이학박사 학위논문

Topological combinatorics in rainbow set problems

(무지개 집합 문제에서의 위상수학적 조합론)

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서울대학교 대학원

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Topological combinatorics in rainbow set problems

A dissertation
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Abstract

Topological combinatorics in rainbow set problems

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Let $\mathscr{F} = \{S_1, \dots, S_m\}$ be a finite family of non-empty subsets on the ground set V. A *rainbow set* of $\mathscr{F}$ is a non-empty set of the form $S = \{s_{i_1}, \dots, s_{i_k}\} \subset V$ with $1 \le i_1 < \cdots < i_k \le m$ such that $s_{i_j} \ne s_{i_{j'}}$ for every $j \ne j'$ and $s_{i_j} \in S_{i_j}$ for each $j \in [k]$. If $k = m$, namely if all S_i is represented, then the rainbow set S is called a *full rainbow set* of $\mathscr{F}$.

Originated from the celebrated Hall's marriage theorem, it has been one of the most fundamental questions in combinatorics and discrete mathematics to find sufficient conditions on set-systems to guarantee the existence of certain rainbow sets. We call problems in this direction the *rainbow set problems*. In this dissertation, we give an overview on two topological tools on rainbow set problems, Aharoni and Haxell's topological Hall theorem and Kalai and Meshulam's topological colorful Helly theorem, and present some results on and rainbow independent sets and rainbow covers in (hyper)graphs.

Key words: Rainbow set, independence complex, non-cover complex, domination parameters, independent set
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Contents

List of Figures

Chapter 1

Introduction

For a positive integer n, we let $[n]$ denote the set $\{1,\ldots,n\}$. Let $\mathscr{F} = \{S_1,\ldots,S_m\}$ be a finite family of non-empty subsets on the ground set V. A *rainbow set* of $\mathscr{F}$ is a non-empty set of the form $S = \{s_{i_1},\ldots,s_{i_k}\} \subset V$ with $1 \leq i_1 < \cdots < i_k \leq m$ such that $s_{i_j} \neq s_{i_{j'}}$ for every $j \neq j'$ and $s_{i_j} \in S_{i_j}$ for each $j \in [k]$. If $k = m$, namely if all S_i is represented, then S is called a *full rainbow set* for $\mathscr{F}$. One of the most fundamental questions in combinatorics is to find sufficient conditions on set-systems to guarantee the existence of rainbow sets satisfying certain properties. We call problems in this direction the *rainbow set problems*. Historically, the first theorem on rainbow sets is the well-known result of Hall [20]: if $|\bigcup_{i\in I} S_i| \geq |I|$ for every nonempty $I \subset [m]$, then there exists a full rainbow set.

In the study of rainbow set problems, topological methods have become an indispensable tool. There are two fundamental results: one is the topological Hall theorem by Aharoni and Haxell [7] and the other is the topological colorful Helly theorem by Kalai and Meshulam [26]. In this dissertation, we give an overview on the topological methods in rainbow set problems and introduce new results on rainbow independent sets and rainbow covers in (hyper)graphs.

Throughout this dissertation, we only consider finite, simple and undirected graphs. For a graph G, we denote the set of all vertices by $V(G)$ and the set of all edges by $E(G)$.

1.1 Topological Hall theorem

A *hypergraph* is a generalization of a graph, and it is defined as a family of non-empty subsets of the ground set. We denote a hypergraph H by $H = (V(H), E(H))$ where $V(H)$ is the ground set (or the vertex set) and $E(H)$ is the edge set of H which consists of non-empty subsets of $V(H)$. In this terminology, graphs can be viewed as hypergraphs such that all of whose edges have cardinality exactly 2. The following extension of Hall's theorem was proved in [7].

Theorem 1.1.1 (Hall's theorem for hypergraphs)**.** *Let $\mathscr{A} = \{H_1, \ldots, H_m\}$ be a family of hypergraphs. If $\mathscr{A}$ satisfies the following condition, then there exists a rainbow matching of size m: for every subfamily $\mathscr{B}$ of $\mathscr{A}$ there exists a matching $M_{\mathscr{B}}$ in $\cup\mathscr{B}$, which cannot be pinned by fewer than $|\mathscr{B}|$ disjoint edges from $\cup\mathscr{B}$.*

The original proof of Theorem 1.1.1 by Aharoni and Haxell is based on an applicaion of Sperner's lemma to a special triangulation of a simplex. Later, Aharoni found that the proof method of [7] can give a more general result, so-called the "topological Hall theorem". In general, the topological Hall theorem proves the existence of "small" rainbow sets when "large" sets are given.

Let X be a finite simplicial complex on V. The *topological connectivity* $\eta(X)$ of X is the maximum integer i such that the j-dimensional reduced homology group $\tilde{H}_j(X;\mathbb{Q})$ is vanishing for every integer $j \leq i-2$. Here, we assume that $\tilde{H}_{-1}(X;\mathbb{Q}) = 0$ if and only if X is non-empty. Thus, for example, $\eta(X) \geq 1$ implies X is non-empty and $\eta(X) \geq 2$ further implies that X is connected. In this dissertation, if there is no confusion, we just use $\tilde{H}_i(X)$ instead of $\tilde{H}_i(X;\mathbb{Q})$ for convenience.

Example 1.1.2. *Let Δ_n be the n-dimensional simplex, i.e. $\Delta_n := \{\sigma : \sigma \subset [n]\}$. Note that the reduced homology group $\tilde{H}_i(\partial\Delta_n)$ of boundary of Δ_n is vanishing if and only if $i \neq n-1$. This implies that $\eta(\partial\Delta_n) = n$.*

Given a partition $V = V_1 \cup \cdots \cup V_m$ of V, we say $\sigma \in X$ is a *colorful simplex* of X (with respect to the partition of V) if $|\sigma \cap V_i| = 1$ for each $i \in [m]$. For every non-empty subset $I \subset [m]$, let us write $V_I = \bigcup_{i \in I} V_i$. The topological Hall theorem gives a topological condition to guarantee the existence of a colorful simplex. Its homotopoic version was first proved by Aharoni and Haxell [7], and later its homological version was given by Meshulam [34].

Theorem 1.1.3 (Topological Hall)**.** *Let X be a simplicial complex on V and let $V = V_1 \cup \cdots \cup V_m$ be a partition of V. If $\eta(X[V_I]) \geq |I|$ for every non-empty $I \subset [m]$, then there exists a colorful simplex of X.*

1.2 Topological colorful Helly theorem

Helly's theorem is a result about intersection patterns of convex sets in Euclidean space, which statement is that for every finite family $\mathscr{F}$ of convex sets in $\mathbb{R}^d$, if every $d+1$ or fewer members of $\mathscr{F}$ have a common point, then all members of $\mathscr{F}$ have a common point. There are many variations of Helly's theorem which shows "global intersection properties" from assumptions on "local intersection properties", and those results are called *Helly type theorems*.

Among the significant number of Helly type theorems, we focus on the colorful generalization of Helly's theorem. Suppose we are given $d+1$ finite families $\mathscr{F}_1, \ldots, \mathscr{F}_{d+1}$ of convex sets in $\mathbb{R}^d$. The colorful Helly theorem by Bárány [11] asserts the following: if $A_1 \cap \ldots \cap A_{d+1} \neq \emptyset$ for every choice of $A_i \in \mathscr{F}_i$ then there exists some $i \in [d+1]$ such that all members of $\mathscr{F}_i$ have a common point. Note that setting $\mathscr{F}_1 = \cdots = \mathscr{F}_{d+1}$ gives us Helly's theorem.

Later, Kalai and Meshulam [26] established topological versions of the colorful Helly theorem. A specific case of the "topological colorful Helly theorem" proves the existence of "large" rainbow sets when "many" of "large" sets are given.

We recommend [37] for an overview of this field.

1.2.1 Collapsibility and Lerayness of simplicial complexes

Let X be a finite simplicial complex. We say a face f of X is *free* if there is a unique maximal face of X containing f. An *elementary d-collapse* of X is the operation deleting all faces containing a free face f of X with $|f| \leq d$. We say a simplicial complex X is *d-collapsible* if we can obtain the void complex from X by a finite sequence of elementray d-collapses.

The notion of d-collapsibility was introduced by Wegner [38]. In the same paper, he proved that the "nerve complex" of a finite family of convex sets in $\mathbb{R}^d$ is d-collapsible (see also [19]). Given a family of non-empty sets $\mathscr{F} = \{F_1, \ldots, F_n\}$, the *nerve* of $\mathscr{F}$ is defined as the simplicial complex

$$N(\mathscr{F}) = \{\sigma \in [n] : \cap_{i \in \sigma} F_i \neq \emptyset\}.$$

In graph theory, the 1-skeleton[1] of $N(\mathscr{F})$ is called as the *intersection graph* of $\mathscr{F}$.

Theorem 1.2.1 ([38])**.** *For every finite family $\mathscr{C}$ of convex sets in $\mathbb{R}^d$, its nerve $N(\mathscr{C})$ is d-collapsible.*

The converse of Theorem 1.2.1 is not true in general. That is, a d-collapsible complex may not be a nerve of some finite family of convex sets in $\mathbb{R}^d$. For $d = 1$, it is well-known that a simplicial complex is 1-collapsible if and only if the *clique complex*, i.e. complex of all cliques, of chordal graphs[2]. Thus the existence of a

[1] The *d-skeleton* of a simplicial complex X is the subcomplex of all faces in X of dimension at most d.

[2] A *chordal graph* is a graph with no induced cycle of length 4 or greater.

chordal graph which is not an intersection graph of intervals implies the existence of 1-collapsible complex which is not the nerve of a finite family of intervals. For $d \geq 2$, let X be the d-skeleton of a $(2d+2)$-simplex. If we consider X as a family of sets, then the nerve $N(X)$ is d-collapsible but $N(X)$ is not the nerve of a family of convex sets in $\mathbb{R}^d$ (see [27]).

1.2.2 Nerve theorem and topological Helly theorem

The nerve $N(\mathscr{F})$ of a family $\mathscr{F}$ of non-empty sets appears frequently in discrete geometry and topological combinatorics, together with "nerve theorems" which show the relation between the topology of $\bigcup_{A\in\mathscr{F}} A$ and that of $N(\mathscr{F})$. The first example of nerve theorems is the following.

Theorem 1.2.2 (Homotopy nerve theorem)**.** *Let X be a simplicial complex and let $\mathscr{F} = \{X_1, \ldots, X_n\}$ be a family of subcomplexes of X whose union is X. If $\bigcap_{i\in\sigma} X_i$ is contractible for every $\sigma \in N(\mathscr{F})$, then $N(\mathscr{F})$ is homotopy equivalent to X.*

The assertion of Theorem 1.2.2 is valid for every family of topological objects whose non-empty intersections are all contractible. As an application of Theorem 1.2.2, we can prove that for every good cover[3] $\mathscr{F} = \{A_1, \ldots, A_n\}$, the nerve $N(\mathscr{F})$ has vanishing homology in dimension d or greater. Note that the same topological property holds for any open, connected subset in $\mathbb{R}^d$. This is the key observation to prove the topological version of Helly's theorem [22].

Theorem 1.2.3 (Topological Helly theorem)**.** *Let $\mathscr{F}$ be a good cover in $\mathbb{R}^d$. If every $d+1$ or fewer members of $\mathscr{F}$ have a common point, then all members of $\mathscr{F}$ have a common point.*

A simplicial complex X is called *d-Leray* if $\tilde{H}_i(L) = 0$ for every integer $i \geq d$ and induced subcomplex L of X. The above argument implies that the nerve

[3]Here, a *good cover* in $\mathbb{R}^d$ is a family of open and contractible sets in $\mathbb{R}^d$ such that non-empty intersection of any subfamily is open and contractible. For example, a family of open convex sets in $\mathbb{R}^d$ is a good cover in $\mathbb{R}^d$.

complex of any good cover in $\mathbb{R}^d$ is d-Leray. In this sense, Theorem 1.2.3 can be written in a more general format.

Theorem 1.2.4. *Let $\mathscr{F}$ be a finite family of non-empty sets whose nerve is d-Leray. If every $d+1$ or fewer members of $\mathscr{F}$ have a common point, then all members of $\mathscr{F}$ have a common point.*

It is important to notice that applying an elementary d-collapse to a simplicial complex X does not affect to the (non-)vanishing property of homology groups of X of dimension d or greater, and that the sequence of elementary d-collapses to X to obtain the void complex can be inherited to any induced subcompelx Y of X. It follows that every d-collapsible complex is d-Leray (see [38]). The converse is not true in general, as an example of a d-Leray complex which is not d-collapsible was found in [36]. (See also [33].)

1.2.3 Topological colorful Helly theorem

In [26], Kalai and Meshulam established the colorful version of the topological Helly theorem. Indeed, they considered a more general situation. Given a matroid M on a ground set V, the *matroidal complex* is a simplicial complex on V such that $\sigma \subset V$ is a face if and only if σ is an independent set in the matroid M.

Theorem 1.2.5. *Let X be a d-collapsible complex on V and M be a matroidal complex on V. If $M \subset X$ then there exists a simplex $\tau \in X$ such that $\rho(\tau) = rk(M)$ and $\rho(V - \tau) \leq d$.*

Theorem 1.2.6. *Let X be a d-Leray complex on V and M be a matroidal complex on V. If $M \subset X$ then there exists a simplex $\tau \in X$ such that $\rho(V - \tau) \leq d$.*

Here is how the topological colorful Helly theorems are related to the rainbow set problems. Consider the special case when the matroid M is a *partition matroid* with respect to a partition $V = V_1 \cup \cdots \cup V_m$ for some integer $m \geq d+1$, i.e. $A \subset V$

is independent in M if and only if A is a rainbow set. If $V_i \notin X$ for all $i \in [m]$, then it follows that $\rho(V-\tau) \geq d+1$ for any $\tau \in X$. Hence the converses of Theorem 1.2.5 and Theorem 1.2.6 imply that there is a rainbow set which is not in X. See [8] for an example of an application of Theorem 1.2.5 to a rainbow set problem.

While the proof of Theorem 1.2.5 is purely combinatorial, the proof of Theorem 1.2.6 requires some topological ideas. It uses Theorem 1.1.3 combined with the homology version of Theorem 1.2.2 [26] and the combinatorial Alexander duality theorem [12].

Theorem 1.2.7 (Homology nerve theorem)**.** *Let X be a finite simplicial complex, and $\mathscr{F} = \{X_1, \ldots, X_m\}$ be a family of non-empty subcomplexes of X such that $\cup_{i\in[m]} X_i = X$. Assume that, for all $\sigma \in N(\mathscr{F})$ with* $\dim\sigma \leq d$, *$\tilde{H}_j(\cap_{i\in\sigma} X_i) = 0$ for each $0 \leq j \leq k - dim\sigma$. Then for all integer $0 \leq j \leq k$,*

$$\tilde{H}_j(X) \cong \tilde{H}_j(N(\mathscr{F})).$$

Moreover, $\tilde{H}_{k+1}(N(\mathscr{F})) \neq 0$ implies $\tilde{H}_{k+1}(X) \neq 0$.

Theorem 1.2.8 (Alexander duality)**.** *Let X be a finite simplicial complex on V. The Alexander Dual $D(X)$ of X is the simplicial complex on V such that $D(X) = \{\sigma \subset V : V \setminus \sigma \notin X\}$. If $V \notin X$, then for all $-1 \leq i \leq |V|-2$,*

$$\tilde{H}_i(D(X)) \cong \tilde{H}_{|V|-i-3}(X).$$

1.3 Domination numbers and non-cover complexes of hypergraphs

Given a hypergraph H on V, a vertex subset $W \subset V$ is said to be *independent* if the induced subhypergraph $H[W]$ has no edge. For a hypergraph H, the *line graph* of

H, denoted by $L(H)$, is the graph on $E(H)$ where two vertices are adjacent if and only if they intersect, as edges of H. In this way, a rainbow matching of a system of hypergraphs can be regarded as a rainbow independent set of system of line graphs. Thus we can express Theorem 1.1.1 in terms of graphs and independent sets, where "pinning edges" corresponds to the "domination numbers".

For every vertex v in G, we denote by $N(v)$ the *open neighbor* of v i.e. the set of all vertices which are adjacent to the vertex v. The *closed neighbor* of v is defined by $N[v] := N(v) \cup \{v\}$. The vertices in $N[v]$ are said to be *dominated* by the vertex v in G, and we say the vertices in $N(v)$ are said to be *strongly dominated* by the vertex v in G. Similarly, for a vertex subset $W \subset V(G)$, we define $N(W) = \bigcup_{v \in W} N(v)$ and $N[W] = \bigcup_{v \in W} N[v]$. A vertex subset $A \subset V$ is said to be *dominated* by $W \subset V$ in G if $A \subset N[W]$ and A is said to be *strongly dominated* by W in G if $A \subset N(W)$. We define the *domination number* of A in G by

$$\gamma(G,A) := \min\{|W| : W \text{ dominates } A \text{ in } G\},$$

and the *strong domination number* of A in G by

$$\gamma_0(G,A) := \min\{|W| : W \text{ strongly dominates } A \text{ in } G\}.$$

Every $A \subset V$ have bounded domination number, but the strong domination number may be unbounded if the graph G contains an isolated vertex. We write $\gamma_0(G,A) = \infty$ if the strong domination number of A is unbounded.

Theorem 1.3.1 (Hall's theorem for hypergraphs, revisited)**.** *Let $G_1, \dots, G_m$ be graphs on V, and let $G_I = \bigcup_{i \in I} G_i$ for every $\emptyset \neq I \subset [m]$. If $\gamma(G_I, A) \geq |I|$ for every $\emptyset \neq I \subset [m]$ and every independent set A in G_I, then there exists a rainbow independent set of size m.*

In the same flavor, by considering various domination parameters of graphs, Theorem 1.1.3 gives many variants of Theorem 1.3.1. Let us define the *indepen-*

dence complex of a graph G on V is as the simplicial complex

$$\mathscr{I}(G) = \{W \subset V : W \text{ is an independent set in } G\}.$$

Note that if G contains an isolated vertex v, then $\mathscr{I}(G)$ is contractible since it is a cone with apex v, implying that $\eta(\mathscr{I}(G)) = \infty$.

Theorem 1.3.2. *Let $G_1, \ldots, G_m$ be graphs on V. If $\eta(\mathscr{I}(G_J)) \geq |J|$ for every $\emptyset \neq J \subset [m]$, then there exists a rainbow independent set of size m.*

There are some known bounds on $\eta(\mathscr{I}(G))$ in terms of the domination numbers of graphs. Here we introduce two domination numbers. Let G be a graph on V.

- $i\gamma(G) := \max\{\gamma_0(G,A) : A \in \mathscr{I}(G).\}$, which is called the *independence domination number* of G.
- $\tilde{\gamma}(G) := \gamma_0(G,V)$, which is called the *total domination number* of G.

Theorem 1.3.3 ([7, 16, 6])**.** $\eta(\mathscr{I}(G)) \geq \max\{i\gamma(G), \frac{\tilde{\gamma}(G)}{2}\}$.

