이학박사 학위논문

Analytic Approach to Invariant Measures and Itô-Stochastic Differential Equations

(불변 측도와 이토 확률미분방정식에 대한 해석적 접근)

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이 논문을 이학박사 학위논문으로 제출함

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Analytic Approach to Invariant Measures and Itô-Stochastic Differential Equations

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Abstract

In this thesis, we study an analytic approach to global well-posedness and long-time behavior for weak solutions to Itô-SDEs with rough coefficients. Using elliptic and parabolic regularity theory and generalized Dirichlet form theory, we show existence of a pre-invariant measure for a large class of elliptic second order partial differential operators and show that these are in fact infinitesimal generators of a Hunt process. Subsequently, this Hunt process is identified for every starting point as a weak solution to an Itô-SDE in $\mathbb{R}^d$ up to its explosion time. The Hunt process has continuous sample paths on the one-point compactification of $\mathbb{R}^d$ and by a known local well-posedness result, it is a pathwise unique and strong solution up to its explosion time to the SDE that it weakly solves. Using analytic and probabilistic methods, we derive general strong Feller properties, including the classical strong Feller property, Krylov type estimates, moment inequalities and various non-explosion criteria. Using a parabolic Harnack inequality, we show irreducibility and strict irreducibility of the process and derive explicit conditions for recurrence and ergodic behavior. Moreover, we investigate well-posedness of weak solutions to Itô-SDEs with degenerate and rough diffusion coefficients whose points of degeneracy form a set of Lebesegue measure zero. In the final part we consider the case where the pre-invariant density is explicitly given. In contrast to the previous case, where we only knew its existence with a certain regularity, we investigate how far our previous methods can be extended and applied in case of a non-degenerate, possibly non-symmetric and discontinuous diffusion matrix. For this, we develop some variational approach to regularity theory for linear parabolic PDEs involving divergence form operators with weight in the term where time derivative appear.

Key words: generalized Dirichlet form, invariant measure, Hunt process, Itô-SDE, elliptic and parabolic regularity, strong Feller property, non-explosion, conservativeness, irreducibility, strict irreducibility, recurrence, transience, ergodicity, weak uniqueness, Krylov type estimate

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Contents

Chapter 1

Introduction

The main subject of our studies is an analytic approach to invariant measures, global well-posedness and long-time behavior of weak solutions to time-homogeneous Itô-Stochastic Differential Equations (Itô-SDEs) with rough coefficients. Different from previous approaches using Krylov type estimates, Girsanov transformation to show weak existence of Itô-SDEs, our main tools are elliptic and parabolic regularity theory and the theory of generalized Dirichlet forms.

This thesis consists of four parts which are closely related to one another. Part I is based on the contents of [49] where the main analytic and probabilistic methods for studying pre-invariant measures and non-degenerate time-homogeneous Itô-SDEs with rough coefficients are developed. For various results of Part II, III, IV, we adapt many methods and techniques from Part I. Throughout, we assume that the dimension d is greater or equal to two, i.e. $d \geq 2$. Consider the following time-homogeneous Itô-SDE with measurable coefficients

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t \mathbf{G}(X_s)ds, \quad 0 \leq t < \zeta,\ x_0 \in \mathbb{R}^d, \tag{1.1}$$

where $W = (W^1, ..., W^l)$ is a standard l-dimensional Brownian motion starting from zero, $A = (a_{ij})_{1\leq i,j\leq d} = \sigma\sigma^T$, $\sigma = (\sigma_{ij})_{1\leq i\leq d, 1\leq j\leq l}$ and $\mathbf{G} = (g_1, ..., g_d)$ and

$$\zeta := \inf\{t \geq 0 : X_t \notin \mathbb{R}^d\} = \lim_{n\to\infty} \inf\{t \geq 0 : X_t \notin B_n\}$$

is the explosion time (or life time) of X, i.e. the time when X has left any Euclidean ball B_n of radius n about the origin.

First, we present some previous results for global and strong well-posedness of (1.1). By a classical result, if $\sigma, \mathbf{G}$ consist of locally Lipschitz continuous functions and satisfy a linear growth condition, then (1.1) with $\zeta = \infty$ has a pathwise unique solution that is strong, i.e. adapted to the filtration generated by W ([34, IV. Theorems 2.4 and 3.1]). Note that the just mentioned reference and most of those below also treat the time inhomogeneous case but we only discuss results in the time homogeneous case, i.e. results related to (1.1). We call a solution that is pathwise unique and strong up to ζ (ζ being possibly finite, cf. [34, IV. Definition 2.1]) strongly unique up to ζ.

Strong uniqueness results for (1.1) with $\zeta = \infty$ for only measurable coefficients were given starting from [86], [80], [81]. In these works σ is non-degenerate and $\sigma, \mathbf{G}$ are bounded. Regarding bounded coefficients one can also mention the later work [4]. To our knowledge the first strong uniqueness results for locally unbounded measurable coefficients start with [30, Theorem 2.1], while weak existence results appeared to exist earlier (cf. introduction of [30]). In [30, Theorem 2.1] σ may be chosen locally Lipschitz, with $\sigma\sigma^T$ globally uniformly strictly elliptic and $g_i \in L^{2(d+1)}_{loc}(\mathbb{R}^d)$ with the following growth condition to ensure non-explosion ([30, Assumption 2.1]): there exists a constant $M \geq 0$ and a non-negative function $F \in L^{d+1}(\mathbb{R}^d)$ such that almost everywhere

$$\|\mathbf{G}\| = \left(\sum_{i=1}^{d} g_i^2 \right)^{1/2} \leq M + F.$$

However, the above condition does not allow for linear growth of drift coefficient.

In [83], the following result was obtained: if σ consists of continuous functions and is globally uniformly non-degenerate, i.e. $A(x) \geq C \cdot \mathrm{Id}$ in the quadratic form sense for some constant $C > 0$ and every $x \in \mathbb{R}^d$ and $g_i, \partial_k \sigma_{ij} \in L^{2(d+1)}_{loc}(\mathbb{R}^d)$ for any i, j, k, then (1.1) has a strongly unique solution up to its explosion time. In [83, Theorem 1.1(i) and (ii)] two non-explosion conditions are given. Both require the global boundedness of σ and then only depend on $\mathbf{G}$. The first one is similar to the one of [30] given above. The second one is as follows: there exist a constant $M \geq 0$, and vector fields $\mathbf{H}$, $\mathbf{F}_i$, with $\|\mathbf{F}_i\| \in L^{p_i}(\mathbb{R}^d)$, $p_i \geq 2(d+1)$, such that almost everywhere

$$\mathbf{G} = \sum_{i=1}^{k} \mathbf{F}_i + \mathbf{H} \quad \text{with} \quad \|\mathbf{H}(x)\| \leq M \left(1 + 1_{\{\|x\|>e\}} \|x\| \log \|x\|\right).$$

This non-explosion condition allows for linear growth and can cover singularities of $\mathbf{G}$, a phenomenon that can not occur for SDEs with continuous coefficients, since these are of course locally bounded.

Prior to [83], the following was obtained in [43]: if σ is the identity matrix, so that the local martingale part in (1.1) is just a d-dimensional Brownian motion $W = (W^1, ..., W^d)$ and $\mathbf{G} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ for some $p > d$, with

$$\int_0^t \|\mathbf{G}(X_s)\|^r ds < \infty \quad \mathbb{P}_{x_0}\text{-almost surely on} \quad \{t < \zeta\}, \tag{1.2}$$

where $r = 2$ and $\mathbb{P}_{x_0}$ is the distribution on the paths starting form x_0, then (1.1) has a strongly unique solution up to its explosion time. Besides a global L_p-integrability condition which does not allow for linear growth a rather special and not really explicit non-explosion condition is presented in [43, Assumption 2.1]. Its formulation is quite long but roughly one can say it is given by assuming that $\mathbf{G}$ is the weak gradient of a function which is a kind of Lyapunov function for (1.1).

The strong uniqueness result of [43] was generalized among others in [84, Theorem 1.3] to the case of non-trivial $d \times d$-dispersion matrix σ with corresponding locally uniformly strictly elliptic diffusion matrix $\sigma\sigma^T$ and $\sigma_{ij} \in H^{1,p}_{loc}(\mathbb{R}^d)$ where $p > d$ is the same as for $\mathbf{G}$, relaxing condition (1.2) to the natural one, i.e. $r = 1$ (although we show here that this does at least in the time-homogeneous case not play a role, since it is always satisfied with $r = 2$, cf. Remark 3.1.7(i)) but no non-explosion condition related to the local conditions of [84, Theorem 1.3] is given. Only a global L_p-integrability condition in space is given in [84, Theorem 1.2], which again does not allow for linear growth. Note that [84, Theorem 1.3] also holds under the conditions of Remark 3.1.3(ii) and that we can handle this case but disregard it for the reasons mentioned in Remark 3.1.3. The strong uniqueness results of [43] were also recovered in [23] using a different method of proof which allowed to obtain additional insight on the solution. For instance, the α-Hölder continuity of the solution for arbitrary $\alpha \in (0, 1)$ and the differentiability in $L^2(\Omega \times [0,T], \mathbb{R}^d)$ (here Ω is the path space) with respect to the initial condition. For the latter result see [24].

Finally, we mention a result from [22]. There, strong uniqueness up to life time is obtained for continuous coefficients $\sigma, \mathbf{G}$ satisfying a log-Lipschitz condition (see [22, Theorem B]). The growth condition ([22, Theorem A]) is for a typical choice of growth function as follows

$$\sum_{i,j} \sigma_{ij}^2(x) \leq C(\|x\|^2 \log(\|x\|) + 1), \quad \|\mathbf{G}(x)\| \leq C(\|x\| \log(\|x\|) + 1), \quad \forall x \in \mathbb{R}^d \setminus B_{N_0}$$

for some $N_0 \in \mathbb{N}$, but $\mathbf{G}$ can of course not have any singularities inside B_{N_0}, because of its continuity. This allows for linear growth but not for more in the sense that there cannot be any compensation since the growth conditions are formulated separately for dispersion and drift coefficient.

In Part I, Chapter 3 we develop the analysis to define rigorously the infinitesimal generator L that a solution to (1.1) should have under our assumptions. We first use a result of [69], i.e. that a strongly continuous semigroup of contractions and a generalized Dirichlet form on some L^2-space associated to an extension of L as in (3.3) below, can be constructed. For this construction, one needs some weak divergence free property of the anti-symmetric part of the drift. Theorem 3.1.2 (from [12, Theorem 2.4.1]) implies that one can obtain this property with respect to a measure $m = \rho\, dx$, where ρ is some strictly positive continuous function, under our basic assumptions on $A = (a_{ij})_{1 \leq i,j \leq d}$ and $\mathbf{G}$ as in Theorem 3.1.2. Typically, the density ρ is not explicit and not a probability density but has the regularity $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{0,1-d/p}_{loc}(\mathbb{R}^d)$. Subsequently, we use the elliptic regularity result Proposition 3.1.4 (from [8, Theorem 5.1]) and our parabolic regularity result Theorem 3.1.8 which we derive from results in [2] to obtain the regularity as stated in Proposition 3.1.10 and $(\mathbf{H2})'$. Following the basic idea from [1], we may then use the Dirichlet form method to obtain the existence of a Hunt process $\mathbb{M}$ with transition function $(P_t)_{t>0}$ associated to the mentioned extension of L, with continuous sample paths on the one point compactification $\mathbb{R}^d_\Delta$ of $\mathbb{R}^d$ with Δ (see Theorem 3.2.1). To obtain its existence we crucially make use of the existence of such a Hunt process having continuous sample paths on $\mathbb{R}^d_\Delta$ for merely almost every starting point which we obtain from [79, 78]. Once $\mathbb{M}$ is constructed, we can use standard methods from [34] (see Theorem 3.2.8 and Remark 3.2.9) to arrive at the identification of a weak solution to (1.1) up to ζ.

In Chapter 4, we first develop non-explosion criteria for $\mathbb{M}$. We proved that the solution is non-explosive, if there exists a constant $M > 0$ and some $N_0 \in \mathbb{N}$, such that

$$-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\text{trace}A(x) + \big\langle \mathbf{G}(x), x\big\rangle \leq M\left(\|x\|^2+1\right)\left(\ln(\|x\|^2+1)+1\right) \tag{1.3}$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$. The conditions (1.3) allow for linear growth, for locally unbounded drifts and an interplay between diffusion and drift coefficients such that (even outside B_{N_0}) superlinear growth of $\mathbf{G}$ is possible if $\langle \mathbf{G}(x), x\rangle$ is non-positive and superlinear growth of $\mathbf{G}$ and A is possible if diffusion and drift coefficients compensate each other. Hence (1.3) allows coefficients of (1.1) to be more general than those of existing literature ([30], [22], [43], [83], [23], [84], [85]) in regard to non-explosion criteria for time-homogeneous Itô-SDEs.

Once we have constructed a weak solution up to its explosion time and we restrict our assumptions further to any set of assumptions as in the papers [84, 43, 22] or vice versa, we must by the pathwise uniqueness results of the mentioned papers have that the solutions coincide. Hence our non-explosion criteria (1.3), can be seen as new non-explosion criteria for all the mentioned papers. This idea was first employed in [62]. As application of this idea, we obtain strong uniqueness of (1.1) up to ∞ just under the additional non-explosion condition (1.3) (see Theorem 4.3.1). But we obtain far more than only new non-explosion results. Namely, the pathwise unique solution $(X_t)_{t\geq 0}$ in Theorem 4.3.1 is not only strong but satisfies all previously derived properties. Our general strong Feller property results improved the previous results obtained in [1, Propositions 3.2 and 3.8] and [8, Theorem 2.8] and show the non-optimality of the results in [85]. There $\mathbb{M}$ should be non-explosive to obtain merely the classical strong Feller property (cf. also Remark 3.1.9(iii)). Also, the irreducibility here is just obtained under the mentioned basic assumptions on A and $\mathbf{G}$, whereas the assumptions to obtain irreducibility in [85] appear to be quite strong (see Remark 4.2.15). Additionally, our method provides implicitly a candidate for an invariant measure as well as for a stationary distribution and we derive several explicit sufficient conditions for recurrence and ergodicity, including existence and uniqueness of invariant measures (see Section 4.2). Moreover, we derive moment inequalities for the solution (see Theorem 4.1.4) which complements [23, Proposition 14] and [52, Lemma 3.2 of Section 2.3, Theorem 4.1 of Section 2.4]. All these are advantages over the methods that were previously employed in [30], [43], [84], [85], [23], and we are able to generalize and even improve many of the classical results in the time-homogeneous case for locally bounded coefficients (see [6] and the standard reference [58]) to the case of locally unbounded coefficients (see for instance Remark 4.2.3 and Theorem 4.2.9).

In Section 4.2 we discuss recurrence and other ergodic properties involving and not involving the density ρ. As previously mentioned, ρ is usually not explicit but can be assumed to be explicit (if needed) as explained in Remark 4.2.1, (see also Remark 5.3.7 and Part IV). Using a pointwise parabolic Harnack inequality from [2, Theorem 5], we then show that the underlying generalized Dirichlet form is strictly irreducible in Corollary 4.2.4(i). Consequently, we can apply explicit volume growth conditions from [29] to obtain not only recurrence (cf. Theorem 4.2.7) but also existence of an invariant measure. In the general case, when ρ is not explicitly known, we can also derive explicit recurrence criteria. Theorem 4.2.9, that is applicable just under our basic assumptions on $A = (a_{ij})_{1\le i,j\le d}$ and $\mathbf{G}$, generalizes [58, Chapter 6, Theorem 1.2] which assumes the drift to be locally bounded. Moreover the proof of Theorem 4.2.9 is different from the one of [58, Chapter 6, Theorem 1.2] and uses basic results of [29], as well as strict irreducibility from Corollary 4.2.4(i) and Proposition 4.2.5. In Proposition 4.2.13, we derive again just under our basic assumptions on A and $\mathbf{G}$ an explicit criterion for ergodicity of $\mathbb{M}$, including the existence of a unique invariant measure. Section 4.3 is devoted to the mentioned application to pathwise uniqueness results and Theorem 4.3.1 is our main result in Part I.

Our work not only presents a new approach to existence of weak solutions to time-homogeneous Itô-SDEs with rough coefficients through a Hunt process, but also complements and improves substantially existing literature in regard to general strong Feller properties, non-explosion, irreducibility, recurrence and ergodicity, including existence as well as uniqueness of invariant measures. This is done by profiting a lot from many authors' previous achievements. The most important are found in [1], [2], [12], [13], [29], [34], [62], [69], [78], [79], [84]. In particular, the transition function of the Hunt process that we construct as a weak solution to (1) has so a nice regularity that then all presumably optimal classical conditions for the properties of a solution to (1) above, carry over to our situation of non-smooth coefficients by using classical probabilistic techniques. In conclusion, our main result, Theorem 4.3.1, seems to be the most general result in non-degenerate time-homogeneous Itô-SDEs.

An important subject of our research is the existence of invariant measures. A locally finite Borel measure m on $\mathbb{R}^d$ is called an invariant measure for a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t)_{t>0}$ on $L^1(\mathbb{R}^d, m)$ if

$$\int_{\mathbb{R}^d} \overline{T}_t f dm = \int_{\mathbb{R}^d} f dm, \qquad f \in L^1(\mathbb{R}^d, m). \tag{1.4}$$

(There also exists a consistent Definition 6.2.2 related to right processes). Invariant measures have been studied since long ago, both through analytic and probabilistic approaches (see [46], [32], [6], [59], [33], [11], [12], [44]). Often, only invariant measures that are probability measures, or finite measures are regarded (see for instance, [59], [33], [11], [12], [44]). Especially in [12], one of the main references that study invariant measures through an analytic approach, those invariant measures are always considered as probability measures. However in our case, we study pre-invariant measures whose existence results from (3.2), (5.8), (8.30), (11.3) and these do not need to be finite or probability measures. Our pre-invariant measures are invariant measures if and only if the dual semigroup to $(\overline{T}_t)_{t>0}$ in (1.4) is conservative, and moreover serve as reference measures to get an $L^1(\mathbb{R}^d, m)$-closed extension of the second order partial differential operator which is formally associated as infinitesimal generator (on the test functions $C_0^\infty(\mathbb{R}^d)$) to the solution to (1.1). The latter is used crucially used for the construction of a generalized Dirichlet form. The existence of a pre-invariant measure is proven by a purely analytic method which is the existence and regularity theory of elliptic partial differential equations. Throughout all parts in this thesis, our pre-invariant measures play a key role to obtain our various results.

Part II consists of the contents in [50] and we investigate a quite general class of divergence form operators with respect to a possibly non-symmetric diffusion matrix

$A = (a_{ij})_{1\le i,j\le d}$ and perturbation $\mathbf{H} = (h_1, ..., h_d)$, which can be written as

$$Lf = \frac{1}{2}\mathrm{div}(A\nabla f) + \langle \mathbf{H}, \nabla f\rangle, \quad f \in C_0^\infty(\mathbb{R}^d). \tag{1.5}$$

Precise conditions on the coefficients are given in assumptions (a) and (b) in Section 5.2, see in particular Remark 5.2.1, where it is also shown that such operators cover a fairly general class of non-divergence form operators.

Our first observation is that just under assumption (a), there exists a pre-invariant density ρ, which further determines a pre-invariant measure $m = \rho\, dx$, and has a nice regularity (see Theorem 5.2.2). This leads by a construction method of [69] to a sub-Markovian C_0-semigroup of contractions $(T_t)_{t\ge 0}$ on $L^1(\mathbb{R}^d, m)$, whose generator is an extension of $(L, C_0^\infty(\mathbb{R}^d))$, i.e. we have found a suitable functional analytic frame for the description of $(L, C_0^\infty(\mathbb{R}^d))$. This functional analytic frame is also described by a generalized Dirichlet form. Subsequently in Section 5.3, we investigate the regularity properties of the semigroup $(T_t)_{t\ge 0}$ and its corresponding resolvent $(G_\alpha)_{\alpha>0}$, which can in fact be considered in every $L^r(\mathbb{R}^d, m)$, $r \in [1, \infty]$. The regularity properties comprise strong Feller properties, i.e. the existence of continuous versions $P_t f$, $f \in L^1(\mathbb{R}^d, m) + L^\infty(\mathbb{R}^d, m)$ and $R_\alpha g$, $g \in L^q(\mathbb{R}^d, m) + L^\infty(\mathbb{R}^d, m)$, q defined as in Section 5.2, of $T_t f$ and $G_\alpha g$, as well as the irreducibility of $(P_t)_{t>0}$ and strict irreducibility of the associated $L^2(\mathbb{R}^d, m)$-semigroup $(T_t)_{t>0}$ (Lemma 4.2.2).

For more general coefficients A, $\mathbf{G}$ than those in [13, Theorem 1 (i)], we prove by different method the existence of a pre-invariant measure of L in Theorem 5.2.2, especially making use of Lemma 5.1.3, Lemma 5.1.4. Although the proofs of Theorem 5.3.1, Theorem 5.3.3, Theorem 5.3.5 seem to be similar to those of (3.9), Theorem 3.1.8, Theorem 4.2.2, the details are slightly different. In contrast to previous results ([10], [1], [8], [62]), where regularity theory of equations whose solutions are measures is used, we use elliptic and parabolic regularity theory for divergence form operators, which allows the diffusion and drift coefficients to be more general.

In Chapter 6, we investigate the stochastic counterpart of $(P_t)_{t>0}$. Adding just assumption (b) to assumption (a) suffices to obtain that $(P_t)_{t>0}$ is the transition function of a Hunt process $\mathbb{M}$ and to carry over most of the probabilistic results from Part I to the more general situation considered here (see Remark 6.1.2 and Theorem 6.1.3 which states that $\mathbb{M}$ solves weakly the stochastic differential equation with coefficients given by L), i.e. for all $x_0 \in \mathbb{R}^d$,

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t \left(\frac{1}{2}\nabla A^T + \mathbf{H}\right)(X_s)ds, \quad \mathbb{P}_{x_0}\text{- a.s.} \quad 0 \le t < \zeta. \tag{1.6}$$

where $\sigma = (\sigma_{ij})_{1\le i,j\le d}$ is any matrix of functions satisfying $\widetilde{A} = \sigma\sigma^T$. Our conditions for weak existence of Itô-SDEs allow the drift vector field to be in $L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $q \in (\frac{d}{2}, d)$. It seems to be the most general condition for drift vector fields in the literature up to now. However our condition to obtain weak existence requires the components of the diffusion coefficient $\widetilde{A}$ to be in $H^{1,2}_{loc}(\mathbb{R}^d)$ and $\nabla A^T \in L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, which is slightly less general than previous results that allow for bounded and continuous diffusion coefficients as in [71, Theorem 7.2.1] or just bounded and measurable diffusion coefficients as in [38, Chapter 2, Theorem 6.1]. Under our assumptions (a), (b), it is not clear at present whether pathwise uniqueness for (6.1) holds or not. We present some new non-explosion condition, which leads to a moment inequality. It also allows for $L^q(\mathbb{R}^d, m)$-singularities outside an arbitrarily large compact set and linear growth at the same time. This is illustrated in the Example 6.1.5. In Section 6.2, we discuss the relation of $L^1(\mathbb{R}^d, m)$-uniqueness from [69], the strong Feller property derived here and uniqueness in law. More precisely, we obtain a result on uniqueness in law among all right processes that have m as sub-invariant measure (see Propositions 6.2.3 and 6.2.4).

In order to obtain the strong Markov property of a weak solution to (1.6) through the method developed by Strook and Varadhan as in [37, Theorem 4.20], the knowledge of uniqueness in law is crucially needed (see [82, Proposition 2]). But since our weak solution to (1.6) is a Hunt process, it automatically satisfies the strong Markov property independently of uniqueness in law. Moreover, different from the previous methods that require the result for uniqueness in law to obtain a local weak solution to Itô-SDEs whose coefficients are locally bounded (see [71, Chapter 10.1]), we directly obtain a local weak solution without using uniqueness in law even in the case of locally unbounded drift vector fields.

Finally, we would like to discuss a special aspect of our work, which we think is remarkable and to relate our work to some other references. The Hunt process $\mathbb{M}$ which is constructed in Part II satisfies the following Krylov type estimate: let $g \in L^r(\mathbb{R}^d, m)$

for some $r \in [q, \infty]$. Then for any Euclidean ball B there exists a constant $c_{B,r,t}$, depending in particular on B, t, and r, but not on $g \in L^r(\mathbb{R}^d, m)$, such that for all $t \geq 0$

$$\sup_{x \in \overline{B}} \mathbb{E}_x \left[\int_0^t g(X_s)\, ds \right] < c_{B,r,t}\, \|g\|_{L^r(\mathbb{R}^d,m)}. \tag{1.7}$$

Using Theorem 5.3.1 below, (1.7) can be shown exactly as in Lemma 3.2.3 (ii). Such type of estimate is an important tool for the analysis of diffusions (see for instance [38] and in particular [38, p.54, 4. Theorem] for the original estimate involving conditional expectation, or also [30] and [84]). A priori (1.7) only holds for the Hunt process $\mathbb{M}$ constructed here. However, if pathwise uniqueness holds (for instance if the coefficients here are locally Lipschitz or under the conditions in [84]), or more generally uniqueness in law holds for the SDE solved by $\mathbb{M}$ with certain given coefficients, then (1.7) holds generally for any diffusion with the given coefficients. If further $g \in L^r(\mathbb{R}^d)$ has compact support, then $\|g\|_{L^r(\mathbb{R}^d,m)}$ in (1.7) can be replaced by $\|g\|_{L^r(\mathbb{R}^d)}$, when $c_{B,r,t}$ is replaced by a constant $c_{B,r,t,\rho}$ that also depends on the values of ρ on the support of g. If $\widetilde{A}$, $\breve{A}$, ρ, $\widetilde{\mathbf{B}}$ are explicitly given, as described in Remark 5.3.7(i), i.e. the case where the generalized Dirichlet form is explicitly given as in [69], then (1.7) holds with explicit ρ and (1.7) can be seen as a Krylov type estimate for a large class of time-homogeneous generalized Dirichlet forms. As a particular example, consider the non-symmetric divergence form case, i.e. the case where $\mathbf{H} \equiv 0$ in (1.5). Then the explicitly given $\rho \equiv 1$ defines a pre-invariant density. Hence m in (1.7) can be replaced by Lebesgue measure in this case. The latter together with some further results of this article complements analytically as well as probabilistically aspects of the works [72], [63], and [75] where also divergence form operators are treated, but where more emphasis is put on the mere measurability of the diffusion matrix and not on the generality of the drift.

In Part III, we present a well-posedness (weak existence and uniqueness in law) result for degenerate Itô-SDEs whose diffusion coefficients and drift vector fields are possibly discontinuous. In the case where the diffusion coefficient is non-degenerate, bounded and uniformly continuous, and the drift vector field is bounded, Strook and Varadhan showed weak existence and uniqueness in law (see [71, Theorem 7.2.1]). However in the case where the diffusion coefficient is degenerate, somewhat restrictive conditions on the diffusion and drift coefficients, namely local Lipschitz continuity and global boundedness are required in [71, Theorem 6.3.4]. On the other hand, the condition for mere weak existence of degenerate Itô-SDEs can be relaxed to bounded and continuous diffusion coefficients and bounded drift vector fields ([37, Theorem 4.22]). To obtain our weak existence, we use the theory of generalized Dirichlet form based on a functional analytic frame and elliptic and parabolic regularity results for PDEs. To do this, we study an analytic theory for second order partial differential operators with possibly degenerate and discontinuous diffusion coefficients, which are given by

$$Lf = \frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 f) + \langle \mathbf{G}, \nabla f \rangle, \qquad f \in C_0^\infty(\mathbb{R}^d), \tag{1.8}$$

where $\widehat{A} := \frac{1}{\psi}A$ and A, ψ, $\mathbf{G}$ satisfy **(A1)** in Section 8.3.

In Chapter 7, we investigate some regularity results for linear parabolic PDEs involving divergence form operators with weight function in the time derivative term as in (7.1). The weight function is bounded below by a positive constant. Developing the arguments in [2], we derive an L^∞-estimate of solutions of weighted parabolic PDEs in terms of the $L^{\frac{2p}{p-2}}$-norm, where $p > d$ is arbitrary but fixed. Besides, we present the standard elliptic Hölder regularity and Hölder estimate of solutions in terms of the L^2-norm which were proved in [67].

In Sections 8.1, 8.2 of Chapter 8, using the main ideas and techniques from [69], we improve the L^1-existence result for elliptic second order partial differential operators with degenerate diffusion coefficients defined as (1.8). Our pre-invariant density is $\rho\psi$ and $\rho\psi\widehat{A} = \rho A$ is non-degenerate since $\rho \in L^\infty_{loc}(\mathbb{R}^d)$ is a positive function satisfying $\frac{1}{\rho} \in L^\infty_{loc}(\mathbb{R}^d)$, so that our arguments are connected with the methods of [69] and regularity results of Chapter 7 involving a non-degenerate matrix of functions. In Section 8.3, we first show in Theorem 8.3.1 the existence of a pre-invariant measure $\rho\psi dx$ for L in (1.8) and ρ has nice regularity. Although we did not derive parabolic Hölder regularity, by combining regularity results of Chapter 7 and our main arguments developed in Part I and II, we derive general strong Feller properties of our semigroup as well as resolvent (Theorem 8.3.3, Lemma 8.3.4, Theorem 8.3.6).

In Chapter 9, using one of the main arguments from Part I, Theorem 3.2.1, we construct a Hunt process associated with a general strong Feller semigroup $(P_t)_{t>0}$. Then we identify our constructed Hunt process with a weak solution to the corresponding degenerate Itô-SDE whose diffusion coefficients are possibly discontinuous. To obtain the existence of a Hunt process as a weak solution to degenerate Itô-SDEs starting at every point in $\mathbb{R}^d$, we should make use of the existence of such a Hunt process for merely almost every starting point, which is showed in Proposition 9.1.1.

Note that Krylov type estimate in Remark 9.1.4 are derived by an elliptic Hölder regularity and an estimate of the resolvent (Theorem 7.2.2), which is distinct from Theorem 5.3.1 that is induced by elliptic $H^{1,p}$-regularity results. The integral orders in the right-hand side of the Krylov type estimate are usually bigger than those of (1.7), but the constant $C_{B,r,t}$ of (9.1) does not depend on the VMO condition of the diffusion coefficient. We mention that some of the conservativeness criteria which are analogous to those of Part I, for instance, Theorem 4.1.2, Theorem 4.1.4 (i) as well as [69, Proposition 1.10](a) also can be applied to our constructed Hunt process. Furthermore if we consider a special weight function like $\psi := \frac{1}{\|x\|^\alpha}$ for some $\alpha > 0$ which has only one singular point in $\mathbb{R}^d$, we can show that strict irreducibility holds (Lemma 9.2.1, Corollary 9.2.2). Therefore recurrence and transience results as in Proposition 4.2.5, Theorem 4.2.7, Lemma 4.2.8, Theorem 4.2.9 can be applied to our constructed Hunt process if $\psi = \frac{1}{\|x\|^\alpha}$. We present a concrete example in Example 9.2.3 that satisfies our results for weak existence and strict irreducibility.

In Section 9.3, assuming **(A4′)**, we show uniqueness in law for our degenerate Itô-SDEs whose dispersion matrix and drift vector field are possibly discontinuous. Our results are new in the sense that examples for uniqueness in law in the case of fully discontinuous dispersion matrix seem to be unknown. The local Krylov type estimate for the solution of our degenerate Itô-SDE plays an important role to derive a time dependent Itô's formula for weak differentiable functions with certain regularity. Moreover, we apply elliptic $H^{2,2d+2}$- regularity results for non-divergence form operators to our resolvent and use the properties of the semigroup which directly solves the Cauchy problem. Since our semigroup is closely related to our resolvent which has nice regularity, parabolic regularity results involving degenerate matrix of functions are not needed in our case. Our result for uniqueness in law allows for fully discontinuous dispersion matrix and it partially improves [41, Theorem 3.11] as well as [74, Theorem 3.1] in the case of time-homogeneous Itô-SDEs.

In Part IV, we generalize the results of Part I, II, considering the case where pre-invariant measures with general conditions are given. Using [29, Lemma 13] which improves [69, Theorem 1.5], more general pre-invariant measures than those in [69] can be investigated. We expect that results in Part IV can be used to show not only general strong Feller properties of transition functions of Hunt processes which have skew reflections or normal reflections, but also to show weak existence for SDEs with reflection terms (see Remark 11.1.1, 11.1.8, 11.2.2). For the sectorial case, one can use the analyticity of the semigroup and the conservativeness of the resolvent to obtain the classical strong Feller property of the semigroup as in [1], [8], [9], [7], [62]. But since we use generalized Dirichlet form techniques and the elliptic and parabolic regularity theory for divergence form operator, it is possible to derive not only general strong Feller properties including the classical strong Feller property but also strict irreducibility and irreducibility of the semigroup without sector condition assumption.

To do this, in Chapter 10, we generalize some parabolic regularity results of [2] in the case where the weight function ψ in the time derivative term is bounded below and above by some positive constants. Different from Part III, since the weight in Part IV is bounded below and above by some positive constants, we can derive a parabolic Harnack inequality as well as the L^{∞}- estimate in terms of the L^2-norm. Thus we can show that the solutions of linear parabolic PDEs involving divergence form operators with a weight function ψ in the time derivative term satisfy a Hölder regularity result and a pointwise parabolic Harnack inequality, which allow us to show general strong Feller properties, irreducibility and strict irreducibility of our semigroup. The proof of the Harnack inequality is based on the fundamental inequality (10.5) and Lemma 10.2.1 which involve the weight function ψ. Then using the technique of the proof of [2, Theorem 3] and [53, Main Lemma], we derive the parabolic Harncak inequality (Theorem 10.2.2), which also partially improves the result [73, Property II] where symmetric Dirichlet forms on abstract spaces are treated and their pre-invariant measures are more general than ours. Since we only treat about weighted parabolic PDEs of linear type and assume boundedness of solutions of our PDEs, some procedures to derive regularity results of solutions are simpler than those in [2] that considers quasi-linear parabolic PDEs and does not assume the boundedness of the solution. However since our parabolic PDEs are always formulated with weight functions in the time derivative term, we rigorously check the details.

In Chapter 11, using methods as in Part I, II, III and regularity results of Chapter 10, we present a weak existence result in the case where a general pre-invariant measure and diffusion coefficients are given, and we obtain analytic and probabilistic results which are analogous to those of Part II, general strong Feller properties including classical strong Felller property, strict irreducibility and irreducibility, non-explosion, recurrence and transience, ergodic properties (see Theorem 11.2.4). We would like to emphasize that the Krylov type estimates (11.8) of our constructed Hunt process also hold. But since we use elliptic Hölder regularity and estimates in Theorem 7.2.2 like in the case of (9.1), the constant $C_{B,r,t} > 0$ does not depend on the VMO condition of its diffusion matrix. We expect that in our later research, this Krylov type estimate would play an important role to study some approximations of stochastic processes with merely measurable diffusion coefficients which have no weak differentiability. In Section 11.2, we consider the case where general diffusion coefficients and drifit vector fields which are possibly discontinuous, are explicitly given. In Theorem 11.3.1, we find a pre-invariant measure using Theorem 5.2.2, hence obtain exactly the same framework as in Part IV where general pre-invariant measures are explicitly given. Through this work, we obtain in Theorem 11.3.2 up to our best knowledge the present most general results for global well-posedness and ergodic properties of non-degenerate time-homogeneous Itô-SDEs whose dispersion coefficients are possibly discontinuous

The work also shown that the previously used techniques to handle the Itô-SDE (1.1) for the last 20 years, mainly based on Krylov type estimates and Girsanov transformation, seem not to be the appropriate and optimal ones. Through the research in this thesis which is an analytic approach to time-homogeneous Itô-SDEs with rough coefficients using generalized Dirichlet form theory and elliptic and parabolic regularity theory, we hope to provide a new tool for the study of Itô-SDEs and their applications.

