$. We put

$$x_k^* = x_k + \frac{x_{m+1}}{m} \qquad (1 \leq k \leq m),$$

whence by assumption

$$|x_1^*| = \cdots = |x_m^*| = \frac{c}{\sqrt{2}}\sqrt{1 - \frac{1}{m}}.$$

Since $|x_k - x_{m+1}| = c$ $(1 \leq k \leq m)$, we get

$$\begin{aligned}\left(1 + \frac{1}{m}\right)\left(|x_k|^2 + \frac{1}{m}|x_{m+1}|^2\right) &= |x_k^*|^2 + \frac{1}{m}|x_k - x_{m+1}|^2 \\ &= \frac{c^2}{2}\left(1 - \frac{1}{m}\right) + \frac{c^2}{m}.\end{aligned}$$

Hence $|x_1| = \cdots = |x_m|$ follows, and thus (since $m \geq 2$ and the enumeration is arbitrary) also $|x_{m+1}| = |x_1|$.

We therefore get for $1 \leq k \leq m+1$

$$\begin{aligned}|x_k|^2 &= \left(\frac{m}{m+1}\right)^2\left(\frac{c^2}{2}\left(1 - \frac{1}{m}\right) + \frac{c^2}{m}\right) \\ &= \frac{c^2}{2}\left(1 - \frac{1}{m+1}\right).\end{aligned}$$

This completes the proof of (3).

We now prove the linear independence. To this end we first observe that because of

$$(x_j, x_k) = \tfrac{1}{2}(|x_j|^2 + |x_k|^2 - |x_j - x_k|^2) \tag{4}$$

we get $(x_j, x_k) = -(c^2/2m)(1 \leq j < k \leq m)$ from (1) and (3). Hence multiplying

$$\lambda_1 x_1 + \cdots + \lambda_{m-1} x_{m-1} = 0 \tag{5}$$

by x_m leads to $\lambda_1 + \cdots + \lambda_{m-1} = 0$, and thus multiplying (5) by x_1 gives $\lambda_1 c^2/2 = 0$, whence $\lambda_1 = 0$.

By symmetry $\lambda_1 = \cdots = \lambda_{m-1} = 0$, and also the linear independence of any $m-1$ of the x_j follows.

We prove (b) by induction on m. The cases $m = 1$ and $m = 2$ being trivial, we assume that we are given two equidistant systems of $m+1$ points $x_1, \ldots, x_m, x_{m+1}$ and $y_1, \ldots, y_m, y_{m+1}$. Without loss of generality we suppose $x_1 + \cdots + x_m = y_1 + \cdots + y_m = 0$.

Then by (3) and (4) we get

$$(x_{m+1}, x_k) = \frac{1}{2}\left[|x_{m+1}|^2 - \frac{c^2}{2}\left(1 + \frac{1}{m}\right)\right] \qquad (1 \leq k \leq m),$$

whence

$$0 = (x_{m+1}, x_1 + \cdots + x_m) = \frac{m}{2}\left[|x_{m+1}|^2 - \frac{c^2}{2}\left(1 + \frac{1}{m}\right)\right],$$

and thus x_{m+1} is orthogonal to the $(m-1)$-dimensional subspace spanned by $x_1, \ldots, x_m$, and $|x_{m+1}| = (c/\sqrt{2})\sqrt{1 + 1/m}$. An analogous reasoning holds for the other system.

By assumption there is an orthogonal transformation A such that

$$Ax_k = y_k \qquad (1 \leq k \leq m).$$

Now let B be an orthogonal transformation reducing to the identity on the $(m-1)$-dimensional subspace spanned by $y_1, \ldots, y_m$ and such that $BAx_{m+1} = y_{m+1}$. The existence of B is obvious, since

$$Ax_{m+1} \perp y_1, \ldots, y_m.$$

Hence BA is an orthogonal transformation such that

$$BAx_k = y_k \qquad (1 \leq k \leq m+1),$$

thus proving the congruence of the two given equidistant systems of $m+1$ points. ∎

3. INFINITE EQUIDISTANT SETS IN REAL HILBERT SPACES

THEOREM 2. *If $(x_\lambda)_{\lambda\in\Lambda}$ is an infinite equidistant family in a real Hilbert space H, i.e.*

$$|x_\lambda - x_\mu| = c > 0 \qquad (\lambda, \mu \in \Lambda, \quad \lambda \neq \mu),$$

then there is an $y \in H$ with the following properties:

(i) *The mean-value relation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_{\lambda_n} = y$$

holds for every countably infinite subfamily $(x_{\lambda_n})_{n\in\mathbf{N}}$.

(ii) *The family*

$$\left(\frac{\sqrt{2}}{c}(x_\lambda - y)\right)_{\lambda\in\Lambda}$$

is orthonormal in H.

REMARK 2. The theorem shows that the only infinite equidistant systems in a real Hilbert space are those obtained by translation and scaling from orthonormal systems.

Proof. We only need to prove assertions (i) and (ii) for countably infinite families. The general case then will follow from a straightforward "zigzag" argument.