In [6], Aharoni et al. found an additional bound $\eta(\mathscr{I}(G)) \geq \gamma_E(G)$, where

$$\gamma_E(G) := \min\{|F| : F \subset E(G), V(F) \text{ strongly dominates } V \text{ in G.}\}\ ^4.$$

Observe that this bound is stronger than the bound $\eta(\mathscr{I}(G)) \geq \frac{\tilde{\gamma}(G)}{2}$. Take $F \subset E$ such that $V(F)$ strongly dominates V in G. Then by the minimality of $\tilde{\gamma}(G)$, we have $\tilde{\gamma}(G) \leq |V(F)| \leq 2|F|$ which means $\tilde{\gamma}(G) \leq 2\gamma_E(G)$.

[4]Here, $V(F)$ is the set of all endpoints of edges in F.

1.3.1 Domination numbers of hypergraphs

Let H be a hypergraph defined on V. For convenience, we will regard a hypergraph H same as its edge set $E(H)$. We say $W \subset V$ *(strongly) dominates*[5] a vertex $v \in V$ if there exists $W' \subset W$ such that $W' \cup \{v\}$ is an edge of H. Here we assume that the empty set dominates the vertex v if H contains $\{v\}$ as a singleton edge, i.e. $\{v\} \in H$. We say $W \subset V$ dominates $A \subset V$ if W dominates every vertex in A. The *domination number of A in H* is defines as an integer

$$\gamma(H,A) := \min\{W \subset V : W \text{ dominates } A\}.$$

In particular, the *total domination number $\tilde{\gamma}(H)$ of H* is defined as the strong domination number of V in H, i.e. $\tilde{\gamma}(H) := \gamma(H,V)$.

An independent set $I \subset V$ in H is said to be *strongly independent* in H if for any two vertices $u, v \in A$, there is no edge of H containing both u and v. The *strong independence domination number* of H is defined as

$$\gamma_{si}(H) := \max\{\gamma(H,A) : A \text{ is a strong independent set in } H\}.$$

Finally, we define $\gamma_E(H) := \min\{|F| : F \subset H, \bigcup_{f \in F} f \text{ dominates } V(H)\}$.

1.3.2 Non-cover complexes of hypergraphs

Throughout this dissertation, we consider only non-empty hypergraphs, and we do not allow an empty set to be an edge of a hypergraph. Let H be a hypergraph on a vertex set V. An *independence complex* of H is defined as

$$\mathscr{I}(H) := \{W \subset V : W \text{ is an independent set in } H\}.$$

[5] Although we defined domination and strong domination for graphs separately, we will use only strong dominations for hypergraphs throughout the rest of this dissertation.

A *cover* in H is a vertex subset $W \subset V$ such that $V \setminus C$ is an independent set in H. W is called a *non-cover* in H if it is not a cover of H. A *non-cover complex* of H is defined as

$$\mathscr{N}\mathscr{C}(H) := \{W \subset V : W \text{ is a non-cover in } H\}.$$

Observe that $\mathscr{N}\mathscr{C}(H)$ is the Alexander dual of $\mathscr{I}(H)$.

Given a simplicial complex K, let us denote by $L(K)$ the *Leray number* of K, that is, the minimum integer d such that K is d-Leray. One of the purpose of this dissertation is to prove some bounds of $\eta(I(H))$ and $L(\mathscr{N}\mathscr{C}(H))$ in terms of $\tilde{\gamma}(H)$, $\gamma_{si}(H)$, and $\gamma_E(H)$. The first main result in this topic is the following.

Theorem 1.3.4. *For every hypergraph H, we have*

$$\tilde{H}_i(\mathscr{N}\mathscr{C}(H)) = 0$$

for all $i \geq |V(H)| - \max\{\left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil, \gamma_{si}(H), \gamma_E(H)\} - 1$.

Applying Theorem 1.2.8 to Theorem 1.3.4 gives us a hypergraph analogue of Theorem 1.3.3. Recall the fact that if H contains an isolatex vertex v, then $I(H)$ is a cone with apex v, thus $\eta(I(H)) = \infty$.

Corollary 1.3.5. $\eta(I(H)) \geq \max\{\left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil, \gamma_{si}(H), \gamma_E(H)\}$ *for every hypergraph H.*

The second main result in this topic is establishing a stronger version of Theorem 1.3.4.

Theorem 1.3.6. *Let H be a hypergraph with no isolated vertices. Then each of the following holds:*

(a) If $|e| \leq 3$ for every $e \in H$, then $L(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1$.

(b) If $|e| \leq 2$ for every $e \in H$, then $L(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \gamma_{si}(H) - 1$.

(c) $L(\mathscr{NC}(H)) \leq |V(H)| - \gamma_E(H) - 1$.

The main contribution of this topic is joint work with Andreas Holmsen and Minki Kim [23]. We also prove a bound for collapsibility numbers[6] of the non-cover complex of graphs in terms of γ_{si}.

Theorem 1.3.7. *$\mathscr{NC}(G)$ is $(|V(G)| - \gamma_{si}(G) - 1)$-collapsible for every graph G.*

This gives another proof of Theorem 1.3.6 (b). The proof for the collapsibility number is joint work with Ilkyoo Choi and Boran Park [15].

1.4 Rainbow independent sets in graphs

Given a graph G, let $\alpha(G)$ be the size of a largest independent set in G and let $\nu(G)$ be the size of a largest matching in G. We are interested in the following type of questions:

Problem 1.4.1. *Given a system $\mathscr{F} = (F_1, \ldots, F_m)$ of independent sets in a graph G and a number q, what conditions on G, on m and on the size of the sets F_i, guarantee the existence of an independent rainbow set of size q?*

A theorem answering an instance of Problem 1.4.1 was proved by Drisko [17]. In a slightly more general version, proved in [2], it states that every $2n-1$ matchings of size n in a bipartite graph have a partial rainbow matching of size n. Since an independent set in $L(H)$ is a matching in H, this can be stated as:

Theorem 1.4.2 (Drisko)**.** *Let H be a bipartite graph and let $G = L(H)$. Every $2n-1$ independent sets of size n in G have a partial rainbow independent set of size n.*

[6] The *collapsibility number* $\mathrm{Coll}(K)$ of a simplicial complex K is the minimum integer d such that K is d-collapsible.

The number $2n-1$ in Theorem 1.4.2 cannot be improved. See Example 1.4.3.

Example 1.4.3. *Take the two matchings of size n in the cycle C_{2n}, each repeated $n-1$ times. These are $2n-2$ matchings of size n in the line graph, having no rainbow matching of size n.*

A generalization of Theorem 1.4.2 to the intersection of two matroids was given in [30]. Theorem 1.4.2 is the special case in which both matroids are partition matroids.

Theorem 1.4.4. *If $\mathscr{M},\mathscr{N}$ are matroids on the same ground set, then any $2n-1$ sets belonging to $\mathscr{M}\cap\mathscr{N}$ have a rainbow set of size n belonging to $\mathscr{M}\cap\mathscr{N}$.*

In [9], a strengthened version of Theorem 1.4.2 was proved:

Theorem 1.4.5. *If $\mathscr{F}=(F_1,\dots,F_{2n-1})$ is a family of matchings in a bipartite graph, and $|F_i|\geq \min(i,n)$ for every $i\leq 2n-1$, then $\mathscr{F}$ has a rainbow matching.*

For a class $\mathscr{C}$ of graphs and integers $m\leq n$, let $f_{\mathscr{C}}(n,m)$ be the minimal number k such that every k independent sets of size n in a graph belonging to $\mathscr{C}$ have a partial rainbow independent set of size m. If there is no such integer k, then we write $f_{\mathscr{C}}(n,m)=\infty$. If K, H are two graphs, we write $H<K$ if K contains an induced copy of H. If $H\not<K$ we say that K is *H-free*.

We shall consider the following classes of graphs:

1. $\mathscr{U}$: the class of all graphs.

2. $\mathscr{B}$: the class of line graphs of bipartite graphs.

3. $\mathscr{G}$: the class of line graphs of all graphs.

4. $\mathscr{X}(k)$: the class of k-colorable graphs.

5. $\mathscr{D}(k)$: the class of graphs with degrees at most k.

6. $\mathscr{T}$: the class of chordal graphs.

7. $\mathscr{F}(H)$: the class of H-free graphs, for a given graph H.

8. $\mathscr{F}(H_1,\dots,H_t) = \bigcap_i \mathscr{F}(H_i)$: the class of graphs which are H_i-free for all $i \in [t]$.

Here is a small example, for practicing the concepts.

Example 1.4.6. *For every k, let G be the complete k-partite graph with all sides of size n, and let F_i be its respective sides. Then there is no independent rainbow set of size* 2*, which shows that $f_{\mathscr{U}}(n,2) = \infty$ for every n.*

Clearly, we have the following trivial lower bound, leaving out graphs with no independent set of size n:

$$f_{\mathscr{C}}(n,m) \geq m \tag{1.4.1}$$

for any subfamily $\mathscr{C}$ of graphs.

For subfamilies $\mathscr{C},\mathscr{D}$ of graphs, if $\mathscr{C} \subseteq \mathscr{D}$ and $m' \leq m \leq n$ then we have:

$$f_{\mathscr{D}}(n,m) \geq f_{\mathscr{C}}(n,m'). \tag{1.4.2}$$

Theorem 1.4.2 and Example 1.4.3 combined yield $f_{\mathscr{B}}(n,n) = 2n-1$. In [2], the authors conjectured that $f_{\mathscr{B}}(n,n-1) = n$. In fact, we do not know a counterexample to the stronger conjecture that $f_{\mathscr{G}}(n,n-1) = n$. In [13] the following was proved. Note that when $k = 1$, we obtain $f_{\mathscr{B}}(n,n-1) \leq \lfloor \frac{3}{2}n \rfloor - 2$.

Theorem 1.4.7. $f_{\mathscr{B}}(n,n-k) \leq \lfloor \frac{k+2}{k+1}n \rfloor - (k+1)$.

The main conjecture for general line graphs is that $f_{\mathscr{G}}(n,n) = 2n$, Guided by examples from [13]. In [3], it was proved that $f_{\mathscr{G}}(n,n) \leq 3n-2$. Although the original conjecture is still open, its fractional version was proved in [8] and a matroidal generalization of the fractional version was proved in [4], based on Theorem 1.2.5. The result of [8] also gives a new proof of Drisko's theorem since the matching number and the fractional matching number are same in bipartite graphs.

In this dissertation, we investigate Problem 1.4.1 for more general graphs, not limited to line graphs. One of our main results is to characterize a graph H where $f_{\mathscr{F}(H)}(n,n) < \infty$ for every positive integer n. We prove that such property can be achieved if and only if $H = K_r$ or K_r^-, where K_r is the complete graph on r vertices and K_r^- is the graph obtained from K_r by deleting one edge.

Theorem 1.4.8. *$f_{\mathscr{F}(H)}(n,n) < \infty$ for every positive integer n if and only if H is either K_r or K_r^- for some r.*

Another important result is to find $f_{\mathscr{D}(k)}(n,n)$ for graphs of bounded maximum degree. We pose a conjecture on $f_{\mathscr{D}(k)}(n,n)$ with a few evidence on our conjecture.

Theorem 1.4.9. *For the class of all graphs with maximum degree at most k, $f_{\mathscr{D}(k)}(n,m) \leq (m-1)k+1$ for all $m \leq n$ and $f_{\mathscr{D}(k)}(n,n) = \lceil \frac{k+1}{2} \rceil + 1$ for $n \leq 3$.*

Some of the results will be given, in addition to purely combinatorial proofs, also topological proofs, based on Theorem 1.2.5. This is joint work with Ron Aharoni, Joseph Briggs, and Minki Kim [5].

1.5 Organization

In Chapter 2, we introduce a proof method of [33] which gives an upper bound on the collapsibility number of simplicial complexes. This technique will be used to prove Theorem 1.3.7. The relations between domination numbers, independence complexes, and non-cover complexes for general hypergraphs, including the proofs of Theorem 1.3.4 and Theorem 1.3.6, will be discussed in Chapter 3. In Chapter 4, we present results on rainbow independent sets. In particular, the proof of Theorem 1.4.8 appears in Section 4.1.4, and Theorem 1.4.9 will be proved in Section 4.3 and Section 4.4.

Chapter 2

Collapsibility of non-cover complexes of graphs

In this chapter, we will prove an upper bound on the collapsibility numbers of non-cover complexes in terms of the independent domination numbers of graphs.

Theorem 1.3.7. *$\mathcal{NC}(G)$ is $(|V(G)| - \gamma_{si}(G) - 1)$-collapsible for every graph G.*

To show Theorem 1.3.7, we first introduce a proof technique of [33]. The proof of Theorem 1.3.7 will be presented in Section 2.2.

2.1 The minimal exclusion sequences

Let X be a finite simplicial complex on vertex set $V = \{v_1, \ldots, v_n\}$. Fix linear orderings

$$\prec: v_1, v_2, \ldots, v_n \quad \text{and} \quad \prec_f: \sigma_1, \sigma_2, \ldots, \sigma_m$$

on the vertices in V and the maximal faces of X, respectively. For a simplex $\sigma \in X$, we write

$$i(\sigma) := \min\{j \in [m] : \sigma \subset \sigma_j\}.$$

If $i(\sigma) = i$, the *minimal exclusion sequence* of σ (with respect to the orderings $\prec$ and $\prec_f$) is a sequence $\mathrm{mes}(\sigma) = (w_1, w_2, \ldots, w_{i-1})$ of length $i-1$ defined as:

(i) If $i = 1$, then $\mathrm{mes}(\sigma)$ is the empty sequence.

(ii) If $i > 1$, then for each $j \in [i-1]$, we define w_j recursively by the following rules:

- If $(\sigma \setminus \sigma_j) \cap \{w_1, \ldots, w_{j-1}\} = \emptyset$, then w_j is the minimal element in $\sigma \setminus \sigma_j$. In this case, we say w_j is *new* at j.
- Otherwise, w_j is the minimal element in $(\sigma \setminus \sigma_j) \cap \{w_1, \ldots, w_{j-1}\}$. In this case, we say w_j is *old* at j.

Let $M(\sigma)$ be the set of vertices in the sequence $\mathrm{mes}(\sigma)$ and let

$$d(X) = \max\{M(\sigma) : \sigma \in X\}.$$

Theorem 2.1.1. *The simplicial complex X is $d(X)$-collapsible.*

Note that $d(X)$ is dependent to linear orderings $\prec, \prec_f$. Thus, to obtain a nice upper bound on the collapsibility number of X by Theorem 2.1.1, it is important to determine linear orderings $\prec, \prec_f$ which can minimize $d(X)$.

Theorem 2.1.1 was implicitly used in the proof of a result in [33] to obtain the upper bound on the collapsibility number of the nerve of a finite family of sets of bounded cardinalities. It was introduced again in [31] in a more general statement. For the completeness of this dissertation, we include the proof of Theorem 2.1.1.

Lemma 2.1.2. *For $\sigma \in X$, $mes(M(\sigma)) = mes(\sigma)$.*

Proof. It is clear that $M(\sigma)$ is a subset of σ, thus we have $M(\sigma) \in X$. Suppose $i(\sigma) = i$ and $i(M(\sigma)) = i'$. Obviously, we have $i \geq i'$ since $M(\sigma) \subset \sigma$. We will first show that $i = i'$. Let $\mathrm{mes}(\sigma) = (w_1, \ldots, w_{i-1})$ and $\mathrm{mes}(M(\sigma)) = (w'_1, \ldots, w'_{i-1})$.

Recall that for each $j \in [i-1]$, the vertex w_j is chosen in $\sigma \setminus \sigma_j$. Since $w_j \in M(\sigma)$, it follows that $w_j \in M(\sigma) \setminus \sigma_j$, implying that $M(\sigma) \setminus \sigma_j \neq \emptyset$. Thus $i = i'$.

Now to show $\text{mes}(\sigma) = \text{mes}(M(\sigma))$, we will prove $w_j = w'_j$ for each $j \in [i-1]$. We proceed by induction on j. Since w_1 is the minimal element in $\sigma \setminus \sigma_1$ and $w_1 \in M(\sigma) \setminus \sigma_1 \subset \sigma \setminus \sigma_1$, we obviously have $w'_1 = w_1$. Now assume $w_k = w'_k$ for $k < j-1$. Suppose w_j is new at j, then it implies that $(\sigma \setminus \sigma_j) \cap \{w_1, \ldots, w_{j-1}\} = \emptyset$ and so w_j is the minimal element in $\sigma \setminus \sigma_j$. Since we know $M(\sigma) \subset \sigma$ and $w_j \in M(\sigma)$, it follows that $(M(\sigma) \setminus \sigma_j) \cap \{w_1, \ldots, w_{j-1}\} = \emptyset$ and w_j is the minimal element in $M(\sigma) \setminus \sigma_j$, which means $w'_j = w_j$. Otherwise, i.e. if w_j is old at j, then $w'_j = w_j$ since $(\sigma \setminus \sigma_j) \cap \{w_1, \ldots, w_{j-1}\}$ and $(M(\sigma) \setminus \sigma_j) \cap \{w_1, \ldots, w_{j-1}\}$ are same. □

Lemma 2.1.3. *$mes(\sigma) = mes(\sigma')$ if and only if $M(\sigma) = M(\sigma')$ for every two simplices σ and σ' in X.*

Proof. If $\text{mes}(\sigma) = \text{mes}(\sigma')$, then it is obvious that $M(\sigma) = M(\sigma')$. For the opposite direction, assume $M(\sigma) = M(\sigma')$. Then by Lemma 2.1.2, we have

$$\text{mes}(\sigma) = \text{mes}(M(\sigma)) = \text{mes}(M(\sigma')) = \text{mes}(\sigma'),$$

which completes the proof. □

Let $M_i = \{M(\sigma) : \sigma \in X, i(\sigma) = i\}$ and $M = \cup_{i=1}^{m} M_i$, and define a linear order $\prec_M$ on the elements of M as follows: First arrange the elements of M so that $x \prec_M y$ whenever $x \in M_a$ and $y \in M_b$ for some $1 \leq a < b \leq m$. Then arrange the elements in M_i for each $i \in [m]$ so that $x \prec_M y$ if and ony if $\text{mes}(x) \prec_{lex} \text{mes}(y)$ where $\prec_{lex}$ is the lexicographic order with repect to the order $\prec$ of vertices of X. The well-definedness of $\prec_M$ is guaranteed by Lemma 2.1.2 and Lemma 2.1.3.