Chapter 2

Notations

Throughout, we consider the Euclidean space $\mathbb{R}^d$, $d \geq 2$, equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle$, the Euclidean norm $\| \cdot \|$ and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. We write $| \cdot |$ for the absolute value in $\mathbb{R}$. For $r \in \mathbb{R}$, $r > 0$ and $x \in \mathbb{R}^d$, let $B_r(x) := \{y \in \mathbb{R}^d \,|\, \|x - y\| < r\}$ and denote its closure by $\overline{B}_r(x)$ (similarly for a subset $A \subset \mathbb{R}^d$, let $\overline{A}$ denote its closure). If $x = 0$, we simply write B_r and $\overline{B}_r$. We call a subset $B \subset \mathbb{R}^d$, for which $B = B_r(x)$ for some $r > 0$ and $x \in \mathbb{R}^d$, a ball. Let $R_x(r)$ denote the open cube in $\mathbb{R}^d$ with edge length $r > 0$ and center $x \in \mathbb{R}^d$ and denote its closure by $\overline{R}_x(r)$. The minimum of two values a and b is denoted by $a \wedge b := \min(a, b)$ and the maximum is denoted by $a \vee b := \max(a, b)$. For two sets A, B, we define $A + B := \{a + b \,|\, a \in A \text{ and } b \in B\}$.

The set of all $\mathcal{B}(\mathbb{R}^d)$-measurable $f : \mathbb{R}^d \to \mathbb{R}$ which are bounded, or nonnegative are denoted by $\mathcal{B}_b(\mathbb{R}^d)$, $\mathcal{B}^+(\mathbb{R}^d)$ respectively. Let $U \subset \mathbb{R}^d$, be an open set. The usual L^q-spaces $L^q(U, \mu)$, $q \in [1, \infty]$ of Borel measurable or classes of Borel measurable functions (depending on the context) are equipped with L^q-norm $\| \cdot \|_{L^q(U,\mu)}$ with respect to the measure μ on U and $L^q_{loc}(\mathbb{R}^d, \mu) := \{f \mid f \cdot 1_U \in L^q(\mathbb{R}^d, \mu), \forall U \subset \mathbb{R}^d, U \text{ relatively compact open}\}$, where 1_A denotes the indicator function of a set $A \subset \mathbb{R}^d$. Define $L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu) := \{\mathbf{G} = (g_1, ..., g_d) : \mathbb{R}^d \to \mathbb{R}^d \mid g_i \in L^q_{loc}(\mathbb{R}^d, \mu), 1 \leq i \leq d\}$. Given any open set U in $\mathbb{R}^d$, define $L^q(U, \mathbb{R}^d, \mu) := \{\mathbf{F} = (f_1, ..., f_d) : U \to \mathbb{R}^d \mid f_i \in L^q(U, \mu), 1 \leq i \leq d\}$, equipped with the norm, $\|\mathbf{F}\|_{L^q(U,\mu)} := \| \|\mathbf{F}\| \|_{L^q(U,\mu)}$, $\mathbf{F} \in L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$. The Lebesgue measure on $\mathbb{R}^d$ is denoted by dx and we write $L^q(U)$, $L^q_{loc}(\mathbb{R}^d)$, $L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $L^q(U, \mathbb{R}^d)$ for $L^q(U, dx)$, $L^q_{loc}(\mathbb{R}^d, dx)$, $L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d, dx)$, $L^q(U, \mathbb{R}^d, dx)$

respectively.
For an open set U in $\mathbb{R}^d$, define $|U| := \int_U 1dx$. For an open interval I in $\mathbb{R}$ and $p, q \in [1, \infty]$, denote by $L^{p,q}(U \times I)$ the set of Borel measurable function f on $U \times I$ such that

$$\|f\|_{L^{p,q}(U\times I)} := \|\|f(\cdot,\cdot)\|_{L^p(U)}\|_{L^q(I)} < \infty.$$

In order to avoid notational complications, we assume that locally integrable functions are whenever necessary pointwisely given (not for instance equivalence classes) and hence measurable. Moreover, whenever a function f possesses a continuous version, we will assume it is given by it. However, if in a situation, it should be necessary or important to distinguish between classes and pointwisely given functions, we will mention it. If $\mathcal{A}$ is a set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, we define $\mathcal{A}_0 := \{f \in \mathcal{A} \mid \text{supp}(f) := \text{supp}(|f|dx) \text{ is compact in } \mathbb{R}^d\}$ and $\mathcal{A}_b := \mathcal{A} \cap L^\infty(\mathbb{R}^d)$. As usual, we also denote the set of continuous functions on $\mathbb{R}^d$, the set of continuous bounded functions on $\mathbb{R}^d$, the set of compactly supported continuous functions in $\mathbb{R}^d$ by $C(\mathbb{R}^d)$, $C_b(\mathbb{R}^d)$, $C_0(\mathbb{R}^d)$, respectively. Two Borel measurable functions f and g are called μ-versions of each other, if $f = g$ μ-a.e.
Given Borel measurable function f on open subset U of $\mathbb{R}^d$, let $\nabla f := (\partial_1 f, \dots, \partial_d f)$, where $\partial_j f$ is the j-th weak partial derivative of f on U of $\mathbb{R}^d$ and $\partial_{ij} f := \partial_i(\partial_j f)$, $i, j = 1, \dots, d$. The Sobolev space $H^{1,q}(U)$, $q \in [1, \infty]$ is defined to be the set of all functions $f \in L^q(U)$ for which $\partial_j f \in L^q(U)$, $j = 1, \dots, d$, and $H^{1,q}_{loc}(U) := \{f : f \cdot \varphi \in H^{1,q}(U), \forall \varphi \in C_0^\infty(U)\}$. Here $C_0^k(U)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the set of all k-fold continuously differentiable functions with compact support in U, and $C_\infty(U)$ denote the set of continuous functions vanishing at infinity, i.e. given $\varepsilon > 0$, there exists a compact set $K \subset U$ such that $|f(x)| < \varepsilon$ for all $x \in U \setminus K$. For Borel measurable function g on open subset Q of $\mathbb{R}^d \times \mathbb{R}$, given $i \in \{1, \dots d\}$, denote by $\partial_i g$ the i-th weak spatial derivative on Q and by $\partial_t g$ the weak time derivative on Q. For $p, q \in [1, \infty]$, let $W^{2,1}_{p,q}(Q)$ be a set of locally integrable functions $g : Q \to \mathbb{R}$ such that $\partial_t g$, $\partial_i g$, $\partial_i \partial_j g \in L^{p,q}(Q)$ for all $1 \leq i, j \leq d$. Let $W^{2,1}_p(Q) := W^{2,1}_{p,p}(Q)$.
Let V be a bounded open set in $\mathbb{R}^d$ and $f : \overline{V} \to \mathbb{R}$ be a continuous function. Define

$\|f\|_{C(\overline{V})} := \sup_{\overline{V}} f$. For $\beta \in (0,1)$ define

$$\text{höl}_\beta(f,\overline{V}) := \sup\left\{\frac{|f(x)-f(y)|}{\|x-y\|^\beta} : x,y\in\overline{V}, x\neq y\right\} \in [0,\infty],$$

and the Hölder continuous functions of order $\beta \in (0,1)$ on $\overline{V}$ by

$$C^{0,\beta}(\overline{V}) := \{f\in C(\overline{V}) : \text{höl}_\beta(f,\overline{V}) < \infty\}.$$

Then $C^{0,\beta}(\overline{V})$ is a Banach space with norm

$$\|f\|_{C^{0,\beta}(\overline{V})} := \sup_{x\in\overline{V}} |f(x)| + \text{höl}_\beta(f,\overline{V}).$$

The space of all locally Hölder continuous functions of order $\beta \in (0,1)$ on $\mathbb{R}^d$ is defined by

$$C^{0,\beta}_{loc}(\mathbb{R}^d) := \{f \,:\, f\in C^{0,\beta}_{loc}(\overline{B}) \text{ for any ball } B\}.$$

Let Q be a bounded open set in $\mathbb{R}^d\times\mathbb{R}$ and $g:\overline{Q}\to\mathbb{R}$ be a function. For $\delta\in(0,1)$ denote

$$\text{phöl}_\delta(g,\overline{Q}) := \sup\left\{\frac{|g(x,t)-g(y,s)|}{\left(\|x-y\|+\sqrt{|t-s|}\right)^\delta} :\ (x,t),(y,s)\in\overline{Q},\ (x,t)\neq(y,s)\right\} \in [0,\infty],$$

and the parabolic Hölder continuous functions of order $\delta\in(0,1)$ on $\overline{Q}$ by

$$C^{\delta;\frac{\delta}{2}}(\overline{Q}) := \{g\in C(\overline{Q}) : \text{phöl}_\delta(g,\overline{Q}) < \infty\}.$$

Then $C^{\delta;\frac{\delta}{2}}(\overline{Q})$ is a Banach space with norm

$$\|g\|_{C^{\delta;\frac{\delta}{2}}(\overline{Q})} := \sup_{(x,t)\in\overline{Q}} |g(x,t)| + \text{phöl}_\delta(g,\overline{Q}).$$

g is called locally parabolic Hölder continuous, if for any bounded and open set Q, there exists $\delta=\delta(Q)$, such that $g\in C^{\delta;\frac{\delta}{2}}(\overline{Q})$. Here δ may be different for different Q. In particular, if $t\in\mathbb{R}$ is fixed, we then say that $g(\cdot,t)$ is locally Hölder continuous with

possibly changing Hölder exponents.
For a matrix A, let A^T denote the transposed matrix of A. If $A = (a_{ij})_{1\le i,j\le d}$ consists of weakly differentiable functions a_{ij}, we define

$$\nabla A = ((\nabla A)_1, \dots, (\nabla A)_d), \qquad (\nabla A)_i := \sum_{j=1}^{d} \partial_j a_{ij}, \quad 1 \le i \le d.$$

If f is two times weakly differentiable, let $\nabla^2 f$ denote the Hessian matrix of second order weak partial derivatives of f. In particular

$$\mathrm{trace}(A\nabla^2 f) = \sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j f.$$

If ρ is weakly differentiable and a.e. positive then

$$\beta^{\rho,A} = (\beta_1^{\rho,A}, \dots, \beta_d^{\rho,A}) := \frac{1}{2}\Big(\nabla A + \frac{A\nabla\rho}{\rho}\Big),$$

is called the logarithmic derivative of ρ associated with A. Hence

$$\beta_i^{\rho,A} = \frac{1}{2}\sum_{j=1}^{d}\Big(\partial_j a_{ij} + a_{ij}\frac{\partial_j\rho}{\rho}\Big), \quad 1 \le i \le d.$$

For a Borel measurable function ψ, define $\beta^{\rho,A,\psi} := \frac{1}{\psi}\beta^{\rho,A}$. For a bounded open subset U of $\mathbb{R}^d$ and a possibly non-symmetric matrix of functions $A = (a_{ij})_{1\le i,j\le d}$ on U, we say that A is *uniformly strictly elliptic and bounded* on U, if there exists $\lambda > 0$ and $M > 0$ such that for any $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, $x \in U$,

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \lambda\|\xi\|^2, \qquad \max_{1\le i,j\le d} |a_{ij}(x)| \le M.$$

In that case, λ is called the elliptic constant and M is called the upper bound constant of A.

Part I

Existence, uniqueness and ergodic properties for time-homogeneous Itô-SDEs with locally integrable drifts and Sobolev diffusion coefficients

Chapter 3

Weak solutions via analytic theory

3.1 Analytic theory of generalized Dirichlet forms

Let $\phi \in H^{1,2}_{loc}(\mathbb{R}^d)$ be such that the measure $m := \rho\, dx$, $\rho := \phi^2$, has full support on $\mathbb{R}^d$. Let $H^{1,2}_0(\mathbb{R}^d, m)$ be the closure of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, m)$ with respect to the norm $(\int_{\mathbb{R}^d} (\|\nabla f\|^2 + f^2)\, dm)^{1/2}$ and $H^{1,2}_{loc}(\mathbb{R}^d, m) := \{f : f \cdot \varphi \in H^{1,2}_0(\mathbb{R}^d, m),\ \forall \varphi \in C_0^\infty(\mathbb{R}^d)\}$. Let $A = (a_{ij})_{1 \le i,j \le d}$ with $a_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d, m)$ be a symmetric matrix of functions and locally uniformly strictly elliptic, i.e. for every (open) ball $B \subset \mathbb{R}^d$ there exist real numbers $\lambda_B, \Lambda_B > 0$, such that

$$\lambda_B \|\xi\|^2 \le \left\langle A(x)\xi, \xi \right\rangle \le \Lambda_B \|\xi\|^2 \ \text{ for all } \ \xi \in \mathbb{R}^d,\ x \in B. \tag{3.1}$$

Let $\mathbf{G} = (g_1, ..., g_d) \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, m)$ be such that with

$$Lf := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \partial_i \partial_j f + \sum_{i=1}^{d} g_i \partial_i f, \qquad f \in C_0^\infty(\mathbb{R}^d),$$

it holds

$$\int_{\mathbb{R}^d} Lf\, dm = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d). \tag{3.2}$$

Then it is shown in [69, Theorem 1.5] that there exists a closed extension $(L_1, D(L_1))$ on $L^1(\mathbb{R}^d, m)$ of $(L, C_0^\infty(\mathbb{R}^d))$ that generates a sub-Markovian C_0-semigroup of contractions $(T_t)_{t>0}$. Restricting $(T_t)_{t>0}$ to $L^1(\mathbb{R}^d, m)_b$, it is well-known by Riesz-Thorin interpolation that $(T_t)_{t>0}$ can be extended to a sub-Markovian C_0-semigroup of contractions on each $L^r(\mathbb{R}^d, m)$, $r \in [1, \infty)$. Denote by $(L_r, D(L_r))$ the corresponding closed generator with graph norm

$$\|f\|_{D(L_r)} := \|f\|_{L^r(\mathbb{R}^d,m)} + \|L_r f\|_{L^r(\mathbb{R}^d,m)},$$

and by $(G_\alpha)_{\alpha>0}$ the corresponding resolvent. For $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ we do not explicitly denote in the notation on which $L^r(\mathbb{R}^d, m)$-space they act. We assume that this is clear from the context. Moreover, $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ can be uniquely defined on $L^\infty(\mathbb{R}^d, m)$, but are no longer strongly continuous there.

Writing

$$Lf \quad = \quad \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j f + \sum_{i=1}^{d} \beta_i^{\rho,A}\partial_i f + \sum_{i=1}^{d}(g_i - \beta_i^{\rho,A})\partial_i f \tag{3.3}$$

with

$$\beta_i^{\rho,A} \quad = \quad \frac{1}{2}\sum_{j=1}^{d}\left(\partial_j a_{ij} + a_{ij}\frac{\partial_j \rho}{\rho}\right), \ 1 \le i \le d, \quad \beta^{\rho,A} = (\beta_1^{\rho,A}, ..., \beta_d^{\rho,A})$$

we observe that (3.2) is equivalent to

$$\int_{\mathbb{R}^d} \langle \mathbf{G} - \beta^{\rho,A}, \nabla f \rangle \, dm = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d), \tag{3.4}$$

hence

$$\int_{\mathbb{R}^d} \widehat{L} f \, dm = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d), \tag{3.5}$$

where

$$\widehat{L}f = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j f + \sum_{i=1}^{d}\beta_i^{\rho,A}\partial_i f - \sum_{i=1}^{d}(g_i - \beta_i^{\rho,A})\partial_i f \tag{3.6}$$

Noting that $\widehat{g}_i := 2\beta_i^{\rho,A} - g_i \in L^2_{loc}(\mathbb{R}^d, m)$, we see that L and $\widehat{L}$ have the same structural properties, i.e. they are given as the sum of a symmetric second order elliptic differential operator and a divergence free first order perturbation with same integrability condition with respect to the measure m. Therefore all what will be derived below for L will hold analogously for $\widehat{L}$. Denote the operators corresponding to $\widehat{L}$ (again defined through [69, Theorem 1.5]) by $(\widehat{L}_r, D(\widehat{L}_r))$ for the co-generator on $L^r(\mathbb{R}^d, m)$, $r \in [1, \infty)$, $(\widehat{T}_t)_{t>0}$ for the co-semigroup, $(\widehat{G}_\alpha)_{\alpha>0}$ for the co-resolvent. By [69, Section 3], we obtain a corresponding bilinear form with domain $D(L_2) \times L^2(\mathbb{R}^d, m) \cup L^2(\mathbb{R}^d, m) \times D(\widehat{L}_2)$ by

$$\mathcal{E}(f,g) := \begin{cases} -\int_{\mathbb{R}^d} L_2 f \cdot g\, dm & \text{for } f \in D(L_2),\ g \in L^2(\mathbb{R}^d, m), \\ -\int_{\mathbb{R}^d} f \cdot \widehat{L}_2 g\, dm & \text{for } f \in L^2(\mathbb{R}^d, m),\ g \in D(\widehat{L}_2). \end{cases}$$

$\mathcal{E}$ is called the *generalized Dirichlet form associated with* $(L_2, D(L_2))$. Using integration by parts, it is easy to see that

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\mathbb{R}^d}\langle A\nabla f, \nabla g\rangle\, dm - \int_{\mathbb{R}^d}\langle \mathbf{G} - \beta^{\rho,A}, \nabla f\rangle g\, dm, \quad f, g \in C_0^\infty(\mathbb{R}^d). \tag{3.7}$$

The following lemma, see [69, Remark 1.7(iii)], will be used later:

Lemma 3.1.1. *Let $u \in D(L_1)_b$. Then $u^2 \in D(L_1)_b$ and*

$$L_1 u^2 = \langle A\nabla u, \nabla u\rangle + 2uL_1 u.$$

We are going to restrict our previous assumptions to the ones of the following theorem. The theorem itself is an immediate consequence of an important result [12, Theorem 2.4.1] (see also [13, Theorem 1] for the original result), which itself is derived by using elliptic regularity results from [76] in an essential way.

Theorem 3.1.2. *Let $p > d$ be arbitrary but fixed. Let $A := (a_{ij})_{1\le i,j\le d}$ be a symmetric $d \times d$ matrix of functions $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^d)$ satisfying (3.1). Let $\mathbf{G} = (g_1, \ldots, g_d) \in$*

$L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Then there exists $\rho \in C^{0,1-d/p}_{loc}(\mathbb{R}^d) \cap H^{1,p}_{loc}(\mathbb{R}^d)$ with $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ and such that

$$\int_{\mathbb{R}^d} \langle \mathbf{G} - \beta^{\rho,A}, \nabla\varphi \rangle \rho \, dx = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d),$$

with

$$\beta^{\rho,A} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d).$$

In particular, setting

$$\mathbf{B} = (b_1, ..., b_d) := \mathbf{G} - \beta^{\rho,A},$$

we have obtained a representation of an arbitrary $\mathbf{G} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ as the sum of the logarithmic derivative $\beta^{\rho,A}$ associated to A and ρ and a ρdx-divergence free vector field $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, namely

$$\mathbf{G} = \beta^{\rho,A} + \mathbf{B}.$$

Remark 3.1.3. *It is possible and not difficult to generalize Theorem 3.1.2 (and basically everything that follows below) in two directions. We do not do this here because it only leads to technical and notational complications, which are better to be investigated and overcome elsewhere. But all necessary tools can be found in this work. The two directions are:*

(i) Theorem 3.1.2 also holds with $\mathbb{R}^d$ replaced by any open set $U \subset \mathbb{R}^d$, $H^{1,p}_{loc}(U)$ defined as in Chapter 2, and

$$L^p_{loc}(U) := \{f : f1_V \in L^p(U), \forall V \text{ relatively compact open with } \overline{V} \subset U\},$$

$$C^{0,1-d/p}_{loc}(U) := \{f : f \in C^{0,1-d/p}(\overline{V}), \forall V \text{ relatively compact open with } \overline{V} \subset U\},$$

by considering an exhaustion with bounded and open sets $(V_n)_{n\geq 1}$ of U, i.e.

$$V_n \subset \overline{V}_n \subset V_{n+1} \text{ for all } n \in \mathbb{N} \text{ and } \cup_{n=1}^\infty V_n = U.$$

(ii) As in [12, Theorem 2.4.1], the regularity conditions on a_{ij}, g_i, $1 \leq i, j \leq d$, can be generalized to $a_{ij} \in H^{1,p_n}(B_n)$ and $g_i \in L^{p_n}(B_n)$ with $p_n > d$. The only interesting case is when $\lim_{n\to\infty} p_n = d$, which leads to a slight but technical improvement of the conditions of Theorem 3.1.2. Note that $(B_n)_{n\geq 1}$ here is a special exhaustion

with bounded and open sets of $\mathbb{R}^d$ *but one can generalize this to an arbitrary exhaustion with bounded and open sets* $(V_n)_{n\geq 1}$ *of* $\mathbb{R}^d$*.*

From now on unless otherwise stated, we fix one density ρ as in Theorem 3.1.2 and hence assume that

$$A := (a_{ij})_{1\leq i,j\leq d},\ \mathbf{G} = (g_1,\dots,g_d),\ \beta^{\rho,A} = (\beta_1^{\rho,A},...,\beta_d^{\rho,A}),\ \mathbf{B} = (b_1,...,b_d),$$

are as in Theorem 3.1.2 with

$$p > d.$$

This implies all assumptions prior to Theorem 3.1.2 and we fix from now on the corresponding generalized Dirichlet form $\mathcal{E}$ associated with $(L_2, D(L_2))$ and all the corresponding objects under the assumptions of Theorem 3.1.2. As before, we set

$$m := \rho\, dx.$$

Note, that due to the properties of ρ in Theorem 3.1.2, we have that $L^p_{loc}(\mathbb{R}^d) = L^p_{loc}(\mathbb{R}^d, m)$ as well as $L^p_{loc}(\mathbb{R}^d,\mathbb{R}^d) = L^p_{loc}(\mathbb{R}^d,\mathbb{R}^d, m)$.

We will use the following result from [8, Theorem 5.1], adapted to our needs.

Proposition 3.1.4. *Let* $d \geq 2$ *and* μ *a locally finite (signed) Borel measure on* $\mathbb{R}^d$ *that is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^d$*. Let* $A = (a_{ij})_{1\leq i,j\leq d}$ *and* $p > d$ *be as in Theorem 3.1.2. Let* $h_i, c, f \in L^p_{loc}(\mathbb{R}^d)$ *and assume that*

$$\int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^d \frac{a_{ij}}{2}\partial_{ij}\varphi + \sum_{i=1}^d h_i\partial_i\varphi + c\varphi\Big)\, d\mu = \int_{\mathbb{R}^d} \varphi f\, dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d),$$

where h_i*,* c *are locally* μ*-integrable. Then* μ *has a density in* $H^{1,p}_{loc}(\mathbb{R}^d)$ *that is locally Hölder continuous.*

We further state a result originally due to Morrey (see the wrong statement in the original monograph [55, Theorem 5.5.5'] and [12, Theorem 1.7.4] and Corollaries for its correction).

Proposition 3.1.5. *Assume $p > d \geq 2$. Let $B' \subset \mathbb{R}^d$ be a ball, $h = (h_1, ..., h_d) : B' \to \mathbb{R}^d$ and $c, e : B' \to \mathbb{R}$ such that*

$$h_i \in L^p(B'), 1 \leq i \leq d, \quad \text{and} \quad c, e \in L^q(B') \quad \text{for} \quad q := \frac{dp}{d+p}.$$

Let $A = (a_{ij})_{1 \leq i,j \leq d}$ be as in Theorem 3.1.2. Assume that $u \in H^{1,p}(B')$ is a solution of

$$\int_{B'} \sum_{i=1}^{d} \Big(\partial_i \varphi \Big(\sum_{j=1}^{d} \frac{a_{ij}}{2} \partial_j u + h_i u \Big) \Big) + \varphi (cu + e) \, dx = 0, \quad \forall \varphi \in C_0^\infty(B'),$$

Then for every ball B with $\overline{B} \subset B'$, we obtain the estimate

$$\|u\|_{H^{1,p}(B)} \leq c_0 (\|e\|_{L^q(B')} + \|u\|_{L^1(B')}),$$

where $c_0 < \infty$ is some constant independent of e and u.

Now, we will apply the standard arguments from [1] whose details have been exposed in a very clear way in [8]. We will briefly explain (until and including Remark 3.1.7) the line of arguments how Propositions 3.1.4 and 3.1.5 lead to elliptic regularity results for $(G_\alpha)_{\alpha>0}$ and $(T_t)_{t>0}$ by using well-known arguments (see for instance [1], [8], or [62]). However, as we will see later, we will slightly improve some regularity results compared to the just mentioned papers. First, we choose an arbitrary $g \in C_0^\infty(\mathbb{R}^d)$, $\alpha > 0$. Applying Proposition 3.1.4 with

$$\mu = -\rho \, G_\alpha g \, dx, \; h_i = \beta_i^{\rho, A} - b_i, \; 1 \leq i \leq d, \; c = -\alpha, \; f = g\rho \in L^p_{loc}(\mathbb{R}^d),$$

we obtain $\rho \, G_\alpha g \in H^{1,p}_{loc}(\mathbb{R}^d)$. Then, we apply Proposition 3.1.5 with

$$u = \rho \, G_\alpha g, \; \; h_i = \sum_{j=1}^{d} \left(\frac{\partial_j a_{ij}}{2} - (\beta_i^{\rho, A} - b_i) \right), \; \; 1 \leq i \leq d,$$

and

$$c = \alpha, \;\; e = \rho g \in L^q(B'),$$

where

$$q := \frac{dp}{d+p} \in (d/2, p/2). \tag{3.8}$$

By the properties of ρ, we obtain

$$\|\rho\, G_\alpha g\|_{H^{1,p}(B)} \le c_0 \left(\|g\|_{L^q(B',m)} + \|G_\alpha g\|_{L^1(B',m)} \right),$$

where c_0 is possibly different form the constant in Proposition 3.1.5, but also doesn't depend on g. The last inequality is easily seen to extend to $g \in L^r(\mathbb{R}^d, m)$, $r \in [q, \infty]$, using the contraction properties of $(G_\alpha)_{\alpha>0}$. From that we then get that for any $r \in [q, \infty]$, $\alpha > 0$

$$\|\rho\, G_\alpha g\|_{H^{1,p}(B)} \le c_0 \left(\|g\|_{L^r(B',m)} + \|G_\alpha g\|_{L^1(B',m)} \right), \quad \forall g \in L^r(\mathbb{R}^d, m), \tag{3.9}$$

where c_0 is a constant that may be different for different α and r, but doesn't depend on g. Using the contraction properties of $(G_\alpha)_{\alpha>0}$, (3.9) immediately implies

$$\|\rho\, G_\alpha g\|_{H^{1,p}(B)} \le c_0 \|g\|_{L^r(\mathbb{R}^d,m)}, \quad \forall g \in L^r(\mathbb{R}^d, m), \tag{3.10}$$

where c_0 in (3.9) may be different from c_0 in (3.10) but has the same properties. Writing $T_0 := id$ and

$$T_t f = G_1(1 - L_r) T_t f, \;\; f \in D(L_r), \;\; r \in [q, \infty), \;\; t \ge 0,$$

we can see by (3.9) that for any $r \in [q, \infty)$, $t \ge 0$

$$\|\rho\, T_t f\|_{H^{1,p}(B)} \le c_0 \|T_t f\|_{D(L_r)}, \quad \forall f \in D(L_r), \tag{3.11}$$

where c_0 is a constant that may be different for different r, but doesn't depend on f. By Morrey's inequality applied to an arbitrary ball B, there exists a constant $c > 0$ independent of f such that

$$\|\widetilde{f}\|_{C^{0,\beta}(\overline{B})} \le c \|f\|_{H^{1,p}(B)}, \qquad \forall f \in H^{1,p}(B),$$

where $\widetilde{f}$ on the left hand side is the unique continuous dx-version of $f \in H^{1,p}(B)$ and

$$\beta := 1 - d/p. \tag{3.12}$$

In our situation $\rho \in C^{0,\beta}(\overline{B})$ for any ball $B \subset \mathbb{R}^d$ and since $\inf_{x \in \overline{B}} \rho(x) > 0$, we obtain that $\frac{1}{\rho} \in C^{0,\beta}(\overline{B})$. Now for $f, g \in C^{0,\beta}(\overline{B})$ it holds $f \cdot g \in C^{0,\beta}(\overline{B})$ and

$$\|f \cdot g\|_{C^{0,\beta}(\overline{B})} \leq \|f\|_{C^{0,\beta}(\overline{B})} \|g\|_{C^{0,\beta}(\overline{B})}. \tag{3.13}$$

For any ball B, $t \geq 0, \alpha > 0, g \in L^r(\mathbb{R}^d, m), r \in [q, \infty], f \in D(L_r), r \in [q, \infty)$

$$\|\rho G_\alpha g\|_{H^{1,p}(B)}, \ \|\rho T_t f\|_{H^{1,p}(B)}$$

are bounded and so by Morrey's inequality applied to each ball B and (3.13) there exist unique locally Hölder continuous m-versions $R_\alpha g, P_t f$ of $G_\alpha g, T_t f$, where we set

$$P_0 := id,$$

with

$$\|R_\alpha g\|_{C^{0,\beta}(\overline{B})} \leq \|\rho^{-1}\|_{C^{0,\beta}(\overline{B})} \|\rho R_\alpha g\|_{C^{0,\beta}(\overline{B})} \leq \|\rho^{-1}\|_{C^{0,\beta}(\overline{B})} c \, \|\rho G_\alpha g\|_{H^{1,p}(B)}$$

and

$$\|P_t f\|_{C^{0,\beta}(\overline{B})} \leq \|\rho^{-1}\|_{C^{0,\beta}(\overline{B})} c \, \|\rho T_t f\|_{H^{1,p}(B)}$$

Applying (3.9), (3.10), (3.11) to the last two inequalities, we get for any $t \geq 0$, $\alpha > 0$, $g \in L^r(\mathbb{R}^d, m)$, $r \in [q, \infty]$, $f \in D(L_r)$, $r \in [q, \infty)$, and any ball B' with $\overline{B} \subset B'$

$$\|R_\alpha g\|_{C^{0,\beta}(\overline{B})} \leq c_0 \left(\|g\|_{L^r(B', m)} + \|G_\alpha g\|_{L^1(B', m)} \right), \tag{3.14}$$

$$\|R_\alpha g\|_{C^{0,\beta}(\overline{B})} \leq c_0 \|g\|_{L^r(\mathbb{R}^d, m)}, \tag{3.15}$$

$$\|P_t f\|_{C^{0,\beta}(\overline{B})} \leq c_0 \|T_t f\|_{D(L_r)}, \tag{3.16}$$

where c_0 is a constant that may be different for different r (and different in each inequality (3.14), (3.15), and (3.16)), but doesn't depend on f, nor on g. We summarize consequences of the derived estimates in the following proposition.

Proposition 3.1.6. *Let $t \geq 0$, $\alpha > 0$ be arbitrary and q, β be defined as in (3.8), (3.12). Then under the conditions of Theorem 3.1.2, it holds:*

(i) $G_\alpha g$ has a locally Hölder continuous m-version

$$R_\alpha g \in C^{0,\beta}_{loc}(\mathbb{R}^d), \quad \forall g \in \bigcup_{r\in[q,\infty]} L^r(\mathbb{R}^d, m).$$

(ii) $T_t f$ has a locally Hölder continuous m-version

$$P_t f \in C^{0,\beta}_{loc}(\mathbb{R}^d), \quad \forall f \in \bigcup_{r\in[q,\infty)} D(L_r).$$

(iii) For any $f \in \bigcup_{r\in[q,\infty)} D(L_r)$ the map

$$(x,t) \mapsto P_t f(x)$$

is continuous on $\mathbb{R}^d \times [0,\infty)$.

Proof (i) and (ii) are direct consequences of (3.14), (3.15), (3.16). In order to show (iii), let $f \in D(L_r)$ for some $r \geq q$ and $((x_n, t_n))_{n\geq 1}$ be a sequence in $\mathbb{R}^d \times [0,\infty)$ that converges to $(x_0, t_0) \in \mathbb{R}^d \times [0,\infty)$. Then there exists a ball B such that $x_n \in \overline{B}$ for all $n \geq 0$. By (3.16) applied with $t = 0$ to $P_{t_n} f - P_{t_0} f \in D(L_r)$, noting that $L_r(P_{t_n} f - P_{t_0} f) = P_{t_n} L_r f - P_{t_0} L_r f$ and using the continuity for each $g \in L^r(\mathbb{R}^d, m)$ of $t \mapsto P_t g$ on $[0,\infty)$, we obtain that $P_{t_n} f \to P_{t_0} f$ in $C^{0,\beta}(\overline{B})$. Then it is clear from (ii) that

$$|P_{t_n} f(x_n) - P_{t_0} f(x_0)| \leq |P_{t_n} f(x_n) - P_{t_0} f(x_n)| + |P_{t_0} f(x_n) - P_{t_0} f(x_0)|$$

converges to zero as $n \to \infty$.

□

Remark 3.1.7. *(i) In comparison to [1], [8], [62], we obtained in Proposition 3.1.6(i) that $(G_\alpha)_{\alpha>0}$ is $L^r(\mathbb{R}^d, m)$-strong Feller for any $r \in [q, \infty]$, which is an improvement to the mentioned papers since there it is only obtained for $r \in [p, \infty]$. This plays a role, since it will imply (1.2) for $r = 2$. Indeed, we will see later in Lemma 3.2.4(ii) that $\int_0^t |f|^2(X_s)ds$ is finite in the sense of (1.2), whenever $f \in L^{2q}_{loc}(\mathbb{R}^d)$. But $2q \in (d, p)$, hence $L^p_{loc}(\mathbb{R}^d) \subset L^{2q}_{loc}(\mathbb{R}^d)$.*

(ii) We can use Proposition 3.1.6(i) to get a resolvent kernel and a resolvent kernel density for any $x \in \mathbb{R}^d$. Indeed, for any $\alpha > 0$, $x \in \mathbb{R}^d$, Proposition 3.1.6(i) implies that

$$R_\alpha(x, A) := \lim_{l\to\infty} R_\alpha(1_{B_l \cap A})(x), \ \ A \in \mathcal{B}(\mathbb{R}^d) \tag{3.17}$$

defines a finite measure $R_\alpha(x, dy)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ (such that $\alpha R_\alpha(x, dy)$ is a sub-probability measure) that is absolutely continuous with respect to m. The Radon-Nikodym derivative

$$r_\alpha(x, \cdot) := \frac{R_\alpha(x, dy)}{m(dy)} \tag{3.18}$$

then defines the desired resolvent kernel density.

(iii) If the $L^2(\mathbb{R}^d, m)$-semigroup $(T_t)_{t>0}$ is analytic (for instance, if the bilinear form in (3.7) satisfies a sector condition) then by Stein interpolation $(T_t)_{t>0}$ is also analytic on $L^r(\mathbb{R}^d, m)$ for any $r \in (2, \infty)$ (cf. [62, Remark 2.5]). Hence by [56, Ch. 2, Theorem 5.2(d)], we have for any $r \in [2, \infty)$, $f \in L^r(\mathbb{R}^d, m)$

$$T_t f \in D(L_r), \quad \textit{and} \quad \|L_r T_t f\|_{L^r(\mathbb{R}^d,m)} \le \frac{const.}{t} \|f\|_{L^r(\mathbb{R}^d,m)}.$$

Therefore, (3.16) can be improved and extended as follows: for any $r \in [q \vee 2, \infty)$, $t > 0$, $f \in L^r(\mathbb{R}^d, m)$ and any ball B

$$\|P_t f\|_{C^{0,\beta}(\overline{B})} \ \ \le \ \ c_0 \left(1 + \frac{const.}{t}\right) \|f\|_{L^r(\mathbb{R}^d,m)}. \tag{3.19}$$

We can then use (3.19) to get a heat kernel and a heat kernel density for any $x \in \mathbb{R}^d$. Indeed, for any $t > 0$, $x \in \mathbb{R}^d$, (3.19) implies that

$$P_t(x, A) := \lim_{l \to \infty} P_t(1_{B_l \cap A})(x), \;\; A \in \mathcal{B}(\mathbb{R}^d) \tag{3.20}$$

defines a sub-probability measure $P_t(x, dy)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ that is absolutely continuous with respect to m. The Radon-Nikodym derivative

$$p_t(x, \cdot) := \frac{P_t(x, dy)}{m(dy)} \tag{3.21}$$

then defines the desired heat kernel density. However, in general $(T_t)_{t>0}$ is not analytic and therefore we cannot impose analyticity. Moreover it is in general difficult to check analyticity, in particular the sector condition of the corresponding bilinear form (see for instance [62, Section 5]).

