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B_{l4} decays and the extraction of $|V_{ub}|$

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The Cabibbo-Kobayashi-Maskawa matrix element $|V_{ub}|$ is not well determined yet. It can be extracted from both inclusive or exclusive decays, like $B \to \pi(\rho) l \bar{\nu}_l$. However, the exclusive determination from $B \to \rho l \bar{\nu}_l$, in particular, suffers from a large model dependence. In this paper, we propose to extract $|V_{ub}|$ from the four-body semileptonic decay $B \to \pi\pi l \bar{\nu}_l$, where the form factors for the pion-pion system are treated in dispersion theory. This is a model-independent approach that takes into account the $\pi\pi$ rescattering effects, as well as the effect of the ρ meson. We demonstrate that both finite-width effects of the ρ meson as well as scalar $\pi\pi$ contributions can be considered completely in this way.

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I. INTRODUCTION

Precisely determining the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1] plays a very important role in testing the Standard Model. Any deviations from the unitarity of the CKM matrix would be viewed as a sign of new physics. The element $|V_{ub}|$ has been measured from inclusive charmless semileptonic B decay as well as from the exclusive decays $B \to \pi(\rho) l \bar{\nu}_l$. For a review on the determination of $|V_{ub}|$, see Ref. [2]. The value of $|V_{ub}|$ preferred by the current global analysis of CKM data is about 15% smaller than the one from inclusive charmless semileptonic B decays [3–5], a problem unresolved to date. Furthermore, the inclusive determinations of $|V_{ub}|$ are about two standard deviations larger than those obtained from $B \to \pi l \bar{\nu}$, presently with a smaller uncertainty. The value of $|V_{ub}|$ predicted from the measured CKM angle $\sin 2\beta$, however, is closer to the exclusive result [6], and it should be stressed that various theoretical extractions based on exclusive decays are remarkably consistent among each other [3,7–10]. These discrepancies prompted a reexamination of the sources of theoretical uncertainty in the inclusive determination [11,12].

In the present paper, we investigate the four-body semileptonic decay mode $B^- \to \pi^+\pi^- l^- \bar{\nu}_l$ (which we will abbreviate as B_{l4} for short) and propose a method that allows one to extract $|V_{ub}|$ in a model-independent way. As a major step forward to a reliable treatment of the hadron-physics aspects of this decay, we use an approach based on dispersion theory without the need to explicitly match on specific resonance contributions or to separate these from the nonresonant background. This presents a significant improvement compared to previous studies of $B \to \rho l \bar{\nu}_l$ [13] and should serve as a valuable cross-check for the inclusive determination. In the future the distributions derived below could be used directly in the Monte Carlo generators of the experiments.

We include the kinematic range for invariant masses of the $\pi\pi$ pair below the $K\bar{K}$ threshold in our analysis and expand the form factors for the full B_{l4} transition matrix element in $\pi\pi$ partial waves up to P waves; D and higher partial waves have been checked to be negligible at these energies. While this model-independent description of the form factor dependence on the $\pi\pi$ invariant mass is in principle general and holds for arbitrary dilepton invariant masses, in practice we make use of matching to heavy-meson chiral perturbation theory to fix the normalization of the matrix element—a prerequisite for the extraction of $|V_{ub}|$. This scheme applies in the kinematics, where heavy-quark effective field theory is valid, i.e. for very large dilepton invariant masses. We point to Ref. [14] for a lucid illustration of the different effective theories applicable in different kinematic regimes for this decay.

This manuscript is organized as follows. In Sec. II the kinematics for the process of the four-body semileptonic B decay is reviewed, and the form factors for the hadronic transition of $B \to \pi\pi l \bar{\nu}_l$ are defined. In Sec. III, we show in detail how to treat these form factors within dispersion theory: the analytic properties are summarized in Sec. III A and the required pole terms calculated in heavy-meson chiral perturbation theory in Sec. III B, before we provide the expressions for the various form factors in the Omnès representation in Sec. III C. We discuss the required matching to leading-order heavy-meson chiral perturbation theory in Sec. III D. Numerical results are discussed in

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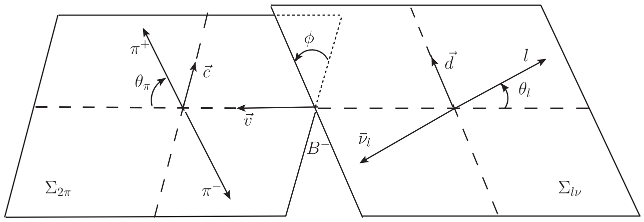

FIG. 1. Illustration of the kinematical variables for B_{l4}.

Sec. IV, and we summarize our findings in Sec. V. Some technical details are relegated to the Appendixes.

II. KINEMATICS, FORM FACTORS, PARTIAL WAVES, DECAY RATES

The kinematics of the process $B^-(p_B) \to \pi^+(p_+)\pi^- \times (p_-)l^-(p_l)\bar{\nu}_l(p_\nu)$ are described in terms of the five variables displayed in Fig. 1 [15–17]:

(i) the effective mass squared of the pion pair $s = (p_+ + p_-)^2 = M_{\pi\pi}^2$;
(ii) the effective mass squared of the dilepton pair $s_l = (p_l + p_\nu)^2$;
(iii) the angle θ_π of the π^+ in the $\pi^+\pi^-$ center-of-mass frame $\Sigma_{2\pi}$ with respect to the dipion line of flight in the B^- rest frame Σ_B;
(iv) the angle θ_l of the charged lepton l in the lepton center-of-mass system $\Sigma_{l\nu}$ with respect to the dilepton line of flight in Σ_B;
(v) the angle ϕ between the dipion and dilepton planes.

Two additional Mandelstam variables are defined as

$$t = (p_B - p_+)^2, \qquad u = (p_B - p_-)^2,$$
$$\Sigma_0 \equiv s + t + u = 2M_\pi^2 + m_B^2 + s_l. \tag{1}$$

We define the combinations of four vectors $P = p_+ + p_-$, $Q = p_+ - p_-$, $L = p_l + p_\nu$, and make use of the kinematical relations

$$(PL) \equiv P \cdot L = \frac{m_B^2 - s - s_l}{2}, \qquad t - u = -2\sigma_\pi X \cos\theta_\pi, \tag{2}$$

where

$$\sigma_\pi = \sqrt{1 - \frac{4M_\pi^2}{s}}, \qquad X = \frac{1}{2}\lambda^{1/2}(m_B^2, s, s_l), \tag{3}$$

and the Källén triangle function is given by $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$.

We decompose the matrix element in terms of form factors according to

$$\begin{aligned} T &= \frac{G_F}{\sqrt{2}} V_{ub}^* \bar{v}(p_\nu)\gamma^\mu(1-\gamma_5)u(p_l)I_\mu, \\ I_\mu &= \langle \pi^+(p_+)\pi^-(p_-)|\bar{u}\gamma_\mu(1-\gamma_5)b|B^-(p_B)\rangle \\ &= -\frac{i}{m_B}(P_\mu F + Q_\mu G + L_\mu R) - \frac{H}{m_B^3}\epsilon_{\mu\nu\rho\sigma}L^\nu P^\rho Q^\sigma, \end{aligned} \tag{4}$$

where $G_F = 1.166365 \times 10^{-5}\ \mathrm{GeV}^{-2}$ is the Fermi constant, and we use the convention $\epsilon_{0123} = 1$. The first three terms correspond to the axial current part, whereas the last term corresponds to the vector current. The dimensionless form factors F, G, H, and R are analytic functions of three independent variables, e.g. s, s_l, and $t - u$. Their partial-wave expansions for fixed s_l read [15,17]

