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# On the quantitative analyses for the random Cucker-Smale model

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# On the quantitative analyses for the random Cucker-Smale model

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# Abstract

In this thesis, we introduce random elements into the Cucker-Smale(C-S) model and provide quantitative analyses for those uncertainties. In real applications of the Cucker-Smale dynamics, we can expect that the C-S model contains some intrinsic uncertainties in itself and misses some extrinsic factors that might affect the dynamics of particles. Thus, to provide a better description for the dynamics of a C-S ensemble, one needs to incorporate such uncertain factors to the model and evaluate their effects on the dynamics or stability of the C-S system.

To fulfill this, we first consider the macroscopic version of the Cucker-Smale model. Namely, we introduce random inputs from communication weights and initial data into the hydrodynamic Cucker-Smale (HCS) model to yield the random HCS model. Furthermore, we address extrinsic uncertainties in the microscopic and mesoscopic level, respectively. For a microscopic model, we introduce a randomly switching network structure to the Cucker-Smale model and investigate sufficient conditions for the emergence of flocking. As a mesoscopic model, we consider the kinetic Cucker-Smale equation perturbed by multiplicative white noise and study the well-posedness and asymptotic dynamics of solutions.

**Key words:** Flocking, Cucker-Smale model, Uncertainty quantification, Local sensitivity analysis, Random dynamical system, Stochastic partial differential equation
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# Contents

# Chapter 1

# Introduction

Collective behaviors in systems of self-propelled particles are widely observed in our nature, e.g. flocking of birds, aggregation of bacteria, synchronous chirps of crickets, schooling of fish, herding of sheep, etc [4, 5, 100, 102, 104]. Among such collective movements, our main interest lies in the so-called *flocking* phenomenon, where self-driven particles adjust their velocities based on simple rules or limited environmental information so that they become organized into an ordered motion. Due to recent applications in unmanned vehicles, sensor networks and robot systems [70, 79, 80], many studies have been dedicated to model such coherent motions. After pioneering works by Viscek and Reynolds [87, 103], several phenomenological models were introduced [8, 20, 23, 75, 76, 99, 102]. In this thesis, we are interested in the model presented by Cucker and Smale [20]. To be specific, let $x_i$ and $v_i$ be the position and velocity of the $i$-th C-S particle in $\mathbb{R}^d$ with unit mass, respectively. Then, the dynamics of C-S particles $(x_i, v_i)$ is governed by the following second order system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in \{1, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \phi(x_j - x_i)(v_j - v_i), \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0). \end{cases} \tag{1.0.1}$$

Here, $\phi = \phi(x)$ is a communication weight function which is nonnegative, bounded, Lipschitz continuous and radially symmetric:

$$\phi(x) = \bar{\phi}(|x|), \quad \forall x \in \mathbb{R}^d,$$

where $\bar{\phi} : [0, \infty) \to \mathbb{R}_+$ is nonnegative, bounded, Lipschitz continuous and monotonically decreasing:

$$0 \le \bar{\phi}(r) \le \bar{\phi}(0) =: \kappa, \quad (\bar{\phi}(r) - \bar{\phi}(s))(r - s) \le 0, \quad \forall r, s \in [0, \infty),$$

$$\text{and} \quad \phi_{Lip} := \sup_{r \neq s} \frac{|\bar{\phi}(r) - \bar{\phi}(s)|}{|r - s|} < \infty.$$

When there is a C-S ensemble with $N$ particles on the phase space $\mathbb{R}^{2d}$ with $N$ very large, it becomes computationally expensive to integrate the infinite number of ODE system (1.0.1). Thus, we introduce a one-particle distribution function $f = f(t, x, v)$ for the infinite ensemble. Via the mean-field limit $N \to \infty$ in (1.0.1), the kinetic density $f$ satisfies the Vlasov equation (see [46, 50] for rigorous justification):

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F_a[f] f) = 0, \quad x, v \in \mathbb{R}^d, \ t > 0, \\ &F_a[f](t, x, v) = -\int_{\mathbb{R}^{2d}} \phi(x - x_*)(v - v_*) f(t, x_*, v_*) dv_* dx_*, \\ &f(0, x, v) = f^{in}(x, v). \end{aligned} \tag{1.0.2}$$

Recently, the particle and kinetic C-S model have been addressed in a lot of extensive research activities from diverse perspectives, i.e. emergence of mono-cluster or multi-cluster flocking [10, 20, 47, 50, 51], effects of white noises [2, 19, 30, 49, 98], time-delay effects [27, 30], application to flight formation [80], collision avoidance [1, 15, 17, 63], generalized network structures [16, 18, 21, 22, 28, 52, 53, 54, 90, 94], mean-field limit [12, 46, 50, 88], kinetic and hydrodynamic description [7, 29, 34, 51, 58, 64, 65, 66, 67, 77, 84], uncertainty quantification (UQ) problems [3, 9, 37, 38, 41], extension of the C-S model [26, 43, 44, 45, 76], etc (see a recent survey [11] for details).

In real applications of C-S systems, modelers or performers determine the communication weight function $\phi$, initial and boundary values based on the phenomenology or their interests. Hence we expect that the C-S model contains some intrinsic uncertainties. On the other hand, the C-S model does not incorporate the influence from the neighboring environment, such as drag forces from the fluid, abrupt disconnection by obstruction, gravitational force, etc. Thus, for a better description of the dynamics of the C-S ensemble, it is necessary to introduce such intrinsic and extrinsic uncertainties to the model (1.0.1) and assess the extent of impacts of these random elements on the flocking dynamics. To fulfill this, the effects of uncertainties need to be quantified, which is the essence of the uncertainty quantification (UQ). During the twenty-first century, UQ has received a lot of attention in diverse disciplines such as the applied mathematics, atmospheric sciences and engineering [3, 9, 55, 56, 57, 59, 60, 62, 71, 72, 74, 78, 81, 82, 83, 85, 89, 92]. Thus, it is natural to synthesize these two emerging disciplines, UQ and emergent flocking dynamics, in a common platform.

In this thesis, we present three works related to the uncertainty quantification for the C-S system. First, we consider a local sensitivity analysis for the hydrodynamic Cucker-Smale model with random inputs from the communication weight and initial data.

Specifically, we consider the pressureless Euler system for the C-S ensemble which is a hyperbolic system with a nonlocal source term. In this case, the nonlocal flocking source term acts like a nonlocal damping which suppresses the appearance of the Delta shocks for small solutions. To incorporate random inputs to the HCS model, we consider a random vector $z$ defined on the sample space $\Omega \subset \mathbb{R}^d$ with the probability density function $\pi = \pi(z)$. For the notational simplicity, we will assume that $z$ is an one-dimensional variable. This random variable $z$ registers the uncertain effects in the initial data and communication weights. To fix the idea, we consider an ensemble of collisionless Cucker-Smale flocking particles on the periodic domain $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, $d \geq 1$, and let $\rho := \rho(t, x, z)$ and $u := u(t, x, z)$ be the local mass and bulk velocity of the C-S fluid at position $x \in \mathbb{T}^d$, random vector $z$ and time $t$, respectively. In this setting, the dynamics of macroscopic observables $(\rho, u)$

is governed by the Cauchy problem to the random HCS model:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad t > 0, \ x \in \mathbb{T}^d, \ z \in \Omega, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) \\ \qquad = \rho \displaystyle\int_{\mathbb{T}^d} \phi(x-y,z)(u(t,y,z) - u(t,x,z))\rho(t,y,z)dy, \\ (\rho, u)(0,x,z) = (\rho_0(x,z), u_0(x,z)), \end{cases} \tag{1.0.3}$$

where $\nabla$ is the spatial gradient. Note that for a frozen $z \in \Omega$, system (1.0.3) becomes the deterministic pressureless Euler system with a flocking dissipation, which has been studied in previous literature, e.g., a rigorous derivation from the kinetic equation [32], the global existence of classical solutions and interaction with incompressible fluids [48] and existence of entropic weak solutions in one-dimension [35].

Here, we would like to see the dynamic properties of $z$-variations $(\partial_z^\alpha \rho, \partial_z^\alpha u)$ to the random HCS model (1.0.3), which is what is called the local sensitivity analysis [91, 95]. Such an analysis is not only of analytical interest. Since it yields regularities in the random space, it is important for numerical methods like stochastic Galerkin or collocation methods [56, 61, 108]. This framework was applied to the particle and kinetic C-S and Kuramoto model in [37, 38, 39, 40], and also to a wide class of random kinetic equations in [55, 56, 57, 59, 60, 62, 71, 72], where the regularity and sensitivity were studied using weighted Sobolev energy estimates and coercivity or hypocoercivity (for perturbative solution near the global equilibrium) of the kinetic operators.

However, the synthesis of local sensitivity analyses and collective dynamics has not been made for the hydrodynamic models from collective dynamics yet. Of course, there are some previous works [60, 74, 81, 82, 83, 85] on the scalar conservation law and Euler system with random inputs from the point of numerics in the context of UQ. It is well known that hydrodynamic models arising from the theory of hyperbolic conservation laws and fluid mechanics do not often allow sufficiently smooth solutions enough to implement a local sensitivity analysis. In particular, hyperbolic conservation laws do not allow a global smooth solution for generic initial data. They instead exhibit discontinuous solutions for generic initial data, which makes a UQ program difficult to implement [24]. This is why the local sensitivity theory has not been well studied in the hyperbolic conservation laws. Despite of this, hyperbolic models arising from the modeling of flocking and synchronization admit smooth solutions for well-prepared initial data thanks to the extra nonlocal flux and source terms, which play the role of regularizing mechanism. Thus, it seems plausible to apply the local sensitivity analysis to the hydrodynamic models for collective dynamics.

On the other hand, it is difficult to provide specific probabilistic estimates in relation with the emergent dynamics via the local sensitivity analysis, since the local sensitivity analysis is performed in an abstract and general framework. Hence, our next goal is to address some probability estimates for (1.0.1) and (1.0.2) with uncertain elements. Here, we focus on the uncertainties in the communication weight since we expect its impact on the dynamics to be stronger than others. In an attempt to obtain such estimates for the particle system, we consider the Cucker-Smale model (1.0.1) with randomly switching topologies.

During the evolution of a C-S flock navigating in the free space $\mathbb{R}^d$, the connection topology might undergo abrupt changes due to unknown external disturbances, obstacles and internal processing mechanisms at unknown instants. In this situation, two natural questions can arise:

- (Q1): How should we model the flocking dynamics of the C-S model with randomly switching network topologies?
- (Q2): If the model is properly set up, then can we find some framework leading to some kind of flocking behavior in terms of system parameters and initial data?

To address the above questions, we assume that the network topology might change along a random sequence of switching times, and at each switching time, we choose a network topology from a given finite set of admissible network topologies randomly, i.e., we employ two random components such as the random switching times and random choice of network topologies. Of course, our chosen network topology may not contain a spanning tree which is necessary for emergence of flocking. Thus, we assume that the union of network topologies in the admissible set contains a spanning tree so that on a suitable time-block with finite size, the union of network topologies contains a spanning tree. Hence, each C-S particle repeatedly communicates with at least one of neighboring particles during each time-block. With this setting in mind, we consider the evolution law for the C-S flocking with randomly switching topologies similar to the model [18]:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad 1 \le i \le N, \quad t > 0, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \chi_{ij}^{\sigma} \phi(x_j - x_i) \left(v_j - v_i\right), \end{cases} \tag{1.0.4}$$

where $(\chi_{ij}^{\sigma(t)})$ denotes the time-dependent network topology corresponding to the switching law $\sigma : [0, \infty) \to \{1, \cdots, N_G\}$. Here, we have the set of admissible (directed) graphs with $N$ vertices $\mathcal{S} := \{\mathcal{G}_1, \cdots, \mathcal{G}_{N_G}\}$. The law $\sigma$, which is piecewise constant and right-continuous, tells which network topology is used to describe the connectivity between C-S particles at a certain instant. Moreover, the sequence of discontinuities $\{t_\ell\}_{\ell \in \mathbb{N}}$ would be called the sequence of *switching* instants (or times). For specific description, once an instant $t$ is given, then $\sigma(t) = \sigma(t_\ell) = k$ for some $1 \le k \le N_G$ and $\ell \in \mathbb{N}$, and the network topology $(\chi_{ij}^{\sigma(t)})$ corresponds to the 0-1 adjacency matrix of $k$-th digraph $\mathcal{G}_k$.

In previous literature [21, 22, 52, 90], the authors considered discretized analogues of the C-S system and $\chi_{ij}$'s in place of $\chi_{ij}^{\sigma}$'s, which are assumed to be nonnegative, independent and identically distributed random variables, to explain the *random failure* of connectivity between C-S particles. In our case, we focus on the continuous system and explore this randomness in connectivity by introducing randomness into the switching law $\sigma$ and the sequence of switching instants $\{t_\ell\}_{\ell \in \mathbb{N}}$. Now, the switching law $\sigma = \sigma(t, \omega)$ ($t \ge 0$, $\omega \in \Omega$) becomes a $\{1, \cdots, N_G\}$-valued jump process and the sequence $\{t_\ell\}_{\ell \in \mathbb{N}}$ has also certain randomness. To describe the random switching times $\{t_\ell\}$, we instead consider the increment process $\{\Delta_\ell := t_{\ell+1} - t_\ell\}$ and we assume that it follows some preassigned distribution $f$ on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. On the other hand, at each switching instant, we choose the network topology $\mathcal{G}_k$ with a probability $p_k$.

Finally, to yield specific probability estimates in the kinetic level, we consider the kinetic equation (1.0.2) perturbed by a multiplicative noise. To fix the idea, we incorporate a stochastic noise into the communication weight, i.e. $\phi \to \phi + \sigma \circ \dot{W}_t$, where $\dot{W}_t$ is a one-dimensional white noise on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\sigma$ denotes the strength of the noise and $\circ$ denotes the stochastic integral in Stratonovich's sense. Then formally, under the unit mass assumption $\int_{\mathbb{R}^{2d}} f(t,x,v)dxdv = 1$, the non-local operator $F_a[f]$ is replaced by a combination of the deterministic part $F_a[f]$ and stochastic part involving with $\dot{W}_t$:

$$F_a[f] \quad \Longrightarrow \quad F_a[f] + \sigma(v_c - v) \circ \dot{W}_t. \tag{1.0.5}$$

Now, we combine (1.0.2) and (1.0.5) to derive the stochastic kinetic C-S equation:

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (F_a[f_t]f_t) = \sigma \nabla_v \cdot ((v - v_c)f_t) \circ \dot{W}_t. \tag{1.0.6}$$

Note that in chapter 5, we use the standard notation for random probability density function $f_t(x,v) := f(t,x,v)$.

As previously mentioned, the effects of white noise perturbations were discussed in [2, 19, 30, 49, 98] at the particle level. Moreover, a rigorous derivation of the equation (1.0.6) as a mean-field limit of the C-S systems with multiplicative noises was recently discussed in [12] based on the propagation of chaos result in [14], and a mean-field limit of the C-S systems with another type of stochastic perturbations was also addressed in [88]. However, as far as we know, the equation (1.0.6) has only been addressed in measure spaces such as $\mathcal{P}_2$, not in other function spaces (e.g. Sobolev spaces). For other types of stochastic kinetic equations, we refer to [33, 86]. In this thesis, we address the following two questions:

- (Well-posedness): Is the stochastic kinetic C-S equation (1.0.6) well-posed in a suitable function space such as Sobolev spaces?

- (Emergence of flocking): If so, does the solution to (1.0.6) exhibit asymptotic flocking dynamics?

Our results in Chapter 5 provide affirmative answers to the above posed questions. First, we introduce a concept of a strong solution to (1.0.6) and then provide a global well-posedness for strong solutions by employing a suitable regularization method and stopping time argument. Second, we provide a stochastic flocking estimate by showing that the expectation of the second velocity moment decays to zero exponentially fast, when the communication weight function $\phi$ has a positive infimum $\phi_m := \inf_{x\in\mathbb{R}^d} \phi(x)$ and noise strength $\sigma$ is sufficiently small compared to $\phi_m$. The main difficulty in our analysis arises, when we prove the existence of a solution to the regularized equation. Here, we obtain $W^{m,\infty}$-estimates for the sequence of functions that approximates the regularized equation. Our $W^{m,\infty}$-estimates contain terms with infinite expectation. Hence, even though we can find a limit function of the sequence from the pathwise estimates, it is not certain that the limit function becomes a solution to the regularized equation. To cope with this problem, we used stopping time argument to get a solution to the regularized equation.

The rest of the thesis is organized as follows. In Chapter 2, we summarize the notation used throughout the thesis and present previous results about the deterministic verision of the particle, kinetic and hydrodynamic C-S models without proofs. In Chapter 3, we present a local sensitivity analysis for the hydrodynamic Cucker-Smale model with random inputs (1.0.3). In Chapter 4, we study the emergent dynamics of the Cucker-Smale flocks (1.0.4) when the network topology changes randomly along time. In Chapter 5, we show the global well-posedness of strong solutions to the equation (1.0.6) and its emergent dynamics. Finally, in Chapter 6, we provide a brief summary of the thesis and discuss the issues which will be addressed in the future. In Appendix A and Appendix B, we present detailed proofs that we omitted in Chapter 3 and 5, respectively.

# Chapter 2

# Preliminaries

In this chapter, we present the notation which will be used throughout this thesis, and review previous results about the deterministic Cucker-Smale model.

## 2.1 Notation

Throughout this thesis, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a generic probability space. For any $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, we set $W^{k,p}(\mathbb{F})$ to be the $k$-th order $L^p$-Sobolev spaces on $\mathbb{F} = \mathbb{T}^d$ or $\mathbb{R}^d$, and $H^k(\mathbb{F}) := W^{k,2}(\mathbb{F})$. If there is no confusion about the choice of the domain $\mathbb{F}$, then we simply write $W^{k,p} := W^{k,p}(\mathbb{F})$ and $H^k := H^k(\mathbb{F})$, respectively. $\mathcal{C}^k(I; \mathcal{B})$ denotes the space of $k$-times continuously differentiable functions from an interval $I$ into a Banach space $\mathcal{B}$. Moreover, $\nabla^k$ denotes any partial derviative $\partial^\alpha$ with respect to $x$-variable with multi-index $\alpha$ with $|\alpha| = k$.

We set

$$X := (x_1, \cdots, x_N), \quad V := (v_1, \cdots, v_N),$$

and $\mathcal{D}(X)$ and $\mathcal{D}(V)$ denote position and velocity diameters:

$$\mathcal{D}(X) := \max_{1 \le i,j \le N} \|x_i - x_j\|, \quad \mathcal{D}(V) := \max_{1 \le i,j \le N} \|v_i - v_j\|.$$

Matrix ordering is meant componentwise, e.g., for matrices $A = (a_{ij})_{N \times N}$ and $B = (b_{ij})_{N \times N}$, $A \ge B$ stands for $a_{ij} \ge b_{ij}$ for all $i$, $j$. For a real number

$c$, denote by $\lfloor c \rfloor$ the floor of c, i.e., the largest integer no greater than $c$. $\mathbb{N}$ denotes the set of all natural numbers (including zero).

For $(x, v) \in \mathbb{R}^{2d}$, $\delta_{(x,v)}$ denotes a point mass concentrated at $(x, v)$. For each $p \in [1, \infty)$, we denote $\mathcal{P}_p(\mathbb{R}^{2d})$ by

$$\mathcal{P}_p(\mathbb{R}^{2d}) := \left\{ \mu \quad : \quad \begin{array}{c} \text{probability measure on } \mathbb{R}^{2d} \text{ such that} \\ \displaystyle\int_{\mathbb{R}^{2d}} |(x, v)|^p d\mu(x, v) < \infty. \end{array} \right\},$$

and we write $p$-Wasserstein distance on $\mathcal{P}_p(\mathbb{R}^{2d})$ as

$$W_p(\mu, \nu) := \left( \inf_{\gamma \in \prod(\mu,\nu)} \int_{\mathbb{R}^{4d}} |(x, v) - (y, w)|^p d\gamma \right)^{1/p},$$

where $\prod(\mu, \nu)$ denotes the collection of all measures on $\mathbb{R}^{4d}$ whose marginals are $\mu$ and $\nu$.

For a probability density function $f = f(t, x, v)$ with $(x, v) \in \mathbb{R}^{2d}$ at time $t \in \mathbb{R}_+$, we set the $p$-th velocity moments $(p = 0, 1, 2)$ of $f$ as

$$M_0[f](t) := \int_{\mathbb{R}^{2d}} f dx dv, \quad M_1[f](t) := \int_{\mathbb{R}^{2d}} v f dx dv,$$

$$M_2[f](t) := \int_{\mathbb{R}^{2d}} |v|^2 f dx dv, \quad t \geq 0,$$

and we also write $v_c[f](t) := M_1[f](t)$. If there is no confusion about the choice of $f$, we write

$$M_p(t) := M_p[f](t), \quad v_c(t) := v_c[f](t).$$

## 2.2 Previous results

In this section, we provide previous results for the deterministic Cucker-Smale model. First, we review the mono-cluster flocking result for (1.0.1) and below, we present the definition for the flocking.

**Definition 2.2.1.** *Let* $\{(x_i, v_i)\}$ *be a C-S ensemble whose dynamics is governed by* (1.0.1)*. Then, it exhibits a mono-cluster flocking if and only if the following two conditions hold.*

$$\sup_{0\le t<\infty} \max_{i,j} \|x_i(t) - x_j(t)\| < \infty, \quad \lim_{t\to\infty} \max_{i,j} \|v_i(t) - v_j(t)\| = 0.$$

For a given configuration $(X, V)$, we set

$$\|X\|_\infty := \max_{1\le i\le N} \|x_i\|, \quad \|V\|_\infty := \max_{1\le i\le N} \|v_i\|.$$

**Theorem 2.2.1.** [1, 37, 50] *Let* $(X, V)$ *be a solution to* (1.0.1) *with the initial data* $(X^0, V^0)$ *satisfying the following conditions:*

$$\sum_{i=1}^N x_i^0 = \sum_{i=1}^N v_i^0 = 0, \quad \|X^0\|_\infty > 0, \quad \|V^0\|_\infty < \frac{1}{2}\int_{\|X^0\|_\infty}^\infty \phi(2r)dr.$$

*Then, there exists a positive constant* $x_M > 0$ *such that*

$$\sup_{t\ge 0} \|X(t)\|_\infty \le x_M, \quad \|V(t)\|_\infty \le \|V^0\|_\infty e^{-\phi(2x_M)t}, \quad t \ge 0.$$

Next, we review the results for the kinetic equation (1.0.2) and its emergent dynamics. Formally, the kinetic equation (1.0.2) can be derived as a mean-field limit of system (1.0.1) by using the standard BBGKY hierarchy under the molecular chaos assumption. For a brief description of BBGKY hierarchy, we refer to [51, 68] and for rigorous derivation of the equation, we refer to [46, 50]. Below, we provide the well-posedness and emergent behaviors of classical solutions to (1.0.2).

**Theorem 2.2.2.** [51] *Suppose that the initial datum* $f_0 \in (\mathcal{C}^1 \cap W^{1,\infty})(\mathbb{R}^{2d})$ *is compactly supported in the phase space, i.e. the* $x$*- and* $v$*-supports of* $f_0$ *in the phase space are bounded. Then for any* $T \in (0, \infty)$*, there exists a unique classical solution* $f \in \mathcal{C}^1([0, T) \times \mathbb{R}^{2d})$ *to* (1.0.2) *satisfying*

$$\Lambda[f](t) \le \Lambda[f_0] e^{-2M_0(t)\int_0^t \varphi(s)ds},$$

*where* $\Lambda[f]$ *and* $\varphi$ *are given by*

$$\Lambda[f](t) := \int_{\mathbb{R}^{2d}} |v - v_c|^2 f(t,x,v)dxdv,$$

$$\varphi(t) := \inf\{\phi(x-y) \ : \ f(t,x,v)f(t,y,v_*) \neq 0 \ \ \textit{for some} \ \ v, v_* \in \mathbb{R}^d\}.$$

**Remark 2.2.1.** *As addressed in [36], we briefly explain the meaning of the zero convergence of* $\Lambda$ *as* $t \to \infty$*. Let* $f$ *be a probability density function over* $\mathbb{R}^{2d}$*. Then we use the Chebyshev inequality to obtain that, for any* $\varepsilon > 0$*,*

$$\begin{aligned}\Lambda[f](t) &= \int_{\mathbb{R}^{2d}} |v-v_c|^2 f dvdx \geq \int_{|v-v_c|>\varepsilon} |v-v_c|^2 f dvdx \\ &\geq \varepsilon^2 \int_{|v-v_c|>\varepsilon} f dvdx = \varepsilon^2 \mathbb{P}[|v - v_c(0)| > \varepsilon].\end{aligned}$$

*This gives*

$$\lim_{t\to\infty} \mathbb{P}[|v-v_c| > \varepsilon] \leq \frac{1}{\varepsilon^2} \lim_{t\to\infty} \Lambda[f](t) = 0,$$

*which implies the formation of velocity alignment in probability sense.*

Finally, we address the deterministic hydrodynamic Cucker-Smale model. To derive a hydrodynamic model from (1.0.2), we introduce the macroscopic observables such as the local mass, momentum and energy densities:

$$\begin{aligned}&\rho(t,x) := \int_{\mathbb{R}^d} f dv, \quad (\rho u)(t,x) := \int_{\mathbb{R}^d} v f dv, \\ &(\rho E)(x,t) := \frac{1}{2}\rho|u|^2 + \rho e, \quad \rho e := \frac{1}{2}\int_{\mathbb{R}^d} |v - u(x,t)|^2 f dv.\end{aligned}$$

We multiply $1, v, |v|^2/2$ to (1.0.2) and integrate the resulting relations with respect to the velocity variable to derive a system of balance laws for the macroscopic observables $(\rho, u, E)$:

$$\begin{aligned}&\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad t > 0, \quad x \in \mathbb{R}^d, \\ &\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u + P) = S^{(1)}, \\ &\partial_t(\rho E) + \nabla \cdot (\rho E u + P u + q) = S^{(2)},\end{aligned} \tag{2.2.1}$$

where $P = (p_{ik})$ and $q = (q_1, \cdots, q_d)$ are the stress tensor and heat flow, respectively:

$$p_{ij}(t,x) := \int_{\mathbb{R}^d} (v_i - u_i)(v_j - u_j) f dv, \quad q_i(t,x) := \int_{\mathbb{R}^d} (v_i - u_i)|v-u|^2 f dv,$$

and the source terms are written as follows:

$$S^{(1)}(t,x) := \rho \int_{\mathbb{R}^d} \phi(x-y)(u(t,y) - u(t,x))\rho(t,y)dy,$$
$$S^{(2)}(t,x) := \rho \int_{\mathbb{R}^d} \phi(x-y)\left(E(t,x) + E(t,y) - u(t,x)\cdot u(t,y)\right)\rho(t,y)dy.$$

Since system (2.2.1) is not closed as it is, one introduces a mono-kinetic ansatz for $f$ as a closure condition:

$$f(t,x,v) = \rho(t,x)\delta_{(v-u(x,t))}(v).$$

With this ansatz, it can be observed that the internal energy, stress tensor and heat flux in (2.2.1) vanish, and we obtain the following Cauchy problem for a pressureless Euler system with flocking dissipation (see [32] for its rigorous derivation):

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho u) = 0, \qquad t > 0,\ x \in \mathbb{T}^d, \\ \partial_t(\rho u) + \nabla\cdot(\rho u \otimes u) = \rho \displaystyle\int_{\mathbb{T}^d} \phi(x-y)(u(t,y) - u(t,x))\rho(t,y)dy, \\ (\rho(0,x), u(0,x)) = (\rho_0(x), u_0(x)), \quad x \in \mathbb{T}^d. \end{cases} \tag{2.2.2}$$

Below, we provide the standing assumptions $(\mathcal{H}1) - (\mathcal{H}2)$ for the well-posedness, stability and flocking estimates for (2.2.2). For an integer $s > \frac{d}{2}+1$,

- $(\mathcal{H}1)$: The communication weight function $\phi : \mathbb{T}^d \to \mathbb{R}$ is in $\mathcal{C}^{s+1}$ and satisfies symmetric, positive conditions: for each $x, y \in \mathbb{T}^d$,

$$\phi(x-y) = \phi(y-x) \quad \text{and} \quad \inf_{x\in\mathbb{T}^d} \phi(x) =: \phi_m > 0.$$

- $(\mathcal{H}2)$: The initial data $(\rho_0, u_0)$ satisfy the non-vacuum, regularity and smallness conditions, i.e. for sufficiently small $\varepsilon > 0$,

$$\inf_{x\in\mathbb{T}^d} \rho_0(x) > 0, \quad (\rho_0, u_0) \in H^s \times H^{s+1}, \quad \|\rho_0\|_{H^s} + \|u_0\|_{H^{s+1}} < \varepsilon.$$

Before we state previous results, we introduce a Lyapunov functional $\mathcal{E}_0$ for flocking:

$$\mathcal{E}_0(t) := \int_{\mathbb{T}^d} \rho|u - u_c(t)|^2 dx, \quad u_c(t) := \frac{\int_{\mathbb{T}^d} \rho u dx}{\int_{\mathbb{T}^d} \rho dx} = u_c(0), \quad t \geq 0. \quad (2.2.3)$$

Then, the deterministic HCS model can be summarized in the following theorem:

**Theorem 2.2.3.** [48] *For a given positive constant $T > 0$, suppose that conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ hold. Then, there exist positive constants $C = C(T)$ and $0 < \varepsilon \ll 1$ such that the Cauchy problem* (2.2.2) *has a unique global-in-time classical solution $(\rho, u)$ satisfying the following properties:*

1. *(Propagation of the Sobolev regularity): The solution $(\rho, u)$ satisfies the following regularity and uniform-in-time boundedness condition:*

$$\inf_{(t,x)\in[0,T]\times\mathbb{T}^d} \rho(t,x) > 0, \quad (\rho(t), u(t)) \in H^s \times H^{s+1}, \quad \textit{for } t \in [0,T],$$
$$\sup_{0\leq t\leq T} (\|\rho(t)\|_{H^s} + \|u(t)\|_{H^{s+1}}) < \sqrt{\varepsilon}.$$

2. *(Finite-in-time stability): For two classical solution processes $(\rho, u)$ and $(\bar{\rho}, \bar{u})$ to* (2.2.2) *with initial data $(\rho_0, u_0)$ and $(\bar{\rho}_0, \bar{u}_0)$ respectively,*

$$\sup_{0\leq t\leq T} \left(\|\rho(t) - \bar{\rho}(t)\|_{L^2}^2 + \|u(t) - \bar{u}(t)\|_{H^1}^2\right)$$
$$\leq C(T)(\|\rho_0 - \bar{\rho}_0\|_{L^2}^2 + \|u_0 - \bar{u}_0\|_{H^1}^2).$$

3. *(Exponential flocking estimate): The functional $\mathcal{E}_0(t)$ decays exponentially pathwise:*

$$\mathcal{E}_0(t) \leq e^{-2\phi_m\|\rho_0\|_{L^1}t}\mathcal{E}_0(0), \quad \forall t > 0.$$

**Remark 2.2.2.** *By Theorem 2.2.3, the local mass $\rho$ stays positive. Moreover, since the solution is classical, the momentum equations of* (2.2.2) *can be rewritten as*

$$\partial_t u + u \cdot \nabla u = \int_{\mathbb{T}^d} \phi(x-y)(u(t,y) - u(t,x))\rho(t,y)dy.$$

# Chapter 3

# A local sensitivity analysis for the hydrodynamic Cucker-Smale model with random inputs

In this chapter, we present a local sensitivity analysis for the hydrodynamic Cucker-Smale model with random inputs. Recall that the HCS model with random inputs explains the dynamics of observables $(\rho, u)$ governed by the following random equation:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad t > 0, \ x \in \mathbb{T}^d, \ z \in \Omega, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) \\ \qquad = \rho \displaystyle\int_{\mathbb{T}^d} \phi(x-y, z)(u(t,y,z) - u(t,x,z))\rho(t,y,z)dy, \end{cases} \tag{3.0.1}$$

subject to random initial data:

$$(\rho, u)(0, x, z) = (\rho_0(x,z), u_0(x,z)), \quad x \in \mathbb{T}^d, \quad z \in \Omega.$$

The main results of this chapter are three-fold. First, we present the propagation of pathwise well-posedness of the random HCS model (3.0.1). For $s > \frac{d}{2} + m + 1$, if the initial processes and their $z$-variations $\{(\partial_z^l \rho_0, \partial_z^l u_0)\}_{l=0}^m$ satisfy the non-vacuum, regularity and smallness conditions, we show that

$z$-variations of solution processes $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^m$ exist in any finite time interval, and satisfy the desired regularity and smallness conditions (see Theorem 3.1.1 and Theorem 3.1.2).

Second, we provide a finite-in-time $L^2$-stability of the $z$-variations to system (3.0.1). More precisely, let $(\rho, u)$ and $(\bar{\rho}, \bar{u})$ be solution processes to (3.0.1) corresponding to initial processes $(\rho_0, u_0)$ and $(\bar{\rho}_0, \bar{u}_0)$, respectively. Then, there exists a positive random function $C = C(T, z)$ such that for each $T \in (0, \infty)$ and $z \in \Omega$,

$$\begin{aligned} &\sup_{0\le t\le T} \sum_{0\le l\le m} \left( \|\partial_z^l \rho(t,z) - \partial_z^l \bar{\rho}(t,z))\|_{H^{m-l}}^2 + \|\partial_z^l u(t,z) - \partial_z^l \bar{u}(t,z))\|_{H^{m-l+1}}^2 \right) \\ &\quad \le C(T,z) \sum_{0\le l\le m} \left( \|\partial_z^l \rho_0(z) - \partial_z^l \bar{\rho}_0(z)\|_{H^{m-l}}^2 + \|\partial_z^l u_0(z) - \partial_z^l \bar{u}_0(z)\|_{H^{m-l+1}}^2 \right). \end{aligned}$$

Third, we show that the bulk velocity process and its $z$-variations $\{\partial_z^l u\}$ exhibit an exponential decay toward the mean-velocity under *a priori* assumptions, which implies the flocking estimate. We assume the uniform-in-time boundedness for solution processes and their $z$-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^m$, and impose an *a priori* condition for the lower bound of the communication weight function to obtain the exponential decay of $\{\partial_z^l u\}$ toward its mean-velocity.

The rest of this chapter is organized as follows. In Section 3.1, we present the pathwise well-posedness for the random HCS model. In Section 3.2, we provide $L^2$-stability estimates for the $z$-variations $\{(\partial_z^l \rho, \, \partial_z^l u)\}_{l=0}^m$. In Section 3.3, we present an exponential decay of the bulk velocity process and its $z$-variations. In Appendix A, we provide tedious and straightforward proofs for Lemma 3.1.2, Lemma 3.1.5, Lemma 3.2.4 and Theorem 3.3.2. Finally, we note that this chapter is based on the joint work [41].

## 3.1 Pathwise well-posedness of $z$-variations

In this section, we present a global existence of $z$-variations $(\partial_z^m \rho, \partial_z^m u)$ to system (3.0.1) using pathwise energy method.

Note that in a non-vacuum regime, system (3.0.1) can be rewritten as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad t > 0,\ x \in \mathbb{T}^d,\ z \in \Omega, \\ \partial_t u + u \cdot \nabla u = \displaystyle\int_{\mathbb{T}^d} \phi(x-y,z)(u(t,y,z) - u(t,x,z))\rho(t,y,z)dy, \\ (\rho, u)(0,x,z) = (\rho_0(x,z), u_0(x,z)). \end{cases} \tag{3.1.1}$$

First, we derive equations for the $z$-variations by applying $z$-derivative to (3.1.1) to obtain

$$\begin{aligned} &\partial_t(\partial_z^m \rho) + \sum_{l=0}^m \binom{m}{l} \nabla \cdot \left(\partial_z^l \rho \partial_z^{m-l} u\right) = 0, \\ &\partial_t(\partial_z^m u) + \sum_{l=0}^m \binom{m}{l} \left(\partial_z^l u \cdot \nabla(\partial_z^{m-l} u)\right) \\ &= \sum_{\alpha+\beta+\gamma=m} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^d} \partial_z^\alpha \phi(x-y,z) \partial_z^\beta [u(t,y,z) - u(t,x,z)]\, \partial_z^\gamma \rho(t,y,z) dy. \end{aligned} \tag{3.1.2}$$

Then, the following estimates directly follow from (3.1.2).

**Proposition 3.1.1.** *Let $(\rho, u)$ be a sufficiently smooth periodic solution to (3.0.1). Then, for $t \geq 0$, $m \geq 0$ and a fixed $z \in \Omega$,*

$$\int_{\mathbb{T}^d} \partial_z^m \rho(t,z)dx = \int_{\mathbb{T}^d} \partial_z^m \rho_0(z)dx, \quad \int_{\mathbb{T}^d} \partial_z^m (\rho u)(t,z)dx = \int_{\mathbb{T}^d} \partial_z^m (\rho_0 u_0)(z)dx.$$

*Proof.* The proofs follow from the direct integration of (3.1.2). □

For a global well-posedness of the $z$-variations, we provide our standing assumptions $(\mathcal{A}1) - (\mathcal{A}2)$ as follows: For an integer $s > \frac{d}{2} + m + 1$,

- $(\mathcal{A}1)$: The communication weight function $\phi : \mathbb{T}^d \times \Omega \to \mathbb{R}$ is in $\mathcal{C}^{s+1}(\mathbb{T}^d \times \Omega)$ and satisfies symmetric, non-negative and boundedness conditions: for each $x, y \in \mathbb{T}^d$ and $z \in \Omega$,

$$\phi(x-y,z) = \phi(y-x,z) \geq 0,$$

$$\|\phi\|_s := \max_{|\alpha|+|\beta| \leq s+1} \sup_{(x,z) \in \mathbb{T}^d \times \Omega} |\partial_z^\alpha \partial_x^\beta \phi(x,z)| < \infty.$$

- ($\mathcal{A}2$): The initial data $(\rho_0, u_0)$ satisfy the non-vacuum, regularity and smallness conditions: for each $z \in \Omega$ and $l = 0, \cdots, m$,

$$\inf_{x\in\mathbb{T}^d} \rho_0(x,z) > 0, \quad (\partial_z^l \rho_0(z), \partial_z^l u_0(z)) \in H^{s-l} \times H^{s-l+1},$$
$$\max_{0\le l\le m} \left(\|\partial_z^l \rho_0(z)\|_{H^{s-l}} + \|\partial_z^l u_0(z)\|_{H^{s-l+1}}\right) < \varepsilon(z),$$

where $\varepsilon = \varepsilon(z)$ is a positive random function such that $\sup\limits_{z\in\Omega} \varepsilon(z) \ll 1$.

For the simplicity of notation, we suppress $z$-dependence in $(\rho, u)$ and $\phi$, i.e.

$$\rho(t,x) := \rho(t,x,z), \qquad u(t,x) := u(t,x,z), \qquad \phi(x) := \phi(x,z).$$

To derive *a priori* estimates, we employ a mathematical induction on $m$.

### 3.1.1 First-order $z$-variations

In this subsection, we consider a global well-posedness for the first-order $z$-variations $(\partial_z\rho, \partial_z u)$ for the initial step of induction process on $m$. To provide a global well-posedness, we construct a sequence of approximated solutions $(\partial_z\rho^{n+1}, \partial_z u^{n+1})$ to (3.1.2). For a given solution $(\rho, u)$ and $m = 1$, we may construct the sequence as follows:

$$\begin{aligned}
&\partial_t(\partial_z\rho^{n+1}) + \nabla\cdot(\partial_z\rho^{n+1}u) + \nabla(\rho\partial_z u^n) = 0, \;\; n = 0,1,2,\cdots\\
&\partial_t(\partial_z u^{n+1}) + \partial_z u^n\cdot\nabla u + u\cdot\nabla(\partial_z u^{n+1})\\
&\qquad = \int_{\mathbb{T}^d} \partial_z\phi(x-y)(u(t,y)-u(t,x))\rho(t,y)dy\\
&\qquad\quad + \int_{\mathbb{T}^d} \phi(x-y)(\partial_z u^n(t,y)-\partial_z u^n(t,x))\rho(t,y)dy\\
&\qquad\quad + \int_{\mathbb{T}^d} \phi(x-y)(u(t,y)-u(t,x))\partial_z\rho^{n+1}(t,y)dy\\
&(\partial_z\rho^0, \partial_z u^0) = (\partial_z\rho_0, \partial_z u_0),
\end{aligned} \tag{3.1.3}$$

subject to the fixed initial data:

$$(\partial_z\rho^{n+1}(0,x), \partial_z u^{n+1}(0,x)) = (\partial_z\rho_0(x), \partial_z u_0(x)).$$

Since the pathwise well-posedness for $(\rho, u)$ can be similarly obtained from Theorem 2.2.3, there is no need for $(\rho, u)$ to be involved in the iteration scheme (3.1.3). Thus, the iteration procedure in (3.1.3) will be carried out only for the $z$-variations $(\partial_z \rho, \partial_z u)$. We proceed by induction on $n$ for the sequence $(\partial_z \rho^n, \partial_z u^n)$. First, we state the results on the uniform-in-$n$ bound estimates.

**Lemma 3.1.1.** *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *and induction hypothesis hold: for each* $z \in \Omega$,

$$\sup_{\substack{0\le j\le n\\ 0\le t\le T}} \|\partial_z u^j(t,z)\|_{H^s} < \sqrt{\varepsilon(z)}.$$

*Then, there exists a unique* $\partial_z \rho^{n+1} = \partial_z \rho^{n+1}(t,z) \in H^{s-1}$ *satisfying relation* $(3.1.3)_1$ *and a bound:*

$$\sup_{0\le t\le T} \|\partial_z \rho^{n+1}(t,z)\|_{H^{s-1}} < \frac{\sqrt{\varepsilon(z)}}{2}.$$

*Proof.* Since system (3.1.3) is linear with respect to $\partial_z \rho^{n+1}$, the existence and uniqueness for $\partial_z \rho^{n+1}$ are obvious. Thus, it suffices to show the boundedness of the solution. Here, we split the estimates into the zeroth-order case and higher-order case.

