Explicit filtering equations for labelled random finite sets

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***Abstract*—We decompose a probability density function (PDF) of a labelled random finite set (RFS) into a probability mass function over a set of labels and a PDF on a vector-valued multitarget state given the labels. Using this decomposition, we write the Bayesian filtering recursion for labelled RFSs in an explicit form. The resulting formulas are of conceptual and practical interest in the RFS approach to multiple target tracking, especially, for track-before-detect particle filter implementations.**

***Keywords*—labelled random finite sets, multitarget tracking, set integral, conditional PDF.**

I. Introduction

In a general dynamic multiple object system, objects move, appear and disappear. It is of interest in many applications, such as multitarget tracking, to infer the current state of this dynamic system based on a sequence of measurements over time. Classic approaches to multiple target tracking are multiple hypothesis tracking (MHT) [1]–[3] and joint probabilistic data association (JPDA) [4]. The random finite set (RFS) formulation enables us to model the system using the Bayesian approach [5]–[7]. Here, the multiobject state is a set whose elements are single object states and the posterior probability density function (PDF) contains all the information regarding the number of targets and their states at the current time step. Based on this PDF or an approximation, we can estimate the current multiobject state by an estimator, e.g., the minimum mean optimal subpattern assignment (MMOSPA) estimator [8], [9].

In the RFS formulation, the prediction and update steps of the Bayesian filtering recursion are analogous to the vector-based case. That is, the prediction step is given by the Chapman-Kolmogorov equation but replacing the vector integral by the set integral and the update step consists of applying Bayes' rule [10]. Therefore, the (unnormalised) posterior is a product of the prior PDF times the likelihood. The prediction step with RFSs is more complicated to perform than in the vector case as it accounts for changes in the target number as well as all possible permutations in the multitarget state.

Labels can be attached to single target states in the RFS formulation [11], [12]. Labels are unique and do not change with time. At each time step, the current labelled set of targets is estimated based on the filtering posterior and, due to the previously mentioned properties of the labels, tracks are built by linking target state estimates with the same label. Importantly, including them in the target state simplifies the set integrals of the prediction step. Basically, the prediction step amounts to weighted vector-based prediction steps. This was shown in [12] for two types of posterior approximations and the radar point detection measurement model.

The contribution of this paper is two-fold. First, we show that a labelled RFS density can be decomposed into a probability mass function (PMF) over the set of labels and a PDF of a vector-valued multitarget state given the labels. This procedure resembles the decomposition of a joint PDF over two vector-valued variables into a marginal PDF and a conditional PDF [13]. In addition, as we will explain, this idea is in fact more general and can also be applied to unlabelled RFS densities. Second, we apply this decomposition to the RFS Bayesian filtering recursion to obtain formulas that are readily implementable and are especially convenient for particle filters (PF) [14] or Markov chain Monte Carlo (MCMC) [15] in track-before-detect [16]–[19]. In this paper, we use the word explicit to mean this readily implementable recursion. For example, an equivalent way to evaluate the posterior PDF is used in the multitarget PF in [20]–[22]. This paper therefore connects the PF developed in [20]–[22], which was not derived explicitly using RFS, to the RFS framework in a clear fashion. This is of theoretical importance due to the mathematical rigor of the RFS framework.

This paper is organised as follows. The general Bayesian filtering recursion for labelled RFS is reviewed in Section II. The proposed decomposition of a labelled RFS PDF and the resulting explicit Bayesian filtering recursion are given in Section III. Section IV provides the proof of the explicit recursion. Finally, conclusions are drawn in Section V.

II. General Bayesian Filtering Recursion

In this paper, labelled RFS densities are denoted as $\boldsymbol{\pi}(\cdot)$ and densities over a vector space as $\pi(\cdot)$, which are referred to as vector densities. A brief introduction to the RFS framework with unlabelled and labelled sets can be found in Sections II and III in [12]. As we only need to consider one prediction and update step, we omit the time index of the filtering recursion for notational simplicity.

At the current step, the collection of target states is given by the labelled set $\mathbf{X} = \{(x_1, l_1), ..., (x_t, l_t)\}$, where $x_j \in \mathbb{R}^{n_x}$ and labels $l_j \in \mathbb{L}$ for $j \in \{1, ..., t\}$ with t being the number of targets. At the previous time step, the labelled set is $\mathbf{X}' = \{(x'_1, l'_1), ..., (x'_{t'}, l'_{t'})\}$, where single target states $x'_j \in \mathbb{R}^{n_x}$ and labels $l'_j \in \mathbb{L}'$ for $j \in \{1, ..., t'\}$ with t' being the number of targets. Therefore, variables $\mathbf{X}'$ and $\mathbf{X}$ belong to the collection of finite subsets of $\mathbb{R}^{n_x} \times \mathbb{L}'$ and $\mathbb{R}^{n_x} \times \mathbb{L}$, respectively. In a labelled set, labels are unique, i.e., no two targets can have the same label, and they do not change with time. In addition, to ensure unique labelling $\mathbb{L} = \mathbb{L}' \cup \mathbb{B}$ where $\mathbb{B}$ is the space of labels of the new born targets and $\mathbb{L}' \cap \mathbb{B} = \emptyset$.