$$\begin{aligned} F &= \sum_{l\geq 0} P_l(\cos\theta_\pi)f_l - \frac{\sigma_\pi(PL)}{X}\cos\theta_\pi G, \\ G &= \sum_{l\geq 1} P_l'(\cos\theta_\pi)g_l, \qquad H = \sum_{l\geq 1} P_l'(\cos\theta_\pi)h_l, \\ R &= \sum_{l\geq 0} P_l(\cos\theta_\pi)r_l + \frac{\sigma_\pi s}{X}\cos\theta_\pi G, \end{aligned} \tag{5}$$

where $P_l(z)$ are the standard Legendre polynomials and $P_l'(z) = dP_l(z)/dz$. An alternative set of form factors is given by

$$\begin{aligned} F_1 &= X \cdot F + \sigma_\pi(PL)\cos\theta_\pi G, \qquad F_2 = G, \qquad F_3 = H, \\ F_4 &= -(PL)F - s_l R - \sigma_\pi X\cos\theta_\pi G, \end{aligned} \tag{6}$$

whose partial-wave expansions

$$\begin{aligned} F_1 &= X\sum_{l\geq 0} P_l(\cos\theta_\pi)f_l, \qquad F_2 = \sum_{l\geq 1} P_l'(\cos\theta_\pi)g_l, \\ F_3 &= \sum_{l\geq 1} P_l'(\cos\theta_\pi)h_l, \qquad F_4 = \sum_{l\geq 0} P_l(\cos\theta_\pi)\tilde{r}_l, \\ \tilde{r}_l &= -((PL)f_l + s_l r_l), \end{aligned} \tag{7}$$

directly follow from Eqs. (5) and (6). Note that all partial waves f_l, g_l, h_l, r_l ($\tilde{r}_l$) are functions of s and s_l. The lowest angular-momentum $\pi\pi$ state contributing to the form factors F_2 and F_3 is the P-wave state, whereas the form factors F_1 and F_4 start with S waves. For the partial-wave decomposition up to P waves, we can therefore write

$$\begin{aligned} F_1 &= X[f_0(s,s_l) + f_1(s,s_l)\cos\theta_\pi + \cdots], \\ F_2 &= g_1(s,s_l) + \cdots, \qquad F_3 = h_1(s,s_l) + \cdots, \\ F_4 &= \tilde{r}_0(s,s_l) + \tilde{r}_1(s,s_l)\cos\theta_\pi + \cdots, \end{aligned} \tag{8}$$

where the ellipses denote higher partial waves. In the following, we sometimes suppress the dependence on s_l in order to ease notation.

The decay rate, after integration over the angles ϕ and θ_l, reads

$$d\Gamma = G_F^2|V_{ub}|^2 N(s,s_l) J_3(s,s_l,\theta_\pi) ds ds_l d\cos\theta_\pi,$$
$$J_3(s,s_l,\theta_\pi) = \frac{2+z_l}{3}|F_1|^2 + z_l|F_4|^2 + \frac{(2+z_l)\sigma_\pi^2 s s_l}{3} \times \left(|F_2|^2 + \frac{X^2}{m_B^4}|F_3|^2\right)\sin^2\theta_\pi, \quad (9)$$

with

$$z_l = \frac{m_l^2}{s_l}, \qquad N(s,s_l) = \frac{(1-z_l)^2\sigma_\pi X}{2(4\pi)^5 m_B^5}. \quad (10)$$

In most of the available phase space (including the kinematic regime where chiral perturbation theory can be applied), the mass of the lepton can be neglected (i.e. $z_l \ll 1$), and the contribution of F_4 to the decay rate is therefore invisible in particular for B_{e4} decays, since it is always associated with a factor of z_l. We will not analyze the form factor F_4 and its partial waves $\tilde{r}_i$ in the following. Integrating Eq. (9) over $\cos\theta_\pi$ yields the partial decay rate $d\Gamma/(ds ds_l)$; neglecting terms of order z_l and inserting the partial-wave expansions Eq. (7), we find

$$\frac{d\Gamma}{ds ds_l} = G_F^2|V_{ub}|^2 N(s,s_l) J_2(s,s_l),$$
$$J_2(s,s_l) = \int_{-1}^{1} d\cos\theta_\pi J_3(s,s_l,\cos\theta_\pi) = \frac{4X^2}{3}\left(|f_0(s)|^2 + \frac{1}{3}|f_1(s)|^2\right) + \frac{8}{9}\sigma_\pi^2 s s_l\left(|g_1(s)|^2 + \frac{X^2}{m_B^4}|h_1(s)|^2\right) + \cdots, \quad (11)$$

where the ellipsis denotes the neglected D and higher waves. Interference terms between different partial waves vanish upon angular integration, such that the partial-wave contributions to the decay rate can be easily read off.

III. FORM FACTORS IN DISPERSION THEORY

A. Analytic properties

The principle of maximal analyticity, which states that amplitudes possess no other singularities than those stemming from unitarity and crossing [18], tells us that the partial-wave amplitudes f_l, g_l, and h_l have the following analytic properties.

(i) At fixed s_l, they are analytic in the complex s plane, cut along the real axis for $s \geq 4M_\pi^2$ and $s \leq 0$. The presence of left-hand cuts $s \leq 0$ follows from the relations

$$t = \frac{\Sigma_0 - s}{2} - \sigma_\pi X\cos\theta_\pi,$$
$$t(\cos\theta_\pi = -1, s < 0) \geq (m_B + M_\pi)^2 \quad (12)$$

(and equivalent expressions for u), since the form factors F, G, and H have cuts for $t,u \geq (m_B + M_\pi)^2$.

(ii) In the interval $0 \leq s \leq 4M_\pi^2$, they are real.

(iii) In the interval $4M_\pi^2 \leq s \leq 16M_\pi^2$, Watson's theorem [19] is satisfied and therefore the phases of the partial-wave amplitudes (f_l, g_l, h_l) coincide with the corresponding pion–pion scattering phases.

(iv) For the crossed (t and u) channels, due to the lack of experimental information on πB phase shifts, we will approximate the πB interaction by B^* pole terms.

In practice, the range of validity of Watson's theorem can be extended to a larger domain, e.g. for the S wave to $s \leq s_K = 4M_K^2 \approx 1$ GeV^2, since inelasticities due to four or more pions are strongly suppressed both by phase space and by chiral symmetry. As pointed out e.g. in Refs. [20,21], chiral perturbation theory predicts the inelasticity parameter of the $\pi\pi$ S and P waves to be of order p^8 below the $K\bar{K}$ threshold, while the corresponding scattering phase shifts are of order p^2. Phenomenological analyses of the $\pi\pi$ interactions show that final states containing more than two particles start playing a significant role only well above the $K\bar{K}$ threshold s_K [22]. Here we refrain from performing a coupled-channel study, which limits the applicability of our approach to the region below s_K. The subtleties associated with the strong onset of inelasticities in the S wave in the vicinity of s_K (very close to the $f_0(980)$ resonance) for scalar form factors of the pion will be briefly discussed in Sec. IV A.