Thus we suppose in what follows that we are given an infinite equidistant sequence $(x_n)_{n\in\mathbf{N}}$ in H.

The sequence $((x_1 + \cdots + x_n)/n)_{n\in\mathbf{N}}$ being bounded, there is a weakly convergent subsequence

$$\left(\frac{x_1 + \cdots + x_{l_n}}{l_n}\right)_{n\in\mathbf{N}}$$

with a weak limit $y \in H$. (This follows from the well-known Eberlein-Shmulyan theorem; see [1] or [3] for the general formulation, and [2, Theorem 4.25] for the simpler Hilbert-space case.)

By (3) and (4) we get for $j, k \leq l_n$, $j \neq k$

$$\left(x_j - \frac{x_1 + \cdots + x_{l_n}}{l_n}, x_k - \frac{x_1 + \cdots + x_{l_n}}{l_n}\right) = -\frac{c^2}{2l_n},$$

whence for $j, k_1, k_2 \leq l_n, \ \ k_1, k_2 \neq j$

$$\left(x_j - \frac{x_1 + \cdots + x_{l_n}}{l_n}, x_{k_1} - x_{k_2}\right) = 0$$

and thus also

$$(x_j - y, x_{k_1} - x_{k_2}) = 0 \qquad (k_1, k_2 \neq j) \tag{6}$$

follows. If we put $k_1 = k$, $k_2 = 1, 2, \ldots, l_n$ in (6), we get

$$\left(x_j - y, x_k - \frac{x_1 + \cdots + x_{l_n}}{l_n}\right) = \left(x_j - y, \frac{x_k - x_j}{l_n}\right)$$

and hence finally

$$(x_j - y, x_k - y) = 0 \qquad \text{for} \quad j \neq k. \tag{7}$$

Observing $c^2 = |x_j - x_k|^2 = |x_j - y|^2 + |x_k - y|^2$ for $j \neq k$, we conclude that all distances $|x_j - y|$ coincide and that $((\sqrt{2}/c)(x_n - y))_{n \in \mathbf{N}}$ is an orthonormal sequence in H.

Now for any orthonormal sequence $(e_n)_{n \in \mathbf{N}}$ in H we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} e_n\right|^2 = \frac{1}{N} \to 0 \qquad (n \to \infty),$$

whence for y we get

$$\left|\frac{1}{N}\sum_{n=1}^{N} x_n - y\right|^2 = \frac{c^2}{2N}.$$

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4. COMPLEX HILBERT SPACES

As is obvious, the theorems have to be modified if we consider complex scalars. For example, if $(e_n)_{n \in \mathbf{N}}$ is an orthonormal sequence in a complex Hilbert space H, then the sequence

$$(e_1, ie_1, e_2, ie_2, \ldots, e_n, ie_n, \ldots)$$

is equidistant in H.

But since the bijection

$$(z_1, \ldots, z_n) \mapsto (x_1, y_1, \ldots, x_n, y_n)$$

(relating coefficients with respect to two orthonormal bases) from the unitary space $\mathbf{C}^n$ onto the Euclidean space $\mathbf{R}^{2n}$ preserves distances, the results for real spaces apply in a straightforward manner to the complex case. We thus have

THEOREM 3.

(a) *A maximal equidistant system in the unitary space* $\mathbf{C}^n$ *consists of* $2n+1$ *points.*

(b) *If* $u_1, \ldots, u_m$ *are equidistant points in a complex Hilbert space* H *such that*

$$|u_j - u_k| = c > 0 \qquad (1 \leq j < k \leq m) \tag{8}$$

and

$$u_1 + \cdots + u_m = 0$$

then

$$|u_1| = \cdots = |u_m| = \frac{c}{\sqrt{2}}\sqrt{1 - \frac{1}{m}},$$

and any $m-1$ *of the points are linearly independent over the reals.*

(c) *If for two systems of* m *points in* H (8) *holds with the same* c, *then there is an isometry* $f : H \to H$ *mapping the points of one system onto those of the other one.* (f *need not be linear but necessarily is additive and homogeneous with respect to real scalars.*)

(d) *If* $(u_\lambda)_{\lambda \in \Lambda}$ *is an infinite equidistant family in a complex Hilbert space* H, *then there is a* $v \in H$ *such that for every countably infinite subfamily* $(u_{\lambda_n})_{n \in \mathbf{N}}$ *we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_{\lambda_n} = v.$$

Moreover, for all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$

$$\Re(u_\lambda - v, u_\mu - v) = 0 \quad \textit{and} \quad |u_\lambda - v| = |u_\mu - v|.$$

REMARK 3. This shows that the only infinite equidistant systems are the trivial ones, namely those obtained from a system $(e_\lambda)_{\lambda \in \Lambda}$ with $|e_\lambda| = 1$ $(\lambda \in \Lambda)$ and $\Re(e_\lambda, e_\mu) = 0$ $(\lambda \neq \mu)$ by translation and scaling; such a system (e_λ) is "almost" orthonormal.

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