There have been two proposals in the literature to define labels. Labels in [12] are two-dimensional vectors that contain the time step when the target is born and a natural number. Labels in [21] are natural numbers. The latter requires the assumption that the number of new born targets at each time step is bounded, which always happens in practice, so that we can ensure that $\mathbb{L} = \mathbb{L}' \cup \mathbb{B}$ [21]. An important property that we will use in this paper for both cases is that given a set $L \subseteq \mathbb{L}$ of labels, we can arrange it in ascending order to form vector $\overrightarrow{L}$, either using the inherent order of natural numbers or lexicographical order for vectors [23]. The remainder of the paper is valid for both labelling approaches taking into account that symbol $<$ refers to lexicographical order if labels are vectors. In addition, it is met that if $l_1 \in \mathbb{L}'$ and $l_2 \in \mathbb{B}$, then $l_1 < l_2$.

The multiobject transition density $\boldsymbol{g}(\cdot |\mathbf{X}')$ encapsulates the underlying models of target dynamics, births and deaths. At each time step, targets are observed through noisy measurements. We denote the resulting multitarget likelihood, which is not a density, as $\tilde{\ell}(\cdot)$ and it does not depend on the labels so we define the unlabelled likelihood $\ell(\cdot)$ as

$$\tilde{\ell}(\{(x_1, l_1), ..., (x_t, l_t)\}) = \ell(\{x_1, ..., x_t\}). \tag{1}$$

Given the posterior PDF $\boldsymbol{\pi}'(\cdot)$, the objective is to compute the posterior PDF $\boldsymbol{\pi}(\cdot)$ at the next time step, which integrates to one using the set integral [12]

$$\begin{aligned}
&\int \boldsymbol{\pi}(\mathbf{X})\, \delta\mathbf{X} \\
&= \sum_{t=0}^{\infty} \frac{1}{t!} \sum_{l_{1:t} \in \mathbb{L}^t} \int \boldsymbol{\pi}(\{(x_1, l_1), ..., (x_t, l_t)\})\, dx_{1:t} \\
&= 1
\end{aligned} \tag{2}$$

where $l_{1:t} = (l_1, ..., l_t)$, $x_{1:t} = (x_1, ..., x_t)$ and

$$\mathbb{L}^t = \underbrace{\mathbb{L} \times ... \times \mathbb{L}}_{t}$$

denotes t Cartesian products over $\mathbb{L}$. Using the prediction and update equations of the Bayesian filtering recursion, we get [5]

$$\boldsymbol{\omega}(\mathbf{X}) = \int \boldsymbol{g}(\mathbf{X} |\mathbf{X}')\, \boldsymbol{\pi}'(\mathbf{X}')\, \delta\mathbf{X}' \tag{3}$$

$$\boldsymbol{\pi}(\mathbf{X}) = \tilde{\ell}(\mathbf{X})\, \boldsymbol{\omega}(\mathbf{X}) / \rho \tag{4}$$

where ρ is the normalising constant

$$\rho = \int \tilde{\ell}(\mathbf{X})\, \boldsymbol{\omega}(\mathbf{X})\, \delta\mathbf{X}. \tag{5}$$

The prior $\boldsymbol{\omega}(\cdot)$ can be written more explicitly by evaluating the set integral [12]

$$\begin{aligned}
\boldsymbol{\omega}(\mathbf{X}) = \sum_{t'=0}^{\infty} \frac{1}{t'!} \sum_{l'_{1:t'} \in (\mathbb{L}')^{t'}} \int \boldsymbol{g}(\mathbf{X} |\{(x'_1, l'_1), ..., (x'_{t'}, l'_{t'})\}) \\
\times \boldsymbol{\pi}'(\{(x'_1, l'_1), ..., (x'_{t'}, l'_{t'})\})\, dx'_{1:t'}.
\end{aligned} \tag{6}$$

We make the usual assumptions: target dynamics are independent with a probability γ of survival and transition density $g(\cdot |\cdot)$, new born targets are born independently of the rest with a PDF $\boldsymbol{\eta}(\cdot)$ and the likelihood does not depend on the labels. Under these assumptions, the objective of this paper is to write (3)-(4) in explicit way suitable for computer implementation. First, we elaborate on (6).

