ON WHAT HILBERT AIMED AT IN THE FOUNDATIONS

1. *Hilbert's overall view of mathematics*

Hilbert describes mathematics as "an organism whose vitality is conditioned upon the connection of its parts".[1] The connection of parts in question calls for a study of specific mathematical theories and their meta-theoretical properties. Such study is to be carried out by means of the axiomatic method. For one reason, axiomatization provides overviews for theories by characterizing the structures that are intended to be studied in them.

Hilbert's foundational investigations display the importance of the application of the axiomatic method to different subject matters. In addition to axiomatization of geometry he worked on the axiomatization of physical theories. According to Hilbert, the axiomatic method guarantees maximum flexibility in research. It must have seemed to him that the method prepares the best conditions for actual (foundational) mathematical work and for its presentation for communicative purposes.

In the historical development of Hilbert's work, it is plain to the eye that his different applications, as well as his approvals of others' axiomatizations, correspond to different periods of heated dispute in the foundations of different fields. His axiomatization of geometry corresponds to that period of epistemological disagreements on Euclidean and non-Euclidean geometries. His encouragement and approval of the axiomatization of set theory corresponds to the period of ontological disagreements partly as a result of the discovery of set theoretical paradoxes. His call for the axiomatization of physical theories corresponds to those dates when theories of special and general

[1] Hilbert 1900.

relativity were about to shake the world of physics. Hilbert came close to be the discoverer of general theory of relativity, which provided a basis for understanding the application of non-Euclidean geometry to reality. Einstein refers to Hilbert in his influential lecture on geometry and experience. These examples are sufficient for seeing Hilbert's quickness to respond different crisis periods.

2. *Hilbert's response to foundational crises*

What was common to the different crisis periods in geometry, set theory and physics is that in each case there appeared epistemological and ontological issues which were taken to be reasons as to admit some of the theories true and some of them false. This whole issue was ill-conceived according to Hilbert's viewpoint. The main source of the ill-conceived issues (especially in mathematics, but also in physics) is due to a lack of appreciation of the study of mathematical models and the absence of epistemological and ontological concerns in it.

There was no talk of mathematical models as such in the nineteenth century. Instead, notions like group and manifold were introduced for the study of different spaces and algebraic domains. These too were structural notions and the new mathematics of the nineteenth century had a structuralist orientation. Mathematics as the study of models is a further generalization of the structuralist orientation in point. Hilbert's *Foundations of Geometry* was perhaps the first example of a model-oriented systematization of a variety of structural approaches to geometry. Weyl calls Hilbert's achievement the first move to the meta-geometrical level.

Being part of a new wave challenging the old conception of mathematics as the study of number and space, the structure-oriented approach to mathematical theories was an attempt to provide new maps and guidelines for a richer description of mathematical (as well as physical) phenomena. In that sense, at least motivationally speaking, there was epistemological and ontological concerns for finding out new truths in mathematics. However, insofar as axiomatization provided the proper logical outlook for mathematical theories, the philosophical concerns in question became byproducts of structural descriptions. Such a proper logical outlook was for Dedekind and Hilbert among others promised a complete analysis of mathematical intuition.[2]

Since Descartes, the notion of intuition has been considered either a supplement or a substitute for logical inference in its traditional Aristotelian sense. From that point on the epistemological motivation behind mathematical constructions remained outside of logical treatments. With the discovery of new constructive possibilities by the means of the logic of quantifiers, the so-called epistemological motivations were partially replaced by purely logical motivations. Such replacement took place in different logicist foundations. Dedekind's and Frege's theories of number provide enough case studies for that. Increase of rigor becomes then elimination of epistemological elements. It is not necessarily in opposition to Kant's philosophy of mathematics where the notion of intuition is interpreted as knowledge of particulars.

The question as to whether one could dispense with epistemic residues in mathematical reasoning was a key element thereof in structural theorizing in mathematics. Most of the reactions against set theoretical conceptualizations were based on the presumption that elimination of epistemic elements in mathematical reasoning was impossible. Poincaré, Kronecker, and Brouwer among others argued that abstract theorizing about infinite sets did not satisfy the requirements of proper

2 Cf. Hilbert's remark in 1889, Introduction.

mathematical construction. Their notions of construction were based on the indispensability of epistemic elements like intuition in mathematics. Infinite sets were beyond the far reaches of mathematical intuition for them. Also, logic was too barren a field for exploring anything properly mathematical.

Hilbert's axiomatic approach was an optimistic take over on the side of the logical foundations. It was also a response to various restrictive views of mathematics supposedly bounded by the reaches of epistemic elements in mathematics. A complete axiomatization should be able to exclude epistemic or ontic elements from mathematical theorizing, according to Hilbert. This exclusion is not necessarily a logicism in similar form to Frege's or Dedekind's projects. That is, intuition can still have a role in mathematical reasoning. Nevertheless, this role is to be given a structural orientation with the help of explications of the underlying logic of axiomatization.

In Euclid's problems and theorems, the underlying logic seems to be epistemic logic. When Descartes argues that traditional syllogism is not the right logic for analyzing mathematical reasoning, he seems to have epistemic primitives in the back of his mind. Likewise, when Kant tries to explain the genesis of geometrical constructions, what he essentially does is to outline the epistemic machinery of pure intuition. In all these historical cases, logic is assumed to be restricted to analytic judgments in some sense. Not only that. But also, that the analytic judgments in point are universal. They neither capture existential instantiations nor particular constructions sufficiently.

In Dedekind's and Frege's logicisms the epistemic bedrock of mathematical reasoning shifts into ontic presuppositions, as a result of their logical framework. They appeal to higher-order elements such as infinite totalities. However, there is a difficulty in speaking of first-order vs. higher-order distinction before Hilbert and Ackermann's mathematical logic. The shift from epistemic bedrock to ontic presuppositions was therefore an implicit semantic shift. It was neither an epistemological nor an ontological shift. From Hilbert's perspective, it was the result of a deeper logical truth which was not worked out sufficiently clearly in the past. In a word, the logic of mathematics was still waiting to be discovered, even after the discovery of quantifiers.

From the twentieth-century logical point of view, the interesting part of mathematical model building seems to be hidden beyond the reaches of first-order quantification. As was shown by Gödel's incompleteness theorems, an uncountable number of independent truths must be captured in case a first-order axiomatization of arithmetic is preferred. This does not necessarily imply the impossibility of a complete axiomatization of arithmetic. It only implies that if a complete description of arithmetic is possible, there will be higher-order elements in its actualized form, as a consequence of the deductive power of first-order logic. That is to say, what is implicit in the so-called first-order axiomatization of a complete mathematical theory is in fact a higher-order logic.

Is higher-order logic the last word then, for the foundations of mathematics? If so, that would be turning back to some ontic presuppositions like Frege's or Dedekind's. But that would definitely not fit into Hilbert's mold. The simultaneous development of logic and mathematics has to make a new mold, in order to give the apparently higher-order elements a new shape for the descriptively complete axiomatization of analysis. In fact, what Gödel tried to do initially before he came up with his incompleteness theorems was proving the consistency of analysis.[3] However, the implicit logical framework in Gödel's attempt was first-order logic as was distinct from higher-order on the basis of

[3] See Zach 2005.

type-theoretical (i.e. ontological) concerns.

3. *The search for foundations without presuppositions*

The possibility of eliminating philosophical concerns in foundational investigations has critical importance for a correct interpretation of Hilbert's thoughts. Mathematics, according to Hilbert, is without presuppositions. The logic that is to be used in analyzing mathematical reasoning therefore has to be made free from epistemic and ontic primitives. It has to have a nominalistic character. It has to surmount the usual type-theoretical and hence ontological hierarchies. Is that impossible? If impossible, impossible on what grounds?

A structure-oriented axiomatization of analysis aims at a complete description of the structure of real numbers. The description in question is a particular way of systematizing the logical consequences of the axioms. The higher-order element which is alien to first-order logical consequence comes in with the least number principle, viz. All nonempty subsets of the set of natural numbers have a least element. That is a form of strong induction. Is it possible to reformulate it in nominalistic terms? If not, then Hilbert was clearly wrong about the reaches of axiomatic method in mathematics.

What was Hilbert's view? He realized that all the logical consequences of a complete description of a mathematical theory should be tautological. That is to say, logical inferences add no new information to what is captured by the axioms. That is Hilbert's formalism. However, if by formalism it is meant that mathematical reasoning can be reduced to logical reasoning about formal structures, then that would be a misunderstanding about Hilbert's view. Hilbert never argued for the view that mathematics is the logical study of formal structures. His view favors the study of the models of formal structures, despite the fact that he called his own investigations proof-theoretical.

Hilbert was not searching for the formal reaches of the axiomatic method. That would plainly be part of epistemological explorations. He emphasized that the aim in the axiomatic method is not to discover new truths. Rather the aim is to capture the necessary and sufficient conditions for proving theorems. The tautological character of the logic to be applied to axioms thereof excludes the possibility of increase of information as a result of formal proof analysis. Logical analyses of proofs are rather to be made for the purpose of extracting nominalistic content from the shorthand assumptions of mathematical reasoning on infinitely complicated structures that are given by axiomatic descriptions.

