FZJ–IKP–TH–2007–11

Remarks on $NN \to NN\pi$ beyond leading order

C. Hanhart and A. Wirzba

Institut für Kernphysik,
Forschungszentrum Jülich GmbH,
D–52425 Jülich, Germany

Abstract

In recent years a two-scale expansion was established to study reactions of the type $NN \to NN\pi$ within chiral perturbation theory. Then the diagrams of some subclasses that are invariant under the choice of the pion field no longer appear at the same chiral order. In this letter we show that the proposed expansion still leads to well defined results. We also discuss the appropriate choice of the heavy baryon propagator.

Key words: Chiral Lagrangians, Effective interactions, Pion production
PACS: 21.30.Fe, 12.39.Fe, 25.10.+s, 25.40.Ve

1 Introduction

Pion production in nucleon-nucleon (NN) collisions is subject of theoretical and experimental investigations already since the 1960s — for a review of the history of the field see Ref. [1]. However, when new high precision data became available due to advanced accelerator technology in the beginning of the 1990s it became clear that all phenomenological studies performed so far were not capable of describing the data. Several mechanisms were proposed to cure the problem; however, no clear picture emerged [2].

There was the hope that chiral perturbation theory (ChPT) could resolve the issue. As the effective field theory for the standard model at low energies it should provide a framework to investigate the reactions $NN \to NN\pi$ in a field-theoretically consistent way. In a first attempt a scheme proposed by

Email addresses: `c.hanhart@fz-juelich.de` (C. Hanhart),
`a.wirzba@fz-juelich.de` (A. Wirzba).

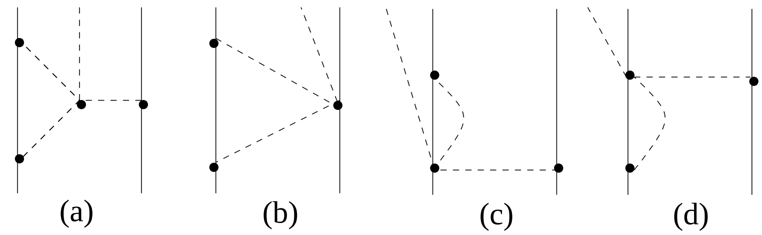

Fig. 1. Some one-loop diagrams that start to contribute to $NN \to NN\pi$ at NLO, (a) & (b), and N^{4}LO, (c) & (d). Dashed lines denote pions and solid lines denote nucleons. The exchange diagrams are not shown. Note that the diagrams (b), (c) and (d) result from diagram (a) under the removal of one internal pion line.

Weinberg to study elastic and inelastic pion reactions on nuclei [3] was applied to investigate also pion production in NN collisions. However, in doing so up to next–to–leading order (NLO) the discrepancy between the calculations and data became even worse [4]. In addition, loop contributions, formally of order NNLO, gave even larger effects [5,6] shedding doubts on an applicability of chiral perturbation theory to $NN \to NN\pi$.

In parallel, already in Refs. [7] it was stressed that the large momentum transfer, typical for meson production in NN collisions, needs to be taken care of in the power counting. This idea was further developed in Refs. [8,9]. The appropriate expansion parameter for $NN \to NN\pi$ therefore is

$$\chi_{\rm prod} = p_{\rm thr}/M = \sqrt{m_\pi/M}\ , \tag{1}$$

where $p_{\rm thr} = \sqrt{Mm_\pi}$ denotes the threshold momentum for pion production in NN collisions. M and m_π are the masses of the nucleon and pion, respectively. Here the leading-order (LO) scales as $\mathcal{O}(\chi^1_{\rm prod})$ and subleading orders N^{n}LO scale as $\mathcal{O}(\chi^{n+1}_{\rm prod})$. For the most recent developments for the reaction $NN \to NN\pi$ within chiral perturbation theory we refer to Refs. [10].

Thus in the reactions $NN \to NN\pi$ one is faced with a two-scale expansion, since both m_π as well as $p_{\rm thr}$ appear explicitly in the expressions. For tree-level diagrams this does not cause any problem. To perform the power counting for loop integrals, however, a rule has to be given what scale to assign to the components of the loop momentum. After subtraction of the nucleon mass M, the residual energy of each external nucleon at threshold is $m_\pi/2$, whereas the corresponding momentum is of order $p_{\rm thr}$. One therefore would be tempted to take over this scaling also for the loop momentum. On the other hand the new power counting is based on two scales, $p_{\rm thr} \gg m_\pi$, and the pions in loops are off-shell. Therefore there is no reason why the scaling of the pion energies in loops should be different from the scaling of the pertinent three-momenta. In Appendix E of Ref. [2] it is shown that for all diagrams that do not have a two-nucleon cut, each component of the loop momentum should be counted of order of the largest external momentum in the loop. The argument there is based on the observation that in time ordered perturbation theory (TOPT) there is no ambiguity for the order assignment of energies since it is a 3 dimensional theory in the first place. On the other hand the leading order of a given TOPT amplitude should agree to that of the corresponding Feynman amplitude. This allows to identify the proper scale for the energy of the loop momentum. The assignment was checked by explicit calculations in Refs. [5,9]. As a result, all components of the loop momentum in diagram (a) and (b) of Fig. 1 scale as $\chi_{\rm prod}M$, but those of diagram (c) and (d) scale as $m_\pi \sim \chi_{\rm prod}^2 M$ and are therefore suppressed [1]. One further consequence of the presence of two scales in the problem is that the individual loops no longer contribute to only a single order, but each loop contributes to infinitely many orders, since $m_\pi/p_{\rm thr} = \chi_{\rm prod}$ appears as the argument of non–analytic functions. The power counting only identifies the lowest order where the particular loop starts to contribute [9].

In Ref. [11] it was shown that the sum of all diagrams of Fig. 1 is independent of the choice of the pion field. However, based on the scheme developed in Refs. [8,9] only diagram (a) and (b) contribute at NLO whereas diagrams (c) and (d) start to contribute not until order N^{4}LO (see Table 11 of Ref. [2]). The main purpose of this letter is to investigate the consistency of these two statements.

As we go along we also need to discuss the appropriate choice of the nucleon propagator in the heavy baryon formulation. This is done in Section 3. Section 4 contains our conclusions. Moreover, for clarification two appendices are added, one is about reparameterizations of the chiral matrix U, the other is about the $1/M$ expansion of the nucleon propagator.