We can write $\mathbf{X} = \mathbf{S} \cup \mathbf{B}$ where $\mathbf{S}$ denotes the labelled set of surviving targets and $\mathbf{B}$ the labelled set of new born targets. As both sets are independent, we can apply the convolution formula to obtain the PDF of their union [5, page 385]:

$$\boldsymbol{\omega}(\mathbf{X}) = \sum_{\mathbf{B} \subseteq \mathbf{X}} \boldsymbol{\eta}(\mathbf{B})\, \boldsymbol{\xi}(\mathbf{X} \setminus \mathbf{B}) \tag{7}$$

where $\setminus$ denotes set subtraction and $\boldsymbol{\xi}(\cdot)$ is the density of $\mathbf{S}$. We can obtain $\boldsymbol{\xi}(\cdot)$ by

$$\boldsymbol{\xi}(\mathbf{S}) = \int \boldsymbol{f}(\mathbf{S} |\mathbf{X}')\, \boldsymbol{\pi}'(\mathbf{X}')\, \delta\mathbf{X}' \tag{8}$$

where $\boldsymbol{f}(\cdot|\cdot)$ is the transition density without accounting for target births and is given by [12, Eq. (25)]

$$\begin{aligned}
&\boldsymbol{f}(\{(x_1, l_1), ..., (x_t, l_t)\} |\{(x'_1, l'_1), ..., (x'_{t'}, l'_{t'})\}) \\
&= \prod_{j'=1}^{t'} \sum_{j=1}^{t} \left[\gamma g(x_j |x'_{j'})\, \delta\left[l_j - l'_{j'}\right]\right. \\
&\quad \left. + (1 - \gamma)\left(1 - \chi_{\{l_1, ..., l_t\}}\left(l'_{j'}\right)\right)\right]
\end{aligned} \tag{9}$$

if $\{l_1, ..., l_t\} \subseteq \{l'_1, ..., l'_{t'}\}$ and there are no repeated elements in $(l_1, ..., l_t)$ and $(l'_1, ..., l'_{t'})$, otherwise $\boldsymbol{f}(\cdot|\cdot) = 0$. In addition, $\delta[\cdot]$ is a Kronecker delta and $\chi_L(\cdot)$ is the indicator function over the set L.

In [12], [24], analytical expressions for $\boldsymbol{\pi}(\cdot)$ are obtained for the radar point detection measurement model, which has a specific type of $\ell(\cdot)$, and assuming that the prior is either generalised labelled multi-Bernoulli (GLMB) or δ-GLMB. In this paper, we provide an explicit form of the general recursion (3)-(4) which provides more insight into the meaning of (3)-(4) and is useful for track-before-detect PF implementations, see Section III-A.

III. Explicit Bayesian Filtering Recursion

The explicit Bayesian filtering equations are based on the following decomposition. Given a labelled RFS density $\boldsymbol{\pi}(\cdot)$ and t different labels $l_1, ..., l_t$, we define the (vector) PDF

$$\pi(x_{1:t}; l_{1:t}) \triangleq \frac{\boldsymbol{\pi}(\{(x_1, l_1), ..., (x_t, l_t)\})}{P_{\pi}(\{l_1, ..., l_t\})} \tag{10}$$

where

$$P_{\pi}(\{l_1,...,l_t\}) = \int \boldsymbol{\pi}(\{(x_1,l_1),...,(x_t,l_t)\})\,dx_{1:t} \quad (11)$$

is the probability of having a labelled set with labels $\{l_1,...,l_t\}$ and $x_{n:m} = (x_n, x_{n+1}, ..., x_m)$. Note that, given distinct labels, by definition, the states and labels in $\pi(\cdot)$ are ordered and $\pi(\cdot; l_{1:t})$ is a vector density on the target states $x_{1:t}$ such that

$$\int \pi(x_{1:t}; l_{1:t})\,dx_{1:t} = 1.$$

In addition,

$$\pi(x_{1:t}; l_{1:t}) = \pi(x_{\sigma_1}, ...x_{\sigma_t}; l_{\sigma_1}, ..., l_{\sigma_t}),\ \forall\sigma \in \Xi_t \quad (12)$$

where $\sigma = (\sigma_1, ..., \sigma_t)$ and Ξ_t contains all permutations of vector $(1, ..., t)$. As proved in Appendix A, the PMF over the labels meets

$$\sum_{L\subseteq\mathbb{L}} P_{\pi}(L) = 1. \quad (13)$$

We also want to highlight that decomposition (10)-(11) can also be applied to more general RFS densities, not only to labelled RFS densities. In order to do so, we need that the single target state contains at least two variables, e.g., position and velocity or unlabelled target state and label. The other condition is that there is probability zero for repeated elements over one variable. Then, we obtain an RFS PDF over this variable (or PMF if it is discrete) and a (vector) PDF over the other variable as in (10).