B. Heavy-meson chiral perturbation theory

In the process $B^- \to \pi^+\pi^- l^- \bar{\nu}_l$, u-channel contributions contain pole terms, while t-channel contributions do not. We obtain the pole terms by computing the leading-order diagrams (b) and (c) of Fig. 2 in the framework of heavy-meson chiral perturbation theory [23–25]. Let us briefly review the heavy-meson chiral Lagrangian. Define the heavy-meson field and its conjugate as

$$H_a = \frac{1+\slashed{v}}{2}(P_{a\mu}^*\gamma^\mu - P_a\gamma_5), \qquad \bar{H}_a = \gamma^0 H_a^\dagger \gamma^0, \quad (13)$$

where $P_{a\mu}^*$ is the field operator that annihilates a P_a^* meson with velocity v, satisfying $v^\mu P_{a\mu}^* = 0$, and P_a annihilates a P_a meson of velocity v. For the B meson family, we have

$$(P_1, P_2, P_3) = (B^-, \bar{B}^0, \bar{B}_s^0),$$
$$(P_1^*, P_2^*, P_3^*) = (B^{*-}, \bar{B}^{*0}, \bar{B}_s^{*0}), \quad (14)$$

which have dimension [mass]$^{3/2}$. The light pseudoscalar Goldstone boson fields are organized in

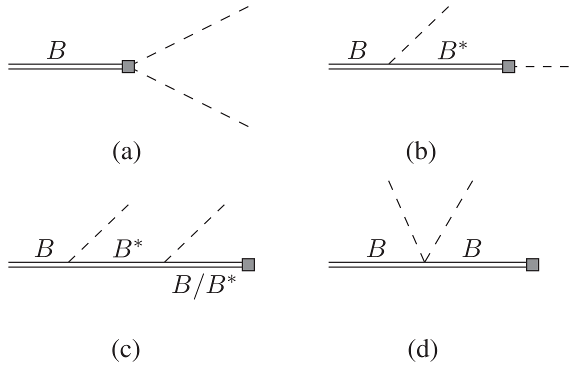

FIG. 2. Leading-order diagrams for $B \to \pi\pi$ matrix elements of the hadronic current. Diagrams (b) and (c) contain u-channel pole terms. Solid double lines and dashed lines represent heavy mesons and pseudo-Goldstone bosons, respectively. The shaded square denotes an insertion of the left-handed leptonic current. Diagram (c) involves both $BB^*\pi$ and $B^*B^*\pi$ vertices. Diagrams (a) and (d) are suppressed in the chiral expansion as long as the lepton mass is neglected.

$$u = \exp\left(\frac{i\phi}{2f_\pi}\right), \tag{15}$$

with

$$\phi = \sqrt{2}\begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{bmatrix}. \tag{16}$$

$f_\pi \simeq 92.2$ MeV is the pion decay constant [26]. Based on these building blocks, the leading-order Lagrangian describing the interactions of the B family and the Goldstone bosons reads [23]

$$\mathcal{L} = -i\mathrm{Tr}\bar{H}_a v_\mu \partial^\mu H_a + \frac{1}{2}\mathrm{Tr}\bar{H}_a H_b v^\mu (u^\dagger \partial_\mu u + u\partial_\mu u^\dagger)_{ba} + \frac{ig}{2}\mathrm{Tr}\bar{H}_a H_b \gamma_\nu \gamma_5 (u^\dagger \partial^\nu u - u\partial^\nu u^\dagger)_{ba}. \tag{17}$$

Determining the coupling $g = g_{B^*B\pi} = g_{B^*B^*\pi}$, using heavy-quark symmetry, from the partial decay width for $D^{*+} \to D^0\pi^+$ leads to $g = g_{D^*D\pi} = 0.58 \pm 0.07$, with the error given by the uncertainty in the width of the D^{*+}. This is in surprisingly good agreement with the most recent lattice simulations, which find $g_{B^*B\pi} = 0.516 \pm 0.052$ [27] and $g_{B^*B\pi} = 0.569 \pm 0.076$ [28] (we have added different error sources in quadrature for simplicity in both cases). In the present analysis, we stick to the experimental number extracted from D^{*+} decays for illustration. The dominant parts of the B_{l4} amplitude will depend on g in a very simple manner (being directly proportional either to g or to g^2), thus suggesting a straightforward strategy towards an extraction of $|V_{ub}|$ via lattice calculations of $g_{B^*B\pi}$.

To improve on the analytic properties of the amplitudes calculated in heavy-meson chiral perturbation theory, we include the effect of the $B^* - B$ mass splitting, defined by $\Delta = m_{B^*} - m_B$ (which is of $\mathcal{O}(1/m_Q)$), in the propagators, which in the heavy-meson approximation are of the form

$$\frac{i}{2v\cdot k} \quad \text{for the pseudoscalar } B \text{ meson}, \qquad \frac{-i(g_{\mu\nu} - v_\mu v_\nu)}{2(v\cdot k - \Delta)} \quad \text{for the vector } B^*, \tag{18}$$

where k is the small residual momentum of the propagating B or B^*. We do not otherwise include heavy-quark-symmetry-breaking effects, and stick to Eq. (17) for the determination of the interaction vertices.

The left-handed current $L_{\nu a} = \bar{q}_a \gamma_\nu (1 - \gamma_5) Q$, with q_a denoting a light and Q the heavy quark, is written in chiral perturbation theory as

$$L_{\nu a} = \frac{i\sqrt{m_B} f_B}{2}\mathrm{Tr}[\gamma_\nu (1-\gamma_5) H_b u^\dagger_{ba}] + \cdots, \tag{19}$$

where the ellipsis denotes terms with derivatives, factors of the light-quark mass matrix m_q, or factors of $1/m_Q$. Computing the trace, one can write it explicitly as

$$L_{\nu a} = i\sqrt{m_B} f_B (P^*_{b\nu} - v_\nu P_b) u^\dagger_{ba} + \cdots. \tag{20}$$

f_B is the B meson decay constant; averaging the most recent lattice calculations with $2+1$ dynamical quark flavors leads to the very precise value $f_B = 190.5 \pm 4.2$ MeV [29]. The whole B_{l4} decay amplitude is proportional to f_B, such that any uncertainty on this parameter directly translates into a contribution to the error in the extraction of $|V_{ub}|$.

We briefly discuss the chiral power counting of the B_{l4} amplitudes and form factors. If we denote soft pion momenta, or derivatives acting on the pion field, by p generically, the current of Eq. (19) is $\mathcal{O}(p^0)$, and so we expect to be the leading-order amplitude resulting from the diagrams in Fig. 2. Equation (4) then suggests the leading contributions to the form factors F, G, H, and R to be of chiral orders p^{-1}, p^{-1}, p^{-2}, and p^0, respectively (remember that the dilepton momentum L_μ is large, of order m_B); the alternative form factors F_1 and F_4 both are $\mathcal{O}(p^0)$.

The results for the individual diagrams of Fig. 2 are given in Appendix A. In order to ensure that we do not miss any effects of the nontrivial analytic structure of triangle graphs, resulting from the B^* pole terms once rescattering between the two outgoing pions is taken into account, we keep the full relativistic form of the denominator part of the propagator. The latter is connected with the above heavy-meson approximation Eq. (18) by [30]

$$\frac{i}{2v\cdot k} \to \frac{-im_B}{(p_B - k)^2 - m_B^2}, \qquad \frac{i}{2(v\cdot k + \Delta)} \to \frac{-im_{B^*}}{(p_B - k)^2 - m_{B^*}^2}, \tag{21}$$

where $p_B = m_B v$ is the on-shell B meson momentum. Written in terms of s and s_l, the pole terms can then be easily identified as

$$
\begin{aligned}
F^{\text{pole}} &= R^{\text{pole}} - G^{\text{pole}}, \qquad R^{\text{pole}} = \frac{\alpha}{u - m_{B^*}^2},\\
F_2^{\text{pole}} &= G^{\text{pole}} = \frac{\beta}{u - m_{B^*}^2}, \qquad F_3^{\text{pole}} = H^{\text{pole}} = \frac{\gamma}{u - m_{B^*}^2},\\
F_1^{\text{pole}} &= X \cdot F_{\text{pole}} + \sigma_\pi (PL)\cos\theta_\pi G_{\text{pole}}\\
&= \frac{X(\alpha - \beta) + \sigma_\pi (PL)\cos\theta_\pi \beta}{u - m_{B^*}^2},
\end{aligned}
\tag{22}
$$

using the abbreviations

$$
\begin{aligned}
\alpha &\equiv -\frac{g^2 f_B m_B^2 m_{B^*}}{f_\pi^2 (m_B^2 - s_l)}(s - 2M_\pi^2),\\
\beta &\equiv -\frac{g f_B m_B^2 m_{B^*}}{2 f_\pi^2}, \qquad \gamma \equiv -\frac{g^2 f_B m_B^3 m_{B^*}^2}{f_\pi^2 (m_{B^*}^2 - s_l)}.
\end{aligned}
\tag{23}
$$