Unfortunately, by what is explained in Remark 3.1.7(iii) the semigroup estimate (3.16) which leads to Proposition 3.1.6(ii) seems just not good enough to obtain a pointwise heat kernel from which one could then try to build a transition function of a nice Markov process. We will proceed by deriving more regularity in the following Theorem 3.1.8.

Theorem 3.1.8. *Let $A := (a_{ij})_{1 \le i,j \le d}$, $\mathbf{G}$, ρ, $\beta^{\rho,A}$, and $\mathbf{B}$ be as in Theorem 3.1.2. For each $s \in [1, \infty]$, consider the $L^s(\mathbb{R}^d, m)$-semigroup $(T_t)_{t>0}$. Then for any $f \in L^s(\mathbb{R}^d, m)$ and $t > 0$, $T_t f$ has a continuous m-version $P_t f$ on $\mathbb{R}^d$. More precisely, $P_\cdot f(\cdot)$ is locally parabolic Hölder continuous on $\mathbb{R}^d \times (0, \infty)$ and for any bounded open sets U, V in $\mathbb{R}^d$ with $\overline{U} \subset V$ and $0 < \tau_3 < \tau_1 < \tau_2 < \tau_4$, i.e. $[\tau_1, \tau_2] \subset (\tau_3, \tau_4)$, we have for some $\gamma \in (0, 1)$ the following estimate for all $f \in \cup_{s \in [1,\infty]} L^s(\mathbb{R}^d, m)$ with $f \ge 0$,*

$$\|P_\cdot f(\cdot)\|_{C^{\gamma; \frac{\gamma}{2}}(\overline{U} \times [\tau_1, \tau_2])} \le C_6 \|P_\cdot f(\cdot)\|_{L^1(V \times (\tau_3, \tau_4), m \otimes dt)}, \tag{3.22}$$

where C_6, γ are constants that depend on $\overline{U} \times [\tau_1, \tau_2], V \times (\tau_3, \tau_4)$, but are independent of f.

Proof First assume $f \in C_0^\infty(\mathbb{R}^d)$, $f \ge 0$ and set $u(x, t) := \rho(x) P_t f(x)$. Then $f \in D(L_p)$ and by Proposition 3.1.6(iii) $P_t f(x)$ is jointly continuous on $\mathbb{R}^d \times [0, \infty)$. Therefore the same is true for $u(x,t)$. Let $\widehat{L}$ be as in (3.6) and $T>0$ be arbitrary. Then exactly as in [10, (4.7)] (note that there the underlying measure $m=\mu$ is a probability measure but it doesn't matter), we get for any $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$

$$0 = -\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + \widehat{L}\varphi\right) u \, dxdt. \tag{3.23}$$

Note that $u \in H^{1,2}(O \times (0,T))$ for any bounded and open set $O \subset \mathbb{R}^d$. We can hence use integration by parts in the right hand term of (3.23) and see that

$$0 = \int_0^T \int_{\mathbb{R}^d} \left(\frac{1}{2}\langle A\nabla u, \nabla \varphi\rangle + u\langle \beta, \nabla\varphi\rangle - u\partial_t \varphi\right) dxdt,$$

where $\beta := \frac{1}{2}\nabla A + \mathbf{G} - 2\beta^{\rho,A} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $(\nabla A)_i := \sum_{j=1}^d \partial_j a_{ij}, 1 \le i \le d$. Let $\tau_2^* := \frac{\tau_2+\tau_4}{2}$ and take $r>0$ so that

$$r < \min\left(\frac{1}{9}\sqrt{\frac{\tau_4-\tau_2}{14}}, \frac{1}{9}\sqrt{\frac{\tau_1}{2}}\right) \quad \text{and} \quad R_{\bar{x}}(9r) \subset V, \ \forall \bar{x} \in \overline{U}.$$

Then for all $(\bar{x},\bar{t}) \in \overline{U} \times [\tau_1, \tau_2^*]$, we have $\bar{t} - 2(9r)^2 > 0$ and

$$R_{\bar{x}}(9r) \times (\bar{t} + 6(9r)^2, \bar{t} + 7(9r)^2)) \subset V \times (\tau_3, \tau_4).$$

Using [2, Theorem 4], for any $(x,t),(y,s) \in R_{\bar{x}}(r) \times (\bar{t}-r^2, \bar{t})$ we have

$$|u(x,t)-u(y,s)| \le C_1 r^{-\gamma}\left(\|x-y\| + \sqrt{|t-s|}\right)^\gamma \sup_{R_{\bar{x}}(3r)\times(\bar{t}-(3r)^2,\bar{t})} u,$$

where C_1 and $\gamma \le 1 - \frac{d}{p}$ are constants independent of f, r and $(\bar{x},\bar{t})$. Thus $u \in C^{\gamma;\frac{\gamma}{2}}\left(\bar{R}_r(\bar{x}) \times [\bar{t}-r^2, \bar{t}]\right)$ and

$$\|u\|_{C^{\gamma;\frac{\gamma}{2}}\left(\bar{R}_r(\bar{x})\times[\bar{t}-r^2,\bar{t}]\right)} \le (1 + C_1 r^{-\gamma}) \sup_{R_{\bar{x}}(3r)\times(\bar{t}-(3r)^2,\bar{t})} u.$$

Using the compactness of $\overline{U} \times [\tau_1, \tau_2]$, there exist $(x_i, t_i) \in \overline{U} \times [\tau_1, \tau_2^*]$, $i = 1, \dots, N$, such that

$$\overline{U} \times [\tau_1, \tau_2] \subset \bigcup_{i=1}^{N} R_{x_i}(r) \times (t_i - r^2, t_i) =: Q.$$

Take a smooth partition of unity $(\phi_i)_{i=1,\ldots,N}$ subordinate to $(R_{x_i}(r) \times (t_i - r^2, t_i))_{i=1,\ldots,N}$. For each $1 \leq i \leq N$, $\phi_i u \in C^{\gamma;\frac{\gamma}{2}}(\overline{Q})$, so that $u = \sum_{i=1}^{N} \phi_i u$ in $\overline{U} \times [\tau_1, \tau_2]$ implies $u \in C^{\gamma;\frac{\gamma}{2}}(\overline{U} \times [\tau_1, \tau_2])$. Furthermore, we have

$$\begin{aligned}
\|u\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} &\leq \sum_{i=1}^{N} \|\phi_i u\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} \leq \sum_{i=1}^{N} \|\phi_i u\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{Q})} \\
&\leq \sum_{i=1}^{N} \|\phi_i\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{Q})} \|u\|_{C^{\gamma;\frac{\gamma}{2}}\left(\bar{R}_r(x_i)\times[t_i-r^2,t_i]\right)} \\
&\leq \underbrace{\left(\sum_{i=1}^{N} \|\phi_i\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{Q})} \cdot (1 + C_1 r^{-\gamma})\right)}_{:=C_2} \cdot \max_{1\leq i\leq N} \left(\sup_{R_{x_i(3r)}\times(t_i-(3r)^2,t_i)} u \right).
\end{aligned} \tag{3.24}$$

Then, by [2, Theorem 2], for each $1 \leq i \leq N$

$$\begin{aligned}
\sup_{R_{x_i}(3r)\times(t_i-(3r)^2,t_i)} u &\leq C_3 \|u\|_{L^2(R_{x_i(9r)}\times(t_i-(9r)^2,t_i))} \\
&\leq C_3 (18r)^{\frac{d}{2}} \cdot (9r) \sup_{R_{x_i}(9r)\times(t_i-(9r)^2,t_i)} u \\
&\leq C_3 (18r)^{\frac{d}{2}} \cdot (9r) \cdot C_4 \inf_{R_{x_i}(9r)\times(t_i+6(9r)^2,t_i+7(9r)^2)} u \\
&\leq C_3 C_4 (18r)^{-\frac{d}{2}} \cdot (9r)^{-1} \|u\|_{L^1(R_{x_i}(9r)\times(t_i+6(9r)^2,t_i+7(9r)^2))} \\
&\leq \underbrace{C_3 C_4 (18r)^{-\frac{d}{2}} (9r)^{-1}}_{:=C_5} \|u\|_{L^1(V\times(\tau_3,\tau_4))},
\end{aligned} \tag{3.25}$$

where C_3 and C_4 are constants which are independent of f and x_i. Combining (3.24), (3.25) we have for $s \in [1,\infty)$

$$\begin{aligned} \|P_\cdot f(\cdot)\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} &\leq \|\rho^{-1}\|_{C^{\gamma}(\overline{U}\times[\tau_1,\tau_2])}\|\rho(\cdot)P_\cdot f(\cdot)\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} \\ &\leq \underbrace{\|\rho^{-1}\|_{C^{\gamma}(\overline{U}\times[\tau_1,\tau_2])}C_2C_5}_{:=C_6}\|P_\cdot f(\cdot)\|_{L^1(V\times(\tau_3,\tau_4),m\otimes dt)} \\ &\leq C_6(\tau_4-\tau_3)\|\rho\|_{L^1(V)}^{\frac{s-1}{s}}\|f\|_{L^s(\mathbb{R}^d,m)}. \end{aligned} \tag{3.26}$$

For $f \in L^s(\mathbb{R}^d, m)$ define

$$P_\cdot f(\cdot) := \lim_{n\to\infty} P_\cdot f_n(\cdot) \ \text{ in } \ C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2]), \tag{3.27}$$

where $(f_n)_{n\geq1} \subset C_0^\infty(\mathbb{R}^d)$ is any sequence converging to f in $L^s(\mathbb{R}^d, m)$. Then $P_\cdot f(\cdot)$ is well-defined, i.e. independent of the choice of $(f_n)_{n\geq1}$, and (3.26) (including all intermediate inequalities) extends to $f \in L^s(\mathbb{R}^d, m)$. In particular, (3.22) holds for $f \in L^s(\mathbb{R}^d, m)$, $f \geq 0$, $s \in [1,\infty)$.
Moreover, given $f \in L^s(\mathbb{R}^d, m)$ and $f_n \in C_0^\infty(\mathbb{R}^d)$ with $f_n \to f$ in $L^s(\mathbb{R}^d, m)$, for each $t > 0$ we have $T_tf_n \to T_tf$ in $L^s(U, m)$ and also $P_tf_n \to P_tf$ in $L^s(U, m)$ by (3.27) holds for $s \in [1,\infty)$. Thus

$$P_tf = T_tf \quad m\text{-a.e. on } U \text{ for each } t > 0. \tag{3.28}$$

This holds for arbitrary bounded open U, hence also on $\mathbb{R}^d$. Thus P_tf is an m-version of T_tf.
For $f \in L^\infty(\mathbb{R}^d, m)$, take $f_n := 1_{B_n} \cdot f$ with $n \geq 1$. Then for each $t > 0$,

$$T_tf = \lim_{n\to\infty} T_tf_n = \lim_{n\to\infty} P_tf_n,\ m\text{-a.e. on } \mathbb{R}^d. \tag{3.29}$$

For each fixed $(x,t) \in V \times (\tau_3, \tau_4)$, $(P_tf_n(x))_{n\geq1}$ is an increasing sequence of real numbers that is bounded by one by the sub-Markovian property and continuity of $z \mapsto P_tf_n(z)$. Thus (3.22) for $s = 1$ and Lebesgue's dominated convergence theorem imply that $(P_\cdot f_n(\cdot))_{n\geq1}$ is a Cauchy sequence in $C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])$. Hence we can again define

$$P_\cdot f(\cdot) := \lim_{n\to\infty} P_\cdot f_n(\cdot) \ \text{ in } \ C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])$$

and (3.22) also holds for $s = \infty$. Moreover for each $t > 0$, $P_t f_n$ converges uniformly to $P_t f$ in U, hence in view of (3.29), (3.28) also holds for $s = \infty$. Since U is an arbitrary bounded open subset in $\mathbb{R}^d$, we have hence shown that for any $f \in \cup_{s\in[1,\infty]} L^s(\mathbb{R}^d, m)$, $P_\cdot f(\cdot)$ is locally parabolic Hölder continuous in $\mathbb{R}^d \times (0,\infty)$ and for each $t > 0$, $P_t f = T_t f$ m-a.e. on $\mathbb{R}^d$.

□

Remark 3.1.9. *(i)* (3.22) *easily implies for any* $s \in [1,\infty]$, $f \in L^s(\mathbb{R}^d, m), t > 0$ *(cf. for instance* (3.26) *for* $s \in [0,\infty)$ *and use the sub-Markovian property for* $s = \infty$*) that*

$$\|P_t f\|_{C^{0,\gamma}(\overline{U})} \;\; \leq \;\; C_6(\tau_4 - \tau_3)\|\rho\|_{L^1(V)}^{\frac{s-1}{s}} \cdot \|f\|_{L^s(\mathbb{R}^d,m)}, \tag{3.30}$$

where $\frac{s-1}{s} := 1$ *for* $s = \infty$. (3.30) *is an improvement over* (3.19) *in regard to analyticity, which is no more required for* (3.30)*, and in regard to the integrability order which is* $s \in [1,\infty]$ *for* (3.30) *but* $r \in [q \vee 2, \infty)$ *for* (3.19). *Only the Hölder exponent* γ *in* (3.30) *depends on the domain and may vary, whereas in* (3.19) *it is always* β *as in* (3.12)*, independently of the domain.*
Using Theorem 3.1.8, we can define $P_t(x, A)$ *as in (3.20) and we see that there exist unique sub-probability measures* $P_t(x, dy)$ *on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$*, absolutely continuous with respect to* m *and with Radon-Nikodym derivatives* $p_t(x,\cdot)$ *defined by (3.21).*
(ii) Let $A := (a_{ij})_{1\leq i,j\leq d}$, $\mathbf{G}$, ρ, $\beta^{\rho,A}$*, and* $\mathbf{B}$ *be as in Theorem 3.1.2, but suppose* $p > d+2$ *and that* m *is a probability measure. In this case similar results to Theorem 3.1.8 and the following Proposition 3.1.10(ii) and some additional structure with respect to duality is derived in [10, Theorem 4.1]. The technique of proof is different to ours but also applies if* m *is not restricted to be a probability measure (cf. [10, Remark 4.2(ii)]). However, we insist that* $K_t(x, dy)$ *as occurring in [10, Remark 4.2(ii)] is in contrast to what is mentioned in [10, Remark 4.2(ii)] always a sub-probability measure and hence finite and moreover in case of merely locally finite measure only the* $L^1(\mathbb{R}^d, m)$*-strong Feller property follows, whereas we derive the* $L^{[1,\infty]}(\mathbb{R}^d, m)$*-strong Feller property (see Theorem 3.1.8 and Proposition 3.1.10 for the definition), that includes the classical strong Feller property.*
(iii) As opposed to [1, Proposition 3.8], we do not need the condition $\alpha R_\alpha 1_{\mathbb{R}^d} \equiv 1$ *in order to derive the classical strong Feller property of* $(P_t)_{t>0}$*. Also in [85], non-explosion*

(see (4.2) below) is used to obtain the classical strong Feller property.

Using Theorem 3.1.8, we obtain the following improvement of Proposition 3.1.6:

Proposition 3.1.10. *Let $t > 0, \alpha > 0$ be arbitrary. Let q, β be defined as in (3.8), (3.12), $r_\alpha(x,y)$ as in Remark 3.1.7, and $p_t(x,y)$ as in Remark 3.1.9. Then under the conditions of Theorem 3.1.2, it holds:*

(i) $G_\alpha g$ has a locally Hölder continuous m-version of order $\beta = 1 - d/p$

$$R_\alpha g = \int_{\mathbb{R}^d} f(y) R_\alpha(\cdot, dy) = \int_{\mathbb{R}^d} f(y) r_\alpha(\cdot, y) m(dy), \quad \forall g \in \bigcup_{r \in [q,\infty]} L^r(\mathbb{R}^d, m). \tag{3.31}$$

In particular, (3.31) extends by linearity to all $g \in L^q(\mathbb{R}^d, m) + L^\infty(\mathbb{R}^d, m)$, i.e. $(R_\alpha)_{\alpha>0}$ is $L^{[q,\infty]}(\mathbb{R}^d, m)$-strong Feller.

(ii) $T_t f$ has a continuous m-version

$$P_t f = \int_{\mathbb{R}^d} f(y) P_t(\cdot, dy) = \int_{\mathbb{R}^d} f(y) p_t(\cdot, y) m(dy), \quad \forall f \in \bigcup_{s \in [1,\infty]} L^s(\mathbb{R}^d, m). \tag{3.32}$$

($P_t f$ is locally Hölder continuous of order $\beta = 1 - d/p$, if $f \in \bigcup_{r \in [q,\infty)} D(L_r)$) and locally Hölder continuous with possibly changing Hölder exponents, if $f \in \bigcup_{s \in [1,\infty]} L^s(\mathbb{R}^d, m) \setminus \bigcup_{r \in [q,\infty)} D(L_r)$. In particular, (3.32) extends by linearity to all $f \in L^1(\mathbb{R}^d, m) + L^\infty(\mathbb{R}^d, m)$, i.e. $(P_t)_{t>0}$ is $L^{[1,\infty]}(\mathbb{R}^d, m)$-strong Feller.

Finally, for any $\alpha > 0, x \in \mathbb{R}^d$, $g \in L^q(\mathbb{R}^d, m) + L^\infty(\mathbb{R}^d, m)$

$$R_\alpha g(x) = \int_0^\infty e^{-\alpha t} P_t g(x)\, dt.$$

Proof Fix $\alpha > 0, t > 0, x \in \mathbb{R}^d$. Let $A \in \mathcal{B}(\mathbb{R}^d)$. Using (3.17), (3.18), monotone integration and (3.14), we can see that

$$\int_{\mathbb{R}^d} 1_A(y) r_\alpha(x,y)\, m(dy) = \int 1_A(y) R_\alpha(x, dy) = \lim_{l \to \infty} R_\alpha(1_{B_l \cap A})(x) = R_\alpha 1_A(x). \tag{3.33}$$

Using (3.20), (3.21), monotone integration and (3.22) (cf. proof of Theorem 3.1.8) , we can see that

$$\int_{\mathbb{R}^d} 1_A(y)p_t(x,y)m(dy) = \int_{\mathbb{R}^d} 1_A(y)P_t(x,dy) = \lim_{l\to\infty} P_t 1_{B_l\cap A}(y) = P_t 1_A(x). \tag{3.34}$$

(3.33), resp. (3.34) extends to $g \in L^r(\mathbb{R}^d, m)$, $r \in [q,\infty]$, resp. $g \in L^s(\mathbb{R}^d, m)$, $s \in [1,\infty]$ in the following way. Split g, f in positive and negative parts. We may hence assume that g, f are positive. Then we use a monotone approximation of g, resp. f with simple functions involving indicator functions like above, i.e. there exists an increasing sequence of simple functions $(g_n)_{n\geq 1}$ with $0 \leq g_n \nearrow g$, resp. $(f_n)_{n\geq 1}$ with $0 \leq f_n \nearrow f$. By this we can use monotone integration for the two left hand terms of (3.33), resp. (3.34), and (3.14), resp. (3.22) for the left hand term. Thus (i) and (ii) follow.
The last statement follows similarly noting that for $A \in \mathcal{B}(\mathbb{R}^d)$

$$R_\alpha 1_A = \int_0^\infty e^{-\alpha t} P_t 1_A \, dt$$

m-a.e. hence everywhere since both sides define continuous functions and we can as before use monotone integration as well as (3.14) and (3.22) to prove the remaining assertion.

□

Remark 3.1.11. *We obtain analogously to [1] that $(P_t)_{t>0}$ defined on*

$$L^\infty(\mathbb{R}^d, m) = L^\infty(\mathbb{R}^d) \supset \mathcal{B}_b(\mathbb{R}^d)$$

determines a (temporally homogeneous) sub-markovian transition function (cf. [17, 1.2]). Thus $(P_t)_{t>0}$ satisfies condition (**H1**) *of [66]. Moreover, $P_t f$, $t > 0$, is by Proposition 3.1.10(ii) independent of the m-version chosen for $f \in L^\infty(\mathbb{R}^d, m)$.*

3.2 Construction of a weak solution

By the results of [78, Section 4.1], the generalized Dirichlet form $\mathcal{E}$ associated with $(L_2, D(L_2))$ is strictly quasi-regular. In particular, by [78, Theorem 6] there exists a Hunt process

$$\tilde{\mathbb{M}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t\geq 0}, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$$

with life time $\tilde{\zeta} := \inf\{t \geq 0 \,|\, \tilde{X}_t = \Delta\}$ and cemetery Δ such that $\mathcal{E}$ is (strictly properly) associated with $\tilde{\mathbb{M}}$.
For some fixed $\varphi \in L^1(\mathbb{R}^d, m)_b$, $0 < \varphi \leq 1$, consider the strict capacity $\text{cap}_{1,\widehat{G}_1\varphi}$ of $\mathcal{E}$ as defined in [78, Definition 1]. Due to the properties of smooth measures with respect to $\text{cap}_{1,\widehat{G}_1\varphi}$ in [78, Section 3] one can consider the work [79] with cap_φ (as defined in [79]) replaced by $\text{cap}_{1,\widehat{G}_1\varphi}$. In particular [79, Theorem 3.10 and Proposition 4.2] apply with respect to the strict capacity $\text{cap}_{1,\widehat{G}_1\varphi}$ and therefore the paths of $\tilde{\mathbb{M}}$ are continuous $\tilde{\mathbb{P}}_x$-a.s. for strictly $\mathcal{E}$-q.e. $x \in \mathbb{R}^d$ on the one-point-compactification $\mathbb{R}^d_\Delta$ of $\mathbb{R}^d$ with Δ as point at infinity, i.e. for strictly $\mathcal{E}$-q.e. $x \in \mathbb{R}^d$,

$$\tilde{\mathbb{P}}_x\bigg(\Big\{\omega \in \tilde{\Omega} \mid \tilde{X}_\cdot(\omega) \in C\big([0,\infty), \mathbb{R}^d_\Delta\big),\, \tilde{X}_\cdot(\omega) = \Delta \ \forall t \geq \zeta(\omega)\Big\} \bigg) = 1.$$

We may hence assume that

$$\tilde{\Omega} = \{\omega = (\omega(t))_{t\geq 0} \in C([0,\infty), \mathbb{R}^d_\Delta) \mid \omega(t) = \Delta \quad \forall t \geq \zeta(\omega)\}$$

and

$$\tilde{X}_t(\omega) = \omega(t), \quad t \geq 0.$$

Now, we can apply the Dirichlet form method of [66, Section 2.1.2]. There it was only developed in a symmetric setting. But here we are in the non-sectorial setting. However one can readily check that it works nearly in the same way using Lemma 3.1.1 instead of [66, Lemma 2.5(i)] and modifying $(\mathbf{H2})'$ of [66, Section 2.1.2] in the following way:

(**H2**)$'$ We can find $\{u_n \mid n \geq 1\} \subset D(L_1) \cap C_0(\mathbb{R}^d)$ satisfying:

(i) For all $\varepsilon \in \mathbb{Q} \cap (0,1)$ and $y \in D$, where D is any given countable dense set in $\mathbb{R}^d$, there exists $n \in \mathbb{N}$ such that $u_n(z) \geq 1$, for all $z \in \overline{B}_{\frac{\varepsilon}{4}}(y)$ and $u_n \equiv 0$ on $\mathbb{R}^d \setminus B_{\frac{\varepsilon}{2}}(y)$,

(ii) $R_1\big([(1-L_1)u_n]^+\big)$, $R_1\big([(1-L_1)u_n]^-\big)$, $R_1\big([(1-L_1)u_n^2]^+\big)$, $R_1\big([(1-L_1)u_n^2]^-\big)$ are continuous on $\mathbb{R}^d$ for all $n \geq 1$,

and

(iii) $R_1 C_0(\mathbb{R}^d) \subset C(\mathbb{R}^d)$,

(iv) For any $f \in C_0(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the map $t \mapsto P_t f(x)$ is right-continuous on $(0, \infty)$.

It is well known that $u \in D(L_2)$ such that $u, L_2 u \in L^r(\mathbb{R}^d, m)$ for some $r \in [1, \infty)$ implies $u \in D(L_r)$. Hence $C_0^2(\mathbb{R}^d) \subset D(L_1) \cap C_0(\mathbb{R}^d)$ and moreover obviously $(1 - L_1)u, (1-L_1)u^2 \in L^p(\mathbb{R}^d)_0$ for any $u \in C_0^2(\mathbb{R}^d)$. Consequently, by Theorem 3.1.8 and Proposition 3.1.10, (**H2**)$'$ is satisfied for some countable subset of $C_0^2(\mathbb{R}^d)$.

Therefore, we obtain:

Theorem 3.2.1. *There exists a Hunt process*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d \cup \{\Delta\}})$$

with state space $\mathbb{R}^d$ and life time

$$\zeta = \inf\{t \geq 0 \,|\, X_t = \Delta\} = \inf\{t \geq 0 \,|\, X_t \notin \mathbb{R}^d\},$$

having the transition function $(P_t)_{t\geq 0}$ as transition semigroup, such that $\mathbb{M}$ has continuous sample paths in the one point compactification $\mathbb{R}^d_\Delta$ of $\mathbb{R}^d$ with the cemetery Δ as point at infinity, i.e. for all $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\bigg(\big\{\omega \in \Omega \mid X_\cdot(\omega) \in C\big([0,\infty), \mathbb{R}^d_\Delta\big),\, X_\cdot(\omega) = \Delta \ \forall t \geq \zeta(\omega)\big\} \bigg) = 1.$$

Remark 3.2.2. *Checking the details of [1, Section 4] one by one with possibly only few modifications one may possibly also obtain Theorem 3.2.1.*

Lemma 3.2.3. *Let $\mathbb{E}_x$ denote the expectation with respect to $\mathbb{P}_x$, $x \in \mathbb{R}^d$.*

(i) For any $x \in \mathbb{R}^d$, $\alpha > 0, t > 0$, we have

$$R_\alpha g(x) = \int_{\mathbb{R}^d} r_\alpha(x,y)g(y)m(dy) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha s} g(X_s)ds\right],$$

for any $g \in L^q(\mathbb{R}^d, m) + L^\infty(\mathbb{R}^d, m)$, and

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y)f(y)m(dy) = \mathbb{E}_x\left[f(X_t)\right],$$

for any $f \in L^1(\mathbb{R}^d, m) + L^\infty(\mathbb{R}^d, m)$.
In particular, integrals of the form $\int_0^\infty e^{-\alpha s} h(X_s)ds$, $\int_0^t h(X_s)ds$, $t \geq 0$ are for any $x \in \mathbb{R}^d$, whenever they are well-defined, $\mathbb{P}_x$-a.s. independent of the measurable m-version chosen for h.

(ii) Let $g \in L^r(\mathbb{R}^d, m)$ for some $r \in [q, \infty]$. Then for any ball B there exists a constant $c_{B,r}$, depending in particular on B and r, such that for all $t \geq 0$

$$\sup_{x \in \overline{B}} \mathbb{E}_x\left[\int_0^t |g|(X_s)\,ds\right] < e^t c_{B,r} \|g\|_{L^r(\mathbb{R}^d,m)}. \tag{3.35}$$

(iii) Let $u \in D(L_r)$, for some $r \in [q, \infty)$ and $\alpha > 0$, $t > 0$. Then for any $x \in \mathbb{R}^d$

$$R_\alpha\big((\alpha - L_r)u\big)(x) = u(x),$$

and

$$P_t u(x) - u(x) = \int_0^t P_s(L_r u)(x)\,ds.$$

Proof (i) By Remark 3.1.11 and Theorem 3.2.1, we have for any $t > 0$, $x \in \mathbb{R}^d$,

$h \in L^\infty(\mathbb{R}^d, m)$

$$P_t h(x) = \int_{\mathbb{R}^d} p_t(x,y)h(y)m(dy) = \mathbb{E}_x\left[h(X_t)\right], \tag{3.36}$$

and the expressions in (3.36) are all well-defined, i.e. do not change in value for any m-version of h. Now the resolvent and semigroup representations follow by splitting functions in $g \in \bigcup_{r\in[q,\infty]} L^r(\mathbb{R}^d, m)$ and $f \in \bigcup_{s\in[1,\infty]} L^s(\mathbb{R}^d, m)$ into their positive and negative parts, using monotone approximations of these with functions in $L^\infty(\mathbb{R}^d, m)$ and finally linearity, which is possible since all expressions are finite by Proposition 3.1.10. In particular, the limits will as the original expressions in (3.36) also not depend on the chosen m-versions, which concludes the proof.
(ii) Using in particular (i) and (3.15), we get

$$\begin{aligned}\sup_{x\in\overline{B}} \mathbb{E}_x\left[\int_0^t |g|(X_s)\,ds\right] &\leq e^t \sup_{x\in\overline{B}} \mathbb{E}_x\left[\int_0^\infty e^{-s}|g|(X_s)\,ds\right] \\ &= e^t \sup_{x\in\overline{B}} R_1|g|(x) \leq e^t c_B \|g\|_{L^r(\mathbb{R}^d,m)}.\end{aligned}$$

Using (i), the proof of (iii) works exactly as in [1, Lemma 5.1]. However, we emphasize that due to the increased regularity $r \geq q$ from (i) (coming from Proposition 3.1.6) in comparison to $r \geq p$ in [1], we obtain more general statements in (ii) and (iii).

□

For $A \in \mathcal{B}(\mathbb{R}^d)$, define

$$\sigma_A := \inf\{t > 0 : X_t \in A\}$$

and

$$\sigma_n := \sigma_{\mathbb{R}^d\setminus B_n}, n \geq 1.$$

Lemma 3.2.4. *(i) For any $x \in \mathbb{R}^d$, we have*

$$\mathbb{P}_x\Big(\lim_{n\to\infty} \sigma_n \geq \zeta\Big) = 1.$$

(ii) For any $x \in \mathbb{R}^d$, $t \geq 0$, we have

$$\mathbb{P}_x\left(\int_0^t |f|(X_s)ds < \infty\right) = 1, \ \textit{if} \quad f \in \bigcup_{r\in[q,\infty]} L^r(\mathbb{R}^d, m)$$

and if $f \in L^q_{loc}(\mathbb{R}^d, m)$

$$\mathbb{P}_x\left(\left\{\int_0^t |f|(X_s)ds < \infty\right\} \cap \{t < \zeta\}\right) = \mathbb{P}_x\left(\{t < \zeta\}\right),$$

Proof (i) By Proposition 3.1.10 and Lemma 3.2.3(i), we have that $\mathbb{E}_\cdot\left[\int_0^\infty e^{-\alpha s} g(X_s)ds\right]$ is an m-version of $G_\alpha g$, for all $\alpha > 0$ and $g \in L^\infty(\mathbb{R}^d, m)$. It hence follows by [68, IV. Theorem 3.1] (or [78, Proposition 2(ii)]) that $\mathcal{E}$ is quasi-regular. Therefore by [68, IV. Definition 1.7] there exists an $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ of compact subsets of $\mathbb{R}^d$. Then [68, IV. Lemma 3.10] implies, $\mathbb{P}_x\Big(\lim_{k\to\infty} \sigma_{\mathbb{R}^d\setminus E_k} \geq \zeta\Big) = 1$ for $\mathcal{E}$-q.e. $x \in \mathbb{R}^d$, hence in particular for m-a.e. $x \in \mathbb{R}^d$ by [68, III. Remark 2.6]. Since $(B_n)_{n\geq 1}$ is an open cover of E_k for each k, and $\sigma_A \leq \sigma_B$ whenever $B \subset A$, we then obtain $\mathbb{P}_x\Big(\lim_{n\to\infty} \sigma_n \geq \zeta\Big) = 1$ for m-a.e. $x \in \mathbb{R}^d$. Now the result follows exactly as in [62, Lemma 3.3].
(ii) The first statement immediately follows from Lemma 3.2.3(ii). For the second statement it is enough to show that for any $t \geq 0$ and $x \in \mathbb{R}^d$

$$\mathbb{P}_x\left(1_{\{t<\zeta\}}\int_0^t |f|(X_s)ds < \infty\right) = 1, \ \text{if} \quad f \in L^q_{loc}(\mathbb{R}^d, m). \tag{3.37}$$

It holds $\mathbb{P}_x(n \wedge \sigma_n < \zeta) = 1$ for any $n \geq 1$ and $x \in \mathbb{R}^d$, since $\mathbb{M}$ has continuous sample paths on the one-point-compactification $\mathbb{R}^d_\Delta$. Thus using (i), we get that the left hand side of (3.37) equals

$$\lim_{n\to\infty} \mathbb{P}_x\left(1_{\{t<n\wedge\sigma_n\}}\int_0^t |f|(X_s)ds < \infty\right). \tag{3.38}$$

Now, fix $x \in \mathbb{R}^d$. Then there exists $N_0 \in \mathbb{N}$ with $x \in B_n$ for any $n \geq N_0$. Consequently, for any $n \geq N_0$ we have $\mathbb{P}_x$-a.s. that $X_s \in B_n$ for any $s \in [0,t]$, if $t < \sigma_n$. It follows with the help of Lemma 3.2.3(ii)

$$\mathbb{E}_x\Big[1_{\{t<n\wedge\sigma_n\}}\int_0^t |f|(X_s)ds\Big] \le \mathbb{E}_x\left[\int_0^t |f|1_{B_n}(X_s)ds\right] < \infty, \quad \forall n \ge N_0.$$

Thus each sequence member in (3.38) is equal to one and therefore (3.37) holds.

□

Proposition 3.2.5. *Let $u \in D(L_r)$, for some $r \in [q, \infty)$. Then*

$$M_t^u := u(X_t) - u(x) - \int_0^t L_r u(X_s)\, ds, \quad t \ge 0.$$

is a continuous $(\mathcal{F}_t)_{t\ge 0}$-martingale under $\mathbb{P}_x$ for any $x \in \mathbb{R}^d$. If $r \ge 2q$, then M^u is square integrable.

Proof The first result is an immediate consequence of Lemma 3.2.3 (see for instance [19, Chapter 7, (1.6) Theorem]). The second follows from Lemma 3.2.3(i) and (ii).

□

Proposition 3.2.6. *Let $u \in C_0^2(\mathbb{R}^d)$, $t \ge 0$. Then the quadratic variation process $\langle M^u\rangle$ of the continuous martingale M^u satisfies $\mathbb{P}_x$-a.s for any $x \in \mathbb{R}^d$, $t \ge 0$*

$$\langle M^u\rangle_t = \int_0^t \langle A\nabla u, \nabla u\rangle(X_s)ds.$$

In particular, by Lemma 3.2.3(ii) $\langle M^u\rangle_t$ is $\mathbb{P}_x$-integrable for any $x \in \mathbb{R}^d$, $t \ge 0$ and so M^u is square integrable.

Proof For $g \in C_0^2(\mathbb{R}^d)$, we have $g \in D(L_r)$ and $L_1 g = L_r g$ for any $r \in [1, p]$. Thus for $u \in C_0^2(\mathbb{R}^d)$, we get by Proposition 3.2.5 and Lemma 3.1.1

$$u^2(X_t) - u^2(x) = M_t^{u^2} + \int_0^t \Big(\langle A\nabla u, \nabla u\rangle(X_s) + 2uL_1 u(X_s)\Big)\, ds.$$

Applying Itô's formula to the continuous semimartingale $(u(X_t))_{t\geq 0}$, we obtain

$$u^2(X_t) - u^2(x) = \int_0^t 2u(X_s)dM_s^u + \int_0^t 2uL_r u(X_s)\, ds + \langle M^u \rangle_t.$$

The last two equalities imply that $\Big(\langle M^u \rangle_t - \int_0^t \langle A\nabla u, \nabla u\rangle (X_s) ds \Big)_{t\geq 0}$ is a continuous $\mathbb{P}_x$-martingale of bounded variation for any $x \in \mathbb{R}^d$. This implies the assertion.

□

For the following result, see for instance [16, Theorem 1.1, Lemma 2.1], that we can apply locally.