Given a set $L = \{l_1, ..., l_t\}$ of labels, we only need to specify $\pi(\cdot)$ for one ordering of the labels as the rest of the PDFs can be obtained from it using (12). Without loss of generality, the vector containing the elements of L arranged in ascending order is denoted by $\overrightarrow{L}$ as indicated in Section II. The PMF $P_{\pi}(\cdot)$ and vector densities $\pi\left(\cdot; \overrightarrow{L}\right)$ for $L \subseteq \mathbb{L}$ characterise the labelled RFS density $\boldsymbol{\pi}(\cdot)$ as both representations contain the same information. Now, we proceed to write the Bayesian filtering equations in terms of this decomposition. We recall that $\boldsymbol{\xi}(\cdot)$ and $\boldsymbol{\eta}(\cdot)$ represent the density of the survival targets and new born targets, respectively, and their decompositions are $P_{\xi}(\cdot)$, $\xi(\cdot;\cdot)$ and $P_{\eta}(\cdot)$, $\eta(\cdot;\cdot)$.

Theorem 1 (Prediction)**.** *The prior $\boldsymbol{\omega}(\cdot)$ is characterised by*

$$P_{\omega}(L) = P_{\xi}(S)\,P_{\eta}(N) \quad (14)$$

$$\omega\left(x_{1:|L|}; \overrightarrow{L}\right) = \xi\left(x_{1:|S|}; \overrightarrow{S}\right)\eta\left(x_{|S|+1:|L|}; \overrightarrow{N}\right) \quad (15)$$

where $L = S \cup N$ with $S \subseteq \mathbb{L}'$ and $N \subseteq \mathbb{B}$ and

$$P_{\xi}(S) = \sum_{L'\supseteq S} p(S|L')\,P_{\pi'}(L') \quad (16)$$

$$\xi\left(x_{1:|S|}; \overrightarrow{S}\right) = \frac{1}{P_{\xi}(S)}\sum_{L'\supseteq S} p(S|L')\,P_{\pi'}(L') \times \int \prod_{j=1}^{|S|} g\left(x_j \left| x'_j\right.\right)\pi'_{\overrightarrow{S}}\left(x'_{1:|S|}; \overrightarrow{L'}\right)dx'_{1:|S|} \quad (17)$$

where

$$p(S|L') = \begin{cases} \gamma^{|S|}(1-\gamma)^{|L'|-|S|} & S \subseteq L' \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

and $\pi'_{\overrightarrow{S}}\left(\cdot; \overrightarrow{L'}\right)$ denotes the marginal PDF of targets with labels $\overrightarrow{S}$ on the density $\pi'\left(\cdot; \overrightarrow{L'}\right)$:

$$\pi'_{\overrightarrow{S}}\left(x'_{1:|S|}; \overrightarrow{L'}\right) = \int \pi'\left(x'_{1:|S|}, y_{1:|L'\setminus S|}; \overrightarrow{S}, \overrightarrow{L'\setminus S}\right)dy_{1:|L'\setminus S|}. \quad (19)$$

Theorem 1 is proved in Section IV. By using the decomposition (10) and Theorem 1, we can evaluate $\boldsymbol{\omega}(\cdot)$, which is given by the set integral (6), as the product of two PMFs over the labels and two vector PDFs. The PMF and PDF that correspond to the new born targets are given by the model and the ones that correspond to the surviving targets are given by (16) and (17). In the PMF of the labels of the surviving targets, which is given by (16), we go through all the possible labels L' that can result in S taking into account its transition probability $p(S|L')$. For the PDF of the surviving targets, which is given by (17), we go through all the possible labels L' that can result in S and apply a usual vector-based prediction step using the single target dynamic model, which is given by $g(\cdot|\cdot)$, on the marginal posterior PDF at the previous time step of the targets that survive, which is represented by $\pi'_{\overrightarrow{S}}\left(\cdot; \overrightarrow{L'}\right)$. The density $\pi'_{\overrightarrow{S}}\left(\cdot; \overrightarrow{L'}\right)$ corresponds to $\pi'\left(\cdot; \overrightarrow{L'}\right)$ but integrating out the states of the targets that do not survive, whose label set is $L' \setminus S$. The prior of the surviving targets is a weighted mixture of the predicted PDFs.