All pole contributions start to contribute at the expected leading chiral orders. We note, though, that $\alpha = \mathcal{O}(p)$ is subleading to $\beta = \mathcal{O}(p^0)$ in F^{pole} and F_1^{pole} and can be neglected; they are indeed partially an artifact of the translation of the heavy-meson formalism back into relativistic kinematics in the calculation of Ref. [24]. We will use the contributions $\propto \alpha$ in the partial waves f_i later on to illustrate potential higher-order effects, although these are neither complete nor necessarily dominant amongst the subleading contributions (cf. the discussion of the scaling behavior of higher-order terms in the current in Ref. [31]). For the purpose of the (s-channel) partial-wave projections to be performed later, the u-channel pole can be written in terms of s and $\cos\theta_\pi$,

$$
\begin{aligned}
u(s, \cos\theta_\pi) - m_{B^*}^2 &= \sigma_\pi X(\cos\theta_\pi + y),\\
y &= \frac{\Sigma_0 - s - 2m_{B^*}^2}{2\sigma_\pi X}.
\end{aligned}
\tag{24}
$$

Finally, the remaining, nonpole, parts of the amplitude can also be extracted from the expressions in Appendix A. There are nonvanishing contributions to the form factor F_1 only, which in view of the required partial-wave expansion, we write as

$$
\begin{aligned}
\frac{F_1(s)^{\chi\text{PT}} - F_1^{\text{pole}}}{X} &= M_0(s)^{\chi\text{PT}} + \frac{2\sigma_\pi \cos\theta_\pi}{X} M_1(s)^{\chi\text{PT}},\\
M_0(s)^{\chi\text{PT}} &= -\frac{(1-g)^2 f_B m_B}{4 f_\pi^2},\\
M_1(s)^{\chi\text{PT}} &= \frac{(1-g^2) f_B m_B}{4 f_\pi^2 (m_B^2 - s_l)} X^2.
\end{aligned}
\tag{25}
$$

$M_0(s)^{\chi\text{PT}}$ and $M_1(s)^{\chi\text{PT}}$ are found to be of chiral orders p^0 and p, respectively, and therefore suppressed by one order compared to the pole terms [31], as explained in Appendix A. We will use these expressions in Sec. III D to match the polynomial parts of the dispersive representations of the corresponding amplitudes, but again rather in order to illustrate potential uncertainties due to subleading effects: these contributions are not complete even at the chiral order at which they occur.

To conclude this section, we point out that in order for the chiral counting scheme to work consistently, we have to assume the lepton invariant mass squared s_l to be large, of the order of m_B^2. This limits the kinematic range of applicability of our approach to match the dispersive representation derived in the following to heavy-meson chiral perturbation theory.

C. Omnès representation

Having fixed the tree-level decay amplitude and in particular the pole terms, we proceed to analyze the effects of pion-pion rescattering using dispersion relations. This will give access to the s dependence of the decay form factors (roughly up to 1 GeV, as detailed in Sec. III A) in a model-independent way. We will resort to the formalism based on Omnès representations as introduced in Ref. [32]. For its application to the closely related process of K_{l4} decays, see Refs. [33,34]. Note, however, that everything discussed in the following is to be understood at fixed s_l: dispersion theory as applied here does not allow us to improve on the form factor dependence on the dilepton invariant mass, beyond what the chiral representation in the previous section includes. We emphasize once more that the dispersive aspect of our analysis is in principle independent of the matching to heavy-meson chiral perturbation theory: the validity of any theoretical description of the different form factors in the soft-pion limit ($s \approx 0$) can be extended at least to the whole kinematic region of elastic $\pi\pi$ scattering with this method.

We may write an alternative form of the partial-wave expansion Eq. (8) for the pole-term-subtracted amplitudes, neglecting terms beyond P waves,

$$
\begin{aligned}
\frac{F_1(s,t,u)}{X} &= \frac{F_1^{\text{pole}}}{X} + M_0(s) - \frac{(t-u)}{X^2} M_1(s),\\
F_2(s,t,u) &= F_2^{\text{pole}} + U_1(s),\\
F_3(s,t,u) &= F_3^{\text{pole}} + V_1(s).
\end{aligned}
\tag{26}
$$

Here and in the following we suppress the dependence on s_l, which is kept fixed. The additional factor of X^2 in the definition of M_1 avoids the introduction of kinematic singularities at the zeros of X (in particular at the limit of the physical decay region $s = (m_B - \sqrt{s_l})^2$). The functions M_0, M_1, U_1, and V_1 defined this way possess right-hand unitarity branch cuts as their only nontrivial analytic structure and no poles. Since the pole terms F_1^{pole}/X, F_2^{pole}, F_3^{pole} are real, one immediately finds

$$\mathrm{Im} f_0(s) = \mathrm{Im} M_0(s), \qquad \mathrm{Im}\left(\frac{X}{2\sigma_\pi} f_1\right) = \mathrm{Im} M_1(s),$$
$$\mathrm{Im} g_1(s) = \mathrm{Im} U_1(s), \qquad \mathrm{Im} h_1(s) = \mathrm{Im} V_1(s), \tag{27}$$

which allows us to write

$$f_0(s) = M_0(s) + \hat{M}_0(s), \quad f_1(s) = \frac{2\sigma_\pi}{X}(M_1(s) + \hat{M}_1(s)),$$
$$g_1(s) = U_1(s) + \hat{U}_1(s), \quad h_1(s) = V_1(s) + \hat{V}_1(s). \tag{28}$$

The real "hat functions" $\hat{M}_0(s)$, $\hat{M}_1(s)$, $\hat{U}_1(s)$, and $\hat{V}_1(s)$ are the partial-wave projections of the pole terms given in Eqs. (22)–(23), which explicitly read

$$\hat{M}_0(s) = \frac{\xi Q_0(y) + (PL)\beta}{X^2}, \qquad \hat{M}_1(s) = -\frac{3\xi}{2\sigma_\pi X} Q_1(y),$$
$$\xi = \frac{X}{\sigma_\pi}(\alpha - \beta) - (PL)y\beta,$$
$$\hat{U}_1(s) = \frac{\beta}{\sigma_\pi X}(Q_0(y) - Q_2(y)),$$
$$\hat{V}_1(s) = \frac{\gamma}{\sigma_\pi X}(Q_0(y) - Q_2(y)), \tag{29}$$

where the $Q_l(y)$ are Legendre functions of the second kind,

$$Q_l(y) = \frac{1}{2}\int_{-1}^{1} \frac{dz}{y - z} P_l(z). \tag{30}$$

Explicitly, the first three of these read

$$Q_0(y) = \frac{1}{2}\log\frac{y+1}{y-1}, \qquad Q_1(y) = yQ_0(y) - 1,$$
$$Q_2(y) = \frac{3y^2 - 1}{2} Q_0(y) - \frac{3}{2}y. \tag{31}$$

We have projected onto the partial waves of F_2 and F_3 [whose partial-wave expansions proceed in derivatives of Legendre polynomials—see Eq. (7)] using

$$\int_{-1}^{1} P_i'(z)[P_{j-1}(z) - P_{j+1}(z)]dz = 2\delta_{ij}. \tag{32}$$

Note that, in order to show that the partial-wave-projected pole terms above indeed are real everywhere along the right-hand cut, i.e. for all $s \geq 4M_\pi^2$, care has to be taken about the correct analytic continuation. For example, X, only defined unambiguously in the physical decay region in Eq. (3), is continued according to [35,36]

$$X = \begin{cases} |X|, & s \in [4M_\pi^2, (m_B - \sqrt{s_l})^2], \\ i|X|, & s \in [(m_B - \sqrt{s_l})^2, (m_B + \sqrt{s_l})^2] \\ -|X|, & s \in [(m_B + \sqrt{s_l})^2, \infty) \end{cases} \tag{33}$$

(where the last range is of no practical relevance for our dispersive integrals). Furthermore, in the range of $(m_B - \sqrt{s_l})^2 < s < (m_B + \sqrt{s_l})^2$, the argument y of the Legendre functions of the second kind becomes purely imaginary; the lowest one can be expressed as $Q_0(y) = i(\pi/2 - \arctan|y|)$. In particular, no singularities arise at the zeros of X, $s = (m_B \pm \sqrt{s_l})^2$. Physically, the reality of the pole terms is based on the fact that the B^* cannot go on its mass shell in any kinematic configuration.