Lemma 3.2.7. *Under the assumptions of Theorem 3.1.2 on the diffusion matrix A, there exists a unique matrix of functions $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ with $\sigma_{ij} \in C(\mathbb{R}^d)$ for all i, j such that*

$$A(x) = \sigma(x)\sigma^T(x), \quad \forall x \in \mathbb{R}^d,$$

i.e.

$$a_{ij}(x) = \sum_{k=1}^d \sigma_{ik}(x)\sigma_{jk}(x), \quad \forall x \in \mathbb{R}^d,\ 1 \leq i, j \leq d.$$

and

$$\det(\sigma(x)) > 0, \quad \forall x \in \mathbb{R}^d.$$

Theorem 3.2.8. *Let $A := (a_{ij})_{1\leq i,j\leq d}$, $\mathbf{G}$, be as in Theorem 3.1.2. Consider the Hunt process $\mathbb{M}$ from Theorem 3.2.1 with coordinates $X_t = (X_t^1, ..., X_t^d)$ and suppose that $\mathbb{M}$ is non-explosive, i.e.*

$$\mathbb{P}_x(\zeta = \infty) = 1 \quad \textit{for any } x \in \mathbb{R}^d.$$

(i) Let $(\sigma_{ij})_{1\leq i,j\leq d}$ be as in Lemma 3.2.7. Then it holds $\mathbb{P}_x$-a.s. for any $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $i = 1, \ldots, d$

$$X_t^i = x_i + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_s)\, dW_s^j + \int_0^t g_i(X_s)\, ds, \quad 0 \leq t < \infty, \tag{3.39}$$

in short

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t \mathbf{G}(X_s)ds, \quad 0 \le t < \infty,$$

where $W = (W^1, \dots, W^d)$ is a standard d-dimensional Brownian motion starting from zero.

(ii) *Let $(\sigma_{ij})_{1\le i\le d, 1\le j\le l}$, $l \in \mathbb{N}$ arbitrary but fixed, be any matrix consisting of continuous functions $\sigma_{ij} \in C(\mathbb{R}^d)$ for all i, j, such that $A = \sigma\sigma^T$ (where A satisfies the assumptions of Theorem 3.1.2), i.e.*

$$a_{ij}(x) = \sum_{k=1}^{l} \sigma_{ik}(x)\sigma_{jk}(x), \quad \forall x \in \mathbb{R}^d, \ 1 \le i, j \le d.$$

Then on a standard extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, that we denote for notational convenience again by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, there exists a standard l-dimensional Brownian motion $W = (W^1, \dots, W^l)$ starting from zero such that (3.39) holds with $\sum_{j=1}^d$ replaced by $\sum_{j=1}^l$.

Proof (i) Consider the stopping times

$$D_n := D_{\mathbb{R}^d\setminus B_n} := \inf\{t \ge 0 \,:\, X_t \in \mathbb{R}^d \setminus B_n\} \quad n \ge 1.$$

Since $\mathbb{M}$ is non-explosive, it follows from Lemma 3.2.4(i) that $D_n \nearrow \infty$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^d$. Let $v \in C^2(\mathbb{R}^d)$. Then we claim that

$$M_t^v := v(X_t) - v(x) - \int_0^t \left(\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j v + \sum_{i=1}^{d} g_i\partial_i v \right)(X_s)\, ds, \quad t \ge 0,$$

is a continuous square integrable local $\mathbb{P}_x$-martingale with respect to the stopping times $(D_n)_{n\ge1}$ for any $x \in \mathbb{R}^d$. Indeed, let $(v_n)_{n\ge1} \subset C_0^2(\mathbb{R}^d)$ be such that $v_n = v$ pointwise on $\overline{B}_n$, $n \ge 1$. Then for any $n \ge 1$, we have $\mathbb{P}_x$-a.s

$$M^v_{t\wedge D_n} = M^{v_n}_{t\wedge D_n}, \quad t \ge 0,$$

and $(M^{v_n}_{t\wedge D_n})_{t\geq 0}$ is a square integrable $\mathbb{P}_x$-martingale for any $x \in \mathbb{R}^d$ by Proposition 3.2.6. Now let $u_i \in C^2(\mathbb{R}^d)$, $i = 1, \ldots, d$, be the coordinate projections, i.e. $u_i(x) = x_i$. Then by Proposition 3.2.6, polarization and localization with respect to $(D_n)_{n\geq 1}$, the quadratic covariation processes satisfy

$$\langle M^{u_i}, M^{u_j}\rangle_t = \int_0^t a_{ij}(X_s)\, ds, \quad 1 \leq i, j \leq d,\ t \geq 0.$$

Using Lemma 3.2.7 we obtain by [34, II. Theorem 7.1] that there exists a d-dimensional Brownian motion $(W_t)_{t\geq 0} = (W^1_t, \ldots, W^d_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, such that

$$M^{u_i}_t = \sum_{j=1}^d \int_0^t \sigma_{ij}(X_s)\ dW^j_s, \quad 1 \leq i \leq d,\ t \geq 0. \tag{3.40}$$

Since for any $x \in \mathbb{R}^d$, $\mathbb{P}_x$-a.s.

$$M^{u_i}_t = X^i_t - x_i - \int_0^t g_i(X_s)\, ds, \quad t \geq 0, \tag{3.41}$$

the assertion follows.
(ii) The proof of (ii) is similar to the proof of (i) but uses [34, II. Theorem 7.1'] instead of [34, II. Theorem 7.1] (see [34, IV. Proposition 2.1])

□

Remark 3.2.9. *Theorem 3.2.8 holds in general only up to ζ, when one does not impose non-explosion. Here, we only sketch in detail the proof in case of Theorem 3.2.8(i). (The case of Theorem 3.2.8(ii) is nearly the same but one has to work on a standard extension of the underlying probability space). One first uses that for $v_k \in C_0^2(\mathbb{R}^d)$, $1 \leq k \leq d$, one has by Proposition 3.2.6*

$$\langle M^{v_k}, M^{v_l}\rangle_t = \int_0^t \Phi_{kl}(X_s)\, ds, \quad 1 \leq k, l \leq d,\ t \geq 0,$$

where $\Phi_{kl} = \sum_{i,j=1}^{d} a_{ij}\partial_j v_k \partial_i v_l$, *so that*

$$\Phi_{kl} = \sum_{m=1}^{d} \Psi_{km}\Psi_{lm}, \quad \text{with} \quad \Psi_{km} = \sum_{i=1}^{d} \sigma_{im}\partial_i v_k, \quad 1 \leq k, l, m \leq d.$$

Note that we then do no longer have

$$\det((\Psi_{km})_{1\leq k,m\leq d}) \neq 0 \tag{3.42}$$

globally as opposed to Lemma 3.2.7. However, choosing $v_k(x) = v_k^n(x) = x_k$ *on* $\overline{B}_n$, $1 \leq k \leq d$, $n \geq 1$, *we can obtain (3.42) locally on* B_n, *hence (3.40) locally on* $\{t \leq D_n\}$ *for each* $n \geq 1$. *Consequently, we also get (3.41) locally on* $\{t \leq D_n\}$ *for each* $n \geq 1$. *Then showing consistency of the local martingale and drift parts, we obtain (3.39) up to* ζ *by Lemma 3.2.4(i).*

Chapter 4

Conservativeness and ergodic properties

In this chapter, we investigate long time behavior like non-explosion (conservativeness), recurrence and ergodicity. We also investigate some moment inequalities that are well-known for classical Itô-SDEs with continuous coefficients. We saw in Theorem 3.2.8 and Remark 3.2.9 that we can obtain a weak solution up to the life time ζ. We first provide explicit non-explosion criteria, i.e. explicit criteria that imply the assumption

$$\mathbb{P}_x(\zeta = \infty) = 1 \quad \text{for any } x \in \mathbb{R}^d$$

of Theorem 3.2.8.

4.1 Non-explosion criteria and moment inequalities

4.1.1 Non-explosion criteria and moment inequalities without involving the density ρ

In this subsection we consider non-explosion criteria that only depend on the coefficients of the underlying SDE. We first derive a lemma that is a variant of the construction in [12, page 197] and then a non-explosion criterion by following a probabilistic technique which traces back at least to [71, 10.2].

Lemma 4.1.1. *Let $f \in C^2(\mathbb{R}^d)$ be a positive, strictly increasing and unbounded radial function, i.e. $f \geq 0$ pointwise, $f(x) \equiv c_r$ on ∂B_r with $0 < c_r < c_{r'}$ whenever $0 < r < r'$, and $\inf_{\partial B_n} f \to \infty$ as $n \to \infty$. Suppose that there exist $M > 0$, $N_0 \in \mathbb{N}$ such that*

$$Lf \leq Mf \quad \text{a.e. on } \mathbb{R}^d \setminus B_{N_0}.$$

Let $\phi \in C^2(\mathbb{R})$, such that $\phi, \phi' \geq 0$ pointwise, such that

$$\phi(t) = \begin{cases} \sup_{B_{N_0}} f & \text{if } t \leq \sup_{B_{N_0}} f, \\ t & \text{if } t \geq \sup_{B_{N_0+1}} f, \end{cases}$$

and let for arbitrary $\alpha \geq 0$

$$\psi := \phi \circ f + C_{\phi,A} + \alpha,$$

where

$$C_{\phi,A} := M\big(c_\phi \sup_{B_{N_0+1}} f + \frac{c_\phi}{2M} \sup_{B_{N_0+1}} \langle A\nabla f, \nabla f\rangle\big)$$

and

$$c_\phi := \sup_{B_{N_0+1}\setminus \overline{B}_{N_0}} \phi' \circ f + \sup_{B_{N_0+1}\setminus \overline{B}_{N_0}} |\phi'' \circ f|.$$

Then $\psi \in C^2(\mathbb{R}^d)$, $\psi > 0$ pointwise, $\inf_{\partial B_n} \psi \nearrow \infty$ as $n \to \infty$, $n \geq N_0$, and

$$L\psi \leq M\psi \quad \text{a.e. on } \mathbb{R}^d.$$

Proof Using the formula

$$L(\phi(f)) = \phi'(f)Lf + \frac{1}{2}\phi''(f)\langle A\nabla f, \nabla f\rangle.$$

the assertion is easily verified.

□

Theorem 4.1.2. *Suppose that (1.3) holds. Then*

$$\mathbb{P}_x(\zeta = \infty) = 1 \quad \text{for any } x \in \mathbb{R}^d.$$

Proof We first show the statement corresponding to (1.3). Let $u_n \in C_0^2(\mathbb{R}^d)$, $n \geq 1$, be positive functions such that

$$u_n(x) = \begin{cases} \|x\|^2 & \text{if } x \in \overline{B}_n, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus B_{n+1}. \end{cases}$$

Then by Proposition 3.2.5

$$Y_t^n := u_n(X_t),\ t \geq 0,$$

is a positive continuous $\mathbb{P}_x$-semimartingale for any $x \in \mathbb{R}^d$, $n \geq 1$.
Let $f(x) = \ln(\|x\|^2 + 1) + 1$, $x \in \mathbb{R}^d$ and let ψ, ϕ and $C_{\phi,A}$ be as in Lemma 4.1.1 with $\alpha = 0$. By Itô's formula applied to Y^n with the function $e^{-Mt}\varphi(y)$,

$$\varphi(y) := \phi(\ln(1+y) + 1) + C_{\phi,A},$$

we obtain $\mathbb{P}_x$-a.s. for any $x \in B_n$

$$e^{-Mt}\varphi(Y_t^n) = \varphi(Y_0^n) + \int_0^t e^{-Ms}\varphi'(Y_s^n)dM_s^{u_n} + \int_0^t e^{-Ms}(L-M)(\varphi \circ u_n)(X_s)\,ds.$$

Note that $(L-M)(\varphi \circ u_n) = (L-M)\psi \leq 0$ m-a.e. on $\overline{B}_n$ for each $n \geq 1$. Therefore, using the last part of Lemma 3.2.3(i), we can see that

$$e^{-Mt\wedge\sigma_n}\varphi \circ u_n(X_{t\wedge\sigma_n}), t \geq 0,$$

is a positive continuous $\mathbb{P}_x$-supermartingale for any $x \in B_n$, $n \geq 1$. Since $\mathbb{M}$ has continuous sample paths on the one-point-compactification $\mathbb{R}^d_\Delta$, we have that $\|X_{t\wedge\sigma_n}\| = n$ $\mathbb{P}_x$-a.s. on $\{\sigma_n \leq t\}$ for any $x \in B_n$. Now let $x \in \mathbb{R}^d$ be arbitrary. Then $x \in B_{k_0}$ for some $k_0 \in \mathbb{N}$ and since supermartingales have decreasing expectations, we get for any

$n > k_0$

$$\begin{aligned}\phi\left(\ln(\|x\|^2+1)+1\right)+C_{\phi,A} &= \mathbb{E}_x[\varphi\circ u_n(X_0)] \\ &\geq \mathbb{E}_x[e^{-Mt\wedge\sigma_n}\varphi\circ u_n(X_{t\wedge\sigma_n})] \\ &\geq e^{-Mt}\mathbb{E}_x[\varphi\circ u_n(X_{t\wedge\sigma_n})1_{\{\sigma_n\leq t\}}] \\ &\geq e^{-Mt}\left(\phi\left(\ln(n^2+1)+1\right)+C_{\phi,A}\right)\mathbb{P}_x(\sigma_n\leq t).\end{aligned}$$

Consequently

$$\mathbb{P}_x(\zeta\leq t)=\lim_{n\to\infty}\mathbb{P}_x(\sigma_n\leq t)=0$$

for any $t\geq 0$, which implies the assertion.

□

Remark 4.1.3. *(i) Suppose that for the semigroup $(T_t)_{t>0}$ defined on $L^\infty(\mathbb{R}^d,m)$ it holds*

$$T_t 1_{\mathbb{R}^d}=1 \ m\text{-a.e. for some (and hence all) } t>0. \tag{4.1}$$

Then, since $T_t1_{\mathbb{R}^d}=P_t1_{\mathbb{R}^d}$ m-a.e. and $P_t1_{\mathbb{R}^d}$ is continuous by the strong Feller property (cf. Proposition 3.1.10(ii))

$$P_t1_{\mathbb{R}^d}(x)=1 \text{ for any } x\in\mathbb{R}^d, t>0, \text{ or equivalently } \mathbb{M} \text{ is non-explosive.} \tag{4.2}$$

(ii) Using (i), the non-explosion criterion (1.3) can be recovered form the dual version of [69, Proposition 1.10]. Indeed, (4.1) holds, if and only if m is invariant for the $L^1(\mathbb{R}^d,m)$-semigroup $(\widehat{T}_t)_{t>0}$. Then Theorem 4.1.2 follows by applying the dual version of [69, Proposition 1.10(b)] to the C^2-function ψ as defined in the proof of Theorem 4.1.2 and then using (4.2).
As a further example consider the following condition: for some $N_0\in\mathbb{N}\cup\{0\}$

$$\begin{aligned}&\left(\frac{\|x\|}{\|x\|-N_0}-\frac{1}{2}-\frac{3(\|x\|-N_0)^2\|x\|}{2(\|x\|-N_0)^3+1}\right)\frac{\langle A(x)x,x\rangle}{\|x\|^2}+\frac{1}{2}\text{trace}(A(x))+\big\langle \mathbf{G}(x),x\big\rangle \\ &\qquad\leq M\left(\|x\|-N_0+\frac{1}{(\|x\|-N_0)^2}\right)\|x\|\left(\ln\left((\|x\|-N_0)^3+1\right)+1\right)\end{aligned} \tag{4.3}$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$ *($B_0 := \emptyset$). Then (4.3) implies conservativeness, i.e.* (4.1) *holds, by applying [69, Proposition 1.10(b)] to the* C^2*-function*

$$\widetilde{\psi}(x) := \ln\Big((\|x\| - N_0)^3 \cdot 1_{\mathbb{R}^d \setminus B_{N_0}}(x) + 1 \Big) + 1, \quad x \in \mathbb{R}^d. \tag{4.4}$$

Indeed (4.3), implies $L\widetilde{\psi} \le M\widetilde{\psi}$ *a.e. so that we can apply [69, Proposition 1.10(b)]. But (4.3) also implies non-explosion, i.e.* (4.2)*, by following the proof of Theorem 4.1.2, replacing the* ψ *there with* $\widetilde{\psi}$ *in* (4.4) *and* u_n *by positive functions* $u_n^{N_0} \in C_0^2(\mathbb{R}^d)$*,* $n > N_0$*, such that*

$$u_n^{N_0}(x) = \begin{cases} (\|x\| - N_0)^3 \cdot 1_{\mathbb{R}^d \setminus B_{N_0}}(x) & \text{if } x \in \overline{B}_n, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus B_{n+1}. \end{cases}$$

(iii) In general, $\mathbb{M}$ *will be non-explosive whenever there exists* $\psi \in C^2(\mathbb{R}^d)$ *and* $M > 0$*, such that* $\inf_{\partial B_n} \psi \to \infty$ *as* $n \to \infty$ *and* $L\psi \le M\psi$ *a.e. on* $\mathbb{R}^d$*. This follows from [69, Proposition 1.10] and (i), and can be shown as well by applying the technique of supermartingales from Theorem 4.1.2, using a generalized version of Lemma 4.1.1 (see [12, page 197]), and noting that* $(M^v_{t\wedge D_n})_{t\ge 0}$*, is a martingale for any* $v \in C^2(\mathbb{R}^d)$ *(see proof of Theorem 3.2.8(i)). Note the subtle difference that [69, Proposition 1.10] is proved by analytic means (starting from the* L^1*-generator or* L^1*-semigroup) and only leads to (4.1), whereas Theorem 4.1.2 is proven by probabilistic means (starting from Proposition 3.2.5) and directly leads to* (4.2) *regardless of the classical strong Feller property.*

Theorem 4.1.4. *(i) Assume for some* $N_0 \in \mathbb{N}$ *and some* $p > 0$*, there exists* $M > 0$ *such that*

$$\left(\frac{p-2}{2}\right) \frac{\langle A(x)x, x\rangle}{\|x\|^2 + 1} + \frac{1}{2}\mathrm{trace}A(x) + \big\langle \mathbf{G}(x), x \big\rangle \le M\left(\|x\|^2 + 1\right), \tag{4.5}$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$*. Then* $\mathbb{M}$ *is non-explosive and for any open ball* B *there exists a constant* $C_B > 0$*, such that*

$$\sup_{x \in \overline{B}} \mathbb{E}_x\left[\|X_t\|^p\right] \le C_B \cdot e^{M\cdot t}, \quad \forall t \ge 0.$$

(ii) Let $\sigma = (\sigma_{ij})_{1\le i,j\le d}$ *be as in Lemma 3.2.7 and and* $\mathbf{G}$ *as in Theorem 3.1.2. Assume that for some* $N_0 \in \mathbb{N}$ *and* $C_1 > 0$

$$\max_{1\le i,j\le d} |\sigma_{ij}(x)| + \max_{1\le i\le d} |g_i(x)| \le C_1(\|x\| + 1) \quad \text{for a.e. } x \in \mathbb{R}^d \setminus B_{N_0}. \tag{4.6}$$

Then $\mathbb{M}$ *is non-explosive and for any* $T > 0$*, and open ball* B*, there exist constants* $C_{T,B}$*,* C_T *such that*

$$\sup_{x\in\overline{B}} \mathbb{E}_x\left[\sup_{s\le t} \|X_s\|^2\right] \le C_{T,B}\cdot e^{C_T\cdot t}, \quad \forall t \le T.$$

Proof (i) Let $f(x) = (\|x\|^2+1)^{\frac{p}{2}}$. Then (4.5) implies $Lf(x) \le Mp\cdot f(x)$ for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$. Let ϕ, ψ, and $C_{\phi,A}$ be as in Lemma 4.1.1 with $\alpha := \sup_{B_{N_0+1}} f$.
Let $\varphi(y) := \phi((y+1)^{\frac{p}{2}}) + C_{\phi,A} + \alpha$. Applying Itô's formula to $u_n(X_\cdot)$, where u_n is as in the proof of Theorem 4.1.2, with the function $e^{-Mp\cdot t}\varphi(y)$, we obtain exactly as in the proof of Theorem 4.1.2 that $\mathbb{M}$ is non-explosive. For arbitrary $n \in \mathbb{N}$ and $x \in B_n$ it holds

$$\big(C_{\phi,A} + 2\sup_{B_{N_0+1}} f\big) f(x) \ge \psi(x) \ge \mathbb{E}_x[e^{-(M\cdot p)t\wedge\sigma_n}\varphi\circ u_n(X_{t\wedge\sigma_n})].$$

Using $f \le \psi$ pointwise, $\sigma_n \nearrow \infty$, Fatou's lemma and the previous inequality, we get

$$e^{-Mp\cdot t}\mathbb{E}_x[f(X_t)] \le \liminf_{n\to\infty} \mathbb{E}_x[e^{-(M\cdot p)t\wedge\sigma_n}\varphi\circ u_n(X_{t\wedge\sigma_n})] \le \big(C_{\phi,A} + 2\sup_{B_{N_0+1}} f\big) f(x).$$

Thus,

$$\mathbb{E}_x[\|X_t\|^p] \le \underbrace{\big(C_{\phi,A} + 2\sup_{B_{N_0+1}} f\big)(\|x\|^2+1)^{\frac{p}{2}}}_{:=C_x}\ e^{Mp\cdot t}.$$

Now set $C_B := \sup_{x\in\overline{B}} C_x$.

(ii) (4.6) implies

$$\mathrm{trace}(A(x)) = \sum_{i,j=1}^{d} \sigma_{ij}(x)^2 \le 2d^2 C_1^2(\|x\|^2+1) \quad \text{for a.e. } x \in \mathbb{R}^d \setminus B_{N_0}$$

and

$$\langle \mathbf{G}(x), x \rangle \le \left(\sum_{i=1}^{d} g_i(x)^2 \right)^{\frac{1}{2}} \|x\| \le 2dC_1(\|x\|^2+1) \quad \text{for a.e. } x \in \mathbb{R}^d \setminus B_{N_0}.$$

Thus (1.3) holds, so that $\mathbb{M}$ is non-explosive by Theorem 4.1.2 and (3.39) holds. Consequently, $\mathbb{P}_x$-a.s. for any $1 \le i \le d$

$$\begin{aligned} &\sup_{0\le s\le t\wedge\sigma_{\mathbb{R}^d\setminus B_n}} |X_s^i|^2 \\ \le\ & (d+2)\left(x_i^2 + \sum_{j=1}^{d} \sup_{0\le s\le t\wedge\sigma_n} \left| \int_0^s \sigma_{ij}(X_u)\, dW_u^j \right|^2 + t\int_0^{t\wedge\sigma_n} |g_i(X_u)|^2\, du \right). \end{aligned} \tag{4.7}$$

Note that $\sum_{i,j=1}^{d} \sigma_{ij}(x)^2 = \mathrm{trace}(A(x)) \le d\cdot\Lambda_{B_{N_0}} \le d\cdot\Lambda_{B_{N_0}}(\|x\|^2+1)$ for a.e. $x \in B_{N_0}$. Thus by Doob's maximal inequality,

$$\begin{aligned} &\sum_{i,j=1}^{d} \mathbb{E}_x\left[\sup_{0\le s\le t\wedge\sigma_n} \left| \int_0^s \sigma_{ij}(X_u)\, dW_u^j \right|^2 \right] \\ \le\ & 4\mathbb{E}_x\left[\sum_{i,j=1}^{d} \left\langle \int_0^{\cdot} \sigma_{ij}(X_u)\, dW_u^j \right\rangle_{t\wedge\sigma_n} \right] \\ \le\ & 4\mathbb{E}_x\left[\sum_{i,j=1}^{d} \int_0^{t\wedge\sigma_n} \sigma_{ij}^2(X_u)du \right] \\ \le\ & \underbrace{4\left(2d^2C_1^2 + d\Lambda_{B_{N_0}}\right)}_{:=C_2} \mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} (\|X_u\|^2+1)\, du \right] \\ \le\ & C_2 \int_0^t \mathbb{E}_x\left[\sup_{0\le s\le u\wedge\sigma_n} \|X_s\|^2 \right] du + C_2 T. \end{aligned} \tag{4.8}$$

Now let $x \in \overline{B}$, and $t \leq T$. Then using (3.35), (4.6), for any $n \in \mathbb{N}$ and $1 \leq i \leq d$

$$
\begin{aligned}
& \mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} |g_i(X_u)|^2\,du\right] \\
\leq\ & \mathbb{E}_x\left[\int_0^{T} |g_i 1_{B_{N_0}}|^2(X_u)\,du\right] + \mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} |g_i 1_{\mathbb{R}^d\setminus B_{N_0}}|^2(X_u)\,du\right] \\
\leq\ & c_{B,p}e^T \left\|g_i 1_{B_{N_0}}\right\|^2_{L^p(\mathbb{R}^d,m)} + 2C_1\mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} (\|X_u\|^2+1)\,du\right] \\
\leq\ & c_{B,p}e^T \sup_{\overline{B}_{N_0}} |\rho|^{\frac{2}{p}}\cdot\|g_i\|^2_{L^p(B_{N_0})} + 2C_1\int_0^t \mathbb{E}_x\left[\sup_{0\leq s\leq u\wedge\sigma_n}\|X_s\|^2\right]du + 2C_1T. \qquad (4.9)
\end{aligned}
$$

Now let $h_n(t) := \mathbb{E}_x\left[\sup_{0\leq u\leq t\wedge\sigma_{\mathbb{R}^d\setminus B_n}}\|X_u\|^2\right]$. Then by (4.7), (4.8), (4.9), we obtain

$$
\begin{aligned}
h_n(t) \ \leq\ & \underbrace{(d+2)\|x\|^2 + C_2T + c_{B,p}e^T\,T\sup_{\overline{B}_{N_0}}|\rho|^{\frac{2}{p}}\cdot\|\mathbf{G}\|^2_{L^p(B_{N_0},\mathbb{R}^d)} + 2dC_1T^2}_{:=C_{T,B}} \\
& +(\underbrace{2dC_1T+C_2}_{:=C_T})\int_0^t h_n(u)du.
\end{aligned}
$$

By Gronwall's inequality, $h_n(t) \leq C_{T,B}\cdot e^{C_T\cdot t}$. Since none of the involved constants depends on n, we can use Fatou's lemma letting $n\to\infty$, and obtain

$$
\mathbb{E}_x\left[\sup_{s\leq t}\|X_s\|^2\right] \leq C_{T,B}e^{C_T\cdot t}, \quad \forall t\leq T.
$$

Since $x \in \overline{B}$ was arbitrary, the desired result follows.

□

4.1.2 Non-explosion criteria involving the density ρ

By [69, Proposition 1.10](a) we know that (4.1) holds, whenever

$$
a_{ij}, g_i - \beta_i^{\rho,A} \in L^1(\mathbb{R}^d, m), \quad 1\leq i,j\leq d. \qquad (4.10)
$$

Thus (4.10) provides a sufficient condition for non-explosion by (4.2) which obviously depends on the knowledge of the density ρ. Furthermore, one can directly check by (4.11) below that if (4.10) holds, then the $L^1(\mathbb{R}^d, m)$-semigroup $(\widehat{T}_t)_{t>0}$ is conservative, hence m is an invariant measure for $(T_t)_{t>0}$.
A systematic study of non-explosion conditions, more precisely results implying (4.1) and involving the density ρ can be found in [28, Corollary 15].

4.2 Recurrence criteria and other ergodic properties involving and not involving the density ρ

The measure $m = \rho\, dx$, where the density ρ is as at the beginning of Section 3.1 or as in Theorem 3.1.2, can be seen to define a stationary distribution. In fact, if the $L^1(\mathbb{R}^d, m)$-semigroup $(\widehat{T}_t)_{t>0}$ is conservative, for instance if there exists a constant $M \geq 0$ and some $N_0 \in \mathbb{N}$, such that

$$-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\mathrm{trace}(A(x)) + \left\langle \left(2\beta^{\rho,A} - \mathbf{G}\right)(x), x\right\rangle \\ \leq M(\|x\|^2+1)(\ln(\|x\|^2+1)+1)$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$, as one can see from the dual version of Theorem 4.1.2 or [69, Proposition 1.10(c)], then m is an invariant measure (for $(T_t)_{t>0}$), i.e. for any $f \in L^1(\mathbb{R}^d, m)$

$$\int_{\mathbb{R}^d} T_t f\, dm = \int_{\mathbb{R}^d} f \widehat{T}_t 1_{\mathbb{R}^d}\, dm = \int_{\mathbb{R}^d} f\, dm \tag{4.11}$$

so that for any $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq 0$

$$\begin{aligned} \mathbb{P}_m(X_t \in A) &:= \int_{\mathbb{R}^d} \mathbb{P}_x(X_t \in A)\, m(dx) = \int_{\mathbb{R}^d} T_t 1_A(x)\, m(dx) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^d} T_t 1_{A\cap B_n}(x)\, m(dx) = \lim_{n\to\infty} \int_{\mathbb{R}^d} 1_{A\cap B_n}(x)\, m(dx) = m(A). \end{aligned}$$

However, usually m is not a probability measure, hence $\mathbb{P}_m$ is also not such a measure. But if it is, then $\mathbb{P}_m$ is a stationary distribution (if $(\widehat{T}_t)_{t>0}$ is conservative). Main parts of the monograph [12] focus on the density ρ or more generally on m, in case m is a probability measure and aim in deriving properties of both (since both are in general not explicit).

We will first consider possibly infinite m and we may assume that ρ is explicit as is explained in the following remark.

Remark 4.2.1. *All results up to now and further hold exactly in the same form, if we assume that $\rho \in C^{0,1-d/p}_{loc}(\mathbb{R}^d) \cap H^{1,p}_{loc}(\mathbb{R}^d)$ for some $p > d$ with $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ is explicitly given from the beginning, that $A := (a_{ij})_{1\le i,j\le d}$ is as in Theorem 3.1.2 and that $\mathbf{B} = (b_1, ..., b_d) \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla f \rangle \, dm = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d).$$

Indeed, we then just have to set $\mathbf{G} := \beta^{\rho,A} + \mathbf{B}$. Then all conclusions of Theorem 3.1.2 hold with the explicitly chosen density from above. Note that this also includes the setting of Theorem 3.1.2 since by its conclusion a ρ like above exists and can hence be "explicitly" chosen.

We want to derive explicit conditions for recurrence involving and not involving the density ρ in two general cases where m is a general σ-finite measure and where m is a finite, yet without loss of generality a probability measure. First, we derive a lemma which leads to irreducibility and strict irreducibility (see Corollary 4.2.4) and as a byproduct leads to a weaker condition for non-explosion (see Remark 4.2.3).

Lemma 4.2.2. *(i) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $P_{t_0}1_A(x_0) = 0$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$. Then $m(A) = 0$.*

(ii) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $P_{t_0}1_A(x_0) = 1$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$. Then $P_t 1_A(x) = 1$ for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$.

Proof (i) Suppose $m(A) > 0$. Choose an open ball $B_r(x_0) \subset \mathbb{R}^d$ such that

$$0 < m\,(A \cap B_r(x_0)) < \infty.$$

Let $u := \rho P_\cdot 1_{A\cap B_r(x_0)}$. Then $0 = u(x_0,t_0) \le \rho(x_0)P_{t_0}1_A(x_0) = 0$. Take $f_n \in C_0^\infty(\mathbb{R}^d)$ with $f_n \ge 0$ such that $f_n \to 1_{A\cap B_r(x_0)}$ in $L^1(\mathbb{R}^d, m)$. Then by (3.26) and the explanation right after it, for arbitrary bounded open set U in $\mathbb{R}^d$ and $[\tau_1,\tau_2] \subset (0,\infty)$, there is some $\gamma \in (0,1)$ such that

$$P_\cdot f_n(\cdot) \to P_\cdot 1_{A\cap B_r(x_0)}(\cdot) \ \text{ in } C^{\gamma;\frac{\gamma}{2}}(\overline{U} \times [\tau_1,\tau_2]),$$

hence

$$u_n := \rho P_\cdot f_n \to u \ \text{ in } C^{\gamma;\frac{\gamma}{2}}(\overline{U} \times [\tau_1,\tau_2]). \tag{4.12}$$

Fix $T > t_0$ and $U \supset \overline{B}_r(x_0)$. Then (see proof of Theorem 3.1.8) for all $\varphi \in C_0^\infty(U \times (0,T))$

$$\int_0^T \int_U \left(\frac{1}{2}\langle A\nabla u_n, \nabla\varphi\rangle + u_n\langle \beta, \nabla\varphi\rangle - u_n\partial_t\varphi \right) dxdt = 0,$$

where β is defined as in the proof of Theorem 3.1.8. Now take arbitrary but fixed $(x,t) \in B_r(x_0) \times (0,t_0)$ By [2, Theorem 5]

$$0 \le u_n(x,t) \le u_n(x_0,t_0)\ \exp\left(C\Big(\frac{\|x_0 - x\|^2}{t_0 - t} + \frac{t_0 - t}{\min(1,t)} + 1 \Big) \right)$$

and (4.12) applied with $U \supset \overline{B}_r(x_0)$, $[\tau_1,\tau_2] \supset [t,t_0]$ then leads to

$$0 \le u(x,t) \le u(x_0,t_0)\ \exp\left(C\Big(\frac{\|x_0 - x\|^2}{t_0 - t} + \frac{t_0 - t}{\min(1,t)} + 1 \Big) \right) = 0.$$

Thus, $P_t 1_{A\cap B_r(x_0)}(x) = 0$ for all $x \in B_r(x_0)$ and $0 < t < t_0$, so that

$$0 = \int_{\mathbb{R}^d} 1_{A\cap B_r(x_0)} P_t 1_{A\cap B_r(x_0)} dm \underset{t\to 0+}{\longrightarrow} m(B_r(x_0)\cap A) > 0,$$

which is contradiction. Therefore, we must have $m(A) = 0$.
(ii) Let $y \in \mathbb{R}^d$ and $0 < s < t_0$ be arbitrary but fixed and let $r := 2\|x_0 - y\|$ and let B be any open ball. Take $g_n \in C_0^\infty(\mathbb{R}^d)$ with $0 \le g_n \le 1$ such that $g_n \to 1_{A\cap B}$ in $L^1(\mathbb{R}^d, m)$. Then by (3.26) and the explanation right after it, there is some $\gamma \in (0,1)$

such that

$$P_\cdot g_n(\cdot) \longrightarrow P_\cdot 1_{A\cap B}(\cdot) \text{ in } C^{\gamma;\frac{\gamma}{2}}(\overline{B}_r(x_0)\times[s/2,2t_0]). \tag{4.13}$$

Fix $T>0$ and $U\supset \overline{B}_r(x_0)$. Using the property

$$\beta = \frac{1}{2}\nabla A + \mathbf{G} - 2\beta^{\rho,A} = \mathbf{B} - \beta^{A,\rho} + \frac{1}{2}\nabla A = \mathbf{B} - \frac{1}{2}A\frac{\nabla\rho}{\rho},$$

and (3.2), we directly get for all $\varphi\in C_0^\infty(U\times(0,T))$

$$\int_0^T\int_U\left(\frac{1}{2}\langle A\nabla\rho,\nabla\varphi\rangle + \rho\langle\beta,\nabla\varphi\rangle - \rho\partial_t\varphi\right)dxdt = \int_0^T\left(\int_U\langle\mathbf{B},\nabla\varphi\rangle\rho dx\right)dt = 0, \tag{4.14}$$

and (cf. the proof of Theorem 3.1.8) we also get

$$\int_0^T\int_U\left(\frac{1}{2}\langle A\nabla\left(\rho P_\cdot g_n\right),\nabla\varphi\rangle + (\rho P_\cdot g_n)\langle\beta,\nabla\varphi\rangle - (\rho P_\cdot g_n)\partial_t\varphi\right)dxdt = 0. \tag{4.15}$$

Now let $u_n(x,t) := \rho(x)\left(1-P_tg_n(x)\right)$. Then $u_n\in H^{1,2}(U\times(0,T))$ and $u_n\geq 0$. Subtracting (4.15) from (4.14) implies

$$\int_0^T\int_U\left(\frac{1}{2}\langle A\nabla u_n,\nabla\varphi\rangle + u_n\langle\beta,\nabla\varphi\rangle - u_n\partial_t\varphi\right)dxdt = 0.$$

Thus, by [2, Theorem 5]

$$0\leq u_n(y,s)\leq u_n(x_0,t_0)\underbrace{\exp\left(C\Big(\frac{\|x_0-y\|^2}{t_0-s}+\frac{t_0-s}{\min(1,s)}+1\Big)\right)}_{=:C_2}.$$

By (4.13)

$$0\leq\rho(y)\left(1-P_s1_{A\cap B}(y)\right)\leq C_2\rho(x_0)\left(1-P_{t_0}1_{A\cap B}(x_0)\right).$$

Note that for all $(x,t)\in\mathbb{R}^d\times(0,\infty)$, $P_t1_{A\cap B_n}1(x)\nearrow P_t1_A(x)$ as $n\to\infty$. Thus,

$$0\leq\rho(y)\left(1-P_s1_A(y)\right)\leq C_2\rho(x_0)\left(1-P_{t_0}1_A(x_0)\right) = 0.$$

Consequently, $P_s 1_A(y) = 1$ for any $(y,s) \in \mathbb{R}^d \times (0,t_0)$ which can be extended on $\mathbb{R}^d \times (0,t_0]$ by continuity. And by sub-Markovian property, $P_{t_0} 1_{\mathbb{R}^d}(y) = 1$ for any $y \in \mathbb{R}^d$. Now let $t \in (0,\infty)$ be given. Then there extist $k \in \mathbb{N} \cup \{0\}$ such that

$$kt_0 < t \leq (k+1)t_0$$

and so $P_t 1_A = P_{kt_0+(t-kt_0)} 1_A = \underbrace{P_{t_0} \circ \cdots \circ P_{t_0}}_{k-times} \circ P_{t-kt_0} 1_A = 1.$

□

Remark 4.2.3. *By Lemma 4.2.2(ii) we know that $\mathbb{M}$ is non-explosive, if $\mathbb{P}_x(\zeta = \infty) = 1$ for some $x \in \mathbb{R}^d$. More precisely, if $\mathbb{P}_{x_0}(X_{t_0} \in \mathbb{R}^d) = 1$ for some $(x_0, t_0) \in \mathbb{R}^d \times (0,\infty)$, then $\mathbb{M}$ is non-explosive. This (together with Proposition 3.1.10, Lemma 3.2.3) generalizes and improves [6, Lemma 2.5] to possibly locally unbounded drift coefficient using a completely different and genuine proof.*

$A \in \mathcal{B}(\mathbb{R}^d)$ is called *weakly invariant* relative to $(T_t)_{t>0}$, if

$$T_t(f \cdot 1_A)(x) = 0, \quad \text{for } m\text{-a.e. } x \in \mathbb{R}^d \setminus A,$$

for any $t > 0$, $f \in L^2(\mathbb{R}^d, m)$. $(T_t)_{t>0}$ is said to be *strictly irreducible*, if for any weakly invariant set A relative to $(T_t)_{t>0}$, we have $m(A) = 0$ or $m(\mathbb{R}^d \setminus A) = 0$.