Theorem 2 (Update)**.** *The posterior $\boldsymbol{\pi}(\cdot)$ is characterised by*

$$P_{\pi}(L) = P_{\omega}(L)\,\rho_L/\rho \quad (20)$$

$$\pi\left(x_{1:|L|}; \overrightarrow{L}\right) = \ell\left(\{x_1, ..., x_{|L|}\}\right)\omega\left(x_{1:|L|}; \overrightarrow{L}\right)/\rho_L \quad (21)$$

$\forall L \subseteq \mathbb{L}$ where

$$\rho = \sum_{L\subseteq\mathbb{L}} P_{\omega}(L)\,\rho_L$$

$$\rho_L = \int \ell\left(\{x_1, ..., x_{|L|}\}\right)\omega\left(x_{1:|L|}; \overrightarrow{L}\right)dx_{1:|L|}.$$

Theorem 2 is proved in Section IV. Given L, the update for the PDF, which is given by (21), is a usual vector-based update. Equation (20) indicates the probability of having label set L after processing the current measurement.

A. Practical Importance

Here, we explain why it can be convenient to write the labelled RFS filtering recursion using Theorems 1 and 2 in track-before-detect PF or MCMC implementations. In a computer program, at a given time step, we usually have a maximum number M of targets and N particles, which are usually stored in two matrices. The first matrix A of dimensions $M \times N$ is used to indicate the labels of the targets that represent the particles. That is, $A_{i,j}=1$ if the target with the ith label in ascending order in the jth particle exists and $A_{i,j}=0$ otherwise. If it exists, its state is stored in components $((i-1)n_x+1,j)$ to $(i\cdot n_x,j)$ of matrix B, whose dimensions are $n_x\cdot M\times N$. In practice, we just need that the multitarget state is ordered according to any order of labels in all particles. For example, if a target dies, we can allocate the freed memory to a new target as long as we keep the association for every particle.

In a PF or MCMC algorithm, we draw N particles $\left\{\mathbf{X}^1,...,\mathbf{X}^N\right\}$ from a density and we evaluate $\boldsymbol{\pi}\left(\mathbf{X}^i\right)$ up to a proportionality constant. Due to the fact that in a computer we have an ordered list of labels and the multitarget state particles ordered according to the labels, the particles can be represented by

$$\left\{\left(\overrightarrow{L}^1,x^1_{1:|L^1|}\right),...,\left(\overrightarrow{L}^N,x^N_{1:|L^N|}\right)\right\}$$

where $\overrightarrow{L^i}$ is the vector of labels of $\mathbf{X}^i$ arranged in ascending order and $x^i_{1:|L^i|}$ their corresponding states. Using (10)-(11), we can evaluate the posterior as

$$\boldsymbol{\pi}\left(\mathbf{X}^i\right)=P_\pi\left(L^i\right)\pi\left(x^i_{1:|L^i|};\overrightarrow{L^i}\right).$$

As a result, we can easily evaluate the posterior for the particles by using Theorems 1 and 2. For example, the PF in [20]–[22] was designed based on a filtering recursion that includes PMFs over the labels and multitarget vector densities given the labels. In particular, we can find analogous PMFs and PDFs in [21, Sec. II.A] to the ones in Theorems 1 and 2, e.g., $p\left(\mathbf{X}^k,\,I^k\left|\mathbf{z}^{1:k}\right.\right)$, $p\left(N^{k+1}\right)$, $p\left(\mathbf{X}^{k+1}\left(N^{k+1}\right)\left|N^{k+1}\right.\right)$ in [21] correspond to $P_{\pi'}\left(\cdot\right)\pi'\left(\cdot;\overrightarrow{L}\right)$, $P_\eta\left(\cdot\right)$ and $\eta\left(\cdot;\overrightarrow{N}\right)$, respectively.

We also want to mention that, as indicated by the δ-GLMB filter, if we use the radar point detection model, the posterior can be written explicitly as a mixture of products of single target densities [12]. That is, multitarget densities given the labels as in (10) can be simplified so it is more convenient to use the δ-GLMB filter equations directly in this case.

IV. Proof of the Result

In this section, we prove Theorems 1 and 2. To do this, in each subsection of this section, we prove (14)-(15), (16)-(17) and (20)-(21), respectively.