Lemma 2.1.4. *For $\sigma \in M$, $M(\sigma) = \sigma$.*

Proof. Since $\sigma \in M$, we have $\sigma = M(\tau)$ for some $\tau \in X$. By Lemma 2.1.2, we obtain $\text{mes}(\sigma) = \text{mes}(M(\tau)) = \text{mes}(\tau)$ and so $M(\sigma) = M(\tau) = \sigma$. □

Lemma 2.1.5. *If $\sigma \subset \sigma' \in X$, then $M(\sigma) \succeq_M M(\sigma')$.*

Proof. Since $\sigma \subset \sigma'$, it is obvious that $i(\sigma) \le i(\sigma')$. If $i(\sigma) < i(\sigma')$, then we have $M(\sigma) \succ_M M(\sigma')$ since $M(\sigma) \in M_{i(\sigma)}$ and $M(\sigma') \in M_{i(\sigma')}$. Thus we may assume that $i(\sigma) = i(\sigma')$. If $\text{mes}(\sigma) = \text{mes}(\sigma')$, then we are done. Hence we may further assume that $\text{mes}(\sigma) \neq \text{mes}(\sigma')$. Let j be the first index where two sequences are different, and let w_j and w'_j be the j-th entries of $\text{mes}(\sigma), \text{mes}(\sigma')$, respectively. Then both w_j and w'_j are new at j, which means that w_j, w'_j are the minimal elements in $\sigma \setminus \sigma_j$ and $\sigma' \setminus \sigma_j$, respectively. Since we have $\sigma \subset \sigma'$ and $w'_j \prec w_j$, it follows that $M(\sigma') \prec_M M(\sigma)$. $\square$

Now for each $\eta \in M$, define

$$T(\eta) = \{v \in V : \text{mes}(\eta \cup \{v\}) = \text{mes}(\eta)\}.$$

Note that we always have $\eta \subset T(\eta)$.

Lemma 2.1.6. *For $\sigma \in X$ and $\eta \in M$, $\eta \subset \sigma \subset T(\eta)$ if and only if $M(\sigma) = \eta$.*

Proof. First assume $M(\sigma) = \eta$. Then we already have $\eta = M(\sigma) \subset \sigma$, and hence it remains to show that $\sigma \subset T(\eta)$. By Lemma 2.1.2, we have

$$\text{mes}(\eta) = \text{mes}(M(\sigma)) = \text{mes}(\sigma),$$

thus it follows that $M(\eta) = M(\sigma) = \eta$. For each $v \in \sigma$, since we have $\eta \subseteq \eta \cup \{v\} \subset \sigma$, Lemma 2.1.5 gives us

$$\eta = M(\eta) \succeq_M M(\eta \cup \{v\}) \succeq_M M(\sigma) = \eta.$$

Thus we have $M(\eta \cup \{v\}) = \eta$, implying that $v \in T(\eta)$. Consequently, $\sigma \subset T(\eta)$.

Now we will show that if $\eta \subset \sigma \subset T(\eta)$ then $M(\sigma) = \eta$. Suppose $\eta \subset \sigma \subset T(\eta)$. It is sufficient to show that $\text{mes}(T(\eta)) = \text{mes}(\eta)$, since if it is true then we obtain $M(T(\eta)) = M(\eta) = \eta$ by Lemma 2.1.4 and then

$$\eta = M(\eta) \succeq_M M(\sigma) \succeq_M M(T(\eta)) = \eta$$

by Lemma 2.1.5, which implies $M(\sigma) = \eta$. Thus we want to show $\mathrm{mes}(T(\eta)) = \mathrm{mes}(\eta)$.

Let $i(\eta) = i$ for some $i \in [m]$. For each $v \in T(\eta)$, we have $\mathrm{mes}(\eta \cup \{v\}) = \mathrm{mes}(\eta)$ by definition. Thus $i(\eta \cup \{v\}) = i(\eta) = i$ and then $\eta \cup \{v\} \subset \sigma_{i+1}$, which implies $v \in \sigma_{i+1}$. Therefore $T(\eta) \subset \sigma_{i+1}$. Then we know $T(\eta) \in X$, and so we can consider $i(T(\eta))$. Obviously $i(T(\eta)) \leq i$ since $T(\eta) \subset \sigma_{i+1}$. On the other hand, we have $i(T(\eta)) \geq i(\eta) = i$ since $\eta \subset T(\eta)$. Therefore $i(T(\eta)) = i$, which means that $\mathrm{mes}(\eta)$ and $\mathrm{mes}(T(\eta))$ have same length.

To show by contradiction, assume that $\mathrm{mes}(T(\eta)) \neq \mathrm{mes}(\eta)$. Let j be the first index where $\mathrm{mes}(T(\eta))$ and $\mathrm{mes}(\eta)$ differ. Let w, w' be the j-th entries of $\mathrm{mes}(\eta), \mathrm{mes}(T(\eta))$, respectively. Then $w' \notin \eta$ and $w' \prec w$. Thus the j-th entry of $\mathrm{mes}(\eta \cup \{w\})$ is also w', which is a contradiction that $w \in T(\eta)$. Therefore $\mathrm{mes}(T(\eta)) = \mathrm{mes}(\eta)$. □

Lemma 2.1.7. *Let $\eta \in M$ and $Y = \{\sigma \in X : M(\sigma) \succeq_M \eta\}$. Then*

(i) $T(\eta)$ is the unique maximal face of Y containing η.

(ii) Let $Y' = Y \setminus \{\sigma \in Y : \eta \subset \sigma \subset T(\eta)\}$. If η is the maximal element of M with respect to $\prec_M$, then $Y' = \emptyset$. Otherwise,

$$Y' = \{\sigma \in X : M(\sigma) \succeq_M \eta'\},$$

where $\eta' \in M$ such that $\eta' \succ_M \eta$ and there is no element between η and η' in the ordering $\prec_M$.

Proof. (i) Take $\sigma \in Y$ such that $\eta \subset \sigma$. We want to show $\sigma \subset T(\eta)$. By Lemma 2.1.5, $M(\sigma) \preceq_M M(\eta) = \eta$. Since $\sigma \in Y$, $M(\sigma) = \eta$. Then $\sigma \subset T(\eta)$ by Lemma 2.1.6.

(ii) By Lemma 2.1.6,

$$\{\sigma \in Y : \eta \subset \sigma \subset T(\eta)\} = \{\sigma \in Y : M(\sigma) = \eta\} = \{\sigma \in X : M(\sigma) = \eta\}.$$

Thus, if η is the maximal element of M with respect to $\prec_M$, then $Y = \{\sigma \in X : M(\sigma) = \eta\}$ and so $Y' = \emptyset$. Otherwise,

$$Y' = \{\sigma \in X : M(\sigma) \succeq_M \eta'\}$$

where $\eta' \in M$ such that $\eta' \succ_M \eta$ and there is no element between η and η' in the order $\prec_M$.

□

2.2 Independent domination numbers and collapsibility numbers of non-cover complexes of graphs

This research was motivated from a question by Aharoni [1]:

Question 2.2.1 ([1])**.** *If G is a graph with no isolated vertices, then is it true that the non-cover complex of G is $(|V(G)| - \max\{i\gamma(G), \frac{\tilde{\gamma}(G)}{2}\} - 1)$-collapsible?*

A partial answer for Question 2.2.1 was given in [15]:

Theorem 2.2.2 ([15])**.** *For a graph G without isolated vertices, the non-cover complex of G is $(|V(G)| - i\gamma(G) - 1)$-collapsible.*

The proof of Theorem 2.2.2 is based on the minimal exclusion sequences technique. Let G be a graph without isolated vertices. For simplicity, we assume $V(G) = [n]$ and denote $\overline{S} := [n] \setminus S$ for $S \subseteq [n]$. Let I be a maximal independent set in G, say $|I| = i$, such that $\gamma(G;I) = i\gamma(G)$. Without loss of generality, we may assume that $I := [n] \setminus [n-i] = \{n-i+1, \ldots, n\}$.

Note that every facet of $\mathcal{NC}(G)$ is the complement of an edge of G. We define a linear ordering $\prec$ of the facets of $\mathcal{NC}(G)$ as follows. For two edges a_1b_1 and a_2b_2, where $a_i < b_i$ for $i \in [2]$, we denote $a_1b_1 <_L a_2b_2$ if either (i) $b_1 < b_2$ or (ii) $b_1 = b_2$ and $a_1 < a_2$. For two distinct facets σ and τ of $\mathcal{NC}(G)$, we denote $\sigma \prec \tau$ if $\overline{\sigma} <_L \overline{\tau}$.

Claim 2.2.3. *For $\sigma, \sigma' \in \mathcal{NC}(G)$, if $\overline{\sigma} \cap \bar{I} = \overline{\sigma'} \cap \bar{I}$ and $G[\overline{\sigma} \cap \bar{I}]$ contains an edge, then $mes(\sigma) = mes(\sigma')$.*

Proof. Let j be the length of $\mathrm{mes}_{\prec}(\sigma)$. Note that since $G[\overline{\sigma} \cap \bar{I}]$ has an edge, for the $(j+1)$th facet σ_{j+1}, $\overline{\sigma_{j+1}}$ is an edge such that $\overline{\sigma_{j+1}} \subseteq \bar{I}$. By the definition of $\prec$, it also follows that for every $k \in [j+1]$, the kth facet σ_k satisfies $\overline{\sigma_k} \subseteq \bar{I}$. Clearly, $\sigma \cap \bar{I} = \sigma' \cap \bar{I}$. Thus, we have

$$\overline{\sigma_k} \cap \sigma = \overline{\sigma_k} \cap \sigma \cap \bar{I} = \overline{\sigma_k} \cap \sigma' \cap \bar{I} = \overline{\sigma_k} \cap \sigma'.$$

Thus the length of $\mathrm{mes}(\sigma')$ is also j and for every $k \in [j]$, the kth entry of $\mathrm{mes}(\sigma)$ is equal to that of $\mathrm{mes}(\sigma')$. □

Claim 2.2.4. *For every $S \subseteq \bar{I}$,*

$$|S| - |N(S) \cap I| \geq i\gamma(G) - |I|,$$

where $N(S) = \{v \in V(G) : \{u, v\} \in E(G)$ for some $u \in S\}$.

Proof. We take $S \subseteq \bar{I}$ so that (1) $|S| - |N(S) \cap I|$ is minimum, and (2) $|S|$ is maximum subject to (1). By the minimality of $|S| - |N(S) \cap I|$, every element in $\bar{I} \setminus S$ has at most one neighbor in $I \setminus N(S)$. If some $v \in \bar{I} \setminus S$ has exactly one neighbor w in $I \setminus N(S)$, then for $T = S \cup \{w\} \subseteq \bar{I}$, we know $|T| - |N(T) \cap I| = |S| - |N(S) \cap I|$ and $|T| > |S|$, which is a contradiction to the maximality of $|S|$. Thus, every element in $\bar{I} \setminus S$ does not have a neighbor in $I \setminus N(S)$. Since G has no isolated vertex, we conclude $N(S) \cap I = I$. Hence, S dominates I and so $|S| \geq i\gamma(G)$. Thus $|S| - |N(S) \cap I| \geq i\gamma(G) - |I|$. □

By Theorem 2.1.1, it is sufficient to show that

$$|M(\sigma)| \leq |V(G)| - i\gamma(G) - 1 \quad \text{for every } \sigma \in \mathscr{N}\mathscr{C}(G). \tag{2.2.1}$$

For a face $\sigma \in \mathscr{N}\mathscr{C}(G)$, by definition, $G[\overline{\sigma}]$ contains an edge of G. Since I is an independent set, $\overline{\sigma} \cap \bar{I} \neq \emptyset$, and moreover, since I is a maximal independent set, every vertex in $\bar{I}$ has a neighbor in I. Thus $\overline{\sigma} \cap \bar{I}$ has a neighbor in I.

For a face $\sigma \in \mathscr{N}\mathscr{C}(G)$, let $\beta(\sigma) = |N(\overline{\sigma} \cap \bar{I}) \cap \overline{\sigma} \cap I|$. Suppose that $\beta(\sigma) = 0$. Then $G[\overline{\sigma} \cap \bar{I}]$ must have an edge. Consider $\sigma' = \sigma \cup I$. Then $\overline{\sigma} \cap \bar{I} = \overline{\sigma'} \cap \bar{I}$. By Claim 2.2.3, $\mathrm{mes}(\sigma) = \mathrm{mes}(\sigma')$ and therefore, $M(\sigma) = M(\sigma')$. On the other hand, we know $\beta(\sigma') \geq 1$ by the definition of σ'. Thus, it is sufficient check (2.2.1) under the assumption $\beta(\sigma) \geq 1$.

Note that for $v \in \sigma \cap I$, if $v \in M(\sigma)$, then v is a neighbor of some vertex in $\overline{\sigma} \cap \bar{I}$. Thus,

$$\begin{aligned}
|M(\sigma)| &\leq |\sigma \cap \bar{I}| + |N(\overline{\sigma} \cap \bar{I}) \cap (\sigma \cap I)| \\
&= |\bar{I}| - |\overline{\sigma} \cap \bar{I}| + |N(\overline{\sigma} \cap \bar{I}) \cap I| - \beta(\sigma) \\
&\leq |\bar{I}| - i\gamma(G) + |I| - \beta(\sigma) \\
&= |V(G)| - i\gamma(G) - \beta(\sigma),
\end{aligned}$$

where the last inequality holds by applying Claim 2.2.4 to the set $\overline{\sigma} \cap \bar{I}$. As we assumed that $\beta(\sigma) \geq 1$, (2.2.1) follows, and this concludes the proof of Theorem 2.2.2.

Chapter 3

Domination numbers and non-cover complexes of hypergraphs

In this chapter, we study further relations between domination numbers and topology of non-cover complexes of hypergraphs. We recall the main results in this research. The first result is the following.

Theorem 1.3.4. *For every hypergraph H, we have*

$$\tilde{H}_i(\mathscr{N}\mathscr{C}(H)) = 0$$

for all $i \geq |V(H)| - \max\{\left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil, \gamma_{si}(H), \gamma_E(H)\} - 1$.

An immediate consequence of Theorem 1.3.4 is a lower bound on the topological connectivity of independence complexes of hypergraphs.

Corollary 1.3.5. $\eta(I(H)) \geq \max\{\left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil, \gamma_{si}(H), \gamma_E(H)\}$ *for every hypergraph H.*

The second main result is establishing a stronger version of Theorem 1.3.4, proving upper bounds of the Leray numbers of non-cover complexes of hypergraphs.

Theorem 1.3.6. *Let H be a hypergraph with no isolated vertices. Then each of the following holds:*

(a) If $|e| \leq 3$ for every $e \in H$, then $L(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1$.

(b) If $|e| \leq 2$ for every $e \in H$, then $L(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \gamma_{si}(H) - 1$.

(c) $L(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \gamma_E(H) - 1$.

The proof of Theorem 1.3.4 will be given in Section 3.1. In Section 3.2, we present the proof of Theorem 1.3.6. In particular, we show by examples that parts (a) and (b) of Theorem 1.3.6 cannot be improved. (See Example 3.2.3 for (a) and Example 3.2.5 for (b)) In Section 3.3, we give some applications of our results (see Section 3.3.3 and Section 3.3.4) and introduce an open problem.

3.1 Proof of Theorem 1.3.4

3.1.1 Edge-annihilation

Given a hypergraph H and an edge $e \in H$, we define an operation which we call an *edge-annihilation* of e in H:

$$H \neg e := \{f \setminus e \neq \emptyset : e \neq f \in H\}.$$

See Figure 3.1 for an example. We first show relations between the domination paramenters of H and those of $H \neg e$. (i)-(iii) for (2-uniform) graphs of the following lemma were shown in the proof of [35, Theorem 1.2].

Lemma 3.1.1. *Let H be a hypergraph defined on V which has no isolated vertices. Then each of the following holds:*

(i) $\tilde{\gamma}(H \neg e) \geq \tilde{\gamma}(H) - 2|e| + 2$ for every edge $e \in H$ with $|e| \geq 2$.

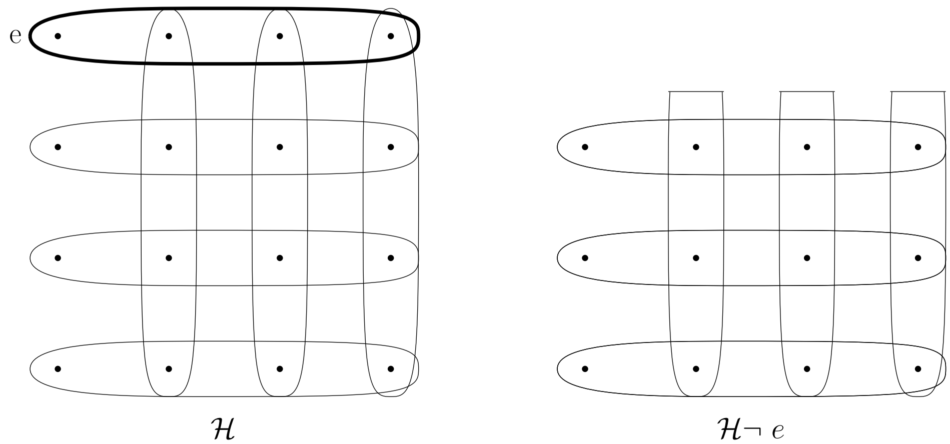


Figure 3.1: $H\neg\ e$ is obtained from H by annihilate the edge e.

(ii) There exists an edge $e \in H$ such that $\gamma_{si}(H\neg\ e) \geq \gamma_{si}(H) - |e| + 1$.

(iii) $\gamma_E(H\neg\ e) \geq \gamma_E(H) - |e| + 1$ for every edge e with $|e| \geq 2$.

(iv) Let e be an edge in H, and let H' be the hypergraph obtained from $H - e$ by deleting all isolated vertices. Then

$$\gamma_E(H') \geq \gamma_E(H) - f(e),$$

where $f(e) = 1$ if there is an isolated vertex in $H - e$ and $f(e) = 0$ otherwise.

Proof. (i) Take any edge $e \in H$ with $|e| \geq 2$, and let $S \subset V(H\neg\ e)$ be a minimum set which dominates $V \setminus e$ in $H\neg\ e$. Then $S \cup e$ dominates V in H, thus we have

$$\tilde{\gamma}(H\neg\ e) + |e| = |S \cup e| \geq \tilde{\gamma}(H).$$

Since $|e| \geq 2$, we obtain $\tilde{\gamma}(H\neg\ e) \geq \tilde{\gamma}(H) - |e| \geq \tilde{\gamma}(H) - 2|e| + 2$.