Corollary 4.2.4. *(i) $(T_t)_{t>0}$ is strictly irreducible.*

(ii) Let $A \in \mathcal{B}(\mathbb{R}^d)$ with $m(A) > 0$. Then $\mathbb{P}_x(X_t \in A) > 0$ for all $x \in \mathbb{R}^d, t > 0$, i.e. $(P_t)_{t>0}$ is irreducible.

Proof (i) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be a weakly invariant set with $m(\mathbb{R}^d \setminus A) \neq 0$. Then by monotone approximation with the L^2-functions 1_{B_n}, $n \geq 1$, we get for any $t > 0$ $P_t 1_A(x) = 0$, for m-a.e. $x \in \mathbb{R}^d \setminus A$. Then there exists $t_0 > 0$ and $x_0 \in \mathbb{R}^d \setminus A$ such that $P_{t_0} 1_A(x_0) = 0$. By Lemma 4.2.2(i), we have $m(A) = 0$, as desired.
(ii) By contraposition of Lemma 4.2.2(i), $\mathbb{P}_x(X_t \in A) = P_t 1_A(x) > 0$, for all $x \in \mathbb{R}^d, t > 0$.

□

4.2.1 Explicit recurrence criteria for possibly infinite m

We continue with some further definitions. Define the last exit time L_A from $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$L_A := \sup\{t \geq 0 : X_t \in A\}, \quad (\sup \emptyset := 0).$$

$\mathbb{M}$ is called *recurrent (in the probabilistic sense)*, if for any $\emptyset \neq U \subset \mathbb{R}^d$, U open, we have

$$\mathbb{P}_x(L_U = \infty) = 1, \quad \forall x \in \mathbb{R}^d. \tag{4.16}$$

Let $(\vartheta_t)_{t \geq 0}$ be the shift operator of $\mathbb{M}$. Using the shift invariance of $\Lambda := \{L_U = \infty\}$, the Markov property and the strong Feller property of $(P_t)_{t>0}$, we get for all $x \in \mathbb{R}^d$, $t > 0$

$$\mathbb{P}_x(\Lambda) = \mathbb{P}_x(\vartheta_t^{-1}(\Lambda)) = \mathbb{E}_x[\mathbb{E}_x[1_\Lambda \circ \vartheta_t \,|\, \mathcal{F}_t]] = \mathbb{E}_x[\mathbb{E}_{X_t}[1_\Lambda]] = P_t \mathbb{E}_\cdot[1_\Lambda](x).$$

Thus

$$(4.16) \iff \mathbb{P}_x(L_U = \infty) = 1 \quad \text{for } m\text{-a.e. } x \in \mathbb{R}^d. \tag{4.17}$$

The following is now a consequence of the results obtained here, in [29] and [26]. Moreover it generalizes [6, Theorem 3.2] that only treats non-explosive weak solutions to time-homogeneous Itô-SDEs whose drift coefficients are locally bounded.

Proposition 4.2.5. $(T_t)_{t>0}$ *(or equivalently* $\mathbb{M}$*) is either transient or recurrent in the sense of [29].*

(i) Suppose $(T_t)_{t>0}$ *is transient in the sense of [29]. Then for any compact* $K \subset \mathbb{R}^d$*, it holds* $\mathbb{P}_x(L_K < \infty) = 1$ *for all* $x \in \mathbb{R}^d$*. In particular*

$$\mathbb{P}_x(\lim_{t \to \infty} X_t = \Delta \text{ in } \mathbb{R}^d_\Delta) = 1 \text{ for any } x \in \mathbb{R}^d. \tag{4.18}$$

(ii) Suppose $(T_t)_{t>0}$ *is recurrent in the sense of [29]. Then* $\mathbb{M}$ *is non-explosive and recurrent (in the probabilistic sense), i.e.* (4.16) *holds for any nonempty open* $U \subset \mathbb{R}^d$*.*

Proof The first assertion follows from Corollary 4.2.4(i) and [29, Remark 3(b)].
(i) Applying [29, Lemma 6] and the last part of Lemma 3.2.3(i) we get the existence of $g \in L^1(\mathbb{R}^d, m) \cap L^\infty(\mathbb{R}^d, m)$ with $g > 0$ everywhere, such that $Rg := \mathbb{E}_\cdot \left[\int_0^\infty g(X_t)dt\right] \in L^\infty(\mathbb{R}^d, m)$. Using that Rg is lower semicontinuous by the strong Feller property and essentially bounded, we deduce $Rg(x) < \infty$ for any $x \in \mathbb{R}^d$. Obviously, $0 < Rg(x)$ for any $x \in \mathbb{R}^d$. Modifying the proof of [29, Proposition 10] (which originates from [26]) with the open sets $U_n := \{Rg > \frac{1}{n}\}$, $n \geq 1$, and using the strong Feller property of $(P_t)_{t>0}$, we obtain $\mathbb{P}_x(L_{U_n} < \infty) = 1$ for all $x \in \mathbb{R}^d, n \geq 1$. Now the first assertion follows easily since $(U_n)_{n\geq 1}$ is an open cover of any compact set $K \subset \mathbb{R}^d$. The second assertion follows from the first since the paths of $\mathbb{M}$ are continuous on the one point compactification $\mathbb{R}^d_\Delta$.
(ii) (4.1) is a consequence of [29, Corollary 20] and $\mathbb{M}$ is hence non-explosive by (4.2). Moreover, the right hand side of (4.17) holds for any $\emptyset \neq U \subset \mathbb{R}^d$, U open, by [29, Proposition 11(d)]. Therefore $\mathbb{M}$ is recurrent in the probabilistic sense.

□

Remark 4.2.6. *In Proposition 4.2.5, we get actually equivalences in (i) and (ii). Namely,* (4.18) *implies that [29, Condition (8) of Proposition 10] is satisfied. Thus* (4.18) *implies transience of* $\mathbb{M}$ *(or equivalently* $(T_t)_{t>0}$*) in the sense of [29] by [29, Proposition 10]. Likewise, if* $\mathbb{M}$ *is recurrent (in the probabilistic sense), then it cannot satisfy* (4.18)*. Therefore, by Proposition 4.2.5(i) and its first part,* $(T_t)_{t>0}$ *must be recurrent in the sense of [29].*

Define for $r \geq 0$,

$$v_1(r) := \int_{B_r} \frac{\langle A(x)x, x\rangle}{\|x\|^2} m(dx), \quad v_2(r) := \int_{B_r} |\langle \mathbf{B}(x), x\rangle|\, m(dx),$$

where $\mathbf{B}$ is defined as in Theorem 3.1.2 and let

$$v(r) := v_1(r) + v_2(r), \quad a_n := \int_1^n \frac{r}{v(r)} dr, \quad n \geq 1.$$

Theorem 4.2.7. *(Corollary of [29, Theorem 21]) Suppose that*

$$\lim_{n\to\infty} a_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\log(v_2(n)\vee 1)}{a_n} = 0.$$

Then $\mathbb{M}$ is recurrent (in the probabilistic sense) and non-explosive. Moreover m is an invariant measure for $(T_t)_{t>0}$.

Proof By [29, Theorem 21] applied with $\rho(x)=\|x\|$ (the ρ of [29] is different from the ρ defined here), the given assumption implies that $(T_t)_{t>0}$ is not transient in the sense of [29]. Then apply Proposition 4.2.5 to show recurrence of $\mathbb{M}$.
Since $v_2(r) := \int_{B_r} |\langle -\mathbf{B}(x), x\rangle|\, m(dx)$, $(\widehat{T}_t)_{t>0}$ is not transient in the sense of [29]. Thus applying Proposition 4.2.5 again, $(\widehat{T}_t)_{t>0}$ is conservative. Using (4.11), m is an invariant measure for $(T_t)_{t>0}$.

□

Lemma 4.2.8. *For any $x\in\mathbb{R}^d$ and $N\in\mathbb{N}$, we have $\mathbb{P}_x(\sigma_N<\infty)=1$.*

Proof Suppose to the contrary that there exists $N\in\mathbb{N}$ and $x\in\overline{B}_N$ such that $\mathbb{P}_x(\sigma_N=\infty)\geq\delta>0$. Then $\mathbb{M}$ is not recurrent in the probabilistic sense. Applying Proposition 4.2.5, we obtain $\mathbb{P}_x(L_K<\infty)=1$ for all $x\in\mathbb{R}^d$ and any compact $K\subset\mathbb{R}^d$. Therefore $\mathbb{P}_x(\sigma_N=\infty)\geq\delta>0$ cannot hold and the assertion follows. □

The following theorem extends [58, Chapter 6, Theorem 1.2] to locally unbounded drift coefficient.

Theorem 4.2.9. *Suppose that there exists a positive $\psi\in C^2(\mathbb{R}^d)$ and some $N_0\in\mathbb{N}$ such that $L\psi\leq 0$ a.e. on $\mathbb{R}^d\setminus B_{N_0}$ and $\inf_{\partial B_n}\psi\to\infty$ as $n\to\infty$. Then $\mathbb{M}$ is recurrent (in the probabilistic sense) and non-explosive. In particular, the assumptions above are satisfied (take $\psi(x)=\ln(\|x\|^2+1)+1$), if there is some $N_0\in\mathbb{N}$, such that*

$$-\frac{\langle A(x)x,x\rangle}{\|x\|^2+1}+\frac{1}{2}\mathrm{trace}A(x)+\big\langle \mathbf{G}(x),x\big\rangle\leq 0$$

for a.e. $x\in\mathbb{R}^d\setminus B_{N_0}$.

Proof Clearly, $\mathbb{M}$ is non-explosive by Remark 4.1.3(iii). Let $n \geq N_0$ and $x \in \mathbb{R}^d \setminus \overline{B}_n$ be arbitrary. Choose any $N \in \mathbb{N}$ with $x \in B_N$. We will first show that $\mathbb{P}_x(\sigma_{B_n} < \infty) = 1$. Using that $L\psi \leq 0$ a.e. on $\mathbb{R}^d \setminus B_{N_0}$ we can see that

$$\mathbb{E}_x[\psi(X_{t \wedge \sigma_{B_n} \wedge \sigma_N})] \leq \psi(x).$$

Since $\mathbb{P}_x(\sigma_N < \infty) = 1$ by Lemma 4.2.8, we can let $t \to \infty$ and obtain with elementary calculations (cf. for instance the proof of Theorem 4.1.2)

$$(\inf_{\partial B_N} \psi) \cdot \mathbb{P}_x(\sigma_{B_n} = \infty) \leq \mathbb{E}_x[\psi(X_{\sigma_N}) 1_{\{\sigma_{B_n} = \infty\}}] \leq \mathbb{E}_x[\psi(X_{\sigma_{B_n} \wedge \sigma_N})] \leq \psi(x).$$

Letting $N \to \infty$ and using the further assumption on ψ, we get $\mathbb{P}_x(\sigma_{B_n} = \infty) = 0$ and the claim is shown. From now on let $n := N_0 + 1$. Then obviously $\mathbb{P}_x(\sigma_{B_n} < \infty) = 1$ for any $x \in B_n$ and by the claim $\mathbb{P}_x(\sigma_{B_n} < \infty) = 1$ for any $\mathbb{R}^d \setminus \overline{B}_n$. If $x \in \partial B_n$, then by the claim again $\mathbb{P}_x(\sigma_{B_{N_0}} < \infty) = 1$ and since $\sigma_{B_{N_0+1}} \leq \sigma_{B_{N_0}}$, we finally get

$$\mathbb{P}_x(\sigma_{B_n} < \infty) = 1 \quad \text{for any } x \in \mathbb{R}^d.$$

Let $z \in \mathbb{R}^d, s > 0$ be arbitrary. Then by the Markov property and since $\mathbb{M}$ is non-explosive

$$\mathbb{P}_z(X_t \in B_n \text{ for some } t \in [s, \infty)) = \mathbb{P}_z(\sigma_{B_n} \circ \vartheta_s < \infty) = \mathbb{E}_z[P_{X_s}(\sigma_{B_n} < \infty)] = 1.$$

Hence $\mathbb{P}_z(L_{\overline{B}_{N_0+1}} < \infty) = 0$ and the assertion now follows from Proposition 4.2.5. $\square$

4.2.2 Uniqueness of invariant measures and ergodic properties in case m is a probability measure

In this subsection, we suppose (except at the very end of it) that m is a finite measure. Dividing by a normalizing constant, which will not change the generator L, we may without loss of generality assume that m is a probability measure. Coming back to the situation at the beginning of Section 4.2, we have the following:

Remark 4.2.10. *If m is a probability measure, then m is $(T_t)_{t>0}$-invariant, if and* *only if it is $(\widehat{T}_t)_{t>0}$-invariant (cf. [69, Proposition 1.10(b)]). In either case $\mathbb{P}_m$ is then a stationary distribution.*

It is clear that the $(\widehat{T}_t)_{t>0}$-invariance of m is equivalent to the conservativeness of $(T_t)_{t>0}$, i.e. to (4.1). Consequently, using Remark 4.2.10, we see that m is an invariant (probability) measure for $(T_t)_{t>0}$, if (4.1) holds. Therefore, (1.3) provides an explicit criterion for m to be an invariant (probability) measure. Now, we have the following:

Theorem 4.2.11. *Suppose that m is a probability measure and that* (4.1) *holds. Then:*

(i) m is strongly mixing (cf. [59]) and for arbitrary $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \in A) = m(A).$$

(ii) m is the unique probability measure that is $(T_t)_{t>0}$-invariant.

(iii) m is equivalent to $\mathbb{P}_x \circ X_t^{-1}$ for any $(x,t) \in \mathbb{R}^d \times (0,\infty)$.

(iv) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $m(A) > 0$ and $(t_n)_{n\geq 1} \subset (0,\infty)$ be any sequence with $\lim_{n\to\infty} t_n = \infty$. Then $\mathbb{P}_x(X_{t_n} \in A$ for infinitely many $n \in \mathbb{N}) = 1 \ \ \forall x \in \mathbb{R}^d$. In particular, $\mathbb{M}$ is recurrent.

Proof By Theorem 3.1.8, Lemma 3.2.3(i) and Corollary 4.2.4(i), $(P_t)_{t>0}$ is strong Feller and irreducible. Then [59, Proposition 4.1.1] implies that $(P_t)_{t>0}$ is regular. Therefore the assertions (i)-(iii) follow by Doob's Theorem, see [59, Theorem 4.2.1]. Then using (i), assertion (iv) follows by [59, Theorem 3.4.5].

□

Remark 4.2.12. *Assume that as in Remark 4.2.1, ρ, A, $\mathbf{B}$ are explicitly given and that $m = \rho\, dx$ is a probability measure such that* (4.1) *holds. Then Theorem 4.2.11 applies. This result seems to be new even if $\mathbf{B} \equiv 0$.*

For the rest of the section we do *not assume that m is a finite measure* and present a condition that is independent of ρ and makes Theorem 4.2.11 applicable. The following proposition is a variant of [58, Chapter 6, Theorem 1.3] which can be applied to locally unbounded drift coefficients.

Proposition 4.2.13. *Suppose that there exists a positive $\psi \in C^2(\mathbb{R}^d)$, some $N_0 \in \mathbb{N}$ and $C > 0$, such that $L\psi \le -C$ a.e. on $\mathbb{R}^d \setminus B_{N_0}$ and $\inf_{\partial B_n} \psi \to \infty$ as $n \to \infty$. Then m is finite and $\mathbb{M}$ is non-explosive. In particular,* (4.1) *holds and by normalizing m if necessary, we can see that the assumptions of Theorem 4.2.11 are satisfied. Thus Theorem 4.2.11(i)-(iv) hold. In particular, the assumptions above are satisfied (take $\psi(x) = \ln(\|x\|^2+1)+1$), if there exists a constant $C > 0$ and some $N_0 \in \mathbb{N}$, such that*

$$-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\text{trace}A(x) + \big\langle \mathbf{G}(x), x\big\rangle \le -C\left(\|x\|^2+1\right) \tag{4.19}$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$.

Proof Using $L\psi(x) \le -C$ for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$, the finiteness of m follows by [12, Corollary 2.3.3] or [13, Theorem 2] for the original result. Since $L\psi(x) \le M\psi(x)$ for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$ for any $M > 0$, $\mathbb{M}$ is non-explosive by Remark 4.1.3(iii). We may hence assume that the conditions of Theorem 4.2.11 are satisfied.

□

In the next example, we shall give a sufficient condition for (4.19) to hold.

Example 4.2.14. *Let I be the identity matrix consisting of ones on the diagonal and zeros outside and set $A(x) := \Psi(x)I$ where $\Psi(x) \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{1-d/p}_{loc}(\mathbb{R}^d)$ with $\Psi(x) > 0$ for all $x \in \mathbb{R}^d$. Let $\phi_1 \in L^p_{loc}(\mathbb{R}^d)$, $\phi_1 \ge 0$ a.e. and $\mathbf{G}(x) := \Big(-\phi_1(x)1_{\mathbb{R}^d\setminus B_{N_0}} + \phi_2(x)1_{B_{N_0}}\Big)x$ for some $\phi_2 \in L^p_{loc}(\mathbb{R}^d)$. Suppose that for some $N_0 \in \mathbb{N} \cup \{0\}$,*

$$\frac{d}{2}\Psi(x) + C(\|x\|^2+1) \le \phi_1(x)\|x\|^2 \ \text{a.e. } x \in \mathbb{R}^d \setminus B_{N_0}\ . \tag{4.20}$$

Then (4.20) *implies* (4.19).

Now we compare our results with results of [85].

Remark 4.2.15. *As one can see from the proof of Theorem 4.2.11 in order to derive the conclusions Theorem 4.2.11(i)-(iv) one needs for instance the classical strong Feller property and the irreducibility. In our case, these are directly implied under the conditions of Theorem 3.1.2 (cf. Theorem 3.1.8 and Corollary 4.2.4(ii)). But the conditions* *to obtain the strong Feller property and irreducibility in [85] are quite strong, and there are many cases where* (4.19) *is satisfied but one cannot obtain the strong Feller property nor irreducibility from the results of [85]. The following provides a comparison of* (4.19) *and the rather strong conditions of [85]:*

(i) a) *If $\mathbf{G}$ is not bounded on an open ball, in order to get the strong Feller property and the irreducibility, [85, Theorem 1.7] needs very strong conditions [85, (H1'), (H2')] such as global uniform ellipticity and boundedness of A and Lipschitz continuity of $A, \mathbf{G}$ and the growth condition $\|G(x)\| \le C(1+\|x\|)$ outside an open ball. For example if we take $A(x) = (1+\|x\|)I$ and $\phi_1(x) = \|x\|^2$, then* (4.20) *holds, but (H1') and (H2') in [85] are both not satisfied. Thus the conditions of [85] do not neither provide global well-posedness, nor strong Feller properties, nor irreducibility and so on, whereas we get the full conclusions of Proposition 4.2.13.*

b) *If $\mathbf{G}$ is locally bounded on $\mathbb{R}^d$, to get the strong Feller property and the irreducibility, [85, Theorem 1.2] also requires quite strong conditions. For example, a diffusion matrix with strong decay such as $A(x) = \exp(-\exp(\|x\|^2))I$ cannot be handled by results of [85], since [85, (1.4)] is not satisfied, but we do not have such restrictions. Moreover, if $A(x) = I$ and $\phi_1(x) = \exp(\exp(\|x\|^2)))$, then clearly* (4.20) *is satisfied, but [85, (1.7)] is not satisfied. Note that [85, (1.6), (1.8)] requires A to be (besides an $H^{1,q}_{loc}$-condition, $q > d+2$) locally Lipschitz outside an open ball, if $b \equiv 0$ in [85]), which is also stronger than our condition $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^d)$ for $1 \le i,j \le d$ for some $p > d$.*

(ii) We will give a simple example which has a global pathwise unique solution satisfying all conclusions of Proposition 4.2.13, but the non-explosion conditions in [85] do even not allow to obtain the existence of global solution. Choose $\Psi(x) = \phi_1(x) = (1+\|x\|)^2$. Then (4.20) *is satisfied, so that by Example 4.2.14 we may apply Proposition 4.2.13 and get a global pathwise unique solution satisfying (i)-(iv) of Theorem 4.2.11. Now consider*

$$\phi_2(x) = \frac{1}{\|x - (\frac{N_0}{2}, 0, \cdots, 0)\|^{d/(p+1)}}, \quad x \in \mathbb{R}^d.$$

Then $\phi_2 \in L^p_{loc}(\mathbb{R}^d)$ *and* $\lim_{x \to (\frac{N_0}{2}, 0, \cdots, 0)} \phi_2(x) = \infty$, *so that* $\mathbf{G}$ *as defined in Example 4.2.14 satisfies*

$$\big\langle \mathbf{G}(x), x \big\rangle \longrightarrow \infty \text{ as } x \to (\frac{N_0}{2}, 0, \cdots, 0).$$

Thus, the non-explosion condition [85, (1.5)] is not satisfied and obviously global boundedness of A *and linear growth of* $\|\mathbf{G}\|$ *do not hold, which means [85, [H1'] [H2']] are not satisfied. In particular, no non-explosion condition of [85] holds.*

(iii) By our method we have directly a candidate for invariant measure, namely m. *In [85] no candidate for invariant measure can be deduced.*

4.3 An application to pathwise uniqueness and strong solutions

In this section, we present an application of our weak existence and non-explosion results to pathwise uniqueness and existence of strong solutions up to ∞.

Theorem 4.3.1. *Let $A = (a_{ij})_{1\leq i,j\leq d}$, $\mathbf{G}$, be as in Theorem 3.1.2 and let $(\sigma_{ij})_{1\leq i,j\leq d}$ be as in Lemma 3.2.7. Suppose that (1.3) holds for A and $\mathbf{G}$. Then the stochastic differential equation*

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t \mathbf{G}(X_s)ds, \qquad t \geq 0,$$

where $W = (W^1, \dots, W^d)$ is a standard d-dimensional Brownian motion starting from zero, has a pathwise unique and strong solution. In particular, and without any further assumption, (X_t) is a Hunt process (by Theorem 3.2.1), satisfies more than classical strong Feller properties (see Theorem 3.1.8, Proposition 3.1.10 and Lemma 3.2.3), has integrability properties as in Lemma 3.2.4, is irreducible (by Corollary 4.2.4), satisfies the long time behavior as in Proposition 4.2.5 and Remark 4.2.6, and has further additional properties like in Lemma 4.2.2, Remark 4.2.3, Lemma 4.2.8. Moreover, there are diverse explicit further conditions to guarantee moment inequalities, recurrence and ergodicity, including existence and uniqueness of invariant measures for (X_t), see Theorems 4.1.4, 4.2.7, 4.2.9 and Proposition 4.2.13.

Proof The existence of a weak solution up to $\zeta = \infty$ under the present assumptions follows from Theorems 3.2.8(i) and 4.1.2. The weak solution is then pathwise unique and strong by [84, Theorem 1.3] and [82, Corollary 1].

□

Part II

Existence and regularity of pre-invariant measures, transition functions and time homogeneous Itô-SDEs

Chapter 5

Analytic results

5.1 Elliptic $H^{1,p}$-regularity and estimates

The $VMO(\mathbb{R}^d)$ space is defined as the space of all locally integrable functions f on $\mathbb{R}^d$ for which there exists a positive continuous function γ on $[0,\infty)$ with $\gamma(0)=0$, such that

$$\sup_{z\in\mathbb{R}^d, r<R} r^{-2d}\int_{B_r(z)\times B_r(z)} |f(x)-f(y)|dxdy \le \gamma(R), \quad \forall R>0. \tag{5.1}$$

If f is uniformly continuous on $\mathbb{R}^d$, we can define

$$\gamma(r) := \Big(\int_{B_1} 1\,dx\Big)^{-2}\cdot \sup_{|x-y|<2r, x,y\in\mathbb{R}^d} |f(x)-f(y)|, \quad \gamma(0):=0.$$

Then γ is continuous on $[0,\infty)$ and (5.1) holds, hence $f\in VMO(\mathbb{R}^d)$. For a bounded open subset U of $\mathbb{R}^d$ and a function g on U, we call $g\in VMO(U)$ if g extends to a function on $\mathbb{R}^d$, again called g, such that $g\in VMO(\mathbb{R}^d)$.

For measurable functions a_{ij}, b_i, β_i, c on $\mathbb{R}^d$, $1\le i,j\le d$, let $A:=(a_{ij})_{1\le i\le d}$, $b:=(b_1,\dots,b_d)$, $\beta:=(\beta_1,\dots,\beta_d)$. Consider the divergence form operator $\mathcal{L}$, defined in distribution sense

$$-\mathcal{L}u := -\left(\sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j u) + \sum_{i=1}^{d} \partial_i(b_i u)\right) + \sum_{i=1}^{d} \beta_i \partial_i u + cu.$$

The following theorem is a simple generalization of (1.2.3) in [12, Theorem 1.2.1], where only symmetric matrices of functions are considered.

Theorem 5.1.1. *Consider a possibly non-symmetric matrix of functions $A = (a_{ij})_{1\le i,j\le d}$ and suppose that $a_{ij} \in VMO(\mathbb{R}^d)$, $1 \le i, j \le d$, and that there exists $\varepsilon, K > 0$ such that*

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \varepsilon\|\xi\|^2_{\mathbb{R}^d} \quad \text{for all } \xi \in \mathbb{R}^d, \quad \text{a.e. } x \in \mathbb{R}^d,$$

$$\sum_{i,j=1}^{d} \|a_{ij}\|_{L^\infty(\mathbb{R}^d)} + \sum_{i=1}^{d} \|b_i\|_{L^\infty(\mathbb{R}^d)} + \sum_{i=1}^{d} \|\beta_i\|_{L^\infty(\mathbb{R}^d)} + \|c\|_{L^\infty(\mathbb{R}^d)} \le K.$$

Then, for every $p \in (1, \infty)$, there are numbers λ_0 and M depending only p, d, K, ε and a common γ that ensures the $VMO(\mathbb{R}^d)$ condition (5.1) simultaneously for all a_{ij}, $1 \le i, j \le d$, such that for all $\lambda \ge \lambda_0$, $v \in H^{1,p}_0(\mathbb{R}^d)$, we have

$$\|v\|_{H^{1,p}(\mathbb{R}^d)} \le M\|\mathcal{L}v - \lambda v\|_{H^{-1,p}(\mathbb{R}^d)}.$$

Proof Take constants λ_0, N as in [42, Theorem 2.8], which depend only on p, d, K, ε. Let $\lambda > \lambda_0$ be given. By [14, Proposition 9.20], there exists $f \in L^p(\mathbb{R}^d)$ and $g = (g_1, \dots, g_d) \in L^p(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\mathcal{L}v - \lambda v = f + \text{div} g \quad \text{ in } H^{-1,p}(\mathbb{R}^d),$$

where

$$\|\mathcal{L}v - \lambda v\|_{H^{-1,p}(\mathbb{R}^d)} = \max(\|f\|_{L^p(\mathbb{R}^d)}, \|g_1\|_{L^p(\mathbb{R}^d)}, \dots, \|g_d\|_{L^p(\mathbb{R}^d)}).$$

Thus

$$\|f\|_{L^p(\mathbb{R}^d)} + \sum_{i=1}^{d} \|g_i\|_{L^p(\mathbb{R}^d)} \le (d+1)\|\mathcal{L}v - \lambda v\|_{H^{-1,p}(\mathbb{R}^d)}.$$

By [42, Theorem 2.8],

$$\begin{aligned}\|v\|_{H^{1,p}(\mathbb{R}^d)} &\leq N\left(\|f\|_{L^p(\mathbb{R}^d)} + \sum_{i=1}^{d} \|g_i\|_{L^p(\mathbb{R}^d)}\right) \\ &\leq \underbrace{N(d+1)}_{=:M} \|\mathcal{L}v - \lambda v\|_{H^{-1,p}(\mathbb{R}^d)}.\end{aligned}$$

□

We shall make a general remark concerning the monograph [12].

Remark 5.1.2. *In what follows, we shall use in particular the statements 1.7.4, 1.7.6, 1.8.3, 2.1.4, 2.1.6, 2.1.8 of [12] which are formulated for a symmetric matrix of functions $A = (a_{ij})_{1\leq i,j\leq d}$ on a bounded smooth domain Ω, such that each function a_{ij} is $VMO(\Omega)$ and A is uniformly strictly elliptic and bounded on Ω. However, a closer look at the corresponding proofs shows that the symmetry is not a neccessary assumption. More precisely, (1.7.10) in the proof of [12, Theorem 1.7.4] follows from (1.2.3) of [12, Theorem 1.2.1]. But we have shown that symmetry of $(a_{ij})_{1\leq i,j\leq d}$ is not essential in Theorem 5.1.1. Consequently, [12, Corollary 1.7.6], whose proof is based on [12, Theorem 1.7.4], also holds for a non-symmetric matrix of functions $(a_{ij})_{1\leq i,j\leq d}$ which is uniformly strictly elliptic and bounded on Ω. The proof of [12, Proposition 2.1.4] is based on the Lax-Milgram Theorem which only uses a coercivity assumption that is well-known to extend to a non-symmetric matrix of functions. [12, Theorem 2.1.8] is taken from [77], where not only non-symmetric matrices of functions are permitted but also even more general conditions on the functions $a_{ij}, 1 \leq i,j \leq d$. [12, Corollary 2.1.6] is a consequence of [12, Corollary 1.7.6 , Proposition 2.1.4 and Theorem 2.1.8]. Finally, the proof of [12, Theorem 1.8.3] follows from [12, Corollary 1.7.6 and Proposition 2.1.4]. Therefore all the above mentioned statements from [12] extend to a non-symmetric matrix of functions $A = (a_{ij})_{1\leq i,j\leq d}$, such that each function a_{ij} is $VMO(\Omega)$ and A is uniformly strictly elliptic and bounded on Ω. However, we will assume more than $VMO(\Omega)$, more precisely $H^{1,2}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, in what follows since we need an integration by parts formula.*

The following Lemma 5.1.3 will be used in the proof of Lemma 5.1.4 for a compactness argument.

Lemma 5.1.3. *Let* $A := (a_{ij})_{1\le i,j\le d}$*,* $A_n := (a^n_{ij})_{1\le i,j\le d}$ *be uniformly strictly elliptic and bounded on an open ball* B*, satisfying* $a^n_{ij} \to a_{ij}$ *in* $L^2(B)$ *as* $n \to \infty$*,* $1 \le i,j \le d$*. Moreover, let* A_n*,* $n \in \mathbb{N}$*, and* A *have the same elliptic constant* $\lambda_n \equiv \lambda$ *and upper bound constant* $M_n \equiv M$*. Let for some* $p > d$*,* $b \in L^p(B,\mathbb{R}^d)$*,* $b_n \in L^p(B,\mathbb{R}^d)$ *such that* $b_n \to b$ *in* $L^p(B,\mathbb{R}^d)$ *as* $n \to \infty$*. Given* $F \in L^2(B,\mathbb{R}^d)$*, suppose that* $u_{n,F} \in H^{1,2}_0(B)$ *satisfies*

$$\int_B \langle A_n \nabla u_{n,F} + b_n u_{n,F}, \nabla\varphi\rangle\, dx = \int_B \langle F, \nabla\varphi\rangle\, dx, \qquad \text{for every } \varphi \in C_0^\infty(B).$$

Then

$$\|u_{n,F}\|_{L^2(B)} \le C\|F\|_{L^2(B,\mathbb{R}^d)},$$

where $C > 0$ *is a constant which is independent of* n *and* F*.*

Proof Assume that the assertion does not hold, i.e. given $k \in \mathbb{N}$ there exists $\widetilde{F}_k \in L^2(B,\mathbb{R}^d)$ and $n_k \in \mathbb{N}$ such that

$$\|u_{n_k,\widetilde{F}_k}\|_{L^2(B)} > k\|\widetilde{F}_k\|_{L^2(B,\mathbb{R}^d)}.$$

Define $F_k := \dfrac{\widetilde{F}_k}{\|u_{n_k,\widetilde{F}_k}\|_{L^2(B)}}$. By [12, Proposition 2.1.4, Theorem 2.1.8] and Remark 5.1.2, and using the maximum principle, we get $u_{n_k,F_k} = \dfrac{u_{n_k,\widetilde{F}_k}}{\|u_{n_k,\widetilde{F}_k}\|_{L^2(B)}}$. Thus we have

$$\|u_{n_k,F_k}\|_{L^2(B)} = 1 \quad \text{and} \quad \|F_k\|_{L^2(B,\mathbb{R}^d)} < \frac{1}{k}.$$

By [12, Corollary 1.7.6] and and Remark 5.1.2,

$$\|u_{n_k,F_k}\|_{H^{1,2}_0(B)} \;\le\; C_1(\|u_{n_k,F_k}\|_{L^2(B)} + \|F_k\|_{L^2(B,\mathbb{R}^d)}) \le 2C_1,$$

where C_1 is independent of k. By the weak compactness of balls in $H^{1,2}_0(B)$ and the Rellich-Kondrachov Theorem, there exists a subsequence $(u_{n_{k_j},F_{k_j}})_j \subset (u_{n_k,F_k})_k$ and

$u \in H_0^{1,2}(B)$ such that

$$u_{n_{k_j},F_{k_j}} \to u \text{ weakly in } H_0^{1,2}(B), \quad u_{n_{k_j},F_{k_j}} \to u \text{ in } L^2(B).$$

In particular, $\|u\|_{L^2(B)} = 1$ and using the assumption, we can see that u satisfies

$$\int_B \langle A\nabla u + bu, \nabla\varphi\rangle dx = 0, \qquad \text{for every } \varphi \in C_0^\infty(B).$$

By [12, Theorem 2.1.8] and Remark 5.1.2, we have $u = 0$ a.e. on B, which is a contradiction. Therefore the assertion must hold.