• Step A (The zeroth-order estimates): First, we multiply $(3.1.3)_1$ by $\partial_z \rho^{n+1}$ and integrate it over $\mathbb{T}^d$ to yield

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z \rho^{n+1}\|_{L^2}^2 \\
&\quad = -\frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot u)|\partial_z \rho^{n+1}|^2 dx - \int_{\mathbb{T}^d} (\nabla \rho \cdot \partial_z u^n)\partial_z \rho^{n+1} dx \\
&\qquad - \int_{\mathbb{T}^d} \rho \nabla\cdot(\partial_z u^n)\partial_z \rho^{n+1} dx \\
&\quad \le \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\partial_z \rho^{n+1}\|_{L^2}^2 + \|\rho\|_{W^{1,\infty}}\|\partial_z u^n\|_{H^1}\|\partial_z \rho^{n+1}\|_{L^2} \\
&\quad \le \left(\frac{\|\nabla\cdot u\|_{L^\infty}}{2} + \frac{\|\rho\|_{W^{1,\infty}}}{2}\right)\|\partial_z \rho^{n+1}\|_{L^2}^2 + \frac{\|\rho\|_{W^{1,\infty}}}{2}\|\partial_z u^n\|_{H^1}^2, \\
&\quad \le \varepsilon^{1/2}\|\partial_z \rho^{n+1}\|_{L^2}^2 + \varepsilon^{3/2},
\end{aligned} \tag{3.1.4}$$

where we used Young's inequality on the second inequality and Theorem 2.2.3 on the last inequality. Then, we integrate the previous relation (3.1.4) to derive

$$\|\partial_z \rho^{n+1}\|_{L^2}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z \rho^{n+1}(s,z)\|_{L^2}^2 ds + \varepsilon^{3/2}\right). \tag{3.1.5}$$

• Step B (Higher-order estimates): For higher-order estimates, let $1 \le k \le s-1$. Then, we apply $\nabla^k$ to $(3.1.3)_1$, multiply by $\nabla^k(\partial_z \rho^{n+1})$ and integrate the resulting relation over $\mathbb{T}^d$ to yield

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}^2\\
&= \frac{1}{2}\int_{\mathbb{T}^d} |\nabla^k(\partial_z\rho^{n+1})|^2(\nabla\cdot u)dx\\
&\quad - \int_{\mathbb{T}^d}\Big[\nabla^k(u\cdot\nabla(\partial_z\rho^{n+1})) - u\cdot\nabla^k(\nabla(\partial_z\rho^{n+1}))\Big]\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad - \int_{\mathbb{T}^d} \partial_z\rho^{n+1}\nabla^k(\nabla\cdot u)\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad - \int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z\rho^{n+1}\nabla\cdot u) - \partial_z\rho^{n+1}\nabla^k(\nabla\cdot u)\Big]\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad - \int_{\mathbb{T}^d} \nabla^k(\nabla\rho)\cdot\partial_z u^n\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad - \int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z u^n\nabla\rho) - \partial_z u^n\cdot\nabla^k(\nabla\rho)\Big]\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad - \int_{\mathbb{T}^d} \rho\nabla^k(\nabla\cdot\partial_z u^n)\nabla^k(\partial_z\rho^{n+1})dx\\
&\quad - \int_{\mathbb{T}^d}\Big[\nabla^k(\rho(\nabla\cdot\partial_z u^n)) - \rho\nabla^k(\nabla\cdot\partial_z u^n)\Big]\nabla^k(\partial_z\rho^{n+1})dx,\\
&=: \sum_{i=1}^{8}\mathcal{I}_{1i}.
\end{aligned}$$

Below, we estimate the terms $\mathcal{I}_{1i}$ separately as follows:

$\diamond$ (Estimates for $\mathcal{I}_{1i}$, $i = 2,4,6,8$) : We use the commutator estimate from Lemma 3.4 in [73] to obtain

$$\mathcal{I}_{12} \le c\Big[\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} + \|\nabla(\partial_z\rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}\Big]\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}$$

$$
\begin{aligned}
&\le C\Big[\|u\|_{H^{s-1}}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} + \|\partial_z\rho^{n+1}\|_{H^{s-1}}\|\nabla^k u\|_{L^2}\Big]\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} \\
&\le C\varepsilon^{1/2}\|\partial_z\rho^{n+1}\|^2_{H^{s-1}},
\end{aligned}
$$

where $c$ and $C$ are positive random functions independent of $n$ and we used the assumptions, Theorem 2.2.3 and the Sobolev embedding:

$$
\|\nabla u\|_{L^\infty} \le C\|u\|_{H^{[\frac{d}{2}]+1}} \le C\|u\|_{H^{s-1}}. \tag{3.1.6}
$$

For other terms, one uses the commutator estimate, (3.1.6), Theorem 2.2.3 and Young's inequality to get

$$
\begin{aligned}
\mathcal{I}_{14} &\le c\Big[\|\nabla(\partial_z\rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2} + \|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2}\Big]\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} \\
&\le C\varepsilon^{1/2}\|\partial_z\rho^{n+1}\|^2_{H^{s-1}}, \\
\mathcal{I}_{16} &\le c\Big[\|\nabla(\partial_z u^n)\|_{L^\infty}\|\nabla^k\rho\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z u^n)\|_{L^2}\Big]\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z\rho^{n+1})\|^2_{L^2} + \varepsilon^{3/2}), \\
\mathcal{I}_{18} &\le c\Big[\|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z u^n)\|_{L^2} + \|\nabla\cdot(\partial_z u^n)\|_{L^\infty}\|\nabla^k\rho\|_{L^2}\Big]\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z\rho^{n+1})\|^2_{L^2} + \varepsilon^{3/2}).
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{1i}$, $i = 1, 3, 5, 7$): By direct calculations, one easily obtains

$$
\begin{aligned}
\mathcal{I}_{11} &\le \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\nabla^k(\partial_z\rho^{n+1})\|^2_{L^2} \le \varepsilon^{1/2}\|\nabla^k(\partial_z\rho^{n+1})\|^2_{L^2}, \\
\mathcal{I}_{13} &\le \|\partial_z\rho^{n+1}\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\partial_z\rho^{n+1}\|^2_{H^{s-1}} + \varepsilon^{3/2}), \\
\mathcal{I}_{15} &\le \|\nabla^{k+1}\rho\|_{L^2}\|\partial_z u^n\|_{L^\infty}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z\rho^{n+1})\|^2_{L^2} + \varepsilon^{3/2}), \\
\mathcal{I}_{17} &\le \|\rho\|_{L^\infty}\|\nabla^{k+1}(\partial_z u^n)\|_{L^2}\|\nabla^k(\partial_z\rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z\rho^{n+1})\|^2_{L^2} + \varepsilon^{3/2}).
\end{aligned}
$$

We combine all results for $\mathcal{I}_{1i}$'s to obtain

$$
\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z\rho^{n+1})\|^2_{L^2} \le C(\varepsilon^{1/2}\|\partial_z\rho^{n+1}\|^2_{H^{s-1}} + \varepsilon^{3/2}). \tag{3.1.7}
$$

Summing (3.1.7) over $1 \le k \le s-1$ and adding these to (3.1.5) yields

$$\frac{\partial}{\partial t}\|\partial_z \rho^{n+1}\|^2_{H^{s-1}} \le C(\varepsilon^{1/2}\|\partial_z \rho^{n+1}\|^2_{H^{s-1}} + \varepsilon^{3/2}).$$

Then, Grönwall's lemma and the smallness of $\varepsilon$ yield the desired estimate:

$$\|\partial_z \rho^{n+1}\|^2_{H^{s-1}} \le e^{\varepsilon^{1/2}CT}\|\partial_z \rho_0\|^2_{H^{s-1}} + \varepsilon(e^{\varepsilon^{1/2}CT} - 1) < \frac{\varepsilon}{4}.$$

□

**Lemma 3.1.2.** *Suppose that assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold and let $(\partial_z \rho^j, \partial_z u^j)$ be the $j$-th iterate satisfying the following assumptions: for each $z \in \Omega$,*

$$\max_{0\le j\le n}\sup_{0\le t\le T}\left(\|\partial_z \rho^j(t,z)\|_{H^{s-1}} + \|\partial_z u^j(t,z)\|_{H^s}\right) < \sqrt{\varepsilon(z)}.$$

*Then for each $z \in \Omega$, there exists a unique $\partial_z u^{n+1} = \partial_z u^{n+1}(t,z) \in H^s$ satisfying relation $(3.1.3)_2$ and the following bound:*

$$\sup_{0\le t\le T}\|\partial_z u^{n+1}(t,z)\|_{H^s} < \frac{\sqrt{\varepsilon(z)}}{2}, \quad \textit{for each} \quad z \in \Omega.$$

*Proof.* Since the proof is similar to that of Lemma 3.1.1, we leave it to Appendix A.1. □

**Remark 3.1.1.** *From Lemmas 3.1.1 and 3.1.2, one can find out that if assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$ hold, the induction on $n$ yields that for every $n$ and $z \in \Omega$:*

$$\sup_{0\le t\le T}(\|\partial_z \rho^n(t,z)\|_{H^{s-1}} + \|\partial_z u^n(t,z)\|_{H^s}) < \sqrt{\varepsilon(z)}.$$

Now, we provide estimates for the convergence of the sequence $(\partial_z \rho^n, \partial_z u^n)$ in $L^2 \times H^1$.

**Lemma 3.1.3.** *Suppose that assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold. Then, for each $z \in \Omega$ and $n \in \mathbb{N}$,*

$$\begin{aligned}
&\|(\partial_z \rho^{n+1} - \partial_z \rho^n)(t,z)\|^2_{L^2} + \|(\partial_z u^{n+1} - \partial_z u^n)(t,z)\|^2_{H^1} \\
&\quad \le C(z)\Bigg(\int_0^t \left(\|(\partial_z \rho^{n+1} - \partial_z \rho^n)(s,z)\|^2_{L^2} + \|(\partial_z u^{n+1} - \partial_z u^n)(s,z)\|^2_{H^1}\right) ds \\
&\qquad + \int_0^t \|(\partial_z u^n - \partial_z u^{n-1})(s,z)\|^2_{H^1} ds\Bigg),
\end{aligned}$$

*where $C = C(z)$ is a positive random function independent of $n$.*

*Proof.* It follows from $(3.1.3)_1$ that

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z\rho^{n+1}-\partial_z\rho^n\|_{L^2}^2\\
&\quad=-\frac{1}{2}\int_{\mathbb{T}^d}\nabla|\partial_z\rho^{n+1}-\partial_z\rho^n|^2\cdot u\ dx\\
&\qquad-\int_{\mathbb{T}^d}\nabla\cdot(\rho(\partial_z u^n-\partial_z u^{n-1}))(\partial_z\rho^{n+1}-\partial_z\rho^n)dx\\
&\quad\le\frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\partial_z\rho^{n+1}-\partial_z\rho^n\|_{L^2}^2\\
&\qquad+\|\rho\|_{W^{1,\infty}}\|\partial_z u^n-\partial_z u^{n-1}\|_{H^1}\|\partial_z\rho^{n+1}-\partial_z\rho^n\|_{L^2}\\
&\quad\le C(\|\partial_z\rho^{n+1}-\partial_z\rho^n\|_{L^2}^2+\|\partial_z u^n-\partial_z u^{n-1}\|_{H^1}^2).
\end{aligned}
$$

We integrate the above relation to see

$$
\begin{aligned}
&\|(\partial_z\rho^{n+1}-\partial_z\rho^n)(t,z)\|_{L^2}^2\\
&\quad\le C(z)\int_0^t\Big[\|(\partial_z\rho^{n+1}-\partial_z\rho^n)(s,z)\|_{L^2}^2+\|(\partial_z u^n-\partial_z u^{n-1})(s,z)\|_{H^1}^2\Big]ds.
\end{aligned}
\tag{3.1.8}
$$

Next, one uses $(3.1.3)_2$ to yield

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z u^{n+1}-\partial_z u^n\|_{L^2}^2\\
&=-\int_{\mathbb{T}^d}(\partial_z u^n-\partial_z u^{n-1})\cdot\nabla u\cdot(\partial_z u^{n+1}-\partial_z u^n)dx\\
&\quad-\frac{1}{2}\int_{\mathbb{T}^d}u\cdot\nabla|\partial_z u^{n+1}-\partial_z u^n|^2dx\\
&\quad+\int_{\mathbb{T}^{2d}}\phi(x-y)\partial_z\left[\begin{array}{c}(u^n-u^{n-1})(y)\\-(u^n-u^{n-1})(x)\end{array}\right]\rho(y)\partial_z(u^{n+1}-u^n)(x)dydx\\
&\quad+\int_{\mathbb{T}^{2d}}\phi(x-y)(u(y)-u(x))(\partial_z\rho^{n+1}-\partial_z\rho^n)(y)(\partial_z u^{n+1}-\partial_z u^n)(x)dydx\\
&\le\|\nabla u\|_{L^\infty}\|\partial_z u^n-\partial_z u^{n-1}\|_{L^2}\|\partial_z u^{n+1}-\partial_z u^n\|_{L^2}\\
&\quad+\frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\partial_z u^{n+1}-\partial_z u^n\|_{L^2}^2\\
&\quad+2\|\phi\|_s\|\rho\|_{L^2}\|\partial_z u^n-\partial_z u^{n-1}\|_{L^2}\|\partial_z u^{n+1}-\partial_z u^n\|_{L^2}\\
&\quad+2\|\phi\|_s\|u\|_{L^2}\|\partial_z\rho^{n+1}-\partial_z\rho^n\|_{L^2}\|\partial_z u^{n+1}-\partial_z u^n\|_{L^2}.
\end{aligned}
$$

We use Young's inequality and integrate the previous relation over $[0,t]$ to get

$$\begin{aligned}
&\|(\partial_z u^{n+1}-\partial_z u^n)(t,z)\|_{L^2}^2\\
\le& C(z)\Bigg(\int_0^t\Big[\|(\partial_z\rho^{n+1}-\partial_z\rho^n)(s,z)\|_{L^2}^2+\|(\partial_z u^{n+1}-\partial_z u^n)(s,z)\|_{L^2}^2\Big]ds\\
&\qquad+\int_0^t\|(\partial_z u^n-\partial_z u^{n-1})(s,z)\|_{L^2}^2 ds\Bigg).
\end{aligned}\tag{3.1.9}$$

For the $H^1$-estimate for $(\partial_z u^{n+1}-\partial_z u^n)$, we use the Cauchy-Schwarz inequality to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla(\partial_z u^{n+1}-\partial_z u^n)\|_{L^2}^2\\
&=-\int_{\mathbb{T}^d}\nabla((\partial_z u^n-\partial_z u^{n-1})\cdot\nabla u):\nabla(\partial_z u^{n+1}-\partial_z u^n)dx\\
&\quad-\int_{\mathbb{T}^d}\nabla(u\cdot\nabla(\partial_z u^{n+1}-\partial_z u^n)):\nabla(\partial_z u^{n+1}-\partial_z u^n)dx\\
&\quad+\int_{\mathbb{T}^{2d}}\nabla\phi(x-y)\partial_z\begin{bmatrix}(u^n-u^{n-1})(y)\\-(u^n-u^{n-1})(x)\end{bmatrix}\rho(y)\nabla(\partial_z u^{n+1}-\partial_z u^n)(x)dydx\\
&\quad-\int_{\mathbb{T}^{2d}}\phi(x-y)\nabla(\partial_z u^n-\partial_z u^{n-1})(x)\rho(y):\nabla(\partial_z u^{n+1}-\partial_z u^n)(x)dydx\\
&\quad+\int_{\mathbb{T}^{2d}}\nabla\{\phi(x-y)(u(y)-u(x))\}\begin{bmatrix}(\partial_z\rho^{n+1}-\partial_z\rho^n)(y)\\:\nabla(\partial_z u^{n+1}-\partial_z u^n)(x)\end{bmatrix}dydx\\
&\le\|\partial_z u^n-\partial_z u^{n-1}\|_{H^1}\|u\|_{W^{2,\infty}}\|\nabla(\partial_z u^{n+1}-\partial_z u^n)\|_{L^2}\\
&\quad+\frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\nabla(\partial_z u^{n+1}-\partial_z u^n)\|_{L^2}^2\\
&\quad+2\|\phi\|_s\|\rho\|_{L^2}\|\partial_z u^n-\partial_z u^{n-1}\|_{H^1}\|\nabla(\partial_z u^{n+1}-\partial_z u^n)\|_{L^2}\\
&\quad+2\|\phi\|_s\|u\|_{H^1}\|\partial_z\rho^{n+1}-\partial_z\rho^n\|_{L^2}\|\nabla(\partial_z u^{n+1}-\partial_z u^n)\|_{L^2}.
\end{aligned}$$

Again, using Young's inequality and integration along $[0,t]$ give

$$
\begin{aligned}
&\|\nabla(\partial_z u^{n+1} - \partial_z u^n)(t,z)\|_{L^2}^2 \\
\le& C(z)\Bigg(\int_0^t \Big[\|(\partial_z\rho^{n+1} - \partial_z\rho^n)(s,z)\|_{L^2}^2 + \|(\partial_z u^{n+1} - \partial_z u^n)(s,z)\|_{H^1}^2\Big] ds \\
&\qquad + \int_0^t \|\nabla(\partial_z u^{n+1} - \partial_z u^n)(s,z)\|_{L^2}^2 ds\Bigg).
\end{aligned}
\tag{3.1.10}
$$

Finally, one combines (3.1.8), (3.1.9) and (3.1.10) to yield the desired result. □

Now, we are ready to state our first result on the well-posedness of a global solution to (3.1.3).

**Theorem 3.1.1.** *Suppose that assumptions* ($\mathcal{A}1$)-($\mathcal{A}2$) *hold. Then for each* $z \in \Omega$*, there exists a unique solution* $(\partial_z\rho(z), \partial_z u(z)) \in H^{s-1} \times H^s$ *satisfying system* (3.1.3) *and uniform bound estimates:*

$$
\sup_{0\le t\le T} (\|\partial_z\rho(t,z)\|_{H^{s-1}} + \|\partial_z u(t,z)\|_{H^s}) < \sqrt{\varepsilon(z)}, \quad \textit{for each} \quad z \in \Omega.
$$

*Proof.* For each $n \in \mathbb{N}$ and $z \in \Omega$, define

$$
\Delta_n(t,z) := \|\partial_z\rho^n - \partial_z\rho^{n-1}\|_{L^2}^2 + \|\partial_z u^n - \partial_z u^{n-1}\|_{H^1}^2.
$$

We can deduce from Lemma 3.1.3 that for each $z \in \Omega$,

$$
\Delta_{n+1}(t,z) \le C(z)\left(\int_0^t \Delta_{n+1}(s,z)ds + \int_0^t \Delta_n(s,z)ds\right), \quad t \in [0,T].
$$

Then, the Grönwall-type lemma in [6] gives, for each $z \in \Omega$,

$$
\sup_{0\le t\le T}\left(\|(\partial_z\rho^n - \partial_z\rho^{n-1})(t,z)\|_{L^2}^2 + \|(\partial_z u^n - \partial_z u^{n-1})(t,z)\|_{H^1}^2\right) \le \frac{(C(z)T)^n}{n!}.
$$

This implies that $\{\partial_z\rho^n\}$ and $\{\partial_z u^n\}$ are Cauchy sequences in $C([0,T];L^2)$ and $C([0,T];H^1)$, respectively. From here, one can follow the proof of Theorem 3.1 in [48] to complete the proof. □

### 3.1.2 Higher-order $z$-variations

In this subsection, we consider higher-order $z$-variations, i.e. the case when $m \geq 2$ in (3.1.2), in order to complete the induction process on $m$. Similar to the case $m = 1$, we again construct a sequence of approximated solutions $(\partial_z^m \rho^{n+1}, \partial_z^m u^{n+1})$ to (3.1.2) as follows:

$$
\begin{aligned}
&\partial_t(\partial_z^m \rho^{n+1}) + \nabla \cdot (\partial_z^m \rho^{n+1} u) + \nabla \cdot (\rho \partial_z^m u^n) \\
&\qquad + \sum_{1 \leq l \leq m-1} \binom{m}{l} \nabla \cdot (\partial_z^l \rho \partial_z^{m-l} u) = 0,
\end{aligned} \tag{3.1.11}
$$

$$
\begin{aligned}
&\partial_t(\partial_z^m u^{n+1}) + \partial_z^m u^n \cdot \nabla u + u \cdot \nabla(\partial_z^m u^{n+1}) \\
&\qquad + \sum_{1 \leq l \leq m-1} \binom{m}{l} \partial_z^l u \cdot \nabla(\partial_z^{m-1} u) \\
&= \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma \neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^d} \partial_z^\alpha \phi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x)) \partial_z^\gamma \rho(y) dy \\
&\quad + \int_{\mathbb{T}^d} \phi(x-y)(\partial_z^m u^n(y) - \partial_z^m u^n(x)) \rho(y) dy \\
&\quad + \int_{\mathbb{T}^d} \phi(x-y)(u(y) - u(x)) \partial_z^m \rho^{n+1}(y) dy, \\
&(\partial_z^m \rho^0, \partial_z^m u^0) = (\partial_z^m \rho_0, \partial_z^m u_0),
\end{aligned} \tag{3.1.12}
$$

subject to the initial data:

$$
(\partial_z^m \rho^{n+1}(0,z), \partial_z^m u^{n+1}(0,z)) = (\partial_z^m \rho_0(z), \partial_z^m u_0(z)).
$$

Similar to the previous subsection, we first show the uniform boundedness of the sequence $\{(\partial_z^m \rho^n, \partial_z^m u^n)\}_{n=0}^\infty$.

**Lemma 3.1.4.** *For $m \geq 2$ and $n \in \mathbb{N}$, suppose that the following conditions hold:*

1. *Assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold.*
2. *For $l \leq m-1$, the $l$-th $z$-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:*

$$
\max_{0 \leq l \leq m-1} \sup_{0 \leq t \leq T} \left( \|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}} \right) < \sqrt{\varepsilon(z)}, \quad \forall z \in \Omega.
$$

3. *The sequence approximating the m-th z-variation of the bulk velocity process satisfies the following boundedness condition:*

$$\max_{0\le j\le n}\sup_{0\le t\le T}\|\partial_z^m u^j(t,z)\|_{H^{s-m+1}} < \sqrt{\varepsilon(z)}, \quad \forall z\in\Omega.$$

*Then, there exists a unique* $\partial_z^m\rho^{n+1}\in H^{s-m}$ *which satisfies relation* (3.1.11) *and the bound:*

$$\sup_{0\le t\le T}\|\partial_z^m\rho^{n+1}(t,z)\|_{H^{s-m}} < \frac{\sqrt{\varepsilon(z)}}{2}, \quad \forall z\in\Omega.$$

*Proof.* We split the estimates into zeroth-order and higher-order cases as follows:

• Step A (The zeroth-order estimates): We multiply (3.1.11) by $\partial_z^m\rho^{n+1}$ and integrate the resulting relation over $\mathbb{T}^d$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z^m\rho^{n+1}\|_{L^2}^2\\
&\quad= -\frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\partial_z^m\rho^{n+1}|^2dx - \int_{\mathbb{T}^d}\nabla\cdot(\rho\partial_z^m u^n)\partial_z^m\rho^{n+1}dx\\
&\qquad-\sum_{1\le l\le m-1}\binom{m}{l}\int_{\mathbb{T}^d}\nabla\cdot(\partial_z^l\rho\partial_z^{m-l}u)\partial_z^m\rho^{n+1}dx\\
&\quad\le \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\partial_z^m\rho^{n+1}\|_{L^2}^2 + \|\rho\|_{W^{1,\infty}}\|\partial_z^m u^n\|_{H^1}\|\partial_z^m\rho^{n+1}\|_{L^2}\\
&\qquad+\sum_{1\le l\le m-1}\binom{m}{l}\|\partial_z^l\rho\|_{W^{1,\infty}}\|\partial_z^{m-l}u\|_{H^1}\|\partial_z^m\rho^{n+1}\|_{L^2}\\
&\quad\le C(\varepsilon^{1/2}\|\partial_z^m\rho^{n+1}\|_{L^2}^2+\varepsilon^{3/2}),
\end{aligned}$$

where $C$ is a positive constant independent of $n$ and we used Young's inequality. Integrating the above relation along $[0,t]$ gives, for each $z\in\Omega$,

$$\|\partial_z^m\rho^{n+1}\|_{L^2}^2 \le C\left(\varepsilon^{1/2}\int_0^t\|\partial_z^m\rho^{n+1}(s)\|_{L^2}^2ds+\varepsilon^{3/2}\right). \tag{3.1.13}$$

• Step B (Higher-order estimates): For $1\le k\le s-m$, we apply $\nabla^k$ to (3.1.11), multiply by $\nabla^k(\partial_z^m\rho^{n+1})$ and integrate the resulting relation over

$\mathbb{T}^d$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^d} \nabla^k(\nabla\cdot(\partial_z^m \rho^{n+1} u))\nabla^k(\partial_z^m \rho^{n+1})dx \\
&\quad -\int_{\mathbb{T}^d} \nabla^k(\nabla\cdot(\rho \partial_z^m u^n)\nabla^k(\partial_z^m \rho^{n+1}))dx \\
&\quad -\sum_{1\le l\le m-1}\binom{m}{l}\int_{\mathbb{T}^d} \nabla^k(\nabla\cdot(\partial_z^l \rho \partial_z^{m-l} u))\nabla^k(\partial_z^m \rho^{n+1})dx \\
&=: \sum_{i=1}^{3}\mathcal{I}_{2i}.
\end{aligned}$$

We separately estimate $\mathcal{I}_{2i}$'s as follows.

◇ (Estimate for $\mathcal{I}_{21}$): We use the commutator estimate, Sobolev embedding theorem, Cauchy-Schwarz inequality and Young's inequality to get

$$\begin{aligned}
\mathcal{I}_{21} &= \frac{1}{2}\int_{\mathbb{T}^d} |\nabla^k(\partial_z^m \rho^{n+1})|^2(\nabla\cdot u)dx \\
&\quad -\int_{\mathbb{T}^d}\Big[\nabla^k(u\cdot\nabla(\partial_z^m \rho^{n+1})) - u\cdot\nabla^k(\nabla(\partial_z^m \rho^{n+1}))\Big]\nabla^k(\partial_z^m \rho^{n+1})dx \\
&\quad -\int_{\mathbb{T}^d} \partial_z^m \rho^{n+1}\nabla^k(\nabla\cdot u)\nabla^k(\partial_z^m \rho^{n+1})dx \\
&\quad -\int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z^m \rho^{n+1}(\nabla\cdot u)) - \partial_z^m \rho^{n+1}\nabla^k(\nabla\cdot u)\Big]\nabla^k(\partial_z^m \rho^{n+1})dx \\
&\le \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2}^2 \\
&\quad + c\Big[\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2} + \|\nabla(\partial_z^m \rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}\Big]\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2} \\
&\quad + \|\nabla^{k+1} u\|_{L^2}\|\partial_z^m \rho^{n+1}\|_{L^2}\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2} \\
&\quad + c\Big[\|\nabla(\partial_z^m \rho^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2} + \|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2}\Big]\|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\partial_z^m \rho^{n+1}\|_{H^{s-m}}^2 + \varepsilon^{3/2}),
\end{aligned}$$

where $c$ and $C$ are positive random functions independent of $n$.

$\diamond$ (Estimate for $\mathcal{I}_{22}$) : Similar to the previous case,

$$
\begin{aligned}
\mathcal{I}_{22} = &-\int_{\mathbb{T}^d} \nabla^k(\nabla\rho)\cdot\partial_z^m u^n \nabla^k(\partial_z^m\rho^{n+1})dx \\
&-\int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z^m u^n\cdot\nabla\rho)-\partial_z^m u^n\cdot\nabla^k(\nabla\rho)\Big]\nabla^k(\partial_z^m\rho^{n+1})dx \\
&-\int_{\mathbb{T}^d}\rho\nabla^k(\nabla\cdot\partial_z^m u^n)\nabla^k(\partial_z^m\rho^{n+1})dx \\
&-\int_{\mathbb{T}^d}\Big[\nabla^k(\rho(\nabla\cdot\partial_z^m u^n))-\rho\nabla^k(\nabla\cdot\partial_z^m u^n)\Big]\nabla^k(\partial_z^m\rho^{n+1})dx \\
\le\ & \|\nabla^{k+1}\rho\|_{L^2}\|\partial_z^m u^n\|_{L^\infty}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&+c\big[\|\nabla(\partial_z^m u^n)\|_{L^\infty}\|\nabla^k\rho\|_{L^2}+\|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z^m u^n)\|_{L^2}\big]\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&+\|\rho\|_{L^\infty}\|\nabla^{k+1}(\partial_z^m u^n)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&+c\big[\|\nabla\rho\|_{L^\infty}\|\nabla^k(\partial_z^m u^n)\|_{L^2}+\|\nabla\cdot\partial_z^m u^n\|_{L^\infty}\|\nabla^k\rho\|_{L^2}\big]\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
\le\ & C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}^2+\varepsilon^{3/2}),
\end{aligned}
$$

where $c$ and $C$ are positive random functions independent of $n$.

$\diamond$ (Estimates for $\mathcal{I}_{23}$): By direct calculation,

$$
\begin{aligned}
\mathcal{I}_{23} = -\sum_{1\le l\le m-1}\binom{m}{l}\Bigg\{&\int_{\mathbb{T}^d}\nabla^k(\nabla(\partial_z^l\rho))\cdot\partial_z^{m-l}u\nabla^k(\partial_z^m\rho^{n+1})dx \\
&+\int_{\mathbb{T}^d}\left[\begin{array}{c}\nabla^k(\partial_z^{m-l}u\cdot\nabla(\partial_z^l\rho)) \\ -\partial_z^{m-l}u\nabla^k(\nabla(\partial_z^l\rho))\end{array}\right]\nabla^k(\partial_z^m\rho^{n+1})dx \\
&+\int_{\mathbb{T}^d}\partial_z^l\rho\nabla^k(\nabla\cdot\partial_z^{m-l}u)\nabla^k(\partial_z^m\rho^{n+1})dx \\
&+\int_{\mathbb{T}^d}\left[\begin{array}{c}\nabla^k(\partial_z^l\rho(\nabla\cdot\partial_z^{m-l}u)) \\ -\partial_z^l\rho\nabla^k(\nabla\cdot\partial_z^{m-l}u)\end{array}\right]\nabla^k(\partial_z^m\rho^{n+1})dx\Bigg\} \\
\le C\sum_{1\le l\le m-1}\Bigg(&\|\nabla^{k+1}\partial_z^l\rho\|_{L^2}\|\partial_z^{m-l}u\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&+\|\nabla(\partial_z^{m-l}u)\|_{L^\infty}\|\nabla^k(\partial_z^l\rho)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&+\|\nabla(\partial_z^l\rho)\|_{L^\infty}\|\nabla^k(\partial_z^{m-l}u)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2} \\
&+\|\partial_z^l\rho\|_{L^\infty}\|\nabla^{k+1}(\partial_z^{m-l}u)\|_{L^2}\|\nabla^k(\partial_z^m\rho^{n+1})\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \|\nabla(\partial_z^l \rho)\|_{L^\infty} \|\nabla^k(\partial_z^{m-l} u)\|_{L^2} \|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2} \\
&\qquad + \|\nabla \cdot \partial_z^{m-l} u\|_{L^\infty} \|\nabla^k(\partial_z^l \rho)\|_{L^2} \|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2} \Big) \\
&\le C(\varepsilon^{1/2} \|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}
$$

where $C$ is a positive random function independent of $n$.

Now, we gather all the results for $\mathcal{I}_{2i}$'s to yield that for each $z \in \Omega$,

$$
\frac{\partial}{\partial t} \|\nabla^k(\partial_z^m \rho^{n+1})\|_{L^2}^2 \le C(\varepsilon^{1/2} \|\partial_z^m \rho^{n+1}\|_{H^{s-m}}^2 + \varepsilon^{3/2}). \tag{3.1.14}
$$

Summing (3.1.14) over $1 \le k \le s-m$, integrating over $[0,t]$ and combining with (3.1.13) give

$$
\|\partial_z^m \rho^{n+1}\|_{H^{s-m}}^2 \le C \left( \varepsilon^{1/2} \int_0^t \|\partial_z^m \rho^{n+1}(s)\|_{H^{s-m}}^2 ds + \varepsilon^{3/2} \right).
$$

Finally, one can use Grönwall's lemma to obtain the desired result.

□

**Lemma 3.1.5.** *For $m \ge 2$ and $n \in \mathbb{N}$, suppose that the following conditions hold:*

1. *Assumptions ($\mathcal{A}1$)-($\mathcal{A}2$) hold.*
2. *For $l \le m-1$, the $l$-th $z$-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:*

$$
\max_{0 \le l \le m-1} \sup_{0 \le t \le T} \left( \|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}} \right) < \sqrt{\varepsilon(z)}, \quad \forall z \in \Omega.
$$

3. *The sequence approximating the $m$-th $z$-variation of the local mass and bulk velocity processes satisfies the following boundedness condition:*

$$
\max_{0 \le j \le n} \sup_{0 \le t \le T} \left( \|\partial_z^m \rho^j(t,z)\|_{H^{s-m}} + \|\partial_z^m u^j(t,z)\|_{H^{s-m+1}} \right) < \sqrt{\varepsilon(z)}.
$$

*Then for each $z \in \Omega$, there exists a unique $\partial_z^m u^{n+1} = \partial_z^m u^{n+1}(t,z) \in H^{s-m+1}$ satisfying relation (3.1.12) and the bound:*

$$
\sup_{0 \le t \le T} \|\partial_z^m u^{n+1}(t,z)\|_{H^{s-m+1}} < \frac{\sqrt{\varepsilon(z)}}{2}, \quad \forall z \in \Omega.
$$

*Proof.* We leave its proof to Appendix A.2. □

**Remark 3.1.2.** *For $m \geq 2$ and $n \in \mathbb{N}$, suppose that the following conditions hold:*

1. *Assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold.*
2. *For $l \leq m-1$, the $l$-th $z$-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:*

$$\max_{0\leq l\leq m-1} \sup_{0\leq t\leq T} \big(\|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}}\big) < \sqrt{\varepsilon(z)}, \quad \forall z \in \Omega.$$

*Then, it follows from Lemmas 3.1.4 and 3.1.5, that for every $n, m \in \mathbb{N}$ and $z \in \Omega$:*

$$\sup_{0\leq t\leq T} (\|\partial_z^m \rho^n(t,z)\|_{H^{s-m}} + \|\partial_z^m u^n(t,z)\|_{H^{s-m+1}}) < \sqrt{\varepsilon(z)}, \quad \forall z \in \Omega.$$

Now, we assert that the sequence is Cauchy under the induction hypothesis on $m$.

**Lemma 3.1.6.** *For $m \geq 2$ and $n \in \mathbb{N}$, suppose that the following conditions hold:*

1. *Assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$ hold.*
2. *For $l \leq m-1$, the $l$-th $z$-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^{m-1}$ satisfy the following boundedness condition:*

$$\max_{0\leq l\leq m-1} \sup_{0\leq t\leq T} \big(\|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}}\big) < \sqrt{\varepsilon(z)}, \quad \forall z \in \Omega.$$

*Then, for each $z \in \Omega$,*

$$\begin{aligned}
&\|(\partial_z^m \rho^{n+1} - \partial_z^m \rho^n)(t,z)\|_{L^2}^2 + \|(\partial_z^m u^{n+1} - \partial_z^m u^n)(t,z)\|_{H^1}^2 \\
&\leq C(z)\Bigg( \int_0^t \big(\|(\partial_z^m \rho^{n+1} - \partial_z^m \rho^n)(s,z)\|_{L^2}^2 + \|(\partial_z^m u^{n+1} - \partial_z u^n)(s,z)\|_{H^1}^2\big)\, ds \\
&\qquad\qquad + \int_0^t \|(\partial_z^m u^{n+1} - \partial_z^m u^n)(s,z)\|_{H^1}^2 ds \Bigg),
\end{aligned}$$

*where $C = C(z)$ is a positive random function independent of $n$.*

*Proof.* We can replace $\partial_z$ in the proof of Lemma 3.1.3 by $\partial_z^m$ to get the desired proof. The details will be omitted. $\square$

Finally, we are ready to present our result on the well-posedness.

**Theorem 3.1.2.** *Suppose that assumptions* $(\mathcal{A}1)$*-*$(\mathcal{A}2)$ *hold. Then for each* $m \in \mathbb{N}$ *and* $z \in \Omega$*, there exists a unique pair* $(\partial_z^m \rho(z), \partial_z^m u(z)) \in H^{s-m} \times H^{s-m+1}$ *satisfying system* (3.1.2) *and the following uniform bound estimates:*

$$\sup_{0 \le t \le T} (\|\partial_z^m \rho(t,z)\|_{H^{s-m}} + \|\partial_z^m u(t,z)\|_{H^{s-m+1}}) < \sqrt{\varepsilon(z)}, \quad \forall z \in \Omega.$$

*Proof.* One can use induction on $m$, Lemma 3.1.6 and follow the proof of Theorem 3.1.1 to show that $\{(\partial_z^m \rho^n(z), \partial_z^m u^n(z))\}_{n=0}^{\infty}$ is a Cauchy sequence in $C([0,T]; L^2) \times C([0,T]; H^1)$ for each $z \in \Omega$. From here, we again refer to [48] to complete the rest of the proof. $\square$

## 3.2 The local sensitivity analysis for stability estimates

In this section, we conduct a local sensitivity analysis for the $L^2$-stability estimates of the solution processes to (3.0.1) and their $z$-variations.

### 3.2.1 Higher-order $L^2$-stability

In this subsection, we derive a higher-order $L^2$-stability estimate of solution processes to (3.0.1) which will be used in the $L^2$-stability of the $z$-variations. First, we begin with the $L^2$-stability estimate for the local mass processes.

**Lemma 3.2.1.** *Suppose that assumptions* $(\mathcal{A}1)$*-*$(\mathcal{A}2)$ *hold, and let* $(\rho, u)$ *and* $(\bar{\rho}, \bar{u})$ *be two classical solution processes to* (3.0.1) *corresponding to the initial data* $(\rho_0, u_0)$ *and* $(\bar{\rho}_0, \bar{u}_0)$*, respectively. Then,*

$$\frac{\partial}{\partial t}\|(\rho - \bar{\rho})(t,z)\|_{H^m}^2 \le C(T,z)(\|(\rho - \bar{\rho})(t,z)\|_{H^m}^2 + \|(u - \bar{u})(t,z)\|_{H^{m+1}}^2),$$

*where* $C = C(T,z)$ *is a positive random function.*

*Proof.* It follows from Theorem 2.2.3 that

$$(\rho, u), (\bar{\rho}, \bar{u}) \in H^s \times H^{s+1} \quad \text{with } s > \frac{d}{2} + m + 1.$$

Since the proof for the case $m = 0$ is analogous to the higher-order case, we only consider the higher-order estimates. So we first apply $\nabla^k$ to $(3.0.1)_1$ for $1 \le k \le m$ to get

$$\partial_t \nabla^k (\rho - \bar{\rho}) + \nabla^k \nabla \cdot ((\rho - \bar{\rho})\bar{u} + \rho(u - \bar{u})) = 0. \tag{3.2.1}$$

Then, we multiply (3.2.1) by $\nabla^k(\rho - \bar{\rho})$ and integrate the resulting relation over $\mathbb{T}^d$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\rho - \bar{\rho})\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})\nabla^k[\nabla \cdot (\rho(u - \bar{u})) + \nabla \cdot ((\rho - \bar{\rho})\bar{u})]dx \\
&\quad = -\sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[\nabla^r(\nabla\rho) \cdot (\nabla^{k-r}(u - \bar{u}))]dx \\
&\qquad - \sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[(\nabla^r \rho)(\nabla^{k-r}(\nabla \cdot (u - \bar{u})))]dx \\
&\qquad - \sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[\nabla^r(\nabla(\rho - \bar{\rho})) \cdot (\nabla^{k-r}\bar{u})]dx \\
&\qquad - \sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[(\nabla^r(\rho - \bar{\rho}))(\nabla^{k-r}(\nabla \cdot \bar{u}))]dx \\
&\quad =: \sum_{i=1}^{4} \mathcal{I}_{3i}.
\end{aligned}$$

Next, we estimate $\mathcal{I}_{3i}$'s one by one as follows:

$\diamond$ (Estimates for $\mathcal{I}_{31}$) : We use the Sobolev embedding theorem to obtain

$$\mathcal{I}_{31} = -\sum_{0 \le r \le k} \binom{k}{r} \int_{\mathbb{T}^d} \nabla^k(\rho - \bar{\rho})[\nabla^r(\nabla\rho) \cdot (\nabla^{k-r}(u - \bar{u}))]dx$$

$$
\begin{aligned}
&\le \sum_{0\le r\le k}\binom{k}{r}\|\nabla^{r+1}\rho\|_{L^\infty}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{k-r}(u-\bar{u})\|_{L^2}\\
&\le C\sum_{0\le r\le k}\binom{k}{r}\|\rho\|_{H^s}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{k-r}(u-\bar{u})\|_{L^2}\\
&\le C(T,z)(\|\nabla^k(\rho-\bar{\rho})\|_{L^2}^2+\|u-\bar{u}\|_{H^k}^2).
\end{aligned}
$$

◇ (Estimates for $\mathcal{I}_{32}$) : Similarly,

$$
\begin{aligned}
\mathcal{I}_{32} &= -\sum_{0\le r\le k}\binom{k}{r}\int_{\mathbb{T}^d}\nabla^k(\rho-\bar{\rho})[(\nabla^r\rho)(\nabla^{k-r}(\nabla\cdot(u-\bar{u})))]dx\\
&\le \sum_{0\le r\le k}\binom{k}{r}\|\nabla^r\rho\|_{L^\infty}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{k-r+1}(u-\bar{u})\|_{L^2}\\
&\le C\sum_{0\le r\le k}\binom{k}{r}\|\rho\|_{H^{s-1}}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{k-r+1}(u-\bar{u})\|_{L^2}\\
&\le C(T,z)(\|\nabla^k(\rho-\bar{\rho})\|_{L^2}^2+\|u-\bar{u}\|_{H^{k+1}}^2).
\end{aligned}
$$

◇ (Estimates for $\mathcal{I}_{33}$) : One has

$$
\begin{aligned}
\mathcal{I}_{33} &= -\sum_{0\le r\le k}\binom{k}{r}\int_{\mathbb{T}^d}\nabla^k(\rho-\bar{\rho})[\nabla^r(\nabla(\rho-\bar{\rho}))\cdot(\nabla^{k-r}\bar{u})]dx\\
&\le \sum_{0\le r\le k-1}\binom{k}{r}\|\nabla^{k-r}\bar{u}\|_{L^\infty}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{r+1}(\rho-\bar{\rho})\|_{L^2}\\
&\quad+\frac{1}{2}\int_{\mathbb{T}^d}|\nabla^k(\rho-\bar{\rho})|^2(\nabla\cdot\bar{u})dx\\
&\le C\sum_{0\le r\le k-1}\binom{k}{r}\|\bar{u}\|_{H^{s-1}}\|\nabla^k(\rho-\bar{\rho})\|_{L^2}\|\nabla^{r+1}(\rho-\bar{\rho})\|_{L^2}\\
&\quad+C\|\nabla^k(\rho-\bar{\rho})\|_{L^2}^2\|\bar{u}\|_{H^{s-m}}\\
&\le C(T,z)\|\rho-\bar{\rho}\|_{H^k}^2.
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{34}$) : We have

$$\begin{aligned}
\mathcal{I}_{34} &= -\sum_{0\le r\le k}\binom{k}{r}\int_{\mathbb{T}^d}\nabla^k(\rho-\bar\rho)[(\nabla^r(\rho-\bar\rho))(\nabla^{k-r}(\nabla\cdot\bar u))]dx \\
&\le \sum_{0\le r\le k}\binom{k}{r}\|\nabla^{k-r+1}\bar u\|_{L^\infty}\|\nabla^k(\rho-\bar\rho)\|_{L^2}\|\nabla^r(\rho-\bar\rho)\|_{L^2} \\
&\le C\sum_{0\le r\le k}\binom{k}{r}\|\bar u\|_{H^s}\|\nabla^k(\rho-\bar\rho)\|_{L^2}\|\nabla^r(\rho-\bar\rho)\|_{L^2} \\
&\le C(T,z)\|\rho-\bar\rho\|_{H^k}^2.
\end{aligned}$$

By collecting all results for $\mathcal{I}_{3i}$'s, summing over $1\le k\le m$ and combining with lower-order estimates, one gets the desired estimate. □

Next, we return to the $L^2$-stability of the bulk velocity processes.