A structure-oriented axiomatization is a particular use of deductive systematization. Models of theories are specified as such by the axioms insofar as they are consistent in the model-theoretical sense. Hilbert's strategy to prove the consistency of arithmetic was to obtain a direct proof of the model-theoretical consistency by way of proving the deductive consistency of arithmetic axioms. In a deductive consistency proof, one has to show only that no inconsistent sentence appears as a result of the deductions from the axioms of a theory. Such proof would suffice for Hilbert's purposes on the assumption that the underlying logic of mathematical reasoning is semantically complete. Hilbert seems to have assumed it, at least tentatively. Unless the underlying logic of mathematical reasoning is semantically incomplete, Hilbert's strategy should have worked. Nevertheless, it turned out as a consequence of Gödel's incompleteness theorems that for a descriptively complete axiomatization of arithmetic the underlying logic of mathematical reasoning could not have been semantically complete. Hilbert had to switch his strategy into another form of direct method for proving the model-theoretical consistency of arithmetic. Another alternative is the admittance of the utter failure of Hilbert's program by reference to incompleteness theorems. Nevertheless, that would have been possible only if the epistemological ground of mathematical reasoning is considerably safer than mathematical reasoning itself. That would be a *petitio principii* in agreement with the ontic infinitistic presuppositions of a would-be higher-order (type-theoretical or set-theoretical) consistency proof. Elimination of philosophical problems from foundational investigations is thus not an elementarily carried out task in the light of the developments in the early twentieth-century logic.

History of logic vindicated Poincaré's predictions up to a point. He was the one who thought that the epistemic element in mathematical reasoning is essential. He argued that the so-called inescapable appeal to complete induction in consistency proofs make purely logical (non-epistemological) axiomatizations circular. Hilbert's response to that argument was that one has to distinguish between different principles of induction. One type of induction takes place on the syntactic level, where Poincaré is right in his observation. However, another type of induction is by structural generalizations of particular constructions. There a more cautious logical investigation is needed in order to see whether consistency proofs are possible by the structural means of particular constructions.

On similar lines there is no need for an appeal to any basic intuition in our foundational theorizing, according to Hilbert. Foundations can be studied mathematically by improving the logical methods. What this means is that the exclusion of certain principles like the axiom of infinity, the axiom of reducibility and the axiom of completeness is for the sake of showing that a logical axiomatic foundation—without making contentual (existential) assumptions about mathematical infinity—is possible. This immediately implies that Hilbert's preference is first-order level theorizing in logical theory, which can be applied to different mathematical domains without making actual assumptions on infinite totalities etc. On this point, epistemological interpretations of Hilbert's views are based on patent misconceptions about Hilbert's philosophy of mathematics. All that is needed for Hilbert's foundational purposes is, first, the determination of models by axiomatic analysis, and then, second, model-theoretical consistency proofs for the axiomatizations.

In his lectures in the 1920s Hilbert used Russell's ramified theory of types as what he considered to be the extended predicate calculus. This treatment included the definition of real numbers and an upper bound as a class of real numbers, which in turn required infinitely many types, since the upper bound (as a class of real numbers) of a set of real numbers has to be a real number of a higher type. Russell's solution for this problem was to introduce an axiom (viz. axiom of reducibility) which reduces the higher types to the lowest compatible type. Hilbert followed Russell's solution in his lectures. Nevertheless, his ultimate aim was to eliminate the axiom of reducibility as a presupposition, which he considered to be an infinitistic assumption. In that sense, Hilbert's aim was still in line with his earlier claims and criticisms against Dedekind and Frege's presuppositions about the application of the universal quantifier.

4. *The priority of model-theoretical concerns to epistemological questions*

Hilbert seems to have considered certain restricted forms of mathematical induction as non-informative. On the other hand, he considered Poincaré's emphasis on the synthetic *a priori* character of the principle of mathematical induction an epistemological presupposition which is not needed for foundational investigations. Likewise, Hilbert considered Russell and Whitehead's axioms of infinity, reducibility, and completeness "actual, contentual assumptions that cannot be compensated for by consistency proofs. ”[4] Nevertheless, he did not consider any epistemological restriction on them as Brouwer or Weyl did. Brouwer's intuitionism was perhaps the last epistemological resistance against logical analyses of mathamatical reasoning on an allegedly Kantian footing.

Partly because Hilbert had to take a stand against Brouwer's intuitionism in the nineteen-twenties, the whole issue of Hilbert’s foundational views is often taken to have been a response on epistemological grounds. Brouwer’s foundational worries were epistemological. Mathematics proper, for him, presupposed an indispensable epistemic element. It was taken as to rest on and generated from a fundamental mathematical act of the mind. According to Brouwer, the derivation of mathematical truths by repeated mental acts take place as a generation of new knowledge from a previous source. The true foundation of mathematics we should seek, therefore, where the original source of the generation of repeated mental acts was activated. In that sense the true intuitionistic foundation is what Brouwer calls the first act of the mind towards mathematical knowledge. In Brouwer 1948 we read: “consciousness in its deepest home seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation”. The creative subject departs from this will-less stage by a move of time. From that stage it passes to the combination of past and present moments of the ur-intuition of two-ity. Iteration of this ur-intuition gives the creative subject, according to Brouwer, sequences. Brouwer calls them causal sequences. Mathematical activity with such sequences is called mathematical attention.

As if Hilbert’s foundational terminology had to have conceptual commitment to Brouwer-like epistemological worries, different works in the philosophy of mathematics literature focus on the epistemological force of Hilbert’s foundational views.[5] It is nevertheless misleading to do so in that

[4] Hilbert 1928, p. 479.
[5] Cf. Kitcher 1976 and Parsons 1998, for example.

Hilbert's model-theoretical concerns were prior to any search for an epistemological foundation. What is commonly misleading in the epistemological interpretations is the meaning assigned to questions like "What are signs?", "What is the epistemological status of finitary objects?", "What kind of intuition is the finitary intuition?" These questions, when they are asked as questions of epistemology, have no significant value for a better understanding of Hilbert's philosophy of mathematics; even though it is true that Hilbert himself sometimes speaks of the *a priori* intuition, and characterizes it as the "frame of the finitary mode of thought". When Hilbert discusses the *a priori*, he does not do it for the sake of explaining his epistemological standpoint. Rather, he wants to emphasize the foundational import of certain mathematical or logical propositions. For example, when he says:

> ...there are...those propositions that are generally held to be *a priori*, but which cannot be achieved within the frame of the finite mode of thought—for example, the principle of *tertim non datur*, as well as the so-called transfinite statements generally. [6]

Hilbert is not primarily taking his epistemological stance and indicating where it differs from Kant's. Hilbert's main point is rather that the applications of the law of excluded middle, for instance, are in some cases not elementary, that is, they are meta-mathematically problematic applications. Such point concerns only the foundational and meta-logical import of a certain logical principle.

Hilbert tried to clear his way from epistemological and metaphysical assumptions about the nature of mathematics. A sharp statement of Hilbert's non-epistemological view can be found, for example, in his 1917 lectures on the principles of mathematics. He says there that his axiomatic approach is not to overcome philosophical difficulties, but to "cut them off".[7] Therefore, the questions

---

6 Hilbert 1931, p. 1150.
7 Cf. Sieg 1999, p. 11.

mentioned just above should not be asked as epistemological questions, but rather be asked in association with a meta-logical sense of the terms occurring in them. They must be treated like questions of meta-logic and meta-mathematics, i.e. as a part of what Hilbert calls the simultaneous development of logical and mathematical methods. Most importantly perhaps, Hilbert's so-called finitism (and hence his view of the *a priori*) should be taken as an object of meta-logical investigation, i.e. by seeking an elementary account of the theory of logic.

Hilbert was interested in what was there in axiomatic mathematics as determination of models for the theories (as much as in their proof-theoretical structure). This is an immediate consequence of his conception of an axiomatic foundation for mathematics. Epistemological problems concerning the cognitive content of symbol structures are a completely different issue. This is not to say that there are no philosophical problems (epistemological or otherwise) concerning the existence and knowledge of the models. Nor it is to say that Hilbert ignores such problems. The point is that epistemological problems are of a different sort. They are of secondary importance for Hilbert's meta-theoretical purposes. It was Brouwer, not Hilbert, who injected epistemology into the discussion of the foundations of mathematics.

5. *On the significance of nominalistic assumptions*

From the very beginning of his foundational studies, it was clear to Hilbert that even the first-order applications of quantifiers with the assumption of infinite operations is a problematic issue. If one wants to clarify the nature of the infinite in mathematics and give a humanly practicable account of universal and existential quantification (i.e. without assuming infinite operations) one has to face the problem of quantification over infinite domains in mathematical reasoning. So not only higher-order reasoning must be reconstructed on the first-order level, but also first-order quantification must be given a practicable (elementary) account.