□

The following is well known in the case where $b \equiv 0$ (see for instance [31, Lemma 4.6]).

Lemma 5.1.4. *Let $A = (a_{ij})_{1\le i,j\le d}$ be uniformly strictly elliptic and bounded on U, which is supposed to a Lipschitz boundary. Let for some $p > d$, $b \in L^p(U, \mathbb{R}^d)$ and assume that $u \in H^{1.2}(U)$ satisfies*

$$\int_U \langle A\nabla u + bu, \nabla\varphi\rangle dx \le 0, \qquad \textit{for every } \varphi \in C_0^\infty(U), \ \varphi \ge 0.$$

Then we have

$$\int_U \langle A\nabla u^+ + bu^+, \nabla\varphi\rangle dx \le 0, \qquad \textit{for every } \varphi \in C_0^\infty(U), \ \varphi \ge 0.$$

Proof Let B be an open ball such that $\overline{U} \subset B$. By [21, Theorem 4.7], $u \in H^{1,2}(U)$ can be extended to a function $u \in H_0^{1,2}(B)$. And by [21, Theorem 4.4], $u^+ \in H_0^{1,2}(B)$ with

$$\nabla u^+ = \begin{cases} \nabla u & \text{a.e. on } \{u > 0\}, \\ 0 & \text{a.e. on } \{u \le 0\}. \end{cases}$$

Given $\varepsilon > 0$ define

$$f_\varepsilon(z) := \begin{cases} (z^2+\varepsilon^2)^{1/2} - \varepsilon & \text{if } z \ge 0, \\ 0 & \text{if } z < 0. \end{cases}$$

Then $f_\varepsilon \in C^1(\mathbb{R}) \cap H^{2,\infty}(\mathbb{R})$ satisfies

$$f'_\varepsilon(z) = \begin{cases} \dfrac{z}{\sqrt{z^2+\varepsilon^2}} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0, \end{cases} \qquad \textit{and} \qquad f''_\varepsilon(z) = \begin{cases} \dfrac{\varepsilon^2}{(z^2+\varepsilon^2)^{3/2}} & \text{if } z > 0, \\ 0 & \text{if } z < 0. \end{cases}$$

Note that $f_\varepsilon(z) \longrightarrow z^+$, $f'_\varepsilon(z) \longrightarrow 1_{(0,\infty)}(z)$ as $\varepsilon \to 0$ for every $z \in \mathbb{R}$. Extend $a_{ij} \in H^{1,2}(U) \cap C(\overline{U})$ to $H^{1,2}_{loc}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ with elliptic constant λ and upper bound constant M and extend $b \in L^p(U, \mathbb{R}^d)$ to $L^p(\mathbb{R}^d, \mathbb{R}^d)$ by setting b zero outside U. Define $F := A\nabla u + bu \in L^2(\mathbb{R}^d, \mathbb{R}^d)$. For $\widetilde{\varepsilon} > 0$ let $\eta_{\widetilde{\varepsilon}} \in C_0^\infty(B_{\widetilde{\varepsilon}})$ be a standard mollifer and let $a^n_{ij} := a_{ij} * \eta_{\frac{1}{n}}$, $A_n := (a^n_{ij})_{1 \leq i,j \leq d}$, $b_n := b * \eta_{\frac{1}{n}}$, $F_n := F * \eta_{\frac{1}{n}}$ on $\mathbb{R}^d$. Then $a^n_{ij} \in C^\infty(\overline{B})$, $b_n, F_n \in C^\infty(\overline{B}, \mathbb{R}^d)$ satisfy

$$a^n_{ij} \longrightarrow a_{ij}, \text{ in } L^2(B), \quad b_n \longrightarrow b \text{ in } L^p(B, \mathbb{R}^d), \quad F_n \longrightarrow F \text{ in } L^2(B, \mathbb{R}^d). \tag{5.2}$$

Moreover, each A_n, $n \in \mathbb{N}$, is uniformly strictly elliptic and bounded on B with same elliptic constant λ and upper bound constant M as A. Let V be a fixed open set with $\overline{V} \subset U$. Choose $\delta > 0$ with $\overline{B}_\delta(z) \subset U$ for all $z \in V$ and take $N \in \mathbb{N}$ with $\frac{1}{N} < \delta$. Then by the assumption, for any $n \geq N$ and $\varphi \in C_0^\infty(V)$ with $\varphi \geq 0$

$$\int_U \langle F_n, \nabla\varphi \rangle dx = \int_U \langle A\nabla u + bu, \nabla(\varphi * \eta_{\frac{1}{n}}) \rangle \, dx \leq 0. \tag{5.3}$$

By [12, Proposition 2.1.4, Theorem 2.1.8] and Remark 5.1.2, there exists $u_n \in H_0^{1,2}(B)$ such that

$$\int_B \langle A_n \nabla u_n + b_n u_n, \nabla\widetilde{\varphi} \rangle dx = \int_B \langle F_n, \nabla\widetilde{\varphi} \rangle dx, \quad \text{ for all } \widetilde{\varphi} \in C_0^\infty(B). \tag{5.4}$$

By [12, Corollary 1.7.6], Remark 5.1.2 and Lemma 5.1.3,

$$\|u_n\|_{H_0^{1,2}(B)} \;\; \leq \;\; C_1 \|F_n\|_{L^2(B,\mathbb{R}^d)} \leq C_1 \|F\|_{L^2(B,\mathbb{R}^d)}.$$

where C_1 is independent of n. By weak compactness of balls in $H_0^{1,2}(B)$, [12, Theorem

2.1.8] and Remark 5.1.2, there exists subsequence $(u_{n_k})_k \subset (u_n)_n$, such that

$$u_{n_k} \to u \ \text{ and } \ u^+_{n_k} \to u^+ \ \text{ weakly in } \ H^{1,2}_0(B). \tag{5.5}$$

Indeed, (5.5) first holds with u replaced by some $\widetilde{u} \in H^{1,2}_0(B)$. Then letting $n \to \infty$ in (5.4) and using the maximum principle, we get $\widetilde{u} = u$. For simplicity, write (u_n) for (u_{n_k}). By [27, Theorem 8.13], we have $u_n \in C^\infty(\overline{B})$. Now define

$$\mathcal{L}_n u_n := \sum_{i,j=1}^d a^n_{ij} \partial_i \partial_j u_n + \langle b_n + \nabla A^T_n, \nabla u_n \rangle + (\mathrm{div}\, b_n) \cdot u_n$$

Then for any $n \geq N$ and $\varphi \in C^\infty_0(V)$ with $\varphi \geq 0$, we obtain using (5.3), (5.4)

$$-\int_U \mathcal{L}_n u_n \, \varphi \, dx \leq 0.$$

Hence $\mathcal{L}_n u_n(x) \geq 0$ for all $x \in V$, $n \geq N$. Define $f^k_\varepsilon := f_\varepsilon * \eta_{\frac{1}{k}}$, $k \in \mathbb{N}$. Then $(f^k_\varepsilon)' \geq 0$, $(f^k_\varepsilon)'' \geq 0$ since $f'_\varepsilon \geq 0$, $f''_\varepsilon \geq 0$. Moreover, $(f^k_\varepsilon)'(u_n) \to f'_\varepsilon(u_n)$ uniformly on U as $k \to \infty$. Then, for any $n \geq N$ and $\varphi \in C^\infty_0(V)$ with $\varphi \geq 0$, we obtain

$$\begin{aligned}
&\int_U \langle A_n \nabla f_\varepsilon(u_n) + b_n f_\varepsilon(u_n), \nabla \varphi \rangle dx \;=\; \lim_{k\to\infty} \int_U \langle A_n \nabla f^k_\varepsilon(u_n) + b_n f^k_\varepsilon(u_n), \nabla \varphi \rangle dx \\
&= \lim_{k\to\infty} \Big(-\int_U \big((f^k_\varepsilon)'(u_n) \mathcal{L}_n u_n + (f^k_\varepsilon)''(u_n) \langle A_n \nabla u_n, \nabla u_n \rangle \big) \cdot \varphi \, dx \Big) \\
&\quad - \lim_{k\to\infty} \int_U \mathrm{div}\, b_n (f^k_\varepsilon(u_n) - u_n (f^k_\varepsilon)'(u_n)) \cdot \varphi \, dx \\
&\leq -\int_U \mathrm{div}\, b_n \big(f_\varepsilon(u_n) - u_n f'_\varepsilon(u_n) \big) \varphi dx.
\end{aligned}$$

Since the latter term converges to zero as $\varepsilon \to 0$, for any $n \geq N$, we obtain

$$\int_U \langle A_n \nabla u^+_n + b_n u^+_n, \nabla \varphi \rangle dx \leq 0, \quad \forall \varphi \in C^\infty_0(V), \ \varphi \geq 0.$$

Consequently, using (5.2), (5.5), we get

$$\int_U \langle A\nabla u^+ + bu^+, \nabla\varphi\rangle dx \leq 0, \quad \forall \varphi \in C_0^\infty(V),\ \varphi \geq 0.$$

Since V is an arbitrary open set with $\overline{V} \subset U$, the assertion follows.

□

5.2 Existence of a pre-invariant measure and construction of a generalized Dirichlet form

Throughout, the real number q shall be given by

$$q := \frac{pd}{p+d}.$$

We consider the following second order partial differential operator

$$Lf \;=\; \frac{1}{2}\sum_{i,j=1}^{d} \widetilde{a}_{ij}\partial_i\partial_j f + \sum_{i=1}^{d} g_i\partial_i f, \quad f \in C_0^\infty(\mathbb{R}^d). \tag{5.6}$$

where $\widetilde{a}_{ij}$ and g_i are throughout as in the following assumption.

(a) $A = (a_{ij})_{1\leq i,j\leq d}$ is a matrix of functions satisfying $a_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ for all $1 \leq i,j \leq d$. Let $\widetilde{A} = (\widetilde{a}_{ij})_{1\leq i,j\leq d} := \frac{A+A^T}{2}$ and $\breve{A} := \frac{A-A^T}{2}$. For every open ball $B \subset \mathbb{R}^d$, there exist positive real numbers λ_B, Λ_B with

$$\lambda_B\|\xi\|^2 \leq \langle \widetilde{A}(x)\xi, \xi\rangle \leq \Lambda_B\|\xi\|^2 \text{ for all } \xi \in \mathbb{R}^d, x \in B. \tag{5.7}$$

$\mathbf{H} = (h_1, \dots, h_d) \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ and let

$$\mathbf{G} = (g_1, \dots, g_d) = \frac{1}{2}\nabla A^T + \mathbf{H},$$

Assumption (a) implies that

$$\mathbf{F} := \frac{1}{2}\nabla A^T - \mathbf{G} = -\mathbf{H} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d).$$

For later purpose we shall also consider the following assumption

(b) $\mathbf{G} = (g_1, \ldots, g_d) \in L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d)$.

Remark 5.2.1. *Under assumption (a), L as in* (5.6) *can be rewritten as non-symmetric divergence form operator with coefficients in $H^{1,2}_{loc}(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ and $L^p_{loc}(\mathbb{R}^d)$-perturbation, i.e. L can be written as in* (1.5)*. Assumption (b) then just means that $\frac{1}{2}(\nabla A^T)_i \in L^s_{loc}(\mathbb{R}^d)$, $1 \le i \le d$, for some $s > \frac{d}{2}$.*
But we can also consider non-divergence form operators. If for instance $\frac{1}{2}(\nabla A^T)_i \in L^p_{loc}(\mathbb{R}^d)$, $1 \le i \le d$, for some $p > d$, then set

$$\mathbf{H} = \widetilde{\mathbf{H}} - \frac{1}{2}\nabla A^T$$

for arbitrarily chosen $\widetilde{\mathbf{H}} = (\widetilde{h}_1, ..., \widetilde{h}_d) \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Then assumptions (a) and (b) both hold and (5.6) *(as well as* (1.5)*) can be rewritten as*

$$Lf \;=\; \frac{1}{2}\sum_{i,j=1}^{d} \widetilde{a}_{ij}\partial_{ij} f + \sum_{i=1}^{d} \widetilde{h}_i \partial_i f, \quad f \in C_0^\infty(\mathbb{R}^d).$$

This covers as a special case the assumptions of [13, Theorem 1] (see also [12, Theorem 2.4.1]).

Theorem 5.2.2 (Existence of a pre-invariant measure)**.** *Suppose assumption (a) holds. Then there exists $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{1-d/p}_{loc}(\mathbb{R}^d)$ with $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} L\varphi \,\rho dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d). \tag{5.8}$$

Proof Using integration by parts, (5.8) is equivalent to

$$\int_{\mathbb{R}^d} \langle \frac{1}{2} A^T \nabla \rho + \rho \, \mathbf{F}, \nabla \varphi \rangle dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d). \tag{5.9}$$

By [12, Proposition 2.1.4, Corollary 2.1.6, Theorem 2.1.8] and Remark 5.1.2, for every $n \in \mathbb{N}$, there exists a unique $v_n \in H_0^{1,p}(B_n) \cap C^{1-d/p}(\overline{B}_n)$ such that

$$\int_{B_n} \langle \frac{1}{2} A^T \nabla v_n + v_n \, \mathbf{F}, \nabla \varphi \rangle dx = \int_{B_n} \langle -\mathbf{F}, \nabla \varphi \rangle dx \quad \text{for all } \varphi \in C_0^\infty(B_n).$$

Let $u_n := v_n + 1$. Then $u_n(x) = 1$ for all $x \in \partial B_n$ and

$$\int_{B_n} \langle \frac{1}{2} A^T \nabla u_n + u_n \, \mathbf{F}, \nabla \varphi \rangle dx = 0, \quad \text{for all } \varphi \in C_0^\infty(B_n).$$

Since $u_n^- \leq v_n^-$, we see $u_n^- \in H_0^{1,p}(B_n) \cap C^{1-d/p}(\overline{B}_n)$. Thus by Lemma 5.1.4, we get

$$\int_{B_n} \langle \frac{1}{2} A^T \nabla u_n^- + u_n^- \, \mathbf{F}, \nabla \varphi \rangle dx \leq 0, \quad \text{for all } \varphi \in C_0^\infty(B_n),\ \varphi \geq 0.$$

By [12, Theorem 2.1.8] and Remark 5.1.2, $u_n^- \leq 0$, so that $u_n \geq 0$. Suppose there exists $x_0 \in B_n$ with $u_n(x_0) = 0$. Then, applying [76, Corollary 5.2 (Harnack inequality)] to u_n on B_n, we get $u_n(x) = 0$ for all $x \in B_n$, which contradicts $u_n \in C^{1-d/p}(\overline{B}_n)$, since $u_n = 1$ on ∂B_n. Hence $u_n(x) > 0$ for all $x \in B_n$. Now let $\rho_n(x) := u_n(0)^{-1} u_n(x)$, $x \in B_n, n \in \mathbb{N}$. Then $\rho_n(0) = 1$ and

$$\int_{B_n} \langle \frac{1}{2} A^T \nabla \rho_n + \rho_n \mathbf{F},\ \nabla \varphi \rangle dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_n).$$

Fix $r > 0$. Then, by [76, Corollary 5.2]

$$\sup_{x \in B_{2r}} \rho_n(x) \leq C_1 \inf_{x \in B_{2r}} \rho_n(x) \quad \text{for all } n > 2r,$$

where C_1 is independent of ρ_n, $n > 2r$. Thus

$$\sup_{x \in B_{2r}} \rho_n(x) \le C_1 \quad \text{for all } n > 2r.$$

By [12, Theorem 1.7.4] and Remark 5.1.2

$$\|\rho_n\|_{H^{1,p}(B_r)} \le C_2 \|\rho_n\|_{L^1(B_{2r})} \le C_1 C_2 |B_{2r}|, \quad \text{for all } n > 2r,$$

where C_2 is independent of $(\rho_n)_{n>2r}$. By weak compactness of balls in $H_0^{1,p}(B_r)$ and the Arzela-Ascoli Theorem, there exists $(\rho_{n,r})_{n\ge1} \subset (\rho_n)_{n>2r}$ and $\rho_{(r)} \in H^{1,p}(B_r) \cap C^{1-d/p}(\overline{B}_r)$ such that

$$\rho_{n,r} \to \rho_{(r)} \text{ weakly in } H^{1,p}(B_r), \qquad \rho_{n,r} \to \rho_{(r)} \text{ uniformly on } \overline{B}_r.$$

Considering $(\rho_{n,k})_{n\ge1} \supset (\rho_{n,k+1})_{n\ge1}$, $k \in \mathbb{N}$, we get $\rho_{(k)} = \rho_{(k+1)}$ on B_k, hence we can well-define ρ as

$$\rho := \rho_{(k)} \text{ on } B_k, k \in \mathbb{N}.$$

Then $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{1-d/p}_{loc}(\mathbb{R}^d)$ with $\rho(x) \ge 0$, $x \in \mathbb{R}^d$, $\rho(0) = 1$ and for any $n \in \mathbb{N}$

$$\int_{B_n} \langle \frac{1}{2} A^T \nabla \rho + \rho \, \mathbf{F}, \nabla \varphi \rangle dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_n).$$

By applying the Harnack inequality to ρ on B_r with $n > r$

$$1 = \rho(0) \le \sup_{x \in B_r} \rho(x) \le C_3 \inf_{x \in B_r} \rho(x),$$

hence $\rho(x) > 0$ for all $x \in B_r$. Therefore $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ and (5.8) holds.

□

From now on unless otherwise stated, we fix ρ as in Theorem 5.2.2. Set

$$m := \rho \, dx.$$

Using integration by parts the following can be easily shown.

Lemma 5.2.3. *If $Q := (q_{ij})_{1\le i,j\le d}$ is a $d\times d$ matrix of functions with $-q_{ji} = q_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d)\cap L^{\infty}_{loc}(\mathbb{R}^d)$, $1\le i,j\le d$. Then $\beta^{\rho,Q}\in L^2_{loc}(\mathbb{R}^d,\mathbb{R}^d;m)$ and $\beta^{\rho,Q}$ is weakly divergence free with respect to m, i.e.*

$$\int_{\mathbb{R}^d}\langle \beta^{\rho,Q},\nabla f\rangle dm = 0, \quad \text{for all } f\in C_0^{\infty}(\mathbb{R}^d).$$

Define

$$\mathbf{B} := \mathbf{G} - \beta^{\rho,A^T}.$$

Note that $\mathbf{B} = \left(\mathbf{G} - \frac{1}{2}\nabla A^T\right) - \frac{A^T\nabla\rho}{2\rho} \in L^p_{loc}(\mathbb{R}^d,\mathbb{R}^d)$. Moreover, using (5.8) and Lemma 5.2.3, we can see that $\beta^{\rho,\check{A}^T}+\mathbf{B}\in L^2_{loc}(\mathbb{R}^d,\mathbb{R}^d;m)$ is weakly divergence free with respect to m, i.e.

$$\int_{\mathbb{R}^d}\langle \beta^{\rho,\check{A}^T}+\mathbf{B},\nabla f\rangle dm = 0 \quad \text{for all } f\in C_0^{\infty}(\mathbb{R}^d).$$

For $f,g\in C_0^{\infty}(\mathbb{R}^d)$, define

$$\mathcal{E}^0(f,g) := \int_{\mathbb{R}^d}\langle \widetilde{A}\nabla f,\nabla g\rangle\, dm.$$

Then $(\mathcal{E}^0, C_0^{\infty}(\mathbb{R}^d))$ is closable in $L^2(\mathbb{R}^d,m)$. We denote its closure by $(\mathcal{E}^0, D(\mathcal{E}^0))$ and its associated generator by $(L^0, D(L^0))$. Since $C_0^{\infty}(\mathbb{R}^d)\subset D(L^0)_{0,b}$ we have that $D(L^0)_{0,b}$ is a dense subset of $L^1(\mathbb{R}^d,m)$, and furthermore

$$L^0 f = \frac{1}{2}\mathrm{trace}(\widetilde{A}\nabla^2 f) + \langle \beta^{\rho,\widetilde{A}},\nabla f\rangle \in L^2(\mathbb{R}^d,m) \quad \text{for all } f\in C_0^{\infty}(\mathbb{R}^d).$$

Define

$$Lf = L^0 f + \langle \beta^{\rho,\check{A}^T}+\mathbf{B},\nabla f\rangle, \quad f\in D(L^0)_{0,b}.$$

Then $(L, D(L^0)_{0,b})$ is an extension of $(L, C_0^{\infty}(\mathbb{R}^d))$ as defined in 5.6. By [69, Theorem 1.5], there exists a $L^1(\mathbb{R}^d,m)$ closed extension $(\overline{L}, D(\overline{L}))$ of $(L, D(L^0)_{0,b})$ in $L^1(\mathbb{R}^d,m)$ which generates sub-Markovian C_0 semigroup of contractions $(T_t)_{t>0}$ on $L^1(\mathbb{R}^d,m)$. Restricting $(T_t)_{t>0}$ to $L^1(\mathbb{R}^d,m)_b$, it is well-known that $(T_t)_{t>0}$ can be extended to a sub-Markovian C_0-semigroup of contractions on each $L^r(\mathbb{R}^d,m)$, $r\in[1,\infty)$. Denote by

$(L_r, D(L_r))$ the corresponding closed generator with graph norm

$$\|f\|_{D(L_r)} := \|f\|_{L^r(\mathbb{R}^d,m)} + \|L_r f\|_{L^r(\mathbb{R}^d,m)},$$

and by $(G_\alpha)_{\alpha>0}$ the corresponding resolvent. For $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ we do not explicitly denote in the notation on which $L^r(\mathbb{R}^d, m)$-space they act. We assume that this is clear from the context. Moreover, $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ can be uniquely defined on $L^\infty(\mathbb{R}^d, m)$, but are no longer strongly continuous there.
For $f \in C_0^\infty(\mathbb{R}^d)$

$$\widehat{L}f \;\; := \;\; L^0 f - \langle \beta^{\rho,\check{A}^T} + \mathbf{B}, \nabla f\rangle = \frac{1}{2}\text{trace}(\widetilde{A}\nabla^2 f) + \langle \widehat{\mathbf{G}}, \nabla f\rangle,$$

with

$$\widehat{\mathbf{G}} := (\widehat{g}_1, \cdots, \widehat{g}_d) = 2\beta^{\rho,\widetilde{A}} - \mathbf{G} = \beta^{\rho,A} - \mathbf{B} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, m).$$

We see that L and $\widehat{L}$ have the same structural properties, i.e. they are given as the sum of a symmetric second order elliptic differential operator and a divergence free first order perturbation with same integrability condition with respect to the measure m. Therefore all what will be derived below for L will hold analogously for $\widehat{L}$. Denote the operators corresponding to $\widehat{L}$ (again defined through [69, Theorem 1.5]) by $(\widehat{L}_r, D(\widehat{L}_r))$ for the co-generator on $L^r(\mathbb{R}^d, m)$, $r \in [1, \infty)$, $(\widehat{T}_t)_{t>0}$ for the co-semigroup, $(\widehat{G}_\alpha)_{\alpha>0}$ for the co-resolvent. By [69, Section 3], we obtain a corresponding bilinear form with domain $D(L_2) \times L^2(\mathbb{R}^d, m) \cup L^2(\mathbb{R}^d, m) \times D(\widehat{L}_2)$ by

$$\mathcal{E}(f,g) := \begin{cases} -\int_{\mathbb{R}^d} L_2 f \cdot g \, dm & \text{for } f \in D(L_2),\ g \in L^2(\mathbb{R}^d, m), \\ -\int_{\mathbb{R}^d} f \cdot \widehat{L}_2 g \, dm & \text{for } f \in L^2(\mathbb{R}^d, m),\ g \in D(\widehat{L}_2). \end{cases}$$

$\mathcal{E}$ is called the *generalized Dirichlet form associated with* $(L_2, D(L_2))$. Using integration by parts, it is easy to see that for $f, g \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} \mathcal{E}(f,g) \;\; &= \;\; \frac{1}{2}\int_{\mathbb{R}^d} \langle \widetilde{A}\nabla f, \nabla g\rangle \, dm - \int_{\mathbb{R}^d} \langle \beta^{\rho,\check{A}^T} + \mathbf{B}, \nabla f\rangle g \, dm \\ &= \;\; \frac{1}{2}\int_{\mathbb{R}^d} \langle A\nabla f, \nabla g\rangle \, dm - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla f\rangle g \, dm, \end{aligned}$$

and

$$L_2 f = \quad \frac{1}{2}\sum_{i,j=1}^{d} \widetilde{a}_{ij}\partial_i\partial_j f + \sum_{i=1}^{d} g_i \partial_i f = \frac{1}{2}\mathrm{trace}(\widetilde{A}\nabla^2 f) + \langle \beta^{\rho,A^T}, \nabla f\rangle + \langle \mathbf{B}, \nabla f\rangle,$$

$$\widehat{L}_2 f = \quad \frac{1}{2}\sum_{i,j=1}^{d} \widetilde{a}_{ij}\partial_i\partial_j f + \sum_{i=1}^{d} \widehat{g}_i \partial_i f = \frac{1}{2}\mathrm{trace}(\widetilde{A}\nabla^2 f) + \langle \beta^{\rho,A}, \nabla f\rangle - \langle \mathbf{B}, \nabla f\rangle.$$

5.3 Regularity results for resolvent and semigroup

Theorem 5.3.1. *Assume (a). Then*

$$\rho G_\alpha g \in H^{1,p}_{loc}(\mathbb{R}^d), \quad \forall g \in \cup_{r\in[q,\infty]} L^r(\mathbb{R}^d, m),$$

and for any open balls B, B' with $\overline{B} \subset B'$,

$$\|\rho\, G_\alpha g\|_{H^{1,p}(B)} \le c_0 \left(\|g\|_{L^q(B',m)} + \|G_\alpha g\|_{L^1(B',m)} \right),$$

where c_0 is independent of g.

Proof Let $g \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$. Then for all $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (\alpha - \widehat{L}_2)\varphi \cdot \big(G_\alpha g\big)\, dm = \int_{\mathbb{R}^d} \widehat{G}_\alpha(\alpha - \widehat{L}_2)\varphi \cdot g\, dm = \int_{\mathbb{R}^d} \varphi g\, dm. \tag{5.10}$$

Note that $G_\alpha g \in D(\overline{L})_b \subset D(\mathcal{E}^0)$ by [69, Theorem 1.5]. Since ρ is locally bounded below and $\widetilde{A}$ satisfies (5.7), we have $D(\mathcal{E}^0) \subset H^{1,2}_{loc}(\mathbb{R}^d)$ and it follows $\rho G_\alpha g \in H^{1,2}_{loc}(\mathbb{R}^d)$. Define

$$\widehat{\mathbf{F}} := \frac{1}{2}\nabla A - \widehat{\mathbf{G}} = -\frac{A\nabla\rho}{2\rho} + \mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d). \tag{5.11}$$

Given any open ball B'' and $\varphi \in C_0^\infty(B'')$, we have using integration by parts in the left hand side of (5.10)

$$\int_{B''}\left[\langle\frac{1}{2}A\nabla(\rho G_\alpha g)+(\rho G_\alpha g)\widehat{\mathbf{F}},\nabla\varphi\rangle+\alpha(\rho G_\alpha g)\varphi\right]dx=\int_{B''}(\rho g)\varphi dx.$$

By [12, Theorem 1.8.3] and Remark 5.1.2, for any open ball B' with $\overline{B'}\subset B''$, we have $\rho G_\alpha g\in H^{1,p}(B')$. Thus by [12, Theorem 1.7.4] and Remark 5.1.2, we obtain for any open ball B with $\overline{B}\subset B'$, $r\in[q,\infty)$

$$\begin{aligned}\|\rho G_\alpha g\|_{H^{1,p}(B)} &\le c_1\left(\|\rho g\|_{L^q(B',dx)}+\|\rho G_\alpha g\|_{L^1(B',dx)}\right)\\ &\le \underbrace{c_1(\sup_{B'}\rho^{\frac{q-1}{q}}\vee 1)}_{=:c_0}\left(\|g\|_{L^q(B',m)}+\|G_\alpha g\|_{L^1(B',m)}\right)\end{aligned}\tag{5.12}$$

By denseness of $C_0^\infty(\mathbb{R}^d)$ in $L^r(\mathbb{R}^d,m)$, (5.12) extends to $g\in L^r(\mathbb{R}^d,m)$, $r\in[q,\infty)$. For $g\in L^\infty(\mathbb{R}^d,m)$, let $g_n:=g1_{B_n}\in L^q(\mathbb{R}^d,m), n\ge 1$. Then $\|g-g_n\|_{L^q(B',m)}+\|G_\alpha(g-g_n)\|_{L^1(B',m)}\to 0$ as $n\to\infty$. Hence (5.12) also extends to $g\in L^\infty(\mathbb{R}^d,m)$.

□

Remark 5.3.2. *Proposition 3.1.6 of Part I holds in our more general situation with exactly the same proof.*

Theorem 5.3.3. *Assume (a). For each $s\in[1,\infty]$, consider the $L^s(\mathbb{R}^d,m)$-semigroup $(T_t)_{t>0}$. Then for any $f\in L^s(\mathbb{R}^d,m)$ and $t>0$, T_tf has a locally Hölder continuous m-version P_tf on $\mathbb{R}^d$. More precisely, $P_\cdot f(\cdot)$ is locally parabolic Hölder continuous on $\mathbb{R}^d\times(0,\infty)$ and for any bounded open sets U, V in $\mathbb{R}^d$ with $\overline{U}\subset V$ and $0<\tau_3<\tau_1<\tau_2<\tau_4$, i.e. $[\tau_1,\tau_2]\subset(\tau_3,\tau_4)$, we have for some $\gamma\in(0,1)$ the following estimate for all $f\in\cup_{s\in[1,\infty]}L^s(\mathbb{R}^d,m)$ with $f\ge 0$,*

$$\|P_\cdot f(\cdot)\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])}\le C_6\|P_\cdot f(\cdot)\|_{L^1(V\times(\tau_3,\tau_4),m\otimes dt)},$$

where C_6,γ are constants that depend on $\overline{U}\times[\tau_1,\tau_2], V\times(\tau_3,\tau_4)$, but are independent of f.

Proof The proof is similar to the corresponding proof in Theorem 3.1.8, but there are some subtle differences. First assume $f \in D(\overline{L})_b \cap D(L_2) \cap D(L_q)$ with $f \geq 0$. Set $u(x,t) := \rho(x) P_t f(x)$. Then $P_t f \in D(L_q)$ and $\rho \in C(\mathbb{R}^d)$ implies $u \in C\big(\mathbb{R}^d \times [0,\infty)\big)$ by Proposition 5.3.2(iii). Let $T > 0$ be arbitrary. Then for any $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$

$$0 = -\int_0^T \int_{\mathbb{R}^d} \Big(\partial_t \varphi + \widehat{L}_2 \varphi\Big)\, u \, dxdt. \tag{5.13}$$

Since $u \in H^{1,2}(O \times (0,T))$ for any bounded and open set $O \subset \mathbb{R}^d$, using integration by parts in the right hand term of (5.13), we get

$$0 = \int_0^T \int_{\mathbb{R}^d} \left(\frac{1}{2} \langle A \nabla u, \nabla \varphi \rangle + u \langle \widehat{\mathbf{F}}, \nabla \varphi \rangle - u \partial_t \varphi \right) dxdt, \tag{5.14}$$

where $\widehat{\mathbf{F}}$ is as in (5.11). Then as in Theorem 3.1.8.

$$\begin{aligned} \|P_\cdot f(\cdot)\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} &\leq \|\rho^{-1}\|_{C^{\gamma}(\overline{U}\times[\tau_1,\tau_2])} \|\rho(\cdot) P_\cdot f(\cdot)\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} \\ &\leq \underbrace{\|\rho^{-1}\|_{C^{\gamma}(\overline{U}\times[\tau_1,\tau_2])} C_2 C_5}_{:=C_6} \|P_\cdot f(\cdot)\|_{L^1(V\times(\tau_3,\tau_4), m\otimes dt)} \\ &\leq C_6 (\tau_4 - \tau_3) \|\rho\|_{L^1(V)}^{\frac{s-1}{s}} \|f\|_{L^s(\mathbb{R}^d,m)}, \quad s \in [1,\infty], \end{aligned} \tag{5.15}$$

where C_2, C_5 is as in Theorem 3.1.8 in Part I.
For $f \in L^1(\mathbb{R}^d,m) \cap L^\infty(\mathbb{R}^d,m)$ with $f \geq 0$ let $f_n := nG_n f$. Then $f_n \in D(\overline{L})_b \cap D(L_2) \cap D(L_q)$ with $f_n \geq 0$ and $f_n \to f$ in $L^s(\mathbb{R}^d,m)$ for any $s \in [1,\infty)$. Thus (5.15) including all intermediate inequalities extend to $f \in L^1(\mathbb{R}^d,m) \cap L^\infty(\mathbb{R}^d,m)$ with $f \geq 0$. If $f \in L^s(\mathbb{R}^d,m)$, $f \geq 0$ and $s \in [1,\infty)$, let $f_n := 1_{B_n} \cdot (f \wedge n)$. Then $f_n \in L^1(\mathbb{R}^d,m) \cap L^\infty(\mathbb{R}^d,m)$ with $f_n \geq 0$ and $f_n \to f$ in $L^s(\mathbb{R}^d,m)$. Thus (5.15) including all intermediate inequalities extend to $f \in L^s(\mathbb{R}^d,m)$ with $f \geq 0$. For $f \in L^\infty(\mathbb{R}^d,m)$, the result follows exactly as in Theorem 3.1.8.

□

Remark 5.3.4. *Besides the possible non-symmetry of A (that also occurs in $\widehat{\mathbf{F}}$), the difference between the proof of Theorem 3.1.8 and Theorem 5.3.3 is the approximation method. The proof of Theorem 3.1.8 uses the denseness of $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d,m)$. The proof of Theorem 5.3.3 uses the denseness of $\cup_{\alpha>0} G_\alpha\big(L^1(\mathbb{R}^d,m) \cap L^\infty(\mathbb{R}^d,m)\big)$ in*

$L^1(\mathbb{R}^d, m)$. Using the latter, we can get the corresponding result to Lemma 4.2.2 in the following Lemma 5.3.5.

Lemma 5.3.5. *(i) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $P_{t_0} 1_A(x_0) = 0$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$. Then $m(A) = 0$.*

(ii) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $P_{t_0} 1_A(x_0) = 1$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$. Then $P_t 1_A(x) = 1$ for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$.

Proof (i) Suppose $m(A) > 0$. Choose an open ball $B_r(x_0) \subset \mathbb{R}^d$ such that

$$0 < m\left(A \cap B_r(x_0)\right) < \infty.$$

Let $u := \rho P_\cdot 1_{A\cap B_r(x_0)}$. Then $0 = u(x_0, t_0) \le \rho(x_0) P_{t_0} 1_A(x_0) = 0$. Set $f_n := nG_n 1_{A\cap B_r}(x_0)$. Then $f_n \in D(\overline{L})_b \cap D(L_2) \cap D(L_q)$ with $f_n \ge 0$ such that $f_n \to 1_{A\cap B_r(x_0)}$ in $L^1(\mathbb{R}^d, m)$. Let $u_n := \rho P_\cdot f_n$. Fix $T > t_0$ and $U \supset \overline{B}_r(x_0)$. Since $u_n \in H^{1,2}(U \times (0,T))$ satisfies (5.13) (see proof of Theorem 5.3.3), (5.14) holds with u replaced by u_n for all $\varphi \in C_0^\infty(U \times (0,T))$. The rest of the proof is then exactly as in Lemma 4.2.2 (i).
(ii) Let $y \in \mathbb{R}^d$ and $0 < s < t_0$ be arbitrary but fixed and let $r := 2\|x_0 - y\|$ and let B be any open ball. Take $g_n := nG_n 1_{B\cap A}$. Then $g_n \in D(\overline{L})_b \cap D(L_2) \cap D(L_q)$ with $0 \le g_n \le 1$ satisfying $g_n \to 1_{A\cap B}$ in $L^1(\mathbb{R}^d, m)$. The rest of the proof is now exactly as in Lemma 4.2.2 (ii).