**Lemma 3.2.2.** *Suppose that assumptions* ($\mathcal{A}1$)-($\mathcal{A}2$) *hold, and let* $(\rho,u)$ *and* $(\bar\rho,\bar u)$ *be two classical solution processes to* (3.0.1) *corresponding to the initial data* $(\rho_0,u_0)$ *and* $(\bar\rho_0,\bar u_0)$*, respectively. Then,*

$$\frac{\partial}{\partial t}\|(u-\bar u)(t,z)\|_{H^{m+1}}^2\le C(T,z)(\|(u-\bar u)(t,z)\|_{H^{m+1}}^2+\|(\rho-\bar\rho)(t,z)\|_{L^2}^2),$$

*where* $C=C(T,z)$ *is a positive random function.*

*Proof.* As in the proof of Lemma 3.2.1, we only consider the higher-order estimates. Applying $\nabla^k$ to $(3.0.1)_2$ for $1\le k\le m+1$ gives

$$\begin{aligned}
&\partial_t\nabla^k(u-\bar u)+\nabla^k((u-\bar u)\cdot\nabla u)+\nabla^k(\bar u\cdot\nabla(u-\bar u)) \\
&\quad=\nabla^k\int_{\mathbb{T}^d}\phi(x-y,z)\left[\begin{array}{c}(u(y)-u(x))(\rho(y)-\bar\rho(y))\\ +\bar\rho(y)(u(y)-\bar u(y))-\bar\rho(y)(u(x)-\bar u(x))\end{array}\right]dy.
\end{aligned}$$

Then, we use commutator estimates, Sobolev embedding and Young's inequality to get

$$\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(u-\bar u)\|_{L^2}^2$$

$$
\begin{aligned}
&= -\int_{\mathbb{T}^d} \nabla^k[(u-\bar{u})\cdot\nabla u]\nabla^k(u-\bar{u})dx \\
&\quad - \int_{\mathbb{T}^d} \nabla^k[\bar{u}\cdot\nabla(u-\bar{u})]\nabla^k(u-\bar{u})dx \\
&\quad + \int_{\mathbb{T}^{2d}} \nabla^k\left[\phi(x-y,z)(u(y)-u(x))(\rho(y)-\bar{\rho}(y))\right]\nabla^k(u(x)-\bar{u}(x))dydx \\
&\quad + \int_{\mathbb{T}^{2d}} \nabla^k\left[\phi(x-y,z)\bar{\rho}(y)(u(y)-\bar{u}(y))\right]\nabla^k(u(x)-\bar{u}(x))dydx \\
&\quad - \int_{\mathbb{T}^{2d}} \nabla^k\left[\phi(x-y,z)\bar{\rho}(y)(u(x)-\bar{u}(x))\right]\nabla^k(u(x)-\bar{u}(x))dydx \\
&\le C\|u\|_{H^s}\|u-\bar{u}\|_{H^k}^2 + C\|\phi\|_s\|u\|_{H^s}(\|\rho-\bar{\rho}\|_{L^2}^2 + \|\nabla^k(u-\bar{u})\|_{L^2}^2) \\
&\quad + C\|\phi\|_s\|\bar{\rho}\|_{L^2}\|u-\bar{u}\|_{H^k}^2 \\
&\le C(T,z)(\|u-\bar{u}\|_{H^k}^2 + \|\rho-\bar{\rho}\|_{L^2}^2).
\end{aligned}
$$

We sum the above relation over $1 \le k \le m+1$ and combine with lower-order estimates, which can be obtained analogously, to get the higher-order estimates. □

Finally, we combine Lemma 3.2.1 and Lemma 3.2.2 to derive our second main result as follows.

**Theorem 3.2.1.** *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *hold, and let* $(\rho, u)$ *and* $(\bar{\rho}, \bar{u})$ *be two classical solution processes to* (3.0.1) *corresponding to the initial data* $(\rho_0, u_0)$ *and* $(\bar{\rho}_0, \bar{u}_0)$*, respectively. Then, there exists a positive random function* $C(T,z)$ *such that*

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|(\rho-\bar{\rho})(t,z)\|_{H^m}^2 + \|(u-\bar{u})(t,z)\|_{H^{m+1}}^2\right) \\
&\qquad \le C(T,z)(\|(\rho_0-\bar{\rho}_0)(z)\|_{H^m}^2 + \|(u_0-\bar{u}_0)(z)\|_{H^{m+1}}^2).
\end{aligned}
$$

*Proof.* We combine Lemma 3.2.1 and Lemma 3.2.2 to get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\|(\rho-\bar{\rho})\|_{H^m}^2 + \|(u-\bar{u})\|_{H^{m+1}}^2\right) \\
&\qquad \le C(T,z)\left(\|(\rho-\bar{\rho})\|_{H^m}^2 + \|(u-\bar{u})\|_{H^{m+1}}^2\right)
\end{aligned}.
$$

Here, one can use Grönwall's lemma to yield the desired result. □

### 3.2.2 $L^2$-stability estimates for $z$-variations

In this subsection, we discuss the $L^2$-stability estimates for the $z$-variations of the solution processes. First, we consider the $L^2$-stability estimates for the $z$-variations $\{\partial_z^l \rho\}$ of local mass processes.

**Lemma 3.2.3.** *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *hold, and let* $(\rho, u)$ *and* $(\bar\rho, \bar u)$ *be two solution processes to* (3.0.1) *corresponding to the initial data* $(\rho_0, u_0)$ *and* $(\bar\rho_0, \bar u_0)$*, respectively. Then,*

$$\frac{\partial}{\partial t}\sum_{0\le l\le m}\|\partial_z^l(\rho-\bar\rho)\|_{H^{m-l}}^2 \le C(T,z)\sum_{0\le l\le m}\Big(\|\partial_z^l(\rho-\bar\rho)\|_{H^{m-l}}^2+\|\partial_z^l(u-\bar u)\|_{H^{m-l+1}}^2\Big),$$

*where* $C = C(T,z)$ *is a positive random function.*

*Proof.* As in Lemma 3.2.1, we only consider the higher-order estimates. For $1 \le k \le m-l$ and $1 \le l \le m$, we apply $\nabla^k \partial_z^l$ to (3.0.1) to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2 \\
&= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\Big(\nabla^{r_2}\nabla(\partial_z^{r_1}(\rho-\bar\rho))\cdot\nabla^{k-r_2}(\partial_z^{l-r_1}u)\Big)\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&\quad -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\Big(\nabla^{r_2}\partial_z^{r_1}(\rho-\bar\rho)\nabla^{k-r_2}\nabla\cdot(\partial_z^{l-r_1}u)\Big)\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&\quad -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\Big(\nabla^{r_2}\nabla(\partial_z^{r_1}\bar\rho)\cdot\nabla^{k-r_2}\partial_z^{l-r_1}(u-\bar u)\Big)\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&\quad -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\Big(\nabla^{r_2}(\partial_z^{r_1}\bar\rho)\nabla^{k-r_2}(\nabla\cdot\partial_z^{l-r_1}(u-\bar u))\Big)\nabla^k(\partial_z^l(\rho-\bar\rho))dx \\
&=: \sum_{i=1}^4 \mathcal{I}_{4i}.
\end{aligned}$$

Below, we estimate $\mathcal{I}_{4i}$'s separately.

$\diamond$ (Estimates for $\mathcal{I}_{41}$): In this case, we have

$$
\begin{aligned}
\mathcal{I}_{41} &= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left[\begin{array}{c}\nabla^{r_2}\nabla(\partial_z^{r_1}(\rho-\bar\rho))\\ \cdot\nabla^{k-r_2}(\partial_z^{l-r_1}u)\end{array}\right]\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left[\begin{array}{c}\nabla^{r_2}\nabla(\partial_z^{r_1}(\rho-\bar\rho))\\ \cdot\nabla^{k-r_2}(\partial_z^{l-r_1}u)\end{array}\right]\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&\quad +\frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\nabla^k(\partial_z^l(\rho-\bar\rho))|^2dx\\
&\le \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2}\|\nabla^{k-r_2}(\partial_z^{l-r_1}u)\|_{L^\infty}\left(\begin{array}{c}\|\nabla^{r_2+1}(\partial_z^{r_1}(\rho-\bar\rho))\|_{L^2}\\ \cdot\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}\end{array}\right)\\
&\quad +\frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\\
&\le C\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2}\|\partial_z^{l-r_1}u\|_{H^{s-l}}\left(\begin{array}{c}\|\nabla^{r_2+1}(\partial_z^{r_1}(\rho-\bar\rho))\|_{L^2}^2\\ +\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\end{array}\right)\\
&\quad +C\|u\|_{H^{s-m}}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\\
&\le C(T,z)\sum_{0\le r\le l}\|\partial_z^r(\rho-\bar\rho)\|_{H^{m-r}}^2.
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{42}$) : By direct calculation, one has

$$
\begin{aligned}
\mathcal{I}_{42} &= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\begin{array}{c}\nabla^{r_2}(\partial_z^{r_1}(\rho-\bar\rho))\\ \cdot\nabla^{k-r_2}(\nabla\cdot\partial_z^{l-r_1}u)\end{array}\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\begin{array}{c}\nabla^{r_2}\partial_z^{r_1}(\rho-\bar\rho)\\ \cdot\nabla^{k-r_2}(\nabla\cdot\partial_z^{l-r_1}u)\end{array}\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&\quad -\int_{\mathbb{T}^d}(\nabla\cdot u)|\nabla^k(\partial_z^l(\rho-\bar\rho))|^2dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k\\(r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2}\|\nabla^{k-r_2+1}\partial_z^{l-r_1}u\|_{L^\infty}\left(\begin{array}{c}\|\nabla^{r_2}(\partial_z^{r_1}(\rho-\bar\rho))\|_{L^2}\\ \cdot\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}\end{array}\right)\\
&\quad+\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\\
&\leq C\sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k\\(r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2}\|\partial_z^{l-r_1}u\|_{H^{s-l}}\left(\begin{array}{c}\|\nabla^{r_2}(\partial_z^{r_1}(\rho-\bar\rho))\|_{L^2}^2\\ +\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\end{array}\right)\\
&\quad+C\|u\|_{H^{s-m}}\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2\\
&\leq C(T,z)\sum_{0\leq r\leq l}\|\partial_z^r(\rho-\bar\rho)\|_{H^{m-r}}^2.
\end{aligned}
$$

⋄ (Estimates for $\mathcal{I}_{43}$): Similarly, one gets

$$
\begin{aligned}
\mathcal{I}_{43} &= -\sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k}} \binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\begin{array}{c}\nabla^{r_2}\nabla(\partial_z^{r_1}\bar\rho)\\ \cdot\nabla^{k-r_2}(\partial_z^{l-r_1}(u-\bar u))\end{array}\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&\leq \sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k}} \binom{l}{r_1}\binom{k}{r_2}\|\nabla^{r_2+1}(\partial_z^{r_1}\bar\rho)\|_{L^\infty}\left(\begin{array}{c}\|\nabla^{k-r_2}\partial_z^{l-r_1}(u-\bar u)\|_{L^2}\\ \cdot\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}\end{array}\right)\\
&\leq C\sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k}} \binom{l}{r_1}\binom{k}{r_2}\|\partial_z^{r_1}\bar\rho\|_{H^{s-l}}\left(\begin{array}{c}\|\nabla^{k-r_2}\partial_z^{l-r_1}(u-\bar u)\|_{L^2}\\ \cdot\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}\end{array}\right)\\
&\leq C(T,z)\sum_{0\leq r\leq l}\left(\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}^2+\|\partial_z^r(u-\bar u)\|_{H^{m-r}}^2\right).
\end{aligned}
$$

⋄ (Estimates for $\mathcal{I}_{44}$): By direct estimates, one obtains

$$
\begin{aligned}
\mathcal{I}_{44} &= -\sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k}} \binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\left(\begin{array}{c}\nabla^{r_2}(\partial_z^{r_1}\bar\rho)\\ \cdot\nabla^{k-r_2}\nabla\cdot(\partial_z^{l-r_1}(u-\bar u))\end{array}\right)\nabla^k(\partial_z^l(\rho-\bar\rho))dx\\
&\leq \sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k}} \binom{l}{r_1}\binom{k}{r_2}\|\nabla^{r_2}(\partial_z^{r_1}\bar\rho)\|_{L^\infty}\left(\begin{array}{c}\|\nabla^{k-r_2+1}(\partial_z^{l-r_1}(u-\bar u))\|_{L^2}\\ \cdot\|\nabla^k(\partial_z^l(\rho-\bar\rho))\|_{L^2}\end{array}\right)
\end{aligned}
$$

$$\leq C \sum_{\substack{0 \leq r_1 \leq l \\ 0 \leq r_2 \leq k}} \binom{l}{r_1}\binom{k}{r_2} \|\partial_z^{r_1}\bar{\rho}\|_{H^{s-l-1}} \begin{pmatrix} \|\nabla^{k-r_2+1}(\partial_z^{l-r_1}(u-\bar{u}))\|_{L^2} \\ \cdot \|\nabla^k(\partial_z^l(\rho-\bar{\rho}))\|_{L^2} \end{pmatrix}$$
$$\leq C(T,z) \sum_{0 \leq r \leq l} \left( \|\nabla^k(\partial_z^l(\rho-\bar{\rho}))\|_{L^2}^2 + \|\partial_z^r(u-\bar{u})\|_{H^{m-r+1}}^2 \right).$$

Finally, we collect all estimates for $\mathcal{I}_{4i}$'s, sum them over $1 \leq k \leq m-l$, $0 \leq l \leq m$ and add the zeroth-order estimate to get the desired result. □

Next, we provide estimates for $z$-variations $\{\partial_z^l u\}$ of the bulk velocity processes.

**Lemma 3.2.4.** *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *hold, and let* $(\rho, u)$ *and* $(\bar{\rho}, \bar{u})$ *be two classical solution processes to* (3.0.1) *corresponding to the initial data* $(\rho_0, u_0)$ *and* $(\bar{\rho}_0, \bar{u}_0)$*, respectively. Then, we have*

$$\frac{\partial}{\partial t} \sum_{0 \leq l \leq m} \|\partial_z^l(u-\bar{u})\|_{H^{m-l+1}}^2 \leq C(T,z) \sum_{0 \leq l \leq m} \left( \|\partial_z^l(\rho-\bar{\rho})\|_{L^2}^2 + \|\partial_z^l(u-\bar{u})\|_{H^{m-l+1}}^2 \right),$$

*where* $C = C(T,z)$ *is a positive random function.*

*Proof.* Since the proof will be straightforward and similar to that of Lemma 3.2.2, we leave its proof to Appendix A.3. □

Finally, we combine Lemma 3.2.3 with Lemma 3.2.4 and use Grönwall's lemma to deduce the following result.

**Theorem 3.2.2.** *Suppose that assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *hold, and let* $(\rho, u)$ *and* $(\bar{\rho}, \bar{u})$ *be two classical solution processes to* (3.0.1) *with initial data* $(\rho_0, u_0)$ *and* $(\bar{\rho}_0, \bar{u}_0)$*, respectively. Then, there exists a positive random function* $C(T,z)$ *such that*

$$\sup_{0 \leq t \leq T} \sum_{0 \leq l \leq m} \left( \|\partial_z^l(\rho-\bar{\rho})(t,z)\|_{H^{m-l}}^2 + \|\partial_z^l(u-\bar{u})(t,z)\|_{H^{m-l+1}}^2 \right)$$
$$\leq C(T,z) \sum_{0 \leq l \leq m} \left( \|\partial_z^l(\rho_0-\bar{\rho}_0)(z)\|_{H^{m-l}}^2 + \|\partial_z^l(u_0-\bar{u}_0)(z)\|_{H^{m-l+1}}^2 \right).$$

## 3.3 A local sensitivity analysis for flocking estimate

In this section, we provide a local sensitivity analysis for the flocking behavior to system (3.0.1).

It follows from Proposition 3.1.1 that

$$\int_{\mathbb{T}^d} \rho(t,z)dx = \int_{\mathbb{T}^d} \rho_0(z)dx, \ \int_{\mathbb{T}^d} (\rho u)(t,z)dx = \int_{\mathbb{T}^d} (\rho_0 u_0)(z)dx, \ \ t \geq 0, \ z \in \Omega.$$

Then, without loss of generality, we may assume that the average bulk velocity is zero:

$$u_c(t,z) := \frac{\int_{\mathbb{T}^d} \rho u dx}{\int_{\mathbb{T}^d} \rho dx} \equiv 0.$$

For a given $z$-variations $\{\partial_z^m u\}$, we introduce a family of flocking functionals $\mathcal{E}_m$:

$$\mathcal{E}_m(t,z) := \int_{\mathbb{T}^d} \rho |\partial_z^m u|^2 dx, \qquad m \geq 1,$$

where $\mathcal{E}_0(t,z)$ is defined in Theorem 2.2.3. Although the functionals $\mathcal{E}_m$ are not $z$-variations of a certain quantity, estimates for these functionals will be of our concern since they play a role in estimating $\|\partial_z^m u\|_{L^2}$. Based on the estimates for $\mathcal{E}_m$, we provide estimates for the exponential decay of $\|\partial_z^m u\|_{L^2}$ under the following *a priori* assumptions $(\mathcal{B})$: for an integer $s > \frac{d}{2} + m + 1$, $T \in (0, \infty)$ and each $z \in \Omega$,

$(\mathcal{B}1)$ The solution process $(\rho, u)$ and their $z$-variations $\{(\partial_z^l \rho, \partial_z^l u)\}_{l=0}^m$ satisfy the following uniform boundedness condition:

$$\max_{0 \leq l \leq m} \sup_{t \in [0,T]} \left( \|\partial_z^l \rho(t,z)\|_{H^{s-l}} + \|\partial_z^l u(t,z)\|_{H^{s-l+1}} \right) \leq \mathcal{U}(z),$$

for some positive random function $\mathcal{U} = \mathcal{U}(z)$.

$(\mathcal{B}2)$ The initial mass $\rho_0$ satisfies the non-vacuum condition:

$$\inf_{x \in \mathbb{T}^d} \rho_0(x,z) > 0, \quad \text{for each } z \in \Omega.$$

($\mathcal{B}3$) The communication weight function $\phi : \mathbb{T}^d \times \Omega \to \mathbb{R}$ is in $\mathcal{C}^{s+1}(\mathbb{T}^d \times \Omega)$ and satisfies symmetric, positive, boundedness conditions: for each $x, y \in \mathbb{T}^d$ and $z \in \Omega$,

$$\begin{aligned}
&\phi(x-y,z) = \phi(y-x,z),\\
&\inf_{x\in\mathbb{T}^d} \phi(x,z) =: \phi_m(z) > \sup_{0\le t\le T}\left(\frac{\|(\nabla\cdot u)(t)\|_{L^\infty}}{2\|\rho_0\|_{L^1}}\right),\\
&\|\phi\|_s := \max_{|\alpha|+|\beta|\le s+1}\ \sup_{(x,z)\in\mathbb{T}^d\times\Omega} |\partial_z^\alpha \partial_x^\beta \phi(x,z)| < \infty.
\end{aligned}$$

**Remark 3.3.1.** *The lower bound assumption for $\phi$ given in ($\mathcal{B}3$) implies a sufficient condition for system* (3.0.1) *to exhibit the decay of the bulk velocity toward the average bulk velocity. To be precise, the condition means the alignment force is so strong that it surpasses the tendency of bulk velocity field to deviate from the mean velocity, and it will lead to the velocity alignment.*

In the following lemma, we study the exponential decay of $\|u\|_{L^2}$.

**Lemma 3.3.1.** *Let $(\rho, u)$ be a classical solution to system* (3.0.1) *and suppose that the a priori assumptions ($\mathcal{B}$) hold for $(\rho, u)$. Then we have*

$$\|u(t,z)\|_{L^2}^2 \le \mathcal{F}_0(z) e^{-2\tilde{\Lambda}(z)t}, \quad t\in[0,T],$$

*where $\mathcal{F}_0(z)$ and $\tilde{\Lambda}(z)$ are positive random functions.*

*Proof.* It follows from (3.0.1) that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}^d} |u|^2 dx\\
&\quad = -\int_{\mathbb{T}^d} (u\cdot\nabla u)\cdot u dx + \int_{\mathbb{T}^d\times\mathbb{T}^d} \phi(x-y)(u(y)-u(x))\cdot u(x)\rho(y) dy dx\\
&\quad = \frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot u)|u|^2 dx + \int_{\mathbb{T}^d\times\mathbb{T}^d} \phi(x-y)(u(y)-u(x))\cdot u(x)\rho(y) dy dx\\
&\quad \le \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|u\|_{L^2}^2 - \phi_m\|\rho_0\|_{L^1}\|u\|_{L^2}^2 + \kappa\|\rho u\|_{L^1}\|u\|_{L^1}\\
&\quad \le \left(-\phi_m\|\rho_0\|_{L^1} + \frac{\sup_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty}}{2} + \delta\right)\|u\|^2\\
&\qquad + \frac{\|\phi\|_s^2}{4\delta}\|\rho\|_{L^2}\mathcal{E}_0(t,z),
\end{aligned}$$

where $\mathcal{E}_0$ is the functional defined in (2.2.3) and the positive constant $\delta > 0$ satisfies the following relation:

$$\phi_m \|\rho_0\|_{L^1} - \frac{\sup\limits_{0\le t\le T} \|(\nabla\cdot u)(t)\|_{L^\infty}}{2} - \delta > 0.$$

We let $\tilde{\Lambda}(z) := \phi_m \|\rho_0\|_{L^1} - \frac{\sup_{0\le t\le T} \|(\nabla\cdot u)(t)\|_{L^\infty}}{2} - \delta$. Then, we have

$$\frac{\partial}{\partial t}\|u\|_{L^2}^2 \le -2\tilde{\Lambda}(z)\|u\|_{L^2}^2 + \hat{F}_0(z)e^{-2\Lambda(z)t}, \tag{3.3.1}$$

where random functions $\Lambda$ and $\hat{F}_0$ are given by the following relations:

$$\Lambda(z) := \phi_m\|\rho_0\|_{L^1}, \quad \hat{F}_0(z) := \frac{\|\phi\|_s^2}{2\delta}\mathcal{U}(z)\|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

Now, we apply Grönwall's lemma to (3.3.1) to obtain

$$\begin{aligned}\|u\|_{L^2}^2 &\le \|u_0\|_{L^2}^2 e^{-2\tilde{\Lambda}(z)t} + \frac{\hat{F}_0(z)}{\sup\limits_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty} + 2\delta}\left(e^{-2\tilde{\Lambda}(z)t} - e^{-2\Lambda(z)t}\right)\\ &\le \mathcal{F}_0(z)e^{-2\tilde{\Lambda}(z)t},\end{aligned}$$

where $\mathcal{F}_0(z)$ is given by

$$\mathcal{F}_0(z) := \|u_0\|_{L^2}^2 + \frac{\hat{F}_0(z)}{\sup\limits_{0\le t\le T}\|(\nabla\cdot u)(t)\|_{L^\infty} + 2\delta}.$$

This implies our desired result.

□

Next, we derive the temporal decay estimates for functional $\mathcal{E}_m(t,z)$ and $\|\partial_z^m u\|$ based on the induction argument.

**Theorem 3.3.1.** *Suppose that a priori assumptions ($\mathcal{B}$) and the following induction hypotheses hold for* $0 \le l \le m-1$*:*

$$\mathcal{E}_l(t,z) \le E_l(z)e^{-\tilde{\Lambda}(z)t}, \quad \|\partial_z^l u\|_{L^2}^2 \le \mathcal{F}_l(z)e^{-\tilde{\Lambda}(z)t},$$

*where $E_l(z)$ and $\mathcal{F}_l(z)$ are positive random functions and $\tilde{\Lambda}(z)$ is given in Lemma 3.3.1. Then, there exists a positive random function $E_m(z)$ such that*

$$\mathcal{E}_m(t,z) \le E_m(z)e^{-\tilde{\Lambda}(z)t}.$$

*Proof.* It follows from (3.1.2) that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{E}_m(t,z)\\
&= \int_{\mathbb{T}^d} \partial_t \rho |\partial_z^m u|^2 dx + 2\int_{\mathbb{T}^d} \rho \partial_z^m u \cdot \partial_t(\partial_z^m u) dx\\
&= -\int_{\mathbb{T}^d} \nabla\cdot(\rho u)|\partial_z^m u|^2 dx - 2\sum_{l=0}^{m}\binom{m}{l}\int_{\mathbb{T}^d} \rho\partial_z^m u\cdot(\partial_z^l u\cdot\nabla\partial_z^{m-l}u)dx\\
&\quad + 2\sum_{\alpha+\beta+\gamma=m}\frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}}\left(\begin{array}{c}\partial_z^\alpha\phi(x-y)(\partial_z^\beta u(y)-\partial_z^\beta u(x))\\ \cdot\,\partial_z^\gamma\rho(y)\rho(x)\cdot\partial_z^m u(x)\end{array}\right)dydx\\
&= -2\sum_{l=1}^{m}\binom{m}{l}\int_{\mathbb{T}^d} \rho\partial_z^m u\cdot(\partial_z^l u\cdot\nabla\partial_z^{m-l}u)dx\\
&\quad + 2\sum_{\alpha+\beta+\gamma=m}\frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}}\left(\begin{array}{c}\partial_z^\alpha\phi(x-y)(\partial_z^\beta u(y)-\partial_z^\beta u(x))\\ \cdot\,\partial_z^\gamma\rho(y)\rho(x)\cdot\partial_z^m u(x)\end{array}\right)dydx\\
&=: \mathcal{I}_{51}+\mathcal{I}_{52}.
\end{aligned}
$$

Next, we separately estimate $\mathcal{I}_{51}$ and $\mathcal{I}_{52}$ as follows:

$\diamond$ (Estimates for $\mathcal{I}_{51}$) : First,

$$
\begin{aligned}
\int_{\mathbb{T}^d}\rho\partial_z^m u\cdot(\partial_z^m u\cdot\nabla)u dx &= -\int_{\mathbb{T}^d}\nabla\cdot(\rho\partial_z^m u\otimes\partial_z^m u)\cdot u dx\\
&\le \|\nabla\cdot(\rho\partial_z^m u\otimes\partial_z^m u)\|_{L^2}\|u\|_{L^2} \le \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)}e^{-\tilde{\Lambda}(z)t}.
\end{aligned}
$$

For $m=1$, one gets

$$\mathcal{I}_{51} \le \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)}e^{-\tilde{\Lambda}(z)t}.$$

For $m \ge 2$,

$$
\begin{aligned}
&2\sum_{l=1}^{m-1}\binom{m}{l}\int_{\mathbb{T}^d}\rho\partial_z^m u\cdot(\partial_z^l u\cdot\nabla\partial_z^{m-l}u)dx\\
&\qquad \le 2\sum_{l=1}^{m-1}\binom{m}{l}\|\nabla(\partial_z^{l-r}u)\|_{L^\infty}\mathcal{E}_m(t,z)\mathcal{E}_l(t,z)
\end{aligned}
$$

$$\leq \delta \mathcal{E}_m^2(t,z) + \frac{m-1}{\delta} \sum_{l=1}^{m-1} \binom{m}{l}^2 \mathcal{U}^2(z) \mathcal{E}_l^2(t,z),$$

where $\delta$ is the same as that in Lemma 3.3.1 and we used Young's inequality. Hence, we can obtain that if $m \geq 2$,

$$\mathcal{I}_{51} \leq \delta \mathcal{E}_m^2(t,z) + \frac{m-1}{\delta} \sum_{l=1}^{m-1} \binom{m}{l}^2 \mathcal{U}^2(z) \mathcal{E}_l^2(t,z) + \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)} e^{-\tilde{\Lambda}(z)t}.$$

$\diamond$ (Estimates for $\mathcal{I}_{52}$): By direct calculation, one has

$$\begin{aligned}
\mathcal{I}_{52} &= 2\int_{\mathbb{T}^{2d}} \phi(x-y)(\partial_z^m u(y) - \partial_z^m u(x))\rho(y)\rho(x)\partial_z^m u(x)\,dydx \\
&\quad + 2 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \begin{pmatrix} \partial_z^\alpha \phi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x)) \\ \cdot\, \partial_z^\gamma \rho(y)\rho(x) \cdot \partial_z^m u(x) \end{pmatrix} dydx \\
&= -\int_{\mathbb{T}^{2d}} \phi(x-y)|\partial_z^m u(y) - \partial_z^m u(x)|^2 \rho(y)\rho(x)\,dydx \\
&\quad + 2 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \begin{pmatrix} \partial_z^\alpha \phi(x-y)(\partial_z^\beta u(y) - \partial_z^\beta u(x)) \\ \cdot\, \partial_z^\gamma \rho(y)\rho(x) \cdot \partial_z^m u(x) \end{pmatrix} dydx \\
&\leq -2\phi_m \|\rho_0\|_{L^1} \mathcal{E}_m(t,z) \\
&\quad + 2 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}} \frac{m!}{\alpha!\beta!\gamma!} \|\phi\|_s \begin{pmatrix} \|\partial_z^\beta u\ \partial_z^\gamma \rho\|_{L^1} \|\rho \partial_z^m u\|_{L^1} \\ + \|\partial_z^\gamma \rho\|_{L^1} \|\rho \partial_z^\beta u \cdot \partial_z^m u\|_{L^1} \end{pmatrix} \\
&\leq -2\phi_m \|\rho_0\|_{L^1} \mathcal{E}_m(t,z) \\
&\quad + 4 \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}} \frac{m!}{\alpha!\beta!\gamma!} \|\phi\|_s \|\partial_z^\gamma \rho\|_{L^2} \sqrt{\|\rho\|}_{L^\infty} \|\partial_z^\beta u\|_{L^2} \sqrt{\mathcal{E}_m(t,z)} \\
&\leq (-2\phi_m \|\rho_0\|_{L^1} + \delta)\mathcal{E}_m(t,z) \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}} \left[ \frac{m!}{\alpha!\beta!\gamma!} \|\phi\|_s \|\partial_z^\gamma \rho\|_{L^2} \sqrt{\|\rho\|}_{L^\infty} \right]^2 \frac{\left(\frac{(m+1)(m+2)}{2} - 1\right)}{\delta} \mathcal{F}_\beta(z) e^{-\tilde{\Lambda}(z)t},
\end{aligned}$$

where we used

$$\sum_{\alpha+\beta+\gamma=m} 1 = \frac{(m+2)(m+1)}{2},$$

and Young's inequality. Therefore, we collect all results for $\mathcal{I}_{51}$ and $\mathcal{I}_{52}$ to yield

$$\frac{\partial}{\partial t}\mathcal{E}_m(t,z) \le -2(\phi_m\|\rho_0\|_{L^1}-\delta)\mathcal{E}_m(t,z) + \hat{E}_m(z)e^{-\tilde{\Lambda}(z)t}, \tag{3.3.2}$$

where $\hat{E}_m(z)$ is given by

$$\begin{aligned}
\hat{E}_1(z) &:= \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)} + \frac{4}{\delta}\|\phi\|_s^2\mathcal{U}^3(z)\mathcal{F}_0(z),\\
\hat{E}_m(z) &:= \mathcal{U}^3(z)\sqrt{\mathcal{F}_0(z)} + \mathcal{E}_m^2(t,z) + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2\mathcal{U}^2(z)E_l(z)\\
&\qquad + \frac{m^2+3m}{2\delta}\sum_{\substack{\alpha+\beta+\gamma=m\\ \beta\neq m}}\left(\frac{m!}{\alpha!\beta!\gamma!}\|\phi\|_s\mathcal{U}^{3/2}(z)\right)^2\mathcal{F}_\beta(z), \quad m\ge 2.
\end{aligned}$$

Now, we integrate (3.3.2) with respect to time to get

$$\begin{aligned}
\mathcal{E}_m(t,z) &\le \mathcal{E}_m(0,z)e^{-2(\phi_m\|\rho_0\|_{L^1}-\delta)t}\\
&\qquad + \frac{\hat{E}_m(z)}{2(\phi_m\|\rho_0\|_{L^1}-\delta)-\tilde{\Lambda}(z)}\left(e^{-\tilde{\Lambda}(z)t} - e^{-2(\phi_m\|\rho_0\|_{L^1}-\delta)t}\right)\\
&\le E_m(z)e^{-\tilde{\Lambda}(z)t},
\end{aligned}$$

where $E_m(z)$ is written as

$$E_m(z) := \mathcal{E}_m(0,z) + \frac{\hat{E}_m(z)}{2(\phi_m\|\rho_0\|_{L^1}-\delta)-\tilde{\Lambda}(z)}.$$

This gives our desired result. □

Finally, we provide all estimates for the $L^2$-decay of the $z$-variations toward the corresponding $z$-variations of the average bulk velcoity process.

**Theorem 3.3.2.** *For a positive constant $T\in(0,\infty)$, let $(\rho,u)$ be a classical solution process on $[0,T]$ and suppose that a priori assumptions $(\mathcal{B})$ hold. Moreover, assume the following induction hypotheses hold for $0\le l\le m$ and $0\le p\le m-1$:*

$$\mathcal{E}_l(t,z)\le E_l(z)e^{-\tilde{\Lambda}(z)t}, \quad \|\partial_z^p u\|_{L^2}^2 \le \mathcal{F}_p(z)e^{-\tilde{\Lambda}(z)t},$$

*where* $E_l(z)$ *and* $\mathcal{F}_p(z)$ *are positive random functions. Then, there exists a positive random function* $\mathcal{F}_m(z)$ *such that*

$$\|\partial_z^m u\|_{L^2}^2 \le \mathcal{F}_m(z) e^{-\tilde{\Lambda}(z)t}.$$

*Proof.* We leave its detailed proof in Appendix A.4. □

# Chapter 4

# On the stochastic flocking of the Cucker-Smale flock with randomly switching topologies

In this chapter, we present an emergent stochastic flocking dynamics of the Cucker-Smale ensemble (1.0.4) under randomly switching topologies. Recall that the evolution of the C-S system with randomly switching topologies is determined by the following second order system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad 1 \le i \le N, \quad t > 0, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \chi_{ij}^{\sigma} \phi(x_j - x_i)\left(v_j - v_i\right), \end{cases} \tag{4.0.1}$$

where $(\chi_{ij}^{\sigma(t)})$ denotes the time-dependent network topology corresponding to the switching law $\sigma : [0, \infty) \times \Omega \to \{1, \cdots, N_G\}$. Note that once $\omega \in \Omega$ is fixed, $\sigma(\cdot, \omega)$ a $\{1, \cdots, N_G\}$-valued, piecewise constant function whose discontinuities are $\{t_\ell(\omega)\}$ and $\mathcal{G}_{\sigma(t_\ell(\omega), \omega)} \in \{\mathcal{G}_1, \cdots, \mathcal{G}_{N_G}\}$ is chosen as the network topology during the interval $[t_\ell(\omega), t_{\ell+1}(\omega))$.

We briefly discuss our main result on the emergence of stochastic flocking of the model (4.0.1). We assume that the probability density function $f$, choice probability $p_k$ and communication weight function $\phi$ satisfy

$$\mathrm{supp}(f) \subset [a,b], \qquad \frac{\kappa b(N-1)}{\min\limits_{1\le k\le N_G} \log \frac{1}{1-p_k}} < 1, \qquad \frac{1}{\bar{\phi}(r)} = \mathcal{O}(r^{\varepsilon}) \quad \text{as} \quad r \to \infty,$$

where $\varepsilon$ is a small positive constant. Then, under the above set of assumptions, we show that any solution process $(X, V)$ to (4.0.1) satisfies the mono-cluster flocking with probability one (Theorem 4.2.1):

$$\mathbb{P}\Big(\omega \in \Omega : \exists\, x^{\infty} > 0 \text{ s.t } \sup_{0\le t<\infty} \mathcal{D}(X(t,\omega)) \le x^{\infty}, \ \lim_{t\to\infty} \mathcal{D}(V(t,\omega)) = 0\Big) = 1.$$

The rest of this chapter is organized as follows. In Section 4.1, we review several basic concepts on directed graphs, scrambling and state transition matrices. In Section 4.2, we present our sufficient framework and main result for the stochastic mono-cluster flocking estimate. In Section 4.3, we first provide a priori flocking estimates along the sample path under two a priori assumptions on the network topologies and position diameter, and then we replace a priori condition for the position diameter by suitable conditions on the system parameters and communication weight, and the a priori condition for the network topology will be shown to hold with probability one for a suitably chosen time-block sequence. Finally, note that this chapter is based on the joint work [25].

# 4.1 Preliminaries

In this section, we first study the dissipative structure of system (4.0.1), and briefly review several notions on the directed graphs, scrambling matrices and state transition matrices.

## 4.1.1 Pathwise dissipative structure

In this subsection, we study the dissipative structure of system (4.0.1) with randomly switching topologies. For the symmetric network topology, the R.H.S. of $(4.0.1)_2$ is skew-symmetric under the exchange symmetry $i \longleftrightarrow j$. Hence, the total momentum $\sum_{i=1}^{N} v_i$ is a constant of motion. In contrast, for a digraph topology, the R.H.S. of system (4.0.1) may not be skew-symmetric under the exchange symmetry. This breaks up the conservation law for the total momentum. Despite of this, we can still see that the velocity diameter is non-increasing pathwise.

**Lemma 4.1.1.** *Let $(X, V)$ be a solution process to (4.0.1). Then, the velocity diameter $\mathcal{D}(V)$ is non-increasing pathwise: for each $\omega \in \Omega$,*

$$\frac{d}{dt}\mathcal{D}(V(t,\omega)) \leq 0, \quad a.e.\ t > 0.$$

*Proof.* For a given $t \geq 0$ and $\omega \in \Omega$, let $i$ and $j$ be indices satisfying the relation:

$$\mathcal{D}(V(t,\omega)) = \|v_i(t,\omega) - v_j(t,\omega)\|. \tag{4.1.1}$$

In the sequel, for a notational simplicity, we suppress $t$ and $\omega$ dependence in $v_i$:

$$v_i = v_i(t,\omega).$$

Then, for such $i$ and $j$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_i - v_j\|^2 &= \left\langle v_i - v_j, \frac{dv_i}{dt} - \frac{dv_j}{dt} \right\rangle \\ &= \left\langle v_i - v_j, \frac{1}{N}\sum_{k=1}^{N} \chi_{ik}^{\sigma}\phi_{ik}(v_k - v_i) \right\rangle \\ &\quad + \left\langle v_j - v_i, \frac{1}{N}\sum_{k=1}^{N} \chi_{jk}^{\sigma}\phi_{jk}(v_k - v_j) \right\rangle \\ &=: \mathcal{J}_1 + \mathcal{J}_2, \end{aligned} \tag{4.1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$, and we wrote

$$\phi_{ij} := \phi(x_i - x_j), \quad i, j = 1, 2, \cdots, N$$

for convenience. Below, we estimate the terms $\mathcal{J}_i,\ i = 1, 2$ one by one.

- (Estimate of $\mathcal{J}_1$): For $k = 1, \cdots, N$, we use the relation (4.1.1) to get

$$\begin{aligned} \langle v_k - v_i, v_i - v_j \rangle &= \frac{\|v_k - v_j\|^2 - \|v_k - v_i\|^2 - \|v_i - v_j\|^2}{2} \\ &\leq \frac{\|v_i - v_j\|^2 - 0 - \|v_i - v_j\|^2}{2} = 0. \end{aligned} \tag{4.1.3}$$

This yields

$$\mathcal{J}_1 = \frac{1}{N}\sum_{k=1}^{N} \chi_{ik}^{\sigma}\phi_{ik}\left\langle v_i - v_j, v_k - v_i\right\rangle \le 0. \tag{4.1.4}$$

• (Estimate of $\mathcal{J}_2$) : Similar to (4.1.3), we also have

$$\begin{aligned}\left\langle v_k - v_j, v_j - v_i\right\rangle &= \frac{\|v_k - v_i\|^2 - \|v_k - v_j\|^2 - \|v_j - v_i\|^2}{2} \\ &\le \frac{\|v_j - v_i\|^2 - 0 - \|v_j - v_i\|^2}{2} = 0.\end{aligned}$$

This again implies

$$\mathcal{J}_2 = \frac{1}{N}\sum_{k=1}^{N} \chi_{jk}^{\sigma}\phi_{jk}\left\langle v_j - v_i, v_k - v_j\right\rangle \le 0. \tag{4.1.5}$$

In (4.1.2), we use $\mathcal{D}(V) = \|v_i - v_j\|$ and combine estimates (4.1.4) and (4.1.5) to get

$$\mathcal{D}(V(t))\frac{d}{dt}\mathcal{D}(V(t)) \le 0, \quad \text{a.e. } t > 0.$$

If $\mathcal{D}(V(t)) > 0$, then we can divide the above inequality by $\mathcal{D}(V(t))$ to obtain the desired estimate.

On the other hand, if $\mathcal{D}(V(t)) = 0$ and diffrentiable at $t$, then $\mathcal{D}(V)$ attains a global minimum at $t$, so $\frac{d}{dt}\mathcal{D}(V(t)) = 0$. Hence we have the following differential inequality:

$$\frac{d}{dt}\mathcal{D}(V(t)) \le 0, \quad \text{a.e. } t > 0.$$

□

**Remark 4.1.1.** *Note that the result of Lemma 4.1.1 illustrates that the velocity diameter is non-increasing in time. Now, our job is to find some conditions leading to the zero convergence of velocity diameter. This will be done in Section 4.3.*

### 4.1.2 A directed graph

In this subsection, we review jargons for network topology modeling by a directed graph (digraph). A digraph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ consists of two sets: a set of vertices (nodes) $\mathcal{V}(\mathcal{G}) = \{1, \cdots, N\}$ with $|\mathcal{G}| = N$, and a set of edges $\mathcal{E}(\mathcal{G}) \subset \mathcal{V} \times \mathcal{V}$ consisting of ordered pairs of vertices:

$$
\begin{aligned}
&(j,i) \in \mathcal{E}(\mathcal{G}) \\
&\quad \Longleftrightarrow \quad \text{vertex } i \text{ receives an information (or signal) from the vertex } j \\
&\quad \Longleftrightarrow \quad j \text{ is a neighbor of } i.
\end{aligned}
$$

In this case, we define a neighbor set $\mathcal{N}_i$ of the vertex $i$:

$$
\mathcal{N}_i := \{j \in \mathcal{V}(\mathcal{G}) : \ (j,i) \in \mathcal{E}(\mathcal{G})\}.
$$

If $(i,i) \in \mathcal{E}(\mathcal{G})$, then we say that $\mathcal{G}$ has a self-loop at $i$. If $\mathcal{G}$ does not have a self-loop at any vertices, then $\mathcal{G}$ is said to be *simple*.