In the elastic regime, the right-hand cut of the partial waves f_i $(i = 0, 1)$, g_1, h_1 for $s > 4M_\pi^2$ is given by discontinuity equations relating them to the elastic $\pi\pi$ partial-wave amplitudes $t_i^i(s)$, $i = 0, 1$,[1] according to

$$\mathrm{disc} f_i(s) = f_i(s + i\epsilon) - f_i(s - i\epsilon) = 2i\,\mathrm{Im} f_i(s)$$
$$= 2i\sigma_\pi f_i(s)[t_i^i(s)]^* = f_i(s)e^{-i\delta_i^i(s)} \sin\delta_i^i(s), \tag{34}$$

where we have expressed $t_i^i(s)$ in terms of the corresponding phase shift $\delta_i^i(s)$ in the usual way. Analogous equations hold for g_1 and h_1. Equation (34) implies Watson's theorem: the phase of the partial wave equals the elastic phase shift. From Eqs. (27) and (28), one finds

$$\mathrm{Im} M_i(s) = (M_i(s) + \hat{M}_i(s))e^{-i\delta_i^i(s)} \sin\delta_i^i(s) \tag{35}$$

and similarly for $U_1(s)$, $V_1(s)$.

Equation (35) demonstrates that the hat functions constitute inhomogeneities in the discontinuity equations. The solution is given by [32]

$$M_i(s) = \Omega_i^i(s)\Bigg\{P_{n-1}(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\hat{M}_i(s')\sin\delta_i^i(s')ds'}{|\Omega_i^i(s')|(s' - s - i\epsilon)s'^n}\Bigg\}, \tag{36}$$

where $P_{n-1}(s)$ is a subtraction polynomial of degree $n - 1$, and the Omnès function is defined as [37]

$$\Omega_l^I(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \frac{\delta_l^I(s')ds'}{s'(s' - s - i\epsilon)}\right\}. \tag{37}$$

The standard Omnès solution $P_{n-1}(s)\Omega_i^i(s)$ of the homogeneous discontinuity equation ($\hat{M}_i = 0$), valid for form factors without any left-hand pole or cut structures, is modified by a dispersion integral over the inhomogeneities $\hat{M}_i$, which in the present case are given by the partial-wave-projected pole terms.

[1] We use this somewhat unusual notation owing to the fact that we only consider S and P waves, and no isospin $I = 2$ is allowed.

The minimal order of the subtraction polynomial is dictated by the requirement of the dispersive integral to converge. First we note that, if the phase $\delta_l^I(s)$ asymptotically approaches a constant value $c\pi$, then the corresponding Omnès function falls off asymptotically $\sim s^{-c}$. We will assume both $\pi\pi$ input phases to approach π for large energies,

$$\delta_0^0(s) \to \pi, \qquad \delta_1^1(s) \to \pi, \tag{38}$$

such that $\Omega_0^0(s)$, $\Omega_1^1(s) \sim 1/s$ for large s.

A more problematic question concerns the behavior of the hat functions for large s. In principle, this is entirely determined by the partial-wave-projected B^* pole terms as given in Eq. (29). However, as we have decided to include the relativistic pole graphs, these explicitly contain the scale m_B, and the asymptotic behavior is only reached for $\sqrt{s} \gg m_B$—far too high a scale, given that we realistically know the pion-pion phase shifts only up to well below 2 GeV, and that we presently neglect all inelastic contributions, which set in above 1 GeV. We can formally remedy this problem by just considering the large-s behavior of the heavy-meson approximation of the pole terms,[2] in which m_B only features parametrically as a prefactor; being aware that corrections to the heavy-meson approximation scale like $\sqrt{s}/m_B$, which is not a very small quantity in the region of 1 GeV $\lesssim \sqrt{s} \lesssim$ 2 GeV, say. In the heavy-meson approximation, i.e., at leading order in an expansion of $1/m_B$, the inhomogeneities of Eq. (29) behave according to

$$\hat{M}_0(s) \sim s^{-1/2}, \qquad \hat{M}_1(s) \sim s^0,$$
$$\hat{U}_1(s) \sim s^{-1/2}, \qquad \hat{V}_1(s) \sim s^{-1/2}. \tag{39}$$

Together with the large-s behavior of the Omnès functions, we conclude that the representation for $M_1(s)$ requires at least two subtractions, while for $M_0(s)$, $U_1(s)$, and $V_1(s)$, one subtraction each seems to be sufficient. However, looking at the behavior of the various hat functions in the low-energy region in Fig. 3 (for a special value of $s_l = (m_B - 1\ \text{GeV})^2$), we note that the falling of $\hat{M}_0(s)$, $\hat{U}_1(s)$, and $\hat{V}_1(s)$ barely seems to set in in the kinematical region $s \lesssim 1\ \text{GeV}^2$ where we have to assume the spectral function to be saturated, while $\hat{M}_1(s)$ even grows at those energies instead of approaching a constant value. It seems therefore advisable to oversubtract all the dispersive representations once, such as to allow for two subtraction constants each for $M_0(s)$, $U_1(s)$, and $V_1(s)$, and three for $M_1(s)$. This way, inelastic contributions at higher energies that we do not take into account explicitly should also be more effectively suppressed. The complete set of dispersion relations of the Omnès type therefore reads

[2]Remember that we made use of the relativistic pole terms mainly to ensure the correct analytic properties at *low* energies, i.e. in the near-threshold region.

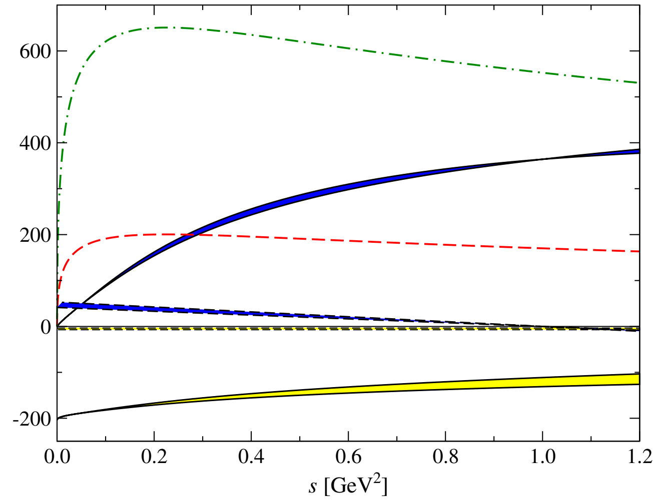

FIG. 3 (color online). Hat functions $\hat{M}_0(s)$ (yellow band with full lines), $\hat{M}_1(s)$ (blue band with full lines), $\hat{U}_1(s)$ (red dashed line), and $\hat{V}_1(s)$ (green dot-dashed line), for $s_l = (m_B - 1\ \text{GeV})^2$. We also show the polynomial contributions to the form factor F_1/X, for S (yellow band with dashed lines) and P wave (blue band with dashed lines), which are seen to be strongly suppressed. $\hat{M}_1(s)$ as well as the P-wave polynomial $M_1(s)^{\chi\text{PT}}$ are given in units of GeV^{-2}, all other functions are dimensionless.