(ii) Let A be a strong independent set in H such that $\gamma_{si}(H) = \gamma(H, A)$. Take a vertex $v \in A$ and an edge $e \in H$ which contains the vertex v. Such an edge exists since H contains no isolated vertex in H. Let A' be the set of all vertices in A which is not dominated by $e\setminus\{v\}$. It is clear that A' is a strong independent set in $H\neg e$. Now let $S\subset V(H\neg e)$ be a minimum set which dominates A' in $H\neg e$. Then $S\cup(e\setminus\{v\})$ dominates A in H, and hence we have

$$\gamma_{si}(H\neg e)+(|e|-1)\geq|S\cup(e\setminus\{v\})|\geq\gamma_{si}(H).$$

(iii) Let e be an edge in H with $|e|\geq 2$, and let $\mathscr{F}'$ be a subgraph of $H\neg e$ such that $\bigcup_{F\in\mathscr{F}'}F$ dominates $V\setminus e$ in $H\neg e$ and $|\mathscr{F}'|=\gamma_E(H\neg e)$. If $\mathscr{F}$ be the corresponding subgraph in H, then $e\cup(\bigcup_{F\in\mathscr{F}}F)$ dominates V in H. Thus we have

$$\gamma_E(H\neg e)+(|e|-1)\geq\gamma_E(H\neg e)+1\geq\gamma_E(H).$$

(iv) If there is no isolated vertex in $H-e$, then $H'=H-e$. Hence it is obvious that $\gamma_E(H')\geq\gamma_E(H)$. Suppose there is an isolated vertex in $H-e$, and let $\mathscr{F}$ be a subgraph of H' such that $\bigcup_{F\in\mathscr{F}}F$ dominates $V(H')$ in H' and $|\mathscr{F}|=\gamma_E(H')$. Clearly, $e\cup(\bigcup_{A\in\mathscr{F}}A)$ dominates $V(H)$, thus we have $\gamma_E(H')+1\geq\gamma_E(H)$.

□

3.1.2 Non-cover complexes for hypergraphs

The proof of Theorem 1.3.4 is based on the Mayer-Vietoris exact sequence. Let X be a simplicial complex and let A and B be complexes such that $X=A\cup B$. Then the following sequence is exact:

$$\cdots\to H_i(A\cap B)\to H_i(A)\oplus H_i(B)\to H_i(X)\to H_{i-1}(A\cap B)\to\cdots$$

In particular, for any integer i_0, if $H_i(A)=H_i(B)=H_{i-1}(A\cap B)=0$ for all $i\geq i_0$ then $H_i(X)=0$ for all $i\geq i_0$.

Lemma 3.1.2. *For every edge e in a hypergraph H, we have*

$$\mathscr{N}\mathscr{C}(H) = \mathscr{N}\mathscr{C}(H-e) \cup \Delta_{e^c},$$

where Δ_X is a simplex on X and e^c is the complement of e.

Proof. It is obvious that each of $\mathscr{N}\mathscr{C}(H-e)$ and Δ_{e^c} is contained in $\mathscr{N}\mathscr{C}(H)$. For every $A \in \mathscr{N}\mathscr{C}(H) \setminus \mathscr{N}\mathscr{C}(H-e)$, we claim that $A \subset e^c$. If not, it must be that $A \cap e \neq \emptyset$. Since $A \in \mathscr{N}\mathscr{C}(H)$, there exists an edge $f \neq e$ such that $A \subset f^c$, which implies that $A \in \mathscr{N}\mathscr{C}(H-e)$. This completes the proof. $\square$

Observation 3.1.3. *Suppose there are two edges e and f in a hypergraph H such that $f \subsetneq e$. Since $\Delta_{e^c} \subset \Delta f^c \subset \mathscr{N}\mathscr{C}(H-e)$, we can deduce that $\mathscr{N}\mathscr{C}(H) = \mathscr{N}\mathscr{C}(H-e)$ from Lemma 3.1.2.*

Lemma 3.1.4. *For any inclusion-minimal edge $e \in H$, we have*

$$\mathscr{N}\mathscr{C}(H-e) \cap \Delta_{e^c} = \mathscr{N}\mathscr{C}(H\neg\, e).$$

Proof. It is obvious that $\mathscr{N}\mathscr{C}(H\neg\, e) \subset \Delta_{e^c}$. If $\sigma \in \mathscr{N}\mathscr{C}(H\neg\, e)$, then there exists an edge $f \nsubseteq e$ in H such that $f \setminus e \in H\neg\, e$ and $f \setminus e \subset V(H\neg\, e) \setminus \sigma$. Since $V(H\neg\, e) = V(H) \setminus e$, we observe that $f \subset V(H) \setminus \sigma$, which implies that $\sigma \in \mathscr{N}\mathscr{C}(H-e)$. For the opposite direction, let $\sigma \in \mathscr{N}\mathscr{C}(H-e) \cap \Delta_{e^c}$. Since $\sigma \in \mathscr{N}\mathscr{C}(H-e)$, there exists an edge $f \nsubseteq e$ such that $f \subset V(H) \setminus \sigma$. Since $\sigma \in \Delta_{e^c}$, it is clear that $f \setminus e \subset V(H\neg\, e) \setminus \sigma$. Then we observe that $\sigma \in \mathscr{N}\mathscr{C}(H\neg\, e)$ since $\emptyset \neq f \setminus e \in H\neg\, e$. $\square$

From Lemma 3.1.2 and Lemma 3.1.4, we obtain the following exact sequence:

$$\cdots \to H_i(\mathscr{N}\mathscr{C}(H-e)) \to H_i(\mathscr{N}\mathscr{C}(H)) \to H_{i-1}(\mathscr{N}\mathscr{C}(H\neg\, e)) \cdots. \quad (3.1.1)$$

Theorem 3.1.5. *For every hypergraph $H \neq \emptyset$, $\tilde{H}_i(\mathscr{N}\mathscr{C}(H)) = 0$ for all $i \geq |V(H)| - \max\{\left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil, \gamma_{si}(H), \gamma_E(H)\} - 1$.*

Proof. We use induction on $|V(H)|+|H|$. Let

$$g(H) = \max\{\left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil, \gamma_{si}(H), \gamma_E(H)\}.$$

The base cases are when there is exactly one edge in H and when $\{v\} \in H$ for every vertex v.

Suppose $H = \{e\}$. If there are isolated vertices in H, then $\mathcal{NC}(H)$ is a simplex which is contractible. If not, then $|V(H)| = |e|$, $g(H) = |e|-1$ and $\mathcal{NC}(H)$ is a void complex. In each case the statement is true. If $\{v\} \in H$ for every vertex v, then $g(H) = 0$ and $\mathcal{NC}(H)$ is the boundary of the simplex on $V(H)$. It is clear that $H_i(\mathcal{NC}(H)) = 0$ for all $i \geq |V(H)|-1$. Therefore we may assume that there exist at least two edges in H and that there is a vertex $v \in V(H)$ with $\{v\} \notin H$.

Suppose v is an isolated vertex in H. Then $\mathcal{NC}(H)$ is a cone with apex v, thus is contractible. Hence we may assume that there is no isolated vertex. In particular, each vertex in a strong independent set in H is contained in an edge of size at least 2. Moreover, we may assume that every edge in H is inclusion-minimal. Suppose H contains an edge e which is not inclusion-minimal. If $H-e$ has an isolated vertex v, then $\mathcal{NC}(H-e)$ is a cone with apex v. Otherwise, we have $g(H) \leq g(H-e) < \infty$ and $|V(H-e)| = |V(H)|$, implying that

$$|V(H-e)| - g(H-e) - 1 \leq |V(H)| - g(H) - 1.$$

By the induction hypothesis, it follows that $H_i(\mathcal{NC}(H-e)) = 0$ for all $i \geq |V(H)| - g(H) - 1$. In either case, by Observation 3.1.3, we have $H_i(\mathcal{NC}(H)) = H_i(\mathcal{NC}(H-e)) = 0$ for all $i \geq |V(H)| - g(H) - 1$.

Now by the exact sequence (3.1.1), it is sufficient to show that for some edge $e \in H$,

$$H_i(\mathcal{NC}(H-e)) = 0 \quad \text{and} \quad H_{i-1}(\mathcal{NC}(H\neg e)) = 0 \qquad (3.1.2)$$

holds for all $i \geq |V(H)| - g(H) - 1$. The first part of (3.1.2) was implicitly proved by the above. It remains to show the second part of (3.1.2).

It is clear that $H \neg\, e$ has no isolated vertex for any edge e. Thus we have $g(H \neg\, e) < \infty$ from the assumption that $g(H) < \infty$. Let A be a strong independent set in H such that $\gamma_{si}(H) = \gamma(H,A)$. Take a vertex $v \in A$ and an edge $e \in H$ which contains the vertex v. Note that $|e| \geq 2$. Then by Lemma 3.1.1, we can find an edge e which satisfies $\tilde{\gamma}(H) - 2|e| + 2 \leq \tilde{\gamma}(H \neg\, e) < \infty$, $\gamma_{si}(H) - |e| + 1 \leq \gamma_{si}(H \neg\, e) < \infty$, and $\gamma_E(H) - |e| + 1 \leq \gamma_E(H) < \infty$. Thus we obtain

$$\begin{aligned} V(H \neg\, e) - g(H \neg\, e) - 1 &\leq (|V(H)| - |e|) - (g(H) - |e| + 1) - 1 \\ &= |V(H)| - g(H) - 2 \end{aligned}$$

By the induction hypothesis, it follows that $H_{i-1}(\mathscr{N}\mathscr{C}(H \neg\, e)) = 0$ for all $i \geq |V(H)| - g(H) - 1$. This completes the proof. $\square$

3.2 Lerayness of non-cover complexes

3.2.1 Total domination numbers

If we take a vertex subset $W \subset V(X)$, then it is clear that the induced subcomplexes satisfies the equalities $X[W] = A[W] \cup B[W]$ and $(A \cap B)[W] = A[W] \cap B[W]$. This implies the following proposition about Leray numbers.

Proposition 3.2.1. *Let $X = A \cup B$ and n be a positive integer. Then*

$$L(X) \leq \max\{L(A), L(B), L(A \cap B) + 1\}.$$

Proof. Let $n = \max\{L(A), L(B), L(A \cap B) + 1\}$. For any vertex subset $W \subset V(X)$, we know that $H_i(A) = H_i(B) = H_{i-1}(A \cap B) = 0$ for all $i \geq n$. By applying the

Mayer-Vietoris sequence for $X[W] = A[W] \cup B[W]$ and $(A \cap B)[W] = A[W] \cap B[W]$, we obtain that $H_i(X[W]) = 0$ for all $i \geq n$. This shows $L(X) \leq n$. $\square$

The proof of the first part of Theorem 1.3.6 is based on the exact sequence (3.1.1) and Proposition 3.2.1.

Theorem 3.2.2. *Let H be a hypergraph defined on V. Suppose H contains no isolated vertices and every $e \in H$ has size $|e| \leq 3$. Then the non-cover complex $\mathscr{N}\mathscr{C}(H)$ is $(|V| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1)$-Leray.*

Proof. We use induction on $|H| + |V(H)|$. If H has no edge, then the statement is true since $\mathscr{N}\mathscr{C}(H)$ is a void complex. If $\{v\} \in H$ for every vertex $v \in H$, then the statement is true since $\tilde{\gamma}(H) = 0$ and $\mathscr{N}\mathscr{C}(H)[W]$ is the boundary of the simplex on W for every $W \subset V(H)$. Note that the boundary of the simplex on W is $(|W| - 1)$-Leray.

Take an edge e with $|e| \geq 2$. If e is not inclusion-minimal, then $\mathscr{N}\mathscr{C}(H) = \mathscr{N}\mathscr{C}(H - e)$ by Observation 3.1.3. If e is inclusion-minimal, then we apply the exact sequence (3.1.1). By Lemma 3.1.1, we have $\tilde{\gamma}(H \neg e) \geq \tilde{\gamma}(H) - 2|e| + 2$, thus $\mathscr{N}\mathscr{C}(H \neg e)$ is $(|V| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 2)$-Leray by the induction hypothesis. Therefore it is sufficient to show that $\mathscr{N}\mathscr{C}(H - e)$ is $(|V| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1)$-Leray.

Assume that $H - e$ has no isolated vertices. Then it is obvious to have $\tilde{\gamma}(H) \leq \tilde{\gamma}(H - e) < \infty$. By the induction hypothesis, $\mathscr{N}\mathscr{C}(H - e)$ is $(|V| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1)$-Leray. Thus we may assume that deleting the edge e from H produces k isolated vertices. Let H' be the hypergraph obtained from $H - e$ by deleting all isolated vertices. We claim that if $|e| \leq 3$, then $\tilde{\gamma}(H') + 2k \geq \tilde{\gamma}(H)$. Then by induction, since $|V(H')| = |V| - k$, we obtain that $\mathscr{N}\mathscr{C}(H')$ is $(|V| - \tilde{\gamma}(H) - 1)$-Leray.

Let $S \subset V(H')$ be the minimum set which dominates $V(H')$, i.e. $|S| = \tilde{\gamma}(H')$. If $k = 1$, let $v \in e$ is the only isolated vertex in $H - e$. Then $S \cup (e \setminus \{v\})$ dominates H, thus $\tilde{\gamma}(H') + |e| - 1 \geq \tilde{\gamma}(H)$. Since $|e| \leq 3$, we obtain

$$\tilde{\gamma}(H') + 2 \geq \tilde{\gamma}(H') + |e| - 1 \geq \tilde{\gamma}(H).$$

If $k \geq 2$, we have $\tilde{\gamma}(H') + |e| \geq \tilde{\gamma}(H)$ since $S \cup e$ dominates H. Since $|e| \leq 3 < 2k$, the claim is true because

$$\tilde{\gamma}(H') + 2k > \tilde{\gamma}(H') + |e| \geq \tilde{\gamma}(H).$$

This completes the proof. $\square$

The following example shows Theorem 3.2.2 is the best possible in the sense that the condition $|e| \leq 3$ for every edge cannot be improved.

Example 3.2.3. *For $r \geq 3$, consider an r-uniform hypergraph*

$$H_r := \{\{(i,1), \ldots, (i,r))\} : 1 \leq i \leq r\} \cup \{\{(1,i), \ldots, (r,i)\} : 2 \leq i \leq r\}$$

defined on $\{1, \ldots, r\} \times \{1, \ldots, r\}$. Clearly $A = \{(i,1) : 1 \leq i \leq r\}$ is a strong independent set in H_r. Since $\{(i,1), \ldots, (i,r))\}$ is the only edge in H_r which contains the vertex $(i,1)$ for each i, the set

$$\{(i,j) : 1 \leq i \leq r, 2 \leq j \leq r\}$$

is the unique set in H_r which dominates A. This shows that $\gamma_{si}(H_r) \geq (r-1)r$, and hence $|V(H_r)| - \gamma_{si}(H_r) - 1 \leq r - 1$. We will show that $\mathscr{NC}(H_r)$ is not $(2r-4)$-Leray. Note that $r - 1 \leq 2r - 4$ for every $r \geq 3$.

However, the induced subcomplex $\mathscr{NC}(H_r)[W]$ is the boundary of $(2r-3)$-simplex where

$$W := \{(i,1) : 1 \leq i \leq r-1\} \cup \{(r,i,) : 2 \leq i \leq n\},$$

thus $H_{2r-4}(\mathscr{NC}(H_r)[W]) \neq 0$. See Figure 3.2 for an illustration when $r = 4$.

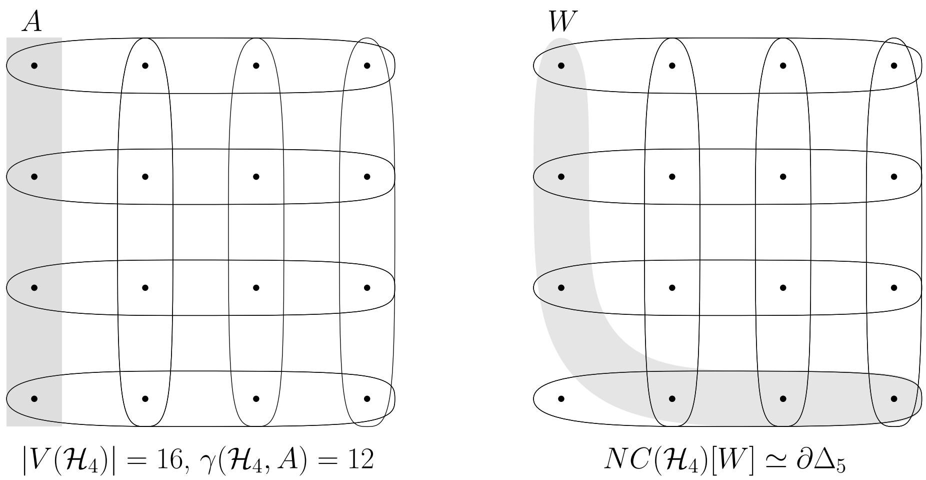


Figure 3.2: $|V(H)_4| - \gamma_{si}(H) - 1 \leq 3$ but $\mathscr{N}\mathscr{C}(H_4)$ is not 4-Leray.

3.2.2 Independent domination numbers

For a hypergraph H defined on V and for each $v \in V$, let

$$H\neg v := \{e \setminus \{v\} : e \in H\}.$$

Note that $\mathscr{N}\mathscr{C}(H\neg v) = \mathscr{N}\mathscr{C}(H)[V \setminus \{v\}]$.

Theorem 3.2.4. *Let H be a hypergraph defined on V. Suppose H contains no isolated vertices and every $e \in H$ has size $|e| \leq 2$. Then the non-cover complex $\mathscr{N}\mathscr{C}(H)$ is $(|V| - \gamma_{si}(H) - 1)$-Leray.*

Proof. We use induction on $|V(H)|$. It is sufficient to show the inequality $\gamma_{si}(H\neg v) \geq \gamma_{si}(H) - 1$ for every $v \in V$. Then we have

$$|V(H\neg v)| - \gamma_{si}(H\neg v) - 1 \leq |V| - \gamma_{si}(H) - 1,$$

and it follows from the induction hypothesis that $\mathscr{N}\mathscr{C}(H)[V(H) \setminus \{v\}]$ is $(|V(H)| - \gamma_{si}(H) - 1)$-Leray.

Let A be a strong independent set in H such that $\gamma(H,A) = \gamma_{si}(H)$. We will show that for every vertex v there exists a set S in $H\neg v$ such that $|S| \leq \gamma_{si}(H\neg v)$ and $|S|+1 \geq \gamma_{si}(H)$. If $v \notin A$, let A' be the set of all vertices in A which are not dominated by v. Let S be the minimum set which dominates A' in $H\neg v$. Then $S \cup \{v\}$ dominates A in H. If $v \in A$, then there must be a vertex $u \notin A$ such that $\{u,v\} \in H$. Let S be the minimum set which dominates $A \setminus \{v\}$ in $H\neg v$. Then $S \cup \{u\}$ dominates A in H. In either case, we have $|S| \leq \gamma_{si}(H\neg v)$ and $|S|+1 \geq \gamma_{si}(H)$. It follows that $\gamma_{si}(H\neg v) \geq \gamma_{si}(H)-1$ as desired. □

The following example shows that the condition $|e| \leq 2$ for every edge $e \in H$ in Theorem 3.2.4 cannot be improved.