2 Dependence on the pion field to NLO

The Lagrangian relevant for our study may be written as [12]

$$\mathcal{L} = \frac{f_\pi^2}{4}\langle u^\mu u_\mu\rangle + \frac{f_\pi^2}{4}\langle \chi_+\rangle + \bar\Psi\left(i\gamma^\mu D_\mu - M + \frac{g_A}{2}\gamma^\mu u_\mu \gamma_5\right)\Psi\ . \qquad (2)$$

Here $\langle\ldots\rangle$ denotes a trace in the isospin-space, Ψ is the relativistic spinor of the nucleon, D_μ is its covariant derivative containing the Weinberg-Tomozawa term [13] and other $\pi^{2n}NN$ terms, g_A is the axial-vector constant, f_π the pion

[1] In this case the large momentum $p_{\rm thr}$ can be removed from the integral by, e.g., a proper choice of the integration variable in full analogy to the discussion given below Eq. (15).

decay constant. Furthermore,

$$u_\mu = i\left(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger\right) \quad \text{and} \quad \chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u$$

are the chiral vielbein and the mass term, respectively, with $\chi = 2B\mathcal{M}$, where $\mathcal{M}$ is the quark mass matrix and B is proportional to the $SU(2)$ quark condensate in chiral limit. In the isospin symmetric case one may write $\chi = m_\pi^2\,\mathbf{1}$.

In order to investigate the dependence of our results on the choice made for the pion field π ($= \vec{\tau}\cdot\vec{\pi}$ in terms of the Pauli matrices for isospin), we start from the following general expression for the chiral matrix $U = u^2$ (see Appendix A)

$$U = \exp\left(\frac{i}{f_\pi}(\vec{\tau}\cdot\vec{\pi})g(\pi^2/f_\pi^2)\right) \ . \tag{3}$$

Here $g(\pi^2/f_\pi^2)$ is an arbitrary regular function with $g(0) = 1$. For our purposes it is sufficient to expand g up to second order in the pion field. We may write

$$g(\pi^2/f_\pi^2) = 1 + \left(\alpha + \frac{1}{6}\right)\frac{\vec{\pi}^2}{f_\pi^2} + \cdots \ . \tag{4}$$

Obviously, for $\alpha = -1/6$ we work with U in the so-called *exponential gauge*. In the σ–gauge one uses

$$U = \sqrt{1 - \frac{\vec{\pi}^2}{f_\pi^2}} + \frac{i}{f_\pi}\vec{\tau}\cdot\vec{\pi} = 1 + \frac{i}{f_\pi}\vec{\tau}\cdot\vec{\pi} - \frac{1}{2f_\pi^2}\vec{\pi}^2 - \frac{1}{8f_\pi^4}\vec{\pi}^4 - \cdots \ . \tag{5}$$

By explicit evaluation one finds, that $\alpha = 0$ reproduces this expression up to and including terms of order $(\pi/f_\pi)^4$. This is sufficient for our purposes. For more details including a justification of the notion "gauge choice" for the pion parameterizations see Appendix A.

All building blocks of the chiral Lagrangian may now be expressed in terms of the field u defined above. One finds for the operators relevant in this work

$$u_\mu = -\frac{1}{f_\pi}\vec{\tau}\cdot\partial_\mu\vec{\pi} - \frac{1}{2f_\pi^3}\left(2\alpha\vec{\pi}^2\left(\vec{\tau}\cdot\partial_\mu\vec{\pi}\right) + (1+4\alpha)\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)\left(\vec{\tau}\cdot\vec{\pi}\right)\right) + \cdots \tag{6}$$

for the chiral vielbein and

$$\chi_+ = m_\pi^2\left(u^\dagger u^\dagger + uu\right) = m_\pi^2\left(2 - \frac{\vec{\pi}^2}{f_\pi^2} - \frac{\vec{\pi}^4}{4f_\pi^4}(1+8\alpha) - \cdots\right) \tag{7}$$

for the mass term. In both cases terms of higher order in the pion field were not displayed, since they are not relevant for the present work.

We use the framework of heavy-baryon chiral perturbation theory (HBChPT) [14,15]. It is then straightforward to find the Feynman rules for the relevant

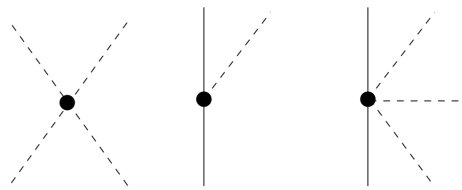

Fig. 2. The different vertices relevant for this study. Dashed lines denote pions and solid lines denote nucleons.

building blocks of the diagrams (see Fig. 2, in all cases the pion momenta $q_i = (q_i^0, \vec{q}_i)$, $i = 1, 2, 3, 4$ are chosen as outgoing):

$$iV_{4\pi} = \frac{i}{f_\pi^2}\Big\{\Big[(q_1+q_2)^2 - m_\pi^2 + 2\alpha\sum_{i=1}^{4}(q_i^2 - m_\pi^2)\Big]\delta^{ab}\delta^{cd} + \Big[\begin{pmatrix} ab;cd \\ 12;34 \end{pmatrix} \to \begin{pmatrix} ac;bd \\ 13;24 \end{pmatrix}\Big] + \Big[\begin{pmatrix} ab;cd \\ 12;34 \end{pmatrix} \to \begin{pmatrix} ad;cb \\ 14;32 \end{pmatrix}\Big]\Big\}\ , \tag{8}$$

$$iV_{NN\pi} = -\frac{g_A}{2f_\pi}\tau^a(\vec{\sigma}\cdot\vec{q}_i)\ , \tag{9}$$

$$iV_{NN3\pi} = -\frac{g_A}{4f_\pi^3}\left\{\delta^{ab}\tau^c\vec{\sigma}\cdot[\vec{q}_1 + \vec{q}_2 + 4\alpha(\vec{q}_1+\vec{q}_2+\vec{q}_3)] + \text{cyclic}\right\}\ , \tag{10}$$