One of the nominalistic assumptions in the philosophy of mathematics is that only individuals are admissible as objects of quantification. In logical terminology this assumption amounts to permitting only to first-order quantification. Hilbert's line of thought is in keeping with such a view:

If logical inference is to be certain, then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else.

In this sense Hilbert is defending here first-order logic, which accepts quantification only over individuals, in contrast to a higher-order one. Hilbert continues:

Because I take this standpoint, the objects [Gegenstände] of number theory are for me— in direct contrast to Dedekind and Frege—the signs themselves, whose shape [Gestalt] can be generally and certainly recognized by us—independently of space and time, of their special conditions of the production of the sign, and of insignificant differences in the finished product.

Hilbert criticizes thereof Frege and Dedekind on their quantification over concepts or their extensions in their logical language. This is in line with Hilbert's overall view on logic and logical reasoning. As was noted it was in Hilbert's school that first-order logic was separated from the higher-order quantification theories of Frege and Russell- Whitehead.

Hence in a wider philosophical perspective Hilbert's opposition to Frege and Dedekind, and operations with general concept scopes is not an opposition of a formalist to a non-formalist. It is rather an opposition of a nominalist to conceptual realism. Under wrong interpretations of Hilbert's philosophical terminology—especially under the attribution of "finitism" and "formalism" to it—the real gist of Hilbert's "philosophical attitude" is poorly obtained. The rest of the passage in Hilbert's paper—what follows below—leads to serious misunderstanding when it is read out of its proper context:

The solid philosophical attitude that I think is required for the grounding of pure mathematics—as well as for all scientific thought, understanding, and communication— is this: *In the beginning was the sign*.

The correct interpretation of this passage should be that Hilbert favored nominalism, and hence first-order quantification in contrast to a higher-order one. In this light, from Hilbert's nominalistic point of view, Frege's conceptual realism was totally ill-advised:

[Frege] fell to some extent into an extreme realism of concepts. ...he believed he was entitled to take [concept scopes] unrestrictedly as things.

In the later editions of Hilbert and Ackermann 1928 second-order logic is considered. Its incompleteness is pointed out. Its relation to set theory is briefly discussed. Higher-order logic is introduced with an indication by examples that it is "the appropriate means of expressing the modes of inference of mathematical analysis". However, just like types and the axiom of reducibility, higher-order quantification does not exactly fit into Hilbert's mold. It involves quantification over a domain of so-called "all" predicates. That is why he preferred first-order logic and tried to surpass the difficulties with universal quantification by means of his epsilon technique. It can be treated as a nominalistic account of quantification theory.

All this is in accordance with Hilbert's concern for concrete content in meta-mathematics. Salvageable domains of concrete objects (i.e. signs with their representative role) which are immediately given in mathematical practice should be the ground to rely on in foundational considerations. From a wider historical perspective, Hilbert is against a commonly accepted view in the philosophy of logic and mathematics. According to this view, logic and mathematics deal with general concepts. And in the last analysis it is sense-perception that grasps particulars. Therefore, the justification of all instantiation and the introduction of particular (concrete) representatives of general concepts must be perceptual. In their foundational works Frege, for example, follows such a view but Hilbert does not. When Frege is trying on the one hand to dispense with intuition, he is on the other trying to reduce number theory to what he takes to be the most general concepts and principles of reasoning. Hilbert notwithstanding treats logic preferably on the first-order level. He criticizes the reliance (especially by Dedekind and Frege) on general concept-scopes. He wants to formulate axiomatic foundations of mathematics in the study of the structures of concrete objects. Accordingly, Hilbert tries to practice his meta-mathematics in nominalistic terms. He believes that logic can cope fully with reasoning about (and with) particular objects, and on the first-order logical level. In this regard his epsilon-technique for example amounts to a method of instantiation. It aims to make systematic use of the particular instances of general concepts in nominalistic terms.

Hilbert's finitism is sometimes seen as the view that the (apparently) actual infinitistic assumptions of mathematical reasoning can be given an epistemological foundation, by reference only to finitary content of mathematical statements (not by going beyond that). As has been pointed out, such conception of finitism makes misleading ways to understanding Hilbert. Hilbert's aim was to provide logical axiomatic foundations, rather than epistemological foundations. He hoped to have reached this aim by detaching the axiomatic investigation from epistemological concerns. In that sense Hilbert's aim amounts to finding out the appropriate logical treatment of the apparently infinitistic assumptions of mathematical reasoning, without permitting any infinitistic technique in the foundational practice. Here the problem is not with the epistemological admissibility of the techniques used. It is more appropriate to say that, in its axiomatic form, Hilbert's finitism amounts to a meta-logical (as well as meta-mathematical) strategy. The right source to decide the admissibility of the techniques involved in this strategy is logical semantics, not epistemology. On this explanation, possible definitions of "Hilbert's finitism" in terms of epistemological or ontological primitives lead to wrong interpretations of Hilbert's ideas. The wrong interpretations are usually implied by the restriction of the so-called big problem about the infinite to that the infinite does not obviously correspond to anything in reality. If the definition of the concept of finitism is restricted to a way out from the lack of correspondence between infinity and reality, then such restriction would lead to misunderstandings. Because, even though it is a part of the problem of foundations to explain how the infinite can come about in actual (real) mathematics, this is not an epistemological concern, according to Hilbert. Its treatment should be accordingly. Otherwise the same mistakes that were made by mathematicians like Poincaré, Weyl and Brouwer would be made. Hilbert's nominalism was for the sake of eliminating "dubious or problematic modes of inference" from foundational studies. "Finitism" is the name he gave his strategy to cope with infinitistic operations in mathematical reasoning. The question here of what the so-called finitistic operations consist of is therefore a tricky one. Yet it should be clear that nothing relevant to Hilbert's views can come out of it, if it is asked as an epistemological question concerning the admissibility of certain recursion techniques.

Two well-known attempts to explain finitism are due to Tait 1981 and Parsons 1998. Tait considers finitism to cover a minimal kind of reasoning presupposed by all reasoning about number.[8] Parsons, on the other hand, argues that finitism determines the domain of intuitive evidence. Thereby, Parsons admits a basic intuition of finite objects. Both approaches try to give an account of epistemic primacy and certainty of finitist mathematical reasoning. From Hilbert's point of view, such an enterprise is pointless. The problem is not how to come up with criteria for an epistemically safe beginning to mathematical reasoning. The criteria are needed rather for meta-logical purposes. For the same reason, asking, for example, whether Hilbert's "finitistic intuition" is the Kantian space-time intuition or it is something else, is a seriously misguided way of approaching the foundational problems. It is being neglected in such mode of questioning that the set of problems concerning finitism and quantification has to be detached from epistemological concerns. From Hilbert's point of view, the solution of foundational problems cannot be dependent on any epistemological preferences.

In logical theory and meta-mathematics no reference to finitude is necessary. We do not need to commit to the finitude of the domain of objects we are dealing with. The characterization of finitistic methods can be maintained entirely in terms of salvageable objects of mathematics. Now basic operations of elementary arithmetic are in principle finite and salvageable. The infinitistic element, as Hilbert seems to have assumed, comes in when we use quantifiers. The central question here is: what kind of operations do we need to clear the quantifiers from committing to infinitistic assumptions (and salvage the entities that are quantified over)? Hilbert considered these operations as follows:

8 Cf. Zach 2001, Ch. 4.

...the modes of inference employing the infinite must be replaced generally by finite processes that have precisely the same results, that is, that permit us to carry out proofs along the same lines and to use the same methods of obtaining formulas and theorems.[9]

Hilbert's aim here is to find out suitable operations that give the same results as those modes of reasoning which appear to have employed the actual infinite in mathematical reasoning. In this sense Hilbert's aim does not involve any epistemologically restrictive (i.e. finitist) condition at all. In Kreisel's way of saying: the eliminability of the infinitistic assumptions "is thought of as a fact (to be discovered), not a doctrinaire restriction".[10] The epistemologically problematic modes of reasoning concerning the infinite can be taken care of by applying logically unproblematic techniques, without making existence claims about any extra-logical (mathematical) entities, other than the ones that are immediately given to our intuitions. For that purpose, all one has to do is to search for logically admissible modes of reasoning that can replace the figure of speech of the apparent infinitism in mathematics. On this point Hilbert remarks sharply in his 1926 paper; he, refers to a certain jargon in mathematics and says playfully:

...if mathematics is to be rigorous, only a finite number of inferences is admissible in a proof—as if anyone had ever succeeded in carrying out an infinite number of them.[11]

What is crucial to Hilbert's purposes is contentual logical inference as he emphasizes in the same paper:

Contentual logical inference is indispensable. It has deceived us only when we accepted arbitrary

[9] Hilbert 1926, p. 370.

[10] Kreisel 1976, p. 98.

[11] Hilbert 1926, p. 370.

abstract notions, in particular those which infinitely many objects are subsumed. What we did, then, was merely to use contentual logical inference in an illegitimate way....

The task is then to find out the legitimate operations of logical inference, to be used in handling mathematical notions which subsume infinitely many objects.