Example 3.2.5. *Let H be a hypergraph defined on $V = \{v_1,\dots,v_9\}$, whose edges are*

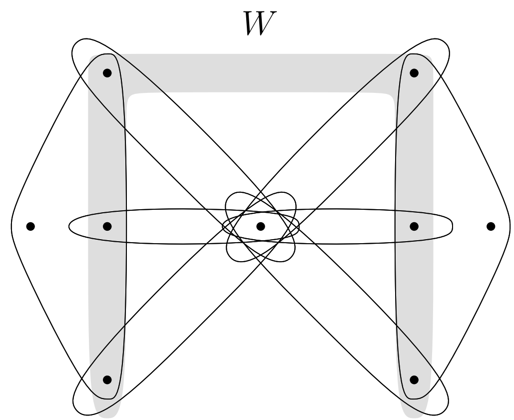
$$\begin{aligned} H = \{&\{v_1,v_2,v_3,v_4\},\{v_2,v_5\},\{v_3,v_5\},\{v_4,v_5\},\\ &\{v_5,v_6\},\{v_5,v_7\},\{v_5,v_8\},\{v_6,v_7,v_8,v_9\}\}. \end{aligned}$$

We will show that $\mathcal{NC}(H)$ is not $(|V(H)| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1)$-Leray.

We first claim that $\tilde{\gamma}(H) \geq 7$. Since $\{v_1,v_2,v_3,v_4\}$ is the only edge which contains the vertex v_1, we need $\{v_2,v_3,v_4\}$ to dominate v_1 in H. Similarly, we need $\{v_6,v_7,v_8\}$ to dominate v_9 in H. Therefore every set $S \subset V$ which dominates V in H should contain the set $T = \{v_2,v_3,v_4,v_6,v_7,v_8\}$. However, $S \setminus T \neq \emptyset$ since T does not dominate each $v_i \in T$ in H, and hence $|S| \geq 7$. This shows that $\tilde{\gamma}(H) \geq 7$, implying that $|V| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1 \leq 4$. However, the induced subcomplex $\mathcal{NC}(H)[T]$ is the boundary of the simplex on T, thus $H_4(\mathcal{NC}(H)[T]) \neq 0$. See Figure 3.3 for an illustration.

3.2.3 Edgewise-domination numbers

For $\gamma_E(H)$, we can prove the following without any restriction on the size of edges in H.



$|V(\mathcal{H})| = 9,\ \tilde{\gamma}(\mathcal{H}) = 7,\ NC(\mathcal{H})[W] \simeq \partial\Delta_5$

Figure 3.3: $|V(H)| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1 = 4$ but $\mathscr{N}\mathscr{C}(H)$ is not 4-Leray.

Theorem 3.2.6. *Let H be a hypergraph with no isolated vertices. Then the non-cover complex $\mathscr{N}\mathscr{C}(H)$ is $(|V| - \gamma_E(H) - 1)$-Leray.*

Proof. Take arbitrary vertex $v \in V$. It is clear that $H \neg v$ has no isolated vertex, thus it is sufficient to show that $\gamma_E(H \neg v) \geq \gamma_E(H) - 1$. Let $\mathscr{F}'$ be a subgraph of $H \neg v$ such that $\bigcup_{F \in \mathscr{F}'} F$ dominates $V \setminus \{v\}$ in $H \neg v$ and $|\mathscr{F}'| = \gamma_E(\mathscr{F} \neg e)$. Let $\mathscr{F}$ be the corresponding subgraph of H. For any edge e which contains v, the union $e \cup (\bigcup_{A \in \mathscr{F}} A)$ dominates V in H. It follows that $\gamma_E(H \neg v) + 1 \geq \gamma_E(H)$. $\square$

3.3 Remarks

3.3.1 Independent domination numbers of hypergraphs

In a graph G, the definitions of an independent set and a strong independent set are equivalent. However, in Theorem 1.3.4 and Corollary 1.3.5, the strong sense of the independent domination number is necessary. Let us define

$$\gamma_i(H) := \max\{\gamma(H,A) : A \text{ is an independent set in } H\}.$$

Then $\gamma_i(H) = \gamma_{si}(H)$ when H is a graph, but if H is a hypergraph, the equality does not hold in general. The following examples shows that Theorem 1.3.4 and Corollary 1.3.5 do not hold if we replace $\gamma_{si}(H)$ with $\gamma_i(H)$.

Example 3.3.1. *Let H be a hypergraph consists of exactly one edge e of cardinality $m \geq 3$. Since every strong indepenent set in H is a single vertex, we have $\gamma_{si}(H) = m-1$. On the other hand, if we take any independent set A of size at least 2, then we need to use all vertices in e to dominate A, hence $\gamma_i(H) = m$.*

We will show that $\eta(I(H)) < \gamma_i(H)$ and $H_{|V(H)|-\gamma_i(H)-1}(\mathscr{N}\mathscr{C}(H)) \neq 0$. Since any proper subset of e is an independent set while e itself is not, it follows that $I(H)$ is the boundary of the simplex on e. Hence we have $\eta(I(H)) = m-1$. On the other hand, $\mathscr{N}\mathscr{C}(H)$ is a void complex, thus we have $H_{-1}(I(H)) \neq \emptyset$, while $|V(H)| - \gamma_i(H) - 1 = -1$.

3.3.2 Independence complexes of hypergraphs

Corollary 1.3.5 can be proved independently, without applying the duality theorem to Theorem 1.3.4. For this, we use a modified definition of edge-annihilations:

$$H \diamond e := \{f \subset V(H) \setminus \Gamma_0(e) : f \neq \emptyset \text{ and } \exists f' \subset e \text{ s.t. } f \cup f' \in H\},$$

where $\Gamma_0(W)$ as the set of all vertices which are strongly dominated by W. The following lemma allows us to apply the Mayer Vietoris exact sequence to show that the sequence

$$\cdots \to H_{i-|e|+1}(I(H \diamond e)) \to H_i(I(H)) \to H_i(I(H-e)) \to \cdots . \tag{3.3.1}$$

is exact. Here, $K * L$ is the *join* of two simplicial complexes K and L which is defined as

$$K * L = \{\sigma \cup \tau : \sigma \in K, \tau \in L\}.$$

Lemma 3.3.2. *For any inclusion-minimal edge $e \in H$, we have*

$$I(H-e) = I(H) \cup (\Delta_e * I(H \diamond e)) \text{ and } I(H) \cap (\Delta_e * I(H \diamond e)) = \partial\Delta_e * I(H \diamond e),$$

where Δ_X is a simplex on X and $\partial \cdot$ is the boundary operation.

The following lemma states an analogue of Lemma 3.1.1. Each can be shown by the proof of Lemma 3.1.1 with a slight modifications.

Lemma 3.3.3. *For every hypergraph H with no isolated vertex, each of the following holds:*

1. $\tilde{\gamma}(H \diamond e) \geq \tilde{\gamma}(H) - 2|e| + 2$ *for every edge $e \in H$ with $|e| \geq 2$.*

2. *There exists an edge $e \in H$ such that $\gamma_{si}(H \diamond e) \geq \gamma_{si}(H) - |e| + 1$.*

3. $\gamma_E(H \diamond e) \geq \gamma_E(H) - |e| + 1$ *for every edge e with $|e| \geq 2$.*

3.3.3 General position complexes

Let P be a set of points in $\mathbb{R}^d$ and let $G(P)$ denote the simplicial complex consisting of those subsets of P which are in general position. Furthermore, let $\varphi(P)$ denote the largest subset of P in general position, that is, $\varphi(P) = \dim(G(P)) + 1$ The following was shown in [24]. Here we give a short proof using Corollary 1.3.5.

Proposition 3.3.4. *Let $d \geq 1$ and $k \geq -1$ be integers.For any set of points $P \subset \mathbb{R}^d$, if $\varphi(P) \geq d\binom{2k+2}{d}$, then $G(P)$ is k-connected.*

Proof. To do this, we define a hypergraph

$$H_P := \{Q \subset P : |Q| = d+1, Q \text{ is contained in a } (|Q|-2)\text{-flat.}\}.$$

Note that $G(P) = I(H_P)$. By Corollary 1.3.5, it is sufficient to show that $\tilde{\gamma}(H_P) > 2k+2$. Let $A \subset P$ be a set of points in general position with $|A| > d\binom{2k+2}{d}$. Observe that if a vertex $v \in P$ can be dominated by S with $|S| \le d$, then v must be contained in a k-flat ($k \le d$) spanned by S. Since any $k+2$ points on a k-flat is not in general position, it follows that every d vertices in H_P can dominate at most d vertices of A. As an immediate consequence, any $2k+2$ vertices in H_P can dominate at most $d\binom{2k+2}{d}$ vertices of A, which shows that $\tilde{\gamma}(H_P) \ge \gamma(H_P, A) > 2k+2$. □

Let $g_d(k)$ be the minimum integer such that for any $P \subset \mathbb{R}^d$, if $\varphi(P) \ge g_d(k)$ then $G(P)$ is k-connected. The proof of Proposition 3.3.4 yields an upper bound on $g_d(k)$ which is in $O(|P|^d)$. Here we give an example which shows that this bound is asymptotically tight, in other words, we show that the function $g_d(k)$ is in $\Theta(|P|^d)$.

Example 3.3.5. *Let A be a set of $n \ge d$ points in $\mathbb{R}^d$ which are in general position. Let H be the set of $N = \binom{n}{d}$ hyperplanes spanned by the d-tuples of points in A. Let B be a set of N points in general position in $\mathbb{R}^d$ such that $|B \cap h| = 1$ for every hyperplane $h \in H$. Let $P = A \cup B$. Notice that $|P| = N + n$ and $\varphi(P) = N + d - 1$.*

Claim 3.3.6. *$\tilde{H}_i(G(P)) = 0$ if and only if $i \ne n-1$.*

Before we prove Claim 3.3.6 we note that $G(P)$ is the independence complex of the $(d+1)$-uniform hypergraph $F = (P, E)$ where each edge of E corresponds to a d-tuple $S \subset A$ together with the corresponding point $x \in B$ lying in the hyperplane spanned by S. It is easily seen that $\gamma(F) = 2n-1 > 2(n-1)$, so by Corollary 1.3.5 we get that $\tilde{H}_k(G(P)) = 0$ for all $k \le n-2$.

Proof of Claim 3.3.6. Consider $\mathscr{N}\mathscr{C}(H_P)$, then by Theorem 1.2.8, we have

$$\tilde{H}_i(\mathscr{N}\mathscr{C}(H_P)) \cong \tilde{H}_{|P|-i-3}(G(P)).$$

We will use the Nerve theorem to determine $\tilde{H}_i(\mathscr{N}\mathscr{C}(H_P))$. The inclusion maximal simplices of $\mathscr{N}\mathscr{C}(H_P)$ are formed by the complements of the edges of H_P which can be labeled by the d-tuples of A. Let $X_1,\ldots,X_N$ denote these simplices. Note that each X_i has dimension $N+n-d-2$. Clearly we have $\mathscr{N}\mathscr{C}(H_P) = \bigcup_{i=1}^N X_i$, $\bigcap_{j\neq i} X_j \neq \emptyset$, and $\bigcap_{i=1}^N X_i = \emptyset$. Therefore, by the Nerve theorem K is homotopy equivalent to the boundary of the $(N-1)$-simplex which gives us $\tilde{H}_i(\mathscr{N}\mathscr{C}(H_P)) = 0$ if and only if $i \neq N-2$. Since $|P| = N+n$, the claim now follows by Alexander duality. $\square$

3.3.4 Rainbow covers of hypergraphs

As an application of Theorem 1.3.6, we can obtain the following result for "rainbow" covers. Given m covers $X_1,\ldots,X_m$ in a hypergraph H, a *rainbow cover* is a cover $X = \{x_{i_1},\ldots,x_{i_l}\}$ in H such that $x_{i_j} \in X_{i_j}$ for each $j \in \{1,\ldots,l\}$.

Corollary 3.3.7. *Let H be a hypergraph with no isolated vertices. Then each of the following holds:*

1. *Suppose that every edge in H has size at most 3. Then for every $|V(H)| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil$ covers in H, there exists a rainbow cover.*
2. *Suppose that every edge in H has size at most 2. Then for every $|V(H)| - \gamma_{si}(H)$ covers in H, there exists a rainbow cover.*
3. *For every $|V(H)| - \gamma_E(H)$ covers in H, there exists a rainbow cover.*

Corollary 3.3.7 follows from the topological colorful Helly theorem. Here we state the specific case of a famous result by Kalai and Meshulam [26].

Theorem 3.3.8 (Topological colorful Helly theorem)**.** *Let X be a d-Leray simpicial complex with a vertex partition $V(X) = V_1 \cup \cdots \cup V_m$ with $m \geq d+1$. If $\sigma \in X$ for every $\sigma \subset V(X)$ with $|\sigma \cap V_i| = 1$, then there exists $I \subset \{1, \ldots, m\}$ of size at least $m-d$ such that $\bigcup_{i \in I} V_i \in X$.*

The following examples shows the tightness of Corollary 3.3.7.

Example 3.3.9. *Let G be a graph on $2n$ vertices where $E(G)$ consists of n pairwise disjoint edges, say $E(G) = \{u_1v_1, \ldots u_nv_n\}$. It is easy to see that $\frac{\tilde{\gamma}(G)}{2} = \gamma_{si}(G) = \gamma_E(G) = n$. However, if we consider $n-1$ covers $A_1 = \ldots = A_{n-1} = \{u_1, \ldots, u_n\}$, then there is no rainbow cover: if v_i be the vertex which is not represented by any of A_j, then the edge u_iv_i is not covered.*

Example 3.3.10. *Let H be a 3-uniform* sunflower *with n petals and the center $\{v\}$. That is, $H = \{e_1, \ldots e_n\}$ such that $e_i \cap e_j = \{v\}$ for all $i \neq j$. Let $e_i = \{v, u_i, w_i\}$. For each i, we need v and w_i to dominate u_i and we need v and u_i to dominate w_i. Thus we should have all vertices to dominate whole hypergraph, i.e. $\tilde{\gamma}(G) = 2n+1$. Hence*

$$|V(H)| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil = (2n+1) - (n+1) = n.$$

Now consider $n-1$ covers $A_1 = \ldots = A_{n-1} = \{u_1, \ldots, u_n\}$. Then there is no rainbow cover: if u_i is the vertex which is not represented by any of A_j, then the edge $\{v, u_i, w_i\}$ is not covered.

3.3.5 Collapsibility of non-cover complexes of hypergraphs

It would be an interesting research problem to establish a generalization of Theorem 2.2.2 for more general hypergraphs. Here we give an explicit statement of the conjecture for the collapsiblity number of the non-cover complex of a hypergraph.

Conjecture 3.3.11. *Let H be a hypergraph with no isolated vertices. Then each of the following holds:*

(a) If $|e| \leq 3$ for every $e \in H$, then $C(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \left\lceil \frac{\tilde{\gamma}(H)}{2} \right\rceil - 1$.

(b) If $|e| \leq 2$ for every $e \in H$, then $C(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \gamma_{si}(H) - 1$.

(c) $C(\mathscr{N}\mathscr{C}(H)) \leq |V(H)| - \gamma_E(H) - 1$.

As the last remark, we point out that proving Conjecture 3.3.11 can enhance Corollary 3.3.7. If we replace d-Leray condition with d-collapsible condition in Theorem 4.4.2, then we have a stronger "rainbow set statement" which follows from Theorem 1.2.5.

Theorem 3.3.12. *Let X be a d-collapsible simpicial complex with a vertex partition $V(X) = V_1 \cup \cdots \cup V_m$ with $m \geq d+1$. If $\sigma \in X$ for every $\sigma \subset V(X)$ with $|\sigma \cap V_i| = 1$, then there exist $I \subset \{1, \ldots, m\}$ of size at least $m-d$ and a set W with $|W| = d$ and $|W \cap V_i| = 1$ for each $i \notin I$ such that $W \cup (\bigcup_{i \in I} V_i) \in X$.*

Chapter 4

Rainbow independent sets

In this chapter, we study Problem 1.4.1 for some graph classes which are different from the class of line graphs. Two most important results are the following.

Theorem 1.4.8. *$f_{\mathscr{F}(H)}(n,n) < \infty$ for every positive integer n if and only if H is either K_r or K_r^- for some r.*

Theorem 1.4.9. *For the class of all graphs with maximum degree at most k, $f_{\mathscr{D}(k)}(n,m) \leq (m-1)k+1$ for all $m \leq n$ and $f_{\mathscr{D}(k)}(n,n) = \lceil \frac{k+1}{2} \rceil + 1$ for $n \leq 3$.*

The proof of Theorem 1.4.8 appears in Section 4.1.4, and Theorem 1.4.9 will be proved in Section 4.3 and Section 4.4.

4.1 Graphs avoiding certain induced subgraphs

In this section, we study some graph classes of the form $\mathscr{F}(H_1, \ldots, H_t)$.

4.1.1 Claw-free graphs

The graph $K_{1,3}$ is called a "claw". Line graphs are claw-free, and some results on line graphs go over also to claw-free graphs. This turns out not to be the case with regard to our problem.

Theorem 4.1.1. *Let $2 \le m \le n$. Then:*

(a) $f_{\mathscr{F}(K_{1,3})}(n,2) = 2$ *for all* $n \ge 3$.

(b) $f_{\mathscr{F}(K_{1,3})}(n,m) = m$ *for* $m \le \lceil \frac{n}{2} \rceil$.

(c) $f_{\mathscr{F}(K_{1,3})}(n,m) = \infty$ *for* $m > \max(\lceil \frac{n}{2} \rceil, 2)$.

Proof.

(a) By (1.4.1) it suffices to prove $f_{\mathscr{F}(K_{1,3})}(n,2) \le 2$. Let G be claw-free, and let I_1, I_2 be independent sets in G. If $I_1 \cap I_2 \neq \emptyset$, then each of I_1, I_2 contains a rainbow independent set of size 2. Hence we may assume that I_1 and I_2 are disjoint. Since G is claw-free and $n \ge 3$, there must be a pair of non-adjacent vertices $v_1 \in I_1$ and $v_2 \in I_2$. Then $\{v_1, v_2\}$ form a rainbow independent set of size 2.

(b) It suffices to prove $f_{\mathscr{F}(K_{1,3})}(n,m) \le m$. We apply induction on $n+m$. The base case $m = 2$ has been proved in (a). Suppose $3 \le m \le \lceil \frac{n}{2} \rceil$. Let G be claw-free, and let $I_1, \ldots, I_m$ be independent sets in G, where $m \le \lceil \frac{n}{2} \rceil$. Pick a vertex $v \in I_m$. For every I_j, $j < m$, either $v \in I_j$ or v has at most two neighbors in I_j. Delete the vertex v and all its neighbors, and consider $I'_j = I_j \setminus N[v]$ for $j < m$. There are $m-1$ independent sets of size at least $n-2$, and we note that $m - 1 < \lceil \frac{n-2}{2} \rceil$ since $m < \frac{n}{2}$. By the induction hypothesis, there exists a rainbow independent set I of size $m-1$. Since $I \subseteq V \setminus N[v]$, $I \cup \{v\}$ is a rainbow independent set of size m.

(c) For every t and $n \ge 5$ we construct a claw-free graph G along with t independent sets of size n, for which every rainbow independent set has size at most $\lceil \frac{n}{2} \rceil$. For odd $n \ge 5$, let G be the t-partite graph $K_t \cup (\frac{n-1}{2})K_{2,2,\ldots,2}$ formed by taking a disjoint union of K_t with $\frac{n-1}{2}$ copies of $K_{2,2,\ldots,2}$, and define $I_1, \ldots, I_t$

to be the t colour classes. Then any rainbow independent set contains at most one vertex from each component, hence at most $\frac{n+1}{2}$ vertices in all. For n even, take G to be $\frac{n}{2}$ copies of $K_{2,2,\ldots,2}$.