where $\vec{\sigma}$ is the Pauli-matrix vector for spin and $a, b, c, d \in \{1, 2, 3\}$ are the isospin indices. [2] In addition we need both the pion propagator and the nucleon propagator. The former is given by the standard expression $iD_\pi(q)^{ab} = i\delta^{ab}(q^2 - m_\pi^2 + i\epsilon)^{-1}$. To leading order we use for the latter

$$iS_N(p-q) = \frac{i}{-q_0 + i\epsilon}\ . \tag{11}$$

We chose the momenta such that the initial, on-shell, nucleon with momentum $P^\mu = Mv^\nu + p^\mu$ is pushed off its mass shell by the emission of a virtual pion with momentum q, where v^μ is a four-vector with the properties $v^2 = 1$ and $v^0 \geq 1$. The standard choice of HBChPT, also used here, is $v^\mu = (1, 0, 0, 0)$. Throughout the paper we follow the convention that uppercase nucleon momenta contain Mv^μ, whereas this term is subtracted out from their lowercase counterparts. Note that for loop momenta temporal (q_0) as well as spacial (q_i) components are assumed of order $p_{\rm thr}$, if not stated otherwise, as outlined in the introduction. The residual energy p_0 of the incoming *on-shell* proton, however, is of order $\chi_{\rm prod}^2 M \sim m_\pi$ because of the on-shell condition. Our rule for the nucleon propagator is different to the one applied in Refs. [5,16], where $i(p_0 - q_0 + i\epsilon)^{-1}$ is used for the propagator. It is justified in the next section and in Appendix B. For a very explicit derivation of the rules of the heavy baryon formalism we refer to Ref. [18] — see chapter 5.5.6 and Eq. (5.112) for another justification that, to leading order in the $1/M$ expansion, $v \cdot p$ has to vanish.

[2] In the sigma gauge all vertices can be found in appendix A of Ref. [17].

With the building blocks at hand we can now evaluate diagram (a) of Fig. 1. Especially let us focus on those terms that are proportional to α. These read

$$
\begin{aligned}
i\tilde{A}^{\mathrm{NLO}}_{(a)} = & -2\frac{i\alpha}{f_\pi^2}\left(\frac{g_A}{2f_\pi}\right)^3 \frac{i^4}{k^2-m_\pi^2}(\vec{\sigma}_2\cdot\vec{k})\tau_2^c \\
& \times\int\frac{d^4l}{(2\pi)^4}\frac{\vec{\sigma}_1\cdot(\vec{p}'-\vec{l}-\vec{p})\tau_1^a(\vec{\sigma}_1\cdot\vec{l})\tau_1^b}{(l^2-m_\pi^2)((p'-l-p)^2-m_\pi^2)(l_0+i\epsilon)} i\tilde{V}^{abcd}_{4\pi}\ ,
\end{aligned}
\tag{12}
$$

where

$$
i\tilde{V}^{abcd}_{4\pi} = (\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})\left[(l^2-m_\pi^2)+((p'-l-p)^2-m_\pi^2)+(k^2-m_\pi^2)\right] . \tag{13}
$$

Here the indices $j=1,2$ of the Pauli matrices τ_j and σ_j refer to the left and right nucleon lines in Fig. 1, respectively; $l \sim p_{\mathrm{thr}}$ denotes the momentum of the pion loop, p and p' are the momenta of the incoming and outgoing leg of the left nucleon line (j=1), respectively, whereas k is the difference of the incoming momentum minus the outgoing one of the right nucleon line (j=2). Note that the temporal components p_0, p_0' and k_0 of the nucleon momenta or, respectively, nucleon-momentum difference scale all as $\chi^2_{\mathrm{prod}}M \ \sim m_\pi$, whereas their spatial counterparts p_i, p_i' and k_i scale as p_{thr}. The uncontracted index d refers to the isospin of the produced pion. Its momentum is equal to $k+p-p'$ and is of course on-shell and scales as m_π. The corresponding term for diagram (b) gives

$$
\begin{aligned}
i\tilde{A}^{\mathrm{NLO}}_{(b)} = & -2i^3\frac{\alpha}{f_\pi^3}\left(\frac{g_A}{2f_\pi}\right)^3 (\vec{\sigma}_2\cdot\vec{k})\,(\delta^{ab}\tau_2^d+\delta^{ad}\tau_2^b+\delta^{bd}\tau_2^a) \\
& \times\int\frac{d^4l}{(2\pi)^4}\frac{\vec{\sigma}_1\cdot(\vec{p}'-\vec{l}-\vec{p})\tau_1^a(\vec{\sigma}_1\cdot\vec{l})\tau_1^b}{(l^2-m_\pi^2)((p'-l-p)^2-m_\pi^2)(l_0+i\epsilon)}\ .
\end{aligned}
\tag{14}
$$

Note that the particular combination of momenta as it appears in the α-dependent terms of the three-pion vertex is independent of the integration variable l and was therefore pulled out of the integral. As a consequence the integral $\tilde{A}^{\mathrm{NLO}}_{(b)}$ exactly cancels that part of $\tilde{A}^{\mathrm{NLO}}_{(a)}$ that corresponds to the last term of Eq. (13). What remains to be studied are the other two terms. Each of them cancels one of the pion propagators inside the integral. We get

$$
\begin{aligned}
i\left(\tilde{A}^{\mathrm{NLO}}_{(b)}+\tilde{A}^{\mathrm{NLO}}_{(a)}\right) = & -10\frac{i\alpha}{f_\pi^2}\left(\frac{g_A}{2f_\pi}\right)^3\frac{i^4}{k^2-m_\pi^2}(\vec{\sigma}_2\cdot\vec{k})\tau_2^d \\
\times\int\frac{d^4l}{(2\pi)^4}\frac{\vec{\sigma}_1\cdot(\vec{p}'-\vec{l}-\vec{p})(\vec{\sigma}_1\cdot\vec{l})}{(l_0+i\epsilon)} & \left\{\frac{1}{l^2-m_\pi^2}+\frac{1}{(p'-l-p)^2-m_\pi^2}\right\}\ .
\end{aligned}
\tag{15}
$$