6. *Consistency as existence in the model of all models*

Broadly speaking, Hilbert's conception of truth and existence in mathematics indicates where the structure-oriented viewpoint cuts off the epistemological and ontological concerns. The information codified by an axiom system specifies the class of its models. So that it becomes a meaningful task to try to understand the contents of mathematical theories by means of axiomatic analysis. Hilbert's conception of truth and existence in mathematics are also along this line. They are envisioned from a structure-oriented viewpoint. In Hilbert 1900a we find a strong statement of this viewpoint:

...the demonstration of the consistency of the axioms [of the real number system] is at the same time the proof of the mathematical existence of the totality of all real numbers or of the continuum. In fact, when the demonstration has been fully achieved, then all objections which hitherto have been raised against the existence of this totality will lose all justification.[12]

Also, in Hilbert's 1899 letter to Frege we read:

If the arbitrary chosen axioms do not contradict each other with all their consequences, then they are true and the things defined by the axioms exist. That for me is the criterion of truth and existence. [13]

Such point of view is almost a refutation of the formalist philosophy of mathematics, which is

[12] Hilbert 1900a, p. 1105.
[13] Kluge 1971, p.12.

sometimes misleadingly attributed to Hilbert. To avoid misunderstandings on this point, Hilbert's approach must be put into a proper context. It has to be taken into consideration against the tacit assumption Hilbert seems to have made when he says that consistency implies existence. The consistency in question is model-theoretical consistency. In line with his general model-theoretical outlook the tacit assumption that comes with Hilbert's criterion of truth (i.e. as the consistency of the axioms) seems to admit the determination of models of potential models for theories, viz. ultimately a model of all models. Indeed, Hilbert's paradoxical sounding claim about truth and existence as implied by consistency is true in the model of all models. The (model- theoretical) consistency of a theory implies the existence of models for it in this model of models.[14]

It would be an oversimplification to assume that axiom systems are generated arbitrarily out of nowhere. New systems are in some way built up on and connected to the previous theories. For such building and connectedness, the notion of a model of all potential models is very useful. In it, quantification provides the same conception of mathematical existence as well as of truth on a preliminary theoretical level for different axiom systems. This kind of view is, for instance, implicit in Hilbert's following statement:

The conception of the continuum, or equally the concept of the system of all functions, exists then in precisely the same sense as does the system of rational numbers or that of the higher Cantorian number-classes and powers.[15]

Purportedly the same sense of mathematical existence is obtained if the model-theoretical consistency of each axiom system is proved. In that sense what Hilbert envisions and hints at in the quoted passage

[14] Cf. Hintikka 2004.
[15] Hilbert 1900a. p. 1105.

is a uniting model of all models for different axiom systems.

The tacit assumption Hilbert seems to have made here shares the same presuppositions as Husserl's notion of definite complete manifolds in which "the concepts of true and formal implication of the axioms are [considered to be] equivalent."[16] The ontology of manifolds in question involves a super-universe of potential models for the theory, a "model of all models". A similar ontology can be imagined in connection with Riemann's work on manifolds, for instance, as a chapter in what might be called a general study of forms of space; since a manifold by definition is a geometrical entity which is a structured totality of all possible solutions of a given polynomial equation. Even though this sense of manifold is not necessarily the same as Husserl's, they are obviously familiar. Likewise, Cantorian universe of sets can be seen as an abstraction from Riemann's geometric notion of manifold.

The so-called model of all models can be considered a natural presupposition of mathematical activity. After all what the mathematician does is to build and connect to each other different structures. The beginning stage of such activity requires the grasp of what might be called a particular relational structure. When the net of relations of such a structure is considered as basis, the task of understanding its models as well as the task of extending its models is a matter of application of structure-preserving rules. Such application presupposes consistency as a ground for its own justification. At that point the model-theoretical consistency of a particular axiom system suffices to justify mathematician's actual intentions to study what there is to be known in the models of the system.

Husserl presented a similar argument to the one just has been sketched in his Göttingen lecture at

[16] Husserl 1913, section 72.

Hilbert's seminar in 1901.[17] Roughly, Husserl's argument goes as follows: Take two axiomatic systems AX_1 and AX_2. Let AX_1 be a subsystem of AX_2, in the sense that AX_2 is the extended system by additional axioms when AX_1 is considered as the original system. Two conditions must hold then, according to Husserl. One is that AX_1 must be a definite manifold. Two is that AX_2 must be consistent. Definite manifold means the intended model of the theory is determined completely. This is suggestive of descriptive completeness. If these two conditions are satisfied then we say AX_2 is a conservative extension of AX_1, in the sense that its models can as well be determined on the basis of the models of AX_1. Here, Husserl's major aim seems to have been to show how an axiom system determines its intended models in a definite way and to justify if possible different extensions of the theories by proving their consistency and completeness.

Assuming that the conservative domain extension of the models of systems say, from AX_1 to AX_n) reaches up to a uniquely determined universe of definite manifolds, the maximal extension that is obtained in the end of such domain-extension procedure can be considered as an analogue to what has been called above "the model of all models". That maximal model is what Husserl and Hilbert seems to have presupposed as a ground for their preliminary theoretical conception of mathematical truth, existence and consistency. For sure, such conception has to be backed up by a proof of the model-theoretical consistency of the systems involved, most notably the continuum or equivalently the system of all functions.

The mathematical investigation of the structure of real numbers falls under a major aspect of the idea of "all models". It requires an explication of the continuity and completeness assumptions for its

17 See Husserl 1970, supplementary texts B, essay III.

logical axiomatic characterization. Historically, the continuity and completeness assumptions in defining the structure of real numbers find their proper treatment in the works of Cantor and Dedekind. Dedekind's characterization of the real line as a densely ordered system which is closed under algebraic operations as well as under limit operations is sometimes called complete in the sense that it determines a model in a definite way for the continuous number line. This kind of completeness is in a different sense from the completeness of axiom systems. It is rather a meta-theoretical property of the models. The considered completeness of the real line is obtained by using what are known as Dedekind cuts. The intuitive idea behind Dedekind's cut-procedure is that the so-called cuts fill in all the gaps in the system of rational numbers. So that each bounded set of reals have a least upper bound. What is remarkable about completeness in the sense just mentioned is that it entails that the structure of real numbers as imagined is uncountable. This is what Cantor's diagonal argument showed.

The results that were reached by Cantor and Dedekind's works were very important discoveries of the nineteenth-century mathematics, according to Hilbert. Nevertheless, one of the main purposes of Hilbert's axiomatic foundations was still to explain how the so-called uncountable infinity can come about without making any assumptions concerning the actual existence of infinite totalities. For that purpose, the completeness and continuity assumptions that are intuitively appealed to in Dedekind's characterization of the real line have to be made explicit with their logical dependence on the axiomatic system. The same task is needed to be accomplished also for understanding mathematical interconnections between different axiom systems and the structures they characterize. Most notably, between algebraic and geometrical structures, interconnections must be studied in the light of the continuity properties. This arises from the geometrical sense of the models in characterizing the real line either as an infinite set of points or as of line segments. On similar lines, to find out meta-theoretical interconnections between the system of real numbers and the Euclidean space, and hence to establish the possibility of the determination of models of geometry, an investigation of their continuity properties is inescapable. By way of disclosing the continuity assumptions of an axiom system one can characterize the space and hence the same sense of existence and truth is obtained in all its models.

7. *Hilbert's treatment of continuity assumptions*

Partly to point out the role of continuity assumptions in the above sense in axiomatized geometry, in addition to the original treatment of the axiomatic foundations of geometry in his 1899 book, Hilbert in Appendix IV gives a different determination of the plane geometry. It can also be generalized to the case of space. Hilbert's determination of the plane is by way of analyzing in an axiomatic way the properties of manifold congruent motions based on the notion of transformation group.[18] Mainly by appealing to the notion of continuous transformation and some axioms of motion; e.g. axiom of the composition of two motions as to form a group, Hilbert presents a determination of a model for the plane.

The continuity assumptions for the characterization in the general case of the space are made at the very beginning. The aim in such analysis is, as Hilbert points out:

> ...to determine the least number of conditions from which to obtain by the most extensive use of continuity the elementary figures of geometry (circle and line) and their properties necessary for the construction of geometry.[19]

[18] Cf. Appendix IV of Hilbert 1899 (Second and later editions) A brief survey of Hilbert's work on geometry can be found in Bernays 1967 and Toretti 1978.

[19] Hilbert 1899, p. 189 (second English edition)

The difference between the main approach in Hilbert 1899 and the group-theoretical approach in the appendix has to do mainly with the role of axioms of continuity in the complete determination of models. In the main axiomatization Hilbert's first four groups of axioms are arranged in such a way that "continuity is required last". This provides a way to clarify which logical consequences of the axioms are independent of the continuity assumptions.