A. Prior PDF with New Born Targets

In this section we prove (14)-(15). We evaluate (7) for

$$\mathbf{X}=\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right),\left(x_{1+r},n_1\right),...,\left(x_{r+q},n_q\right)\right\}$$

where $N=\{n_1,...,n_q\}\subseteq\mathbb{B}$ and $S=\{s_1,...,s_r\}\subseteq\mathbb{L}'$ and $s_1<...<s_r<n_1<...<n_q$. As these two sets are disjoint, a condition which was necessary to ensure unique labelling, and $\boldsymbol{\eta}\left(\cdot\right)$ is equal to zero unless all the labels belong to $\mathbb{B}$, there is only one term in (7) which is different from zero. Therefore, we get

$$\begin{aligned}&\boldsymbol{\omega}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right),\left(x_{1+r},n_1\right),...,\left(x_{r+q},n_q\right)\right\}\right)\\&\quad=\boldsymbol{\xi}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\right)\\&\qquad\times\boldsymbol{\eta}\left(\left\{\left(x_{1+r},n_1\right),...,\left(x_{r+q},n_q\right)\right\}\right).\end{aligned}\tag{22}$$

Using (10) and the fact that we are evaluating the density for $s_1<...<s_r<n_1<...<n_q$, we obtain

$$\begin{aligned}&\boldsymbol{\omega}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right),\left(x_{1+r},n_1\right),...,\left(x_{r+q},n_q\right)\right\}\right)\\&\quad=\omega\left(x_{1:|N|+|S|};s_{1:r},n_{1:q}\right)P_\omega\left(\{s_1,...,s_r\}\cup\{n_1,...,n_q\}\right)\\&\quad=\omega\left(x_{1:|N|+|S|};\overrightarrow{S\cup N}\right)P_\omega\left(S\cup N\right).\end{aligned}$$

Performing the same operation for $\boldsymbol{\xi}\left(\cdot\right)$ and $\boldsymbol{\eta}\left(\cdot\right)$ in (22), we finish the proof

$$\begin{aligned}&\omega\left(x_{1:|N|+|S|};\overrightarrow{S\cup N}\right)P_\omega\left(S\cup N\right)\\&\quad=\xi\left(x_{1:|S|};\overrightarrow{S}\right)P_\xi\left(S\right)\eta\left(x_{1+|S|:|N|+|S|};\overrightarrow{N}\right)P_\eta\left(N\right).\end{aligned}$$

B. Prior PDF of Survival Targets

In this section, we prove (16)-(17). We evaluate $\boldsymbol{\xi}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\right)$ in (8) for $s_1<...<s_r$ to obtain

$$\begin{aligned}&\boldsymbol{\xi}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\right)\\&=\sum_{t'=r}^{\infty}\frac{1}{t'!}\sum_{l'_{1:t'}}\int\boldsymbol{\pi}'\left(\left\{\left(x'_1,l'_1\right),...,\left(x'_{t'},l'_{t'}\right)\right\}\right)\\&\times\boldsymbol{f}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\left|\left\{\left(x'_1,l'_1\right),...,\left(x'_{t'},l'_{t'}\right)\right\}\right.\right)dx'_{1:t'}.\end{aligned}\tag{23}$$

Due to (9), the integral in (23) is different from zero only if $\{s_1,...,,s_r\}\subseteq\{l'_1,...,,l'_{t'}\}$. In addition, the integral is permutation invariant w.r.t. any combination of labels $l'_1...l'_{t'}$. There are

$$r!\begin{pmatrix}t'\\r\end{pmatrix}$$

possible (ordered) ways of arranging the labels $\{s_1,...,,s_r\}\subseteq\{l'_1,...,,l'_{t'}\}$. Therefore, as the integral is permutation invariant, we can select one ordering and multiply by this quantity. We get (24) on top of the next page.

Using (9) in (24), we get

$$\begin{aligned}&\boldsymbol{\xi}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\right)\\&\quad=\sum_{t'=r}^{\infty}\frac{1}{(t'-r)!}\sum_{l'_{r+1:t'}}\gamma^r\left(1-\gamma\right)^{t'-r}\int\prod_{j=1}^{r}g\left(x_j\left|x'_j\right.\right)\\&\qquad\times\boldsymbol{\pi}'\left(\left\{\left(x'_1,s_1\right),...,\left(x'_t,s_r\right),\right.\right.\\&\qquad\quad\left.\left.\left(x'_{r+1},l'_{r+1}\right),...,\left(x'_{t'},l'_{t'}\right)\right\}\right)dx'_{1:t'}.\end{aligned}\tag{25}$$