□

4.1.2 $\{C_4, C_5, \ldots, C_s\}$-free graphs

In general, $f_{\mathscr{F}(C_4,C_5,\ldots,C_s)}(n,n)$ is not always finite. This can be shown by a generalization of Example 1.4.3.

Example 4.1.2. *For integers n,t, let $G := G_{t,n}$ be obtained from a cycle of length tn by adding all edges connecting any two vertices of distance $< t$ in the cycle. There are precisely t independent sets of size n, say $I_1, \ldots, I_t$. As the family of independent sets, take $n-1$ copies of each I_j, yielding $t(n-1)$ colours in total. (Setting $t = 2$ gives Example 1.4.3).*

Since the only independent sets of size n are the I_j's themselves, and each repeats only $n-1$ times, there is no rainbow independent set of size n.

Theorem 4.1.3. *$f_{\mathscr{F}(C_4,C_5,\ldots,C_s)}(n,n) = \infty$ for $n \geq s$.*

Proof of Theorem 4.1.3. The proof will be complete if we show that if $n \geq i \geq 4$ then $G_{t,n}$ is C_i-free. This will mean that $f_{\mathscr{F}(C_4,C_5,\ldots,C_n)}(n,n) \geq t(n-1)+1$ for every t.

To see the last claim, suppose for contradiction that $x_1, \ldots, x_s$ are the vertices of a copy of C_s in $G_{t,n}$. Then their consecutive distances $d_i := x_{i+1} - x_i \in \{\pm 1, \pm 2, \ldots, \pm(t-1)\}$ for every $i \in \mathbb{Z}_s$ as x_i and x_{i+1} are adjacent. Furthermore, d_i, d_{i+1} have the same sign, as otherwise $|x_{i+2} - x_i| = |d_{i+1} - d_i| < t-1$ contradicts the non-adjacency of x_i and x_{i+2}. Reversing the order if necessary, we may assume $d_i > 0$ for every i. Then $x_1 \equiv x_2 - d_1 \equiv x_3 - d_2 - d_1 \cdots \equiv x_1 - \sum_i d_i \pmod{nt}$ implies that $\sum_i d_i$ is a positive multiple of nt, hence at least nt. But

$$\sum_{i=1}^{s} d_i \leq \sum_{i=1}^{s} (t-1) \leq s(t-1) < nt,$$

a contradiction. $\square$

However, for any $s \geq 4$, observe that $f_{\mathscr{F}(C_4,C_5,\dots,C_s)}(2,2) = 2$. In a similar spirit, and in sharp contrast to Theorem 4.1.3, the forbidding of C_{n+1} among a collection of independent sets of size n can make a large difference:

Theorem 4.1.4. *If $s > n \geq 3$ then $f_{\mathscr{F}(C_4,C_5,\dots,C_s)}(n,n) < \infty$.*

A *sunflower* is a collection of sets $S_1,\dots,S_k$ with the property that, for some set Y, $S_i \cap S_j = Y$ for every pair $i \neq j$. The set Y is called the *core* of the sunflower and the sets $S_i \setminus Y$ are called *petals*. In particular, a collection of pairwise disjoint sets is a sunflower with $Y = \emptyset$.

Lemma 4.1.5 (Erdős-Rado Sunflower Lemma, [18])**.** *Any collection of $n!(k-1)^n$ sets of cardinality n contains a sunflower with k petals.*

We use the sunflower lemma to reduce Theorem 4.1.4 to the case where all independent sets are disjoint:

Theorem 4.1.6. *For all numbers $n \geq 3$ there is some large N_n, increasing with n, satisfying the following property. Suppose $s > n$ and $G \in \mathscr{F}(C_4,\dots,C_s)$ has N_n* disjoint *independent sets of size n. Then G contains a rainbow independent set of size n.*

Proof of Theorem 4.1.4. We claim that $f_{\mathscr{F}(C_4,\dots,C_s)}(n,n) \leq n!(N_n-1)^n$. Indeed, if a graph G has this many independent sets of size n, then by the Sunflower Lemma, some N_n of them form a sunflower $S_1,\dots,S_{N_n}$ with core Y. If $\ell \leq n < s$ is the size of the resulting petals $S_i \setminus Y$, then these $N_n \geq N_\ell$ sets form disjoint independent sets of size ℓ in G. So applying Theorem 4.1.6 to the induced graph

$G[\sqcup_i(S_i \setminus Y)]$ gives a rainbow independent set I of size ℓ among these petals $S_i \setminus Y$. But extending I to $I \cup Y$ also produces an independent set, now of size n, as the core Y is nonadjacent to all vertices in the sunflower. The additional $n-\ell$ vertices in Y can all be assigned distinct new colours not used in I, merely needing $N_n \geq n$, as Y is contained in every S_i. So $I \cup Y$ is rainbow, as desired. □

Definition 4.1.1. Let A and B be finite ordered sets with $|B| = b$. Let G be a b-partite graph on $A \times B$ whose parts are columns $A \times \{j\}$. We say G is *repeating* if the following holds:

For every i_1, i_2 in A (not necessarily distinct) and two pairs $j_1 < j_2$ and $j_1' < j_2'$, the vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if the vertices (i_1, j_1') and (i_2, j_2') are adjacent. That is, all bipartite graphs of the form $G[A \times \{j, j'\}]$ with $j \neq j'$ (induced by two parts of G) are isomorphic a bipartite graph Γ in a fashion consistent between parts. In particular, if $G[A \times \{j, j'\}]$ is isomorphic to a bipartite graph Γ for all $j \neq j'$, we say G is Γ-*repeating*.

Note that every row $\{i\} \times B$ is either a clique or an independent set in any repeating graph H on $A \times B$, and that the subgraphs $H[A' \times B']$ obtained by inducing H on the subgrid $A' \times B'$ of $A \times B$ is also repeating.

We next appeal to a Ramsey-type result. Recall that $R(r_1, \dots, r_t)$ denotes the smallest number of vertices in a complete graph for which any t-edge-colouring contains in some colour i a monochromatic K_{r_i}, and that Ramsey's Theorem guarantees the existence of such a number.

Lemma 4.1.7. *For each $n \in \mathbb{N}$, there is some R with the following property. Suppose G is an R-partite graph on $[n] \times [R]$, whose parts are the R columns. Then there is some $B \subset [R]$ with $|B| = n+1$ for which the induced subgraph $G[[n] \times B]$ is repeating.*

Proof. This is a direct consequence of Ramsey's Theorem, with

$$R := R(\overbrace{n+1, \dots, n+1}^{2^{n^2}}).$$

□

This immediately reduces Theorem 4.1.6 to repeating graphs:

Theorem 4.1.8. *Let s and n be integers with $s > n \geq 3$. Suppose $G \in \mathscr{F}(C_4, \ldots, C_s)$ is a Γ-repeating graph on vertex set $[n] \times [n+1]$. Then G contains an independent set of size n which is rainbow with respect to the columns.*

Proving this will need two graph-theoretic lemmas as follows.

Definition 4.1.2. For a path or cycle H and a matching M in a bipartite graph Γ, we say H is *M-alternating* if $E(H) \setminus M$ is a matching.

Lemma 4.1.9. *Let Γ be a bipartite graph containing a perfect matching M. Suppose that Γ has minimum degree ≥ 2. Then Γ contains an* induced *M-alternating cycle C.*

Proof. First we see that Γ contains some M-alternating cycle, not necessarily induced. Indeed, let $P = v_1 v_2 \ldots v_{s-1} v_s$ be any maximal M-alternating *path*. If the last edge $v_{s-1}v_s \notin M$, then v_s is matched to some vertex v by M. Otherwise the last edge $v_{s-1}v_s \in M$, so as $d(v_s) \geq 2$, v_s is adjacent to another vertex v. In either case, by maximality of P, v is a previous vertex v_i from the path. Then the cycle $v_i v_{i+1} \ldots v_{s-1} v_s v_i$ is M-alternating.

Now, let C be any inclusion-minimal M-alternating cycle in Γ. We wish to show C is in fact induced, so suppose not. Then we may write $C = u_1 u_2 \ldots u_{2t-1} u_{2t} u_1$, where the odd edges $u_{2i-1}u_{2i}$ are in M, in such a way that the offending chord is $u_1 u_{2i}$ for some i strictly between 1 and t. But now $u_1 u_2 \ldots u_{2i-1} u_{2i} u_1$ is a strictly smaller M-alternating cycle. □

Lemma 4.1.10. *Let G be a Γ-repeating graph where every row is a clique. Let M be the matching in Γ given by these rows. Suppose, for some k, Γ has an induced M-alternating cycle of length $2k$, and G has at least $k+1$ columns. Then G has an induced cycle of length $k+1$.*

Proof. Reordering the rows and columns if necessary, we may assume that the induced cycle of Γ is $(1,1),(1,2),(2,1),\ldots,(k,1),(k,2),(1,1)$. Then one such desired cycle in G is $(k,1),(k-1,2),(k-2,3),\ldots,(2,k-1),(1,k),(k,k+1),(k,1)$. See Figure 4.1 for an illustration when $k=3$. $\square$

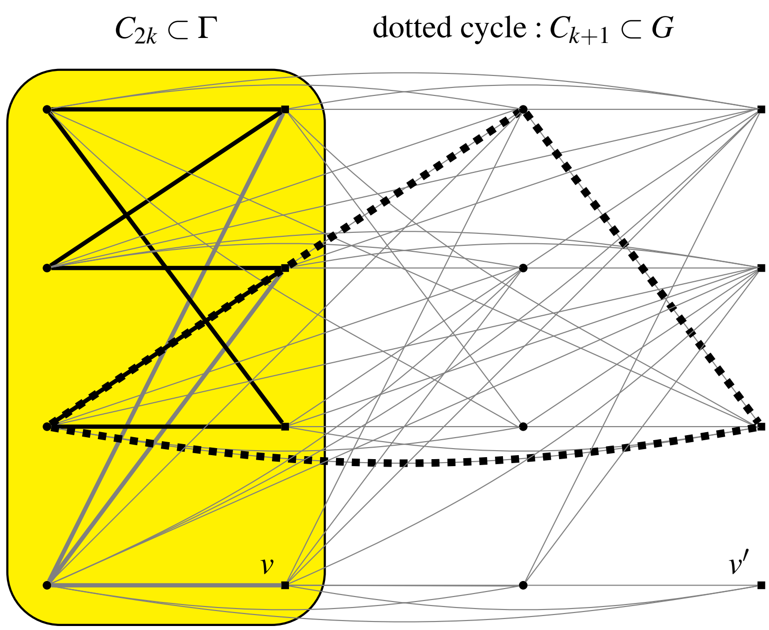


Figure 4.1: Example for Lemma 4.1.10 when $k=3$

Proof of Theorem 4.1.8. We proceed by induction on n.

Every row of G is necessarily a rainbow set of size $n+1>n$, so if any row is independent, we are already done. So as G is repeating we may assume that every row is a clique. Write M for the matching of size n in Γ induced by these rows.

Suppose for contradiction Γ has minimum degree at least 2. Then Lemma 4.1.9 yields an induced M-alternating cycle in Γ of some even length $2k$. @@Crudely, $2k \leq |V(\Gamma)| = 2n$, so if $k \geq 3$ then Lemma 4.1.10 produces an induced cycle C of some length between 4 and $n+1$ in G, contradicting $G \in \mathscr{F}(C_4,\ldots,C_s)$. Mean-while, if $k = 2$, then C was an induced $C_{2k} = C_4$ in Γ, hence in G, contradicting our choice of G without even needing Lemma 4.1.10.

Thus, in fact, Γ contains a vertex v of degree 1. That is, v only has its M-edge in Γ. If $v = (i,1)$ for some i, namely v is in the first column of Γ and G, then let $v' := v$. If v was in the second column of Γ, namely $v = (i,2)$ for some i, then replace $v = (i,2)$ by $v' := (i,n+1)$-the last vertex in row i (see Figure 4.1). In either case, v' is only adjacent to vertices in row j as G is Γ-repeating.

Note that both $G[([n] \setminus \{i\}) \times ([n+1] \setminus \{1\})]$ and $G[([n] \setminus \{i\}) \times [n]]$ are also repeating. Thus, by the induction hypothesis, we may find a rainbow independent $(n-1)$-set I in the remaining columns obtained by deleting the row and column containing v'. To obtain the desired rainbow independent n-set we simply add v' to I. $\square$

Remark 4.1.11. *By the observation* (1.4.2)*, Theorem 4.1.4 can be extended as follows: if $s > m \geq 3$ and $n \geq m$ then $f_{\mathscr{F}(C_4,C_5,\ldots,C_s)}(n,m) < \infty$.*

4.1.3 Chordal graphs

A graph is called *chordal* if it does not contain an induced cycle of length larger than 3. Recall that the class of chordal graphs is denoted by $\mathscr{T}$. By Theorem 4.1.4 (applying monotonicity), $f_{\mathscr{T}}(n,n) \leq f_{C_4,\ldots,C_{n+1}}(n,n) < \infty$ for every positive integer n. In fact, in this case the exact value of $f_{\mathscr{T}}(n,m)$ is known:

Theorem 4.1.12. *If $m \leq n$ then $f_{\mathscr{T}}(n,m) = m$.*

A useful property of chordal graphs was given in [14, Theorem 8.11]:

Theorem 4.1.13. *Any chordal graph contains a simplicial vertex, namely a vertex whose neighbors form a clique.*

Proof of Theorem 4.1.12. By (1.4.1) it suffices to show that $f_{\mathscr{T}}(n,m) \leq m$. The proof is by induction on m. For $m = 0$ there is nothing to prove. Assume that the result is valid for $m-1$. Let G be a chordal graph and let $I_1,\dots,I_m$ be independent sets in G of size n. We may assume that $V(G) = I_1 \cup \cdots \cup I_m$.

Let v be a simplicial vertex. Without loss of generality, $v \in I_m$. Consider the induced subgraph $G' = G[V \setminus N[v]]$ and the $m-1$ independent sets $I'_j = I_j \setminus N[v]$, $1 \le j \le m-1$, in G'. Since $N[v]$ is a clique, any independent set in G contains at most one vertex from $N[v]$, hence each I'_j has cardinality at least $n-1$. Since G' is also chordal, by induction we may assume that there is a rainbow independent set $\{v_1,\dots,v_{m-1}\}$ in G', where $v_j \in I_j$ for each $i \le j \le m-1$. Since the vertex v is not adjacent to any v_j, the set $\{v_1,\dots,v_{m-1},v\}$ is a rainbow independent set in G. $\square$

4.1.4 K_r-free graphs and K_r^--free graphs

Recall that K_r^- denotes K_r minus an edge. In particular, the graph K_4^- is called "diamond". It is known [10] that a graph is claw-free and diamond-free if and only if it is the line graph of a triangle-free graph, and that if in addition it does not contain an induced odd cycle then it is the line graph of a bipartite graph [21].

To establish upper bounds on $f_{\mathscr{F}(K_r^-)}(n,m)$, we shall use a Ramsey-type result on multipartite graphs. As usual, we denote by $R(a,b)$ the smallest number q such that if the edges of K_q are coloured red and blue, there necessarily exist a red K_a or a blue K_b.

Lemma 4.1.14. *For every integers $n,r \in \mathbb{N}$ with $r \ge 2$, there exists $M = M(n,r) \in \mathbb{N}$ for which the following holds. Let G be an $M-$partite graph with classes $V_1,\dots,V_M$ such that $|V_i| = n$ for every i. If G does not contain an induced copy of K_r^-, then it contains a rainbow independent set of size n.*

Proof. We proceed by induction on n. For $n = 1$, we clearly have $M(1,r) = 1$. We claim that for $n \ge 2$, it is sufficient to set

$$M = M(n,r) = R(\overbrace{t,\dots,t}^{2^{n^2}}),$$

where $t = \max\{r-1, M(n-1,r)\}$. This can be proved by the proof of Theorem 4.1.8 with a slight modification.

Let G be an $M-$partite graph where each part has size $n \geq 2$. By Ramsey's theorem, G contains a Γ-repeating subgraph H, say on vertex set $[n] \times [t]$. Since every row of H is a rainbow set of size $t > n$, we have a rainbow independent set of size n if any row is independent. Since H is repeating, we may assume that every row is clique.

Assume that every vertex in Γ has degree at least 2. Then the vertex $(1,2)$ is adjacent to $(1,1)$ and $(j,1)$ for some $j \in [n] \setminus \{1\}$. Since H is Γ-repeating, every vertex $(1,i)$ is adjacent to both $(1,1)$ and $(j,1)$. Since $(1,1)$ and $(j,1)$ are not adjacent, then the vertex set

$$\{(1,i) : 1 \leq i \leq r-1\} \cup \{(1,j)\}$$

induces K_r^- in H, which is a contradiction. Hence there must be a vertex in Γ which has degree 1. Since $t \geq M(n-1,r)$, applying the last paragraph of the proof of Theorem 4.1.8 gives us a rainbow independent set of size n. $\square$

Corollary 4.1.15. *For every integer r, if $n \geq m$ then*

$$f_{\mathscr{F}(K_r^-)}(n,m) < \infty \quad \textit{and} \quad f_{\mathscr{F}(K_r)}(n,m) < \infty.$$

Proof. We show $f_{\mathscr{F}(K_r^-)}(n,n) \leq N$ for some large enough $N = N(n,r)$. The result for K_{r-1} and for $m < n$ follows by the monotonicity relation (1.4.2), since $\mathscr{F}(K_r^-) \subseteq \mathscr{F}(K_{r-1})$.

Suppose a graph G is equipped with N independent sets of size n (colour classes). By Lemma 4.1.5, if $N \geq n!M^{n+1}$ for some M, then there are some classes $I_1, \ldots, I_M$ which form a sunflower in $V(G)$. That is, some $S \subseteq V$ has $I_i \cap I_j = S$ for every distinct $i, j \in [M]$. Now, if $S \neq \emptyset$, then we may inductively find a rainbow independent set of size $n - |S|$ among $\left(\cup_{j=1}^{M} I_j\right) \setminus S$, provided that $M \geq N_{n-|S|}$.

Together with S this will generate a rainbow set of size n. So we may assume $S=\emptyset$, namely that the sets I_j are disjoint.

By Lemma 4.1.14, if M is large enough, then we either find an independent n-transversal among the I_j's or an induced K_r^-. $\square$

In fact, when $m=n$, the graphs K_r and K_r^- are the only (nonempty) graphs for which Corollary 4.1.15 holds. We prove this in a series of lemmas.

Lemma 4.1.16. *Let H,K be graphs. If H is an induced subgraph of K and $f_{\mathscr{F}(H)}(n,n)=\infty$, then $f_{\mathscr{F}(K)}(n,n)=\infty$.*

This follows from the transitivity of the "subgraph" relation.