$$\begin{aligned}
M_0(s) &= \Omega_0^0(s)\bigg\{a_0 + a_1 s \\
&\quad + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{\hat{M}_0(s')\sin\delta_0^0(s')ds'}{|\Omega_0^0(s')|(s'-s-i\epsilon)s'^2}\bigg\},\\
M_1(s) &= \Omega_1^1(s)\bigg\{a_0' + a_1' s + a_2' s^2 \\
&\quad + \frac{s^3}{\pi}\int_{4M_\pi^2}^\infty \frac{\hat{M}_1(s')\sin\delta_1^1(s')ds'}{|\Omega_1^1(s')|(s'-s-i\epsilon)s'^3}\bigg\},\\
U_1(s) &= \Omega_1^1(s)\bigg\{b_0 + b_1 s \\
&\quad + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{\hat{U}_1(s')\sin\delta_1^1(s')ds'}{|\Omega_1^1(s')|(s'-s-i\epsilon)s'^2}\bigg\},\\
V_1(s) &= \Omega_1^1(s)\bigg\{c_0 + c_1 s \\
&\quad + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{\hat{V}_1(s')\sin\delta_1^1(s')ds'}{|\Omega_1^1(s')|(s'-s-i\epsilon)s'^2}\bigg\}.
\end{aligned}\tag{40}$$

The subtraction constants are *a priori* unknown, and need to be determined either by further theoretical input, or by fitting to experimental data. It is easy to check that the functions $M_0(s), \ldots, V_1(s)$ themselves do not satisfy Watson's theorem; however, taking into account Eq. (28), the partial-wave amplitudes f_0 (f_1, g_1, h_1) do; i.e., their phases equal the elastic scattering phases δ_0^0 (δ_1^1).

We add a few further remarks concerning Fig. 3. All of the partial-wave-projected pole terms display singular behavior of square-root type at $s = 0$ (suppressed as $s^{3/2}$ in the case of $\hat{M}_1(s)$; note that also $\hat{M}_0(s)$ has a square-root singularity, which is hard to discern in Fig. 3 due to the axis scaling). These left-hand singularities obviously carry over to the partial waves: close to the $\pi\pi$ threshold, the partial-wave amplitudes cannot be represented by simple scalar or vector form factors.

The uncertainty bands for $\hat{M}_i(s)$, $i = 0, 1$, in Fig. 3 indicate the effect of the (incomplete) higher-order contribution $\propto \alpha$ in Eq. (29), suppressed by $1/m_B$ and found to be surprisingly small. We do *not* include the uncertainty due to the overall scaling with the coupling constant g, which translates directly into an uncertainty of a projected extraction of $|V_{ub}|$, but does not (at this order) affect the shape of the distributions. The inhomogeneities scale with g according to $\hat{M}_i(s)$, $\hat{U}_1(s) \propto g$, $\hat{V}_1(s) \propto g^2$.

The dispersive method using inhomogeneities as described above has by now been used for a variety of low-energy processes, such as $\eta \to 3\pi$ [32,38], $\omega/\phi \to 3\pi$ [39], $K \to \pi\pi$ [40], K_{l4} [33,34], $\gamma\gamma \to \pi\pi$ [41,42], or $\gamma\pi \to \pi\pi$ [43,44]. In several of those cases, the inhomogeneities (given in terms of hat functions), which incorporate left-hand-cut structures, and the amplitudes given in terms of Omnès-type solutions with a right-hand cut only are calculated iteratively from each other, until convergence is reached. In our present analysis, the ansatz is comparably simpler, as the left-hand cut is approximated by pole terms, whose partial-wave projections then determine the inhomogeneities. This is closely related to the method of Ref. [41] for $\gamma\gamma \to \pi\pi$, where the left-hand structures are approximated by Born terms and resonance contributions to $\gamma\pi \to \gamma\pi$.

D. Matching the subtraction constants

We need to consider two essentially different contributions to the subtraction constants in the representation Eq. (40), writing them formally as

$$a_i = \bar{a}_i + \hat{a}_i, \tag{41}$$

and similar decompositions for the a_i', b_i, and c_i. We discuss the contributions $\hat{a}_i$ etc. first. We argue in Appendix B that for inhomogeneities of essentially constant ($\hat{M}_0, \hat{U}_1, \hat{V}_1$) or approximately linear ($\hat{M}_1$) behavior over a large part of the kinematical region of interest, the coefficients of the highest power in the subtraction polynomials (a_1, a_2', b_1, and c_1) need to be adjusted in order to provide a reasonable high-energy behavior.[3] These coefficients are given by the derivative of the corresponding Omnès function at $s = 0$, multiplied with the constant/the derivative of the inhomogeneity in question. Obviously, the hat functions are not exactly constant/linear: to the contrary, they include square-root singularities at $s = 0$ due to the left-hand cut. There is, therefore, necessarily an uncertainty due to the choice of a "matching point" s_m at which to evaluate these "constants,"

$$\hat{a}_1 = \hat{M}_0(s_m) \times \dot{\Omega}_0^0(0), \qquad \hat{a}_2' = \frac{\hat{M}_1(s_m)}{s_m} \times \dot{\Omega}_1^1(0),$$
$$\hat{b}_1 = \hat{U}_1(s_m) \times \dot{\Omega}_1^1(0), \qquad \hat{c}_1 = \hat{V}_1(s_m) \times \dot{\Omega}_1^1(0). \tag{42}$$

We choose $s_m = M_\rho^2$, due to the expected strong enhancement of the distribution at the ρ resonance peak. Here, $\dot{\Omega}_l^I(0) = d\Omega_l^I(s)/ds|_{s=0}$. All other subtraction constants do not receive "hat" contributions.

The second contribution to the subtraction constants, dominantly to those of *low* polynomial order in s, stems from matching to the nonpole part of the chiral amplitude Eq. (25), which yields (for fixed s_l) a polynomial contribution in s. In this exploratory study we use the leading-order expressions only. We expect the chiral expansion to converge best at the sub-threshold point $s = 0$, as opposed to, e.g., the $\pi\pi$ threshold [47].

As we match the dispersive representation Eq. (40) to the leading chiral tree-level amplitude, which does not contain any rescattering or loop corrections, we identify the subtraction constants $\bar{a}_{0-1}$, $\bar{a}_{0-2}'$ by setting the scattering phases to zero, i.e., $\Omega_i^i(s) \equiv 1$, and the dispersive integrals over the inhomogeneities vanish. At $s = 0$, we find from Eq. (25)

$$\bar{a}_0 = -\frac{(1-g)^2 f_B m_B}{4f_\pi^2}, \quad \bar{a}_1 = 0,$$
$$\bar{a}_0' = \frac{(1-g^2) f_B m_B}{16 f_\pi^2}(m_B^2 - s_l),$$
$$\bar{a}_1' = -\frac{(1-g^2) f_B m_B}{8 f_\pi^2}\frac{m_B^2 + s_l}{m_B^2 - s_l}, \quad \bar{a}_2' = \frac{(1-g^2) f_B m_B}{16 f_\pi^2 (m_B^2 - s_l)}. \tag{43}$$

The term $\propto \bar{a}_2' s^2$, stemming from the expansion of X^2, is chirally suppressed and could as well be neglected. F_2 and F_3 at leading order coincide with their pole terms, thus the matching implies that the parameters $\bar{b}_i$ and $\bar{c}_i$ vanish.

In order to illustrate the relative importance of the (partial-wave-projected) pole terms relative to the subtraction polynomial—that is, the decompositions $M_i(s) + \hat{M}_i(s)$ on tree level, for $i = 0, 1$—we also show these for $s_l = (m_B - 1\ \text{GeV})^2$ in Fig. 3. We verify the expected dominance of the pole terms/the hat functions in $f_0(s)$ and $f_1(s)$, as suggested by power counting arguments. For the uncertainty bands of the polynomial corrections with mixed dependence on g, we have varied this coupling within its assumed uncertainty, $g = 0.58 \pm 0.07$.