For a given digraph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$, we consider its $(0,1)$-adjacency matrix $\chi = (\chi_{ij})$:

$$
\chi_{ij} := \begin{cases} 1 & \text{if} \quad (j,i) \in \mathcal{E}(\mathcal{G}), \\ 0 & \text{if} \quad (j,i) \notin \mathcal{E}(\mathcal{G}). \end{cases}
$$

A path in $\mathcal{G}$ from $i$ to $j$ is a sequence of ordered distinct vertices $(i_0 = i, \cdots, i_n = j)$:

$$
i = i_0 \longrightarrow i_1 \longrightarrow \quad \cdots \quad \longrightarrow i_n = j
$$

$$
\text{such that } (i_{m-1}, i_m) \in \mathcal{E}(\mathcal{G}) \quad \text{for every } 1 \le m \le n.
$$

If there is a path from $i$ to $j$, then we say $j$ is reachable from $i$. Moreover, a digraph $\mathcal{G}$ is said to have a spanning tree if $\mathcal{G}$ has a vertex $i$ from which any other vertices are reachable. As long as there is no confusion, we suppress $\mathcal{G}$-dependence in $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ throughout this chapter:

$$
\mathcal{V} = \mathcal{V}(\mathcal{G}), \quad \mathcal{E} = \mathcal{E}(\mathcal{G}).
$$

### 4.1.3 A scrambling matrix

Next, we recall the concept of scrambling matrices. First, we introduce several concepts of nonnegative matrices in the following definition.

**Definition 4.1.1.** *Let $A = (a_{ij})$ be a nonnegative $N \times N$ matrix, i.e. a matrix whose entries are nonnegative.*

1. *$A$ is a* stochastic *matrix, if its row-sum is equal to unity:*

$$\sum_{j=1}^{N} a_{ij} = 1, \quad 1 \le i \le N.$$

2. *$A$ is a* scrambling *matrix, if for each pair of indices $i$ and $j$, there exist an index $k$ such that*

$$a_{ik} > 0 \quad \text{and} \quad a_{jk} > 0.$$

3. *$A$ is an* adjacency matrix *of a digraph $\mathcal{G}$ if the following holds:*

$$a_{ij} > 0 \quad \Longleftrightarrow \quad (j, i) \in \mathcal{E}.$$

*In this case, we write $\mathcal{G} = \mathcal{G}(A)$.*

**Remark 4.1.2.** *Define the* ergodicity coefficient *of $A$ as follows.*

$$\mu(A) := \min_{i,j} \sum_{k=1}^{N} \min\{a_{ik}, a_{jk}\}. \tag{4.1.6}$$

*Then, it is easy to see that*

1. *$A$ is scrambling if and only if $\mu(A) > 0$.*
2. *For nonnegative matrices $A$ and $B$,*

$$A \ge B \quad \Longrightarrow \quad \mu(A) \ge \mu(B). \tag{4.1.7}$$

For a $N \times N$ matrix $A = (a_{ij})$, the Frobenius norm of $A$ is defined as follows.

$$\|A\|_F := \sqrt{\text{trace}(AA^*)} = \sqrt{\text{trace}(A^*A)}.$$

In the following lemma, we state some properties of scrambling matrices without proofs.

**Lemma 4.1.2.** (Lemma 2.2, [26]) *Suppose that a nonnegative $N \times N$ matrix $A = (a_{ij})$ is stochastic, and let $B = (b_i^j)$, $Z = (z_i^j)$ and $W = (w_i^j)$ be $N \times d$ matrices such that*

$$W = AZ + B.$$

*Then, we have*

$$\max_{i,k} \|w_i - w_k\| \le (1 - \mu(A)) \max_{l,m} \|z_l - z_m\| + \sqrt{2}\|B\|_F,$$

*where*

$$z_i := (z_i^1, \cdots, z_i^d), \quad b_i := (b_i^1, \cdots, b_i^d), \quad w_i := (w_i^1, \cdots, w_i^d), \quad i = 1, \cdots, N.$$

**Proposition 4.1.1.** (Theorem 5.1, [107]) *Let $A_i$ be nonnegative $N \times N$ matrices with positive diagonal elements. Suppose that $\mathcal{G}(A_i)$ has a spanning tree for all $1 \le i \le N-1$. Then, one has*

$$A_1 A_2 \dots A_{N-1} \text{ is a scrambling matrix.}$$

### 4.1.4 A state transition matrix

In this subsection, we discuss the notion and properties of state transition matrices. Let $t_0 \in \mathbb{R}$ and $A : [t_0, \infty) \to \mathbb{R}^{N \times N}$ be an $N \times N$ matrix of piecewise continuous function.

Consider the following Cauchy problem for the time-dependent linear ODE:

$$\begin{aligned} \frac{d\xi(t)}{dt} &= A(t)\xi(t), \quad t > t_0, \\ \xi|_{t=t_0} &= \xi(t_0). \end{aligned} \tag{4.1.8}$$

Then, the solution of (4.1.8) is given by

$$\xi(t) = \Phi(t, t_0)\xi(t_0), \quad t \geq t_0,$$

where $\Phi(t, t_0)$ is called the state transition matrix or the fundamental matrix for (4.1.8).

Note that we can write the state transition matrix $\Phi(t, t_0)$ corresponding to system (4.1.8) as the Peano-Baker series (see [96]):

$$\Phi(t, t_0) = I + \sum_{n=1}^{\infty} \int_{t_0}^{t} \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{n-1}} A(\tau_1)A(\tau_2)\cdots A(\tau_n)d\tau_n \cdots d\tau_2 d\tau_1,$$

where $I$ is the $N \times N$ identity matrix.

Let $t_0 \in \mathbb{R}$, $c \in \mathbb{R}$ and $A : [t_0, \infty) \to \mathbb{R}^{N \times N}$ be an $N \times N$ matrix of continuous functions. Then, for such time-dependent matrix $A$, we set $\Phi(t, t_0)$ and $\Psi(t, t_0)$ to be the state transition matrices corresponding to the following linear ODEs, respectively:

$$\frac{d\xi(t)}{dt} = A(t)\xi(t) \quad \text{and} \quad \frac{d\xi(t)}{dt} = [A(t) + cI]\xi(t), \quad t > t_0.$$

In the next lemma, we study a relation between $\Phi(t, t_0)$ and $\Psi(t, t_0)$ to be used in Lemma 4.3.1.

**Lemma 4.1.3.** [26] *The following relation holds.*

$$\Phi(t, t_0) = e^{-c(t-t_0)}\Psi(t, t_0), \quad \text{or} \quad \Psi(t, t_0) = e^{c(t-t_0)}\Phi(t, t_0), \quad t \geq t_0.$$

*Proof.* The proof can be found in Lemma 2.3 of [26]. □

### 4.1.5 Previous results

Before we present our main result, we review the previous result [18] about the Cucker-Smale system with deterministic switching topologies. We consider the following deterministic version of (4.0.1):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad 1 \le i \le N, \quad t > 0, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} \chi_{ij}^{\sigma} \psi(x_j - x_i)\left(v_j - v_i\right), \end{cases} \tag{4.1.9}$$

where $\psi(x)$ is given by

$$\psi(x) := \frac{K}{(1 + \|x\|^2)^{\beta}}.$$

Note that the switching law $\sigma : [0, \infty) \to \{1, \cdots, N_G\}$ becomes deterministic. Now, we assume the following condition on the switching law:

> The sequence of switching times $\{t_\ell\}_{\ell \in \mathbb{N}}$ satisfies $\tau_0 \le t_{\ell+1} - t_\ell \le T$ for some positive constants $\tau_0 < T$ for all $\ell \in \mathbb{N}$.

For convenience, we also set the following quantities:

$$\alpha := 2T(N-1), \quad \eta := e^{-NK\alpha} \min_{2 \le q \le \lfloor \frac{2T}{\tau_0} \rfloor} \tau_0^{q(N-1)} \left(\frac{K}{2}\right)^{(q-1)(N-1)},$$

$$\gamma := 2\beta(N-1), \quad b_1 := \frac{\alpha \mathcal{D}(V(0))}{\eta K^{N-1}}, \quad b_2 := \mathcal{D}(X(0)) + 1.$$

Now, we state the main result for the deterministic model (4.1.9).

**Theorem 4.1.1.** [18] *Assume that one of the following Assume that one of the following three hypotheses holds:*

*(i)* $\beta < 1/(2(N-1))$,

*(ii)* $\beta = 1/(2(N-1))$ *and* $b_1 < 1$,

*(iii)* $\beta > 1/(2(N-1))$ *and*

$$\left(\frac{1}{b_1}\right)^{\frac{1}{\gamma-1}} \left(\left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}} - \left(\frac{1}{\gamma}\right)^{\frac{\gamma}{\gamma-1}}\right) > b_2.$$

*Then the agents converge to flocking exponentially fast.*

## 4.2 A description of main result

In this section, we present a framework and main result for the emergence of stochastic flocking to the C-S model with randomly switching topologies.

### 4.2.1 Standing assumptions

Let $\{t_\ell\}_{\ell\in\mathbb{N}}$ be an increasing sequence of "random switching times" such that the increment sequence $\{t_{\ell+1} - t_\ell\}_{\ell\in\mathbb{N}}$ is a sequence of i.i.d. positive random variables on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with probability density function $f$. We also assume that the switching law $\{\sigma_t\}_{t\geq 0}$ satisfies the following conditions:

- For each $\ell \geq 0$ and $\omega \in \Omega$, $\sigma_t(\omega) = \sigma(t, \omega)$ is constant on the interval $t \in [t_\ell(\omega), t_{\ell+1}(\omega))$.
- $\{\sigma_{t_\ell}\}_{\ell\geq 0}$ is a sequence of i.i.d. random variables such that for any $\ell \geq 0$,
$$\mathbb{P}(\sigma_{t_\ell} = k) = p_k, \quad \text{for each } k = 1, \cdots, N_G,$$
where $p_1, \cdots, p_{N_G}$ are given positive constants with $p_1 + \cdots + p_{N_G} = 1$.

For each $k = 1, \cdots, N_G$, let $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$ be the $k$-th admissible digraph, and for each $t \geq 0$ and $\omega \in \Omega$, the time-dependent network topology $(\chi_{ij}^{\sigma}) = (\chi_{ij}^{\sigma_t(\omega)})$ is determined by

$$\chi_{ij}^{\sigma_t(\omega)} := \begin{cases} 1 & \text{if} \quad (j, i) \in \mathcal{E}_{\sigma_t(\omega)}, \\ 0 & \text{if} \quad (j, i) \notin \mathcal{E}_{\sigma_t(\omega)}. \end{cases}$$

For technical reasons and without loss of generality, we assume that each $\mathcal{G}_k$ has a self-loop at each vertex. For later use, we define the union graph of $\mathcal{G}_{\sigma_t(\omega)}$ for $t \in [s_0, s_1)$ and $\omega \in \Omega$ as

$$\mathcal{G}([s_0, s_1))(\omega) := \bigcup_{t\in[s_0,s_1)} \mathcal{G}^{\sigma_t(\omega)} = \left(\mathcal{V}, \bigcup_{t\in[s_0,s_1)} \mathcal{E}_{\sigma_t(\omega)}\right).$$

Note that the network topology might not actually 'switch' at the (possibly) switching instants. In other words, it might happen that $\sigma_{t_{\ell+1}}(\omega) =$

$\sigma_{t_\ell}(\omega)$ for some $\ell \geq 0$ and $\omega \in \Omega$. Now, we are ready to provide a framework for stochastic flocking to the random dynamical system (4.0.1).

For a set of admissible digraphs and the probability density function $f$ of increments of switching times, we impose the following assumption $(\mathcal{A})$ as our standing assumption throughout this chapter.

- $(\mathcal{A}1)$: The union digraph of all available network topologies in the set $\mathcal{S}$ has a spanning tree:

$$\bigcup_{1 \leq k \leq N_G} \mathcal{G}_k := \left( \mathcal{V}, \bigcup_{1 \leq k \leq N_G} \mathcal{E}_k \right) \quad \text{has a spanning tree.}$$

- $(\mathcal{A}2)$: $f$ is supported on some bounded interval with a positive lower bound, say

$$\text{supp} f \subset [a, b] \subset (0, \infty).$$

### 4.2.2 Main result

Below, we first briefly sketch our proof strategy and then present our main result. Basically, we will use matrix theory discussed in the previous section as key tools for the flocking estimate along sample paths. More precisely, we delineate our proof strategy in four steps.

- Step A (Matrix formulation): In order to use matrix theory, we rewrite the momentum equation $(4.0.1)_2$ as a matrix form:

$$\frac{d}{dt} V(t) = -\frac{1}{N} L_{\sigma_t}(t) V(t),$$

  where $L_{\sigma_t}(t)$ is the Laplacian matrix to be defined in (4.3.3) - (4.3.4).

- Step B (A priori velocity alignment estimate along a sample path): For each sample point $\omega \in \Omega$, we introduce a priori conditions:

1. ($\mathcal{P}1$): there exist $n \in \mathbb{N}$ and $c > 0$ such that $\kappa b(N-1)c < 1$, and the subsequence $\{t_\ell^*\}_{\ell\in\mathbb{N}} \subset \{t_\ell\}_{\ell\in\mathbb{N}}$ defined by $t_\ell^* := t_{a_\ell(n,c)}$ satisfies

$$\mathcal{G}([t_\ell^*, t_{\ell+1}^*))(\omega) \text{ has a spanning tree for all } \ell \geq 0,$$

   where the explicit construction of $a_\ell(n,c)$ will be given in (4.3.7).

2. ($\mathcal{P}2$): the position diameter is uniformly bounded pathwise:

$$\sup_{0\leq t<\infty} \mathcal{D}(X(t,\omega)) \leq x^\infty < \infty.$$

   Note that the constant $x^\infty$ can be chosen independent of $\omega$ in this step.

Under the above two a priori assumptions, we show that the velocity alignment estimate can emerge (Proposition 4.3.1):

$$\lim_{t\to\infty} \mathcal{D}(V(t,\omega)) = 0.$$

- Step C (Flocking along a sample path): We replace the a priori assumption ($\mathcal{P}2$) by a suitable condition on the system parameters and communication weight, and derive flocking estimates along sample path: for each $\omega \in \Omega$ satisfying ($\mathcal{P}1$),

$$\sup_{0\leq t<\infty} \mathcal{D}(X(t,\omega)) \leq x^\infty < \infty, \quad \lim_{t\to\infty} \mathcal{D}(V(t,\omega)) = 0.$$

- Step D (Stochastic flocking): We look for a suitable condition for the choice probability $p_k$ for the network selection, and construct a suitable time-block guaranteeing an existence of spanning tree in each time-block, and then under these well-prepared setting, the a priori assumption ($\mathcal{P}1$) can be attained with probability one.

We perform the above outlined strategy one by one to derive our main result on the flocking estimate of (4.0.1) .

**Theorem 4.2.1.** *Suppose that the framework* $(\mathcal{A}1)-(\mathcal{A}2)$ *holds, and system parameters* $b$, $N$, $p_k$*'s and communication weight* $\phi$ *satisfy the following conditions:*

$$\frac{\kappa b(N-1)}{\min\limits_{1\le k\le N_G}\log\frac{1}{1-p_k}}<1 \qquad \text{and} \qquad \frac{1}{\bar{\phi}(r)}=\mathcal{O}(r^{\varepsilon}) \quad \text{as} \quad r\to\infty,$$

*where* $\varepsilon$ *is a positive constant satisfying the following relation:*

$$0\le\varepsilon<\frac{1}{N-1}-\frac{\kappa b}{\min\limits_{1\le k\le N_G}\log\frac{1}{1-p_k}}.$$

*Then, for any solution process* $(X,V)$ *to* (4.0.1)*, the asymptotic flocking emerges with probability one:*

$$\mathbb{P}\Big(\omega\in\Omega:\ \exists\, x^{\infty}>0\ s.t\ \sup_{0\le t<\infty}\mathcal{D}(X(t,\omega))\le x^{\infty},\ \ \lim_{t\to\infty}\mathcal{D}(V(t,\omega))=0\Big)=1.$$

**Remark 4.2.1.** *1. The first condition on system paramters in Theorem 4.2.1 implies that* $b$ *should be small enough:*

$$\frac{\kappa b(N-1)}{\min\limits_{1\le k\le N_G}\log\frac{1}{1-p_k}}<1.$$

*Indeed, the network topology should switch frequently enough so that even if each element of* $\mathcal{S}$ *has very few edges, chances of the union digraph* $\mathcal{G}([s_0,s_1))$ *containing a spanning tree will be good enough for* $[s_0,s_1)$*'s with small length, meaning that the network topology will be 'connected enough' in some sense.*

*2. The second condition*

$$\frac{1}{\bar{\phi}(r)}=\mathcal{O}(r^{\varepsilon}) \quad \text{as} \quad r\to+\infty \quad \text{for some} \quad 0\le\varepsilon<\frac{1}{N-1}-\frac{\kappa b}{\min\limits_{1\le k\le N_G}\log\frac{1}{1-p_k}}$$

*asserts that the rate of decrease of* $\bar{\phi}$ *should be slow enough so that the strength of the interaction between each pair of agents does not decay too fast as the distance between them increases.*

## 4.3 Emergent behavior of the randomly switching system

In this section, we present a proof for Theorem 4.2.1 following the outline depicted in Section 4.2.2.

### 4.3.1 A matrix formulation

In this subsection, we first reformulate the momentum equations in $(4.0.1)_2$ so that we can use tools from matrix theory documented in Section 4.1.

Consider the momentum equations:

$$\dot{v}_i = \frac{1}{N} \sum_{j=1}^{N} \chi_{ij}^{\sigma} \phi(x_j - x_i)\left(v_j - v_i\right), \quad 1 \le i \le N. \tag{4.3.1}$$

We rearrange the terms in (4.3.1) as follows.

$$\dot{v}_i = -\frac{1}{N}\Big[\Big(\sum_{j=1}^{N} \chi_{ij}^{\sigma}\phi(x_i - x_j)\Big)v_i - \sum_{j=1}^{N} \chi_{ij}^{\sigma}\phi(x_i - x_j)v_j\Big]. \tag{4.3.2}$$

For the matrix formulation of (4.3.2), we introduce $N \times N$ Laplacian matrices $L_k(t)$ $(k = 1, \cdots, N_G)$ as follows:

$$L_k(t) := D_k(t) - A_k(t), \tag{4.3.3}$$

where $A_k(t) = \left(a_{ij}^k(t)\right)$ and $D_k(t) = \mathrm{diag}\left(d_1^k(t), \cdots, d_N^k(t)\right)$ are written as

$$a_{ij}^k(t) := \chi_{ij}^k \phi(x_i(t) - x_j(t)) \quad \text{and} \quad d_i^k(t) = \sum_{j=1}^{N} \chi_{ij}^k \phi(x_i(t) - x_j(t)). \tag{4.3.4}$$

Thus, system (4.3.2) can be rewritten as

$$\frac{d}{dt}V(t) = -\frac{1}{N}L_{\sigma_t}(t)V(t). \tag{4.3.5}$$

Let $\Phi(t_2, t_1)$ be the state transition matrix associated with (4.3.5) on the interval $[t_1, t_2]$. Then we have the representation formula for $V$:

$$V(t_2) = \Phi(t_2, t_1)V(t_1), \quad t_2 \ge t_1 \ge 0. \tag{4.3.6}$$

### 4.3.2 Pathwise flocking under a priori assumptions

In this subsection, we study the emergence of stochastic flocking estimate under a priori assumption on the uniform bound for position diameter. From now on, we present *a priori* flocking estimates for each fixed sample $\omega \in \Omega$. In the sequel, as long as there is no confusion, we frequently suppress the $\omega$-dependence of solution processes or parameters for convenience.

**A priori assumptions**: For each positive integer $n$ and positive real number $c > 0$, we define an increasing sequence $\{a_\ell(n,c)\}_{\ell\in\mathbb{N}}$ of integers by the following recurrence relation:

$$a_0(n,c) = 0, \qquad a_{\ell+1}(n,c) = a_\ell(n,c)+n+\lfloor c\log(\ell+1)\rfloor, \qquad (\ell \in \mathbb{N}). \quad (4.3.7)$$

Let $\omega \in \Omega$ be fixed, and let $(X, V)$ be a solution process to (4.0.1). Then, our two a priori assumptions are as follows:

- ($\mathcal{P}1$): there exist $n \in \mathbb{N}$, $n > 0$ and $c > 0$ such that $\kappa b(N-1)c < 1$, and the subsequence $\{t^*_\ell\}_{\ell\in\mathbb{N}} \subset \{t_\ell\}_{\ell\in\mathbb{N}}$ defined by $t^*_\ell := t_{a_\ell(n,c)}$ in (4.3.7) satisfies

$$\mathcal{G}([t^*_\ell, t^*_{\ell+1}))(\omega) \text{ has a spanning tree for all } \ell \geq 0,$$

- ($\mathcal{P}2$): the position diameter is uniformly bounded in time:

$$\sup_{0\leq t<\infty} \mathcal{D}(X(t,\omega)) \leq x^\infty < \infty.$$

**Lemma 4.3.1.** *Suppose that $\omega \in \Omega$ satisfies the a priori assumptions ($\mathcal{P}1$) and ($\mathcal{P}2$) and let $x^\infty \geq \sup_{0\leq t<\infty} \mathcal{D}(X(t,\omega))$ be given. Then, the transition matrix $\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})$ is stochastic and its ergodicity coefficient satisfies*

$$\mu\left(\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})\right) \geq e^{-\kappa(t^*_{r(N-1)}-t^*_{(r-1)(N-1)})}\left(\frac{a}{N}\right)^{N-1}\bar{\phi}(x^\infty)^{N-1}. \quad (4.3.8)$$

*Proof.* First, we focus on the second assertion, and we claim:

$$\begin{aligned}&\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})\\ &\quad\geq e^{-\kappa(t^*_{r(N-1)}-t^*_{(r-1)(N-1)})}\left(\frac{a}{N}\right)^{N-1}\bar{\phi}(x^\infty)^{N-1}\prod_{i=1}^{N-1}F_i,\end{aligned} \tag{4.3.9}$$

where, for each $i = 1, \ldots, N-1$, $F_i$ is the $(0,1)$-adjacency matrix of the union digraph

$$\mathcal{G}([t^*_{(r-1)(N-1)+i-1}, t^*_{(r-1)(N-1)+i})).$$

*Proof of claim* (4.3.9): Let $\{t_{\ell_1}, t_{\ell_2}, \ldots, t_{\ell_{q+1}}\}$ be the subsequence of $\{t_\ell\}_{\ell\geq 0}$ contained in the interval $[t^*_{(r-1)(N-1)+i-1}, t^*_{(r-1)(N-1)+i}]$ such that

$$t_{\ell_1} = t^*_{(r-1)(N-1)+i-1} \quad \text{and} \quad t_{\ell_{q+1}} = t^*_{(r-1)(N-1)+i}.$$

We set

$$\sigma_t = k_p \quad \text{for } t \in [t_{\ell_p}, t_{\ell_{p+1}}) \text{ and } p = 1, \ldots, q.$$

Then we have

$$\Phi(t^*_{(r-1)(N-1)+i}, t^*_{(r-1)(N-1)+i-1}) = \Phi_{k_q}(t_{\ell_{q+1}}, t_{\ell_q})\cdots\Phi_{k_1}(t_{\ell_2}, t_{\ell_1}), \tag{4.3.10}$$

where, for $p = 1, \ldots, q$, $\Phi_{k_p}(t_{\ell_{p+1}}, t_{\ell_p})$ is the state transition matrix corresponding to system (4.3.6) on $[t_{\ell_p}, t_{\ell_{p+1}})$. We need to estimate $\Phi_{k_p}(t_{\ell_{p+1}}, t_{\ell_p})$ and for this, we estimate the coefficient matrix for (4.3.5) as follows:

$$-\frac{1}{N}L_{k_p}(t) = \frac{1}{N}(A_{k_p}(t) - D_{k_p}(t)) \geq \frac{1}{N}\underline{A}_{k_p} - \kappa I, \tag{4.3.11}$$

where $\underline{A}_{k_p} = (\underline{a}^{k_p}_{ij})$ is given by

$$\underline{a}^{k_p}_{ij} := \begin{cases}\chi^{k_p}_{ij}\bar{\phi}(x^\infty), & i \neq j,\\ \kappa, & i = j.\end{cases}$$

Then, the relation (4.3.11) implies

$$-\frac{1}{N}L_{k_p}(t) + \kappa I \geq \frac{1}{N}\underline{A}_{k_p} \geq 0. \tag{4.3.12}$$

On the other hand, let $\Psi_{k_p}(t_{\ell_{p+1}}, t_{\ell_p})$ be the state transition matrix of

$$-\frac{1}{N}L_{k_p}(t) + \kappa I \quad \text{on } [t_{\ell_p}, t_{\ell_{p+1}}).$$

Then it follows from Lemma 4.1.3 that

$$\Phi_{k_p}\left(t_{\ell_{p+1}}, t_{\ell_p}\right) = e^{-\kappa(t_{\ell_{p+1}} - t_{\ell_p})}\Psi_{k_p}\left(t_{\ell_{p+1}}, t_{\ell_p}\right). \tag{4.3.13}$$

Now, we can apply (4.3.12) to the Peano-Baker series to obtain

$$\begin{aligned}
&\Psi_{k_p}\left(t_{\ell_{p+1}}, t_{\ell_p}\right) \\
&= I + \sum_{n=1}^{\infty}\int_{t_{\ell_p}}^{t_{\ell_{p+1}}}\int_{t_{\ell_p}}^{\tau_1}\cdots\int_{t_{\ell_p}}^{\tau_{n-1}} \begin{bmatrix} \left(-\frac{1}{N}L_{k_p}(\tau_1) + \kappa I\right) \\ \cdots\left(-\frac{1}{N}L_{k_p}(\tau_n) + \kappa I\right) \end{bmatrix} d\tau_n \cdots d\tau_1 \\
&\geq I + \sum_{n=1}^{\infty}\int_{t_{\ell_p}}^{t_{\ell_{p+1}}}\int_{t_{\ell_p}}^{\tau_1}\cdots\int_{t_{\ell_p}}^{\tau_{n-1}} \left(\frac{1}{N}\underline{A}_{k_p}\right)^n d\tau_n \cdots d\tau_1 \\
&= I + \sum_{n=1}^{\infty}\frac{1}{n!}(t_{\ell_{p+1}} - t_{\ell_p})^n\left(\frac{1}{N}\underline{A}_{k_p}\right)^n \\
&\geq I + \frac{a}{N}\underline{A}_{k_p}.
\end{aligned} \tag{4.3.14}$$

We combine (4.3.13) with (4.3.14) to obtain

$$\Phi_{k_r}\left(t_{\ell_{p+1}}, t_{\ell_p}\right) \geq e^{-\kappa(t_{\ell_{p+1}} - t_{\ell_p})}\left(I + \frac{a}{N}\underline{A}_{k_p}\right). \tag{4.3.15}$$

Then, the relation (4.3.15) and (4.3.10) yield

$$\begin{aligned}
&\Phi(t^*_{(r-1)(N-1)+i}, t^*_{(r-1)(N-1)+i-1}) \\
&\quad \geq e^{-\kappa(t_{\ell_{q+1}} - t_{\ell_1})}\left(I + \frac{a}{N}\underline{A}_{k_q}\right)\cdots\left(I + \frac{a}{N}\underline{A}_{k_1}\right) \\
&\quad \geq e^{-\kappa(t^*_{(r-1)(N-1)+i} - t^*_{(r-1)(N-1)+i-1})}\frac{a}{N}(\underline{A}_{k_q} + \cdots + \underline{A}_{k_1}).
\end{aligned} \tag{4.3.16}$$

Here, one has

$$\underline{A}_{k_q} + \cdots + \underline{A}_{k_1} \geq \bar{\phi}(x^\infty)F_i. \tag{4.3.17}$$

Now, we combine (4.3.16) with (4.3.17) to obtain

$$\Phi(t^*_{(r-1)(N-1)+i}, t^*_{(r-1)(N-1)+i-1}) \geq e^{-\kappa(t^*_{(r-1)(N-1)+i} - t^*_{(r-1)(N-1)+i-1})} \frac{a}{N} \bar{\phi}(x^\infty) F_i.$$

This implies

$$\begin{aligned} &\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)}) \\ &\quad = \prod_{i=1}^{N-1} \Phi(t^*_{(r-1)(N-1)+i}, t^*_{(r-1)(N-1)+i-1}) \\ &\quad \geq e^{-\kappa(t^*_{r(N-1)} - t^*_{(r-1)(N-1)})} \left(\frac{a}{N}\right)^{N-1} \bar{\phi}(x^\infty)^{N-1} \prod_{i=1}^{N-1} F_i. \end{aligned} \tag{4.3.18}$$

This verifies the claim (4.3.9). Since the union digraph

$$\mathcal{G}([t^*_{(r-1)(N-1)+i-1}, t^*_{(r-1)(N-1)+i})) = \mathcal{G}(F_i)$$

has a spanning tree, we apply Proposition 4.1.1 to see that $F_1 F_2 \dots F_{N-1}$ is scrambling and moreover, (4.1.6) yields

$$\mu\left(\prod_{i=1}^{N-1} F_i\right) \geq 1. \tag{4.3.19}$$

Hence, we use (4.1.7) and (4.3.19) to get

$$\mu\left(\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})\right) \geq e^{-\kappa(t^*_{r(N-1)} - t^*_{(r-1)(N-1)})} \left(\frac{a}{N}\right)^{N-1} \bar{\phi}(x^\infty)^{N-1}.$$

This verifies the relation (4.3.8).

For the first assertion, $\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})$ is nonnegative by (4.3.18). So it remains to show that each of its rows sums to 1. Note that the constant state $\xi(t) := [\xi_1(t), \cdots, \xi_N(t)]^\top \equiv [1, \cdots, 1]^\top$ is a solution to (4.3.5):

$$\frac{d}{dt}\xi(t) = -\frac{1}{N} L_{\sigma_t}(t)\xi(t).$$

Hence,

$$[1, \cdots, 1]^\top = \Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})[1, \cdots, 1]^\top.$$

This implies that $\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})$ is stochastic.

□

**Proposition 4.3.1.** (A priori velocity alignment) *Suppose that $\omega \in \Omega$ satisfies the a priori assumptions $(\mathcal{P}1) - (\mathcal{P}2)$ and let $x^\infty \geq \sup_{0 \leq t < \infty} \mathcal{D}(X(t,\omega))$ be given. Then, for all $t \in [t^*_{r(N-1)}, t^*_{(r+1)(N-1)})$ with $r \in \mathbb{N}$, we have*

$$\mathcal{D}(V(t)) \leq \mathcal{D}(V(0)) \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)e^{-\kappa bn}(N-1)^{-\kappa bc}}{N}\right)^{N-1} \frac{(r+1)^{1-\kappa b(N-1)c}-1}{1-\kappa b(N-1)c}\right].$$

*Proof.* Since $\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})$ is stochastic (Lemma 4.3.1), we combine Lemma 4.1.2 and Lemma 4.3.1 to obtain that for $t \in [t^*_{r(N-1)}, t^*_{(r+1)(N-1)})$,

$$\begin{aligned}
&\mathcal{D}\left(V(t)\right) \\
&\quad \leq \mathcal{D}\left(V(t^*_{r(N-1)})\right) \\
&\quad \leq \left[1 - \mu\Big(\Phi(t^*_{r(N-1)}, t^*_{(r-1)(N-1)})\Big)\right] \mathcal{D}(V(t^*_{(r-1)(N-1)})) \\
&\quad \leq \left[1 - e^{-\kappa(t^*_{r(N-1)} - t^*_{(r-1)(N-1)})} \left(\frac{a\bar{\phi}(x^\infty)}{N}\right)^{N-1}\right] \mathcal{D}(V(t^*_{(r-1)(N-1)})) \\
&\quad \leq \exp\left[-e^{-\kappa(t^*_{r(N-1)} - t^*_{(r-1)(N-1)})} \left(\frac{a\bar{\phi}(x^\infty)}{N}\right)^{N-1}\right] \mathcal{D}(V(t^*_{(r-1)(N-1)})) \\
&\quad \leq \cdots \leq \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)}{N}\right)^{N-1} \sum_{i=1}^{r} e^{-\kappa(t^*_{i(N-1)} - t^*_{(i-1)(N-1)})}\right] \mathcal{D}(V(0)) \\
&\quad \leq \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)}{N}\right)^{N-1} \sum_{i=1}^{r} e^{-\kappa b(a_{i(N-1)}(n,c) - a_{(i-1)(N-1)}(n,c))}\right] \mathcal{D}(V(0)) \\
&\quad = \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)}{N}\right)^{N-1} \sum_{i=1}^{r} e^{-\kappa b\left((N-1)n + \sum_{j=(i-1)(N-1)+1}^{i(N-1)} \lfloor c\log j \rfloor\right)}\right] \mathcal{D}(V(0)) \\
&\quad \leq \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)}{N}\right)^{N-1} \sum_{i=1}^{r} e^{-\kappa b(N-1)(n + c\log(i(N-1)))}\right] \mathcal{D}(V(0)) \\
&\quad = \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)e^{-\kappa bn}(N-1)^{-\kappa bc}}{N}\right)^{N-1} \sum_{i=1}^{r} i^{-\kappa b(N-1)c}\right] \mathcal{D}(V(0)) \\
&\quad \leq \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)e^{-\kappa bn}(N-1)^{-\kappa bc}}{N}\right)^{N-1} \int_1^{r+1} x^{-\kappa b(N-1)c} dx\right] \mathcal{D}(V(0))
\end{aligned}$$

$$\leq \exp\left[-\left(\frac{a\bar{\phi}(x^\infty)e^{-\kappa bn}(N-1)^{-\kappa bc}}{N}\right)^{N-1}\frac{(r+1)^{1-\kappa b(N-1)c}-1}{1-\kappa b(N-1)c}\right]\mathcal{D}(V(0)).$$

□

Next, we assert that our a priori condition on the uniform boundedness of distances between C-S particles can be obtained from other existing a priori conditions. Before we move on, we present a technical lemma.

**Lemma 4.3.2.** *For any $x > 0$ and $\delta > 0$, we have the following inequality:*

$$e^{-x} \leq \left(\frac{\delta}{e}\right)^{\delta} x^{-\delta}.$$

*Proof.* By differentiation, we can check that the function $x \mapsto -x + \delta \log x$ attains its maximal value at $x = \delta$. Hence

$$-x + \delta \log x \leq -\delta + \delta \log \delta, \quad x > 0,\ \delta > 0.$$

We take the exponential of both sides to get

$$e^{-x}x^{\delta} \leq e^{-\delta}\delta^{\delta} \quad \Longrightarrow \quad e^{-x} \leq \left(\frac{\delta}{e}\right)^{\delta} x^{-\delta}.$$

□

Next, we show that the a priori assumption ($\mathcal{P}2$) on the position diameter can be replaced by the condition on the initial data so that we can establish pathwise flocking estimate only under the a priori condition ($\mathcal{P}1$).

**Proposition 4.3.2.** *Suppose that $\omega \in \Omega$ satisfies a priori condition ($\mathcal{P}1$) and there exist $\delta > 0$ and $x^\infty > 0$ independent of a sample point such that*

$$\begin{aligned}&\mathcal{D}(X(0)) + \mathcal{D}(V(0))b(N-1)(n + c\log((N-1)))\\ &+ \mathcal{D}(V(0))b(N-1)\left(\frac{\delta}{e}\right)^{\delta}\left(\frac{a\bar{\phi}(x^\infty)e^{-\kappa bn}(N-1)^{-\kappa bc}}{N}\right)^{-(N-1)\delta}\\ &\times\sum_{r=1}^{\infty}\left[(n+c\log((r+1)(N-1)))\left(\frac{(r+1)^{1-\kappa b(N-1)c}-1}{1-\kappa b(N-1)c}\right)^{-\delta}\right] < x^\infty,\end{aligned} \tag{4.3.20}$$

*and let $(X, V)$ be a solution process to (4.0.1). Then, a priori condition ($\mathcal{P}2$) holds: for $\omega \in \Omega$,*

$$\sup_{0\leq t<\infty} \mathcal{D}(X(t,\omega)) < x^\infty < \infty.$$

*Proof.* We use a contradiction argument for the desired estimate. For this, we define a set $\mathcal{T}$ and its supremum as follows:

$$\mathcal{T} := \Big\{ T > 0 : \quad \max_{0 \le t \le T} \mathcal{D}(X(t)) < x^{\infty} \Big\}, \quad T^* := \sup \mathcal{T}.$$

By assumption (4.3.20) and the continuity of $\mathcal{D}(X)$, the set $\mathcal{T}$ is nonempty. Now, we claim:

$$\sup \mathcal{T} = \infty.$$

Suppose not, i.e. $T^* := \sup \mathcal{T} < \infty$. Then, we have

$$\mathcal{D}(X(T^*)) = x^{\infty}. \tag{4.3.21}$$

It follows from Proposition 4.3.1 and Lemma 4.3.2 that we have

$$\begin{aligned}
x^{\infty} &= \mathcal{D}(X(T^*)) \le \mathcal{D}(X(0)) + \int_0^{T^*} \mathcal{D}(V(t))dt \\
&\le \mathcal{D}(X(0)) + \mathcal{D}(V(0)) \sum_{r=0}^{\infty} \Bigg[ (t^*_{(r+1)(N-1)} - t^*_{r(N-1)}) \\
&\quad \times \exp\left( - \left( \frac{a\bar{\phi}(x^{\infty}) e^{-\kappa b n} (N-1)^{-\kappa b c}}{N} \right)^{N-1} \frac{(r+1)^{1-\kappa b (N-1) c} - 1}{1 - \kappa b (N-1) c} \right) \Bigg] \\
&\le \mathcal{D}(X(0)) + \mathcal{D}(V(0)) \sum_{r=0}^{\infty} \Bigg[ b(N-1)(n + c \log((r+1)(N-1))) \\
&\quad \times \exp\left( - \left( \frac{a\bar{\phi}(x^{\infty}) e^{-\kappa b n} (N-1)^{-\kappa b c}}{N} \right)^{N-1} \frac{(r+1)^{1-\kappa b (N-1) c} - 1}{1 - \kappa b (N-1) c} \right) \Bigg] \\
&\le \mathcal{D}(X(0)) + \mathcal{D}(V(0)) b(N-1)(n + c\log((N-1))) \\
&\quad + \mathcal{D}(V(0)) \sum_{r=1}^{\infty} \Bigg[ b(N-1)(n + c\log((r+1)(N-1))) \left( \frac{\delta}{e} \right)^{\delta} \\
&\quad \times \left( \left( \frac{a\bar{\phi}(x^{\infty}) e^{-\kappa b n} (N-1)^{-\kappa b c}}{N} \right)^{N-1} \frac{(r+1)^{1-\kappa b (N-1) c} - 1}{1 - \kappa b (N-1) c} \right)^{-\delta} \Bigg] \\
&< x^{\infty}.
\end{aligned}$$

This yields a contradiction to (4.3.21). Therefore we have $\sup \mathcal{T} = \infty$. □

As a corollary, we can use Proposition 4.3.2 to prove that a priori condition on network structures together with conditions in Theorem 4.2.1 implies the uniform boundedness of distances between particles and the velocity relaxation estimates for any initial configuration.

**Corollary 4.3.1.** *Suppose that $\omega \in \Omega$ satisfies a priori condition $(\mathcal{P}1)$ holds, and in addition, the communication weight $\phi$ satisfies*

$$\frac{1}{\bar{\phi}(r)} = \mathcal{O}(r^{\varepsilon}) \quad \text{as } r \to \infty,$$

*where $\varepsilon$ is a positive constant satisfying the relation $0 \le \varepsilon < \frac{1-\kappa b(N-1)c}{N-1}$. Then the mono-cluster flocking emerges pathwise for any initial configuration:*

$$\sup_{0 \le t < \infty} \mathcal{D}(X(t,\omega)) < \infty \quad \text{and} \quad \lim_{t \to \infty} \mathcal{D}(V(t,\omega)) = 0.$$

*Proof.* We choose a positive number $\delta > 0$ such that

$$\frac{1}{1-\kappa b(N-1)c} < \delta < \frac{1}{(N-1)\varepsilon}. \tag{4.3.22}$$

The left-hand side in (4.3.22) implies

$$\sum_{r=1}^{\infty} \left[ (n + c\log((r+1)(N-1))) \left( \frac{(r+1)^{1-\kappa b(N-1)c} - 1}{1-\kappa b(N-1)c} \right)^{-\delta} \right] < \infty.$$

Moreover, the right-hand side in (4.3.22) implies

$$\bar{\phi}(r)^{-(N-1)\delta} = O(r^{(N-1)\delta\varepsilon}) \quad \text{as } r \to \infty.$$

Hence, one has

$$\lim_{|x| \to \infty} \frac{\bar{\phi}(r)^{-(N-1)\delta}}{r} = 0.$$

This implies the existence of $x^{\infty}$ satisfying (4.3.20) for $\delta$ chosen in (4.3.22). Hence the condition (4.3.20) is satisfied, and the results follow from Proposition 4.3.1 and Proposition 4.3.2. □

### 4.3.3 Emergence of stochastic flocking

In the previous section, we verified the emergence of pathwise flocking under the a priori assumption on the network structure ($\mathcal{P}1$). In the sequel, we will show that the a priori assumption ($\mathcal{P}1$) can be guaranteed with probability one.

Next step is to prove that *a priori* assumption ($\mathcal{P}1$) on network structure can be satisfied for most of $\omega \in \Omega$, once we determine appropriate values for $n$ and $c$.