Using the variable transformation $l \to l' = p'-l-p$ in the second term we find

$$
i\left(\tilde{A}^{\rm NLO}_{(b)}+\tilde{A}^{\rm NLO}_{(a)}\right) = -10\frac{i\alpha}{f_\pi^2}\left(\frac{g_A}{2f_\pi}\right)^3\frac{i^4}{k^2-m_\pi^2}(\vec{\sigma}_2\cdot\vec{k})\tau_2^d
$$
$$
\times\int\frac{d^4l}{(2\pi)^4}\frac{\vec{\sigma}_1\cdot(\vec{w}-\vec{l})(\vec{\sigma}_1\cdot\vec{l})}{l^2-m_\pi^2}\left\{\frac{1}{l_0+i\epsilon}+\frac{1}{w_0-l_0+i\epsilon}\right\}\ , \qquad (16)
$$

where we defined $w\equiv p'-p$ and renamed l' back to l. The integrand now does not contain the large scale $|\vec{p}|$ anymore in the denominator. Consequently, l will now be of order m_π and no longer of order $p_{\rm thr}$. This is why w_0, which is also of order m_π (while $|\vec{w}|\sim p_{\rm thr}$), is to be kept in the denominator of the last term. The angular integration leads to

$$
\vec{\sigma}_1\cdot(\vec{p}'-\vec{l}-\vec{p})(\vec{\sigma}_1\cdot\vec{l}) \ \rightarrow\ -\vec{l}^2=(l^2-m_\pi^2)-(l_0^2-m_\pi^2)\ .
$$

Inserting the first bracket into the above integral leads to a vanishing result, since the spatial integration is free of scales. We may therefore write

$$
i\left(\tilde{A}^{\rm NLO}_{(b)}+\tilde{A}^{\rm NLO}_{(a)}\right) = 10\frac{i\alpha}{f_\pi^2}\left(\frac{g_A}{2f_\pi}\right)^3\frac{i^4}{k^2-m_\pi^2}(\vec{\sigma}_2\cdot\vec{k})\tau_2^d
$$
$$
\times w_0\int\frac{d^4l}{(2\pi)^4}\frac{l_0^2-m_\pi^2}{l^2-m_\pi^2}\left\{\frac{1}{(l_0+i\epsilon)(w_0-l_0+i\epsilon)}\right\}\ . \qquad (17)
$$

The only external scales in the integral are m_π and $w_0\sim m_\pi$. In addition, the integral is quadratically divergent. Therefore, when being evaluated in dimensional regularization, the resulting expression will scale as $|\vec{k}|^{-1}\times w_0\times m_\pi^2\sim 1/p_{\rm thr}\times m_\pi^3$.

As was explained in Ref. [9], the leading loops (including pion-field independent terms) can be well estimated by identifying all momentum/energy scales by $p_{\rm thr}$. Thus the α-dependent terms of the sum of diagram (a) and (b) are suppressed by a power of $(m_\pi/p_{\rm thr})^3=\chi^3_{\rm prod}$ compared to the leading loops that start to contribute at order NLO. This implies that the pion-field dependent terms start to contribute only at order N^{4}LO.[3] At this order the sum of Eq. (17) cancels against the sum of the α-dependent contributions of diagrams (c) and (d) of Fig. 1 which are separately of order N^{4}LO, when the same Feynman rules are applied in the calculation. This, however, does not exclude the possibility that there may exist other α-dependent terms of order N^{4}LO that result from subleading Feynman rules. This we will investigate in the next section.

[3] The power counting can only account for a parametric suppression of diagrams relative to each other. Obviously the expression of Eq. (17) can be enhanced artificially by choosing a gauge that corresponds to a very large value α. Note that for all standard choices $|\alpha|\leq 1/4$ (see Eqs. (A.3)–(A.5)). For practical purposes, the σ-gauge is of course the most efficient one, since each α-dependent term trivially vanishes individually.

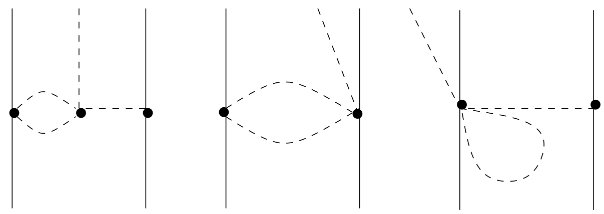

Fig. 3. Some one-loop diagrams that start to contribute to to $NN \to NN\pi$ at NNLO, (a) & (b), and N^{4}LO, (c). Dashed lines denote pions and solid lines denote nucleons. The exchange diagrams are not shown. Note that the diagrams (b) and (c) result from diagram (a) under the removal of one internal pion line. The $\pi\pi NN$ vertex my either be taken from $\mathcal{L}_{\pi N}^{(1)}$ — Weinberg–Tomozawa term [13] — or from those vertices of $\mathcal{L}_{\pi N}^{(2)}$ that depend on the pion momenta (e.g. the c_3 term) [12].

3 Beyond leading order

We found that to NLO the sum of diagram (a) and (b) (and – trivially – the sum of diagram (c) and (d)) of Fig. 1 is invariant under the choice of the pion field. All terms that depend on the pion field vanish to this order. In this section we investigate the pion-field dependent terms of the diagrams shown in Fig. 1 to NNLO. Please note that there are several additional diagrams contributing to this order that are potentially pion-field dependent — one example being the so–called football diagrams shown in Fig. 3. The proof that to order lower than N^{4}LO, i.e. to $\mathcal{O}(\chi_{\rm prod}^n)$ with $n \leq 4$, there are no α-dependent terms resulting from the additional diagrams is analogous to the one given here for the diagrams of Fig. 1 and thus we do not present it in detail.

To order NNLO the only thing that needs to be considered is the subleading contribution to the nucleon propagator, which is suppressed by one power in $\chi_{\rm prod}$ compared to (11).[4] The subleading πNN vertices on the other hand are already down by $m_\pi/M \sim \chi_{\rm prod}^2$. There are two pieces to the subleading propagator: one from treating p_0 as subleading in Eq. (11)

$$i\Delta_1 S_N(p-q) \equiv \frac{i}{p_0 - q_0 + i\epsilon} - \frac{i}{-q_0 + i\epsilon} = -i\frac{p_0}{(q_0 - i\epsilon)^2}\left[1 + \mathcal{O}\left(\frac{p_0}{q_0}\right)\right] ,$$

and one coming from the $1/M$ corrections [17] given by

$$i\Delta_2 S_N(p-q) = \frac{i}{2M}\left(1 - \frac{(p-q)^2}{(p_0 - q_0 + i\epsilon)^2}\right) = i\frac{(\vec{p} - \vec{q})^2}{2M(q_0 - i\epsilon)^2}\left[1 + \mathcal{O}\left(\frac{p_0}{q_0}\right)\right] .$$

[4] Of course the subleading contribution to the propagator can be interpreted as an $\mathcal{O}(q^2/M)$ insertion between two leading-order propagators $(-q_0 + i\epsilon)^{-1}$.