Let K_3^{--} be the graph on three vertices with exactly one edge.

Lemma 4.1.17. *If H does not contain K_3^{--} as an induced subgraph, then H is a complete r-partite graph $K_{s_1,s_2,\dots,s_r}$ for some r.*

Proof. The condition implies that the relation of "not being connected in the graph" on the vertex set of the graph is an equivalence relation, which is just the conclusion of the lemma. $\square$

Lemma 4.1.18. *If $f_{\mathscr{F}(H)}(n,n)<\infty$ for every n, then:*

(a) H is claw-free,

(b) H is C_4-free.

(c) H is $\bar{K}_{n+1}$-free, and

(d) H is K_3^{--}-free.

Proof. (a) and (b) follow from Lemma 4.1.16 combined with Theorem 4.1.1 and Theorem 4.1.3. For (c) and (d), consider the complete t-partite graph $G=K_{n,\dots,n}$. It is obvious that G does not contain $\bar{K}_{n+1}$ and K_3^{--} as induced subgraphs. On the other hand, since $K_{n,t}$ has t pairwise disjoint independent sets of size n, and there is no rainbow independent set of size n (or even of size 2) where t is arbitrarily large. This proves that $f_{\mathscr{F}(\bar{K}_{n+1})}(n,n) = \infty$ and that $f_{\mathscr{F}(K_3^{--})}(n,n) = \infty$ for every $n \geq 2$. Hence (c) and (d) follow from Lemma 4.1.16. $\square$

Part (d) and Lemma 4.1.17 imply:

Lemma 4.1.19. *If $f_{\mathscr{F}(H)}(n,n) < \infty$, then H is complete multipartite.*

We can now prove the main result of this section:

Theorem 4.1.20. *$f_{\mathscr{F}(H)}(n,n) < \infty$ for every positive integer n if and only if H is either K_r or K_r^- for some r.*

Proof. The "if" part is proved by Corollary 4.1.15. For the "only if" part, we have shown that if $f_{\mathscr{F}(H)}(n,n) < \infty$ then H is multipartite, avoiding C_4's or claws as induced subgraphs. The absence of induced C_4's implies that at most one class in the partition of the graph is of size 2 or more, and the absence of induced $K_{1,3}$'s implies that if there is a class of size larger than 1 it is of size 2. If there is no such class, H is complete. If there is a single class of size 2, then H is K_r^-. $\square$

To complete this section, we use the Ramsey numbers $R(s,t)$ to find a nontrivial lower bound on $f_{\mathscr{F}(K_r)}$ and $f_{\mathscr{F}(K_{r+1}^-)}$.

Theorem 4.1.21. *For any numbers r and $m \leq n$:*

$$R(r,m) - 1 \leq f_{\mathscr{F}(K_r)}(n,m) \leq f_{\mathscr{F}(K_{r+1}^-)}(n,m).$$

Proof. We exhibit a graph G showing $f_{\mathscr{F}(K_r)}(n,m) \geq N := R(r,m) - 1$. The second inequality is due to monotonicity (1.4.2) as $K_r \subseteq K_{r+1}^-$ implies $\mathscr{F}(K_r) \subseteq \mathscr{F}(K_{r+1}^-)$.

Take a K_r-free graph H with no independent set of size m on N vertices, as guaranteed by the definition of $R(r,m)$.

Let G be the graph blowup $H^{(n)}$. That is, replace each $v \in V(H)$ with an independent set of size n, and replace each edge in H with the corresponding complete bipartite graph in G (specifically a copy of $K_{n,n}$). Then G is K_r-free since H was. Now defining the colour classes to be the N blown up vertices yields no rainbow independent set of size m in G (since the classes are disjoint, and H had no independent set of size m). $\square$

We close this section with a remark that well-known bounds on $R(3,t)$ (for example [28]) tell us that f is superlinear for the class of diamond-free graphs:

Corollary 4.1.22.

$$\Omega\left(\frac{m^2}{\log m}\right) \leq R(3,m)-1 \leq f_{\mathscr{F}(K_4^-)}(n,m) < \infty.$$

4.2 k-colourable graphs

Recall that the class of k-colourable graphs is denoted by $\mathscr{X}(k)$.

Theorem 4.2.1. *If $m \leq n$ then $f_{\mathscr{X}(k)}(n,m) = (m-1)k+1$.*

Proof. To show that $f_{\mathscr{X}(k)}(n,m) > k(m-1)$, let G be the complete k-partite graph with all sides of size n, and take a family of $k(m-1)$ independent sets, consisting of each side of the graph repeated $m-1$ times. A rainbow set of size m must include vertices from two different sides of the graph, and hence cannot be independent. To bound $f_{\mathscr{X}(k)}(n,m)$ from above, let G be a k-colourable graph and let $I_1,\dots,I_{k(m-1)+1}$ be independent sets in G of size n. colour G by colours V_i $(i \leq k)$, so $V_1,\dots,V_k$ are independent sets covering $V(G)$.

Let M be an inclusion-maximal rainbow set. If M represents all sets I_j, then $|M| = k(m-1)+1$, and by the pigeonhole principle M contains m vertices from the same set V_j. Since V_j is independent this means that M contains a rainbow set of size m, as required.

Thus we may assume that I_j is not represented in M for some j. By the maximality of M, this implies that $M \supseteq I_j$, implying in turn that I_j is a rainbow independent set of size $n \geq m$. $\square$

Theorem 4.2.1 can be strengthened, in the spirit of Theorem 1.4.5: a family of independent sets F_i, $i \leq k(m-1)+1$ in a k-chromatic graph, where $|F_i| \geq \min(i,n)$, has a rainbow independent set of size m. To prove this, follow the same proof as above, choosing the elements of M greedily from the sets $F_1, \ldots, F_n$.

4.3 Graphs with bounded degrees

Recall that $\mathscr{D}(k)$ is the class of graphs having vertex degrees no larger than k. Theorem 4.2.1, together with Brooks' theorem (stating that the chromatic number does not exceed the maximal degree, unless the graph is complete or an odd cycle), imply $f_{\mathscr{D}(k)}(n,n) \leq k(n-1)+1$. This is probably not best possible.

Conjecture 4.3.1. $f_{\mathscr{D}(k)}(n,n) = \left\lceil \frac{k+1}{2} \right\rceil (n-1)+1$.

One inequality, $f_{\mathscr{D}(k)}(n,n) \geq \left\lceil \frac{k+1}{2} \right\rceil (n-1)+1$, is shown by Example 4.1.2. As noted, in the graph $G_{t,n}$ constructed in that example, there is no rainbow independent set of size n. To establish the desired bound, note every $v \in G_{t,n}$ has degree $2t-2 \leq k$ if we choose $t := \left\lceil \frac{k+1}{2} \right\rceil$.

Theorem 4.3.2. *Conjecture 4.3.1 is true for $k \leq 2$.*

Proof. Consider first the case $k \leq 1$, n general. Let G be a graph with $\Delta(G) \leq 1$, namely a matching. Then $\mathscr{I}(G)$ is a partition matroid whose parts are the edges of the matching. Greedy choice shows that in a matroid every n independent sets of size n have a rainbow independent set of size n.

Next consider the case $k=2$. We have to show that if $\Delta(G) \leq 2$ and $I_1, \ldots, I_{2n-1} \in \mathscr{I}(G)$ are of size n, then there is a rainbow independent set of size n. Take an inclusion-maximal rainbow set M that induces a bipartite graph in G. If $|M| = 2n-1$, then M contains a subset of size n in one of the sides of the bipartite graph, which is independent in G. Thus we may assume that $|M| < 2n-1$. Then one of the sets I_j, say I_1, is not represented by M.

Since M is inclusion-maximal rainbow bipartite, each vertex $v \in I_1 \setminus M$ is contained in an odd cycle C_v of odd length, say of length $2t(v)+1$, that is a connected component of G, such that $V(C_v) \setminus \{v\} \subseteq M$. For each such v replace in I_1 the set $I_1 \cap V(C_v)$ (which is of size at most $t(v)$, since I_1 is independent) by any independent set J_v of size $t(v)$ not containing v (note that $J_v \subseteq M$). The result of these replacements is an independent set of size n, contained in M, yielding the desired rainbow set.

□

4.3.1 The case $m < n$

Here is a generalization of Conjecture 4.3.1:

Conjecture 4.3.3. *If $m \leq n$ then $f_{\mathscr{D}(k)}(n,m) = \left\lceil \frac{k+1}{n-m+2} \right\rceil (m-1)+1$.*

The bound in the conjecture cannot be improved. The examples showing this follow the pattern of Example , with some modifications.

Let $r = n-m+2$, and write $k = r(t-1)+\alpha$ for $\alpha \in [0,r)$. Then $t = \frac{k-\alpha}{r}+1 = \left\lfloor \frac{k}{r}+1 \right\rfloor = \left\lceil \frac{k+1}{r} \right\rceil = \left\lceil \frac{k+1}{n-m+2} \right\rceil$.

Let G be the graph obtained from a cycle of length nt by adding edges between any two vertices whose distance from each other is not a multiple of t and is not larger than $\lfloor \frac{rt-1}{2} \rfloor$.

We then have:

$$\Delta(G) = 2(\left\lfloor \frac{rt}{2} - \langle \frac{r}{2} \rangle \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor) \tag{4.3.1}$$

(Here $\langle x \rangle = x - \lfloor x \rfloor$.)

We divide into two cases:

Case I. $r(t-1)$ is even. In this case, (4.3.1) yields $\Delta(G) \leq r(t-1) \leq k$ (to see this, check separately the subcases "r is even" and "t is odd"). So, $G \in \mathscr{D}(k)$.

Observe that the modulo t residue classes $\mathbb{Z}_{tn}/\mathbb{Z}_n := \{a + t\mathbb{Z}_n : a = 0, 1, \ldots, (t-1)\}$ are independent sets of size n in G, as no two vertices at a distance some multiple of t are adjacent. We claim that any independent set $I \subseteq G$ of size m is fully contained in one residue class. This will entail the desired lower bound $f_{\mathscr{D}(k)}(n,m) \geq t(m-1)+1 = \left\lceil \frac{k+1}{n-m+2} \right\rceil (m-1)+1$, by taking $m-1$ copies of each independent set of size m in G.

To prove the claim, suppose that $I \subseteq V(G)$ has size m, and write $d_1, \ldots, d_m$ for the consecutive distances in I.

Then $\sum d_i = |V(G)| = nt$.

Assume for contradiction that there exists an independent set I of size m that is not contained in some residue class $\pmod{t}$. Then:

- $d_i \geq t$ for every i, by independence, and
- for some distinct pair j, j'; both $d_j, d_{j'} > \frac{rt}{2}$.

Indeed, if the $\{x_i\}$ are not all in the same residue class, then there is some consecutive pair $x_j \not\equiv x_{j+1} \pmod{t}$. We may further find one other such index j' with $x_{j'} \not\equiv x_{j'+1} \pmod{t}$ for otherwise $x_{j+1} \equiv x_{j+2} \equiv \cdots \equiv x_{j-1} \equiv x_j \pmod{t}$. Then, by nonadjacency, $d_j, d_{j'} > \left\lfloor \frac{rt-1}{2} \right\rfloor$ and hence (as r is even or t is odd) $d_j, d_{j'} > \frac{rt}{2}$.

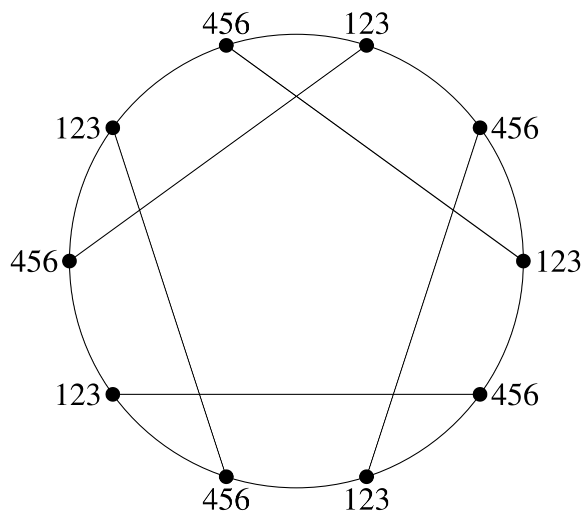
Putting these two facts together, we obtain

$$\sum_{i=1}^{m} d_i \geq (m-2)t + d_j + d_{j'} > (m-2)t + rt = (m-2)t + (n-m+2)t = nt,$$

a contradiction.

Case II. $r(t-1)$ **is odd**. In this case, (4.3.1) yields $\Delta(G) \leq k-1$. This means that we can add a matching to the graph, without violating the condition $\Delta(G) \leq k$, which enables solving the problem, that it is possible for two nonadjacent vertices of different $\pmod{t}$ classes to be at a distance exactly $\frac{rt}{2}$. Form a graph G' by adding an edge from x to $x + r\frac{t}{2}$ whenever $x \in \{0, 1, \ldots, \frac{t}{2} - 1\} + t\mathbb{Z}_n$, or equivalently from y to $y - r\frac{t}{2}$ whenever $y \in \{\frac{t}{2}, \ldots, t-1\} + t\mathbb{Z}_n$.

Again, we show that if $I \subseteq G'$ is an independent m-set, then it is contained in a $\pmod{t}$ class. Assume that this is not the case, and let the consecutive distances in I be $d_1, \ldots, d_m$. As above, $\sum d_i = nt$, $d_i \geq t$ for every i, and $d_j, d_{j'} \geq \frac{rt}{2}$ for some pair j, j'. Without loss of generality we may assume $j' = m$. As above, equality must hold everywhere, so that $I = x_1 + \{0, t, 2t, \ldots, (j-1)t, -\frac{t}{2} + (j + \frac{r-1}{2})t, -\frac{t}{2} + (j + \frac{r+1}{2})t, \ldots, -\frac{t}{2} - \frac{r-1}{2}t\}$, for some $x_1 \in \mathbb{Z}_{nt}$. In both cases a contradiction is reached. If $x_1 \in \{0, 1, \ldots, \frac{t}{2} - 1\} + t\mathbb{Z}_n$, then the two elements $x_1 + (j-1)t$ and $x_1 - \frac{t}{2} + (j + \frac{r-1}{2})t$ are adjacent. If $x_1 \in \{\frac{t}{2}, \ldots, t-1\} + \mathbb{Z}_n$, then $x_1 - \frac{t}{2} - \frac{r-1}{2}t$ and x_1 are adjacent vertices in I.



$f_{\mathscr{D}(3)}(5,4) > (m-1)t = 6$: 6 independent sets of size 5 in a graph of max degree 3 do not guarantee a rainbow independent 4-set.

Theorem 4.3.4. $f_{\mathscr{D}(k)}(n,2) = \lceil \frac{k+1}{n} \rceil + 1$.

Proof. Let G be a graph with maximum degree k and let $I_1, \ldots, I_{\lceil \frac{k+1}{n} \rceil + 1}$ be the independent sets of size n in G. We first note that $\lceil \frac{k+1}{n} \rceil + 1 \geq 2$. If $I_i \cap I_j \neq \emptyset$ for some $i \neq j$, then both I_i and I_j contain at least one rainbow independent set of size 2. Thus we may assume that the sets $I_1, \ldots, I_{\lceil \frac{k+1}{n} \rceil + 1}$ are mutually disjoint.

Now take $u \in I_1$. Note that there are $\lceil \frac{k+1}{n} \rceil \times n \geq k+1$ vertices in $\bigcup_{j>1} I_j$. Since the degree of u in G is at most k, there must be a vertex $v \in I_j$ for some $j \neq 1$ which is not adjacent to u. Then $\{u,v\}$ form a rainbow independent set of size 2 in G. □

Theorem 4.3.5. $f_{\mathscr{D}(k)}(n,3) = 2 \times \lceil \frac{k+1}{n-1} \rceil + 1$.

The proof is similar to that in the proof of Theorem **??**, needing just a little more case analysis. We give a sketch of the proof, starting with the following:

Observation 4.3.6. *Let I_1, I_2, I_3 be independent sets of size $n \geq 3$ in a graph G. If $I_1 \cap I_2 \neq \emptyset$, then we have the following.*

1. *If $|I_1 \cap I_2| \geq 2$ and $I_1 \cap I_3 \neq \emptyset$, then there is a subset of I_1 which forms a rainbow independent set of size 3.*
2. *If $|I_1 \cap I_2| = 1$, say $I_1 \cap I_2 = \{x\}$, and I_3 meets I_1 at a vertex $y \neq x$, then there is a subset of I_1 which forms a rainbow independent set of size 3.*
3. *If $I_1 \cap I_2 = I_2 \cap I_3 = I_1 \cap I_3 = \{u\}$ and there is a vertex $v \in I_i$ and $w \in I_j$ for some $i \neq j$ which are non-adjacent, then $\{u,v,w\}$ forms a rainbow independent set.*
4. *If I_3 is disjoint from $I_1 \cup I_2$ and there is a vertex $u \in I_3$ such that there exist*
   (a) *a vertex $v \in I_1 \cap I_2$ which is not adjacent to u, and*
   (b) *a vertex $w \neq v$ in $I_1 \cup I_2$ which is not adjacent to u,*

   then $\{u,v,w\}$ forms a rainbow independent set.

Proof of Theorem 4.3.5. We use induction on k and n. Let $n > 3$ and denote $\lceil \frac{k+1}{n-1} \rceil$ by q. Let G be a graph with maximum degree k and let $I_1, \ldots, I_{2q+1}$ be the independent sets of size n in G. Assume, for contradiction, that there is no rainbow independent set of size 3.

Let $t = \max\{|I_i \cap I_j : i \neq j\}$ and suppose $t > 0$. Without loss of generality, we may assume that $|I_1 \cap I_2| = t$. Let

$$A = \{v \in \bigcup_{j \geq 3} I_j : I_1 \cap I_2 \subseteq N(v)\} \text{ and } B = \left(\bigcup_{j \geq 3} I_j\right) \setminus A.$$

Claim 4.3.7. *$t < 3$ for every $i \neq j$.*

Proof. By Observation 4.3.6(1), the union $\bigcup_{j \geq 3} I_j$ is disjoint form $I_1 \cup I_2$. By Observation 4.3.6(4), every vertex in B is adjacent to $2n - t - 1$ vertices in $I_1 \cup I_2$: all vertices in $(I_1 \cup I_2) \setminus (I_1 \cap I_2)$ and $t - 1$ vertices in $I_1 \cap I_2$.

By a double counting on the degree sum of the vertices in $I_1 \cap I_2$, we obtain the inequality

$$t|A| + (t-1)|B| \leq \sum_{u \in I_1 \cap I_2} \deg_G(u) \leq tk. \tag{4.3.2}$$

On the other hand, one can see the lower bound

$$\left|\bigcup_{j \geq 3} I_j\right| \geq (2n - t)\frac{2q - 2}{2} + n \geq (2n - t)\frac{k+1}{n-1} - n + t, \tag{4.3.3}$$

where the equality holds when $|I_{2s-1} \cap I_{2s}| = t$ for each $2 \leq s \leq q$ and I_{2q+1} is disjoint from all others. Here we may assume that $t < n$ because if not, then we have

$$n(k+1) \leq (n-1)(|A| + |B|) \leq n|A| + (n-1)|B| \leq nk,$$

which is a contradiction.