**Proposition 4.3.3.** *Let $(X, V)$ be a solution process to (4.0.1), and let $n \in \mathbb{N}$ and $c > 0$ be such that*

$$\sum_{k=1}^{N_G} (1-p_k)^n \le \frac{1}{2} \quad \text{and} \quad c > \frac{1}{\min\limits_{1\le k\le N_G} \log \frac{1}{(1-p_k)}}.$$

*Then, the following assertions hold.*

1. *The subsequence $\{t_\ell^*\}_{\ell\in\mathbb{N}} \subset \{t_\ell\}_{\ell\in\mathbb{N}}$, defined by $t_\ell^* := t_{a_\ell(n,c)}$ in (4.3.7), satisfies*

$$\begin{aligned}&\mathbb{P}\Big(\omega : \mathcal{G}([t_\ell^*, t_{\ell+1}^*))(\omega) \text{ has a spanning tree for any } \ell \ge 0\Big)\\ &\qquad \ge \exp\left(-(2\log 2)\sum_{k=1}^{N_G}(1-p_k)^n \sum_{\ell=0}^{\infty}(1-p_k)^{\lfloor c\log(\ell+1)\rfloor}\right).\end{aligned}$$

2. *The series $\sum\limits_{\ell=0}^{\infty}(1-p_k)^{\lfloor c\log(\ell+1)\rfloor}$ converges for all $k = 1, \cdots, N_G$.*

*Proof.* (i) For any $q$, $r \in \mathbb{N}$, we have the following estimate:

$$\begin{aligned}&\mathbb{P}(\omega : \mathcal{G}([t_q, t_{q+r}))(\omega) \text{ does not have a spanning tree})\\ &\quad \le \mathbb{P}(\omega : \exists 1 \le k \le N_G \ \text{ such that } \sigma_{t_{q+i}}(\omega) \ne k \text{ for } \forall\ 0 \le i \le r-1)\\ &\quad \le \sum_{k=1}^{N_G} \mathbb{P}(\omega : \sigma_{t_{q+i}}(\omega) \ne k \text{ for } \forall\ 0 \le i \le r-1) = \sum_{k=1}^{N_G}(1-p_k)^r,\end{aligned}$$

where the last inequality follows from the independence of $\{t_{\ell+1} - t_\ell\}_{\ell\in\mathbb{N}}$.

This implies

$$\mathbb{P}(\omega : \mathcal{G}([t_q, t_{q+r}))(\omega) \text{ has a spanning tree}) \geq 1 - \sum_{k=1}^{N_G}(1-p_k)^r.$$

Here, we substitute $a_\ell(n,c)$ and $a_{\ell+1}(n,c)$ for $q$ and $q+r$, respectively, and take the product over $\ell \in \mathbb{N}$ to see the following relations:

$$\begin{aligned}
&\mathbb{P}(\omega : \mathcal{G}([t^*_\ell, t^*_{\ell+1}))(\omega) \text{ has a spanning tree for any } \ell \in \mathbb{N}) \\
&= \prod_{\ell=0}^{\infty} \mathbb{P}(\omega : \mathcal{G}([t^*_\ell, t^*_{\ell+1}))(\omega) \text{ has a spanning tree }) \\
&\geq \prod_{\ell=0}^{\infty}\left(1 - \sum_{k=1}^{N_G}(1-p_k)^{a_{\ell+1}-a_\ell}\right) \\
&= \exp\left(\sum_{\ell=0}^{\infty}\log\left(1 - \sum_{k=1}^{N_G}(1-p_k)^{n+\lfloor c\log(\ell+1)\rfloor}\right)\right) \\
&\geq \exp\left(-(2\log 2)\sum_{\ell=0}^{\infty}\sum_{k=1}^{N_G}(1-p_k)^{n+\lfloor c\log(\ell+1)\rfloor}\right) \\
&= \exp\left(-(2\log 2)\sum_{k=1}^{N_G}(1-p_k)^n\sum_{\ell=0}^{\infty}(1-p_k)^{\lfloor c\log(\ell+1)\rfloor}\right),
\end{aligned}$$

where we used the following inequality:

$$\log(1-x) \geq -(2\log 2)x, \quad 0 \leq x \leq \frac{1}{2}.$$

(ii) The convergence of the series $\sum_{\ell=0}^{\infty}(1-p_k)^{\lfloor c\log(\ell+1)\rfloor}$ can be shown as follows: by comparison test, it suffices to show that

$$\sum_{\ell=0}^{\infty}(1-p_k)^{c\left(\log(\ell+1)-1\right)} < \infty \quad \Longleftrightarrow \quad \sum_{\ell=1}^{\infty}(1-p_k)^{c\log\ell} < \infty.$$

By Cauchy's condensation test, the right-hand side of the above is equivalent to

$$\sum_{\ell=1}^{\infty} 2^{\ell}(1-p_k)^{c\log(2^{\ell})} = \sum_{\ell=1}^{\infty} \left(2(1-p_k)^{c\log 2}\right)^{\ell} < \infty.$$

The condition $c > \frac{1}{\log \frac{1}{(1-p_k)}}$ is equivalent to $0 < 2(1-p_k)^{c\log 2} < 1$. Thus, we have the desired result. □

**The proof of Theorem 4.2.1**: We choose $\varepsilon$ to satisfy

$$0 \le \varepsilon < \frac{1}{N-1} - \frac{\kappa b}{\min\limits_{1\le k\le N_G} \log \frac{1}{1-p_k}},$$

$$\text{or equivalently} \quad \frac{1}{\min\limits_{1\le k\le N_G} \log \frac{1}{1-p_k}} < \frac{1-\varepsilon(N-1)}{\kappa b(N-1)},$$

and we set

$$c := \frac{1}{2}\left( \frac{1}{\min\limits_{1\le k\le N_G} \log \frac{1}{1-p_k}} + \frac{1-\varepsilon(N-1)}{\kappa b(N-1)} \right).$$

Then, it is easy to see that the constant $c$ defined above satisfies

$$c > \frac{1}{\min\limits_{1\le k\le N_G} \log \frac{1}{1-p_k}}, \quad \kappa b(N-1)c < 1, \quad \text{and} \quad 0 \le \varepsilon < \frac{1-\kappa b(N-1)c}{N-1}.$$

Now, we choose any $n \in \mathbb{N}$ such that

$$\sum_{k=1}^{N_G} (1-p_k)^n \le \frac{1}{2},$$

and we define $p(n)$ as

$$p(n) := \exp\left[ -(2\log 2)\sum_{k=1}^{N_G}(1-p_k)^n \sum_{\ell=0}^{\infty}(1-p_k)^{\lfloor c\log(\ell+1)\rfloor} \right].$$

With this choice of $n$ and $c$, Proposition 4.3.3 implies, for $t_\ell^* := t_{a_l(n,c)}$,

$$\mathbb{P}\Big\{\omega : \mathcal{G}([t_\ell^*, t_{\ell+1}^*))(\omega) \text{ has a spanning tree for any } \ell \ge 0\Big\} \ge p(n).$$

Hence, it follows from Corollary 4.3.1 that

$$\mathbb{P}\Big\{\omega : \exists\, x^{\infty} > 0 \text{ s.t } \sup_{0\le t<\infty} \mathcal{D}(X(t,\omega)) \le x^{\infty} \text{ and } \lim_{t\to\infty} \mathcal{D}(V(t,\omega)) = 0\Big\} \ge p(n).$$

Since $n$ can be arbitrarily large and $p(n) \to 1$ as $n \to \infty$, our desired result follows.

# Chapter 5

# Collective stochastic dynamics of the Cucker-Smale ensemble under uncertain communication

In this chaper, we consider the kinetic Cucker-Smale equation perturbed by a multiplicative noise (1.0.6):

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (F_a[f_t] f_t) = \sigma \nabla_v \cdot ((v - v_c) f_t) \circ \dot{W}_t, \quad t > 0, \quad (5.0.1)$$

subject to deterministic initial data

$$f_0(x, v) = f^{in}(x, v), \quad (x, v) \in \mathbb{R}^{2d}.$$

Main results of this chapter are two-fold. First, we prove a global well-posedness for strong solutions by employing three tools, i.e. regularization of initial data, $W^{m,\infty}$-estimates along the stochastic characteristics and a suitable choice of a stopping time. Second, we show the emergence of flocking in the kinetic level by showing dissipation estimates for the second velocity moment. If the communication weight function $\phi$ has a positive infimum, then the second velocity moment converges to 0 for each sample path. Moreover, once noise strength $\sigma$ is sufficiently smaller than $\phi_m$, then the expectation of the second velocity moment converges to 0 at an exponential rate.

The rest of this chapter is organized as follows. In Section 5.1, we provide a rigorous derivation of equation (5.0.1) from the C-S system with a multiplicative noise, and then briefly discuss our main results on the global well-posedness for strong solutions and asymptotic flocking estimates of classical solutions. In Section 5.2, we provide several a priori estimates for classical solutions to (5.0.1). In Section 5.3, we show our global well-posedness and emergent dynamics for strong solutions to (5.0.1). Finally, note that this chapter is based on the joint work [42].

## 5.1 Preliminaries

In this section, we provide a rigorous derivation of the equation (5.0.1) and present our main results on the global well-posedness of strong solutions to (5.0.1) and emergent flocking dynamics.

### 5.1.1 Derivation of the SPDE

In this subsection, following [12], we present a derivation of (5.0.1) from the C-S system perturbed by a multiplicative noise. To be specific, we begin our discussion with the C-S model [20].

Let $(x_t^i, v_t^i) \in \mathbb{R}^d \times \mathbb{R}^d$ be the position and velocity of the $i$-th particle at time $t \geq 0$, respectively. Then, the ensemble of C-S particles is governed by the following system:

$$\begin{aligned} dx_t^i &= v_t^i dt, \quad t > 0, \quad 1 \leq i \leq N, \\ dv_t^i &= F_a[\mu_t^N](x_t^i, v_t^i)dt, \quad \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{(x_t^i, v_t^i)}, \end{aligned} \tag{5.1.1}$$

where the flocking force $F_a$ is given in $(1.0.2)_2$. However, in a real world situation, the communication among particles is subject to the neighboring environment, which is an extrinsic randomness missing in the model. To reflect these effects in the communication, stochastic noises can be incorporated into the communication weight $\phi$ appearing in system (5.1.1). To address a stochastic perturbation in system (5.1.1), we replace $\phi$ by $\phi + \sigma \circ \dot{W}_t$ and yield the following system of stochastic differential equations:

$$\begin{aligned} dx_t^i &= v_t^i dt, \quad t > 0, \ 1 \le i \le N, \\ dv_t^i &= F_a[\mu_t^N](x_t^i, v_t^i)dt + \sigma(\bar{v}_t - v_t^i) \circ dW_t^i, \quad \bar{v}_t := \frac{1}{N}\sum_{i=1}^N v_t^i. \end{aligned} \tag{5.1.2}$$

Let us compare (5.1.2) with the model presented in [2], where the authors replaced $\phi$ in (5.1.1) by $\phi+\sigma\dot{W}_t$ to obtain the C-S system with a multiplicative noise in Itô's sense. However, we adopt the integral in Stratonovich's sense rather than Itô's sense, since it enables us to use the method of stochastic characteristics once we derive a stochastic partial differential equation from system (5.1.2). Moreover, it is natural in the following sense: for each $1 \le i \le N$, let $W_t^{i,\varepsilon}$ be a smooth approximation to the Wiener process $W_t^i$ (e.g. approximation by using a mollifier). Now, we consider the following system of deterministic equations:

$$\begin{aligned} dx_t^{i,\varepsilon} &= v_t^{i,\varepsilon} dt, \quad t > 0, \ 1 \le i \le N, \\ dv_t^{i,\varepsilon} &= F_a[\mu_t^{N,\varepsilon}](x_t^{i,\varepsilon}, v_t^{i,\varepsilon})dt + \sigma\left(\bar{v}_t^{\varepsilon} - v_t^{i,\varepsilon}\right) dW_t^{i,\varepsilon}, \end{aligned} \tag{5.1.3}$$

Then, the Wong-Zakai theorem [97, 105, 106] implies that the solution to system (5.1.3) converges in probability to the solution to system (5.1.2). Here, we note that system (5.1.2) is equivalent to the following Itô equation [31]:

$$\begin{aligned} dx_t^i &= v_t^i dt, \\ dv_t^i &= \left[F_a[\mu_t^N](x_t^i, v_t^i) - \frac{1}{2}\sigma^2(\bar{v}_t - v_t^i)\right] dt + \sigma(\bar{v}_t - v_t^i) dW_t^i. \end{aligned} \tag{5.1.4}$$

When $W^i$'s are i.i.d Wiener processes, a similar analysis as in [36] yields the mean field limit of system (5.1.4) as $N \to \infty$, which is the following Fokker-Planck type equation: for $t > 0$ and $(x, v) \in \mathbb{R}^{2d}$,

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left[\left(F_a[f] - \frac{1}{2}\sigma^2(v_c - v)\right) f\right] = \sigma \Delta_v(|v - v_c|^2 f),$$

where $v_c := \int_{\mathbb{R}^d \times \mathbb{R}^d} vf\,dxdv$. However, if each $W_t^i$ is identical to a single Wiener process $W$, i.e. $W^i \equiv W_t$, we can use a propagation of chaos result [12] to obtain that the empirical measure $\mu_t^N$ associated with system (5.1.2) converges to a measure-valued solution to (5.0.1). Let us summarize the results on the mean-field limit and asymptotic flocking estimates in [12] as follows.

**Theorem 5.1.1.** *Suppose that $T > 0$ and consider a communication weight $\phi$ with $\bar{\phi} \in \mathcal{C}_b^1(\mathbb{R}_+)$, and let $\mu_0, \tilde{\mu}_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ be compactly supported in velocity. Then, the following assertions hold.*

1. *If $\mu$ and $\tilde{\mu}$ are two measure-valued solutions to* (5.0.1) *with compactly supported initial data $\mu_0$ and $\tilde{\mu}_0$ in velocity, then*

$$W_2(\mu_t, \tilde{\mu}_t) \le C W_2(\mu_0, \tilde{\mu}_0) e^{C(1+W_2(\mu_0,\tilde{\mu}_0)}, \quad \textit{for a.s. } t \in [0, T],$$

*where the constant $C$ depends only on $\phi$, $T$, $\sigma$, $\sup_{t\in[0,T]} |B_t|$, and the support in velocity of $\mu_0$ and $\tilde{\mu}_0$.*

2. *If $\mu_0^N := \frac{1}{N} \sum_{i=1}^N \delta_{(x_0^i, v_0^i)}$ is an initial atomic measure such that*

$$W_2(\mu_0, \mu_0^N) \to 0 \quad \textit{as } N \to \infty,$$

*then the empirical measure $\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{(x_t^i, v_t^i)}$ associated with system* (5.1.2) *is a measure-valued solution to* (5.0.1) *with initial data $\mu_0^N$. Moreover, it converges almost surely to the measure-valued solution $\mu_t$ corresponding to the initial measure $\mu_0$:*

$$\sup_{0\le t\le T} W_2(\mu_t, \mu_t^N) \le C W_2(\mu_0, \mu_0^N) e^{C(1+W_2(\mu_0,\mu_0^N))} \to 0, \quad \textit{as} \ \ N \to \infty,$$

Note that the stability estimate in Wasserstein metric implies the uniqueness of (measure-valued) solutions in $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$.

**Theorem 5.1.2.** *Suppose that the communication weight function $\phi$ satisfies*

$$0 < \phi_m \le \phi(x) \le \kappa \quad \textit{for } x \in \mathbb{R}^d,$$

*and let $\mu_t$ be a measure-valued solution to* (5.0.1)*. Then we have*

$$\mathbb{E}[E_0] e^{-2(\kappa-\sigma^2)t} \le \mathbb{E}[E_t] \le \mathbb{E}[E_0] e^{-2(\phi_m-\sigma^2)t},$$

*where $E_t$ is defined as*

$$E_t := \int_{\mathbb{R}^d\times\mathbb{R}^d} |\bar{v}_0 - v|^2 \mu_t(dx, dv), \quad \bar{v}_0 := \int_{\mathbb{R}^d\times\mathbb{R}^d} v \mu_0(dx, dv).$$

**Remark 5.1.1.** *The results in [12] imply that the equation* (5.0.1) *can be derived as a mean-field limit of the particle system* (5.1.2). *Now, our objective is to establish solutions with higher regularity than measure-valued solutions.*

### 5.1.2 Presentation of main results

In this subsection, we provide our main results on the global well-posedness of (5.0.1) and emergent flocking dynamics. First, we provide a definition for a strong solution to the Cauchy problem (5.0.1) as follows.

**Definition 5.1.1.** *For a given* $T \in (0, \infty]$, $f_t = f_t(x, v)$ *is a strong solution to* (5.0.1) *on* $[0, T]$ *if it satisfies the following relations:*

1. *(Regularity): For* $k \geq 1$, $f_t \in \mathcal{C}([0,T]; W^{k,\infty}(\mathbb{R}^{2d}))$ *a.s.* $\omega \in \Omega$.

2. *(Integral relation):* $f_t$ *satisfies the equation* (5.0.1) *in distribution sense: for* $\psi \in \mathcal{C}_c^\infty([0,T] \times \mathbb{R}^{2d})$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} f_t \psi \; dvdx \\
&= \int_{\mathbb{R}^{2d}} f^{in} \psi \; dvdx + \int_0^t \int_{\mathbb{R}^{2d}} f_s \left( v \cdot \nabla_x \psi + F_a[f_s] \cdot \nabla_v \psi \right) dvdxds \quad (5.1.5) \\
&\quad - \sigma \int_0^t \Big( \int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi \; dvdx \Big) \circ dW_s, \quad \text{a.s. } \omega \in \Omega.
\end{aligned}$$

**Remark 5.1.2.** *1. We say* $f_t$ *is a classical solution to* (5.0.1) *if it is a* $\mathcal{F}_t$*-semimartingale satisfying relation* (5.0.1) *pointwise and the regularity condition* $f_t \in L^\infty(\Omega; \mathcal{C}([0,T]; \mathcal{C}^{3,\delta}(\mathbb{R}^{2d})))$ *for some* $\delta \in (0, 1)$. *We require this regularity condition to use Itô's formula and the relation between Itô and Stratonovich integration without any restriction.*
*2. As can be seen later, the representation of a classical solution to* (5.0.1) *via the stochastic characteristics shows that* $f_t$ *can not satisfy the* $L^\infty$*-boundedness over* $\Omega$ *due to the exponential Wiener process. To handle this, we would use a suitable stopping time.*

Next, we are ready to provide a framework $(\mathcal{F})$ and main results below:

- $(\mathcal{F}1)$: The initial datum $f^{in}$ is nonnegative, compactly supported in $x$ and $v$ and independent of $\omega$.

- $(\mathcal{F}2)$: For $k \geq 1$, $f^{in}$ and $\phi$ are assumed to be in $W^{k,\infty}(\mathbb{R}^{2d})$ and $\mathcal{C}^\infty(\mathbb{R}^{2d})$, respectively.

- $(\mathcal{F}3)$: The first two moments of $f^{in}$ are normalized as follows:

$$\int_{\mathbb{R}^{2d}} f^{in} dvdx = 1, \quad \int_{\mathbb{R}^{2d}} v f^{in} dvdx = 0.$$

Under the framework $(\mathcal{F})$, our main results can be summarized as follows.

**Theorem 5.1.3.** *Let $T \in (0,\infty)$ and assume that $f^{in}$ and $\phi$ satisfies the framework $(\mathcal{F})$. Then, there exists a strong solution $f_t$ to* (5.0.1) *on* $[0,T]$ *such that*

$$\begin{aligned} &\mathbb{E}\|f_t\|_{L^\infty} \leq \|f^{in}\|_{L^\infty} \exp\left\{ \left( d\kappa + \frac{(\sigma d)^2}{2} \right) t \right\}, \\ &\mathbb{E}[M_2](t) \leq M_2(0) \exp(2\sigma^2 t), \quad t \in [0,T). \end{aligned}$$

*Moreover, if a strong solution $f_t$ exists on $(0,\infty)$ and $\phi_m := \inf_{x\in\mathbb{R}^N} \phi(x) > \sigma^2$, then one obtains an asymptotic flocking estimate:*

$$\mathbb{E}[M_2](t) \leq M_2(0) \exp(-2(\phi_m - \sigma^2)t), \quad t > 0.$$

*Proof.* For a proof, we first regularize the initial datum using the standard mollification and then solve the linearized system for (5.0.1) to get a sequence of approximate solutions. Then, we use the stopping time argument to get a strong solution for (5.0.1) with the given initial datum. The detailed proof will be presented in Section 5.3. □

**Remark 5.1.3.** *1. Note that for $k > 3$, a strong solution $f_t$ to* (5.0.1) *can be shown to satisfy the equation* (5.0.1) *pointwise within our framework.*
*2. Since the uniqueness of measure-valued solution is guaranteed, it suffices to prove the existence result for the global well-posedness of strong solutions.*

### 5.1.3 Elementary lemmas

Before we move on, we provide two useful lemmas used throughout this chapter. First, we begin with estimate on a variant of geometric brownian motion.

**Lemma 5.1.1.** *Let* $\{X_t\}_{t\geq 0}$ *be a solution satisfying the following Cauchy problem:*

$$\begin{cases} dX_t = (a_t + b_t X_t)dt + cX_t dW_t, \quad t > 0, \\ X_0 = x \geq 0, \end{cases}$$

*where* $\{a_t\}_{t\geq 0}$ *and* $\{b_t\}_{t\geq 0}$ *are stochastic processes with continuous sample paths, and c is a constant. Then one has*

$$\begin{aligned} X_t = x \exp\Big[\int_0^t \Big(b_s - \frac{c^2}{2}\Big)ds + cW_t\Big] \\ + \int_0^t a_s \exp\Big[\int_s^t \Big(b_\tau - \frac{c^2}{2}\Big)d\tau + c(W_t - W_s)\Big]ds. \end{aligned}$$

*Proof.* The proof is exactly given in Example 19.7 from [93]. So, we refer to [93] for its proof.

□

**Lemma 5.1.2.** (Comparision principle) *Suppose that two stochastic proceeses* $\{X_t\}_{t\geq 0}$ *and* $\{Y_t\}_{t\geq 0}$ *satisfy*

$$\begin{aligned} &dX_t \leq (a_t + bX_t)dt + cX_t dW_t, \quad X_0 = x \geq 0, \\ &dY_t = (a_t + bY_t)dt + cY_t dW_t, \quad Y_0 = x, \end{aligned}$$

*where* $\{a_t\}_{t\geq 0}$ *is a stochastic process with continuous sample paths. Then, we have*

$$X_t \leq Y_t, \quad \forall t \geq 0.$$

*Proof.* Let $\{Y_t^\delta\}_{t\geq 0}$, $(\delta > 0)$ be a stochastic process satisfying

$$\begin{cases} dY_t^\delta = (a_t + bY_t^\delta)dt + cY_t^\delta dW_t, \ t > 0, \\ Y_0^\delta = x + \delta, \end{cases}$$

and we set

$$Z_t^\delta := Y_t^\delta - X_t.$$

Then, we have

$$dZ_t^\delta \geq bZ_t^\delta dt + cZ_t^\delta dW_t, \quad t > 0 \quad \text{and} \quad Z_0 = \delta, \quad t = 0.$$

We use Itô's lemma to get

$$d(\ln Z_t^\delta) = \frac{dZ_t^\delta}{Z_t^\delta} - \frac{1}{2}\frac{1}{(Z_t^\delta)^2}(dZ_t^\delta)\cdot(dZ_t^\delta) \geq \left(b_t - \frac{c^2}{2}\right) dt + c\, dW_t.$$

Again, we integrate the above relation to get

$$Z_t^\delta \geq \delta \exp\left\{\int_0^t \left(b_s - \frac{c^2}{2}\right) ds + cW_t\right\} \geq 0.$$

This yields

$$X_t \leq Y_t^\delta \quad \text{for all } t \geq 0.$$

It follows from the representation formula in Lemma 5.1.1 that

$$\begin{aligned} Y_t^\delta &= (x+\delta)\exp\left\{\int_0^t \left(b_s - \frac{c^2}{2}\right) ds + cW_t\right\} \\ &\quad + \int_0^t a_s \exp\Big[\int_s^t \Big(b_\tau - \frac{c^2}{2}\Big) d\tau + c(W_t - W_s)\Big] ds, \\ Y_t &= x\exp\left\{\int_0^t \left(b_s - \frac{c^2}{2}\right) ds + cW_t\right\} \\ &\quad + \int_0^t a_s \exp\Big[\int_s^t \Big(b_\tau - \frac{c^2}{2}\Big) d\tau + c(W_t - W_s)\Big] ds. \end{aligned}$$

This yields the desired result:

$$Y_t = \liminf_{\delta\to 0} Y_t^\delta \geq X_t.$$

□

## 5.2 A priori estimates for classical solutions

In this section, we study a priori estimates for classical solutions to (5.0.1). First, we study several equivalent relations to the weak formulation (5.1.5), when a strong solution satisfies suitable conditions.

**Lemma 5.2.1.** *Suppose that for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ and a random process $f_t \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^{2d})$, $\int_{\mathbb{R}^{2d}} f_t\psi dvdx$ has a continuous $\mathcal{F}_t$-adapted modification, where $\{\mathcal{F}_t\}$ is a family of $\sigma$-field generated by the Wiener process. Then, $f_t$ is a $\mathcal{F}_t$-semimartingale satisfying relation* (5.1.5) *if and only if for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$,*

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} f_t\psi \; dvdx \\
&= \int_{\mathbb{R}^{2d}} f^{in}\psi \; dvdx + \int_0^t \int_{\mathbb{R}^{2d}} f_s \left(v \cdot \nabla_x \psi + F_a[f_s] \cdot \nabla_v \psi\right) dvdxds \\
&\quad - \sigma \int_0^t \Big( \int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi \; dvdx \Big) dW_s \\
&\quad + \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^{2d}} (v - v_c) f_s \cdot \Big[ \nabla_v \big( (v - v_c) \cdot \nabla_v \psi \big) \Big] dvdxds \quad a.s.\ \omega \in \Omega.
\end{aligned} \tag{5.2.1}$$

*Proof.* The proof is almost the same as in Lemma 13 from [33], but we provide a sketch for a proof for readers' convenience. Note that the following relation between Itô and Stratonovich integrals holds:

$$\int_0^t h_s \circ dW_s = \int_0^t h_s dW_s + \frac{1}{2}\langle h, W \rangle_t,$$

where $\langle \cdot, \cdot \rangle$ denotes the joint quadratic variation (see [69]). In our case, $h_s$ corresponds to $\int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi dvdx$. Then, to deal with $\langle h, W \rangle_t$,, one needs to specify the stochastic part of $h_s$. Here, if we replace $\psi$ in (5.1.5) by $(v - v_c) \cdot \nabla_v \psi$, we can find out that the stochastic part of $h_s$ becomes $-\sigma \int_0^t \Big[ \int_{\mathbb{R}^{2d}} ((v - v_c) f_s) \cdot \nabla_v (v \cdot \nabla_v \psi) dvdx \Big] dW_s$. This means

$$\begin{aligned}
&\Big\langle \int_{\mathbb{R}^{2d}} [(v - v_c) f_\cdot] \cdot \nabla_v \psi dvdx, W \Big\rangle_t \\
&\quad = -\sigma \int_0^t \int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v [(v - v_c) \cdot \nabla_v \psi] dvdxds,
\end{aligned}$$

and we may conclude the proof here. ∎

Once we reformulate relation (5.1.5) to Itô form (5.2.1), we can show that the solution process $f_t$ satisfies the following pointwise relation under the regularity condition for $f$.

**Lemma 5.2.2.** *Suppose that* $f_t \in L^\infty(\Omega;\mathcal{C}([0,T];\mathcal{C}^2(\mathbb{R}^{2d})))$ *has a continuous* $\mathcal{F}_t$*-adapted modification and has a compact support in* $x$ *and* $v$*. Then,* $f_t$ *satisfies relation* (5.2.1) *if and only if* $f_t$ *satisfies the following relation:*

$$\begin{aligned} f_t(x,v) &= f^{in}(x,v) - \int_0^t \Big(v\cdot\nabla_x f_s + \nabla_v\cdot(F_a[f_s]f_s)\Big)ds \\ &\quad + \sigma\int_0^t \Big[\nabla_v\cdot\big((v-v_c)f_s\big)\Big]dW_s \\ &\quad + \frac{\sigma^2}{2}\int_0^t \nabla_v\cdot\Big[(v-v_c)\nabla_v\cdot\big((v-v_c)f_s\big)\Big]ds, \quad \mathbb{P}\otimes dx\otimes dv\text{-a.s.} \end{aligned} \tag{5.2.2}$$

*Proof.* First, we assume that $f$ satisfies (5.2.1). Since $f_t$ is smooth and compactly supported, we use Fubini's theorem to show that (5.2.1) is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} f_t\psi\ dvdx \\ &= \int_{\mathbb{R}^{2d}} f^{in}\psi\ dvdx - \int_0^t\int_{\mathbb{R}^{2d}} \Big[v\cdot\nabla_x f_s + \nabla_v\cdot(F_a[f_s]f_s)\Big]\psi\ dvdxds \\ &+ \sigma\int_0^t\Big(\int_{\mathbb{R}^{2d}}\nabla_v\cdot[(v-v_c)f_s]\psi\ dvdx\Big)dW_s \\ &+ \frac{\sigma^2}{2}\int_0^t\int_{\mathbb{R}^{2d}}\nabla_v\cdot\Big[(v-v_c)\nabla_v\cdot\big((v-v_c)f_s\big)\Big]dvdxds \quad \text{a.s. } \omega\in\Omega. \end{aligned} \tag{5.2.3}$$

Note that for each $\psi\in\mathcal{D}(\mathbb{R}^{2d})$, it satisfies the relation (5.2.3) outside $\mathbb{P}$-zero set depending on the choice of $\psi$. We recall from standard functional analysis that $\mathcal{D}(\mathbb{R}^{2d})$ is separable, i.e. there exists $\{\psi_i\}_{i=1}^\infty \subseteq \mathcal{D}(\mathbb{R}^{2d})$ which is dense in $\mathcal{D}(\mathbb{R}^{2d})$. Here, we choose $\Omega_i\subset\Omega$ such that $\mathbb{P}(\Omega_i)=1$ and (5.1.5) holds for $f_t$ and $\psi_i$ over $\Omega_i$. Let $\tilde{\Omega} := \cap_{i=1}^\infty \Omega_i$. Then $\mathbb{P}(\tilde{\Omega})=1$ and (5.1.5) holds for any $\psi_i$ and $f_t$ over $\tilde{\Omega}$.

Now, we show $f_t$ satisfies the relation (5.2.2). For this, we define functionals $\mathscr{L}_t[f]$ and $\mathscr{M}[f_t]$ as follows:

$$\begin{aligned}\mathscr{L}[f_t](x,v) &:= f_t - f^{in} + \int_0^t \Big(v\cdot\nabla_x f_s + \nabla_v\cdot(F_a[f_s]f_s)\Big)ds\\ &\quad - \frac{\sigma^2}{2}\int_0^t \nabla_v\cdot\Big[(v-v_c)\nabla_v\cdot\big((v-v_c)f_s\big)\Big]ds,\\ \mathscr{M}[f_t] &:= \nabla_v\cdot[(v-v_c)f_t].\end{aligned}$$

For a given $(x^*,v^*)\in\mathbb{R}^{2d}$, we can choose a sequence $\{\rho_i\}\subset\mathcal{D}(\mathbb{R}^{2d})$, using the standard mollifier technique or other tools, such that for any $i\in\mathbb{N}$,

$$\begin{pmatrix}\Big|(\rho_i*\mathscr{L}[f_t])(x^*,v^*)-\mathscr{L}[f_t](x^*,v^*)\Big|\\ +\displaystyle\int_0^t\Big|(\rho_i*\mathscr{M}[f_s])(x^*,v^*)-\mathscr{M}[f_s](x^*,v^*)\Big|^2ds\end{pmatrix}\le\frac{1}{2^{i+1}},$$

where the regularity and compact support of $f$ can be used to guarantee the above inequality. We also use the denseness of $\{\psi_i\}$ to obtain $\{\tilde{\psi}_i\}\subseteq\{\psi_i\}$ which satisfies, for any $i\in\mathbb{N}$,

$$\Big|(\rho_i-\tilde{\psi}_i)*\mathscr{L}_t[f](x^*,v^*)\Big|+\int_0^t\Big|(\rho_i-\tilde{\psi}_i)*\mathscr{M}[f_s](x^*,v^*)\Big|^2ds\le\frac{1}{2^{i+1}}.$$

Thus, we have

$$\Big(\tilde{\psi}_i*\mathscr{L}_t[f]\Big)(x^*,v^*)\longrightarrow\mathscr{L}_t[f](x^*,v^*).\tag{5.2.4}$$

Moreover, we use Itô isometry to get

$$\begin{aligned}&\mathbb{E}\left[\Big(\int_0^t(\tilde{\psi}_i*\mathscr{M}[f_s]-\mathscr{M}[f_s])dW_s\Big)^2\right]\\ &=\mathbb{E}\left[\int_0^t(\tilde{\psi}_i*\mathscr{M}[f_s]-\mathscr{M}[f_s])^2ds\right]\longrightarrow 0.\end{aligned}\tag{5.2.5}$$

Hence, we can obtain the convergence of (5.2.3) with $\psi=\tilde{\psi}_i(x^*-x,v^*-v)$ towards (5.2.2) at $(x^*,v^*)$ as $i\to\infty$, by combining (5.2.4) and (5.2.5). We perform this procedure to obtain that for every $(x^*,v^*)\in\mathbb{R}^{2d}$, $f$ satisfies relation (5.2.2) $\mathbb{P}$-a.s. and this gives

$$\mathbb{E}\left[\left|\mathcal{L}[f_t]-\int_0^t\mathscr{M}[f_s]dW_s\right|(x,v)\right]=0,$$

for every $(x,v) \in \mathbb{R}^{2d}$. Thus, we use Fubini theorem to get

$$\mathbb{E}\left[\int_{\mathbb{R}^{2d}} \left|\mathcal{L}[f_t] - \int_0^t \mathscr{M}[f_s]dW_s\right| dvdx\right] = 0.$$

This implies our first assertion.

Next, we assume that $f$ satisfies (5.2.2) $\mathbb{P} \otimes dx \otimes dv$-a.s. Then by (deterministic) Fubini's theorem, the following relation is easily obtained: for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} f_t\psi \; dvdx \\
&= \int_{\mathbb{R}^{2d}} f^{in}\psi \; dvdx + \int_0^t \int_{\mathbb{R}^{2d}} f_s \left(v \cdot \nabla_x \psi + F_a[f_s] \cdot \nabla_v \psi\right) \; dvdxds \\
&\quad + \sigma \int_{\mathbb{R}^{2d}} \Big(\int_0^t \nabla_v \cdot [(v - v_c) f_s]\psi \; dW_s\Big) dvdx \\
&\quad + \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}^{2d}} (v - v_c) f_s \cdot \Big[\nabla_v \big((v - v_c) \cdot \nabla_v \psi\big)\Big] dvdxds \quad \text{a.s. } \omega \in \Omega.
\end{aligned}$$

Since $f_t$ is in $L^\infty(\Omega; \mathcal{C}([0,T]; \mathcal{C}^2(\mathbb{R}^{2d})))$ and compactly supported, we have

$$\int_{\mathbb{R}^{2d}} \left(\int_0^t \Big|\nabla_v \cdot [(v - v_c) f_s]\psi\Big|^2 ds\right)^{1/2} dvdx < \infty, \quad \text{a.s. } \omega \in \Omega.$$

Then, we can use the stochastic Fubini theorem (see [101] and references therein) and deterministic Fubini's theorem to get

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \Big(\int_0^t \nabla_v \cdot [(v - v_c) f_s]\psi \; dW_s\Big) dvdx \\
&= \int_0^t \Big(\int_{\mathbb{R}^{2d}} \nabla_v \cdot [(v - v_c) f_s]\psi \; dvdx\Big) dW_s \\
&= -\int_0^t \Big(\int_{\mathbb{R}^{2d}} [(v - v_c) f_s] \cdot \nabla_v \psi \; dvdx\Big) dW_s.
\end{aligned}$$

This implies our desired result. $\square$

**Remark 5.2.1.** *1. If a strong solution $f_t$ to* (5.0.1) *satisfies conditions in Lemma 5.2.2, then $f_t$ satisfies the relation* (5.2.2).
*2. If $f_t$ is a classical solution to* (5.0.1), *we may use Lemma 5.2.2 in [13] to obtain that the Itô relation* (5.2.2) *is equivalent to* (5.0.1).

### 5.2.1 Quantitative estimates for classical solutions

We provide several properties of classical solutions $f$ to (5.0.1). First, we study the propagation of velocity moments along the stochastic flow of $(5.0.1)_1$. For a random density function $f_t$, we set velocity moments:

$$M_0(t) := \int_{\mathbb{R}^{2d}} f_t dv dx, \quad M_1(t) := \int_{\mathbb{R}^{2d}} v f_t dv dx,$$
$$M_2(t) := \int_{\mathbb{R}^{2d}} |v|^2 f_t dv dx, \quad t \geq 0.$$

Consider the following stochastic characteristics $\varphi_t(x,v) := (X_t(x,v), V_t(x,v))$:

$$\begin{cases} dX_t = V_t dt, \quad t > 0, \\ dV_t = (F_a[f_t](X_t, V_t))\,dt + \sigma(v_c - V_t) \circ dW_t, \end{cases} \tag{5.2.6}$$

subject to the initial data:

$$(X_0(x,v), V_0(x,v)) = (x,v).$$

Note that if $f_t$ is compactly supported in $x$ and $v$, and satisfies the regularity condition for classical solutions, the system (5.2.6) has a unique solution and the family $\{\varphi_{s,t}(x,v) := \varphi_t(\varphi_s^{-1}(x,v))\}$, $0 \leq s \leq t \leq T$, forms a stochastic flow of smooth diffeomorphisms (we refer to Lemma 4.1 in Chapter 2 of [13] for details). Furthermore, we define the functionals that measure spatial and velocity supports of $f_t$, respectively:

$$\mathcal{X}(t) := \sup\{|x| \;:\; f_t(x,v) \neq 0 \quad \text{for some } v \in \mathbb{R}^d\},$$
$$\mathcal{V}(t) := \sup\{|v| \;:\; f_t(x,v) \neq 0 \quad \text{for some } x \in \mathbb{R}^d\}.$$

**Lemma 5.2.3.** *Let $f_t$ be a classical solution to* (5.0.1) *which is compactly supported in $x$ and $v$ and satisfies*

$$M_0(0) = 1, \quad M_1(0) = 0.$$

*Then for $t \geq 0$,*

$$M_0(t) = 1, \quad M_1(t) = 0, \quad M_2(t) \leq M_2(0) \exp\left(-2\int_0^t \bar{\phi}(2\mathcal{X}(s))ds - 2\sigma W_t\right).$$

*Proof.* • (Conservation of mass): It follows from Remark 5.2.1 that

$$\begin{aligned} f_t(x,v) = f^{in}(x,v) &- \int_0^t \Big(v\cdot\nabla_x f_s + \nabla_v\cdot(F_a[f_s]f_s)\Big)ds \\ &+ \sigma\int_0^t \Big(\nabla_v\cdot((v-v_c)f_s)\Big)dW_s \\ &+ \frac{\sigma^2}{2}\int_0^t \nabla_v\cdot\Big[(v-v_c)\nabla_v\cdot\big((v-v_c)f_s\big)\Big]ds. \end{aligned} \tag{5.2.7}$$

We integrate (5.2.7) over $(x,v)\in\mathbb{R}^{2d}$ to get

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} f_t(x,v)dvdx \\ &\quad = \int_{\mathbb{R}^{2d}} f^{in}(x,v)dvdx - \int_{\mathbb{R}^{2d}}\int_0^t \Big(v\cdot\nabla_x f_s + \nabla_v\cdot(F_a[f_s]f_s)\Big)dsdvdx \\ &\quad\quad + \sigma\int_{\mathbb{R}^{2d}}\Big[\int_0^t \Big(\nabla_v\cdot((v-v_c)f_t)\Big)dW_s\Big]dvdx \\ &\quad\quad + \frac{\sigma^2}{2}\int_{\mathbb{R}^{2d}}\int_0^t \nabla_v\cdot\Big[(v-v_c)\nabla_v\cdot\big((v-v_c)f_s\big)\Big]dsdvdx \\ &\quad =: \int_{\mathbb{R}^{2d}} f^{in}(x,v)dvdx + J_{11} + J_{12} + J_{13}. \end{aligned}$$

Next, we show that the terms $J_{1i}$ are zero using deterministic and stochastic Fubini's theorems.

◇ (Estimate of $J_{11}$ and $J_{13}$): Since $f_t$ has a compact support in $(x,v)$, we can use deterministic Fubini's theorem to see

$$J_{11} + J_{13} = -\int_0^t\int_{\mathbb{R}^{2d}} \Big(\nabla_x\cdot(vf_s) + \nabla_v\cdot(F_a[f_s]f_s)\Big)dvdxds$$

$$
\begin{aligned}
&+\frac{\sigma^2}{2}\int_0^t\int_{\mathbb{R}^{2d}}\nabla_v\cdot\Big[(v-v_c)\nabla_v\cdot\big((v-v_c)f_s\big)\Big]dvdxdx \\
&=0.
\end{aligned}
$$

$\diamond$ (Estimate of $J_{12}$): As in the proof of Lemma 5.2.2, we can use the stochastic Fubini theorem to get

$$
J_{12}=\int_0^t\Big(\int_{\mathbb{R}^{2d}}\nabla_v\cdot((v-v_c)f_t)dvdx\Big)dW_s=0.
$$

• (Conservation of momentum): In this case, we multiply $v$ to (5.2.7) and use the same argument for conservation of mass to derive

$$
M_1(t)=M_1(0)=0,\quad t\geq 0.
$$

• (Dissipation estimate): We multiply (5.2.7) by $|v|^2$ and use stochastic Fubini's theorem to have

$$
dM_2(t)=\Big(2\sigma^2M_2(t)+\int_{\mathbb{R}^{2d}}2v\cdot F_a[f_s]f_sdvdx\Big)dt-2\sigma M_2(t)dW_t, \tag{5.2.8}
$$

where we used the relation $M_1(t)=0$.

We use (5.2.8) to get

$$
\begin{aligned}
&M_2(t)\\
&=M_2(0)+\int_0^t\Big[\Big(\int_{\mathbb{R}^{2d}}2v\cdot F_a[f_s]f_sdvdx\Big)+2\sigma^2M_2(s)\Big]ds\\
&\quad-2\sigma\int_0^tM_2(s)dW_s\\
&=M_2(0)+2\int_0^t\int_{\mathbb{R}^{4d}}\phi(x_*-x)(v_*-v)\cdot vf_s(x_*,v_*)f_s(x,v)dv_*dx_*dvdxds\\
&\quad+2\sigma^2\int_0^tM_2(s)ds-2\sigma\int_0^tM_2(s)dW_s\\
&=M_2(0)-\int_0^t\int_{\mathbb{R}^{4d}}\phi(x_*-x)|v-v_*|^2f_s(x_*,v_*)f_s(x,v)dv_*dx_*dvdxds
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\sigma^2 \int_0^t M_2(s)ds - 2\sigma \int_0^t M_2(s)dW_s \\
&\le M_2(0) - \int_0^t \bar{\phi}(2\mathcal{X}(s)) \Big[ \int_{\mathbb{R}^{4d}} |v - v_*|^2 f_s(x_*, v_*) f_s(x, v) dv_* dx_* dv dx \Big] ds \\
&+ 2\sigma^2 \int_0^t M_2(s)ds - 2\sigma \int_0^t M_2(s)dW_s \\
&\le M_2(0) - 2 \int_0^t \left( \bar{\phi}(2\mathcal{X}(s)) - \sigma^2 \right) M_2(s)ds - 2\sigma \int_0^t M_2(s)dW_s.
\end{aligned}
$$

Then we use Lemma 5.1.1 and Lemma 5.1.2 to get

$$
M_2(t) \le M_2(0) \exp\left( -2 \int_0^t \bar{\phi}(2\mathcal{X}(s))ds - 2\sigma W_t \right).
$$

□

**Remark 5.2.2.** *In Lemma 3.3, we observe that the first momentum is preserved. Thus, without loss of generality, we may assume that* $v_c(t) = 0$.