First one of Hilbert's continuity axioms is what is called the Archimedean axiom. This axiom says that, given two line segments AB and CD, either one of them, let us say, AB can be extended by multiplied measure of the other segment CD such that it exceeds the length of CD. The algebraic structure that might be superimposed on space with the help of the Archimedean axiom here is obtained by reference to the system of coordinates that satisfies Hilbert's axioms of incidence, order and congruence, and the axiom of parallels. An instance of this algebraic structure is the system of algebraic numbers and rational operations on them with the exclusion of square roots.[20] Hence with the help of the Archimedean axiom, continuity is obtained only up to a point. An additional second axiom, which connects the geometric continuity to the real continuum, is necessary. That second continuity axiom is Hilbert's axiom of line completeness, which says:

> An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from [the axioms of incidence, order and congruence] and from [the Archimedean axiom] is impossible.[21]

From this axiom Hilbert derives the theorem of completeness which states that the extension of the

[20] See Hilbert's Theorem 65 in Hilbert 1899 (second English edition).
[21] Hilbert 1899, p. 26.

elements (viz. points, lines, planes) of geometry is not possible without violating the axioms of incidence, order, congruence and Archimedes.[22]

The theorem of completeness provides the appropriate perspective to consider the foundations of analysis in relation to the foundations of geometry. In particular, Hilbert's consistency proof for the axioms of geometry, which is relative to the consistency of analysis, can be positioned in the proper foundational basis. Most notably, as Hilbert also points out, the existence of infinitely many geometries which satisfy the first four groups of Hilbert's axioms plus the Archimedean axiom is shown. And when the axiom of line completeness is added to the axioms, a uniquely determination of the Cartesian geometry is obtained.[23] This signifies almost a simultaneous development in the foundations of geometry and of analysis, which is due to the additional of the continuity axioms. They are added to the axioms of number theory in Hilbert 1900, as follows:

(Archimedean axiom) If $a > 0$ and $b > 0$ are two arbitrary numbers, then it is always possible to add a to itself so often that the resulting sum has the property that $a+a+...+a>b$

(Axiom of completeness) It is not possible to add to the system of numbers another system of things so that the axioms [of linking, calculation and ordering with the Archimedean axiom] are all satisfied in the combined system; in short, the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms.[24]

Hilbert in his 1900 paper defines the system of real numbers as a complete ordered Archimedean field. And the models that he constructed in Hilbert 1899 to prove the consistency of geometry can be

[22] ibid. p. 27; the axiom of line completeness is added to Hilbert's book in the later editions. In the first edition there is no axiom of completeness. In the second edition there is the axiom of completeness for the general case. Later the axiom of line completeness is added so as to suffice to prove what is referred above as the theorem of completeness. For different completeness axioms see Peckhaus 1990, pp. 29-35.

[23] Cf. Hilbert 1899, p. 32.

[24] Hilbert 1900a, par. 6 (p. 1094)

considered as the relevant subfields of the system of real numbers for different sets of geometry axioms.[25] In general terms, it seems fair to say that Hilbert's completeness axiom (or theorem) provides a way of translating Euclidean geometry to the Cartesian geometry.[26] By doing that it specifies an ordered Archimedean field, for which if there were a combinatorial way to show its consistency that would also lay the foundations of analysis. What is further needed for the consistency proof is to eliminate the appeal to arbitrary sets, for instance in the application of Dedekind cuts and correspondingly in making a combinatorial sense of arbitrary sets of points in the continuity axioms.[27]

For the same reasons as in geometry, continuity assumptions and hence completeness play a crucial role also in physics. To give an example, in his mechanics lectures Hilbert considers the addition of vectors as a continuous operation, in the sense of the Archimedean axiom.[28] For example, given a domain D around the vector sum A + B, one can always find other domains D_1, D_2... around the endpoints of A and B such that any considered sum of two vectors in these domains has endpoints falling inside the domain D. The intuitive idea here seems to be closer to the notion of connectedness. What Hilbert had in mind though about continuity is fairly easy to understand. The punchline of the assumed principle is that we can move from any point of the domain to any other point of it through a continuous line, which remains in the same domain. It is plain to the eye here that Hilbert's major aim is to specify a particular class of models for physical forces, i.e. which obeys the continuity axiom. This does not mean that Hilbert's view excludes systems with certain discontinuities or systems without the Archimedean property; since an axiom system in Hilbert's sense does not express a fixed set of states of

---

25 Hilbert 1899, Chapter II.

26 Cf. Bernays 1967.

27 Cf. Kreisel 1976, p. 101.

28 Hilbert's 1905/06 lectures; see Corry 1997.

affairs. It only defines a “possible form of a system of connections, a system which is to be investigated according to its internal properties.”[29] Hilbert’s view simply suggests the study of different physical systems. In his 1900 Paris address, he states it straightforwardly:

> As he has in geometry, the mathematician will not merely have to take account of those theories coming near to reality, but also of all logically possible theories.

All that matters here is the determination of models up to isomorphism. And hence what matters in a logical axiomatization is the model-theoretical consistency of the axiom system. And for that purpose, as was indicated above the underlying logic must be capable of allowing the intended models in question to be captured completely. This is suggestive, in the first place, of a descriptive completeness. Nevertheless, if a deductive consistency proof could be achieved, that also could serve as a way to capture the intended models of the theory. Of course, provided that the underlying logical theory is semantically complete. Otherwise the deductive consistency of the theory does not imply its model-theoretical consistency.

As can be seen from the remarks up to this point, the interconnections between completeness, continuity and consistency properties of mathematical systems are closely related with their model-theoretical characterizations. If one uses a logical axiomatization these characterizations can be handled by means of two requirements of the axiomatic method: First, the purely logical character of inferences from axioms to the truths of the theories is needed. Second, a complete logic which provides means to obtain deductively or descriptively complete representations of the theories must be formulated.

At some point Hilbert might have assumed the semantic completeness of the underlying logic of

[29] Cf. Hallett 1995, p. 137.

axiomatization. Nevertheless, even if this is true, it does not mean that he was arguing for a mechanical procedure to prove the consistency of mathematical theories. The model-theoretical character of his viewpoint excludes such an approach as an ultimate foundational aim for Hilbert. Whatever "comes near to reality", whatever is logically possible are at bottom all depending on their determination up to isomorphism and hence on the meta-theoretical level, on the model-theoretical consistency of the axiom systems.

As is presented in his sixth Paris problem, probability as part of physics provides a strong case for Hilbert's views. Hilbert considered probability as a part of the physical sciences and his main interest in the probability was the problem of how to avoid and eliminate observational errors in measuring physical magnitudes.[30] Hilbert's application of probabilistic reasoning to the physical measurement proves that the continuity assumptions for Hilbert—however appears to involve infinitistic operations—always had a combinatorial and model-theoretical basis:

The validity of the Archimedean axiom in nature stands in just as much need of confirmation by experiment as does the familiar proposition about the sum of angles of a triangle.[31]

In this regard any view stating that the infinite (as well as the continuity assumptions about infinite systems) in mathematics is part of mere formal manipulations for Hilbert, misses the essential connection of Hilbert's mathematical ideas with his general structure-oriented view of physics and physical continuum:

In general, I [Hilbert] shall like to formulate the axiom of continuity in physics as follows: 'If for the

[30] Corry 1997, p. 160-161.
[31] Hilbert 1918, p. 1110.

validity of a proposition of physics we prescribe any degree of accuracy whatsoever, then it is possible to indicate small regions within which the presuppositions that have been made for the proposition may vary freely, without the deviation of the proposition exceeding the prescribed degree of accuracy.' This axiom basically does nothing more than express something that already lies in the essence of experiment; it is constantly presupposed by the physicists, although it has not previously been formulated.[32]

To considerable extent Hilbert's work on physics is devoted to the purpose of searching suitable ways of axiomatizing different theories. As also is seen in the statement of his sixth problem Hilbert's central emphasis is on the logical axiomatization of theories. As has been sketched here, the determination of models by means of logical axiomatization is obtained by investigating the continuity properties of the systems in consideration. Thereby, above all, the streamline of Hilbert's foundational investigations is to be found where the continuity assumptions for different mathematical and physical fields meet, viz. in the meta-theoretical study of the system of real numbers and in its model- theoretical consistency. For that purpose, development of the meta-theory for logical axiomatization is also required; presumably, on the basis of suitable model-theoretical consideration of continuity and completeness properties of algebraic and geometrical structures on the meta-theoretical level.

8. *The role of logical axiomatization in consistency proofs*

The problem of model-theoretical consistency of analysis and arithmetic has to be approached by means of logical axiomatization. For the primary purposes of a logical axiomatization, it suffices that the theorems of arithmetic, for example, are all logical semantic consequences of the axioms. As has been pointed out, this does not require that these consequence relations can be implemented by mechanical

[32] ibid.

rules of inference. Thus, for example a second-order axiomatization can serve these primary purposes as well as a first-order one, even though second-order logical truths are not recursively enumerable. For this reason, it cannot be conclusively said that Hilbert's consistency program was made impossible by Gödel's results.