Putting both pieces together we get the following next-to-leading contribution of the nucleon propagator in HBChPT, using the on-shell condition $p_0 = \vec{p}^2/2M + \mathcal{O}(\vec{p}^4/M^3)$ and neglecting higher-order terms:

$$i\Delta S_N(p-q) = i\Delta_1 S_N(p-q) + i\Delta_2 S_N(p-q) = i\frac{\vec{q}^2 - 2\vec{p}\cdot\vec{q}}{2M(q_0 - i\epsilon)^2} \,. \tag{18}$$

This illustrates nicely why p_0 should be treated as order $\chi^2_{\rm prod}M$: if the nucleon leg attached to the propagator is on-shell, the p_0 term gets canceled by the $\vec{p}^2/(2M)$ term of the $1/M$ corrections, as soon as both contributions are treated on equal footing. Note that each of the two steps of the derivation was based on $p_0/q_0 \sim \chi_{\rm prod}$, whereas the total result holds in general. Therefore we present in Appendix B a straight forward derivation of this result – based on the covariant propagator – that still is valid even when $q_0 \sim |\vec{q}| \sim p_0 \sim m_\pi$.

Using Eq. (18) for the nucleon propagator we get for the NNLO contribution of the α-dependent terms of diagram (a) of Fig. 1 with $\tilde{V}^{abcd}_{4\pi}$ as in Eq. (13):

$$\begin{aligned} i\tilde{A}^{\rm NNLO}_{(a)} &= -2\frac{i\alpha}{f_\pi^2}\left(\frac{g_A}{2f_\pi}\right)^3 \frac{i^4}{k^2-m_\pi^2}(\vec{\sigma}_2\cdot\vec{k})\tau_2^c \int\frac{d^4l}{(2\pi)^4}\left(\frac{2\vec{l}\cdot\vec{p}+\vec{l}^2}{2M}\right) \\ &\times \frac{\vec{\sigma}_1\cdot(\vec{p}'-\vec{l}-\vec{p})\tau_1^a(\vec{\sigma}_1\cdot\vec{l})\tau_1^b}{(l^2-m_\pi^2)((p'-l-p)^2-m_\pi^2)(l_0+i\epsilon)^2} i\tilde{V}^{abcd}_{4\pi} \,. \end{aligned} \tag{19}$$

As before, the integral that emerges when introducing the last term of Eq. (13) into Eq. (19) gets canceled by the corresponding term for diagram (b) and we refrain from showing the expression explicitly. After the same variable transformation ($l \to l' = p' - l - p$) as above, the remainder reads

$$\begin{aligned} i\left(\tilde{A}^{\rm NNLO}_{(b)} + \tilde{A}^{\rm NNLO}_{(a)}\right) &= -10\frac{i\alpha}{f_\pi^2}\left(\frac{g_A}{2f_\pi}\right)^3 \frac{i^4}{k^2-m_\pi^2}(\vec{\sigma}_2\cdot\vec{k})\tau_2^d\frac{1}{2M} \\ \times\int\frac{d^4l}{(2\pi)^4}&\frac{\vec{\sigma}_1\cdot(\vec{w}-\vec{l})(\vec{\sigma}_1\cdot\vec{l})}{l^2-m_\pi^2}\left\{\frac{2\vec{l}\cdot\vec{p}+\vec{l}^2}{(l_0+i\epsilon)^2} + \frac{2(\vec{w}-\vec{l})\cdot\vec{p}+(\vec{w}-\vec{l})^2}{(w_0-l_0+i\epsilon)^2}\right\}, \end{aligned} \tag{20}$$

again using $w = p' - p$. As before, this integral diverges (at least) quadratically with the only scale in the denominator given by $m_\pi \sim w_0$. In addition there is now an overall scale of order $\vec{p}^2/M$ present, which is also of order m_π. Therefore the integral given in Eq. (20) also starts to contribute only at order N^{4}LO. Again this sum cancels against the summed α-dependent contributions of diagram (c) and (d) of Fig. 1, if the subleading nucleon propagator (18) is inserted into the latter diagrams.

Finally note that the next-to-subleading correction to the nucleon propagator and vertices would necessarily involve an additional factor $\chi_{\rm prod}$ relative to the above presented N^{4}LO result. The summed α-dependent contributions of diagram (a) and (b) (and of (c) and (d)) of Fig. 1 resulting from this next-to-subleading order should therefore contribute only at order N^{5}LO. In other words, the proof that the α-dependent terms in the sum of all diagrams of Fig. 1 cancels to order N^{4}LO is now complete.

4 Conclusions

Of course, the fact that there is a cancellation of the summed α-dependent terms of the four diagrams of Fig. 1 does not come as a surprise, see Ref. [11], since these terms would cancel also in a relativistic calculation of the type [12] where the nucleon propagator (11) is replaced by the non-expanded covariant form (B.1) and where the terms $-\vec{\sigma}\cdot\vec{q}_i$ appearing in the vertices (9) and (10) are replaced by their covariant Dirac-analogs $\gamma_\mu\gamma_5(q_i)^\mu$. In fact, this cancellation between the α-dependent contributions of diagram (a) on the one hand and the ones of diagrams (b), (c) and (d) on the other hand is solely based on the cancellations of the inverse pion-propagators appearing in Eq. (13) and the various pion propagators appearing in diagram (a) which all are of covariant nature – even in HBChPT. The point, however, is that now it is clear that this cancellation is also consistent with the two-scale expansion scheme of Refs. [8,9]: (i) we have explicitly shown that the pion-field dependent contributions of the diagrams (a) and (b) of Fig. 1, although both are NLO diagrams, cancel at NLO and at N^{2}LO when calculated with leading and next-to-leading input, respectively, for the nucleon-propagators and vertices. (ii) We have shown that the remainders are only of N^{4}LO, the very same order at which the diagrams (c) and (d) start to contribute. (iii) At this order, the pion-field dependent contributions of the sum of diagram (a) and (b) indeed cancel against the corresponding contributions of diagram (c) and (d). (iv) We have argued that further subleading orders of the nucleon propagator and vertices will lead to pion-field dependent terms that are at least of N^{5}LO.