Now since $(t-1)\left|\bigcup_{j \geq 3} I_j\right| \leq t|A| + (t-1)|B|$, combining the inequalities (4.3.2) and (4.3.3) gives us

$$(t-1)\left((2n - t)\frac{k+1}{n-1} - n + t\right) \leq tk,$$

which is equivalent to the inequality

$$(n-t)(t-2)k \le (t-1)((2-n)t+n(n-3)).$$

Since $(2-n)t+n(n-3) < (n-t)(n-2)$ and $t < n$, we obtain

$$(t-2)k < (t-1)(n-2). \tag{4.3.4}$$

Next we claim that $k \ge 2n-t-1$ by showing $B \neq \emptyset$. Note that every vertex in B has $2n-t-1$ neighbors in $I_1 \cup I_2$. Applying this to the inequality (4.3.4), we obtain

$$(2n-t-1)(t-2) < (t-1)(n-2) \iff (n-t)(t-3) < 0.$$

Since $n > t$, it must be that $t < 3$.

Suppose $|B| = 0$. Then we have

$$(2n-t)\frac{k+1}{n-1} - n + t \le \left|\bigcup_{j\ge 3} I_j\right| = |A| \le k,$$

which implies $(n-t+1)k \le n(n-3)-(n-2)t$. Since $0 < n-t < n-t+1$ and $n(n-3)-(n-2)t < (n-t)(n-2)$, we obtain $k < n-2$. However, since $\left|\bigcup_{j\ge 3} I_j\right| \ge n$, we have an inequality $n \le k < n-2$, which is a contradiction. This completes the proof. $\square$

Claim 4.3.8. *$t < 2$ for every pair $i \neq j$.*

Proof. Suppose not, say $|I_1 \cap I_2| = 2$. As in the previous claim, we have

$$\begin{cases} 2|A| + |B| \le 2k \\ |A| + |B| = \left|\bigcup_{j\ge 3} I_j\right| \ge (2n-2)\frac{2q-2}{2} + n \ge 2k - n + 4. \end{cases}$$

Combining the two inequalities, we obtain

$$|A| \le n-4 \quad \text{and} \quad |B| \ge 2k-2n+8.$$

If $B=\emptyset$, then $n \le \left|\bigcup_{j\ge 3} I_j\right| = |A| \le n-4$, contradiction. If $B \neq \emptyset$, then we have $|B| \le k$ since every vertex in $(I_1 \cup I_2) \setminus (I_1 \cap I_2)$ is adjacent to every vertex in B. This implies that $2k-2n+8 \le k$, i.e. $k \le 2n-8$. On the other hand, we have $k \ge 2n-3$ since every vertex in B has at least $2n-3$ neighbors in $I_1 \cup I_2$, which is contradiction. This completes the proof. □

Claim 4.3.9. *If $|I_{i_1} \cap \cdots \cap I_{i_p}| = 1$, then $p \le 2$.*

Proof. Without loss of generality, let us assume that $I_1, \ldots, I_p$ is the maximal collection such that $|I_1 \cap \cdots \cap I_p| = 1$, say $I_1 \cap \cdots \cap I_p = \{u\}$. Suppose $p \ge 3$. By Observation 4.3.6(3), we can further assume that $\left(\bigcup_{1\le j\le p} I_j\right) \setminus \{u\}$ induces a complete p-partite graph, whose parts are $I_j \setminus \{u\}$, $1 \le j \le p$. Let

$$C = \{v \in \bigcup_{j\ge p+1} I_j : u \in N(v)\} \text{ and } D = \left(\bigcup_{j\ge p+1} I_j\right) \setminus C.$$

We observe that $|C| \le k$ since u has at most k neighbors and that $|D| \le k-(n-1)(p-1)$ since every vertex in $\left(\bigcup_{1\le j\le p} I_j\right) \setminus \{u\}$ has at most k neighbors. Hence we obtain

$$\left|\bigcup_{j\ge p+1} I_j\right| = |C| + |D| \le 2k-(n-1)(p-1).$$

On the other hand, we have the lower bound

$$\left|\bigcup_{p+1\le j\le 2q+1} I_j\right| > (n-1)(2q+1-p) \ge 2k+2-(n-1)(p-1)$$

which is a contradiction. Therefore, it must be that $p \le 2$. □

Now let v be a vertex in $\bigcup_j I_j$ such that

$$M(v) := \max \left| N(v) \cap (I_{i_1} \cup \cdots \cup I_{i_q}) \right|$$

is maximum among all choices of v and I_{i_j}'s. Without loss of generality, we may assume that $v \in I_1$ and $M(v) = |N(v) \cap (I_2 \cup \cdots \cup I_{q+1})| = l \leq k$. Since we have

$$|I_2 \cup \cdots \cup I_{q+1}| \geq (2n-2)\frac{q}{2} + 1 \geq k+2,$$

there exists a vertex $v' \in (I_2 \cup \cdots \cup I_{q+1}) \setminus \{v\}$ which is not adjacent to v.

Now suppose

$$|N(v) \cap (I_2 \cup \cdots \cup I_{q+1}) \cap (I_{q+2} \cup \cdots \cup I_{2q+1})| = s$$

for some $s \geq 0$. Then it is clear that

$$|N(v) \cap (I_{q+2} \cup \cdots \cup I_{2q+1})| \leq k - l + s.$$

If we let

$$s' = |(I_2 \cup \cdots \cup I_{q+1}) \cap (I_{q+2} \cup \cdots \cup I_{2q+1})|,$$

then we notice that $s' \geq s$ if $v' \notin I_{q+2} \cup \cdots \cup I_{2q+1}$, and $s' \geq s+1$ otherwise since $v' \notin N(v)$.

By Claim 4.3.9 and Observation 4.3.6, it must be that $I_{q+2}, \ldots, I_{2q+1}$ contains s' pairwise disjoint sets that are disjoint from all others. This gives us

$$\begin{aligned}
|I_{q+2} \cup \cdots \cup I_{2q+1}| &\geq (n-1)q + s' + \left\lceil \frac{q-s'}{2} \right\rceil \\
&\geq k + 1 + s' + \left\lceil \frac{q-s'}{2} \right\rceil
\end{aligned}$$

If $s < q$, then we have $|I_{q+2} \cup \cdots \cup I_{2q+1}| \geq k + s' + 2$. Since we have

$$|N(v) \cap (I_{q+2} \cup \cdots \cup I_{2q+1})| \leq k - l + s \text{ and } |N(v') \cap (I_{q+2} \cup \cdots \cup I_{2q+1})| \leq l,$$

there exists a vertex $v'' \in (I_{q+2} \cup \cdots \cup I_{2q+1}) \setminus \{v, v'\}$ which is not adjacent to any of v and v', providing a rainbow independent set $\{v, v', v''\}$ of size 3. Note that

$$k + s' + 2 \geq \begin{cases} k + s + 3 > k + s + 2 & \text{if } v' \in I_{q+2} \cup \cdots \cup I_{2q+1} \\ k + s + 2 > k + s + 1 & \text{otherwise.} \end{cases}$$

If $s = q$, the only possible case is when $I_{q+2}, \ldots, I_{2q+1}$ are pairwise disjoint and $v, v' \notin I_{q+2} \cup \cdots \cup I_{2q+1}$. In this case, there is a vertex $v'' \in I_{q+2} \cup \cdots \cup I_{2q+1}$ which is not adjacent to any of v and v' since

$$|I_{q+2} \cup \cdots \cup I_{2q+1}| = nq \geq k + 1 + q > k + q.$$

This completes the proof. $\square$

Remark 4.3.10. *If the independent sets are pairwise disjoint, then we need only* $2 \times \lceil \frac{k+1}{n} \rceil + 1 < 2 \times \lceil \frac{k+1}{n-1} \rceil + 1$ *independent sets of size n to have a rainbow independent set of size* 3.

4.4 A topological approach

In this section we give alternative proofs for some of the above results, using a topological tool developed by Kalai and Meshulam [26].

Theorem 4.4.1 (Kalai-Meshulam)**.** *If X is d-Leray, then every $d+1$ sets in X^c have a rainbow set belonging to X^c.*

Theorem 4.4.2. *If X is d-collapsible, then every $d+1$ sets in X^c have a rainbow set belonging to X^c.*

For a graph G and a positive integer n, we write $\alpha(G) = \max\{|A| : A \in \mathscr{I}(G)\}$, and let

$$I_n(G) := \{W \subset V(G) : \alpha(G[W]) < n\}.$$

By Theorem 4.4.2, to prove $f_{\mathscr{C}}(n,n) \leq t$ for a class $\mathscr{C}$ and an integer n, it suffices to prove that $I_n(G)$ is $(t-1)$-Leray (or even stronger, $(t-1)$-collapsible) for every graph $G \in \mathscr{C}$. Thus Theorem 4.1.12 will follow from:

Theorem 4.4.3. *If G is chordal, then the complex $I_n(G)$ is $(n-1)$-collapsible.*

An example showing tightness of Theorem 4.4.3 is the graph consisting of n isolated vertices. Here $I_n(G)$ is the boundary of a simplex on n vertices, so it is $n-1$ -collapsible, but not even $n-2$-Leray.

For the proof we shall need to recall the following definition:

Definition 4.4.1. The *star* $\mathrm{st}_X(v)$ of a vertex v in a complex X is $\{\sigma \in X : v \in \sigma\}$.

Lemma 4.4.4. *Let v be a simplicial vertex in a graph G, i.e. $N[v]$ is a clique. For any integers $n \geq 2$ and $d \geq 0$, if $I_n(G-v)$ is $(d+1)$-collapsible and $I_{n-1}(G[V \setminus N[v]])$ is d-collapsible, then $I_n(G)$ is $(d+1)$-collapsible.*

Proof. Suppose $I_n(G-v)$ is $(d+1)$-collapsible and $I_{n-1}(G[V \setminus N[v]])$ is d-collapsible. Note that $I_n(G) = \mathrm{st}_{I_n(G)}(v) \cup I_n(G-v)$. Since the vertex v is simplicial, a subset W of V not containing v belongs to $\mathrm{st}_{I_n(G)}(v)$ if and only if $\alpha(G[W \setminus N[v]]) < n-1$, namely $W \setminus N[v] \in I_{n-1}(G)$. Hence we have

$$\mathrm{st}_{I_n(G)}(v) = \Delta_{N[v]} * I_{n-1}(G[V \setminus N[v]]). \tag{4.4.1}$$

Consider a sequence $\{[\sigma_i, \tau_i]\}_i$ of elementary d-collapses that reduces $I_{n-1}(G[V \setminus N[v]])$ to $\{\emptyset\}$. We claim that $\{[\{v\} \cup \sigma_i, N[v] \cup \tau_i]\}_i$ is a sequence of elementary $(d+1)$-collapses that reduces $I_n(G)$ to $I_n(G-v)$. Then it is obvious that $I_n(G)$ is $(d+1)$-collapsible since $I_n(G-v)$ is $(d+1)$-collapsible.

It is clear that $|\{v\} \cup \sigma_i| \leq d+1$ and that $\{v\} \cup \sigma_i \in \mathrm{st}_{I_n(G)}(v) \subset I_n(G)$ is contained in $N[v] \cup \tau_i \in \mathrm{st}_{I_n(G)}(v) \subset I_n(G)$. For $i > 1$, let $I_n^{i-1}(G) = I_n(G) \setminus \cup_{j=1}^{i-1} [\{v\} \cup \sigma_j, N[v] \cup \tau_j]$. If $i = 0$, let $I_n^0(G) = I_n(G)$. First, we will show that $N[v] \cup \tau_i \in I_n^{i-1}(G)$. If $i = 1$, then it is trivial. Thus assume $i > 1$. Suppose $N[v] \cup \tau_i \notin I_n^{i-1}(G)$, then $N[v] \cup \tau_i \in [\{v\} \cup \sigma_j, N[v] \cup \tau_j]$ for some $1 \leq j \leq i-1$. Since $\sigma_j, \tau_j \subset V \setminus N[v]$, we obtain $\tau_i \in [\sigma_j, \tau_j]$, which is a contradiction. Thus $N[v] \cup \tau_i \in I_n^{i-1}(G)$.

To show the $N[v] \cup \tau_i$ is a unique maximal face of $I_n^{i-1}(G)$ containing $\{v\} \cup \sigma_i$, we will prove that for every vertex $w \notin N[v] \cup \tau_i$ in G, $\{v, w\} \cup \sigma_i$ is not a face in $I_n(G)$. Since $w \notin N[v]$, it must be that $w \in I_{n-1}(G[V \setminus N[v]]$. Since $w \notin \tau_i$, by the uniqueness of τ_i, it must be that $\{w\} \cup \sigma_i$ is not a face in $I_{n-1}(G[V \setminus N[v]])$. In other words, $\alpha(G[\{w\} \cup \sigma_i]) \geq n-1$. Since $(\{w\} \cup \sigma_i) \cap N[v] = \emptyset$, it follows that $\alpha(G[\{v, w\} \cup \sigma_i]) \geq n$, and therefore $\{v, w\} \cup \sigma_i \notin I_n(G)$. $\square$

Proof of Theorem 4.4.3. This can be done by induction on $n + |V(G)|$. The theorem is obvious when $n = 1$ or $|V(G)| = 1$. Suppose $n > 1$ and $|V(G)| > 1$. By Theorem 4.1.13, there exists a simplicial vertex v in G. By the induction hypothesis, $I_n(G - v)$ is $(n-1)$-collapsible and $I_{n-1}(G[V \setminus N[v]])$ is $(n-2)$-collapsible. An immediate consequence of Lemma 4.4.4 is that $I_n(G)$ is $(n-1)$-collapsible. $\square$

Theorem 4.4.5. *For every graph $G \in \mathscr{X}(k)$, the complex $I_n(G)$ is $k(n-1)$-collapsible.*

Proof. We regard the graph G as a k-partite graph with partition $V(G) = V_1 \cup \cdots \cup V_k$. If $W \in I_n(G)$, then $|W \cap V_i| \leq n-1$ for each $i \in [k]$. In particular, $|W| \leq k(n-1)$. Thus $I_n(G)$ is $k(n-1)$-collapsible. $\square$

In the following theorem, $q = \lceil \frac{k+1}{2} \rceil$, as above.

Theorem 4.4.6. *For every graph $G \in \mathscr{D}(k)$, the complex $I_2(G)$ is q-collapsible.*

Proof. Take a vertex $v \in V(G)$. We will first collapse $I_2(G)$ to $I_2(G - v)$ using elementary q-collapses. Note that the link of v in $I_2(G)$ is $I_2(G[N(v)])$. If $\Delta(G[N(v)]) < \Delta(G) - 1$, then $I_2(G[N(v)])$ is $(q-1)$-collapsible. For a sequence

$\{[\sigma_k, \tau_k]\}_k$ of elementary $(q-1)$-collapses of $I_2(G[N(v)])$, we consider $\{[\sigma_k \cup \{v\}, \tau_k \cup \{v\}]\}_k$ of elementary q-collapses of $I_2(G[N[v]])$. By adding $[\{v\},\{v\}]$ at the end, we removed all faces containing the vertex v, so we are done.

If $\Delta(G[N(v)]) = \Delta(G) - 1$, then there is a vertex $w \in N(v)$ which is connected to all other vertices in $N(v)$. In particular, $N[v] = N[w]$ and $\Delta(G[N(v) \setminus \{w\}]) < \Delta(G) - 1$. Then we are done by the above argument, because v and w can be regarded as a duplicate. $\square$

4.5 Concluding remark

A topological version of Conjecture 4.3.1 can be given as the following.

Conjecture 4.5.1. *For every $G \in \mathscr{D}(k)$ and $n \geq 2$, $I_n(G)$ $\left(\left\lceil \frac{k+1}{2} \right\rceil (n-1)\right)$-collapsible.*

The case $n = 2$ of Conjecture 4.5.1 was shown in Theorem 4.4.6, and a partial solution for the case $n = 3$ when k is even was given by Kim and Lew [29]. Here we also suggest a conjecture on the topology of $I_n(G)$ when $G \in \mathscr{G}$, i.e. G is a line graph.

Conjecture 4.5.2. *For every line graph G, $I_n(G)$ $(3n-3)$-collapsible.*

Note that, by Theorem 4.4.1, proving Conjecture 4.5.2 provides an alternative proof of a result in [3] that $f_{\mathscr{G}}(n,n) \leq 3n-2$. However, this kind of topological approach cannot prove the conjecture $f_{\mathscr{G}}(G) = 2n$ since the number $3n-3$ in Conjecture 4.5.2 cannot be improved: in [32], it was shown that

$$\tilde{H}_{3n-4}(L(K_m)) \neq 0 \text{ for sufficiently large } m,$$

while $\tilde{H}_i(L(K_m)) = 0$ for every m and $i \geq 3n-3$.

It is also an interesting question to ask whether the number $3n-3$ can be improved in a subclass of $\mathscr{G}$. For example, if G is a line graph of a bipartite graph, then $I_n(G)$ is $(2n-2)$-collapsible [8]. Here we conlcude this section with a questions in this direction.

Question 4.5.3. *Is there a function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) < 3n-3$ such that the following holds? Let G be a line graph of a graph with no K_3 as a subgraph. Then $I_n(G)$ $f(n)$-collapsible.*

국문초록

$\mathscr{F} = \{S_1, \ldots, S_m\}$를 V의 공집합이 아닌 부분 집합들의 모임이라 할 때, $\mathscr{F}$의 무지개 집합이란 공집합이 아니며 $S = \{s_{i_1}, \ldots, s_{i_k}\} \subset V$와 같은 형태로 주어지는 것으로 다음 조건을 만족하는 것을 말한다. $1 \leq i_1 < \cdots < i_k \leq m$이고 $j \neq j$이면 $s_{i_j} \neq s_{i'_j}$를 만족하며 각 $j \in [m]$에 대해 $s_{i_j} \in S_{i_j}$이다. 특히 $k = m$인 경우, 즉 모든 S_i들이 표현되면, 무지개 집합 S를 $\mathscr{F}$의 완전 무지개 집합이라고 한다.

주어진 집합계가 특정 조건을 만족하는 무지개 집합을 가지기 위한 충분조건을 찾는 문제는 홀의 결혼 정리에서 시작되어 최근까지도 조합수학에서 가장 대표적 문제 중 하나로 여겨져왔다. 이러한 방향으로의 문제를 무지개 집합 문제라고 부른다. 본 학위논문에서는 무지개 집합 문제와 관련하여 위상수학적 홀의 정리와 위상수학적 다색 헬리 정리를 소개하고, (하이퍼)그래프에서의 무지개 덮개와 무지개 독립 집합에 관한 결과들을 다루고자 한다.

주요어휘: 무지개 집합, 독립 복합체, 비(非)덮개 복합체, 지배 매개변수, 독립 집합

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