As is well known, a crucial first step to achieve Hilbert's principal aims for the foundations of mathematics is to prove that the usual set of axioms of arithmetic is consistent. Gödel's second incompleteness result showed that if any such set of formal axioms AX that can codify elementary arithmetic is consistent, then the consistency of AX cannot be proved in AX. That is to say, the sentence coded in the language of AX, which says that AX is consistent, cannot be derived in the formal system AX. Gödel's argument implicitly assumes that ordinary first-order logic is used in the axiomatization. It also seems to assume that we are dealing solely with proof-theoretical consistency in meta-mathematics. This result led some logicians to immediately give up hope about Hilbert's program. However, Hilbert himself never admitted that it contradicted his conception of the problem of foundation. Hilbert was right in not giving up his foundational aims. One can base AX on a richer logic than the ordinary first-order logic, and then a proof of the consistency of arithmetic which is acceptable by Hilbert's standards can be given. What one has to do is to find out whether there are elementary logical operations that can be formulated in a second-order axiomatization, to carry out a proof of the consistency of arithmetic.

Presumably, Hilbert would not consider the underlying logic of an axiomatization elementary, if the logic allows quantification over all predicates without restriction. Yet this does not mean that parts of second-order logic which permit quantification over definable predicates as well as their possible reductions to first-order level of reasoning are excluded:

We have to ask ourselves the question, what does it mean when we say "There is a predicate P"? In axiomatic set theory, the "there is" always refers to the domain B we take to be there at the foundation. In logic, we could think of the predicates as collected together in a domain. But this domain of predicates cannot be considered as something given from the beginning; rather it must be formed through logical operations. Only through the rules of logical construction is the predicate-domain subsequently determined.

And now it becomes obvious that, in the rules of the logical construction of predicates, reference to the domain of predicates can be permitted.[33]

Therefore, it would be a mistake to think that Hilbert's model-theoretical aims are not realizable by means of semantically incomplete logics that are strong enough to codify mathematical reasoning. The idea of purely logical axiomatization does not necessarily presuppose that the underlying logic is semantically complete. What is necessarily presupposed is a demarcation between the logical and extra-logical structures. That does not require all valid formulas to be recursively enumerated, by deriving them from a recursive set of axioms. An axiomatization can be purely logical even when the derivation of theorems from axioms is carried out by semantically valid inferences instead of formal derivations. It can even be called "formal" in that semantically valid inferences depend only on the logical form of the premises and the conclusions.

To explicate this point further we can distinguish between the formalist view of mathematics, and the formal character of logical inference. Formalist view of mathematics is the view that mathematical reasoning consists primarily of the manipulation of formal symbols. It is a separate view from the doctrine of the formal character of logical inference. Formal character of a logical inference means that the inference from a sentence to another is independent of the non-logical constants occurring in them. That is, an inference from S_1 to S_2 depends only on the logical structure of S_1 and S_2.

33 Quoted from Hilbert's 1920 lectures, in Hallett 1995, p. 165.

One way of seeing the difference between these two meanings of the term “formalist” is to imagine a framework in which philosophical formalism fails but formal character of logic is retained. Second-order logic provides such a framework. In second-order axiomatizations mathematical inferences cannot be reduced to the manipulation of formulas, such as mechanical deductions. Yet the validity of second-order inferences depends only on the logical structure of the inferences.

It can be said that most of actual mathematical reasoning can be thought of as being carried out in second-order logic.[34] And such an enterprise cannot be restricted to mechanical deduction. The reason is that there is no semantically complete axiomatization of second-order logic. Hence from the point of view of logical theory, philosophical formalism cannot yield an adequate account of mathematical reasoning. Deduction must be complemented by an additional of new principles of proof, presumably on the basis of suitable model-theoretical considerations which fits into Hilbert’s mold. The crucial point here is that deductive incompleteness does not make any difference to the formal character of relations of logical consequence. A sentence S_1 logically implies another one, say S_2, if and only if the same relation holds between any two sentences of the same logical form but with different non-logical constants. In this sense the formal character of logical reasoning is an obvious truth.

What Hilbertian formalization amounts to then is a reduction of all derivation of theorems from axioms to purely logical inferences. Such inference is formal only in the sense of being independent of the interpretation of the basic concepts of the axiom system. In this sense, all the proofs are intended to be independent of the domain of objects that is being considered. Here the fact that deduction is independent of interpretation is compatible with Hilbert’s insistence that the choice of axioms is guided

[34] For some aspects of the second-order logical foundations of mathematics, see Väänänen 2001.

by the intuitive content of the concepts involved. That is why for example Hilbert and Bernays 1934 discuss two kinds of axiomatization: formal and contentual.[35] This point makes it conclusively clear that the attribution of formalism to Hilbert's foundational ideas is missing an essential distinction between form and content in Hilbert's axiomatic approach.

The distinction between form and content in mathematics, and the fact that inferences are independent of interpretation means that all mathematical results considered in an axiomatic framework are intended to have a structural meaning. This is an obvious truth from the model-theoretical viewpoint. As a problem of mathematical logic, the topic arises in Hilbert's 1920 lectures:

…we have to interpret our signs of our calculus when representing separately symbolically the premises from which we start and when understanding the results obtained by formal operations.

The logical signs are interpreted as before according to the prescribed linguistic reading; and the occurrence of indeterminate statement-signs and function-signs in a formula is to be understood as follows: for arbitrary replacements by determinate statements and functions…the claim that results from the formula is correct.[36]

Here the leading idea is that a correct symbolism constitutes an isomorphic replication of what it represents. This is seen from the fact that Hilbert intends to obtain in the quoted passage arbitrary instantiations of the structures that are described by the logical axiomatization with the prescribed linguistic reading of the logical signs give model-theoretically correct results. In that sense the proof-theoretical analysis of mathematical inference is not enough for Hilbert's model-theoretical purposes. Correct interpretation of symbolic framework is essential:

We have analyzed the language (of the logical calculus proper) in its function as a universal instrument

[35] See Hilbert and Bernays 1934, § 1.

[36] Cf. Sieg 1999, p. 18.

of human reasoning and revealed the mechanism of argumentation. However, the kind of viewpoint we have taken is incomplete in so far as the application of the logical calculus to a particular domain of knowledge requires an axiom system as its basis. I.e. a system (or several systems) of objects must be given and between them certain relations with particular assumed basic properties are considered.[37]

When Hilbert's starting point, i.e. that number symbols themselves as objects of number theory, is combined with the idea that correct symbolism is an isomorphic replication of what it represents, models in Hilbert-style axiomatization can safely consist of any objects, including number symbols. What it brings about, as was indicated earlier, is that mathematical symbols themselves (for example number-theoretical symbols) can be used as the contentual extra-logical elements of the mathematical proofs. In that sense Hilbert's signs or symbols, as in the case of algebraic manipulations and symmetries in abstract algebra, can share the same common models up to isomorphism with their subject matter, whatever that matter might be. Meta-mathematics in that sense can be seen as the combinatorial study of certain symbol structures. The so-called nineteenth century arithmetization of analysis can be included in that, assuming that int had similar motives to meta-mathematical investigations. Of course, this combinatorial study presupposes its own determination of models, and its own model-theoretical consistency.

Such a determination of models up to isomorphism requires that the underlying logic of axiomatization is semantically complete. As was shown by Gödel 1931—since it shows the impossibility of a categorical characterization of arithmetic by using first-order axiomatization—there is no hope for determining a unique model, and also no hope for proving the consistency of arithmetic, by using the ordinary-first-order logic as the underlying logic. That is the case, although the proof-

37 Quoted from Hilbert's 1920 lectures, in Sieg 1999, p. 24

theoretical consistency of an axiom system implies its model-theoretical consistency in virtue of the completeness of first-order logic. Hilbert's aim to prove the proof-theoretical consistency of arithmetic cannot be achieved due to the deductive incompleteness of first-order arithmetic (based on the ordinary first-order logic). This impossibility calls for an investigation of the possible continuity principles underlying Hilbert's assumption that structures of symbol combinations can be used as instantiations of mathematical structures. However, without a proof of model-theoretical consistency such investigation would be a *petitio principii*. Therefore, in order to carry out the desired consistency proofs, by means of suitable alternative logical and algebraic techniques, it is more appropriate as much as it is inevitable that alternative continuity and completeness assumptions for these techniques must be introduced in tandem with those techniques.

9. *The idea of the quantifier as a choice term*

It is a characteristic feature of some of the developments in the nineteen-twenties that quantifiers were considered to be closely related to the choice functions.[38] In Skolem's work, for example, this was the case. According to Skolem, quantifiers serve no better than choice functions.[39] Like Skolem, Hilbert recognized the close interrelation between quantifiers and choice functions. In fact, he realized that the basic idea underlying the axiom of choice and quantification was one and the same. Later this basic idea is outlined in Hilbert and Bernays 1934 as that a finitistic interpretation of a universal statement is an assertion about any given object from a domain, whereas an existential statement amounts to a series of operations that have a definite bound. So, for example,

[38] Cf. Goldfarb 1979.
[39] ibid. p. 357-358.