These results can be generalized in the following way: as long as the order in the expansion of the nucleon propagator in diagram (a) matches those of diagrams (b), (c) and (d) and as long as the order in the expansion of the $NN\pi$ vertex matches those of the $NN3\pi$ vertex, the following cancellations are bound to happen: first, the cancellation between the α-dependent contribution of diagram (b) and the one of diagram (a) that results from the insertion of the last term of Eq. (13); at this stage the remainder of the α-dependent contribution of diagram (a) has now the same order in the two-scale expansion as the α-dependent contributions of diagram (c) and (d) calculated with the same input; secondly, since the cancellation is based on covariant input from the (inverse) pion-propagators and since the rest of the input is the same, the sum of these remaining α-dependent contributions has to vanish. Of course, at the same order in the chiral expansion, say at N^{n}LO with a fixed $n > 4$, the diagrams of Fig. 1 might generate additional α-dependent terms resulting from further subleading orders in the expansion of the nucleon propagators and vertices, as it was e.g. the case at the leading and subleading order in the expansion of the nucleon propagator. Nevertheless, for the same reasons as above, also these additional contributions have to sum to zero. Eventually at an even higher order in the expansion of the nucleon propagator and vertices no more α-dependent terms of N^{n}LO can appear in the summation; instead contributions of the next order $n + 1$ will arise which again sum to zero and so on. As indicated, our proof linked to the diagrams of Fig. 1 can easily be generalized to other classes of potentially pion-field dependent diagrams as e.g. given in Fig. 3. In summary, the two-scale expansion scheme of Refs. [8,9] is consistent with pion-field independence to all orders in the expansion.

As by-products of the investigation we could show that the *parameterizations* of the pion field indeed correspond to *gauge choices*, and could clarify the structure of the heavy-baryon propagator connected to an on-shell nucleon leg. Contrary to a naive interpretation of the heavy-baryon rules the on-shell residual energy of the external nucleon is of the same order as the kinetic recoil term of the nucleon – in fact, to the very same order, they cancel each other.

Acknowledgements

We thank Fred Myhrer for fruitful discussions at an early stage of this research.

A Reparameterizations of the chiral matrix U

The theorem that *on-shell* matrix elements do not dependent on the specific parameterization of the local interpolating field(s) has a long history reaching back to the LSZ reduction formula [19] and the work of Haag [20], see also Refs. [21,22,23,24] etc. In the context of non-linear realizations of chiral Lagrangians this general theorem of axiomatic field theory was confirmed in Ref. [25]. The more restricted question of the general reparameterizations of the chiral matrix U for the chiral group $SU(2) \times SU(2)$, more specifically, the general reparameterization of the pion field under nonlinear transformations induced by chiral $SU(2) \times SU(2)$ was first studied by Weinberg [26]. From the parity of the pion and the transformation properties of the pion field under vector and axial–vector transformations combined with Jacobi-identity constraints Weinberg could show that the most general redefinition of the nonlinearly realized pion field is of the form

$$\pi'^a = \pi^a g(\pi^2), \quad a \in \{1,2,3\} \tag{A.1}$$

where the π^a's are the usual isospin components of the pion field $\pi \equiv \vec{\tau}\cdot\vec{\pi} = \sum_{a=1}^3 \tau^a\pi^a$ and where $g(\pi^2)$ is regular in $\pi^2 = \vec{\pi}^2 = \sum_{a=1}^3(\pi^a)^2$. In terms of the chiral matrix and the dimensionful version of the pion field this corresponds to

$$U' \equiv \exp\left(\frac{i}{f_\pi}\vec{\tau}\cdot\vec{\pi}'\right) = \exp\left(\frac{i}{f_\pi}\vec{\tau}\cdot\vec{\pi}\, g(\pi^2/f_\pi^2)\right) \ . \tag{A.2}$$

This is the result of Eq. (3) under the additional condition that $g(0) = 1$, which follows from fixing the wave function normalization of the pion at tree-level or, in other words, the free-particle part of the Lagrangian (2), see Refs. [22,23].

The various known parameterizations, the exponential one, the so-called σ-gauge, the Weinberg one [26], etc. follow from (A.2) with the help of the following choices of $g(\pi^2/f_\pi^2)$ functions:

$$g(\pi^2/f_\pi^2) = 1 \qquad \text{(exponential parameterization)}, \tag{A.3}$$

$$g(\pi^2/f_\pi^2) = \frac{1}{\sqrt{\pi^2/f_\pi^2}}\arcsin\left(\sqrt{\pi^2/f_\pi^2}\right) = 1 + \frac{\pi^2}{6f_\pi^2} + \cdots \quad (\sigma\text{-gauge}), \tag{A.4}$$

$$g(\pi^2/f_\pi^2) = \frac{1}{\sqrt{\pi^2/f_\pi^2}}\arcsin\left(\frac{\sqrt{\pi^2/f_\pi^2}}{1+\pi^2/(4f_\pi^2)}\right) = 1 - \frac{\pi^2}{12f_\pi^2} + \cdots \qquad \text{(Weinberg parameterization)}. \tag{A.5}$$

In fact, the transformation (A.2) can be simplified by the following rescaling

$$U' = \exp\left(\frac{i}{f_\pi}\vec{\tau}\cdot\vec{\pi}\, g(\pi^2/f_\pi^2)\right) = \exp\left(i\vec{\tau}\cdot\hat{\vec{\pi}}\, F\left(\sqrt{\pi^2/f_\pi^2}\right)\right) \tag{A.6}$$

where $\hat{\vec{\pi}} = \vec{\pi}/\sqrt{\pi^2}$ is the pion unit vector in isospin-space and $F(x) = xg(x^2)$ is an odd analytic function of the variable x with a normalized first coefficient in the Taylor expansion, $F(x) = x + \sum_{n=2}^\infty c_{2n-1}x^{2n-1}$, see e.g. [27,28]. In terms of this function the various parameterizations become especially simple [27,28]: [5]

$$F(x) = x \qquad \text{(exponential parameterization)}, \tag{A.7}$$

$$F(x) = \arcsin(x) \qquad (\sigma\text{-gauge parameterization}), \tag{A.8}$$

$$F(x) = \arcsin\left(x/(1+x^2/4)\right) \qquad \text{(Weinberg parameterization)}. \tag{A.9}$$

[5] In $SU(2)$: $\exp\left(i\vec{\tau}\cdot\hat{\vec{\pi}}F\left(\sqrt{\pi^2/f_\pi^2}\,\right)\right) = \cos\left(F\left(\sqrt{\pi^2/f_\pi^2}\,\right)\right) + i\vec{\tau}\cdot\hat{\vec{\pi}}\sin\left(F\left(\sqrt{\pi^2/f_\pi^2}\,\right)\right)$.