Next, we discuss the size of spatial and velocity supports of $f_t$.

**Lemma 5.2.4.** *The support functionals* $\mathcal{X}$ *and* $\mathcal{V}$ *satisfy the following estimates:*

$$
\begin{aligned}
\mathcal{X}(t) &\le \mathcal{X}_0 + \sqrt{2} \int_0^t \left( \mathcal{V}_0 + \kappa \sqrt{dM_2(0)} s \right) \exp\Big[ - \int_0^s \bar{\phi}(2\mathcal{X}(\tau))d\tau - \sigma W_s \Big] ds, \\
\mathcal{V}(t) &\le \sqrt{2} \left( \mathcal{V}_0 + \kappa \sqrt{dM_2(0)} t \right) \exp\left[ - \int_0^t \bar{\phi}(2\mathcal{X}(s))ds - \sigma W_t \right], \quad t \ge 0.
\end{aligned}
$$

*Moreover, if* $\phi_m > 0$, *then*

$$
\begin{aligned}
\mathcal{X}(t) &\le \mathcal{X}_0 + \sqrt{2} \int_0^t \left( \mathcal{V}_0 + \kappa \sqrt{dM_2(0)} s \right) \exp(-\phi_m s - \sigma W_s) ds, \\
\mathcal{V}(t) &\le \sqrt{2} \left( \mathcal{V}_0 + \kappa \sqrt{dM_2(0)} t \right) \exp(-\phi_m t - \sigma W_t), \quad t \ge 0, .
\end{aligned}
$$

*Proof.* First, we consider the case when $\phi_m > 0$ may not hold.

$\diamond$ (Estimate of $\mathcal{V}$): Note that the stochastic characteristics $(X_t, V_t) = \{(x_t^i, v_t^i)\}_{i=1}^d$ starting from $(x, v) \in \mathrm{supp} f^{in}$ satisfy

$$\begin{cases} dx_t^i = v_t^i dt, \quad 1 \le i \le d, \\ dv_t^i = \left( \left( F_a[f_t](X_t, V_t) \right)^i + \frac{1}{2}\sigma^2 v_t^i \right) dt - \sigma v_t^i dW_t. \end{cases} \tag{5.2.9}$$

Now, we rewrite $(5.2.9)_2$ to have

$$\begin{aligned} dv_t^i = \Bigg[ & \left( -\int_{\mathbb{R}^{2d}} \phi(x_* - X_t) f_t(x_*, v_*) dv_* dx_* + \frac{1}{2}\sigma^2 \right) v_t^i \\ & + \int_{\mathbb{R}^{2d}} \phi(x_* - X_t) v_*^i f_t(x_*, v_*) dv_* dx_* \Bigg] dt - \sigma v_t^i dW_t. \end{aligned} \tag{5.2.10}$$

Thus, we apply Lemma 5.1.1 and Lemma 5.2.3 to (5.2.10) to get

$$\begin{aligned} |v_t^i| = & \Bigg| v_0^i \exp\left[ -\int_0^t \left\{ \int_{\mathbb{R}^{2d}} \phi(x_* - X_s) f_s(x_*, v_*) dv_* dx_* \right\} ds - \sigma W_t \right] \\ & + \int_0^t \left\{ \int_{\mathbb{R}^{2d}} \phi(x_* - X_s) v_*^i f_s(x_*, v_*) dv_* dx_* \right\} \\ & \times \exp\left[ -\int_s^t \left\{ \int_{\mathbb{R}^{2d}} \phi(x_* - X_\tau) f_\tau(x_*, v_*) dv_* dx_* \right\} d\tau - \sigma(W_t - W_s) \right] ds \Bigg| \\ \le & |v_0^i| \exp\left[ -\int_0^t \bar{\phi}(2\mathcal{X}(s)) ds - \sigma W_t \right] \\ & + \kappa \int_0^t \sqrt{M_2(s)} \exp\left[ -\int_s^t \bar{\phi}(2\mathcal{X}(\tau)) d\tau - \sigma(W_t - W_s) \right] ds \\ \le & \left( |v_0^i| + \kappa \sqrt{M_2(0)} t \right) \exp\left[ -\int_0^t \bar{\phi}(2\mathcal{X}(s)) ds - \sigma W_t \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} |V_t|^2 &= \sum_{i=1}^d |v_t^i|^2 \\ &\le \sum_{i=1}^d \left( |v_0^i| + \kappa \sqrt{M_2(0)} t \right)^2 \exp\left[ -2\int_0^t \bar{\phi}(2\mathcal{X}(s)) ds - 2\sigma W_t \right] \end{aligned}$$

$$\leq 2\left(|V_0|^2 + d\kappa^2 M_2(0)t^2\right)^2 \exp\left[-2\int_0^t \bar{\phi}(2\mathcal{X}(s))ds - 2\sigma W_t\right],$$

where we used Young's inequality, and this yields

$$\mathcal{V}(t) \leq \sqrt{2}\left(\mathcal{V}_0 + \kappa\sqrt{dM_2(0)}t\right)\exp\left[-\int_0^t \bar{\phi}(2\mathcal{X}(s))ds - \sigma W_t\right].$$

This gives the desired estimate.

$\diamond$ (Estimate of $\mathcal{X}$): We use Itô's formula and the Cauchy-Schwarz inequality to get

$$d|X_t|^2 = 2X_t \cdot dX_t + dX_t \cdot dX_t = 2X_t \cdot V_t dt \leq 2|X_t| \cdot |V_t| dt.$$

This and the estimates for $\mathcal{V}(t)$ yield

$$\frac{d|X_t|}{dt} \leq |V_t| \leq \sqrt{2}\left(\mathcal{V}_0 + \kappa\sqrt{dM_2(0)}t\right)\exp\left[-\int_0^t \bar{\phi}(2\mathcal{X}(s))ds - \sigma W_t\right].$$

We integrate the above differential inequality to get

$$|X_t| \leq |X_0| + \sqrt{2}\int_0^t \left(\mathcal{V}_0 + \kappa\sqrt{dM_2(0)}s\right)\exp\left[-\int_0^s \bar{\phi}(2\mathcal{X}(\tau))d\tau - \sigma W_s\right] ds,$$

and this implies our desired estimate for $\mathcal{X}(t)$.

When $\phi_m > 0$, we can use $\phi_m \leq \phi(2\mathcal{X}(s))$ to get the desired results.

□

**Remark 5.2.3.** *Here, we discuss the necessity of the lower bound condition $\phi_m > 0$ for flocking estimates. As observed in [51], the equation* (5.0.1) *without noise exhibits flocking without the condition $\phi_m > 0$, but it was attainable since the sizes of $x$- and $v$-supports increase at most in an algebraic order, which is not the case for* (5.0.1) *due to the exponential Wiener process. Here, it is well known that*

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log t}} = 1, \quad \textit{for a.s. } \omega \in \Omega.$$

*Thus, for the pathwise flocking estimate, we require*

$$\limsup_{t\to\infty} \frac{\int_0^t \bar{\phi}(2\mathcal{X}(s))ds}{\sqrt{t\log\log t}} = 0, \quad \textit{for a.s. } \omega \in \Omega. \tag{5.2.11}$$

*However, as observed in Lemma 5.2.4, it becomes difficult to estimate $\mathcal{X}(t)$ without the lower bound assumption $\phi_m > 0$. Accordingly, it is hard to find a condition weaker than $\phi_m > 0$ which entails the estimate* (5.2.11)*.*

Now, we are ready to state the stability results for (5.0.1).

**Theorem 5.2.1.** ($L^\infty$-stability) *Let $f_t$ and $\tilde{f}_t$ be two classical solutions to* (5.0.1) *corresponding to regular initial data $f^{in}$ and $\tilde{f}^{in}$, respectively, which are compactly supported in $x$ and $v$. Moreover, let $\varphi_t = \varphi_t(x,v)$ and $\tilde{\varphi}_t = \tilde{\varphi}_t(x,v)$ be the stochastic characteristics associated to $f$ and $\tilde{f}$, respectively. Then, we have*

$$\|f_t - \tilde{f}_t\|^2_{\mathcal{C}^0} + \|\varphi_t - \tilde{\varphi}_t\|^2_{\mathcal{C}^0} \le \mathcal{D}_t \|f^{in} - \tilde{f}^{in}\|^2_{\mathcal{C}^0},$$

*where $\mathcal{D}_t$ is a non-negatvie process with continuous sample paths and*

$$\|\varphi_t - \tilde{\varphi}_t\|_{\mathcal{C}^0} := \sup\left\{ |\varphi_t(x,v) - \tilde{\varphi}_t(x,v)| \ : \ (x,v) \in (supp f^{in}) \cup (supp \tilde{f}^{in}) \right\}.$$

*Proof.* Since the proof is rather lengthy, we postpone it to Appendix B.1. □

## 5.3 Global well-posedness and asymptotic dynamics of strong solutions

In this section, we provide global well-posedness and asymptotic flocking estimates for strong solutions to (5.0.1). Here, we show our desired estimates for (5.0.1) corresponding to regularized initial data. Then, based on the stability estimates for classical solutions that we obtained in the previous section, we conclude that solutions to (5.0.1) with regularized initial data converge to a strong solution to (5.0.1). Moreover, we show that a strong solution obtained as above satisfies the asymptotic flocking estimates.

Let $f^{in,\varepsilon}$ be a smooth mollification of the given initial datum $f^{in}$ satisfying the framework $(\mathcal{F})$. Then, consider the Cauchy problem (5.0.1) with these regularized initial data:

$$\begin{aligned}&\partial_t f_t^\varepsilon + v\cdot\nabla_x f_t^\varepsilon + \nabla_v\cdot(F_a[f_t^\varepsilon]f_t^\varepsilon) = \sigma\nabla_v\cdot(vf_t^\varepsilon)\circ\dot{W}_t,\\ &f_0^\varepsilon(x,v) = f^{in,\varepsilon}(x,v).\end{aligned} \tag{5.3.1}$$

Note that due to the framework $(\mathcal{F})$, the initial datum $f^{in}$ and its partial derivatives up to order $k$ are uniformly continuous on $\mathbb{R}^{2d}$ and there exists a constant $R_0 > 0$, such that

$$\mathrm{supp} f^{in} \subseteq B_{R_0}(0),$$

where $B_{R_0}(0)$ is a ball of radius $R_0$ centered at $0 \in \mathbb{R}^{2d}$. As mentioned above, we use a mollifier to obtain a family of regularized initial data $f^{in,\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, $\varepsilon \in (0,1)$, so that the regularized datum satisfies the following conditions:

- $(\mathcal{F}^\varepsilon 1)$: $\{f^{in,\varepsilon}\}$ are nonnegative, compactly supported, uniformly converge to $f^{in}$ in $\mathcal{C}^0(\mathbb{R}^{2d})$ and

$$\|f^{in,\varepsilon}\|_{W^{k,\infty}} \le \|f^{in}\|_{W^{k,\infty}}.$$

- $(\mathcal{F}^\varepsilon 2)$: $\{M_2^\varepsilon\}(0)$ is uniformly bounded with respect to $\varepsilon$ and converges to $M_2(0)$ as $\varepsilon \to 0$.

- $(\mathcal{F}^\varepsilon 3)$: The zeroth and first moment of $f^{in,\varepsilon}$ are initially constrained:

$$\int_{\mathbb{R}^{2d}} f^{in,\varepsilon}dxdv = 1, \quad \int_{\mathbb{R}^{2d}} vf^{in,\varepsilon}dvdx = 0.$$

- $(\mathcal{F}^\varepsilon 4)$: $f^{in,\varepsilon}$ has a compact support in $x$ and $v$, and satisfy

$$\mathrm{supp} f^{in,\varepsilon} \subseteq B_{R_0+1}(0).$$

In the following three subsections, we will provide a global existence for system (5.3.1).

### 5.3.1 Construction of approximate solutions

In this subsection, we provide a sequence of approximate solutions to (5.3.1) using successive approximations.

First, the zeroth iterate $f_t^{0,\varepsilon}$ is simply defined as the mollified initial datum:

$$f_t^{0,\varepsilon}(x,v) := f^{in,\varepsilon}(x,v), \quad (x,v) \in \mathbb{R}^{2d}.$$

For $n \geq 1$, suppose that the $(n-1)$-th iterate $f_t^{n-1,\varepsilon}$ is given. Then, the $n$-th iterate is defined as the solution to the linear equation with fixed initial datum: for each $n \geq 1$,

$$\begin{cases} \partial_t f_t^{n,\varepsilon} + v \cdot \nabla_x f_t^{n,\varepsilon} + \nabla_v \cdot (F_a[f_t^{n-1,\varepsilon}] f_t^{n,\varepsilon}) = \sigma \nabla_v \cdot (v f_t^{n,\varepsilon}) \circ \dot{W}_t, \\ f_0^{n,\varepsilon}(x,v) = f^{in,\varepsilon}(x,v). \end{cases} \tag{5.3.2}$$

The linear system (5.3.2) can be solved by the method of stochastic characteristics. Let $\varphi_t^{n,\varepsilon}(x,v) := (X_t^{n,\varepsilon}(x,v), V_t^{n,\varepsilon}(x,v))$ be the forward stochastic characteristics, which is a solution to the following SDE:

$$\begin{cases} dX_t^{n,\varepsilon} = V_t^{n,\varepsilon} dt, \\ dV_t^{n,\varepsilon} = F_a[f_t^{n-1,\varepsilon}](X_t^{n,\varepsilon}, V_t^{n,\varepsilon}) dt - \sigma V_t^{n,\varepsilon} \circ dW_t, \\ (X_t^{n,\varepsilon}(0), V_t^{n,\varepsilon}(0)) = (x,v) \in \text{supp} f^{in,\varepsilon}. \end{cases} \tag{5.3.3}$$

Note that the SDE (5.3.3) is equivalent to the following Itô SDE [31]:

$$\begin{cases} dX_t^{n,\varepsilon} = V_t^{n,\varepsilon} dt, \\ dV_t^{n,\varepsilon} = \Big( F_a[f_t^{n-1,\varepsilon}](X_t^{n,\varepsilon}, V_t^{n,\varepsilon}) + \dfrac{\sigma^2}{2} V_t^{n,\varepsilon} \Big) dt - \sigma V_t^{n,\varepsilon} dW_t, \\ (X_t^{n,\varepsilon}(0), V_t^{n,\varepsilon}(0)) = (x,v) \in \text{supp} f^{in,\varepsilon}. \end{cases} \tag{5.3.4}$$

Here, we can deduce from Lemma 3.1 and Theorem 3.2 in [13] and our framework that for any $m \geq 3$, (5.3.3) has a unique solution $f_t^{n,\varepsilon}$ which is a $\mathcal{C}^m$-semimartingale for every $n \geq 0$ and the characteristics (5.3.3) becomes a $\mathcal{C}^m$-diffeomorphism. Then, $f_t^{n,\varepsilon}$ can also be represented by the following integral formula:

$$\begin{aligned} &f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}(x,v)) \\ &\quad = f^{in,\varepsilon}(x,v)\exp\Big[-\int_0^t \nabla_v\cdot F_a[f^{n-1,\varepsilon}](s,\varphi_s^{n,\varepsilon}(x,v))ds + d\sigma W_t\Big]. \end{aligned} \tag{5.3.5}$$

Note that if $f^{in,\varepsilon}$ is nonnegative, then surely $f_t^{n,\varepsilon}$ is also nonnegative as well. Before we finish this subsection, we also remark that the linear, first-order Stratonovich equation (5.3.2) is equivalent to the following parabolic Itô equation (see Corollary 3.3. in Chapter 2 from [13]):

$$\begin{cases} \partial_t f_t^{n,\varepsilon} + v\cdot\nabla_x f_t^{n,\varepsilon} + \nabla_v\cdot(F_a[f_t^{n-1,\varepsilon}]f_t^{n,\varepsilon}) \\ \qquad = \sigma\nabla_v\cdot(vf_t^{n,\varepsilon})\dot W_t + \dfrac{\sigma^2}{2}\nabla_v\cdot\Big[v\nabla_v\cdot(vf_t)\Big], \quad n\ge 1, \\ f_0^{n,\varepsilon}(x,v) = f^{in,\varepsilon}(x,v). \end{cases} \tag{5.3.6}$$

### 5.3.2 Estimates on approximate solutions

In this subsection, we provide several estimates for the approximate solutions for (5.3.2). To be more precise, we would try to obtain $n$ and $\varepsilon$-independent estimates for the later sections. Before we move on, we define $p$-th velocity moments $M_p^{n,\varepsilon}(t)$, $p = 0,1,2$:

$$\begin{aligned} &M_0^{n,\varepsilon}(t) := \int_{\mathbb{R}^{2d}} f_t^{n,\varepsilon}dvdx, \quad M_1^{n,\varepsilon}(t) := \int_{\mathbb{R}^{2d}} vf_t^{n,\varepsilon}dvdx, \\ &M_2^{n,\varepsilon}(t) := \int_{\mathbb{R}^{2d}} |v|^2 f_t^{n,\varepsilon}dvdx, \quad M_p^{n,\varepsilon}(0) := M_{p0}^{\varepsilon}. \end{aligned}$$

Before we provide the uniform estimates for the $p$-th ($p=0,1,2$) moments, we set

$$M_{20}^\infty := \sup_{\varepsilon\in(0,1)} M_2^\varepsilon(0), \qquad \gamma := \max\{M_{20}^\infty, \kappa\}. \tag{5.3.7}$$

We also present a technical lemma from [6] for a later discussion.

**Lemma 5.3.1.** *[6] Let $T\in(0,\infty]$ and $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative continuous functions on $[0,T]$ satisfying*

$$a_n(t) \le A + B\int_0^t a_{n-1}(s)ds + C\int_0^t a_n(s)ds, \quad t\in[0,T], \quad n\ge 1,$$

*where $A$, $B$ and $C$ are nonnegative constants.*

1. *If $A = 0$, there exists a constant $\Lambda \geq 0$ depending on $B$, $C$ and $\sup_{t\in[0,T]} a_0(t)$ such that*

$$a_n(t) \leq \frac{(\Lambda t)^n}{n!}, \quad t \in [0,T], \quad n \in \mathbb{N}.$$

2. *If $A > 0$ and $C = 0$, there exists a constant $\Lambda \geq 0$ depending on $A$, $B$ and $\sup_{t\in[0,T]} a_0(t)$ such that*

$$a_n(t) \leq \Lambda \exp(\Lambda t), \quad t \in [0,T], \quad n \in \mathbb{N}.$$

**Remark 5.3.1.** *1. In (2) of Lemma 5.3.1, $\Lambda$ can be explicitly written as*

$$\Lambda := \max\left\{ A,\ B,\ \sup_{t\in[0,T]} a_0(t) \right\}.$$

*2. We can also use the similar argument to obtain the following estimate for (2):*

$$a_n(t) \leq (\Lambda + K_t)\exp(\Lambda t), \quad t \in [0,T], \quad n \in \mathbb{N},$$

*where $\Lambda := \max\{A, B\}$ and $K_t := \sup_{0\leq s\leq t} a_0(s)$.*

**Proposition 5.3.1.** *For every $n \in \mathbb{N}$ and $T \in (0,\infty)$, let $f_t^{n,\varepsilon}$ be a solution to (5.3.2). Then, for any $t \in (0,T)$ we have*

$$M_0^{n,\varepsilon}(t) = 1, \quad M_1^{n,\varepsilon}(t) = 0, \quad M_2^{n,\varepsilon}(t) \leq (\gamma + \mathcal{K}_t)\exp\{(\gamma+\kappa)t - 2\sigma W_t\},$$

*where $\gamma$ is a constant in (5.3.7) and $\mathcal{K}_t$ is defined as*

$$\mathcal{K}_t := M_{20}^{\infty} \sup_{0\leq s\leq t} \exp(-\kappa s + 2\sigma W_s).$$

*Proof.* Note that $f_t^{n,\varepsilon}$ satisfies relation (5.3.6) and $f_t^{n,\varepsilon}$ is compactly supported in $x$ and $v$, since $f^{in,\varepsilon}$ is compactly supported in the phase space and $\varphi_t^{n,\varepsilon}$ is a $\mathcal{C}^m$-diffeomorphisim. Thus, we may follow the arguments in Lemma 5.2.3 to derive the conservation estimates.

For the dissipation estimate of $M_2^{n,\varepsilon}$, we use a similar argument to Lemma 5.2.3 to have

$$\begin{aligned}
M_2^{n,\varepsilon}(t) &= M_2^{\varepsilon}(0) + 2\sigma^2 \int_0^t M_2^{n,\varepsilon}(s)ds - 2\sigma \int_0^t M_2^{n,\varepsilon}(s)dW_s \\
&\quad + 2\int_0^t \int_{\mathbb{R}^{4d}} \phi(x_* - x)(v_* - v)\cdot v f_s^{n-1,\varepsilon}(x_*, v_*) f_s^{n,\varepsilon}(x,v) dv_* dx_* dv dx ds \\
&\le M_2^{\varepsilon}(0) + 2\sigma^2 \int_0^t M_2^{n,\varepsilon}(s)ds - 2\sigma \int_0^t M_2^{n,\varepsilon}(s)dW_s \\
&\quad + 2\int_0^t \int_{\mathbb{R}^{4d}} \phi(x_* - x) v_* \cdot v f_s^{n-1,\varepsilon}(x_*, v_*) f_s^{n,\varepsilon}(x,v) dv_* dx_* dv dx \\
&\le M_2^{\varepsilon}(0) + \kappa \int_0^t M_2^{n-1,\varepsilon}(s)ds + (\kappa + 2\sigma^2)\int_0^t M_2^{n,\varepsilon}(s)ds \\
&\quad - 2\sigma \int_0^t M_2^{n,\varepsilon}(s)dW_s,
\end{aligned}$$

where we used Young's inequality on the second inequality. In a differential form, we have

$$dM_2^{n,\varepsilon}(t) \le \left\{ \begin{array}{l} \kappa M_2^{n-1,\varepsilon}(t) \\ +(\kappa + 2\sigma^2) M_2^{n,\varepsilon}(t) \end{array} \right\} dt - 2\sigma M_2^{n,\varepsilon}(t) dW_t. \tag{5.3.8}$$

Then, it follows from (5.3.8) and comparison theorem (in Lemma 5.1.2) that

$$M_2^{n,\varepsilon}(t) \le X_t,$$

where the process $X_t$ satisfies

$$\begin{cases} dX_t = \left\{\kappa M_2^{n-1,\varepsilon}(t) + (\kappa + 2\sigma^2) X_t\right\} dt - 2\sigma X_t dW_t, \quad t > 0, \\ X_0 = M_2^{\varepsilon}(0). \end{cases}$$

It follows from Lemma 5.1.1 that $X_t$ can be represented as

$$\begin{aligned}
X_t &= X_0 \exp(\kappa t - 2\sigma W_t) \\
&\quad + \kappa \int_0^t \exp\{\kappa(t-s) - 2\sigma(W_t - W_s)\} M_2^{n-1,\varepsilon}(s) ds.
\end{aligned}$$

This implies

$$\begin{aligned} M_2^{n,\varepsilon}(t) &\le M_2^{\varepsilon}(0)\exp(\kappa t - 2\sigma W_t) \\ &\qquad + \kappa \int_0^t \exp\{\kappa(t-s) - 2\sigma(W_t - W_s)\} M_2^{n-1,\varepsilon}(s)ds. \end{aligned}$$

Now, we set

$$a_n(t) := M_2^{n,\varepsilon}(t)\exp\{-\kappa t + 2\sigma W_t\}.$$

Then, it satisfies

$$a_{n+1}(t) \le M_{20}^{\varepsilon} + \kappa \int_0^t a_n(s)ds.$$

We use Lemma 5.3.1 in the way from Remark 5.3.1 to get

$$a_n(t) \le (\gamma + \mathcal{K}_t)e^{\gamma t}, \quad t \in (0,T).$$

This yields the desired result. □

We also provide uniform estimates for the stochastic characteristic flows.

**Proposition 5.3.2.** *For each $n \in \mathbb{N}$ and $T \in (0,\infty)$, let $(X_t^{n,\varepsilon}, V_t^{n,\varepsilon})$ be the stochastic characteristic flow for* (5.3.2) *with the initial data:*

$$(X_0^{n,\varepsilon}, V_0^{n,\varepsilon}) = (x,v) \in supp f^{in,\varepsilon}.$$

*Then for $t \in (0,T)$, we have*

$$(i)\ |V_t^{n,\varepsilon}|^2 \le \left\{ |v|^2 + \kappa \int_0^t (\gamma + \mathcal{K}_s)\exp(\gamma s)ds \right\} \exp(\kappa t - 2\sigma W_t).$$

$$(ii)\ |X_t^{n,\varepsilon}|^2 \le 2\left[ |x|^2 + t\int_0^t \left\{ \begin{array}{c} \left( |v|^2 + \kappa \displaystyle\int_0^s (\gamma + \mathcal{K}_\tau)\exp(\gamma\tau)d\tau \right) \\ \times \exp(\kappa s - 2\sigma W_s) \end{array} \right\} ds \right].$$

*Proof.* (i) It follows from Itô's lemma and (5.3.4) that

$$\begin{aligned} d|V_t^{n,\varepsilon}|^2 &= 2V_t^{n,\varepsilon}\cdot dV_t^{n,\varepsilon} + dV_t^{n,\varepsilon}\cdot dV_t^{n,\varepsilon} \\ &= 2\left(F_a[f_t^{n-1,\varepsilon}](X_t^{n,\varepsilon}, V_t^{n,\varepsilon})\cdot V_t^{n,\varepsilon} + \sigma^2|V_t^{n,\varepsilon}|^2\right)dt - 2\sigma|V_t^{n,\varepsilon}|^2 dW_t \\ &\le \left[ 2\int_{\mathbb{R}^{2d}} \phi(x_* - X_t^{n,\varepsilon})(v_*\cdot V_t^{n,\varepsilon}) f_t^{n-1,\varepsilon}(x_*,v_*)dv_*dx_* + \sigma^2|V_t^{n,\varepsilon}|^2 \right] dt \end{aligned}$$

$$
\begin{aligned}
&- 2\sigma d|V_t^{n,\varepsilon}|^2 dW_t \\
&\le \left(\kappa M_2^{n-1,\varepsilon}(t) + (\kappa + 2\sigma^2)|V_t^{n,\varepsilon}|^2\right) dt - 2\sigma |V_t^{n,\varepsilon}|^2 dW_t,
\end{aligned}
$$

where $dV_t^{n,\varepsilon} \cdot dV_t^{n,\varepsilon}$ denotes a handy notation for a quadratic variation of $V_t^{n,\varepsilon}$.

We use Proposition 5.3.1 and Lemmas 5.1.1-5.1.2 to get

$$
\begin{aligned}
|V_t^{n,\varepsilon}|^2 &\le |v|^2 \exp(\kappa t - 2\sigma W_t) \\
&\quad + \kappa \int_0^t \exp\{\kappa(t-s) - 2\sigma(W_t - W_s)\} M_2^{n-1,\varepsilon}(s) ds \\
&\le \left\{ |v|^2 + \kappa \int_0^t (\gamma + \mathcal{K}_s) \exp(\gamma s) ds \right\} \exp(\kappa t - 2\sigma W_t).
\end{aligned}
$$

(ii) For the estimate of spatial process, we use Cauchy-Schwarz inequality to get

$$
\begin{aligned}
|X_t^{n,\varepsilon}|^2 &\le \left( |x|^2 + \int_0^t |V_s^{n,\varepsilon}|^2 ds \right)^2 \le 2\left( |x|^2 + t\int_0^t |V_s^{n,\varepsilon}|^2 ds \right) \\
&\le 2\left[ |x|^2 + t\int_0^t \left\{ \begin{array}{c} \left( |v|^2 + \kappa \int_0^s (\gamma + \mathcal{K}_\tau)\exp(\gamma\tau) d\tau \right) \\ \times \exp(\kappa s - 2\sigma W_s) \end{array} \right\} ds \right].
\end{aligned}
$$

This yields the desired result.

□

As a corollary of Proposition 5.3.2, we have estimates for the sizes of velocity and spatial supports: We set

$$
\begin{aligned}
\mathcal{X}^{n,\varepsilon}(t) &:= \sup\{|x| \ : \ f_t^{n,\varepsilon}(x,v) \neq 0 \quad \text{for some } v \in \mathbb{R}^d\}, \\
\mathcal{V}^{n,\varepsilon}(t) &:= \sup\{|v| \ : \ f_t^{n,\varepsilon}(x,v) \neq 0 \quad \text{for some } x \in \mathbb{R}^d\}.
\end{aligned}
$$

**Corollary 5.3.1.** *For each $n \in \mathbb{N}$ and $T \in (0,\infty]$, let $(X_t^{n,\varepsilon}, V_t^{n,\varepsilon})$ be the stochastic characteristic flow for* (5.3.2) *with the initial data:*

$$
(X_0^{n,\varepsilon}, V_0^{n,\varepsilon}) = (x,v) \in supp f^{in,\varepsilon}.
$$

*Then for $t \in (0, T)$, we have*

$$|\mathcal{V}^{n,\varepsilon}(t)| \le |\mathcal{V}^{\infty}(t)| \quad \text{and} \quad |\mathcal{X}^{n,\varepsilon}(t)| \le |\mathcal{X}^{\infty}(t)|,$$

*where $\mathcal{X}^{\infty}(t)$ and $\mathcal{V}^{\infty}(t)$ are given by the following relations:*

$$|\mathcal{X}^{\infty}(t)|^2 := 2\left[(R_0+1)^2 + t\int_0^t \left\{\begin{matrix}\left((R_0+1)^2 + \kappa\displaystyle\int_0^s(\gamma+\mathcal{K}_\tau)\exp(\gamma\tau)d\tau\right)\\ \times\exp(\kappa s - 2\sigma W_s)\end{matrix}\right\}ds\right],$$

$$|\mathcal{V}^{\infty}(t)|^2 := \left\{(R_0+1)^2 + \kappa\int_0^t(\gamma+\mathcal{K}_s)\exp(\gamma s)ds\right\}\exp(\kappa t - 2\sigma W_t).$$

*Proof.* It follows from Proposition 5.3.2 that

$$\begin{aligned}|\mathcal{V}^{n,\varepsilon}(t)|^2 &\le \left\{|\mathcal{V}^{n,\varepsilon}(0)|^2 + \kappa\int_0^t(\gamma+\mathcal{K}_s)\exp(\gamma s)ds\right\}\exp(\kappa t - 2\sigma W_t)\\ &\le \left\{(R_0+1)^2 + \kappa\int_0^t(\gamma+\mathcal{K}_s)\exp(\gamma s)ds\right\}\exp(\kappa t - 2\sigma W_t)\\ &= |\mathcal{V}^{\infty}(t)|^2.\end{aligned}$$

This yields the first estimate for velocity support. On the other hand, we also use Proposition 5.3.2 to get

$$\begin{aligned}&|\mathcal{X}^{n,\varepsilon}(t)|^2\\ &\le 2\left[|\mathcal{X}^{n,\varepsilon}(0)|^2 + t\int_0^t \left\{\begin{matrix}\left(|\mathcal{V}^{n,\varepsilon}(0)|^2 + \kappa\displaystyle\int_0^s(\gamma+\mathcal{K}_\tau)\exp(\gamma\tau)d\tau\right)\\ \times\exp(\kappa s - 2\sigma W_s)\end{matrix}\right\}ds\right]\\ &\le 2\left[(R_0+1)^2 + t\int_0^t \left\{\begin{matrix}\left((R_0+1)^2 + \kappa\displaystyle\int_0^s(\gamma+\mathcal{K}_\tau)\exp(\gamma\tau)d\tau\right)\\ \times\exp(\kappa s - 2\sigma W_s)\end{matrix}\right\}ds\right]\\ &=: |\mathcal{X}^{\infty}(t)|^2.\end{aligned}$$

□

**Remark 5.3.2.** *Note that $f_t^{n,\varepsilon}$ has compact supports in $x$ and $v$ for every sample path which are bounded uniformly in $n$ and $\varepsilon$.*

Now, we are ready to state the results on the uniform bound for the sequence $\{f_t^{n,\varepsilon}\}$.

**Proposition 5.3.3.** *For every $n$, $m \in \mathbb{N}$ and $t \in (0,T)$, there exists a nonnegative process $\mathcal{A}_t^m$ which has continuous sample paths and is independent of $n$ and $\varepsilon$ such that*

$$\|f_t^{n,\varepsilon}\|_{W^{m,\infty}} \leq \mathcal{A}_t^m \cdot \|f^{in,\varepsilon}\|_{W^{m,\infty}}.$$

*Proof.* Since the proof is quite lengthy, we postpone it to Appendix B.2. □

**Remark 5.3.3.** *It is easy to see that for fixed $t$ and $\omega$, $\mathcal{A}_t^m$ is monotonically increasing with respect to $m$.*

Next, we prove that the sample paths of approximate solutions become a Cauchy sequence in a suitable functional space.

**Proposition 5.3.4.** *For every $n$ and $t \in (0,T)$, there exists a nonnegative process $\tilde{\mathcal{D}}_t$ which has continuous sample paths and is independent of $n$ and $\varepsilon$ such that*

$$\begin{aligned}
&\|f_t^{n,\varepsilon} - f_t^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 + \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 \\
&\quad \leq \tilde{\mathcal{D}}_t \left[ \int_0^t \left( \|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{\mathcal{C}^0}^2 \right) ds \right], \quad n \geq 2
\end{aligned}$$

*Proof.* Since the proof is almost the same as that of Theorem 5.2.1, we only point out some differences. In the proof of Theorem 5.2.1, we just replace $\mathcal{R}(t)$, $\mathcal{P}(t)$, $\max(\|f^{in}\|_{L^\infty}, \|\tilde{f}^{in}\|_{L^\infty})$ and $\max(\|f_t\|_{W^{1,\infty}}, \|\tilde{f}_t\|_{W^{1,\infty}})$ by $\mathcal{X}^\infty(t)$, $\mathcal{V}^\infty(t)$, $\|f^{in}\|$ and $\|f^{in}\|_{W^{1,\infty}}\mathcal{A}_t^1$, respectively. Then it becomes our desired estimate and hence, we can actually get

$$\begin{aligned}
&\|f_t^{n,\varepsilon} - f_t^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 + \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 \\
&\quad \leq \mathcal{B}_t^1 \int_0^t \mathcal{C}_s^1 \left( \|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 + \|f_s^{n,\varepsilon} - f_s^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 \right) ds \\
&\qquad + (1+2\|f^{in}\|_{W^{1,\infty}}\mathcal{A}_t^1)\mathcal{B}_t^2 \left[ \int_0^t \mathcal{C}_s^2 (\|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{\mathcal{C}^0}^2) ds \right]
\end{aligned}$$

$$\leq \tilde{\mathcal{D}}_t \int_0^t \Big( \|\varphi_s^{n,\varepsilon} - \varphi_s^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 + \|f_s^{n-1,\varepsilon} - f_s^{n-2,\varepsilon}\|_{\mathcal{C}^0}^2 \Big) ds,$$

where

$$\begin{aligned}
\mathcal{B}_t^1 &:= 6\left[1 + t\left(d\|\phi\|_{W^{1,\infty}}\|f^{in}\|_{L^\infty}\exp(d\kappa t + d\sigma W_t)\right)^2\right],\\
\mathcal{C}_t^1 &:= (1 + (4\mathcal{X}^\infty(t)\mathcal{V}^\infty(t))^{2d}),\\
\mathcal{B}_t^2 &:= 1 + 2\|\phi\|_{W^{1,\infty}}\exp\left(4\sigma \sup_{0\leq s\leq t}|W_s|\right),\\
\mathcal{C}_t^2 &:= 1 + \mathcal{V}^\infty(t)(4\mathcal{X}^\infty(t)\mathcal{V}^\infty(t))^d,\\
\tilde{\mathcal{D}}_t &:= \mathcal{B}_t^1\left(\sup_{0\leq s\leq t}\mathcal{C}_s^1\right) + \left(1 + 2\|f^{in}\|_{W^{1,\infty}}\mathcal{A}_t^1)\right)\mathcal{B}_t^2\left(\sup_{0\leq s\leq t}\mathcal{C}_s^2\right).
\end{aligned}$$

This gives the desired result.

□

For each $t$ and $\omega \in \Omega$, we define

$$\Delta_n^\varepsilon(t,\omega) := \|f_t^{n,\varepsilon} - f_t^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2 + \|\varphi_t^{n,\varepsilon} - \varphi_t^{n-1,\varepsilon}\|_{\mathcal{C}^0}^2.$$

**Corollary 5.3.2.** *The functional $\Delta_n^\varepsilon(t)$ satisfies*

$$\Delta_n^\varepsilon(t,\omega) \leq \frac{(\mathcal{K}(\omega)t)^n}{n!}, \quad \text{for each } \ t \in [0,T] \quad \text{and a.s. } \omega \in \Omega,$$

*where $\mathcal{K} = \mathcal{K}(\omega)$ is a nonnegative random variable.*

*Proof.* It follows from Proposition 5.3.4 that

$$\Delta_{n+1}^\varepsilon(t) \leq \tilde{\mathcal{D}}_t\left(\int_0^t (\Delta_n^\varepsilon(s) + \Delta_{n+1}^\varepsilon(s))ds\right).$$

Since $\tilde{\mathcal{D}}_t$ is a nonnegative process with continuous sample paths, there exists a nonnegative random variable $D = D(\omega)$ such that

$$\sup_{0\leq t\leq T}\tilde{\mathcal{D}}_t(\omega) \leq D(\omega) < \infty, \quad \text{for each} \quad \omega \in \Omega.$$

Thus, we can use the Grönwall-type lemma in Lemma 5.3.1 to deduce

$$\Delta_n(t,\omega) \le \frac{(\hat{D}(\omega)t)^n}{n!}, \quad \text{for each } \ t \in [0,T], \ \omega \in \Omega,$$

where $\hat{D} = \hat{D}(\omega)$ depends on $D(\omega)$. □

**Remark 5.3.4.** *Corollary 5.3.2 implies that for everry $\omega$,*

$$f_t^{n,\varepsilon}(\omega) \to f_t^{\varepsilon}(\omega) \quad \textit{in } \mathcal{C}([0,T]\times\mathbb{R}^{2d}).$$

*Since $f_t^{n,\varepsilon}$ is $\mathcal{F}_t$-adapted (where $\mathcal{F}_t$ is a filtration generated by the Wiener process) and $f_t^{\varepsilon}$ is a pointwise limit of $f^{n,\varepsilon}$ over $\Omega$, we have $f$ is $\mathcal{F}_t$-adapted. Moreover, we have a uniform boundedness of $f_t^{n,\varepsilon}$ in $L^\infty([0,T];W^{m,p}(\mathbb{R}^{2d}))$ for any $p \in [1,\infty)$. By the property of reflexive Banach space, there exists a subsequence $\{f^{n_k,\varepsilon}(\omega)\} \subseteq \{f^{n,\varepsilon}(\omega)\}$ which is weakly convergent to $\tilde{f}_t(\omega)$ in $L^\infty([0,T];W^{m,p}(\mathbb{R}^{2d}))$ for each $\omega \in \Omega$ and every $p \in [1,\infty)$. Since we have already a strong convergence in the lower order, we can conclude that $f_t^{\varepsilon}(\omega) = \tilde{f}_t(\omega)$. However, we can not proceed further, since it is not clear whether $f_t^{\varepsilon}$ satisfies the equation (5.0.1) at this moment. This is due to the noise term in the right-hand side of (5.3.2). It is not certain whether the Stratonovich integral of $f_t^{\varepsilon}$ can be defined or not. In addition, even if the noise term can be well-defined, it is also not clear whether the Stratonovich integral of $f^{n,\varepsilon}$ converges to that of $f_t^{\varepsilon}$ or not.*

### 5.3.3 Proof of Theorem 5.1.3

In this subsection, we prove a global well-posedness of a solution to system (5.3.2) by showing that the limit of the sequence $\{f_t^{n,\varepsilon}\}$ exists as $n \to \infty$ for each $\varepsilon$, and that this limit is indeed a strong solution to (5.0.1) corresponding to the regularized initial datum $f^{in,\varepsilon}$.

In order to cope with the problems discussed in Remark 5.3.4, we employ a stopping time argument. First, we define a sequence of stopping times $\{\tau_M\}_{M\in\mathbb{N}}$ as follows:

$$\begin{aligned}
&\tau_M^1(\omega) := \inf\{t \ge 0 \mid \mathcal{A}_t^{k_*}(\omega) > M\} \wedge T,\\
&\tau_M^2(\omega) := \inf\{t \ge 0 \mid \tilde{\mathcal{D}}_t(\omega) > M\} \wedge T,\\
&\tau_M^2(\omega) := \inf\{t \ge 0 \mid \mathscr{D}_t(\omega) > M\} \wedge T, \quad \tau_M := \tau_M^1 \wedge \tau_M^2 \wedge \tau_M^3,
\end{aligned}$$

where $k_* := \max\{k, 4\}$ and $\mathscr{D}_t$ is a nonnegative process with continuous sample paths which will be specified later. Now, we verify the existence of regularized solutions step by step.

• (Step A: The limit $n \to \infty$): First, we obtain the limit function $f^{\varepsilon}_{t\wedge\tau_M}$ which is a classical solution to equation (5.0.1) with the regularized initial data, based on the estimates in the previous subsection.