(1) $(\forall x)(A(x) \supset (\exists y) B(x, y))$

means a series of operations, which for any given x that is A makes it possible to find a y on the basis of x that is related to x by B.[40] Later developments in logic makes it sufficiently clear that the operations Hilbert and Bernays considered are based on the idea of operating with choice functions.

In his 1961 paper, Henkin introduced the first-order partially-ordered branching quantifiers, e.g.:

(2)
$$\begin{matrix} (\forall x)(\exists y) & \\ & A(x, y, z, u) \\ (\forall z)(\exists u) & \end{matrix}$$

If we use Skolem functions, (2) is equivalent with:

(3) $(\exists f)(\exists g)(\forall x)(\forall y) A(x, f(x), z, g(y))$

If quantifiers are interpreted as choice functions in the way Hilbert also seems to have done, Henkin's quantifiers amount to expressing different dependency relations between quantified objects (compare (3) and (5)) from the linearly-ordered quantified versions— such as of (1):

40 Hilbert and Bernays 1934, pp. 32-33.

(4) $(\forall x)\ (\exists y)\ (\forall z)\ (\exists u)\ A\ (x, y, z, u)$

If we use Skolem functions (3) is equivalent with:

(5) $(\exists f)\ (\exists g)\ (\forall x)\ (\forall y)\ A\ (x, f(x), z, g(x, z))$

Here in (5) the choice of a value for u depends both on (∀x) and (∀z), whereas in (3) u depends only on (∀z). What is relevant here to Hilbert's views on quantification theory is that Henkin quantifiers unseals the connections between quantifiers and quantifier-dependence, and choice functions, when one comes to interpret their meaning. Henkin suggested in his 1961 paper to treat the alternation between quantifiers as choices dependently or independently made from a domain. Accordingly, a given formula, say (2), can be evaluated by means of a procedure of choices made by two players. In order for keeping with Hilbert's approach, one has to find the appropriate operations for the evaluation in the sense that infinitistic assumptions about quantifying "all" must be eliminated. In the general case, say for all sequence of choices $c_1, c_2, c_3, \ldots, c_n$, the existence of a function s viz. a winning strategy which is correlated to $c_1, c_2, c_3, \ldots, c_n$ in the given formula determines, albeit not necessarily in the actual cases, the winning and hence truth.

The recognition of the close interrelation between quantifiers and choice functions in fact made it more visible that the basic idea underlying the axiom of choice and quantification was one and the same. For example, Hilbert introduced his epsilon technique in order to capture the usual instatiation rules and the so-called axiom of choice. In the epsilon technique, an epsilon term $\varepsilon x A(x)$ stands for an individual x of which A(x) holds if there is such an individual. And the logical axiom $A(x) \supset A(\varepsilon(A))$ contains according to Hilbert "the core of...the axiom of choice".[41] It is missing, however, in Hilbert's axiom, what the choice in question depends on. So it does not actually cover the core of the axiom of choice.

Hilbert's aim to treat the axiom of choice and quantification in tandem has its roots in Hilbert's 1923 lectures. There he points out the close connection and his proposed solution i.e. the epsilon technique to the problems arising from quantification and choice:

We have not yet addressed the question of the applicability of these concepts ["all" and "there is"] to infinite totalities. ...The objections...are directed against the choice principle. But they should likewise be directed against "all" and "there is" which are based on the same basic idea.[42]

In line with his aim concerning the axiom of choice, Hilbert's main concern seems to have been to point out the need for a logic which is based on the same basic idea as the axiom of choice. In that sense the status of the axiom of choice is the paradigm case for Hilbertian investigations in the foundations of mathematics.

Hilbert tried to give the axiom of choice as well as to the application of quantifiers a firm footing by the "logical ε-axiom":

(6) $A(x) \supset A(\varepsilon(A))$

Hilbert put the ε-function or strictly, ε-functional to use for different purposes. His main goal was to use

41 Hilbert 1925, p. 382.
42 Quoted in Zach 2001, pp. 70-71.

it in consistency proofs. By its means he defined universal and existential quantifiers:

(7) $(\forall x)\, A(x) \leftrightarrow A(\varepsilon(\neg A))$

(8) $(\exists x)\, A(x) \leftrightarrow A(\varepsilon(A))$

On this basis Hilbert formulated universal instantiation, and the law of excluded middle as a quantifier rule:

(9) $(\forall x)\, A\, (x) \supset A\, (x)$

(10) $\neg\, (\forall x)\, A\, (x) \supset (\exists x) \neg\, A\, (x)$

The ε-function could also serve to pick witness individuals for those propositions which hold for one and only one individual. If A(x) is one such proposition, then there obtains

(11) $x = \varepsilon\, (A)$

Most notably the ε-function could purportedly take the role of a choice function. This is not completely true though; for as was pointed out, it is not indicated in the epsilon term what the choice is based on. In case A(x) holds for more than one object, ε(A) is one of those objects x of which A(x) holds. This is where Hilbert's logical ε-axiom was intended to cover the main idea behind the axiom of choice. At the same time it was also a tool for instantiation in the sense that the value of an ε-function for a predicate A is an individual for which A holds if it holds for any.[43] Here, based on Hilbert's epsilon definition of the existential quantifier (viz. (8) above) the following can be stated:

(12) $(\forall x)\,(\exists y)\,A\,(x, y) \supset (\forall x)\,A\,(x, \varepsilon\,(A\,(x, y)))$

Here (12) can be read as to capture a nominalistic formulation of the axiom of choice, since it asserts a choice from any given domain {y: A (x, y)}, where ε (A (x, y)) designates the arbitrarily chosen individual. The problem with Hilbert's epsilon calculus is that it assumes in its day that ordinary first-order logic is the basic logic. The definition of quantifiers and instantiation rules are given by Hilbert for the ordinary Hilbert-Ackermann first-order logic.[44] More specifically, it assumes that an epsilon term depends on all the outside universal quantifiers; since an epsilon term does not indicate what it formally depends on. Because of this assumption epsilon functions, although they seem to capture the intended force of the axiom of choice in meaning, they cannot serve its intended purpose as the paradigm case for developing a combinatorial interpretation of quantification theory that Hilbert seems to have aimed at.

Practically, Hilbert put his epsilon technique in use for several aims, including: formulating the axiom of choice as a logical principle, explaining applications of the quantifiers, and proving the consistency of mathematical theories. The fundamental idea of the epsilon technique for consistency proofs is to make use of epsilon functions in producing quantifier-free true formulas. Any consistency proof has to include a proof that each such quantifier-free formula is correct:

---

43 Cf. Hilbert and Bernays 1939, p. 12.

44 See Hilbert 1925 and 1928.

In proving consistency for the ε-function the point is to show that from a given proof of $0 \neq 0$ the ε-function can be eliminated, in the sense that the arrays formed by means of it can be replaced by numerals in such a way that the formulas resulting from the logical axiom of choice by substitution, "the critical formulas", go over into "true" formulas in virtue of these replacements.[45]

Given a mathematical proof formulated in the epsilon calculus, each epsilon term occurring in the proof is assigned a numerical value. The aim of this procedure is to transform all the uses of epsilon axioms as well as the axioms of AX of the theory in question into quantifier-free formulas in finitely many steps. Since epsilon-terms are used in a proof finitely many times, this must have seemed to Hilbert to be possible. However, values that are assigned to different epsilon terms depend on each other due to the nested structure of epsilon terms in some formulas. Since in the usual notation of first-order logic scopes are nested, quantifier dependence is eventually packed into epsilon dependence and it creates difficulties in assigning numerical terms for nested terms. For example, the values that we assign to the inner epsilon terms might necessitate changes in the previous assignments. Later assignments might turn the correct formulas into incorrect formulas. Thereby the nested structure of the assignment process might divide into branches and loops on the branches so that the substitution procedure might never come to an end.

In any case, due to Gödel's second incompleteness result we cannot reach true numerically correct formulas by means of the epsilon technique. For if we could, then this would give us a consistency proof for the axioms of number theory; since the theorems of number theory would then also be numerically correct. Such a consistency proof assuming that the underlying logic is the ordinary first-order logic is impossible to carry out. Therefore, Hilbert's epsilon calculus cannot serve its

45 Hilbert 1928, p. 477.

intended purpose.

One can follow a similar line of thought to Hilbert’s epsilon technique, in second-order logic too; since the job of the epsilon functions can be done by Skolem functions as well. One can start with the general observation that for each choice value x a natural number y can be found such that y is correlated in some way to x:

(13) $(\forall x)\,(\exists y)\,A\,(x, y)$

This can be taken here as to imply the existence of a function f such that for every x, f produces a term out of x. Thereby one can obtain a second-order form of the axiom of choice:

(14) $(\forall x)\,(\exists y)\,A\,(x, y) \supset (\exists f)\,(\forall x)\,A\,(x, f(x))$

which is a second-order logically valid formula. In fact, it is also the same general formulation of the axiom of choice as in Hilbert's and Bernays' work.[46] The same line of thought can be even traced back to operating only with first-order quantifiers. If we allow functional instantiation in (13) and write:

(15) $(\forall x)\,A\,(x, f(x))$

The step from (13) to (15) is enough in principle to capture Hilbert’s main idea in putting epsilon

[46] Hilbert and Bernays 1934, p. 41.

functions in use. Just like Hilbert's epsilon function, any arbitrary function-name can be considered as to pick ideally an individual from a given domain.