In fact, with the help of the machinery of Ref. [25] the various parameterizations can be transformed into each other by the following axial gauge transformations

$$U \to U' = U_A(\pi)\, U\, U_A(\pi) \tag{A.10}$$

in terms of the local $SU(2)$ matrix $U_A(\pi) = \exp\left((i/2f_\pi)\vec{\tau}\cdot\vec{\pi}\ (g(\pi^2/f_\pi^2)-1)\right)$. The backtransformation follows then from the inverse gauge transformation $U' \to U = U_A(\pi)^\dagger\, U'\, U_A(\pi)^\dagger$. Transitions between other representations or *gauges* can be found as compositions of gauge transformations from and to the exponential gauge, say. The *σ-gauge* indeed results from a gauge transformation (A.10) of the exponential "gauge" $U = \exp\left(i\vec{\tau}\cdot\vec{\pi}/f_\pi\right)$ when the *gauge choice* (A.4) is inserted into $U_A(\pi)$. In addition, the various parameterizations of the matrix $u = \sqrt{U}$ transform into each other as

$$u \to u' = U_A(\pi)\, u\, h(U_A,\pi)^{-1} = h(U_A,\pi)\, u\, U_A(\pi), \tag{A.11}$$

where $h(U_A,\pi) \in SU(2)_V$ is the so-called "compensator" or "hidden" matrix [25] which cancels in $U' = u'u' = U_A(\pi)uuU_A(\pi)$ and in the Lagrangian (2).

Whereas the transformations of the type (A.2) or (A.6) are $SU(2)$ specific, the gauge transformation as such – whether in the form (A.10) or (A.11) – can be generalized to $SU(3)$ with a suitably selected $SU(3)$ gauge matrix $U_A(\pi) \sim 1 + i\frac{\alpha}{2}\pi^3/f_\pi^3 + \cdots$ that does not spoil the wave function normalization at tree level, where $\pi = \sum_{a=1}^{8}\lambda^a\pi^a$ in terms of the Gell-Mann matrices λ^a.

B On the $1/M$ expansion of the nucleon propagator

In the main section the rules of HBChPT were applied directly. For illustration, we show in this appendix that the same expressions can be recovered by a straight forward expansion of the nucleon propagator. We start from the covariant expression for the nucleon propagator

$$iS_N^{\rm cov}(P-q) = i\frac{\not{P} - \not{q} + M}{(P-q)^2 - M^2 + i\epsilon}\ , \tag{B.1}$$

where the momenta are defined in Fig. B.1. We now want to expand this propagator in powers of $1/M$. The easiest way to proceed is via the decomposition

$$iS_N^{\rm cov}(K) = i\frac{M}{E_K}\sum_s\left\{\frac{u(\vec{K},s)\bar{u}(\vec{K},s)}{K_0 - E_K + i\epsilon} + \frac{v(-\vec{K},s)\bar{v}(-\vec{K},s)}{K_0 + E_K - i\epsilon}\right\}\ ,$$

where $E_K = \sqrt{M^2 + \vec{K}^2}$. First observe that the second term, corresponding to the contribution of anti–nucleons, does not propagate in HBChPT. It therefore gets absorbed into local counter terms at the Lagrangian level. The spinors get

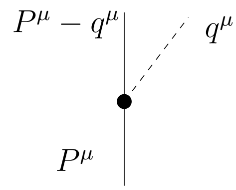

Fig. B.1. Definition of the various momenta used in Appendix B. The initial nucleon is supposed to be on-shell ($P^2 = M^2$).

part of the vertex functions and we may therefore focus on the denominator of the first term. In the kinematics chosen we have $K = P - q$. To make contact to the expressions of HBChPT, we write, in accordance with the notation of the main text, $P^\mu = Mv^\mu + p^\mu$ with $v^\mu = (1, 0, 0, 0)$. Thus

$$\begin{aligned} P_0 - q_0 - E_{(P-q)} &= M + p_0 - q_0 - \sqrt{M^2 + (\vec{p} - \vec{q})^2} \\ &= -q_0 + \frac{2\vec{p} \cdot \vec{q} - \vec{q}^2}{2M} + \mathcal{O}\left(\frac{(\vec{p} - \vec{q})^4}{M^3}, \frac{\vec{p}^4}{M^3}\right) , \end{aligned} \tag{B.2}$$

where the on-shell condition $p_0 = \vec{p}^2/2M + \mathcal{O}(\vec{p}^4/M^3)$ was used in the last step. Observe that p_0 has disappeared from the expression (B.2). However, this happens only, if p_0 is put into the same order as $\vec{p}^2/2M$, as advocated in the main section. In the power counting relevant for pion production, the pion energies in loops are to be counted as order $p_{\rm thr}$ as in case (a) and (b) of Fig. 1. Therefore, in line with Eqs. (11) and (18), the expression for the propagator in HBChPT is simply

$$\begin{aligned} iS_N(p - q) &= \frac{i}{-q_0 + i\epsilon}\left(1 - \frac{2\vec{p} \cdot \vec{q} - \vec{q}^2}{2M(-q_0 + i\epsilon)} + \mathcal{O}\left(\frac{p^2, p \cdot q, q^2}{M^2}\right)\right) \\ &= \frac{i}{-v \cdot q + i\epsilon}\left(1 + \frac{2p \cdot q_\perp - q_\perp^2}{2M(-v \cdot q + i\epsilon)} + \mathcal{O}\left(\frac{p^2, p \cdot q, q^2}{M^2}\right)\right) \ . \end{aligned} \tag{B.3}$$

The last relation refers to the general "velocity" case ($v^2 = 1$ and $v_0 \geq 1$) with the definition $q_\perp \equiv q - v(v \cdot q)$ and the on-shell condition $v \cdot p = -p^2/2M$ [18].