◇ (Step A-1: Extracting a limit function): We can find out that for each $n \in \mathbb{N}$,

$$
\begin{aligned}
&(i)\ \|f^{n,\varepsilon}_{t\wedge\tau_M}\|_{W^{m,\infty}} \le M\|f^{in,\varepsilon}\|_{W^{m,\infty}}.\\
&(ii)\ \|f^{n,\varepsilon}_{t\wedge\tau_M} - f^{n-1,\varepsilon}_{t\wedge\tau_M}\|^2_{\mathcal{C}^0} + \|\varphi^{n,\varepsilon}_{t\wedge\tau_M} - \varphi^{n-1,\varepsilon}_{t\wedge\tau_M}\|^2_{\mathcal{C}^0}\\
&\qquad \le M\Big[\int_0^t \Big(\|\varphi^{n,\varepsilon}_{s\wedge\tau_M} - \varphi^{n-1,\varepsilon}_{s\wedge\tau_M}\|^2_{\mathcal{C}^0} + \|f^{n-1,\varepsilon}_{s\wedge\tau_M} - f^{n-2,\varepsilon}_{s\wedge\tau_M}\|^2_{\mathcal{C}^0}\Big) ds\Big].
\end{aligned}
$$

Thus, we can use the same argument as in Corolllary 5.3.2 to yield that as $n \to \infty$, there exists a limit function $f^{\varepsilon}_{t\wedge\tau_M}$ such that, up to a subsequence,

$$
\begin{aligned}
&f^{n,\varepsilon}_{t\wedge\tau_M} \to f^{\varepsilon}_{t\wedge\tau_M} \quad \text{in} \quad L^\infty(\Omega; \mathcal{C}([0,T]\times\mathbb{R}^{2d})),\\
&f^{n,\varepsilon}_{t\wedge\tau_M} \rightharpoonup f^{\varepsilon}_{t\wedge\tau_M} \quad \text{in} \quad L^\infty(\Omega\times[0,T]; W^{m,p}(\mathbb{R}^{2d})), \quad \forall\, p \in [1,\infty).
\end{aligned}
$$

◇ (Step A-2: Verification of relation (5.1.5)): Now, we need to show that $f^{\varepsilon}_{t\wedge\tau_M}$ satisfies (5.3.2) in the sense of Definition 5.1.1. Since $f^{n,\varepsilon}_{t\wedge\tau_M}$ satisfies (5.3.6) and conditions of Lemma 5.2.2, it satisfies the following relation:

$$
\begin{aligned}
\int_\Sigma f^{n,\varepsilon}_{t\wedge\tau_M}\psi dz &= \int_\Sigma f^{in,\varepsilon}\psi dz\\
&\quad + \int_0^t\int_\Sigma f^{n,\varepsilon}_{s\wedge\tau_M}\Big[v\cdot\nabla_x\psi + \Big(F_a[f^{n-1,\varepsilon}_{s\wedge\tau_M}] + \frac{1}{2}\sigma^2 v\Big)\cdot\nabla_v\psi\Big] dzds\\
&\quad + \frac{1}{2}\sigma^2\int_0^t\int_\Sigma v f^{n,\varepsilon}_{s\wedge\tau_M}\cdot(D_v^2\psi)v dzds\\
&\quad - \sigma\int_0^t\int_\Sigma f^{n,\varepsilon}_{s\wedge\tau_M} v\cdot\nabla_v\psi dzdW_s,
\end{aligned}
\tag{5.3.9}
$$

where $\Sigma := \mathbb{R}^{2d}$ and $dz = dvdx$.

Next, our job is to pass $n \to \infty$ in the integral relation (5.3.9) to derive an integral relation (5.1.5) for $f^\varepsilon_{t\wedge\tau_M}$. For this, note that the $x$- and $v$-supports of $f^{n,\varepsilon}_{t\wedge\tau_M}$ and $f^\varepsilon_{t\wedge\tau_M}$ are uniformly bounded by $|\mathcal{X}^\infty_{t\wedge\tau_M}|$ and $|\mathcal{V}^\infty_{t\wedge\tau_M}|$ (see Corollary 5.3.1). Moreover, we can find out that $|\mathcal{X}^\infty_{t\wedge\tau_M}|$ and $|\mathcal{V}^\infty_{t\wedge\tau_M}|$ are bounded by $\mathcal{A}^{k_*}_{t\wedge\tau_M}$, and hence by $M$. We combine the strong convergence on the lower order with these facts to yield

(i) $\displaystyle\int_\Sigma (f^{n,\varepsilon}_{t\wedge\tau_M} - f^\varepsilon_{t\wedge\tau_M})\psi dz \to 0.$

(ii) $\displaystyle\int_0^t\int_\Sigma (f^{n,\varepsilon}_{s\wedge\tau_M} - f^\varepsilon_{s\wedge\tau_M})\left[v\cdot\nabla_x\psi + \left(F_a[f^{n-1,\varepsilon}_{s\wedge\tau_M}] + \frac{1}{2}\sigma^2 v\right)\cdot\nabla_v\psi\right]dzds \to 0.$

(iii) $\displaystyle\int_0^t\int_\Sigma f^\varepsilon_{s\wedge\tau_M}\left(F_a[f^{n-1,\varepsilon}_{s\wedge\tau_M}] - F_a[f^\varepsilon_{s\wedge\tau_M}]\right)\nabla_v\psi dzds \to 0.$

(iv) $\displaystyle\frac{1}{2}\sigma^2\int_0^t\int_\Sigma v(f^{n,\varepsilon}_{s\wedge\tau_M} - f^\varepsilon_{s\wedge\tau_M})\cdot(D^2_v\psi)vdzds \to 0,$

uniformly in $\omega$, as $n$ goes to infinity.

Now it remains to check with the stochastic integral term in (5.3.9). For this term, one has

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t\int_\Sigma (f^{n,\varepsilon}_{s\wedge\tau_M} - f^\varepsilon_{s\wedge\tau_M})v\cdot\nabla_v\psi dzdW_s\right)^2\right] \\
&\quad = \mathbb{E}\left[\int_0^t\left(\int_\Sigma (f^{n}_{s\wedge\tau_M} - f^\varepsilon_{s\wedge\tau_M})v\cdot\nabla_v\psi dz\right)^2 ds\right] \\
&\quad \le \mathbb{E}\left[\int_0^t \|f^{n,\varepsilon}_{s\wedge\tau_M} - f^\varepsilon_{s\wedge\tau_M}\|^2_{\mathcal{C}^0} ds\right]\left(\int_\Sigma |v\cdot\nabla_v\psi|dz\right)^2 \longrightarrow 0, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

This $L^2$-convergence over $\Omega$ implies that there exists a subsequence $\{f^{n_l,\varepsilon}_{t\wedge\tau_M}\}$ such that

$$
\left(\int_0^t\int_\Sigma f^{n_l,\varepsilon}_{s\wedge\tau_M} v\cdot\nabla_v\psi dzdW_s\right)(\omega) \longrightarrow \left(\int_0^t\int_\Sigma f^\varepsilon_{s\wedge\tau_M} v\cdot\nabla_v\psi dzdW_s\right)(\omega),
$$

for a.s. $\omega$, as $l$ goes to infinity. Thus, we can conclude that for a.s. $\omega \in \Omega$, $f^\varepsilon_{s\wedge\tau_M}$ satisfies

$$\begin{aligned}\int_\Sigma f^\varepsilon_{t\wedge\tau_M}\psi dz &= \int_\Sigma f^{in,\varepsilon}\psi dz \\ &\quad - \int_0^t \int_\Sigma f^\varepsilon_{s\wedge\tau_M}\left(v\cdot\nabla_x\psi + \left(F_a[f^\varepsilon_{s\wedge\tau_M}] + \frac{1}{2}\sigma^2 v\right)\cdot\nabla_v\psi\right)dzds \\ &\quad - \frac{1}{2}\sigma^2\int_0^t\int_\Sigma v f^\varepsilon_{s\wedge\tau_M}\cdot(D_v^2\psi)vdzds + \int_0^t\int_\Sigma f^\varepsilon_{s\wedge\tau_M} v\cdot\nabla_v\psi dzdW_s,\end{aligned}$$

for every $\psi \in \mathcal{D}(\mathbb{R}^{2d})$. One also has $f^\varepsilon_{t\wedge\tau_M}$ is a $\mathcal{F}_t$-semimartingale. Here, we use Lemma 5.2.1 to obtain that $f^\varepsilon_{t\wedge\tau_M}$ satisfies (5.0.1) in the sense of distribution.

• (Step B: The limit $\varepsilon \to 0$): Here, we address the convergence of solutions to the regularized system (5.3.1). Since $k_* \geq 4$, one uses the Sobolev embedding theorem to get $f^\varepsilon_{t\wedge\tau_M} \in L^\infty(\Omega;\mathcal{C}([0,T];\mathcal{C}^{3,\delta}(\mathbb{R}^{2d})))$. Thus, it follows from Lemma 5.2.2 and Remark 5.2.1 that $f^\varepsilon_{t\wedge\tau_M}$ becomes a classical solution to (5.3.1) corresponding to the regularized initial datum $f^{in,\varepsilon}$ .

⋄ (Step B-1: Extracting a limit function): Note that the strong convergence in Step A implies that the $x$-support and the $v$-support of $f^\varepsilon_{t\wedge\tau_M}$ are bounded by $\mathcal{X}^\infty$ and $\mathcal{V}^\infty$, respectively, uniformly in $\varepsilon$. Thus, we can follow the stability estimate in Theorem 5.2.1 to get

$$\begin{aligned}&\|f^\varepsilon_{t\wedge\tau_M} - f^{\varepsilon'}_{t\wedge\tau_M}\|^2_{\mathcal{C}^0} + \|\varphi^\varepsilon_{t\wedge\tau_M} - \varphi^{\varepsilon'}_{t\wedge\tau_M}\|^2_{\mathcal{C}^0} \\ &\qquad \leq \mathscr{D}_{t\wedge\tau_M}\|f^{in,\varepsilon} - f^{in,\varepsilon'}\|^2_{\mathcal{C}^0} \leq M\|f^{in,\varepsilon} - f^{in,\varepsilon'}\|^2_{\mathcal{C}^0},\end{aligned} \tag{5.3.10}$$

where $\mathscr{D}_t$ can be obtained if $\mathcal{R}(t)$, $\mathcal{P}(t)$, $\max(\|f^{in}\|_{L^\infty}, \|\tilde{f}^{in}\|_{L^\infty})$ and $\max(\|f_t\|_{W^{1,\infty}}, \|\tilde{f}_t\|_{W^{1,\infty}})$ in the formulation of $\mathcal{D}_t$ from Theorem 5.2.1 are substituted by $\mathcal{X}^\infty(t)$, $\mathcal{V}^\infty(t)$, $\|f^{in}\|$ and $\|f^{in}\|_{W^{1,\infty}}\mathcal{A}^1_t$, respectively.

Since $f^{in,\varepsilon}$ converges uniformly to $f^{in}$, it follows from the stability estimate (5.3.10) that there exists $f_{t\wedge\tau_M}$ such that

$$f^\varepsilon_{t\wedge\tau_M} \to f_{t\wedge\tau_M} \quad \text{in} \quad L^\infty(\Omega;\mathcal{C}([0,T]\times\mathbb{R}^{2d})).$$

Moreover, it follows from the weak convergence and ($\mathcal{F}^\varepsilon 1$) that

$$\|f^{\varepsilon}_{t\wedge\tau_M}\|_{W^{k,\infty}} \le \mathcal{A}^k_t \|f^{in,\varepsilon}\|_{W^{k,\infty}} \le M\|f^{in}\|_{W^{k,\infty}}.$$

Hence, we can follow the arguments in Step A to yield that $f_{t\wedge\tau_M}$ satisfies relation (5.2.1) and hence (5.1.5). Moreover, $f_{t\wedge\tau_M}$ is compactly supported in $x$ and $v$.

$\diamond$ (Step B-2: Regularity of a strong solution): Now, we prove that $f_{t\wedge\tau_M}$ has the desired regularity. Since $f^{\varepsilon}_{t\wedge\tau_M}$ is a classical solution to (5.0.1) with initial data $f^{in,\varepsilon}$, it can be uniquely written as

$$\begin{aligned} &f^{\varepsilon}_{t\wedge\tau_M}(\varphi^{\varepsilon}_{t\wedge\tau_M}(x,v)) \\ &\quad = f^{in,\varepsilon}(x,v)\exp\left[-\int_0^{t\wedge\tau_M} \nabla_v\cdot F_a[f_s](\varphi^{\varepsilon}_s)ds + d\sigma W_{t\wedge\tau_M}\right], \end{aligned}$$

(for detail, we refer to Appendix B.1). Since we also obtain the uniform convergence of the characteristics $\varphi^{\varepsilon}_{t\wedge\tau_M}$ as $\varepsilon\to 0$ from (5.3.10), the solution $f_{t\wedge\tau_M}$ satisfies the following relation:

$$\begin{aligned} &f_{t\wedge\tau_M}(\varphi_{t\wedge\tau_M}(x,v)) \\ &\quad = f^{in}(x,v)\exp\left[-\int_0^{t\wedge\tau_M} \nabla_v\cdot F_a[f_s](\varphi_s)ds + d\sigma W_{t\wedge\tau_M}\right], \end{aligned} \tag{5.3.11}$$

and the limit $\varphi_{t\wedge\tau_M}(x,v) = (X_{t\wedge\tau_M}(x,v), V_{t\wedge\tau_M})$ is a solution to the following SDE:

$$\begin{cases} X_{t\wedge\tau_M} = x + \displaystyle\int_0^{t\wedge\tau_M} V_s ds, \\ V_{t\wedge\tau_M} = v + \displaystyle\int_0^{t\wedge\tau_M} \left(F_a[f_s](X_s,V_s)\right) ds + \int_0^{t\wedge\tau_M} \sigma(v_c - V_t)\circ dW_s. \end{cases}$$

Since the kernel $F_a[f_t]$ is smooth, $\varphi_{t\wedge\tau_M}(x,v)$ can be shown to be a $\mathcal{C}^m$-diffeomorphism for any $m\in\mathbb{N}$, and so is its inverse $\psi_{t\wedge\tau_M}(x,v) := (\varphi_t(x,v))^{-1}$. Thus, if we write

$$\begin{aligned} &f_{t\wedge\tau_M}(x,v) \\ &\quad = f^{in}(\psi_{t\wedge\tau_M}(x,v))\exp\left[-\int_0^{t\wedge\tau_M} \nabla_v\cdot F_a[f_s](\varphi_s(\psi_{t\wedge\tau_M}(x,v))ds + d\sigma W_{t\wedge\tau_M}\right], \end{aligned}$$

it directly follows from the regularity of $f^{in}$ and $\psi_t$ that $f_{t\wedge\tau_M}$ has the desired regularity.

• (Step C: Properties of a strong solution): We recall several properties of regularized solutions. First, it is obvious from (5.3.11) that

$$\|f_{t\wedge\tau_M}\|_{L^\infty} \le \|f^{in}\|_{L^\infty}\exp(d\kappa t\wedge\tau_M + d\sigma W_{t\wedge\tau_M})$$

Since $f^\varepsilon_{t\wedge\tau_M}$ is a classical solution to (5.3.1) corresponding to the regularized initial datum $f^{in,\varepsilon}$, Lemma 5.2.3 gives

$$M_2^\varepsilon(t\wedge\tau_M) \le M_2^\varepsilon(0)\exp(-2\phi_m t\wedge\tau_M - 2\sigma W_{t\wedge\tau_M}),$$

and the strong convergence together with compact supports gives

$$M_2(t\wedge\tau_M) \le M_2(0)\exp(-2\phi_m t\wedge\tau_M - 2\sigma W_{t\wedge\tau_M}).$$

Moreover, it is obvious that

$$\tau_M(\omega)\to T \quad \text{as } M\to\infty \text{ for a.s. } \omega.$$

Thus, we choose a sufficiently large $M$ for each $\omega\in\Omega$ such that $f_{t\wedge\tau_M}(\omega)$ satisfies the relation (5.1.5) on $[0,T]$.

For the expectation estimates of the solution, we use Fatou's lemma to get, for any $p\in(1,\infty)$,

$$\begin{aligned}
\mathbb{E}\|f_t\|_{L^\infty} &\le \liminf_{M\to\infty}\mathbb{E}\|f_{t\wedge\tau_M}\|_{L^\infty}\\
&\le \liminf_{M\to\infty}\|f^{in}\|_{L^\infty}\mathbb{E}\Big[\exp(d\kappa t\wedge\tau_M + d\sigma W_{t\wedge\tau_M})\Big]\\
&= \liminf_{M\to\infty}\|f^{in}\|_{L^\infty}\mathbb{E}\left[\begin{array}{c}\exp\left(d\sigma W_{t\wedge\tau_M} - \dfrac{p(d\sigma)^2}{2}t\wedge\tau_M\right)\\ \times\exp\left(\Big(d\kappa + \dfrac{p(d\sigma)^2}{2}\Big)t\wedge\tau_M\right)\end{array}\right]\\
&\le \liminf_{M\to\infty}\|f^{in}\|_{L^\infty}\mathbb{E}\left[\exp\left(\frac{p}{p-1}\Big(d\kappa + \frac{p(d\sigma)^2}{2}\Big)t\wedge\tau_M\right)\right]^{(p-1)/p}\\
&= \|f^{in}\|_{L^\infty}\exp\left(\Big(d\kappa + \frac{p(d\sigma)^2}{2}\Big)t\right),
\end{aligned}$$

where we used the fact $X_t = \exp(aW_t - a^2t/2)$ is a martingale, Hölder's inequality and Lebesgue's dominated convergence theorem. Then we take the limit $p \to 1$ on both sides to obtain the desired result. For the dissipation of the second velocity moment, we use a similar argument to get the desired estimate.

# Chapter 6

# Conclusion and future works

In this thesis, we covered three topics related to quantitative estimates for intrinsic and extrinsic uncertainties in the Cucker-Smale model.

First, we have presented the local sensitivity analysis for the random hydrodynamic Cucker-Smale model describing the emergence of flocking in the ensemble of Cucker-Smale flocking particles. In previous works, quantitative estimates for the variations of the solutions in random space were derived from particle and kinetic models for the C-S flocking. We extended the aforementioned quantitative estimates to the hydrodynamic Cucker-Smale model, e.g., the propagation of the $z$-variations of spatial and velocity process, where $z$ is the random input variable, the $L^2$-stability and flocking estimates along the sample path. Thanks to the regularity analysis of the deterministic HCS model, we can lift regularity estimates to the random solution process along the sample path. As mentioned in the Introduction, the synthesis of flocking dynamics and local sensitivity analysis is not that mature yet. There are many open questions, for example, the effect of uncertainties on the formation of multi-cluster flocking and extension of the local sensitivity to the initial and boundary problems in the context of flocking. Moreover, as far as the authors know, the initial and boundary value problems are not well studied in the flocking problems even for the deterministic flocking models. We leave these issues for future works.

Second, we have introduced the Cucker-Smale model with randomly switching topologies for flocking phenomena and provided a sufficient framework leading to the stochastic flocking in terms of system parameters and communication weight function. For the stochastic flocking modeling, we employed two random components for the switching times and selection of network topology at switching instant. Our flocking analysis took two procedures: First, we derived flocking estimates along the sample path in a priori setting on the network topologies and position diameter. Second, we replaced a priori assumption on the position diameter by suitable assumptions on the system parameters and communication weight, and moreover, we also showed that the a priori assumption on the network topology can be attained by imposing some condition on the network selection probability. There are still many questions to be investigated for the proposed model. For example, what if the support of the probability density function $f$ for the sequence $\{t_{\ell+1} - t_\ell\}_{\ell \geq 0}$ is not compactly supported, say $(0, \infty)$? Our analysis employed in the proof of the main result breaks down for unbounded support cases. However, it seems that our methodology and framework is quite general so that it can be applied to other C-S type flocking and Kuramoto type synchronization models. These issues will be addressed in our future works.

Finally, we studied a global well-posedness of strong solutions and their asymptotic emergent dynamics for the stochastic kinetic Cucker-Smale equation perturbed by multiplicative white noise. For a global well-posedness, we first derive a sequence of classical solutions to the stochastic kinetic C-S equation with regularized initial data. Then, by using the properties of classical solutions, we obtained the well-posedness of a strong solution corresponding to the original initial data and asymptotic emergent stochastic dynamics of strong solutions. Of course, there are lots of interesting issues to be addressed in a future work, e.g., a global existence of weak solutions, emergent dynamics under other types of random perturbations and zero noise limit, etc. These topics will be discussed in future works.

# Appendix A

# Detailed proof of Chapter 3

## A.1 Proof of Lemma 3.1.2

Similarly to Lemma 3.1.1, it suffices to provide the upper-bound estimates. Again, we split the cases into the zeroth-order and higher-order estimates.

• Step A (The zeroth-order estimates) : We multiply $(3.1.3)_2$ by $\partial_z u^{n+1}$ to get

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z u^{n+1}\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{T}^d} \partial_z u^n \cdot \nabla u \cdot \partial_z u^{n+1} dx + \frac{1}{2}\int_{\mathbb{T}^d} (\nabla \cdot u)|\partial_z u^{n+1}|^2 dx \\
&\quad\quad + \int_{\mathbb{T}^{2d}} \partial_z \phi(x-y)(u(y)-u(x))\rho(y)\partial_z u^{n+1}(x) dydx \\
&\quad\quad + \int_{\mathbb{T}^{2d}} \phi(x-y)(\partial_z u^n(y)-\partial_z u^n(x))\rho(y)\partial_z u^{n+1} u(x) dydx \\
&\quad\quad + \int_{\mathbb{T}^{2d}} \phi(x-y)(u(y)-u(x))\partial_z \rho^{n+1}(y)\partial_z u^{n+1}(x) dydx \\
&\quad \le \|\nabla u\|_{L^\infty}\|\partial_z u^n\|_{L^2}\|\partial_z u^{n+1}\|_{L^2} + \|\nabla u\|_{L^\infty}\|\partial_z u^{n+1}\|_{L^2}^2 \\
&\quad\quad + 2\|\phi\|_s\|\rho\|_{L^2}\Big(\|u\|_{L^2}\|\|\partial_z u^{n+1}\|_{L^2} + \|\partial_z u^n\|_{L^2}\|\partial_z u^{n+1}\|_{L^2}\Big) \\
&\quad\quad + 2\|\phi\|_s\|u\|_{L^2}\|\partial_z \rho^{n+1}\|_{L^2}\|\partial_z u^{n+1}\|_{L^2} \\
&\quad \le C(\varepsilon^{1/2}\|\partial_z u^{n+1}\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}
\tag{A.1.1}
$$

where $C$ is a positive random function independent of $n$ and we used the Sobolev embedding theorem, Young's inequality and Lemma 3.1.1. Now, we apply Grönwall's lemma for (A.1.1) to yield

$$\|\partial_z u^{n+1}\|_{L^2}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z u^{n+1}(s,z)\|_{L^2}^2 ds + \varepsilon^{3/2}\right). \tag{A.1.2}$$

• Step B (Higher-order estimates) : For $1 \le k \le s$, we apply $\nabla^k$ to $(3.1.3)_2$, multiply by $\nabla^k(\partial_z u^{n+1})$ and integrate the resulting relation over $\mathbb{T}^d$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{T}^d} \partial_z u^n \cdot \nabla(\nabla^k u)\cdot \nabla^k(\partial_z u^{n+1})dx \\
&\qquad - \int_{\mathbb{T}^d}\Big[\nabla^k(\partial_z u^n \cdot \nabla u) - \partial_z u^n \cdot \nabla^k(\nabla u)\Big]\nabla^k(\partial_z u^{n+1})dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\nabla^k(\partial_z u^{n+1})|^2 dx \\
&\qquad - \int_{\mathbb{T}^d}\Big[\nabla^k(u\cdot\nabla(\partial_z u^{n+1})) - u\cdot\nabla^k(\nabla(\partial_z u^{n+1})\Big]\nabla^k(\partial_z u^{n+1})dx \\
&\qquad + \int_{\mathbb{T}^d}\nabla^k\left(\int_{\mathbb{T}^d}\partial_z\phi(x-y)(u(y)-u(x))\rho(y)dy\right)\nabla^k(\partial_z u^{n+1})(x)dx \\
&\qquad + \int_{\mathbb{T}^d}\nabla^k\left(\int_{\mathbb{T}^d}\phi(x-y)(\partial_z u^n(y)-\partial_z u^n(x))\rho(y)dy\right)\nabla^k(\partial_z u^{n+1})(x)dx \\
&\qquad + \int_{\mathbb{T}^d}\nabla^k\left(\int_{\mathbb{T}^d}\phi(x-y)(u(y)-u(x))\partial_z\rho^{n+1}(y)dy\right)\nabla^k(\partial_z u^{n+1})(x)dx \\
&\quad =: \sum_{i=1}^7 \mathcal{I}_{6i}.
\end{aligned}$$

Here, we separately estimate $\mathcal{I}_{6i}$'s as follows.

⋄ (Estimates for $\mathcal{I}_{6i}$, $i = 1, 2, 3, 4$) : We use the Cauchy-Schwarz inequality, commutator estimates and Young's inequality to get

$$\mathcal{I}_{61} \le \|\partial_z u^n\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \le C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}),$$

$$
\begin{aligned}
\mathcal{I}_{62} &\leq c\Big[\|\nabla(\partial_z u^n)\|_{L^\infty}\|\nabla^k u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z u^n)\|_{L^2}\Big]\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \\
&\leq C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}), \\
\mathcal{I}_{63} &\leq \frac{\|\nabla\cdot u\|_{L^\infty}}{2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 \leq \varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2, \\
\mathcal{I}_{64} &\leq c\Big[\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} + \|\nabla(\partial_z u^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2}\Big]\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \\
&\leq C\varepsilon^{1/2}\|\partial_z u^{n+1}\|_{H^s}^2.
\end{aligned}
$$

$\diamond$ (Estimates for $\mathcal{I}_{6i}$, $i = 5, 6, 7$) : For $\mathcal{I}_{65}$, one gets

$$
\begin{aligned}
\mathcal{I}_{65} &= \int_{\mathbb{T}^{2d}} \nabla^k(\partial_z\phi(x-y))(u(y)-u(x))\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\
&\quad - \sum_{0\leq r\leq k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \nabla^r(\partial_z\phi(x-y))\nabla^{k-r}u(x)\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\
&\leq C\|\phi\|_s\|u\|_{H^k}\|\rho\|_{L^2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \\
&\leq C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality and Young's inequality.

For $\mathcal{I}_{66}$, we use the same arguments as $\mathcal{I}_{65}$ to get

$$
\begin{aligned}
\mathcal{I}_{66} &= \int_{\mathbb{T}^{2d}} \nabla^k\phi(x-y)(\partial_z u^n(y)-\partial_z u^n(x))\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\
&\quad - \sum_{0\leq r\leq k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \nabla^r\phi(x-y)\nabla^{k-r}(\partial_z u^n)(x)\rho(y)\nabla^k(\partial_z u^{n+1})(x)dydx \\
&\leq C\|\phi\|_s\|\rho\|_{L^2}\|\partial_z u^n\|_{H^k}\|\nabla^k(\partial_z u^{n+1})\|_{L^2} \\
&\leq C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}
$$

For $\mathcal{I}_{67}$, we have

$$
\begin{aligned}
\mathcal{I}_{67} &= \int_{\mathbb{T}^{2d}} \nabla^k\phi(x-y)(u(y)-u(x))\partial_z\rho^{n+1}\nabla^k(\partial_z u^{n+1})(x)dydx \\
&\quad - \sum_{0\leq r\leq k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \nabla^r\phi(x-y)\nabla^{k-r}u(x)\partial_z\rho^{n+1}(y)\nabla^k(\partial_z u^{n+1})(x)dydx
\end{aligned}
$$

$$\le C\|\phi\|_s\|\partial_z\rho^{n+1}\|_{L^2}\|u\|_{H^k}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}$$
$$\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).$$

We gather all results for $\mathcal{I}_{6i}$'s, sum over $1 \le k \le s$, integrate the resulting relation and combine with (A.1.2) to get

$$\|\partial_z u^{n+1}\|_{H^s}^2 \le C\left(\varepsilon^{1/2}\int_0^t \|\partial_z u^{n+1}(s,z)\|_{H^s}^2 ds + \varepsilon^{3/2}\right).$$

Finally, we use Grönwall's lemma to obtain the desired result.

## A.2 Proof of Lemma 3.1.5

We split the estimates into the zeroth-order and the higher-order cases.

• Step A (The zeroth-order estimates): It follows from (3.1.12) that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\partial_z^m u^{n+1}\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^d} \partial_z^m u^n \cdot \nabla u \cdot \partial_z^m u^{n+1} dx + \frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot u)|\partial_z^m u^{n+1}|^2 dx \\
&\quad - \sum_{1\le l\le m-1} \binom{m}{l} \int_{\mathbb{T}^d} \partial_z^l u \cdot \nabla(\partial_z^{m-l} u)\cdot \partial_z^m u^{n+1} dx \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m\\ \beta,\gamma\ne m}} \frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}} \partial_z^\alpha\phi(x-y)(\partial_z^\beta u(y)-\partial_z^\beta u(x))\partial_z^\gamma\rho(y)\partial_z^m u^{n+1}(x)dydx \\
&\quad + \int_{\mathbb{T}^{2d}} \phi(x-y)(\partial_z^m u^n(y)-\partial_z^m u^n(x))\rho(y)\partial_z^m u^{n+1}(x)dydx \\
&\quad + \int_{\mathbb{T}^{2d}} \phi(x-y)(u(y)-u(x))\partial_z^m \rho^{n+1}(y)\partial_z^m u^{n+1}(x)dydx \\
&\le \|\nabla u\|_{L^\infty}\|\partial_z^m u^n\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2} + \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\partial_z^m u^{n+1}\|_{L^2}^2 \\
&\quad + C\sum_{1\le l\le m-1}\|\partial_z^l u\|_{L^\infty}\|\nabla(\partial_z^{m-l}u)\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2} \\
&\quad + C\sum_{\substack{\alpha+\beta+\gamma=m\\ \beta,\gamma\ne m}}\|\partial_z^\beta u\|_{L^2}\|\partial_z^\gamma\rho\|_{L^2}\|\partial_z^m u^{n+1}\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+ 2\|\phi\|_s \|\rho\|_{L^2} \|\partial_z^m u^n\|_{L^2} \|\partial_z^m u^{n+1}\|_{L^2} \\
&+ 2\|\phi\|_s \|u\|_{L^2} \|\partial_z^m \rho^{n+1}\|_{L^2} \|\partial_z^m u^{n+1}\|_{L^2} \\
&\le C(\varepsilon^{1/2} \|\partial_z^m u^{n+1}\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality and Young's inequality.

• Step B (Higher-order estimates): For $1 \le k \le s - m + 1$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t} \|\nabla^k (\partial_z^m u^{n+1})\|_{L^2}^2 \\
&= -\int_{\mathbb{T}^d} \nabla^k (\partial_z^m u^n \cdot \nabla u) \nabla^k (\partial_z^m u^{n+1}) dx \\
&\quad - \int_{\mathbb{T}^d} \nabla^k (u \cdot \nabla(\partial_z^m u^{n+1})) \nabla^k (\partial_z^m u^{n+1}) dx \\
&\quad - \sum_{1 \le l \le m-1} \binom{m}{l} \int_{\mathbb{T}^d} \nabla^k (\partial_z^l u \cdot \nabla(\partial_z^{m-l} u)) \nabla^k (\partial_z^m u^{n+1}) dx \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma \ne m}} \frac{m!}{\alpha!\beta!\gamma!} \int_{\mathbb{T}^{2d}} \nabla^k \begin{bmatrix} \partial_z^\alpha \phi(x-y) \\ \cdot (\partial_z^\beta u(y) - \partial_z^\beta u(x)) \end{bmatrix} \partial_z^\gamma \rho(y) \nabla^k (\partial_z^m u^{n+1})(x) dy dx \\
&\quad + \int_{\mathbb{T}^{2d}} \nabla^k \Big( \phi(x-y)(\partial_z^m u^n(y) - \partial_z^m u^n(x)) \Big) \rho(y) \nabla^k (\partial_z^m u^{n+1})(x) dy dx \\
&\quad + \int_{\mathbb{T}^{2d}} \nabla^k \Big( \phi(x-y)(u(y) - u(x)) \Big) \partial_z^m \rho^{n+1}(y) \nabla^k (\partial_z^m u^{n+1})(x) dy dx \\
&=: \sum_{i=1}^{6} \mathcal{I}_{7i}.
\end{aligned}
$$

In the sequel, we estimate the terms $\mathcal{I}_{7i}$'s separately.

◇ (Estimates for $\mathcal{I}_{71}$ and $\mathcal{I}_{72}$) : For $\mathcal{I}_{71}$,

$$
\begin{aligned}
\mathcal{I}_{71} &= -\int_{\mathbb{T}^d} \nabla^k (\partial_z^m u^n) \cdot \nabla u \cdot \nabla^k (\partial_z^m u^{n+1}) dx \\
&\quad - \int_{\mathbb{T}^d} \Big[ \nabla^k (\partial_z^m u^n \cdot \nabla u) - \partial_z^m u^n \cdot \nabla^k (\nabla u) \Big] \nabla^k (\partial_z^m u^{n+1}) dx \\
&\le \|\nabla u\|_{L^\infty} \|\nabla^k (\partial_z^m u^n)\|_{L^2} \|\nabla^k (\partial_z^m u^{n+1})\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ c\begin{pmatrix} \|\nabla(\partial_z^m u^n)\|_{L^\infty}\|\nabla^k u\|_{L^2} \\ +\|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z^m u^n)\|_{L^2} \end{pmatrix}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}),
\end{aligned}
$$

where $c$ and $C$ are positive random functions independent of $n$. Similarly,

$$
\begin{aligned}
\mathcal{I}_{72} &= \frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\nabla^k(\partial_z^m u^{n+1})|^2dx \\
&\quad - \int_{\mathbb{T}^d}\Big[\nabla^k(u\cdot\nabla(\partial_z^m u^{n+1})) - u\cdot\nabla^k(\nabla(\partial_z^m u^{n+1}))\Big]\cdot\nabla^k(\partial_z^m u^{n+1})dx \\
&\le \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 \\
&\quad + c\begin{pmatrix} \|\nabla u\|_{L^\infty}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\ +\|\nabla(\partial_z^m u^{n+1})\|_{L^\infty}\|\nabla^k u\|_{L^2} \end{pmatrix}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\partial_z^m u^{n+1}\|_{H^{s-m+1}}^2 + \varepsilon^{3/2}),
\end{aligned}
$$

where $c$ and $C$ are positive random functions independent of $n$.

◇ (Estimates for $\mathcal{I}_{73}$): One gets

$$
\begin{aligned}
\mathcal{I}_{73} &= -\sum_{1\le l\le m-1}\binom{m}{l}\Bigg\{\int_{\mathbb{T}^d}\partial_z^l u\cdot\nabla^k(\nabla(\partial_z^{m-l}u))\cdot\nabla^k(\partial_z^m u^{n+1})dx \\
&\qquad\qquad + \int_{\mathbb{T}^d}\begin{bmatrix} \nabla^k(\partial_z^l u\cdot\nabla(\partial_z^{m-l}u)) \\ -\partial_z^l u\cdot\nabla^k(\nabla(\partial_z^{m-l}u)) \end{bmatrix}\cdot\nabla^k(\partial_z^m u^{n+1})dx\Bigg\} \\
&\le C\sum_{1\le l\le m-1}\begin{pmatrix} \|\partial_z^l u\|_{L^\infty}\|\nabla^{k+1}(\partial_z^{m-l}u)\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\ +\|\nabla(\partial_z^l u)\|_{L^\infty}\|\nabla^k(\partial_z^{m-l}u)\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\ +\|\nabla(\partial_z^{m-l}u)\|_{L^\infty}\|\nabla^k(\partial_z^l u)\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \end{pmatrix} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}
$$

◇ (Estimates for $\mathcal{I}_{74}$): By direct calculation,

$$
\mathcal{I}_{74} = \sum_{\substack{\alpha+\beta+\gamma=m\\ \beta,\gamma\ne m}}\frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}}\begin{pmatrix} \nabla^k\Big(\partial_z^\alpha\phi(x-y)\Big)(\partial_z^\beta u(y)-\partial_z^\beta u(x)) \\ \cdot\,\partial_z^\gamma\rho(y)\nabla^k(\partial_z^m u^{n+1})(x) \end{pmatrix}dydx
$$

$$
\begin{aligned}
&- \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma\neq m \\ 0\le r\le k-1}} \frac{m!}{\alpha!\beta!\gamma!}\binom{k}{r}\int_{\mathbb{T}^{2d}} \left(\begin{array}{l} \nabla^r\big(\partial_z^\alpha\phi(x-y)\big)\nabla^{k-r}\big(\partial_z^\beta u(x)\big) \\ \cdot\, \partial_z^\gamma\rho(y)\nabla^k(\partial_z^m u^{n+1})(x) \end{array}\right) dydx \\
&\le \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma\neq m}} \frac{m!}{\alpha!\beta!\gamma!} 2\|\phi\|_s\|\partial_z^\gamma\rho\|_{L^2}\|\partial_z^\beta u\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta,\gamma\neq m \\ 0\le r\le k-1}} \frac{m!}{\alpha!\beta!\gamma!}\binom{k}{r}\|\phi\|_s\|\partial_z^\gamma\rho\|_{L^2}\|\nabla^{k-r}(\partial_z^\beta u)\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}
$$

◇ (Estimates for $\mathcal{I}_{75}$): In this case, we get

$$
\begin{aligned}
\mathcal{I}_{75} &= \int_{\mathbb{T}^{2d}} \nabla^k\phi(x-y)(\partial_z^m u^n(y) - \partial_z^m u^n(x))\rho(y)\nabla^k\big(\partial_z^m u^{n+1}\big)(x)dydx \\
&\quad - \sum_{r=0}^{k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \left(\begin{array}{l} \nabla^r\phi(x-y)\nabla^{k-r}\big(\partial_z^m u^n\big)(x) \\ \cdot\, \rho(y)\nabla^k(\partial_z^m u^{n+1})(x) \end{array}\right) dydx \\
&\le 2\|\phi\|_s\|\rho\|_{L^2}\|\partial_z^m u^n\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\quad + \sum_{r=0}^{k-1}\binom{k}{r}\|\phi\|_s\|\rho\|_{L^2}\|\nabla^{k-r}(\partial_z^m u^n)\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\le C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).
\end{aligned}
$$

◇ (Estimates for $\mathcal{I}_{76}$) : One has

$$
\begin{aligned}
\mathcal{I}_{76} &= \int_{\mathbb{T}^{2d}} \nabla^k\phi(x-y)(u(y) - u(x))\partial_z^m\rho^{n+1}(y)\nabla^k(\partial_z^m u^{n+1})(x)dydx \\
&\quad - \sum_{r=0}^{k-1}\binom{k}{r}\int_{\mathbb{T}^{2d}} \left(\begin{array}{l} \nabla^r\phi(x-y)\nabla^{k-r}u(x) \\ \cdot\, \partial_z^m\rho^{n+1}(y)\nabla^k(\partial_z^m u^{n+1})(x) \end{array}\right) dydx \\
&\le 2\|\phi\|_s\|u\|_{L^2}\|\partial_z^m\rho^{n+1}\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2} \\
&\quad + \sum_{r=0}^{k-1}\binom{k}{r}\|\phi\|_s\|\nabla^{k-r}u\|_{L^2}\|\partial_z^m\rho\|_{L^2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}
\end{aligned}
$$

$$\leq C(\varepsilon^{1/2}\|\nabla^k(\partial_z^m u^{n+1})\|_{L^2}^2 + \varepsilon^{3/2}).$$

Now, we gather all results for $\mathcal{I}_{7i}$'s, sum over $1 \leq k \leq s-m+1$ and combine with the zeroth-order estimate to yield that for each $z \in \Omega$,

$$\frac{\partial}{\partial t}\|\partial_z^m u^{n+1}\|_{H^{s-m+1}}^2 \leq C(\varepsilon^{1/2}\|\partial_z^m u^{n+1}\|_{H^{s-m+1}}^2 + \varepsilon^{3/2}).$$

Thus, we integrate the above relation over $[0,t]$ and use Grönwall's lemma to get the desired result.

## A.3 Proof of Lemma 3.2.4

We consider only higher-order estimates. For $1 \leq k \leq m-l+1$ and $1 \leq l \leq m$, we apply $\nabla^k\partial_z^l$ to (3.0.1) to get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|\nabla^k(\partial_z^l(u-\bar{u}))\|_{L^2}^2 \\
&= -\sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\nabla^{r_2}(\partial_z^{r_1}(u-\bar{u}))\cdot\nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}u))\cdot\nabla^k(\partial_z^l(u-\bar{u}))dx \\
&\quad -\sum_{\substack{0\leq r_1\leq l\\0\leq r_2\leq k}}\binom{l}{r_1}\binom{k}{r_2}\int_{\mathbb{T}^d}\nabla^{r_2}(\partial_z^{r_1}\bar{u})\cdot\nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}(u-\bar{u}))\cdot\nabla^k(\partial_z^l(u-\bar{u}))dx \\
&\quad +\sum_{\alpha+\beta+\gamma=l}\frac{l!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}}\Bigg[\left\{\begin{array}{l}\nabla^k\Big(\partial_z^\alpha\phi\;\partial_z^\beta((u-\bar{u})(y)-(u-\bar{u})(x))\Big)\\ \quad\cdot\;\partial_z^\gamma\rho(y)\nabla^k(\partial_z^l(u-\bar{u}))(x)\end{array}\right\} \\
&\qquad\qquad +\left\{\begin{array}{l}\nabla^k\Big(\partial_z^\alpha\phi\;\partial_z^\beta(\bar{u}(y)-\bar{u}(x))\Big)\\ \quad\cdot\;\partial_z^\gamma(\rho-\bar{\rho})(y)\nabla^k(\partial_z^l(u-\bar{u}))(x)\end{array}\right\}\Bigg]dydx \\
&=: \sum_{i=1}^4\mathcal{I}_{8i}.
\end{aligned}$$

In the sequel, we estimate the terms $\mathcal{I}_{8i}$'s one by one.

$\diamond$ (Estimates for $\mathcal{I}_{81}$) : One has

$$\begin{aligned}
\mathcal{I}_{81} &\le \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}u))\|_{L^\infty} \left( \begin{array}{c} \|\nabla^{r_2}(\partial_z^{r_1}(u-\bar{u}))\|_{L^2} \\ \cdot \|\nabla^k(\partial_z^l(u-\bar{u})\|_{L^2} \end{array} \right) \\
&\le C \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k}} \|\partial_z^{l-r_1}u\|_{H^{s-l+1}} \|\nabla^{r_2}(\partial_z^{r_1}(u-\bar{u}))\|_{L^2} \|\nabla^k(\partial_z^l(u-\bar{u})\|_{L^2} \\
&\le C \sum_{r=0}^{l} \|\partial_z^r(u-\bar{u})\|_{H^{m-r+1}}^2.
\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{82}$) : Similarly, we have

$$\begin{aligned}
\mathcal{I}_{82} &= -\sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \int_{\mathbb{T}^d} \nabla^{r_2}(\partial_z^{r_1}\bar{u}) \cdot \left( \begin{array}{c} \nabla(\nabla^{k-r_2}(\partial_z^{l-r_1}(u-\bar{u}))) \\ \cdot\nabla^k(\partial_z^l(u-\bar{u})) \end{array} \right) dx \\
&\quad + \frac{1}{2}\int_{\mathbb{T}^d} (\nabla\cdot\bar{u})|\nabla^k(\partial_z^l(u-\bar{u}))|^2 dx \\
&\le \sum_{\substack{0\le r_1\le l\\ 0\le r_2\le k\\ (r_1,r_2)\neq(l,k)}} \binom{l}{r_1}\binom{k}{r_2} \|\nabla^{k-r_2}(\partial_z^{l-r_1}\bar{u})\|_{L^\infty} \left( \begin{array}{c} \|\nabla^{r_2+1}(\partial_z^{r_1}(u-\bar{u}))\|_{L^2} \\ \cdot\|\nabla^k(\partial_z^l(u-\bar{u})\|_{L^2} \end{array} \right) \\
&\quad + \frac{\|\nabla\cdot\bar{u}\|_{L^\infty}}{2} \|\nabla^k(\partial_z^l(u-\bar{u}))\|_{L^2}^2 \\
&\le C \sum_{r=0}^{l} \|\partial_z^r(u-\bar{u})\|_{H^{m-r+1}}^2.
\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{83}$ and $\mathcal{I}_{84}$) : One gets

$$\begin{aligned}
\mathcal{I}_{83} &\le C \sum_{\substack{\alpha+\beta+\gamma=l\\ 0\le r\le k}} \|\nabla^{k-r}(\partial_z^\beta(u-\bar{u}))\|_{L^2} \|\partial_z^\gamma\rho\|_{L^2} \|\nabla^k(\partial_z^l(u-\bar{u}))\|_{L^2} \\
&\le C \sum_{0\le r\le l} \|\partial_z^r(u-\bar{u})\|_{H^{m-r+1}}^2,
\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{84} &\le C \sum_{\substack{\alpha+\beta+\gamma=l\\ 0\le r\le k}} \|\nabla^{k-r}(\partial_z^\beta \bar{u})\|_{L^2}\|\partial_z^\gamma(\rho-\bar{\rho})\|_{L^2}\|\nabla^k(\partial_z^l(u-\bar{u}))\|_{L^2}\\ &\le C\left(\|\nabla^k(\partial_z^l(u-\bar{u}))\|_{L^2}^2 + \sum_{0\le r\le l}\|\partial_z^r(\rho-\bar{\rho})\|_{L^2}^2\right).\end{aligned}$$

Now, we combine all the estimates for $\mathcal{I}_{8i}$, sum over $1 \le k \le m-l+1$, $0 \le l \le m$ and add the zeroth-order estimate to get the desired result.