□

Remark 5.3.6. *Using the Lemma 5.3.5, Corollary 4.2.4 holds in our more general situation with exactly the same proof.*

Remark 5.3.7. *(i) (cf. Remark 4.2.1 in Part I) Consider A, ρ, $\widetilde{\mathbf{B}}$ which are explicitly given by following assumptions. Let $A = (a_{ij})_{1\le i,j\le d}$ be a matrix of functions as in assumption (a) and $\check{A} = (\check{a}_{ij})_{1\le i,j\le d} := \frac{A - A^T}{2}$. Suppose that for some $p > d$, we are given $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{0,1-d/p}(\mathbb{R}^d)$, $\rho(x) > 0$ for all $x \in \mathbb{R}^d$, such that for some $\widetilde{\mathbf{B}} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ it holds*

$$\int_{\mathbb{R}^d} \langle \widetilde{\mathbf{B}}, \nabla f \rangle \rho dx = 0 \quad \text{for all } f \in C_0^\infty(\mathbb{R}^d). \tag{5.16}$$

Let

$$\widetilde{L}f = L^0 f + \langle \beta^{\rho,\check{A}^T} + \widetilde{\mathbf{B}}, \nabla f\rangle, \quad f \in D(L^0)_{0,b}.$$

Then (5.8) *holds for* L *replaced with* $\widetilde{L}$. *Moreover, everything that was developed for* $(L, D(L^0)_{0,b})$ *right after Theorem 5.2.2 until and including Corollary 5.3.6 (and even beyond until the end of this article if additionally* $\beta^{\rho,\check{A}^T} + \widetilde{\mathbf{B}} \in L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, *i.e. assumption (b) holds, cf. Remark 6.1.2) holds analoguously for* $(\widetilde{L}, D(L^0)_{0,b})$. *Now suppose again that assumption (a) holds. Then by Theorem 5.2.2, there exists* ρ *as right above such that* $\widetilde{\mathbf{B}} := \mathbf{B} = \frac{1}{2}\nabla A^T + \mathbf{H} - \beta^{\rho,A^T} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ *and such that* $\widetilde{\mathbf{B}}$ *satisfies* (5.16). *Thus all that has been done up to now is in fact a special realization of the just explained explicit case.*

(ii) (cf. Remark 3.1.3) It is possible to realize the results of this article with $\mathbb{R}^d$ *replaced by an arbitrary open set* $U \subset \mathbb{R}^d$. *Moreover as it is well-known the* L^p_{loc}*-condition can be relaxed by an* $L^{p_n}_{loc}$*-condition on an exhaustion* $(V_n)_{n\in\mathbb{N}}$ *of* $\mathbb{R}^d$ *(or* U*), where* $p_n > d$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} p_n = d$.

Chapter 6

Probabilistic results

6.1 The underlying SDE

Additionally to assumption (a) we assume throughout this section that assumption (b) holds. Then $C_0^2(\mathbb{R}^d) \subset D(L_1) \cap D(L_q)$ and assumption $(\mathbf{H2})'$ of Part I holds. Here, assumption (b) was needed to get the continuity property of the resolvent in $(\mathbf{H2})'$(ii) of Part I. Thus, through the exactly same method as in Theorem 3.2.1, we arrive at the following theorem:

Theorem 6.1.1. *There exists a Hunt process*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$$

with state space $\mathbb{R}^d$ *and life time*

$$\zeta = \inf\{t \geq 0 \,:\, X_t = \Delta\} = \inf\{t \geq 0 \,:\, X_t \notin \mathbb{R}^d\},$$

having the transition function $(P_t)_{t\geq 0}$ *as transition semigroup, such that* $\mathbb{M}$ *has continuous sample paths in the one point compactification* $\mathbb{R}^d_\Delta$ *of* $\mathbb{R}^d$ *with the cemetery* Δ *as point at infinity.*

Remark 6.1.2. *Actually, under assumptions (a) and (b) most of the results from Part I generalize to the more general coefficients considered here, i.e. the analogues of Lemmas*

3.14, 3.15, 3.18, Propositions 3.16, 3.17 Theorem 3.19, Remark 3.20 and the analogues of the results in Chapter 4 of Part I hold. These results include, various non-explosion criteria, moment inequalities, a general Krylov type estimate, recurrence criteria and criteria for ergodicity including uniqueness of the invariant measure ρdx.

According to Remark 6.1.2, we obtain.

Theorem 6.1.3. *Consider the Hunt process $\mathbb{M}$ from Theorem 6.1.1 with coordinates $X_t = (X_t^1, ..., X_t^d)$. Let $(\sigma_{ij})_{1\le i\le d, 1\le j\le l}$, $l \in \mathbb{N}$ arbitrary but fixed, be any matrix consisting of continuous functions $\sigma_{ij} \in C(\mathbb{R}^d)$ for all $1 \le i \le d$, $1 \le j \le l$, such that $\widetilde{A} = \sigma\sigma^T$, i.e.*

$$\widetilde{a}_{ij}(x) = \sum_{k=1}^{l} \sigma_{ik}(x)\sigma_{jk}(x), \quad \forall x \in \mathbb{R}^d, \ 1 \le i, j \le d.$$

Then on a standard extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, that we denote for notational convenience again by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, there exists a standard l-dimensional Brownian motion $W = (W^1, \dots, W^l)$ starting from zero such that $\mathbb{P}_x$-a.s. for any $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $i = 1, \dots, d$

$$X_t^i = x_i + \sum_{j=1}^{l} \int_0^t \sigma_{ij}(X_s)\, dW_s^j + \int_0^t g_i(X_s)\, ds, \quad 0 \le t < \zeta, \tag{6.1}$$

in short

$$X_t = x_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t \mathbf{G}(X_s) ds, \quad 0 \le t < \zeta.$$

The non-explosion result in the following theorem is new and allows for linear growth together with $L^q(\mathbb{R}^d, m)$-singularities of the drift. It completes various other non-explosion results form Part I and existing literature.

Theorem 6.1.4. *Let $\sigma = (\sigma_{ij})_{1\le i,j\le d}$ be as in Theorem 6.1.3, i.e. $l = d$ (such σ always exists, cf. Lemma 3.2.7) and assume that for some $h_1 \in L^p(\mathbb{R}^d, m)$, $h_2 \in L^q(\mathbb{R}^d, m)$ and $C > 0$ it holds for a.e. $x \in \mathbb{R}^d$*

$$\max_{1\le i,j\le d} |\sigma_{ij}(x)| \le |h_1(x)| + C(\sqrt{\|x\|} + 1), \quad \max_{1\le i\le d} |g_i(x)| \le |h_2(x)| + C(\|x\| + 1).$$

Then $\mathbb{M}$ *is non-explosive and for any* $T > 0$*, and any open ball* B*, there exist constants* $C_{5,T}$*,* C_6 *such that*

$$\sup_{x\in\overline{B}} \mathbb{E}_x\left[\sup_{s\le t}\|X_s\|\right] \le C_{5,T}\cdot e^{C_6\cdot t},\quad \forall t\le T.$$

Proof Let $x \in \overline{B}$ and $n \in \mathbb{N}$ such that $x \in B_n$ (B_n is the open ball about zero with radius n in $\mathbb{R}^d$). Let $0 \le t \le T$. Then with $\sigma_n := \inf\{t > 0 : X_t \in \mathbb{R}^d \setminus B_n\}$, $n \ge 1$, we obtain $\mathbb{P}_x$-a.s. for any $1 \le i \le d$

$$\sup_{0\le s\le t\wedge\sigma_n} |X_s^i| \le |x_i| + \sum_{j=1}^d \sup_{0\le s\le t\wedge\sigma_n}\left|\int_0^s \sigma_{ij}(X_u)\,dW_u^j\right| + \sup_{0\le s\le t\wedge\sigma_n}\int_0^s |g_i(X_u)|\,du.$$

By the Burkholder-Davis-Gundy inequality [61, Chapter IV. (4.2) Corollary] and (1.7), there exists a constant $c_{x,B,q}$, depending on q and the open ball B, such that

$$\begin{aligned}
&\sum_{j=1}^d \mathbb{E}_x\left[\sup_{0\le s\le t\wedge\sigma_n}\left|\int_0^s \sigma_{ij}(X_u)\,dW_u^j\right|\right]\\
\le\ &\sum_{j=1}^d \sqrt{32}\mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} \sigma_{ij}^2(X_u)du\right]^{1/2}\\
\le\ & d\sqrt{96}\mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} |h_1^2(X_u)|du\right]^{1/2} + dC\sqrt{96T} + C\sqrt{96}d\cdot\mathbb{E}_x\left[\int_0^{t\wedge\sigma_n}\|X_u\|du\right]^{1/2}\\
\le\ & \underbrace{d\sqrt{96}e^T c_{x,B,q}\|h_1\|_{L^{2q}(\mathbb{R}^d,m)} + dC(\sqrt{96T}+\sqrt{32})}_{=:C_{3,T}} + C\sqrt{96}d\int_0^t \mathbb{E}_x\left[\sup_{0\le s\le u\wedge\sigma_n}\|X_s\|\right]du,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_x\left[\sup_{0\le s\le t\wedge\sigma_n}\int_0^s |g_i(X_u)|du\right]\\
\le\ & \mathbb{E}_x\left[\int_0^{t\wedge\sigma_n} |h_2(X_u)|du\right] + C\mathbb{E}_x\Big[\int_0^{t\wedge\sigma_n}\big(\|X_u\|+1\big)\,du\Big]\\
\le\ & \mathbb{E}_x\left[\int_0^{T} |h_2(X_u)|du\right] + CT + C\mathbb{E}_x\left[\int_0^t \sup_{0\le s\le u\wedge\sigma_n}\|X_s\|du\right]\\
\le\ & \underbrace{e^T c_{x,B,q}\|h_2\|_{L^q(\mathbb{R}^d,m)} + CT}_{=:C_{4,T}} + C\int_0^t \mathbb{E}_x\left[\sup_{0\le s\le u\wedge\sigma_n}\|X_s\|\right]du.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}_x\left[\sup_{0\le s\le t\wedge\sigma_n}\|X_s\|\right] \le \sum_{i=1}^d \mathbb{E}_x\left[\sup_{0\le s\le t\wedge\sigma_n}|X_s^i|\right]\\
\le\ & \underbrace{(\sqrt{d}\|x\| + dC_{3,T} + dC_{4,T})}_{=:C_{5,T}} + \underbrace{dC(1+\sqrt{96}d)}_{=:C_6}\int_0^t \mathbb{E}_x\left[\sup_{0\le s\le u\wedge\sigma_n}\|X_s\|\right]du. \qquad (6.2)
\end{aligned}$$

Now let $p_n(t) := \mathbb{E}_x\left[\sup_{0\le s\le t\wedge\sigma_n}\|X_s\|\right]$. Then by (6.2), we obtain

$$p_n(t) \ \le\ C_{5,T} + C_6\int_0^t p_n(u)du, \qquad 0\le t\le T.$$

By Gronwall's inequality, $p_n(t)\le C_{5,T}\cdot e^{C_6\cdot t}$ for any $t\in[0,T]$. By the Markov inequality,

$$\begin{aligned}
\mathbb{P}_x(\sigma_n\le T) &= P_x\left(\sup_{s\le T}|X_s|>n\right)\\
&\le P_x\left(\sup_{s\le T\wedge\sigma_n}|X_s|\ge n\right)\\
&\le \frac{1}{n}\mathbb{E}_x\left[\sup_{s\le T\wedge\sigma_n}|X_s|\right]\\
&\le \frac{1}{n}C_{5,T}\cdot e^{C_6\cdot T}\to 0 \quad \text{ as } n\to\infty.
\end{aligned}$$

Therefore $\mathbb{P}_x(\zeta = \infty) = 1$. Finally applying Fatou's lemma to $p_n(t)$, we obtain

$$\mathbb{E}_x\left[\sup_{s\leq t}\|X_s\|\right] \leq C_{5,T}\cdot e^{C_6\cdot t}, \quad \forall t \leq T.$$

$\square$

Example 6.1.5. *Let $\eta \in C_0^\infty(B_{1/4})$ be given. Define $w : \mathbb{R}^d \to \mathbb{R}$ by*

$$w(x_1,\dots,x_d) := \eta(x_1,\dots,x_d)\cdot\int_{-2}^{x_1}\frac{1}{|y_1|^{1/d}}1_{[-1,1]}(y_1)dy_1.$$

Then $w \in H^{1,q}(\mathbb{R}^d)\cap C_0(B_{1/4})$ but $\partial_1 w \notin L^d_{loc}(\mathbb{R}^d)$. Define $v : \mathbb{R}^d \to \mathbb{R}$ by

$$v(x_1,\dots,x_d) := w(x_1,\dots,x_d) + \sum_{i=1}^{\infty}\frac{1}{2^i}w(x_1-i,\dots,x_d)$$

Then $v \in H^{1,q}(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ but $\partial_1 v \notin L^d_{loc}(\mathbb{R}^d)$. Now define $P = (p_{ij})_{1\leq i,j\leq d}$ as

$$p_{1d} := v,\;\; p_{d1} := -v, \quad p_{ij} := 0 \text{ if } (i,j)\notin\{(1,d),(d,1)\}.$$

Let $Q = (q_{ij})_{1\leq i,j\leq d}$ be a matrix of functions such that $q_{ij} = -q_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ for all $1 \leq i,j \leq d$ and assume there exists a constant $C > 0$ satisfying

$$\|\nabla Q\| \leq C(\|x\|+1), \quad \text{for a.e. on } \mathbb{R}^d.$$

Let $\widetilde{A} := id$, $\check{A} := P + Q$ and $\mathbf{H} \equiv 0$. Then $\widetilde{A}$ and $\check{A}$ satisfy assumption (a) with $\mathbf{G} := \frac{1}{2}\nabla A^T$ and $\mathbf{G}$ satisfies assumption (b). Define $\rho \equiv 1$ on $\mathbb{R}^d$. Then ρ satisfies (5.8) *and $\mathbf{B} \equiv 0$. Obviously $\sigma = id$ and $\mathbf{G}$ satisfy the conditions of Theorem 6.1.4. Thus $\mathbb{M}$ from Theorem 6.1.1 is non-explosive. Note that the non-explosion criterion of this example can not be derived from [69, Proposition 1.10], nor from* (1.3) *or for instance [30, Assumption 2.1] (one of the pioneering works on local and global well posedness of SDEs with unbounded merely measurable drifts), since $\mathbf{G}$ has a part with infinitely many singular points outside an arbitrarily large compact set and may have a part with linear growth.*

6.2 Uniqueness in law under low regularity

Let $\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$ be a right process (see for instance [78]). For a σ-finite or finite Borel measure ν on $\mathbb{R}^d$ we define

$$\widetilde{\mathbb{P}}_\nu(\cdot) := \int_{\mathbb{R}^d} \widetilde{\mathbb{P}}_x(\cdot)\, \nu(dx).$$

Consider $(L, C_0^\infty(\mathbb{R}^d))$ as defined in (5.6). According to [69, Definition 2.5], we define:

Definition 6.2.1. *A right process $\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$ with state space $\mathbb{R}^d$ and natural filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ is said to solve the martingale for $(L, C_0^\infty(\mathbb{R}^d))$, if for all $u \in C_0^\infty(\mathbb{R}^d)$:*

(i) $\int_0^t Lu(\widetilde{X}_s)\, ds$, $t \geq 0$, is $\widetilde{\mathbb{P}}_m$-a.e. independent of the measurable m-version chosen for Lu.

(ii) $u(\widetilde{X}_t) - u(\widetilde{X}_0) - \int_0^t Lu(\widetilde{X}_s)\, ds$, $t \geq 0$, is a continuous $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$-martingale under $\widetilde{\mathbb{P}}_{vm}$ for any $v \in \mathcal{B}_b^+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} v\, dm = 1$.

Definition 6.2.2. *A σ-finite Borel measure ν on $\mathbb{R}^d$ is called sub-invariant measure for a right process $\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$ with state space $\mathbb{R}^d$, if*

$$\int_{\mathbb{R}^d} \widetilde{\mathbb{E}}_x[f(\widetilde{X}_t)]\nu(dx) \leq \int_{\mathbb{R}^d} f(x)\nu(dx) \tag{6.3}$$

for any $f \in L^1(\mathbb{R}^d, \nu) \cap \mathcal{B}_b(\mathbb{R}^d)$, $f \geq 0$, $t \geq 0$. ν is called invariant measure for $\widetilde{\mathbb{M}}$, if "$\leq$" can be replaced by "$=$" in (6.3)

Part (i) of the following proposition is proven in [69, Proposition 2.6]. And part (ii) is a simple consequence of part (i) and the strong Feller property of $(p_t^{\mathbb{M}})_{t\geq 0}$, $\mathbb{M}$ as in Theorem 3.2.1.

Proposition 6.2.3. *(i) Let $\widetilde{\mathbb{M}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$ solve the martingale for $(L, C_0^\infty(\mathbb{R}^d))$ such that m is a sub-invariant measure for $\widetilde{\mathbb{M}}$ and let $(L, C_0^\infty(\mathbb{R}^d))$ be L^1-unique. Then $p_t^{\widetilde{\mathbb{M}}} f(x) := \widetilde{\mathbb{E}}_x[f(\widetilde{X}_t)]$ is an m-version of $T_t f$ for all $f \in L^1(\mathbb{R}^d, m) \cap \mathcal{B}_b(\mathbb{R}^d)$, $t \geq 0$ and m is an invariant measure for $\widetilde{\mathbb{M}}$.*

(ii) If additionally $(p_t^{\widetilde{\mathbb{M}}})_{t\geq 0}$ *is strong Feller, then* $\widetilde{\mathbb{P}}_x = \mathbb{P}_x$ *for any* $x \in \mathbb{R}^d$.

Proposition 6.2.4. *Suppose that (a) and (b) hold, and that for any compact set* K *in* $\mathbb{R}^d$*, there exist* $L_K \geq 0, \alpha_K \in (0,1)$ *with*

$$|\widetilde{a}_{ij}(x) - \widetilde{a}_{ij}(y)| \leq L_K |x-y|^{\alpha_K}, \quad \forall x, y \in K, \, 1 \leq i,j \leq d.$$

Suppose further that m *is an invariant measure for* $\mathbb{M}$*. Let* $\widetilde{\mathbb{M}}$ *be a right process with strong Feller transition function* $(p_t^{\widetilde{\mathbb{M}}})_{t\geq 0}$ *that solves the martingale problem for* $(L, C_0^{\infty}(\mathbb{R}^d))$ *and such that* m *is a sub-invariant measure for* $\widetilde{\mathbb{M}}$*. Then* $\widetilde{\mathbb{P}}_x = \mathbb{P}_x$ *for any* $x \in \mathbb{R}^d$.

Proof By [69, Corollary 2.2] $(L, C_0^{\infty}(\mathbb{R}^d))$ is L^1-unique, iff m is an invariant measure for $\mathbb{M}$. Then appy Proposition 6.2.3.

□

Remark 6.2.5. *Note that* m *is an invariant measure for* $\mathbb{M}$ *as in Theorem 6.1.1, if and only if the co-semigroup* $(\widehat{T}_t)_{t>0}$ *of* $(T_t)_{t>0}$ *is conservative. One advantage of our approach is that we can use all previously derived conservativeness results for generalized Dirichlet forms (see for instance [69, Proposition 1.10], [28], Part I, but also Example 6.2.6).*

Example 6.2.6. *(i) Assume (a), (b) holds and that the* $\widetilde{a}_{ij}$ *are locally Hölder continuous on* $\mathbb{R}^d$ *as in Proposition 6.2.4. If there exists a constant* $C > 0$ *and some* $N_0 \in \mathbb{N}$*, such that*

$$-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\text{trace}A(x) + \big\langle \mathbf{G}(x), x \big\rangle \leq -C\left(\|x\|^2+1\right)$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$*, then* $\mathbb{M}$ *as in Theorem 6.1.1 solves the martingale problem for* $(L, C_0^{\infty}(\mathbb{R}^d))$ *and* m *is an invariant measure for* $\mathbb{M}$ *by the analogue of Proposition 4.2.13 (see Remark 6.1.2). In this situation Proposition 6.2.4 applies.*

(ii) *Let* $\widetilde{A}$*,*$\check{A}$ *and* $\mathbf{G}$ *be as in Example 6.1.5. By Theorem 6.1.4, not only* $\mathbb{M}$ *but also its co-process* $\widehat{M}$ *is non-explosive. Hence* dx *is an invariant measure for* $\mathbb{M}$*. Now if* $\widetilde{a}_{ij}$ *are locally Hölder continuous on* $\mathbb{R}^d$ *as in Proposition 6.2.4, then Proposition 6.2.4 also applies.*

(iii) *Suppose that in the situation of Remark 5.3.7(i) the conditions of Theorem 4.2.7 hold with* $\mathbf{B} = \widetilde{\mathbf{B}}$ *and that the* $\widetilde{a}_{ij}$ *are locally Hölder continuous on* $\mathbb{R}^d$ *as in Proposition 6.2.4. Then* $\rho\, dx$ *is an invariant measure for* $\mathbb{M}$ *and Proposition 6.2.4 again applies.*

Part III

Well-posedness for Itô-SDEs with degenerate and rough diffusion coefficients

Chapter 7

Regularity of solutions

7.1 Regularity results for linear parabolic equation with singular weight in the time derivative term

The following Lemma which will lead to our main result, is a slight modification of [2, Lemma 6] and involves a weight function ψ.

Lemma 7.1.1. *Let U be a bounded open subset of $\mathbb{R}^d$ and $T > 0$. Let $w \in L^2(U \times (0,T))$ be such that $\operatorname{supp}(w) \subset U \times (0,T]$ and assume $\partial_t w \in L^2(U \times (0,T))$, $\psi \in L^2(U)$. Then for a.e. $\tau \in (0,T)$, it holds*

$$\int_0^\tau \int_U \partial_t w \cdot \psi \, dxdt = \int_U w|_{t=\tau} \, \psi dx.$$

Proof Let $\psi_n \in C_0^\infty(U)$, $n \geq 1$, satisfy $\lim_{n\to\infty} \psi_n = \psi$ in $L^2(U)$. Then $w\psi \in L^{1,2}(U \times$

$(0,T))$ and for any $\varphi \in C_0^\infty(U \times (0,T))$, we have

$$\begin{aligned}\iint_{U\times(0,T)} \partial_t\varphi \cdot w\psi \, dxdt &= \lim_{n\to\infty} \iint_{U\times(0,T)} \partial_t\varphi \cdot w\psi_n \, dxdt \\ &= \lim_{n\to\infty} \iint_{U\times(0,T)} \partial_t(\varphi\psi_n) \cdot w \, dxdt \\ &= -\lim_{n\to\infty} \iint_{U\times(0,T)} \varphi\psi_n \cdot \partial_t w \, dxdt \\ &= -\iint_{U\times(0,T)} \varphi \cdot (\partial_t w \cdot \psi) dxdt.\end{aligned}$$

Thus $\partial_t(w\psi) = \partial_t w \cdot \psi \in L^{1,2}(U \times (0,T))$. Now let $f(t) := \int_U w(x,t)\psi(x)dx$. Then $f(t)$ is defined for a.e. $t \in (0,T)$ and is in $L^1((0,T))$. Let $g \in C_0^\infty\big((0,T)\big)$ be given. Take $\tau_0 \in (0,T)$ satisfying $\mathrm{supp}(g) \subset (0,\tau_0)$. Let V be a bounded open subset of $\mathbb{R}^d$ such that $\overline{V} \subset U$ and $\mathrm{supp}(w) \cap \big(U \times (0,\tau_0)\big) \subset V \times (0,\tau_0)$. Let $\chi \in C_0^\infty(U)$ with $\chi \equiv 1$ on V. Then

$$\begin{aligned}\int_0^T \partial_t g \cdot f \, dt &= \iint_{U\times(0,\tau_0)} \partial_t g \cdot w \, \psi dxdt \\ &= \iint_{V\times(0,\tau_0)} \partial_t(g\chi) \cdot (w\psi) dxdt \\ &= -\iint_{U\times(0,T)} g\chi \, \partial_t w \cdot \psi dxdt \\ &= -\int_0^T g \cdot \left(\int_U \partial_t w \cdot \psi dx\right) dt.\end{aligned}$$

Thus $\partial_t f = \int_U \partial_t w \cdot \psi dx \in L^1\big((0,T)\big)$. Then by [21, Theorem 4.20], f has an absolutely continuous dx-version on $(0,T)$ and by the Fundamental Theorem of Calculus, for a.e $\tau_1, \tau \in (0,T)$ it holds

$$\int_{\tau_1}^{\tau} \int_U \partial_t w \cdot \psi dxdt = \int_{\tau_1}^{\tau} \partial_t f dt = \int_{\tau_1}^{\tau} f' dt = f(\tau) - f(\tau_1) = \int_U (w|_{t=\tau} - w|_{t=\tau_1}) \, \psi dx.$$

Choosing τ_1 near 0, our assertion holds.

□

Consider the following condition.

(I) $U \times (0,T)$ is a bounded open set in $\mathbb{R}^d \times \mathbb{R}$, $T > 0$. $A = (a_{ij})_{1 \le i,j \le d}$ is a matrix of functions on U that is uniformly strictly elliptic and bounded, i.e. there exists constants $\lambda > 0$, $M > 0$, such that for all $\xi = (\xi_1, \dots, \xi_d)$, $x \in U$, it holds

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \lambda \|\xi\|^2, \qquad \max_{1 \le i,j \le d} |a_{ij}(x)| \le M,$$

and let $\mathbf{B} \in L^p(U, \mathbb{R}^d)$ with $p > d$, $\psi \in L^q(U)$, $q \in [2 \vee \frac{p}{2}, p)$. There exists $c_0 > 0$ such that $c_0 \le \psi$ on U, and finally

$$u \in H^{1,2}(U \times (0,T)) \cap L^\infty(U \times (0,T)).$$

Assuming **(I)**, we consider a divergence form linear parabolic equation with a singular weight in the time derivative term as follows.

$$\iint_{U \times (0,T)} (u\partial_t \varphi)\psi dxdt = \iint_{U \times (0,T)} \big\langle A\nabla u, \nabla \varphi \big\rangle + \langle \mathbf{B}, \nabla u \rangle \varphi \, dxdt, \quad \text{for all } \varphi \in C_0^\infty(U \times (0,T)). \tag{7.1}$$

Using integration by parts in the left hand term, (7.1) is equivalent to

$$-\iint_{U \times (0,T)} (\partial_t u)\, \varphi \psi dxdt = \iint_{U \times (0,T)} \big\langle A\nabla u, \nabla \varphi \big\rangle + \langle \mathbf{B}, \nabla u \rangle \varphi dxdt, \quad \text{for all } \varphi \in C_0^\infty(U \times (0,T)). \tag{7.2}$$

Define $\mathcal{A} := \{v \in L^\infty(U \times (0,T)) \mid \nabla v \in L^2(U \times (0,T)) \text{ and } \operatorname{supp}(v) \subset U \times (0,T)\}$. Using the standard mollification on $\mathbb{R}^d \times \mathbb{R}$ to approximate functions in $\mathcal{A}$, (7.2) extends to

$$-\iint_{U \times (0,T)} (\partial_t u)\, \varphi \psi dxdt = \iint_{U \times (0,T)} \big\langle A\nabla u, \nabla \varphi \big\rangle + \langle \mathbf{B}, \nabla u \rangle \varphi dxdt, \quad \text{for all } \varphi \in \mathcal{A}. \tag{7.3}$$

Fix $\beta \geq 1$. For $t \in \mathbb{R}$, define functions $G(t) := (t^+)^\beta$, $H(t) := \frac{1}{\beta+1}(t^+)^{\beta+1}$, where $t^+ := \max(0,t)$. Then by [21, Theorem 4.4], $G'(t) = \beta(t^+)^{\beta-1}1_{[0,\infty)}(t)$ and $H'(t) = G(t)$. Let $\eta \in C_0^\infty(U \times (0,T])$ with $\eta \geq 0$. Given $\tau \in (0,T)$, define $\widetilde{\varphi} := \eta^2 G(u) 1_{(0,\tau)}$. Then by [21, Theorem 4.4] (or [2, Lemma 4]),

$$\nabla \widetilde{\varphi} = \begin{cases} \eta^2 G'(u)\nabla u + 2\eta \nabla \eta \, G(u), & 0 < t < \tau, \\ 0, & \tau \leq t < T. \end{cases}$$

Thus $\widetilde{\varphi} \in \mathcal{A}$ and by (7.3), we have

$$-\iint_{U\times(0,T)} (\partial_t u)\, \widetilde{\varphi}\psi dxdt = \iint_{U\times(0,T)} \left\langle A\nabla u, \nabla\widetilde{\varphi}\right\rangle + \langle \mathbf{B}, \nabla u\rangle \widetilde{\varphi} dxdt. \tag{7.4}$$

Observe that by [21, Theorem 4.4] (or [2, Lemma 4]),

$$\partial_t(\eta^2 H(u)) = 2\eta\partial_t\eta\, H(u) + \eta^2 G(u)\partial_t u.$$

Thus by Lemma 7.1.1

$$\begin{aligned}
&\iint_{U\times(0,T)} \widetilde{\varphi}\, (\partial_t u)\, \psi dxdt \\
= \;& \iint_{U\times(0,\tau)} \eta^2 G(u)\partial_t u \cdot \psi dxdt \\
= \;& \int_0^\tau \int_U \partial_t(\eta^2 H(u))\psi dxdt - 2\int_0^\tau \int_U \eta\partial_t\eta\, H(u)\psi dxdt \\
= \;& \int_U \eta^2 H(u)\mid_{t=\tau} \psi dx - \int_0^\tau \int_U 2\eta\partial_t\eta\, H(u)\psi dxdt, \quad \text{for a.e. } \tau \in (0,T).
\end{aligned} \tag{7.5}$$

By (7.4) and (7.5), we get

$$\begin{aligned}
\int_U \eta^2 H(u)\mid_{t=\tau} \psi dxdt + \int_0^\tau \int_U \left\langle A\nabla u, \nabla\widetilde{\varphi}\right\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi} dxdt& \\
= \int_0^\tau \int_U 2\eta\, \partial_t\eta\, H(u)\, \psi dxdt, \qquad \text{for a.e. } \tau \in (0,T).&
\end{aligned} \tag{7.6}$$

On $\{\widetilde{\varphi} > 0\}$, it holds $u > 0$, so that $\nabla u = \nabla u^+$. Thus on $\{\widetilde{\varphi} > 0\}$, we have

$$\begin{aligned}
&\big\langle A\nabla u, \nabla\widetilde{\varphi}\big\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi} \\
&= \big\langle A\nabla u^+, \eta^2 G'(u)\nabla u^+\big\rangle + \big\langle A\nabla u^+, 2\eta\nabla\eta\, G(u)\big\rangle + \langle \mathbf{B}, \nabla u^+\rangle\eta^2 G(u) \\
&\geq \eta^2 G'(u)\lambda\,\|\nabla u^+\|^2 - 2\eta G(u) dM\|\nabla\eta\|\|\nabla u^+\| - \eta^2 G(u)\|\mathbf{B}\|\|\nabla u^+\|.
\end{aligned}$$

Note that on $\{\widetilde{\varphi} > 0\}$

$$(u^+)^{-\beta-1}\, G(u)^2 \leq\, G'(u),$$

hence using Young's inequality, we obtain

$$\begin{aligned}
&2\eta G(u) dM\|\nabla\eta\|\|\nabla u^+\| \\
&\leq 2\cdot\frac{1}{4}\frac{\Big(\sqrt{\lambda}\,(u^+)^{-\frac{\beta+1}{2}} G(u)\,\eta\,\|\nabla u^+\|\Big)^2}{2} + 2\cdot 4\frac{\Big(dM\sqrt{\lambda^{-1}}\,(u^+)^{\frac{\beta+1}{2}}\,\|\nabla\eta\|\Big)^2}{2} \\
&= \frac{\lambda}{4}\eta^2 G'(u)\|\nabla u\|^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\,(u^+)^{\beta+1},
\end{aligned}$$

and

$$\begin{aligned}
\eta^2 G(u)\|\mathbf{B}\|\|\nabla u^+\| \;&\leq\; \frac{1}{2}\cdot\frac{\Big(\sqrt{\lambda}\,(u^+)^{-\frac{\beta+1}{2}}\, G(u)\eta\|\nabla u^+\|\Big)^2}{2} + 2\cdot\frac{\Big(\sqrt{\lambda^{-1}}\,(u^+)^{\frac{\beta+1}{2}}\|\mathbf{B}\|\eta\Big)^2}{2} \\
&\leq\; \frac{\lambda}{4}\eta^2 G'(u)\|\nabla u^+\|^2 + \frac{1}{\lambda}\|\mathbf{B}\|^2 (u^+)^{\beta+1}\eta^2.
\end{aligned}$$

Therefore on $\{\widetilde{\varphi} > 0\}$, it holds

$$\begin{aligned}
&\frac{\lambda}{2}\eta^2 G'(u)\|\nabla u^+\|^2 \\
&\leq \big\langle A\nabla u, \nabla\widetilde{\varphi}\big\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi} + \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big)(u^+)^{\beta+1}. \qquad (7.7)
\end{aligned}$$

Note that $\{\widetilde{\varphi} = 0\} \cap \Big(U \times (0,\tau)\Big) = \{\eta = 0\} \cup \{u \leq 0\}$ and $\nabla u^+ = 0$ on $\{u \leq 0\}$.

Thus (7.7) holds on $U \times (0,\tau)$. Combining (7.7) and (7.6), we obtain for a.e. $\tau \in (0,T)$

$$\frac{1}{\beta+1}\int_U \eta^2 (u^+)^{\beta+1} |_{t=\tau} \psi dx + \frac{\lambda\beta}{2}\int_0^\tau \int_U \eta^2 (u^+)^{\beta-1}\|\nabla u^+\|^2 dxdt$$
$$\leq \int_0^\tau \int_U \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big)(u^+)^{\beta+1} dxdt + \frac{2}{\beta+1}\int_0^\tau \int_U \eta|\partial_t \eta|(u^+)^{\beta+1}\psi dxdt. \tag{7.8}$$

Now let $(\bar{x},\bar{t})$ be an arbitrary but fixed point in $U \times (0,T)$. Let $R_{\bar{x}}(r)$ be the open cube in $\mathbb{R}^d$ of edge length $r > 0$ centered at $\bar{x}$. Define $Q(r) := R_{\bar{x}}(r) \times (\bar{t} - r^2, \bar{t})$.

Theorem 7.1.2. *Assume* **(I)** *and $Q(3r) \subset U \times (0,T)$. Then it holds*

$$\|u\|_{L^\infty(Q(r))} \leq C\|u\|_{L^{\frac{2p}{p-2},2}(Q(2r))}, \tag{7.9}$$

where $C > 0$ is a constant depending only on r, λ, M and $\|\mathbf{B}\|_{L^p(R_{\bar{x}}(3r))}$.