[3] This can be corroborated to some extent by arguments from Brodsky-Lepage quark counting rules [45] and soft-collinear effective theory [46], albeit in kinematic regions with completely different scaling of s_l with respect to m_B^2 (taken as fixed and not particularly large here). Assuming the large-s behavior of the different form factors and partial waves is independent thereof, we indeed need to require the leading powers in s to cancel between the dispersion integrals over the inhomogeneities and the subtraction polynomial.

Remember that $g_1(s)$ and $h_1(s)$ consist of B^* pole terms only at leading order: this pole dominance should have very favorable consequences for the reliability of the form factor prediction, as the pole contributions are essentially fixed by the coupling constant g (as well as f_B) beyond the chiral expansion; the latter affects only the precision of the polynomial contribution. Next-to-leading-order corrections to the residues of the pole terms seem to have surprisingly little effect.

IV. RESULTS

A. Scattering phase input

The $\pi\pi$ phase shifts are known to sufficient accuracy in the region $s \lesssim s_0 \equiv (1.4\ \mathrm{GeV})^2$ (cf. Refs. [48,49]). In order to ensure the assumed asymptotic behavior $\delta_0^0(s)$, $\delta_1^1(s) \to \pi$ for $s \to \infty$, we continue the phases beyond s_0 according to the prescription [50]

$$\delta_i^i(s \geq s_0) = \pi + (\delta_i^i(s_0) - \pi) f\left(\frac{s}{s_0}\right), \quad f(x) = \frac{2}{1+x^{3/2}}. \tag{44}$$

There is a further subtlety concerning the S-wave phase shift: as we have discussed in Sec. III A, the elastic approximation breaks down at the $K\bar{K}$ threshold s_K with the occurrence of the $f_0(980)$ resonance. Both the phase of the partial wave $\arg t_0^0(s)$ and, e.g., the phase of the nonstrange scalar form factor of the pion $\arg F_\pi^S(s)$ differ significantly from $\delta_0^0(s)$ in this region: they quickly drop and then roughly follow the energy dependence of $\delta_0^0(s)$ again, with $\delta_0^0(s) - \arg t_0^0(s) \approx \delta_0^0(s) - \arg F_\pi^S(s) \approx \pi$ [51]. Therefore a single-channel approximation to the pion scalar form factor only works for $s < s_K$ if a phase of the form of either $\arg t_0^0(s)$ or $\arg F_\pi^S(s)$ are used as input to the Omnès function instead of $\delta_0^0(s)$. We use such a form factor phase taken from Ref. [52]. Obviously, we cannot provide a reliable description of pion-pion rescattering effects where the inherent two-channel nature of the problem becomes important, hence our dispersive description is confined to below s_K.

With the phase shift input thus continued formally up to infinity, the Omnès integrals can be fully performed. We have checked that different continuation prescriptions from the one given in Eq. (44) above s_0 have very little impact on the physics at low energies, i.e., below 1 GeV.

The phase input allows us to evaluate the derivatives of the Omnès functions required in Eq. (42) via the sum rules

$$\dot{\Omega}_l^I(0) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_l^I(s')}{s'^2}, \tag{45}$$

leading to $\dot{\Omega}_0^0(0) = 2.5\ \mathrm{GeV}^{-2}$, $\dot{\Omega}_1^1(0) = 1.8\ \mathrm{GeV}^{-2}$. This corresponds to squared radii of the pion scalar and vector form factors $\langle r_S^2\rangle = 0.58\ \mathrm{fm}^2$, $\langle r_V^2\rangle = 0.42\ \mathrm{fm}^2$, both only around 5% below the central values of more sophisticated evaluations [53–55].

In order to ensure numerically stable results, we perform the dispersion integrals over the inhomogeneities Eq. (40) up to $\sqrt{s} = 3$ GeV. This upper limit of the integration does not have any real physical significance: it merely represents an attempt to sum up the high-energy remainder of the integral to reasonable approximation, and does not mean we pretend to understand $\pi\pi$ interactions at such scales.

B. Subtraction constants, spectrum

We illustrate the results of our discussion for a sample value of $s_l = (m_B - 1\ \mathrm{GeV})^2$, which means the kinematically allowed range in the invariant mass of the pion pair extends to $\sqrt{s} = 1$ GeV. Evaluating the (nonvanishing) subtraction constants obtained from matching to the non-pole, polynomial parts of the chiral tree-level amplitude, Eq. (43), we find

$$\bar{a}_0 = -5.3 \pm 1.8, \qquad \bar{a}_0' = (48 \pm 6)\ \mathrm{GeV}^2,$$
$$\bar{a}_1' = -48 \pm 6, \qquad \bar{a}_2' = (0.5 \pm 0.1)\ \mathrm{GeV}^{-2}, \tag{46}$$

where the errors refer to the uncertainty in g only. The "hat" contributions to the subtractions of Eq. (42), at $s_l = (m_B - 1\ \mathrm{GeV})^2$, are found to be

$$\hat{a}_1 = (-363 \ldots -330)\left(\frac{g}{0.58}\right)\ \mathrm{GeV}^{-2},$$
$$\hat{a}_2' = (888 \ldots 924)\left(\frac{g}{0.58}\right)\ \mathrm{GeV}^{-2},$$
$$\hat{b}_1 = 332\left(\frac{g}{0.58}\right)\ \mathrm{GeV}^{-2}, \qquad \hat{c}_1 = 1078\left(\frac{g}{0.58}\right)^2\ \mathrm{GeV}^{-2}, \tag{47}$$

where we have displayed the scaling with g explicitly and shown the range of parameters in the F_1 partial waves due to the higher-order corrections discussed above.

For demonstration, we plot the partial decay rate in Fig. 4 for the dilepton invariant mass $s_l = (m_B - 1\ \mathrm{GeV})^2$. We find that the S-wave contribution leads to a significant enhancement of the spectrum at low $\pi\pi$ invariant masses, beyond what might be considered ρ dominance. The near-threshold dominance of the S wave was already pointed out in Ref. [31] in the context of heavy-meson chiral perturbation theory. Concerning the different P waves, we find that the kinematical prefactor X^2/m_B^4 strongly suppresses the partial wave h_1 or the form factor F_3 for the values of s_l considered here. Of the other two, g_1 yields a contribution to the differential rate roughly twice as large as f_1.

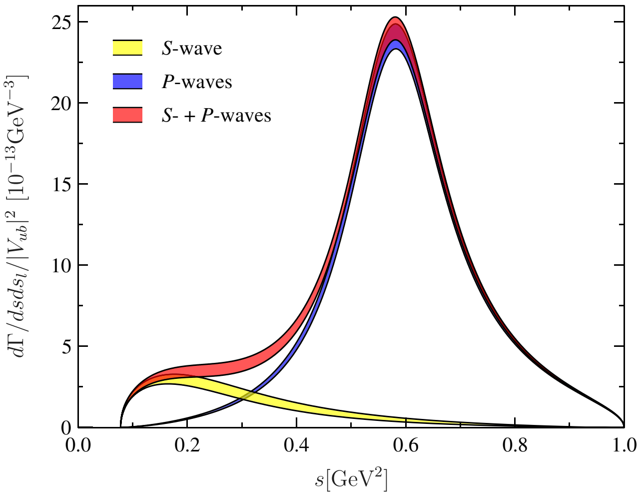

FIG. 4 (color online). Differential decay width $d\Gamma/dsds_l$ divided by $|V_{ub}|^2$ for the example value of $s_l = (m_B - 1\text{ GeV})^2$, decomposed into S- and P-wave contributions. For details, see discussion in main text.

V. DISCUSSION AND SUMMARY

We wish to emphasize that matching to chiral perturbation theory at leading order can only be considered an estimate, and mainly serves for illustration purposes here. Higher-order corrections are expected to be significant. Ultimately, the subtraction constants that influence the shape ought to be determined by fits to experimental data; they can be thought of as parametrizing a "background polynomial," beyond the dominant pole terms, albeit with completely correct rescattering corrections, obeying Watson's theorem. The necessary theoretical normalization of the form factors is essentially provided at $s = M_\rho^2$, via Eq. (42); its stability under higher-order corrections still merits further investigation in order to provide a theoretical uncertainty for $|V_{ub}|$ extracted from B_{l4} decays.