In independence-friendly logic and hence on the first-order level it can be shown that such principle is in fact logically true. In the second-order formulation of a general choice principle such as (14) the existentially quantified function f can be cashed in by independent choices of individuals. That is, the second-order sentence $(\exists f)\ (\forall x)\ A\ (x, f(x))$ is translated to

(16) $(\forall x_1)\ (\forall x_2)\ (\exists y_1/\forall x_2)\ (\exists y_2/\forall x_1)\ (((x_1 = x_2) \supset (y_1 = y_2))\ \&\ A\ [x_1, y_1]\ \&\ A\ [x_2, y_2])$

Here if we use Skolem functions (16) is equivalent with:

(17) $(\exists f_1)\ (\exists f_2)\ (\forall x_1)\ (\forall x_2)\ (((x_1 = x_2) \supset f_1(x_1) = f_2(x_2)) \supset A\ (x_1, f_1(x_1))\ \&\ A\ (x_2, f_2(x_2)))$

Thereby the choices which are expressed by Skolem functions in (17) are reduced to suitable operations by means of quantifiers and their dependency relations. Then we have the following independence-friendly formulation of the axiom of choice:

(18) $(\forall x)\ (\exists y)\ A(x,y) \supset (\forall x_1)(\forall x_2)(\exists y_1/\forall x_2)(\exists y_2/\forall x_1)(((x_1{=}x_2) \supset (y_1{=}y_2))\ \&\ A[x_1,y_1]\ \&\ A[x_2,y_2])$

What has been achieved in (18) is the conclusion that the way quantifiers operate on the first-order level provides a suitable framework, as Hilbert seems to have thought, to place the reasoning behind the axiom of choice in its appropriate logical context. Thereof the apparently second-order reasoning behind the axiom of choice is translated to combinatorial first-order reasoning.

To that extent Hilbert's aim to show that Zermelo's axiom of choice is as a logical truth can be thus achieved, although by a technique different from his. In fact, this result is the paradigm case for independence-friendly logic as much as it seems to have been for Hilbert's intended theory of quantification which is beyond the reaches of the epsilon technique in its original form. The reason for that seems to be the existence of Skolem functions which depends on the status of the axiom of choice itself. Nevertheless, the argument that has been carried out shows that quantification and the axiom of choice are really based on the same basic idea, as Hilbert aimed to have obtained. A further task to be carried out thus is to have a look at the behavior of Hilbert's epsilon symbol in the presence of independence-friendly quantifiers.

REFERENCES

Bernays, Paul, 1967, "David Hilbert", in Paul Edwards, editor, *The Encyclopedia of Philosophy*, New York, pp. 496-505.

Brouwer, Luitzen, E. J., (1948) "Consciousness, philosophy and mathematics", in Paul Benacerraf and Hilary Putnam, editors, *Philosophy of Mathematics*, Cambridge University Press, 2nd edition, pp. 90-96.

Caicedo, Xavier, 1991, "Hilbert's eps'lon symbol in the presence of generalized quantifiers", *Bulletin of the Section of Logic*, Vol. 20: 3-4, 85-86.

Corry, Leo, 1997, Corry, Leo, 1997, "David Hilbert and the axiomatization of physics (1894-1905)", *Archive for the History of Exact Sciences*., vol. 51, pp. 83-198.

Goldfarb, Warren, 1979, “Logic in the Twenties: The Nature of the Quantifier”, *Journal of Symbolic Logic*, vol. 44, pp. 351-368.

Gödel, Kurt, 1931, "On formally undecidable propositions of Principia Mathematica and other related systems", in Jean van Heijenoort, editor, *From Frege to Gödel*, Cambridge, MA, Harvard University Press, 1967, pp. 596-616.

Hallett, Michael, 1995, Hallett, Michael, 1995, “Hilbert and Logic”, in M. Marion and R. S. Cohen, editors, *Quebec Studies in the Philosophy of Science I*, Kluwer Academic Publlishers, pp. 135-187.

Henkin, Leon, 1961, “Some remarks on the infinitely long formulas, in P. Lorentzen, editor, *Infinitistic Methods, Proc. Of a Symposium on Foundations of Mathematics*, Warsaw, Pergamon Press, London, pp. 167-183.

Hilbert, David, 1889, *Foundations of Geometry*, translated by L. Unger, Opencourt, LaSalle, 1971.

Hilbert, David, 1900, “Mathematical Problems”, *Bulletin of the American Mathematical Society*, vol. 8, pp. 437-479.

Hilbert, David, 1918, "Axiomatic thought", in W. B. Ewald, editor, *From Kant to Hilbert (vol. 2)*, Oxford: Oxford University Press, 1996, pp. 1105-1115

Hilbert, David, 1926, "On the infinite", in Jean van Heijenoort, editor, *From Frege to Gödel*, Cambridge, MA; Harvard University Press, 1967, pp, 367-392.

Hilbert, David, 1928, "The foundations of mathematics", in Jean van Heijenoort, editor, *From Frege to Gödel*, Cambridge, MA: Harvard University Press, 1967, pp. 464-479.

Hilbert, David, 1931, "The grounding of elementary number theory", in W. B. Ewald, editor, *From Kant to Hilbert (volume 2)*, Oxford: Oxford University Press, 1996, pp. 1148- 1157.

Hilbert, David, and Paul Bernays, 1934, *Grundlagen der Mathematik (volume 1)*, 2nd edition, Berlin: Springer.

Hilbert, David and Paul Bernays, 1939, *Grundlagen der Mathematik (volume 2)*, 2nd edition, Berlin: Springer.

Hintikka, Jaakko, 2004, "Independence-friendly logic and axiomatic set theory", *Annals of Pure and Applied Logic*, vol. 126, pp. 313-333.

Husserl, Edmund, 1900, *Logical Investigations I*, English translation by J. N. Findlay, Humanities Press, New York, 1970.

Husserl, Edmund, 1913, *Ideas: General Introduction to Pure Phenomenology*, translated by Boyce Gibson, Humanities Press, New York.

Husserl, Edmund, 1970, *Philosophy of Arithmetic: Psychological and Logical Investigations* – with supplementary texts from 1887 – 1901 (Edmund Husserl Collected Works), translated by Dallas Willard, Kluwer Academic Publishers.

Kitcher, Philip, 1976, "Hilbert's epistemology", *Philosophy of Science*, vol 43, pp. 99-115.

Kluge, E-H.W., editor, 1971, *Gottlob Frege on the Foundations of Geometry and Formal Theories of Arithmetic*, Yale University Press, New Haven.

Kreisel, Georg, 1976, "What have we learned from Hilbert's second problem", in Felix E. Brouder, editor, *Mathematical Developments Arising from Hilbert's Problems* (Proceedings of Symposia in Pure Mathematics, vol. 28, Part I), American Mathematical Society, Providence, Rhode

Island, pp. 93-131.

Kreisel, Georg, 1983, “Hilbert’s programme”, in Paul Benacerraf and Hilary Putnam, editors, *Philosophy of Mathematics*, Cambridge University Press, 1983, 2nd edition, pp. 207-238.

Leisenring, A. C., 1969, *Mathematical Logic and Hilbert’s ε-symbol*, Macdonald & Co. (Publishers) Ltd., London.

Majer, Ulrich, 1997, “Husserl and Hilbert on completeness: a neglected chapter in early twentieth century foundations of mathematics”, *Synthese*, vol. 110, no.1, pp.37- 56.

Majer, Ulrich, 2001, “The axiomatic method and the foundations of science: historical roots of mathematical physics in Göttingen (1900-1930), *Vienna Circle Institute Yearbook*, vol. 9, pp. 213-224.

Parsons, Charles, 1998, "Finitism and intuitive knowledge", M. Schirn, editor, *The Philosophy of Mathematics Today*, Oxford: Oxford University Press, 1998, pp. 249-270.

Peckhaus, Volker, 1990, *Hilbertprogramm und Kritische Philosophie*, Vandenhoeck und Ruprecht, Göttingen, 1990.

Sieg, Wilfrid, 1999, “Hilbert programs 1917-1922”, *Bulletin of Symbolic Logic*, vol 5, pp. 1- 44.

Toretti, Roberto, 1978, *Philosophy of Mathematics from Riemann to Poincaré*, D. Reidel, Dordrecht.

Väänänen, Jouko, 2001, "Second-order logic and foundations of rnathematics", *Bulletin of Symbolic Logic*, vol. 7, pp. 504-520.

Zach, Richard, 2001, *Hilbert’s Finitism*, Dissertation, University of California, Berkeley.

Zach, Richard, 2006, “Hilbert’s program then and now”, in Dale Jacquette, editor, *Philosophy of Logic*, Elsevier, pp. 411-447.