Applying decomposition (10)-(11), we obtain

$$\begin{aligned}&\boldsymbol{\xi}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\right)\\&\quad=\sum_{t'=r}^{\infty}\frac{1}{(t'-r)!}\sum_{l'_{r+1:t'}}\gamma^r\left(1-\gamma\right)^{t'-r}\int\prod_{j=1}^{r}g\left(x_j\left|x'_j\right.\right)\\&\qquad\times P_{\pi'}\left(\left\{s_1,...,s_r,l'_{r+1},...,l'_{t'}\right\}\right)\\&\qquad\times\pi'\left(x'_{1:t'};s_{1:r},l'_{r+1:t'}\right)dx'_{1:t'}.\end{aligned}\tag{26}$$

$$\boldsymbol{\xi}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\right)=\sum_{t'=r}^{\infty}\frac{r!}{t'!}\left(\begin{array}{c}t'\\r\end{array}\right)\sum_{l'_{t+1:t'}}\int\boldsymbol{\pi}'\left(\left\{\left(x'_1,s_1\right),...,\left(x'_r,s_r\right),\left(x'_{r+1},l'_{r+1}\right),...,\left(x'_{t'},l'_{t'}\right)\right\}\right)$$
$$\times\boldsymbol{f}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\left|\left\{\left(x'_1,s_1\right),...,\left(x'_r,s_r\right),\left(x'_{r+1},l'_{r+1}\right),...,\left(x'_{t'},l'_{t'}\right)\right\}\right.\right)dx'_{1:t'}. \quad (24)$$

The integral in (26) is permutation invariant w.r.t. $l'_{r+1},...,l'_{t'}$, therefore, we can just sum over one order $l'_{r+1}<...<l'_{t'}$ and multiply by the number $(t'-r)!$ of possible combinations.

$$\begin{aligned}&\boldsymbol{\xi}\left(\left\{\left(x_1,s_1\right),...,\left(x_r,s_r\right)\right\}\right)\\&=\sum_{L\subseteq\mathbb{L}:\{s_1,...,s_r\}\cap L=\emptyset}\gamma^r\left(1-\gamma\right)^{|L|}\int\prod_{j=1}^{r}g\left(x_j\left|x'_j\right.\right)\\&\times P_{\pi'}\left(\{s_1,...,s_r\}\cup L\right)\\&\times\pi'\left(x'_{1:|L|+r};s_{1:r},\overrightarrow{L}\right)dx'_{1:|L|+r}.\end{aligned} \quad (27)$$

We calculate $P_\xi(\cdot)$ using (11) and substituting

$$S=\{s_1,...,s_r\}$$

and $\overrightarrow{S}=s_{1:r}$ into (27)

$$\begin{aligned}P_\xi(S)=&\sum_{L\subseteq\mathbb{L}:S\cap L=\emptyset}\gamma^{|S|}\left(1-\gamma\right)^{|L|}\int\int\prod_{j=1}^{|S|}g\left(x_j\left|x'_j\right.\right)\\&\times P_{\pi'}\left(S\cup L\right)\pi'\left(x'_{1:|L|+|S|};\overrightarrow{S},\overrightarrow{L}\right)\\&\times dx'_{1:|L|+|S|}dx_{1:|S|}\\=&\sum_{L\subseteq\mathbb{L}:S\cap L=\emptyset}\gamma^{|S|}\left(1-\gamma\right)^{|L|}P_{\pi'}\left(S\cup L\right)\\=&\sum_{L'\supseteq S}\gamma^{|S|}\left(1-\gamma\right)^{\left|L'\right|-|S|}P_{\pi'}\left(L'\right)\end{aligned}$$

which finishes the proof of (16).

We calculate $\xi\left(x_{1:|S|};\overrightarrow{S}\right)$ using (15) and (27):

$$\begin{aligned}&\xi\left(x_{1:|S|};\overrightarrow{S}\right)\\&=\frac{1}{P_\xi(S)}\sum_{L\subseteq\mathbb{L}:S\cap L=\emptyset}\gamma^{|S|}\left(1-\gamma\right)^{|L|}\int\prod_{j=1}^{|S|}g\left(x_j\left|x'_j\right.\right)\\&\times P_{\pi'}\left(S\cup L\right)\pi'\left(x'_{1:|L|+|S|},\overrightarrow{S},\overrightarrow{L}\right)dx'_{1:|L|+|S|}\\&=\frac{1}{P_\xi(S)}\sum_{L'\supseteq S}p\left(S\left|L'\right.\right)P_{\pi'}\left(L'\right)\\&\times\int\prod_{j=1}^{|S|}g\left(x_j\left|x'_j\right.\right)\pi'\left(x'_{1:|L'|},\overrightarrow{S},\overrightarrow{L'\setminus S}\right)dx'_{1:|L'|}\\&=\frac{1}{P_\xi(S)}\sum_{L'\supseteq S}p\left(S\left|L'\right.\right)P_{\pi'}\left(L'\right)\\&\times\int\prod_{j=1}^{|S|}g\left(x_j\left|x'_j\right.\right)\pi'_{\overrightarrow{S}}\left(x'_{1:|S|};\overrightarrow{L'}\right)dx'_{1:|S|}\end{aligned}$$

which finishes the proof of (17).