To summarize, we have provided a description of the form factors for the decay $B^- \to \pi^+\pi^- l^- \bar{\nu}_l$ using dispersion theory, which should lead to an improved method to measure $|V_{ub}|$. Pion-pion final-state interactions have been included nonperturbatively in the elastic approximation, while left-hand-cut structures in the πB interaction are approximated by B^* pole terms. We stress that our formalism allows, for the first time, to use the full information for $\pi\pi$ invariant masses below 1 GeV, without the need to refer to particular parametrization for selected resonances such as the $\rho(770)$ [or the $f_0(980)$]; it allows for a full exhaustion of the corresponding spectra. Improved experimental data to allow for such an analysis to be performed in practice is therefore highly desirable.

As an outlook concerning theoretical improvement, we have hinted at the possibility to extend the present analysis to lower values of the dilepton invariant mass s_l, beyond the range of applicability of heavy-meson chiral perturbation theory, but still making use of dispersion relation for the dependence on the dipion invariant mass s. One promising constraint could be obtained from soft-pion theorems [56], which relate linear combinations of B_{l4} form factors at $s = M_\pi^2$, but arbitrary s_l, to $B \to \pi l \nu$ (B_{l3}) form factors at same s_l. Given reliable phenomenological information on the form factors for B_{l3}, this may provide precisely (part of) the matching information needed to extend the dispersive method of this paper to lower values of s_l.

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APPENDIX A: TREE-LEVEL AMPLITUDES IN HEAVY-MESON CHIRAL PERTURBATION THEORY

Calculating the tree-level diagrams in Fig. 2 in heavy-meson chiral perturbation theory, one obtains the corresponding amplitudes [24] [$\mathcal{A} - \mathcal{D}$, in obvious correspondence to diagrams (a)–(d)],

$$\begin{aligned}
\mathcal{A} &= \frac{if_B}{4f_\pi^2} p_B^\mu, \qquad \mathcal{B} = ip_-^\mu \mathcal{B}^{(1)} + ip_B^\mu \mathcal{B}^{(2)}, \\
\mathcal{B}^{(2)} &= -\frac{gf_B}{2f_\pi^2} \frac{v\cdot p_-}{v\cdot p_- + \Delta} = -\frac{v\cdot p_-}{m_B}\mathcal{B}^{(1)}, \\
\mathcal{C} &= ip_B^\mu \mathcal{C}^{(1)} + \epsilon^{\mu\alpha\beta\gamma} p_{B\alpha} p_{-\beta} p_{+\gamma} \mathcal{C}^{(2)}, \\
\mathcal{C}^{(1)} &= -\frac{g^2 f_B}{2f_\pi^2} \frac{p_+\cdot p_- - (v\cdot p_+)(v\cdot p_-)}{[v\cdot(p_+ + p_-)][v\cdot p_- + \Delta]}, \\
\mathcal{C}^{(2)} &= -\frac{g^2 f_B}{2f_\pi^2} \frac{1}{[v\cdot(p_+ + p_-) + \Delta][v\cdot p_- + \Delta]}, \\
\mathcal{D} &= ip_B^\mu D^{(1)}, \qquad \mathcal{D}^{(1)} = -\frac{f_B}{4f_\pi^2} \frac{v\cdot(p_+ - p_-)}{v\cdot(p_+ + p_-)}. \qquad \text{(A1)}
\end{aligned}$$

Identifying the contributions to the individual decay form factors, we find for these as the leading-order (LO) results

$$F^{\rm LO} = R^{\rm LO} - G^{\rm LO}, \qquad G^{\rm LO} = \frac{m_B}{2}\mathcal{B}^{(1)}, \qquad H^{\rm LO} = -\frac{m_B^3}{2}\mathcal{C}^{(2)},$$
$$R^{\rm LO} = -\frac{m_B f_B}{4f_\pi^2} - m_B(\mathcal{B}^{(2)} + \mathcal{C}^{(1)} + \mathcal{D}^{(1)}). \qquad \text{(A2)}$$

From these, it is then straightforward to identify the pole contributions given in Eq. (22), as well as the nonpole pieces of Eq. (25).

It is obvious that all diagrams (a)–(d) are formally of $\mathcal{O}(p^0)$ in terms of soft pion momenta. Note, however, that all pieces proportional to $p_B^\mu = P^\mu + L^\mu$ are effectively suppressed: the part $\propto L^\mu$ enters the form factor R, which is suppressed by the small lepton mass and neglected throughout the main text, while the part $\propto P^\mu$ leads to a chiral suppression by one order (and is at least partially an artifact of the heavy-meson approximation anyway). As a consequence, the only leading contributions are given by the amplitudes $\mathcal{B}^{(1)}$ and $\mathcal{C}^{(2)}$ in the above, and hence the B^* pole graphs. This was already pointed out in Ref. [31].

APPENDIX B: DISPERSVE REPRESENTATION FOR POLYNOMIAL INHOMOGENEITIES

Consider a partial wave $f(s)$ given at tree level as a constant, $f^{\rm tree}(s) = A$. In this case, we can write down the dispersive representation including final-state interactions right away, *if* we assume a certain high-energy behavior of the amplitude, as

$$f(s) = A\Omega(s), \tag{B1}$$

with the Omnès function $\Omega(s)$. Here, we assume (as in the main text) an Omnès function falling according to $1/s$, i.e. given by a phase shift approaching π asymptotically, and a partial wave that vanishes in the same way for large s. This assumption prevents us from multiplying $\Omega(s)$ with a polynomial of higher degree.

However, in the spirit of the solution discussed in the main text, it should also be possible to treat this constant as an inhomogeneity and reconstruct the same solution from the corresponding formalism. Our solution is then of the form

$$f(s) = A + \Omega(s)\left\{a + a's + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{A\sin\delta(s')ds'}{|\Omega(s')|s'^2(s'-s)}\right\}, \tag{B2}$$

where we have chosen the minimal number of subtractions (two) required to make the dispersion integral converge. Note that the subtraction constants a, a' are not *a priori* fixed from the tree-level input; we can set $a = 0$ by requiring the normalization of the amplitude at $s = 0$ to match the tree-level input. The integral in Eq. (B2) can be performed explicitly, using a dispersive representation of the inverse of the Omnès function,

$$\Omega^{-1}(s) = 1 - \dot{\Omega}(0)s - \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{\sin\delta(s')ds'}{|\Omega(s')|s'^2(s'-s)}, \tag{B3}$$

where $\dot{\Omega}(0) = d\Omega(s)/ds|_{s=0}$. As a result, we find

$$f(s) = \Omega(s)\{A + [a' - A\dot{\Omega}(0)]s\}. \tag{B4}$$

Therefore, Eq. (B1) is reproduced if we choose $a = 0$, $a' = A\dot{\Omega}(0)$. We essentially apply the same requirement on the high-energy behavior as in Eq. (B1): terms that do not vanish for large s are only canceled for this specific choice of a'.

More generally, if we match to a tree-level amplitude of the form As^n, demanding the same leading behavior near $s = 0$ such that all subtraction terms $\propto s^{m\leq n}$ can be put to zero, the solution using this tree-level input as an inhomogeneity,

$$As^n + \Omega(s)\left\{a's^{n+1} + \frac{s^{n+2}}{\pi}\int_{4M_\pi^2}^\infty \frac{As'^n\sin\delta(s')ds'}{|\Omega(s')|s'^{n+2}(s'-s)}\right\}, \tag{B5}$$

agrees with the "canonical" solution $As^n\Omega(s)$, with the "correct" high-energy behavior, only if $a' = A\dot{\Omega}(0)$.

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