## A.4 Proof of Theorem 3.3.2

It follows from (3.1.2) that

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}^d}|\partial_z^m u|^2 dx\\ &= -\int_{\mathbb{T}^d}(\partial_z^m u\cdot\nabla u)\cdot\partial_z^m u dx - \int_{\mathbb{T}^d}(u\cdot\nabla(\partial_z^m u))\cdot\partial_z^m u dx\\ &\quad -\sum_{l=1}^{m-1}\binom{m}{l}\int_{\mathbb{T}^d}(\partial_z^l u\cdot\nabla(\partial_z^{m-l}u))\cdot\partial_z^m u dx\\ &\quad +\sum_{\substack{\alpha+\beta+\gamma=m\\ \beta\ne m}}\frac{m!}{\alpha!\beta!\gamma!}\int_{\mathbb{T}^{2d}}\partial_z^\alpha\phi(x-y)(\partial_z^\beta u(y)-\partial_z^\beta u(x))\partial_z^\gamma\rho(y)\cdot\partial_z^m u(x)dydx\\ &\quad +\int_{\mathbb{T}^{2d}}\phi(x-y)(\partial_z^m u(y)-d\partial_z^m u(x))\rho(y)\cdot\partial_z^m u(x)dydx\\ &=: \sum_{i=1}^{5}\mathcal{I}_{9i}.\end{aligned}$$

Next, we estimate each $\mathcal{I}_{9i}$ separately.

⋄ (Estimates for $\mathcal{I}_{91}$ and $\mathcal{I}_{92}$) : One gets

$$\begin{aligned}\mathcal{I}_{91} &= \frac{1}{2}\int_{\mathbb{T}^d}(\nabla\cdot u)|\partial_z^m u|^2 dx \le \frac{\|\nabla\cdot u\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^d)}}{2}\|\partial_z^m u\|_{L^2}^2,\\ \mathcal{I}_{92} &\le \|\nabla(\partial_z^m u)\|_{L^\infty}\|u\|_{L^2}\|\partial_z^m u\|_{L^2} \le \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{\mathcal{U}(z)^2}{\delta}\|u\|_{L^2}^2\end{aligned}$$

$$\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{\mathcal{U}(z)^2}{\delta}\mathcal{F}_0(z)e^{-2\tilde{\Lambda}(z)t}.$$

$\diamond$ (Estimates for $\mathcal{I}_{93}$) : Note that this term does not appear when $m = 1$. If $m \geq 2$,

$$\begin{aligned}
\mathcal{I}_{93} &\leq \sum_{l=1}^{m-1}\binom{m}{l}\|\nabla(\partial_z^{m-l}u)\|_{L^\infty}\|\partial_z^l u\|_{L^2}\|\partial_z^m u\|_{L^2} \\
&\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2\mathcal{U}^2(z)\|\partial_z^l u\|_{L^2}^2 \\
&\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2\mathcal{U}^2(z)\mathcal{F}_l(z)e^{-\tilde{\Lambda}(z)t}.
\end{aligned}$$

$\diamond$ (Estimates for $\mathcal{I}_{94}$ and $\mathcal{I}_{95}$) : One has

$$\begin{aligned}
\mathcal{I}_{94} &\leq \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}}\frac{m!}{\alpha!\beta!\gamma!}\|\phi\|_s\left(\|\partial_z^\beta u\ \partial_z^\gamma\rho\|_{L^1}\|\partial_z^m u\|_{L^1} + \|\partial_z^\gamma\rho\|_{L^1}\|\partial_z^\beta u\cdot\partial_z^m u\|_{L^1}\right) \\
&\leq 2\sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}}\frac{m!}{\alpha!\beta!\gamma!}\|\phi\|_s\|\partial_z^\gamma\rho\|_{L^2}\|\partial_z^\beta u\|_{L^2}\|\partial_z^m u\|_{L^2} \\
&\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + 4\sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}}\left(\|\phi\|_s\frac{m!}{\alpha!\beta!\gamma!}\mathcal{U}(z)\right)^2\frac{\frac{(m+1)(m+2)}{2}-1}{\delta}\|\partial_z^\beta u\|_{L^2}^2 \\
&\leq \frac{\delta}{4}\|\partial_z^m u\|_{L^2}^2 + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}}\left(\|\phi\|_s\frac{m!}{\alpha!\beta!\gamma!}\mathcal{U}(z)\right)^2\frac{2m^2+6m}{\delta}\mathcal{F}_\beta(z)e^{-\tilde{\Lambda}(z)t}, \\
\mathcal{I}_{95} &= -\int_{\mathbb{T}^d}\phi(x-y)|\partial_z^m u(x)|^2\rho(y)dydx \\
&\quad + \int_{\mathbb{T}^d}\phi(x-y)\partial_z^m u(y)\cdot\partial_z^m u(x)\rho(y)dydx \\
&\leq -\phi_m\|\rho_0\|_{L^1}\|\partial_z^m u\|_{L^2}^2 + \|\phi\|_s\|\rho\partial_z^m u\|_{L^1}\|\partial_z^m u\|_{L^1}
\end{aligned}$$

$$\begin{aligned}
&\leq -\left(\phi_m\|\rho_0\|_{L^1} - \frac{\delta}{4}\right)\|\partial_z^m u\|_{L^2} + \frac{\|\phi\|_s^2\|\rho\|_{L^2}}{\delta}\mathcal{E}_m(t,z) \\
&\leq -\left(\phi_m\|\rho_0\|_{L^1} - \frac{\delta}{4}\right)\|\partial_z^m u\|_{L^2} + \frac{\|\phi\|_s^2\|\rho\|_{L^2}}{\delta}E_m(z)e^{-\tilde{\Lambda}(z)t}.
\end{aligned}$$

Finally, we gather the estimates for $\mathcal{I}_{9i}$ $(i = 1, \cdots, 5)$ to obtain

$$\frac{\partial}{\partial t}\|\partial_z^m u\|_{L^2}^2 \leq -2\tilde{\Lambda}(z)\|\partial_z^m u\|_{L^2}^2 + \hat{F}_m(z)e^{-\tilde{\Lambda}(z)t}, \tag{A.4.1}$$

where $\hat{F}_m(z)$ is given by, for $m \geq 2$,

$$\begin{aligned}
\hat{F}_1(z) &:= \frac{\mathcal{U}(z)^2}{\delta}\mathcal{F}_0(z) + \frac{16}{\delta}\mathcal{U}^2(z)\mathcal{F}_0(z) + \frac{\|\phi\|_s^2\mathcal{U}(z)}{\delta}E_1(z), \\
\hat{F}_m(z) &:= \frac{\mathcal{U}(z)^2}{\delta}\mathcal{F}_0(z) + \frac{m-1}{\delta}\sum_{l=1}^{m-1}\binom{m}{l}^2\mathcal{U}^2(z)\mathcal{F}_l(z) \\
&\quad + \sum_{\substack{\alpha+\beta+\gamma=m \\ \beta\neq m}}\left(\|\phi\|_s\frac{m!}{\alpha!\beta!\gamma!}\mathcal{U}(z)\right)^2\frac{2m^2+6m}{\delta}\mathcal{F}_\beta + \frac{\|\phi\|_s^2\mathcal{U}(z)}{\delta}E_1(z).
\end{aligned}$$

We apply Grönwall's lemma for (A.4.1) to yield

$$\begin{aligned}
\|\partial_z^m u\|_{L^2}^2 &\leq \|\partial_z^m u_0\|_{L^2}^2 e^{-2\tilde{\Lambda}(z)t} + \frac{\hat{F}_m(z)}{\tilde{\Lambda}(z)}(e^{-\tilde{\Lambda}(z)t} - e^{-2\tilde{\Lambda}(z)t}) \\
&\leq \mathcal{F}_m(z)e^{-\tilde{\Lambda}(z)t},
\end{aligned}$$

where $\mathcal{F}_m(z)$ is defined as

$$\mathcal{F}_m(z) := \|\partial_z^m u_0\|_{L^2}^2 + \frac{\hat{F}_m(z)}{\tilde{\Lambda}(z)}.$$

This implies the desired estimate.

# Appendix B

# Detailed proof of Chapter 5

## B.1 A proof of Theorem 5.2.1

First, we define $\eta$ and $\tilde{\eta}$ as follows:

$$\eta_t(x,v) := f^{in}(x,v) - \int_0^t \eta_s(x,v)(\nabla_v \cdot F_a[f_s])(\varphi_s)ds + \sigma d\int_0^t \eta_s(x,v) \circ dW_s,$$
$$\tilde{\eta}_t(x,v) := \eta_t((\varphi_t)^{-1}).$$

We use the generalized Itô's formula from Theorem 3.3.2 in [69] to obtain that $\tilde{\eta}_t$ satisfies the relation (5.0.1). Since the classical solutions can become measure-valued solutions and the uniqueness of measure-valued solutions is guaranteed in Theorem 5.1.1, we have

$$\tilde{\eta}_t(x,v) = f_t(x,v).$$

Moreover, since $\eta$ is a geometric Brownian motion, a unique classical solution $f$ corresponding to the initial datum $f^{in}$ can be represented by

$$\eta_t(x,v) = f_t(\varphi_t(x,v)) = f^{in}(x,v)\exp\left[-\int_0^t \nabla_v \cdot F_a[f_s](\varphi_s)ds + d\sigma W_t\right].$$

Now, we consider another classical solution $\tilde{f}$ corresponding to the initial datum $\tilde{f}^{in}$ and the associated stochastic flow $\tilde{\varphi}_t(x,v)$. Moreover, we set

$$\mathcal{R}(t) := \sup\left\{|x| \ : \ f_t(x,v) \neq 0 \quad \text{or} \quad \tilde{f}_t(x,v) \neq 0 \quad \text{for some} \ \ v \in \mathbb{R}^d\right\},$$

$$\mathcal{P}(t) := \sup\Big\{|v| \ : \ f_t(x,v) \neq 0 \quad \text{or} \quad \tilde{f}_t(x,v) \neq 0 \quad \text{for some} \ \ x \in \mathbb{R}^d\Big\}.$$

Then, we claim

$$\begin{aligned}
&(i)\ \|f_t - \tilde{f}_t\|^2_{L^\infty} \leq \mathcal{B}^1_t \left[ \begin{array}{l} \|f^{in} - \tilde{f}^{in}\|^2_{L^\infty} \\ + \displaystyle\int_0^t \mathcal{C}^1_s \Big( \|\varphi_s - \tilde{\varphi}_s\|^2_{L^\infty} + \|f_s - \tilde{f}_s\|^2_{L^\infty} \Big) ds \end{array} \right] \\
&\qquad\qquad + 2\max(\|f_t\|_{W^{1,\infty}}, \|\tilde{f}_t\|_{W^{1,\infty}}) \|\varphi_t - \tilde{\varphi}_t\|^2_{L^\infty}, \\
&(ii)\ \|\varphi_t - \tilde{\varphi}_t\|^2_{L^\infty} \leq \mathcal{B}^2_t \left( \int_0^t \mathcal{C}^2_s (\|\varphi_s - \tilde{\varphi}_s\|^2_{L^\infty} + \|f_s - \tilde{f}_s\|^2_{L^\infty}) ds \right),
\end{aligned} \tag{B.1.1}$$

where $\mathcal{B}^i_t$ and $\mathcal{C}^i_t$ $(i = 1, 2)$ are nonnegative processes which have continuous sample paths.

(i) First, we derive the $L^\infty$-estimates for classical solutions:

$$\begin{aligned}
& f(\varphi_t(x,v)) - \tilde{f}_t(\varphi(x,v)) \\
&\quad = \Big( f(\varphi_t(x,v)) - \tilde{f}(\tilde{\varphi}_t(x,v)) \Big) + \Big( \tilde{f}(\tilde{\varphi}_t(x,v)) - \tilde{f}(\varphi_t(x,v)) \Big) \\
&\quad =: J_{21} + J_{22}.
\end{aligned}$$

• (Estimate of $J_{21}$): By direct estimate, one has

$$\begin{aligned}
\mathcal{I}_{21} &= f^{in}(x,v) \exp\left[ -\int_0^t \nabla_v \cdot F_a[f_s](\varphi_s) ds + d\sigma W_t \right] \\
&\quad - \tilde{f}^{in}(x,v) \exp\left[ -\int_0^t \nabla_v \cdot F_a[\tilde{f}_s](\varphi_s) ds + d\sigma W_t \right] \\
&\leq \|f^{in} - \tilde{f}^{in}\|_{L^\infty} \exp\left[ -\int_0^t \nabla_v \cdot F_a[f_s](\varphi_s) ds + d\sigma W_t \right] \\
&\quad + \|\tilde{f}^{in}\|_{L^\infty} \exp(d\sigma W_t) \left[ \begin{array}{l} \exp\Big( -\displaystyle\int_0^t \nabla_v \cdot F_a[f_s](\varphi_s) ds \Big) \\ -\exp\Big( -\displaystyle\int_0^t \nabla_v \cdot F_a[\tilde{f}_s](\tilde{\varphi}_s) ds \Big) \end{array} \right] \\
&\leq \|f^{in} - \tilde{f}^{in}\|_{L^\infty} \exp\left( d\kappa t + d\sigma W_t \right) \\
&\quad + \|\tilde{f}^{in}\|_{L^\infty} \exp(d\kappa t + d\sigma W_t) \left| \int_0^t \Big( \nabla_v \cdot F_a[f_s](\varphi_s) - \nabla_v \cdot F_a[\tilde{f}_s](\tilde{\varphi}_s) \Big) ds \right|,
\end{aligned}$$

where we used the mean-value theorem, and we have

$$\begin{aligned}
&\left|\nabla_v \cdot F_a[f_s](\varphi_s) - \nabla_v \cdot F_a[\tilde{f}_s](\tilde{\varphi}_s)\right| \\
&\quad \le d \int_{\mathbb{R}^{2d}} \left|\phi(x_* - X_s) - \phi(x_* - \tilde{X}_s)\right| f_s dv_* dx_* \\
&\qquad + d \int_{\mathbb{R}^{2d}} \phi(x_* - \tilde{X}_t)|f_s - \tilde{f}_s| dv_* dx_* \\
&\quad \le d\phi_{Lip}|X_s - \tilde{X}_s| + d\kappa(4\mathcal{R}(s)\mathcal{P}(s))^d \|f_s - \tilde{f}_s\|_{L^\infty}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
J_{21} &\le \|f^{in} - \tilde{f}^{in}\|_{L^\infty} \exp\left(d\kappa t + d\sigma W_t\right) \\
&\quad + \|\tilde{f}^{in}\|_{L^\infty} \exp(d\kappa t + d\sigma W_t) \int_0^t d\phi_{Lip}|X_s - \tilde{X}_s| ds \\
&\quad + \|\tilde{f}^{in}\|_{L^\infty} \exp(d\kappa t + d\sigma W_t) \int_0^t d\kappa (4\mathcal{R}(s)\mathcal{P}(s))^d \|f_s - \tilde{f}_s\|_{L^\infty} ds.
\end{aligned}$$

• (Estimate of $J_{22}$): By direct estimate, one has

$$J_{22} \le \|\tilde{f}_t\|_{W^{1,\infty}} \|\varphi_t - \tilde{\varphi}_t\|_{L^\infty}.$$

Hence, we take the supremum over $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, and use Young's inequality and the Cauchy-Schwarz inequality to get

$$\begin{aligned}
&\|f_t - \tilde{f}_t\|_{L^\infty} \\
&\le 2J_{21}^2 + 2J_{22}^2 \\
&\le 6\|f^{in} - \tilde{f}^{in}\|_{L^\infty}^2 \exp\left(2d\kappa t + 2d\sigma W_t\right) \\
&\quad + 6\|\tilde{f}^{in}\|_{L^\infty}^2 \exp\left(2d\kappa t + 2d\sigma W_t\right) \left(\int_0^t d\phi_{Lip}|X_s - \tilde{X}_s| ds\right)^2 \\
&\quad + 6\|\tilde{f}^{in}\|_{L^\infty}^2 \exp\left(2d\kappa t + 2d\sigma W_t\right) \left(\int_0^t d\kappa (4\mathcal{R}(s)\mathcal{P}(s))^d \|f_s - \tilde{f}_s\|_{L^\infty} ds\right)^2 \\
&\quad + 2\|\tilde{f}_t\|_{W^{1,\infty}}^2 \|\varphi_t - \tilde{\varphi}_t\|_{L^\infty}^2 \\
&\le 6\|f^{in} - \tilde{f}^{in}\|_{L^\infty}^2 \exp\left(2d\kappa t + 2d\sigma W_t\right) \\
&\quad + 6t\left(d\phi_{Lip}\|\tilde{f}^{in}\|_{L^\infty} \exp\left(d\kappa t + d\sigma W_t\right)\right)^2 \int_0^t \|\varphi_s - \tilde{\varphi}_s\|_{L^\infty}^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+ 6t \left(\|\tilde{f}^{in}\|_{L^\infty} \exp\left(d\kappa t + d\sigma W_t\right)\right)^2 \int_0^t d\kappa (4\mathcal{R}(s)\mathcal{P}(s))^{2d} \|f_s - \tilde{f}_s\|_{L^\infty}^2 ds \\
&+ 2\|\tilde{f}_t\|_{W^{1,\infty}}^2 \|\varphi_t - \tilde{\varphi}_t\|_{L^\infty}^2.
\end{aligned}
$$

Setting

$$
\begin{aligned}
&\mathcal{B}_t^1 := 6\left[1 + t\left(d\|\phi\|_{W^{1,\infty}} \max(\|f^{in}\|_{L^\infty}, \|\tilde{f}^{in}\|_{L^\infty}) \exp(d\kappa t + d\sigma W_t)\right)^2\right], \\
&\mathcal{C}_t^1 := (1 + (4\mathcal{R}(t)\mathcal{P}(t))^{2d}),
\end{aligned}
$$

we obtain the desired result (i) of (B.1.1).

(ii) Now, we estimate $\|\varphi_t - \tilde{\varphi}_t\|_{L^\infty}$. It follows from (5.2.6) and Itô's lemma that

$$
\begin{aligned}
d|V_t - \tilde{V}_t|^2 &= 2(V_t - \tilde{V}_t) d(V_t - \tilde{V}_t) + d(V_t - \tilde{V}_t) d(V_t - \tilde{V}_t) \\
&= 2\left(\underbrace{(V_t - \tilde{V}_t)(F_a[f_t](\varphi_t) - F_a[\tilde{f}_t](\tilde{\varphi}_t))}_{=:J_{23}.} + \sigma^2 |V_t - \tilde{V}_t|^2\right) dt \\
&\quad - 2\sigma |V_t - \tilde{V}_t|^2 dW_t.
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
J_{23} \le &\int_{\mathbb{R}^{2d}} \left|\phi(x_* - X_t) - \phi(x_* - \tilde{X}_t)\right| |(v_* - V_t) \cdot (V_t - \tilde{V}_t)| f_t dv_* dx_* \\
&- \int_{\mathbb{R}^{2d}} \phi(x_* - \tilde{X}_t) |V_t - \tilde{V}_t|^2 f_t dv_* dx_* \\
&+ \int_{\mathbb{R}^{2d}} \phi(x_* - \tilde{X}_t)) |(v_* - \tilde{V}_t) \cdot (V_t - \tilde{V}_t)| |f_t - \tilde{f}_t| dv_* dx_* \\
&=: J_{231} + J_{232} + J_{233}.
\end{aligned}
$$

We separately estimate the $J_{23i}$'s as follows:

$$
J_{231} \le 2\phi_{Lip} \mathcal{P}(t) |X_t - \tilde{X}_t| |V_t - \tilde{V}_t| \le 2\phi_{Lip} \mathcal{P}(t) \|\varphi_t - \tilde{\varphi}_t\|_{L^\infty}^2, \quad J_{232} \le 0,
$$

$$
\begin{aligned}
J_{233} &\le 2\kappa\mathcal{P}(t)|V_t-\tilde{V}_t|\|f_t-\tilde{f}_t\|_{L^\infty}(4\mathcal{R}(t)\mathcal{P}(t))^d \\
&\le \kappa\mathcal{P}(t)(4\mathcal{R}(t)\mathcal{P}(t))^d\Big(\|f_t-\tilde{f}_t\|_{L^\infty}^2+\|\varphi_t-\tilde{\varphi}_t\|_{L^\infty}\Big)^2.
\end{aligned}
$$

Thus, by Lemma 5.1.1 we get

$$
\begin{aligned}
|V_t-\tilde{V}_t|^2 \le \kappa\int_0^t \mathcal{P}(s)(4\mathcal{R}(s)\mathcal{P}(s))^d\|f_s-\tilde{f}_s\|_{L^\infty}^2\exp(-2\sigma(W_t-W_s))ds& \\
+(2\phi_{Lip}+\kappa)\int_0^t\begin{bmatrix}\mathcal{P}(s)(4\mathcal{R}(s)\mathcal{P}(s))^d\|\varphi_s-\tilde{\varphi}_s\|_{L^\infty}^2\\ \times\exp(-2\sigma(W_t-W_s))\end{bmatrix}ds& \\
\le 2\|\phi\|_{W^{1,\infty}}\exp\left(4\sigma\sup_{0\le s\le t}|W_s|\right)& \\
\times\int_0^t\mathcal{P}(s)(4\mathcal{R}(s)\mathcal{P}(s))^d(\|\varphi_s-\tilde{\varphi}_s\|_{L^\infty}^2+\|f_s-\tilde{f}_s\|_{L^\infty}^2)ds.&
\end{aligned}
$$

Moreover, it is easy to obtain that

$$
d|X_t-\tilde{X}_t|^2=2(X_t-\tilde{X}_t)\cdot(V_t-\tilde{V}_t)\le 2\|\varphi_t-\tilde{\varphi}_t\|_{L^\infty}^2.
$$

Thus, if we define $\mathcal{B}_t^2$ and $\mathcal{C}_t^2$ as

$$
\mathcal{B}_t^2:=1+2\|\phi\|_{W^{1,\infty}}\exp\left(4\sigma\sup_{0\le s\le t}|W_s|\right),\quad \mathcal{C}_t^2:=1+\mathcal{P}(t)(4\mathcal{R}(t)\mathcal{P}(t))^d,
$$

then (ii) of (B.1.1) can be fulfilled with the above $\mathcal{B}_t^2$ and $\mathcal{C}_t^2$.

Therefore, we add $(i)$ in $(\text{B.1.1})_1$ to $(1+2\max(\|f_t\|_{W^{1,\infty}},\|\tilde{f}_t\|_{W^{1,\infty}}))$ times $(ii)$ in $(\text{B.1.1})_2$ and obtain

$$
\begin{aligned}
&\|f_t-\tilde{f}_t\|_{L^\infty}^2+\|\varphi_t-\tilde{\varphi}_t\|_{L^\infty}^2 \\
&\quad\le \mathcal{B}_t^1\Big[\|f^{in}-\tilde{f}^{in}\|_{L^\infty}^2+\int_0^t\mathcal{C}_s^1\Big(\|\varphi_s-\tilde{\varphi}_s\|_{L^\infty}^2+\|f_s-\tilde{f}_s\|_{L^\infty}^2\Big)ds\Big] \\
&\qquad+(1+2\max(\|f_t\|_{W^{1,\infty}},\|\tilde{f}_t\|_{W^{1,\infty}})\mathcal{B}_t^2\Big[\int_0^t\mathcal{C}_s^2(\|\varphi_s-\tilde{\varphi}_s\|_{L^\infty}^2+\|f_s-\tilde{f}_s\|_{L^\infty}^2)ds\Big] \\
&\quad\le \mathcal{B}_t^1\|f^{in}-\tilde{f}^{in}\|_{L^\infty}^2+\tilde{\mathcal{B}}_t\int_0^t\Big(\|\varphi_s-\tilde{\varphi}_s\|_{L^\infty}^2+\|f_s-\tilde{f}_s\|_{L^\infty}^2\Big)ds,
\end{aligned}
$$

where $\tilde{\mathcal{B}}_t$ is given by

$$\tilde{\mathcal{B}}_t := \mathcal{B}_t^1 \left( \sup_{0\le s\le t} \mathcal{C}_s^1 \right) + \Big(1+2\max(\|f_t\|_{W^{1,\infty}}, \|\tilde{f}_t\|_{W^{1,\infty}})\Big) \mathcal{B}_t^2 \left( \sup_{0\le s\le t} \mathcal{C}_s^2 \right).$$

Then, letting $y_t := \int_0^t \Big( \|\varphi_s - \tilde{\varphi}_s\|_{L^\infty}^2 + \|f_s - \tilde{f}_s\|_{L^\infty}^2 \Big) ds$, we have

$$dy_t \le \Big( \mathcal{B}_t^1 \|f^{in} - \tilde{f}^{in}\|_{L^\infty}^2 + \tilde{\mathcal{B}}_t y_t \Big) dt.$$

Then, by Grönwall's lemma we get

$$y_t \le \|f^{in} - \tilde{f}^{in}\|_{L^\infty}^2 \int_0^t \mathcal{B}_s^1 \exp\left( \int_s^t \tilde{\mathcal{B}}_\tau d\tau \right) ds,$$

and this gives

$$\begin{aligned} &\|f_t - \tilde{f}_t\|_{L^\infty}^2 + \|\varphi_t - \tilde{\varphi}_t\|_{L^\infty}^2 \\ &\quad \le \|f^{in} - \tilde{f}^{in}\|_{L^\infty}^2 \left[ \mathcal{B}_t^1 + \tilde{\mathcal{B}}_t \int_0^t \mathcal{B}_s^1 \exp\left( \int_s^t \tilde{\mathcal{B}}_\tau d\tau \right) ds \right]. \end{aligned}$$

Hence, defining

$$\mathcal{D}_t := \mathcal{B}_t^1 + \tilde{\mathcal{B}}_t \int_0^t \mathcal{B}_s^1 \exp\left( \int_s^t \tilde{\mathcal{B}}_\tau d\tau \right) ds,$$

we arrive at the desired estimate.

## B.2 A proof of Proposition 5.3.3

Recall that $f_t^{n,\varepsilon}$ satisfies a differential form:

$$\partial_t f_t^{n,\varepsilon} = -v \cdot \nabla_x f_t^{n,\varepsilon} - \nabla_v \cdot (F_a[f_t^{n-1,\varepsilon}] f_t^{n,\varepsilon}) + \sigma \nabla_v \cdot (v f_t^{n,\varepsilon}) \circ \dot{W}_t,$$

i.e., it satisfies

$$\begin{aligned} f_t^{n,\varepsilon} = f^{in,\varepsilon} &- \int_0^t v \cdot \nabla_x f_s^{n,\varepsilon} ds - \int_0^t \nabla_v \cdot (F_a[f_t^{n-1,\varepsilon}] f_t^{n,\varepsilon}) ds \\ &+ \sigma \int_0^t \nabla_v \cdot (v f_t^{n,\varepsilon}) \circ dW_s. \end{aligned} \tag{B.2.1}$$

Next, we claim: there exists a nonnegative process $\mathcal{A}_t^m$ with continuous sample paths such that

$$\|f_t\|_{W^{m,\infty}} \le \|f^{in}\|_{W^{m,\infty}} \mathcal{A}_t^m.$$

In the sequel, we provide $L^\infty$-estimate of $f_t$ and its derivatives to provide a proof of Proposition 5.3.3.

• (Zeroth-order estimate): It follows the formula (5.3.5) that

$$\begin{aligned} f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}(x,v)) &= f^{in,\varepsilon}(x,v) \exp\left[ -\int_0^t \nabla_v \cdot F_a[f_s^{n-1,\varepsilon}](\varphi_s^{n,\varepsilon}(x,v))ds + d\sigma W_t \right] \\ &\le \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\kappa t + d\sigma W_t). \end{aligned}$$

This implies the zeroth-order estiamte:

$$\|f_t^{n,\varepsilon}\|_{L^\infty} \le \|f^{in,\varepsilon}\|_{L^\infty} \exp(d\kappa t + d\sigma W_t). \tag{B.2.2}$$

• (Higher-order estimates): Let $\alpha$ and $\beta$ be multi-indices satisfying

$$1 \le |\alpha| + |\beta| \le m.$$

Then, we apply $\partial_x^\alpha \partial_v^\beta$ to the relation (B.2.1) using Theorem 3.1.2 in [69]:

$$\begin{aligned} \partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon} &= \partial_x^\alpha \partial_v^\beta f^{in,\varepsilon} \\ &\quad - \sum_{|\mu_1| \le 1} \binom{\beta}{\mu_1} \int_0^t \partial_v^{\mu_1}(v) \cdot \nabla_x (\partial_x^\alpha \partial_v^{\beta-\mu_1} f_s^{n,\varepsilon}) ds \\ &\quad - \sum_{\substack{\mu_2 \le \alpha \\ |\mu_3| \le 1}} \binom{\alpha}{\mu_2} \binom{\beta}{\mu_3} \int_0^t \nabla_v \cdot (\partial_x^{\mu_2} \partial_v^{\mu_3} F_a[f_s^{n-1,\varepsilon}] \partial_x^{\alpha-\mu_2} \partial_v^{\beta-\mu_3} f_s^{n,\varepsilon}) ds \\ &\quad + \sigma \sum_{|\mu_4| \le 1} \binom{\beta}{\mu_4} \int_0^t \nabla_v \cdot (\partial_v^{\mu_4}(v) \partial_x^\alpha \partial_v^{\beta-\mu_4} f_s^{n,\varepsilon}) \circ dW_s, \end{aligned} \tag{B.2.3}$$

where we used the relation:

$$\partial_v^{\mu_3} F_a[f_t^{n-1,\varepsilon}] = 0, \quad \text{for } |\mu_3| \ge 2.$$

Note that the differentiation equality (B.2.3) is only true outside a $\mathbb{P}$-zero set in $\Omega$ which depends on $(x,v)$, according to Theorem 3.1.2 in [69].

However, we can use the argument in Lemma 5.2.2 to obtain that the equality also holds $\mathbb{P}\otimes dx\otimes dv$-a.s. Now, we rearrange the previous relation to obtain

$$\begin{aligned}\partial_x^\alpha\partial_v^\beta f_t^{n,\varepsilon} &= \partial_x^\alpha\partial_v^\beta f^{in,\varepsilon}\\ &\quad-\int_0^t\Big[v\cdot\nabla_x(\partial_x^\alpha\partial_v^\beta f_s^{n,\varepsilon})+F_a[f_s^{n-1,\varepsilon}]\cdot\nabla_v(\partial_x^\alpha\partial_v^\beta f_s^{n,\varepsilon})\Big]ds\\ &\quad+\sigma\int_0^t v\cdot\nabla_v(\partial_x^\alpha\partial_v^\beta f_s^{n,\varepsilon})\circ dW_s\\ &\quad-\frac{d+|\beta|}{d}\int_0^t\nabla_v\cdot F_a[f_s^{n-1,\varepsilon}]\partial_x^\alpha\partial_v^\beta f_s^{n,\varepsilon}ds\\ &\quad+\sigma(d+|\beta|)\int_0^t\partial_x^\alpha\partial_v^\beta f_s^{n,\varepsilon}\circ dW_s-\int_0^t\mathcal{L}_{\alpha,\beta}(s)ds,\end{aligned}\tag{B.2.4}$$

for $\mathbb{P}\otimes dx\otimes dv$-a.s., where the process $\mathcal{L}_{\alpha,\beta}$ is given by the following relation:

$$\begin{aligned}\mathcal{L}_{\alpha,\beta} &:= \sum_{|\mu_1|=1}\binom{\beta}{\mu_1}\partial_v^{\mu_1}(v)\cdot\nabla_x(\partial_x^\alpha\partial_v^{\beta-\mu_1}f_s^{n,\varepsilon})\\ &\quad+\sum_{0\neq\mu_2\le\alpha}\binom{\alpha}{\mu_2}\nabla_v\cdot(\partial_x^{\mu_2}F_a[f_s^{n-1,\varepsilon}])\partial_x^{\alpha-\mu_2}\partial_v^\beta f_s^{n,\varepsilon}\\ &\quad+\sum_{\substack{0\neq\mu_2\le\alpha\\|\mu_3|=1}}\binom{\alpha}{\mu_2}\binom{\beta}{\mu_3}\partial_x^{\mu_2}\partial_v^{\mu_3}F_a[f_s^{n-1,\varepsilon}]\cdot\nabla_v(\partial_x^{\alpha-\mu_2}\partial_v^{\beta-\mu_3}f_s^{n,\varepsilon})\\ &\quad+\sum_{0\neq\mu_2\le\alpha}\binom{\alpha}{\mu_2}\partial_x^{\mu_2}F_a[f_s^{n-1,\varepsilon}]\cdot\nabla_v(\partial_x^{\alpha-\mu_2}\partial_v^\beta f_s^{n,\varepsilon}).\end{aligned}$$

Next, we define $\lambda$ and $\tilde{\lambda}$ as follows:

$$\begin{aligned}\lambda_t(x,v) &:= \partial_x^\alpha\partial_v^\beta f^{in,\varepsilon}(x,v)-\frac{d+|\beta|}{d}\int_0^t\lambda_s(x,v)(\nabla_v\cdot F_a[f_s^{n-1,\varepsilon}])(\varphi_s^{n,\varepsilon})ds\\ &\qquad+\sigma(d+|\beta|)\int_0^t\lambda_s(x,v)\circ dW_s-\int_0^t\mathcal{L}_{\alpha,\beta}(\varphi_s^{n,\varepsilon})ds,\\ \tilde{\lambda}_t(x,v) &:= \lambda_t((\varphi_t^{n,\varepsilon})^{-1}).\end{aligned}$$

By using generalized Itô's formula from Theorem 3.3.2 in [69], $\tilde{\lambda}_t$ satisfies the relation (B.2.4). Thus, by the uniqueness,

$$\tilde{\lambda}_t = \partial_x^\alpha\partial_v^\beta f_t^{n,\varepsilon},$$

and we use Itô's formula on $\lambda_t$ to get

$$
\begin{aligned}
&\partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}) \\
&= \partial_x^\alpha \partial_v^\beta f^{in,\varepsilon}(x,v) \exp\Big[ -\frac{d+|\beta|}{d} \int_0^t \nabla_v \cdot F_a[f_s^{n-1,\varepsilon}](\varphi_s^{n,\varepsilon}) ds + \sigma(d+|\beta|) W_t \Big] \\
&\quad - \int_0^t \exp\left[ \begin{array}{c} -\frac{d+|\beta|}{d} \int_s^t \nabla_v \cdot F_a[f_\tau^{n-1,\varepsilon}](\varphi_\tau^{n,\varepsilon}) d\tau \\ +\sigma(d+|\beta|)(W_t - W_s) \end{array} \right] \mathcal{L}_{\alpha,\beta}(s, \varphi_s^{n,\varepsilon}) ds.
\end{aligned}
$$

For detailed explanation for the above realtion, we refer to the proof of Theorem 3.2 in [13].

Note that the following estimates hold:

- If $|\beta| = 1$, one has
$$|\partial_x^\alpha \partial_v^\beta F_a[f_t^{n-1,\varepsilon}]| \le \|\phi\|_{\mathcal{C}^m}.$$
- If $|\alpha| \ge 1$, one gets
$$
\begin{aligned}
|\partial_x^\alpha F_a[f_t^{n-1,\varepsilon}](\varphi_t^{n,\varepsilon})| &\le \|\phi\|_{\mathcal{C}^m} \int_{\mathbb{R}^{2d}} |v_* \cdot V_t^{n,\varepsilon}| f_t^{n-1,\varepsilon}(x_*, v_*) dv_* dx_* \\
&\le \|\phi\|_{\mathcal{C}^m} (\mathcal{V}^{n-1,\varepsilon}(t)) |V_t^{n,\varepsilon}| \le \|\phi\|_{\mathcal{C}^m} (\mathcal{V}^\infty(t))^2.
\end{aligned}
$$

We set $C_{\alpha,\beta}(t)$ to be

$$C_{\alpha,\beta}(t) := \|\phi\|_{\mathcal{C}^m} \left[ \sum_{|\mu_1|=1} \binom{\beta}{\mu_1} + \sum_{0 \ne \mu_2 \le \alpha} \binom{\alpha}{\mu_2} + \sum_{\substack{0 \le \mu_2 \le \alpha \\ |\mu_3|=1}} \binom{\alpha}{\mu_2} \binom{\beta}{\mu_3} \right] (1 + (\mathcal{V}^\infty(t))^2).$$

This yields

$$|\mathcal{L}_{\alpha,\beta}(t, \varphi_t^{n,\varepsilon})| \le C_{\alpha,\beta}(t) \|f_t^{n,\varepsilon}\|_{W^{m,\infty}}.$$

Thus, we have

$$
\begin{aligned}
&\partial_x^\alpha \partial_v^\beta f_t^{n,\varepsilon}(\varphi_t^{n,\varepsilon}) \\
&\le \|\partial_x^\alpha \partial_v^\beta f^{in,\varepsilon}\|_{L^\infty} \exp((d+|\beta|)(\kappa t + \sigma W_t)) \\
&\quad + \int_0^t \exp((d+|\beta|)\{\kappa(t-s) + \sigma(W_t - W_s)\} C_{\alpha,\beta}(s) \|f_s^{n,\varepsilon}\|_{W^{m,\infty}} ds.
\end{aligned}
\tag{B.2.5}
$$

Now, we take the supremum over all characteristic flow, sum (B.2.5) over all $1 \le |\alpha| + |\beta| \le m$ and combine this with (B.2.2) to obtain

$$\begin{aligned} \|f_t^{n,\varepsilon}\|_{W^{m,\infty}} &\le \|f^{in,\varepsilon}\|_{W^{m,\infty}} \mathcal{M}_t^m \\ &\quad + \mathcal{M}_t^m \int_0^t \left[ \begin{array}{c} \sum\limits_{|\alpha|+|\beta|\le m} \exp(-\sigma(d+|\beta|)W_s) C_{\alpha,\beta}(s) \\ \times \exp(-(d+m)\kappa s) \|f_s^{n,\varepsilon}\|_{W^{m,\infty}} \end{array} \right] ds, \end{aligned}$$

where the process $\mathcal{M}_t^m$ is given by the following relation:

$$\mathcal{M}_t^m := \exp((d+m)\kappa t) \sum_{|\beta|\le m} \exp(\sigma(d+|\beta|)W_t).$$

Note that $\mathcal{M}_t^m$ is independent of $n$ and $\varepsilon$. We set

$$b_n(t) := \|f_t^{n,\varepsilon}\|_{W^{m,\infty}} (\mathcal{M}_t^m)^{-1}.$$

Then, one gets

$$b_{n+1}(t) \le b_0 + \int_0^t \tilde{\mathcal{N}}_s^m b_{n+1}(s) ds,$$

where the process $\tilde{\mathcal{N}}_s^m$ is

$$\begin{aligned} \tilde{\mathcal{N}}_s^m := &\left\{ \sum_{|\beta|\le m} \exp(\sigma(N+|\beta|)W_s) \right\} \left\{ \sum_{|\beta|\le m} \exp(-\sigma(N+|\beta|)W_s) \right\} \\ &\times \left( \sum_{|\alpha|+|\beta|\le m} C_{\alpha,\beta}(s) \right). \end{aligned}$$

Thus, we can use Grönwall's lemma to obtain

$$\|f_t^{n,\varepsilon}\|_{W^{m,\infty}} \le \|f^{in,\varepsilon}\|_{W^{m,\infty}} \mathcal{A}_t^m,$$

where the process $\mathcal{A}_t^m$ is given by the following relation:

$$\mathcal{A}_t^m := \exp((d+m)\kappa t) \sum_{|\beta|\le m} \exp(\sigma(d+|\beta|)W_t) \exp\left[ \int_0^t \tilde{\mathcal{N}}_s^m ds \right].$$

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# 국문초록

본 학위 논문에서는, 쿠커-스메일 모형에 임의적 요소를 도입하여 그러한 불확실성에 대한 정량적 해석을 제시한다. 쿠커-스메일 모형의 다이나믹스를 실제로 응용함에 있어 우리는 쿠커-스메일 모형 자체가 몇몇 내적 불확실성을 포함하고 있으며 입자들의 다이나믹스에 영향을 줄 수 있는 몇 가지 외부적 요인을 놓치고 있음을 예상할 수 있다. 그러므로 쿠커-스메일 총체의 다이나믹스를 더 잘 서술하기 위해, 이러한 불확실성이 있는 요소를 모형에 도입하여 그것들이 쿠커-스메일 계의 다이나믹스나 안정성에 주는 영향을 평가할 필요가 있다.

이를 달성하기 위해, 우리는 우선 쿠커-스메일 모형의 거시적인 형태를 고려한다. 즉, 우리는 통신 가중치 함수와 초기값에서 오는 임의적 입력치를 유체역학 쿠커-스메일 모형에 포함시켜 임의적 유체역학 쿠커-스메일 모형을 유도한다. 더 나아가 미시적 그리고 중간보기적 단계에서 외적 불확실성에 대해 다룬다. 미시적 모형에 대해서, 쿠커-스메일 모형에 임의로 변하는 네트워크 구조를 도입하여 플로킹의 창발에 대한 충분조건을 알아본다. 중간보기적 단계의 모형으로서, 우리는 곱셈 백색 소음으로 동요된 쿠커-스메일 운동방정식을 고려하고 해의 존재성 및 유일성과 점근적 다이나믹스를 공부한다.

**주요어휘:** 플로킹, 쿠커-스메일 모형, 불확실성 정량화, 국소 민감도 분석, 임의적 동역학계, 확률편미분방정식
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