C. Update Equation

In this section, we prove Theorem 2. We have from (4)

$$\begin{aligned}&\boldsymbol{\pi}\left(\left\{\left(x_1,l_1\right),...,\left(x_t,l_t\right)\right\}\right)\\&\quad=\ell\left(\{x_1,...,x_t\}\right)\boldsymbol{\omega}\left(\left\{\left(x_1,l_1\right),...,\left(x_t,l_t\right)\right\}\right)/\rho\end{aligned}$$

where we have used that $\ell(\cdot)$ does not depend on the labels, see assumptions in Section II. We use (10) in the previous equation so that

$$\begin{aligned}P_\pi\left(\{l_1,...,l_t\}\right)\pi\left(x_{1:t};l_{1:t}\right)=\ell\left(\{x_1,...,x_t\}\right)P_\omega\left(\{l_1,...,l_t\}\right)\\\times\omega\left(x_{1:t};l_{1:t}\right)/\rho.\end{aligned} \quad (28)$$

Using (5), the normalising constant is

$$\begin{aligned}\rho=&\sum_{t=0}^{\infty}\frac{1}{t!}\sum_{l_{1:t}}P_\omega\left(\{l_1,...,l_t\}\right)\\&\times\int\ell\left(\{x_1,...,x_t\}\right)\omega\left(x_{1:t};l_{1:t}\right)dx_{1:t}.\end{aligned}$$

The integral is permutation invariant, w.r.t. any permutation of the labels. Therefore, we can just use one permutation an multiply by $t!$. This gives us

$$\begin{aligned}\rho&=\sum_{t=0}^{\infty}\sum_{L:|L|=t}\int\ell\left(\{x_1,...,x_t\}\right)\omega\left(x_{1:t};\overrightarrow{L}\right)P_\omega(L)\,dx_{1:t}\\&=\sum_{L\subseteq\mathbb{L}}P_\omega(L)\,\rho_L.\end{aligned}$$

We obtain $P_\pi(\cdot)$ by integrating w.r.t. $x_{1:t}$ on both sides of (28) and, then, $\pi(\cdot;l_{1:t})$ is also obtained from (28), which completes the proof of Theorem 2.

V. Conclusions

In this paper, we have obtained an explicit form of the Bayesian filtering recursion for labelled random finite sets. This recursion is based on the decomposition of a labelled RFS density into a PMF over the labels and a PDF over the target states given the labels. Apart from the theoretical importance of this contribution, the explicit recursion is useful to develop multiple target tracking particle filters using labelled random finite sets in track-before-detect applications, such as in [21].

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Appendix A

In this appendix, we prove (13). We calculate the set integral of density $\boldsymbol{\pi}(\cdot)$ taking into account (10)-(11)

$$\begin{aligned}
&\int \boldsymbol{\pi}(\mathbf{X})\,\delta\mathbf{X} \\
&= \sum_{t=0}^{\infty}\frac{1}{t!}\sum_{l_{1:t}\in\mathbb{L}^t}\int \boldsymbol{\pi}\left(\left\{(x_1,l_1),...,(x_t,l_t)\right\}\right)dx_{1:t} \\
&= \sum_{t=0}^{\infty}\frac{1}{t!}\sum_{l_{1:t}\in\mathbb{L}^t}P_{\pi}\left(\{l_1,...,l_t\}\right)\int \pi\left(x_{1:t};l_{1:t}\right)dx_{1:t} \\
&= \sum_{t=0}^{\infty}\frac{1}{t!}\sum_{l_{1:t}\in\mathbb{L}^t}P_{\pi}\left(\{l_1,...,l_t\}\right).
\end{aligned}$$

As $P_{\pi}(\cdot)$ is permutation invariant w.r.t. $l_1,...,l_t$, we can sum over the label set $\{l_1,...,l_t\}$ and multiply by $t!$ to get

$$\begin{aligned}
&\int \boldsymbol{\pi}(\mathbf{X})\,\delta\mathbf{X} \\
&= \sum_{t=0}^{\infty}\sum_{\{l_1,...,l_t\}\subseteq\mathbb{L}}P_{\pi}\left(\{l_1,...,l_t\}\right) \\
&= \sum_{t=0}^{\infty}\sum_{L\subseteq\mathbb{L}:|L|=t}P_{\pi}(L) \\
&= \sum_{L\subseteq\mathbb{L}}P_{\pi}(L) \\
&= 1
\end{aligned}$$

where the last equality follows from (2).

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