Note that the above equations hold even for more general kinematics. In all cases of relevance here, the loop momenta are at least of the order of the pion mass. Thus the components of q^μ either scale as $p_{\rm thr}$, as used in the previous paragraph, or as m_π — as in the integral of Eqs. (16) and (17) or in Fig. 1 (c) and (d). Let us stress that also in the latter case the expansion of Eq. (B.3) holds, since the other terms of order m_π, namely p_0 and $\vec{p}^2/2M$, canceled and the remaining recoil terms are suppressed by at least one power of $\chi_{\rm prod}$.

It is also instructive to derive the $1/M$ expansion of the propagator directly from the covariant expression of Eq. (B.1). Using again the on-shell condition for the incoming nucleon, $P^2 = M^2$, we may write $P_0 = M\left(1 + \mathcal{O}(\vec{p}^2/M^2)\right)$ – note that $\vec{P} \equiv \vec{p}$. We are interested in the case $|\vec{p}| \sim p_{\rm thr}$, where $p_{\rm thr}$ was defined below Eq. (1). Therefore $\mathcal{O}(\vec{p}^2/M^2)$ corresponds to $\mathcal{O}(\chi^2_{\rm prod})$. We thus identify $-i/q_0$ as the leading term for the propagator in accordance with Eq. (11). All other terms that still appear in the denominator are corrections. After a Taylor expansion to next–to–leading order we get

$$
\begin{aligned}
iS_N^{\rm cov}(P-q) = \frac{i}{-q_0+i\epsilon}\Bigg\{ & \tfrac{1}{2}\left(\mathbf{1}+\gamma_0\right)\left(1-\frac{2\vec{p}\cdot\vec{q}-\vec{q}^2}{2M(-q_0+i\epsilon)}\right) \\
& -\underbrace{\frac{1}{2M}\vec{\gamma}\cdot(\vec{p}-\vec{q})}_{\text{into vertices}} \;+\; \underbrace{\frac{q_0}{2M}\tfrac{1}{2}\left(\mathbf{1}-\gamma_0\right)}_{\text{effect of anti–nucleon}} \Bigg\}\left(1+\mathcal{O}\left(\frac{p^2, p\cdot q, q^2}{M^2}\right)\right) .
\end{aligned}
$$

As indicated, this expression contains the leading and next–to–leading piece of the propagator and, in addition, a piece that can give momentum dependence to the vertices (this contribution can be mapped onto the effect of the spinors in the previous derivation), and, finally, a contact term that is the leading term for the effects of the anti–nucleon in the intermediate state.

References

[1] H. Garcilazo and T. Mizutani, πNN Systems, World Scientific (Singapore, 1990).

[2] C. Hanhart, Phys. Rept. **397** (2004) 155.

[3] S. Weinberg, Phys. Lett. B **295** (1992) 114.

[4] B. Y. Park, F. Myhrer, J. R. Morones, T. Meissner and K. Kubodera, Phys. Rev. C **53** (1996) 1519; C. Hanhart, J. Haidenbauer, M. Hoffmann, U.-G. Meißner and J. Speth, Phys. Lett. B **424** (1998) 8.

[5] V. Dmitrasinović, K. Kubodera, F. Myhrer and T. Sato, Phys. Lett. B **465** (1999) 43.

[6] S. I. Ando, T. S. Park and D. P. Min, Phys. Lett. B **509** (2001) 253.

[7] T. D. Cohen, J. L. Friar, G. A. Miller and U. van Kolck, Phys. Rev. C **53** (1996) 2661; C. da Rocha, G. Miller and U. van Kolck, Phys. Rev. C **61** (2000) 034613.

[8] C. Hanhart, U. van Kolck, and G. Miller, Phys. Rev. Lett. **85** (2000) 2905.

[9] C. Hanhart and N. Kaiser, Phys. Rev. C **66** (2002) 054005.

[10] V. Lensky, V. Baru, J. Haidenbauer, C. Hanhart, A. E. Kudryavtsev and U.-G. Meißner, Eur. Phys. J. A **27** (2006) 37; Y. Kim, T. Sato, F. Myhrer and K. Kubodera, arXiv:0704.1342 [nucl-th].

[11] V. Bernard, N. Kaiser and U.-G. Meißner, Eur. Phys. J. A **4** (1999) 259.

[12] J. Gasser, M. E. Sainio and A. Švarc, Nucl. Phys. B **307** (1988) 779.

[13] S. Weinberg, Phys. Rev. Lett. **17** (1966) 616; Y. Tomozawa, Nuovo Cim. A **46** (1966) 707.

[14] E. Jenkins and A. V. Manohar, Phys. Lett. B **255** (1991) 558.

[15] V. Bernard, N. Kaiser, J. Kambor and U.-G. Meißner, Nucl. Phys. B **388** (1992) 315.

[16] F. Myhrer, nucl-th/0611051.

[17] V. Bernard, N. Kaiser and U.-G. Meißner, Int. J. Mod. Phys. E **4** (1995) 193.

[18] S. Scherer, Adv. Nucl. Phys. **27** (2003) 277.

[19] H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cim. **1** (1955) 205; Nuovo Cim. **6** (1957) 319.

[20] R. Haag, Phys. Rev. **112** (1958) 669.

[21] H. J. Borchers, Nuovo Cim. **15** (1960) 784.

[22] J. S. R. Chisholm, Nucl. Phys. **26** (1961) 469.

[23] S. Kamefuchi, L. O'Raifeartaigh and A. Salam, Nucl. Phys. **28** (1961) 529.

[24] D. Ruelle, Helv. Phys. Acta **35** (1962) 147.

[25] S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2239; C. G. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2247.

[26] S. Weinberg, Phys. Rev. **166** (1968) 1568.

[27] J. Delorme, G. Chanfray and M. Ericson, Nucl. Phys. A **603** (1996) 239.

[28] G. Chanfray, M. Ericson and J. Wambach, Phys. Lett. B **388** (1996) 673.