Proof Let $\eta \in C_0^\infty(R_{\bar{x}}(r) \times (\bar{t} - 9r^2, \bar{t}])$. Then (7.8) holds with $U \times (0,T)$ replaced by $Q(3r)$. Using appropriate scaling arguments(cf. [2, proof of Theorem 2]), we may assume $r = \frac{1}{3}$. Set $v := (u^+)^\gamma$ with $\gamma := \frac{\beta+1}{2}$. Then $\|\nabla v\|^2 = \gamma^2 (u^+)^{\beta-1}\|\nabla u^+\|^2$. By (7.8), it holds for a.e. $\tau \in (\bar{t}-1, \bar{t})$

$$\frac{c_0}{2\gamma}\int_{R_{\bar{x}}(1)} \eta^2 v^2 |_{t=\tau} dx + \frac{\lambda}{2\gamma^2}\int_{\bar{t}-1}^\tau \int_{R_{\bar{x}}(1)} \eta^2 \|\nabla v\|^2 dxdt$$
$$\leq \iint_{Q(1)} \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big) v^2 dxdt + \iint_{Q(1)} \eta|\partial_t\eta| v^2 \psi dx.$$

Let l and l' be positive numbers satisfying $\frac{1}{3} < l' < l \leq \frac{2}{3}$. Assume that $\eta \equiv 1$ in $Q(l')$, $\eta \equiv 0$ outside $Q(l)$, $0 \leq \eta \leq 1$, and $|\partial_t \eta|, \|\nabla\eta\| \leq 2d(l-l')^{-1}$. Then

$$\iint_{Q(1)} \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big) v^2 dxdt$$
$$\leq \frac{4d^2}{\lambda}(l-l')^{-2}\iint_{Q(l)} \left(\|\mathbf{B}\|^2 + 4d^2M^2\right) v^2 dxdt$$
$$\leq \frac{4d^2}{\lambda}(l-l')^{-2}(\|\mathbf{B}\|^2_{L^p(R_{\bar{x}}(1))} + 4d^2M^2)\|v\|^2_{L^{\frac{2p}{p-2},2}(Q(l))},$$

and

$$\begin{aligned}\int_{\bar{t}-1}^{\bar{t}}\int_{R(1)}\eta|\partial_t\eta|v^2\,\psi dx &\leq 2d(l-l')^{-1}\|\psi\|_{L^q(R_{\bar{x}}(1))}\|v\|^2_{L^{\frac{2q}{q-1}},2(Q(l))}\\ &\leq 2d(l-l')^{-2}\|\psi\|_{L^q(R_{\bar{x}}(1))}\|v\|^2_{L^{\frac{2p}{p-2}},2(Q(l))}.\end{aligned}$$

Thus we obtain

$$\lambda\|\eta\nabla v\|^2_{L^2(Q(1))} \leq 2C_1(l-l')^{-2}\gamma^2\|v\|^2_{L^{\frac{2p}{p-2}},2(Q(l))}$$

and

$$\|\eta v\|^2_{L^{2,\infty}(Q(1))} \leq 2c_0^{-1}C_1(l-l')^{-2}\gamma^2\|v\|^2_{L^{\frac{2p}{p-2}},2(Q(l))},$$

where $C_1 = \frac{4d^2}{\lambda}(\|\mathbf{B}\|^2_{L^p(R_{\bar{x}}(1))} + 4d^2M^2) + 2d\|\psi\|_{L^q(R_{\bar{x}}(1))}$.

Now set $\theta := 1 - \frac{d}{p}$, and $\sigma := 1 + \frac{\theta}{2}$ if $d = 2$, $\quad \sigma := 1 + \frac{2\theta}{d}$ if $d \geq 3$.
Set $p_\sigma := \left(\frac{\sigma p}{p-2}\right)' = \frac{\sigma p}{\sigma p - p + 2}$, $q_\sigma := \sigma' = \frac{\sigma}{\sigma - 1}$. Then

$$\frac{d}{2p_\sigma} + \frac{1}{q_\sigma} < 1 \;\text{ if } d = 2, \qquad \frac{d}{2p_\sigma} + \frac{1}{q_\sigma} = 1 \;\text{ if } d \geq 3.$$

By [2, Lemma 3],

$$\begin{aligned}\|v^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2}},2(Q(l'))} &\leq \|(\eta v)^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2}},2(Q(1))}\\ &= \|\eta v\|^2_{L^{\frac{2\sigma p}{p-2}},2\sigma(Q(1))}\\ &= \|\eta v\|^2_{L^{2(p_\sigma)',2(q_\sigma)'}(Q(1))}\\ &\leq K\Big(\|\eta v\|^2_{L^{\infty,2}(Q(1))} + \|\nabla(\eta v)\|^2_{L^2(Q(1))}\Big)\\ &\leq K\Big(\|\eta v\|^2_{L^{\infty,2}(Q(1))} + 2\|\eta\nabla v\|^2_{L^2(Q(1))} + 8d^2(l-l')^{-2}\|v\|^2_{L^2(Q(l))}\Big)\\ &\leq C_2(l-l')^{-2}\gamma^2\|v\|^2_{L^{\frac{2p}{p-2}},2(Q(l))}, \qquad (7.10)\end{aligned}$$

where $C_2 = K(4C_1\lambda^{-1} + 2C_1c_0^{-1} + 8d^2)$.

Now for $m \in \mathbb{N} \cup \{0\}$, set $l = l_m := 3^{-1}(1 + 2^{-m})$, $\quad l' = l'_m := 3^{-1}(1 + 2^{-m-1})$, $\varphi_m := \|(u^+)^{\sigma^m}\|^{2/\sigma^m}_{L^{\frac{2p}{p-2},2}(Q(l_m))}$. Taking $\gamma = \sigma^m$ and $1/3 < l' = l'_m < l = l_m \leq 2/3$ for $m \in \mathbb{N} \cup \{0\}$, we obtain using (7.10)

$$\varphi_{m+1} \leq (36C_2)^{\frac{1}{\sigma^m}} (2\sigma)^{\frac{2m}{\sigma^m}} \varphi_m. \tag{7.11}$$

Iterating (7.11), we get

$$\begin{aligned} \varphi_{m+1} &\leq (36C_2)^{\sum_{i=0}^m \frac{1}{\sigma^i}} (2\sigma)^{\sum_{i=0}^m \frac{2i}{\sigma^i}} \varphi_0 \\ &\leq \underbrace{(36C_2)^{\frac{\sigma}{\sigma-1}} (2\sigma)^{\frac{2\sigma}{(\sigma-1)^2}}}_{=:C_3} \|u\|^2_{L^{\frac{2p}{p-2},2}(Q(2/3))}. \end{aligned}$$

Letting $m \to \infty$, we get

$$\|u^+\|_{L^\infty(Q(1/3))} \leq \sqrt{C_3} \|u\|_{L^{\frac{2p}{p-2},2}(Q(2/3))}.$$

Exactly in the same way, but with u replaced by $-u$, we obtain (7.9) with $C = 2\sqrt{C_3}$.

□

7.2 Elliptic Hölder regularity and estimates

Lemma 7.2.1. *Let U be a bounded open ball in $\mathbb{R}^d$. Let $f \in L^{\widetilde{q}}(U)$ with $\frac{d}{2} < \widetilde{q} < d$. Then there exists $\mathbf{F} = (f_1, \ldots, f_d) \in H^{1,\widetilde{q}}(U, \mathbb{R}^d)$ such that $\mathrm{div}\mathbf{F} = f$ in U and*

$$\sum_{i=1}^{d} \|f_i\|_{H^{1,\widetilde{q}}(U)} \le C\|f\|_{L^{\widetilde{q}}(U)},$$

where $C > 0$ only depends on $\widetilde{q}$, U. In particular, applying the Sobolev inequality, we get

$$\sum_{i=1}^{d} \|f_i\|_{L^{\frac{d\widetilde{q}}{d-\widetilde{q}}}(U)} \le C'\|f\|_{L^{\widetilde{q}}(U)},$$

where $C' > 0$ only depends on $\widetilde{q}$, U.

Proof By [27, Theorem 9.15 and Lemma 9.17], there exists $u \in H^{2,\widetilde{q}}(U) \cap H_0^{1,\widetilde{q}}(U)$ such that $\Delta u = f$ in U and

$$\|u\|_{H^{2,\widetilde{q}}(U)} \le C_1\|f\|_{L^{\widetilde{q}}(U)},$$

where $C_1 > 0$ is a constant only depending on $\widetilde{q}$, U. Let $\mathbf{F} := \nabla u$. Then $\mathbf{F} \in H^{1,\widetilde{q}}(U, \mathbb{R}^d)$ with $\mathrm{div}\mathbf{F} = f$ in U and it holds

$$\begin{aligned}
\sum_{i=1}^{d} \|f_i\|_{H^{1,\widetilde{q}}(U)} &= \sum_{i=1}^{d} \|\partial_i u\|_{H^{1,\widetilde{q}}(U)} = \sum_{i=1}^{d} \left(\|\partial_i u\|_{L^{\widetilde{q}}(U)}^{\widetilde{q}} + \sum_{j=1}^{d} \|\partial_j\partial_i u\|_{L^{\widetilde{q}}(U)}^{\widetilde{q}} \right)^{\frac{1}{\widetilde{q}}} \\
&= \sum_{i=1}^{d} \|\partial_i u\|_{L^{\widetilde{q}}(U)} + \sum_{i=1}^{d}\sum_{j=1}^{d} \|\partial_j\partial_i u\|_{L^{\widetilde{q}}(U)} \\
&\le \left(d + d^2\right)^{\frac{\widetilde{q}-1}{\widetilde{q}}} \left(\sum_{i=1}^{d} \|\partial_i u\|_{L^{\widetilde{q}}(U)}^{\widetilde{q}} + \sum_{i=1}^{d}\sum_{j=1}^{d} \|\partial_j\partial_i u\|_{L^{\widetilde{q}}(U)}^{\widetilde{q}} \right)^{\frac{1}{\widetilde{q}}} \\
&\le \left(d + d^2\right)^{\frac{\widetilde{q}-1}{\widetilde{q}}} \|u\|_{H^{2,\widetilde{q}}(U)} \\
&\le C_1 \left(d + d^2\right)^{\frac{\widetilde{q}-1}{\widetilde{q}}} \|f\|_{L^{\widetilde{q}}(U)}.
\end{aligned}$$

□

Theorem 7.2.2. *Let U be a bounded open ball in $\mathbb{R}^d$. Let $A = (a_{ij})_{1 \le i,j \le d}$ be a matrix of bounded functions on U that is uniformly strictly elliptic. Assume $\mathbf{B} \in L^p(U, \mathbb{R}^d)$, $c \in L^q(U)$, $f \in L^{\widetilde{q}}(U)$ for some $p > d$, q, $\widetilde{q} > \frac{d}{2}$. If $u \in H^{1,2}(U)$ satisfies*

$$\int_U \langle A\nabla u, \nabla\varphi\rangle + (\langle \mathbf{B}, \nabla u\rangle + cu)\,\varphi\, dx = \int_U f\varphi\, dx, \quad \textit{for all } \varphi \in C_0^\infty(U), \tag{7.12}$$

then for any open ball U_1 in $\mathbb{R}^d$ with $\overline{U}_1 \subset U$, we have $u \in C^{0,\gamma}(\overline{U}_1)$ and

$$\|u\|_{C^{0,\gamma}(\overline{U}_1)} \le C\left(\|u\|_{L^1(U)} + \|f\|_{L^{\widetilde{q}}(U)}\right),$$

where $\gamma \in (0,1)$ and $C > 0$ are constants which are independent of u and f.

Proof Without loss of generality, we may assume $\frac{d}{2} < \widetilde{q} < d$. Let U_2 be an open ball in $\mathbb{R}^d$ satisfying $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$. By Lemma 7.2.1, we can find $\mathbf{F} = (f_1, \cdots, f_d) \in H^{1,\widetilde{q}}(U_2, \mathbb{R}^d) \subset L^{\frac{d\widetilde{q}}{d-\widetilde{q}}}(U_2, \mathbb{R}^d)$ such that

$$\mathrm{div}\mathbf{F} = f \ \text{ in } U_2, \qquad \sum_{i=1}^d \|f_i\|_{L^{\frac{d\widetilde{q}}{d-\widetilde{q}}}(U_2)} \le C_1 \|f\|_{L^{\widetilde{q}}(U_2)},$$

where $C_1 > 0$ is a constant only depending on $\widetilde{q}$ and U_2. Then (7.12) implies

$$\int_{U_2} \langle A\nabla u, \nabla\varphi\rangle + (\langle \mathbf{B}, \nabla u\rangle + cu)\,\varphi\, dx = \int_{U_2} \langle -\mathbf{F}, \nabla\varphi\rangle\, dx \ \text{ for all } \varphi \in C_0^\infty(U_2).$$

Given $x \in U_1$, $r > 0$ with $r < dist(x, U_2)$, set $\omega_x(r) := \sup_{B_x(r)} u - \inf_{B_x(r)} u$. By [67, Théorème 7.2] and Lemma 7.2.1,

$$\begin{aligned}\omega_x(r) &\le K\left(\|u\|_{L^2(U_2)} + \sum_{i=1}^d \|f_i\|_{L^{\frac{d\widetilde{q}}{d-\widetilde{q}}}(U_2)}\right) r^\gamma \\ &\le K(1 + C')\left(\|u\|_{L^2(U_2)} + \|f\|_{L^{\widetilde{q}}(U_2)}\right) r^\gamma,\end{aligned}$$

where $\gamma \in (0,1)$ and $K, C' > 0$ are constants which are independent of x, r, u, $\mathbf{F}$, f.

Thus we have

$$\int_{B_r(x)} |u(y) - u_{x,r}|^2 dy \le (K')^2 \left(\|u\|_{L^2(U_2)} + \|f\|_{L^{\tilde{q}}(U_2)}\right)^2 r^{d+2\gamma},$$

where $u_{x,r} := \frac{1}{|B_r(x)|}\int_{B_r(x)} u(u)\,dy$ and $(K')^2 := K^2 \cdot \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}(1+C')^2$. Finally by [31, Theorem 3.1], [12, Theorem 1.7.4], we obtain

$$\begin{aligned}
\|u\|_{C^{0,\gamma}(\overline{U}_1)} &\le c\Big(K'\left(\|u\|_{L^2(U_2)} + \|f\|_{L^{\tilde{q}}(U_2)}\right) + \|u\|_{L^2(U_2)}\Big) \\
&\le (cK' \vee c)\left(\|u\|_{H^{1,2}(U_2)} + \|f\|_{L^{\tilde{q}}(U_2)}\right) \\
&\le (cK' \vee c)\Big(C_1\|u\|_{L^1(U)} + C_1\|f\|_{L^{\tilde{q}}(U)} + \|f\|_{L^{\tilde{q}}(U_2)}\Big) \\
&\le (C_1+1)\,(cK' \vee c)\Big(\|u\|_{L^1(U)} + \|f\|_{L^{\tilde{q}}(U)}\Big),
\end{aligned}$$

where $c > 0$, $C_1 > 0$ are constants which are independent of u and f.

□

Chapter 8

Analytic theory for degenerate second order partial differential operators

8.1 Framework

Let $\rho \in H^{1,2}_{loc}(\mathbb{R}^d) \cap L^\infty_{loc}(\mathbb{R}^d)$, $\psi \in L^1_{loc}(\mathbb{R}^d)$ be a.e positive functions satisfying $\frac{1}{\rho}$, $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$ and set $\widehat{\rho} := \rho\psi$, $\mu := \widehat{\rho}\,dx$. If U is any open subset of $\mathbb{R}^d$, then the bilinear form $\int_U \langle \nabla u, \nabla v\rangle dx$, $\quad u, v \in C_0^\infty(U)$ is closable in $L^2(U,\mu)$ by [51, Subsection II.2a)]. Define $\widehat{H}^{1,2}_0(U,\mu)$ as the closure of $C_0^\infty(U)$ in $L^2(U,\mu)$ with respect to the norm $\left(\int_U \|\nabla u\|^2 dx + \int_U u^2 d\mu\right)^{1/2}$. Thus $u \in \widehat{H}^{1,2}_0(U,\mu)$, if and only if there exists $(u_n)_{n\geq 1} \subset C_0^\infty(U)$ such that

$$\lim_{n\to\infty} u_n = u \text{ in } L^2(U,\mu), \qquad \lim_{n,m\to\infty} \int_U \|\nabla(u_n - u_m)\|^2 dx = 0, \tag{8.1}$$

and moreover $\widehat{H}^{1,2}_0(U,\mu)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{\widehat{H}^{1,2}_0(U,\mu)} = \lim_{n\to\infty} \int_U \langle \nabla u_n, \nabla v_n\rangle dx + \int_U uv\, d\mu,$$

where $(u_n)_{n\geq 1}, (v_n)_{n\geq 1} \subset C_0^\infty(U)$ are arbitrary sequences that satisfy (8.1).

If $u \in \widehat{H}_0^{1,2}(V,\mu)$ for some bounded open subset V of $\mathbb{R}^d$, then $u \in H_0^{1,2}(V) \cap L^2(V,\mu)$ and there exists $(u_n)_{n\geq 1} \subset C_0^\infty(V)$ such that

$$\lim_{n\to\infty} u_n = u \quad \text{in } H_0^{1,2}(V) \text{ and in } L^2(V,\mu).$$

Consider a symmetric matrix of functions $A = (a_{ij})_{1\leq i,j\leq d}$ satisfying

$$a_{ij} = a_{ji} \in H^{1,2}_{loc}(\mathbb{R}^d), \quad 1 \leq i,j \leq d,$$

and assume A is locally uniformly strictly elliptic, i.e. for every open ball B, there exist constants $\lambda_B, \Lambda_B > 0$ such that

$$\lambda_B \|\xi\|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda_B \|\xi\|^2, \quad \text{for all } \xi \in \mathbb{R}^d,\ x \in B. \tag{8.2}$$

Define $\widehat{A} := \frac{1}{\psi}A$. By [51, Subsection II.2b)], the symmetric bilinear form

$$\mathcal{E}^0(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla f, \nabla g\rangle d\mu, \quad f,g \in C_0^\infty(\mathbb{R}^d),$$

is closable in $L^2(\mathbb{R}^d,\mu)$ and its closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric Dirichlet form in $L^2(\mathbb{R}^d,\mu)$ (see [51, (II. 2.18)]). Denote the corresponding generator of $(\mathcal{E}^0, D(\mathcal{E}^0))$ by $(L^0, D(L^0))$. Let $f \in C_0^\infty(\mathbb{R}^d)$. Using integration by parts, for any $g \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
\mathcal{E}^0(f,g) &= \frac{1}{2}\int_{\mathbb{R}^d} \langle \rho A\nabla f, \nabla g\rangle dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^d} \big(\rho\,\mathrm{trace}(A\nabla^2 f) + \langle \rho\nabla A + A\nabla\rho, \nabla f\rangle\big)\, g\, dx \\
&= -\int_{\mathbb{R}^d} \Bigg(\frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 f) + \langle \underbrace{\frac{1}{2\psi}\nabla A + \frac{A\nabla\rho}{2\rho\psi}}_{=\beta^{\rho,A,\psi}}, \nabla f\rangle\Bigg) g\, d\mu.
\end{aligned}$$

Thus $f \in D(L^0)$. This implies $C_0^\infty(\mathbb{R}^d) \subset D(L^0)$ and

$$L^0 f = \frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 f) + \langle \beta^{\rho,A,\psi}, \nabla f\rangle \in L^2(\mathbb{R}^d,\mu).$$

Let $(T_t^0)_{t>0}$ be the sub-Markovian C_0-semigroup of contractions on $L^2(\mathbb{R}^d,\mu)$ associated with $(L^0, D(L^0))$. By Proposition 8.4.1, $T_t^0|_{L^1(\mathbb{R}^d,\mu)\cap L^\infty(\mathbb{R}^d,\mu)}$ can be uniquely extended to a sub-Markovian C_0-semigroup of contractions $(\overline{T_t^0})_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$.
Now let $\mathbf{B}\in L^2_{loc}(\mathbb{R}^d,\mathbb{R}^d,\mu)$ be weakly divergence free with respect to μ, i.e.

$$\int_{\mathbb{R}^d}\langle \mathbf{B},\nabla u\rangle d\mu = 0, \qquad \text{for all } u\in C_0^\infty(\mathbb{R}^d). \tag{8.3}$$

Moreover assume

$$\rho\psi\mathbf{B}\in L^2_{loc}(\mathbb{R}^d,\mathbb{R}^d). \tag{8.4}$$

Then using Lemma 8.4.3, (8.3) can be extended to all $u\in \widehat{H}_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}$ and

$$\int_{\mathbb{R}^d}\langle \mathbf{B},\nabla u\rangle v d\mu = -\int_{\mathbb{R}^d}\langle \mathbf{B},\nabla v\rangle u d\mu, \qquad \text{for all } u,v\in \widehat{H}_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}.$$

Define $Lu := L^0 u + \langle \mathbf{B},\nabla u\rangle,\ u\in D(L^0)_{0,b}$. Then $(L, D(L^0)_{0,b})$ is an extension of

$$\frac{1}{2}\text{trace}(\widehat{A}\nabla^2 u) + \langle \beta^{\rho,\psi,A}+\mathbf{B},\nabla u\rangle, \quad u\in C_0^\infty(\mathbb{R}^d).$$

For any bounded open subset V of $\mathbb{R}^d$,

$$\mathcal{E}^{0,V}(f,g) := \frac{1}{2}\int_V \langle \widehat{A}\nabla f,\nabla g\rangle d\mu, \quad f,g\in C_0^\infty(V).$$

is also closable on $L^2(V,\mu)$ by [51, Subsection II.2b)]. Denote by $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$ the closure of $(\mathcal{E}^{0,V}, C_0^\infty(V))$ in $L^2(V,\mu)$. Using (8.2) and $0<\inf_V\rho\le\sup_V\rho<\infty$, it is clear that $D(\mathcal{E}^{0,V}) = \widehat{H}_0^{1,2}(V,\mu)$ since the norms $\|\cdot\|_{D(\mathcal{E}^{0,V})}$ and $\|\cdot\|_{\widehat{H}_0^{1,2}(V,\mu)}$ are equivalent. Denote by $(L^{0,V}, D(L^{0,V}))$ the generator of $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$, by $(G_\alpha^{0,V})_{\alpha>0}$ the associated sub-Markovian C_0-resolvent of contractions on $L^2(V,\mu)$, by $(T_t^{0,V})_{t>0}$ the associated sub-Markovian C_0-semigroup of contractions on $L^2(V,\mu)$ and by $(\overline{T}_t^{0,V})_{t>0}$ the unique extension of $(T_t^{0,V}|_{L^1(V,\mu)\cap L^\infty(V,\mu)})_{t>0}$ on $L^1(V,\mu)$, which is a sub-Markovian C_0-semigroup of contractions on $L^1(V,\mu)$. Let $(\overline{L}^{0,V}, D(\overline{L}^{0,V}))$ be the generator corresponding to $(\overline{T}_t^{0,V})_{t>0}$. By Proposition 8.4.1, $(\overline{L}^{0,V}, D(\overline{L}^{0,V}))$ is the closure of $(L^{0,V}, D(L^{0,V}))$ on $L^1(\mathbb{R}^d,\mu)$.

8.2 L^1-existence results

In this section, we use all notations and assumptions from Section 8.1. All ideas and techniques used here are based on [69, Chapter 1]. But the structure of the given symmetric Dirichlet form differs from that of [69] which will enable us to cover a degenerate diffusion matrix. Because of that subtle difference, we check the details one by one that the methods of [69, Chapter 1] can be adapted to our situation. The main difference between [69, Chapter 1] and what is treated here is that we consider local convergence in the space $\widehat{H}_0^{1,2}(V,\mu)$, while [69, Chapter 1] considers the space $H_0^{1,2}(V,\mu)$ where the pre-invariant density of [69, Chapter 1] does not need to be locally bounded. Since $\widehat{H}^{1,2}(V,\mu)$ is naturally included in the Sobolev space $H^{1,2}(V)$, the arguments to derive our results are at times even easier than the ones of [69, Chapter 1]. For instance, we can use the prodcut and chain rules in $\widehat{H}^{1,2}(V,\mu)$ inherited from the Sobolev space structure (see Remark 8.2.3). Moreover, assumption (8.4) will play an important role to apply the methods of [69, Chapter 1].

Lemma 8.2.1. *Let V be a bounded open subset of $\mathbb{R}^d$. Then*

(i) $D(\overline{L}^{0,V})_b \subset \widehat{H}_0^{1,2}(V,\mu)$.

(ii) $\lim_{t\to 0+} T_t^{0,V} u = u$ *in* $\widehat{H}_0^{1,2}(V,\mu)$ *for all* $u \in D(\overline{L}^{0,V})_b$.

(iii) $\mathcal{E}^0(u,v) = -\int_V \overline{L}^{0,V} u\, v\, d\mu$ *for all* $u \in D(\overline{L}^{0,V})_b$, $v \in \widehat{H}_0^{1,2}(V,\mu)_b$.

(iv) Let $\varphi \in C^2(\mathbb{R}^d)$, $\varphi(0)=0$, *and* $u \in D(\overline{L}^{0,V})_b$. *Then* $\varphi(u) \in D(\overline{L}^{0,V})_b$ *and*

$$\overline{L}^{0,V}\varphi(u) = \varphi'(u)\overline{L}^{0,V}u + \frac{1}{2}\varphi''(u)\langle \widehat{A}\nabla u, \nabla u\rangle.$$

Proof Let $u \in D(\overline{L}^{0,V})_b$. Since $(T_t^{0,V})_{t>0}$ is an analytic semigroup on $L^2(V,\mu)$, we get

$$\overline{T}_t^{0,V} u = T_t^{0,V} u \in D(L^{0,V}) \quad \text{for all } t>0,$$

hence by Proposition 8.4.1,

$$L^{0,V}T_t^{0,V}u = L^{0,V}\overline{T}_t^{0,V}u = \overline{L}^{0,V}\overline{T}_t^{0,V}u = \overline{T}_t^{0,V}\overline{L}^{0,V}u.$$

Therefore

$$\begin{aligned}
&\mathcal{E}^{0,V}\left(T_t^{0,V}u - T_s^{0,V}u, T_t^{0,V}u - T_s^{0,V}u\right)\\
&\qquad = -\int_V L^{0,V}\left(T_t^{0,V}u - T_s^{0,V}u\right)\cdot\left(T_t^{0,V}u - T_s^{0,V}u\right)d\mu\\
&\qquad = -\int_V \left(\overline{T}_t^{0,V}\overline{L}^{0,V}u - \overline{T}_s^{0,V}\overline{L}^{0,V}u\right)\cdot\left(T_t^{0,V}u - T_s^{0,V}u\right)d\mu\\
&\qquad \le \left\|\overline{T}_t^{0,V}\overline{L}^{0,V}u - \overline{T}_s^{0,V}\overline{L}^{0,V}u\right\|_{L^1(V,\mu)}\cdot 2\|u\|_{L^\infty(V,\mu)}\\
&\qquad\quad \longrightarrow 0 \quad \text{as } \ t,s\to 0+\,.
\end{aligned}$$

Thus $(T_t^{0,V}u)_{t>0}$ is an $\widehat{H}_0^{1,2}(V,\mu)$-Cauchy sequence as $t\to 0+$, which implies $u\in\widehat{H}_0^{1,2}(V,\mu)$ and $\lim_{t\to 0+}T_t^{0,V}u = u$ in $\widehat{H}_0^{1,2}(V,\mu)$. Thus (i), (ii) are proved.

Let $v\in\widehat{H}_0^{1,2}(V,\mu)_b$. Then

$$\begin{aligned}
\mathcal{E}^{0,V}(u,v) &= \lim_{t\to 0+}\mathcal{E}^{0,V}(T_t^{0,V}u,v) = \lim_{t\to 0+} -\int_V\left(L^{0,V}T_t^{0,V}u\right)v\,d\mu\\
&= \lim_{t\to 0+} -\int_V\left(\overline{T}_t^{0,V}\overline{L}^{0,V}u\right)\ v\,d\mu = -\int_V \overline{L}^{0,V}u\,v\,d\mu,
\end{aligned}$$

hence (iii) is proved.

(iv): Note $u\in D(\overline{L}^{0,V})_b\subset\widehat{H}_0^{1,2}(V,\mu)_b$. Set $u_n := nG_n^{0,V}u$, $M := \|u\|_{L^\infty(V)}$. Then $\|u_n\|_{L^\infty(V)}\le M$. By strong continuity, $\lim_{n\to\infty}u_n = u$ in $\widehat{H}_0^{1,2}(V,\mu)$ and there exists a subsequence of $(u_n)_{n\ge 1}$, say $(u_n)_{n\ge 1}$ again, such that $\lim_{n\to\infty}u_n = u$ μ-a.e. on V. Thus by Lebesgue's Theorem, $\lim_{n\to\infty}\varphi(u_n)=\varphi(u)$ in $L^2(V,\mu)$. Observe that

$$\begin{aligned}
\sup_{n\ge 1}\|\nabla\varphi(u_n)\|_{L^2(V,\mathbb{R}^d)} &= \sup_{n\ge 1}\|\varphi'(u_n)\nabla u_n\|_{L^2(V,\mathbb{R}^d)}\\
&\le \|\varphi'\|_{L^\infty([-M,M])}\sup_{n\ge 1}\|u_n\|_{\widehat{H}_0^{1,2}(V,\mu)} < \infty.
\end{aligned}$$

Thus by Banach-Alaoglu Theorem, $\varphi(u)\in\widehat{H}_0^{1,2}(V,\mu)$. Similarly, we get $\varphi'(u)\in\widehat{H}_0^{1,2}(V,\mu)$.

Let $v \in \widehat{H}_0^{1,2}(V,\mu)_b$. Then by [51, I. Corollary 4.15], $v\varphi'(u) \in \widehat{H}_0^{1,2}(V,\mu)_b$ and

$$\begin{aligned}
\mathcal{E}^{0,V}(\varphi(u),v) &= \frac{1}{2}\int_V \langle \widehat{A}\nabla\varphi(u),\nabla v\rangle d\mu \\
&= \frac{1}{2}\int_V \langle \widehat{A}\nabla u,\nabla v\rangle \varphi'(u) d\mu \\
&= \frac{1}{2}\int_V \langle \widehat{A}\nabla u,\nabla (v\varphi'(u))\rangle d\mu - \frac{1}{2}\int_V \langle \widehat{A}\nabla u,\nabla u\rangle \varphi''(u) v d\mu \\
&= -\int_V \big(\varphi'(u)\overline{L}^{0,V}u + \frac{1}{2}\varphi''(u)\langle \widehat{A}\nabla u,\nabla u\rangle\big) v\, d\mu.
\end{aligned}$$

Since $\varphi'(u)\overline{L}^{0,V}u + \frac{1}{2}\varphi''(u)\langle \widehat{A}\nabla u,\nabla u\rangle \in L^1(V,\mu)$, (iv) holds by [5, I. Lemma 4.2.2.1]. □

Recall that a densely defined operator $(L, D(L))$ on a Banach space X is called *dissipative* if for any $u \in D(L)$, there exists $l_u \in X'$ such that

$$\|l_u\|_{X'} = \|u\|_X,\ \ l_u(u) = \|u\|_X^2 \ \text{ and } \ l_u(Lu) \le 0. \tag{8.5}$$

Proposition 8.2.2. *Let V be a bounded open subset of $\mathbb{R}^d$.*

(i) The operator $(L^V, D(L^{0,V})_b)$ on $L^1(V,\mu)$ defined by

$$L^V u := L^{0,V}u + \langle \mathbf{B},\nabla u\rangle,\ \ u \in D(L^{0,V})_b,$$

is dissipative, closable on $L^1(V,\mu)$. The closure $(\overline{L}^V, D(\overline{L}^V))$ generates a sub-Markovian C_0-semigroup of contractions $(\overline{T}_t^V)_{t>0}$ on $L^1(V,\mu)$.

(ii) $D(\overline{L}^V)_b \subset \widehat{H}_0^{1,2}(V,\mu)$ and

$$\mathcal{E}^{0,V}(u,v) - \int_V \langle \mathbf{B},\nabla u\rangle\, v d\mu = \int_V \overline{L}^V u\cdot v d\mu, \ \ \textit{for all } u \in D(\overline{L}^V)_b,\ v \in \widehat{H}_0^{1,2}(V,\mu)_b. \tag{8.6}$$

Proof (i) **Step 1:** For $u \in D(L^{0,V})_b$, we have $\int_V L^V u 1_{\{u>1\}} d\mu \le 0$.
Let $\varphi_\varepsilon \in C^2(\mathbb{R})$, $\varepsilon > 0$, be such that $\varphi_\varepsilon'' \ge 0$, $0 \le \varphi_\varepsilon \le 1$ and $\varphi_\varepsilon(t) = 0$ if $t < 1$,

$\varphi'_\varepsilon(t) = 1$ if $t \geq 1+\varepsilon$. Then $\varphi_\varepsilon(u) \in D(\overline{L}^{0,V})$ by Lemma 8.2.1(iv) and

$$\begin{aligned}\int_V L^{0,V} u\,\varphi'_\varepsilon(u)d\mu &\leq \int_V L^{0,V} u\,\varphi'_\varepsilon(u)\,d\mu + \int_V \frac{1}{2}\varphi''_\varepsilon(u)\langle \widehat{A}\nabla u, \nabla u\rangle d\mu \\ &= \int_V \overline{L}^{0,V}\varphi_\varepsilon(u)d\mu \\ &= \lim_{t\to 0+}\int_V \frac{\overline{T}_t^{0,V}\varphi_\varepsilon(u) - \varphi_\varepsilon(u)}{t}d\mu \\ &\leq 0, \end{aligned} \tag{8.7}$$

where the last inequality followed by the $L^1(V,\mu)$-contraction property of $(\overline{T}_t^{0,V})_{t>0}$. Since $\lim_{\varepsilon\to 0+}\varphi'_\varepsilon(t) = 1_{(0,\infty)}(t)$ for every $t \in \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0+}\varphi'_\varepsilon(u) = 1_{\{u>1\}} \quad \mu\text{-a.e. on } V \quad \text{and} \quad \|\varphi'_\varepsilon(u)\|_{L^\infty(V)} \leq 1.$$

Thus by Lebesgue's Theorem

$$\int_V L^{0,V} u\, 1_{\{u>1\}}d\mu = \lim_{\varepsilon\to 0+}\int_V L^{0,V} u\;\varphi'_\varepsilon(u)d\mu \leq 0.$$

Similarly, since $\varphi_\varepsilon(u) \in \widehat{H}_0^{1,2}(V,\mu)$, using (8.3) we get

$$\int_V \langle \mathbf{B}, \nabla u\rangle 1_{\{u>1\}}d\mu = \lim_{\varepsilon\to 0+}\int_V \langle \mathbf{B}, \nabla u\rangle \varphi'_\varepsilon(u)d\mu = \lim_{\varepsilon\to 0+}\int_V \langle \mathbf{B}, \nabla \varphi_\varepsilon(u)\rangle d\mu = 0.$$

Therefore $\int_V L^V u 1_{\{u>1\}}d\mu \leq 0$ and Step 1 is proved. Observe that by Step 1, for any $n \geq 1$

$$\int_V \left(L^V nu\right) 1_{\{nu>1\}}d\mu \leq 0 \implies \int_V L^V u\, 1_{\{u>\frac{1}{n}\}}d\mu \leq 0.$$

Letting $n \to \infty$ it follows from Lebesgue's Theorem that $\int_V L^V u\, 1_{\{u>0\}}d\mu \leq 0$. Replacing u with $-u$, we have

$$-\int_V L^V u\, 1_{\{u<0\}}d\mu = \int_V L^V(-u)\, 1_{\{-u>0\}}d\mu \leq 0,$$

hence

$$\int_V L^V u\,(1_{\{u>1\}} - 1_{\{u<0\}})d\mu \le 0.$$

Setting $l_u := \|u\|_{L^1(V,\mu)}(1_{\{u>1\}} - 1_{\{u<0\}}) \in L^\infty(V,\mu) = (L^1(V,\mu))'$, (8.5) is satisfied. Since $(L^{0,V}, D(L^{0,V})_b)$ is densely defined on $L^1(V,\mu)$ becasue $C_0^\infty(V) \subset D(L^{0,V})_b$, $(L^{0,V}, D(L^{0,V})_b)$ is dissipative.

Step 2: We have $(1-L^V)(D(L^{0,V})_b) \subset L^1(V,\mu)$ densely.
Let $h \in L^\infty(V,\mu) = (L^1(V,\mu))'$ be such that $\int_V (1-L^V)u\,h d\mu = 0$ for all $u \in D(L^{0,V})_b$. Then $u \mapsto \int_V (1-L^{0,V})u\,h\,d\mu$ is continuous with respect to the norm on $\widehat{H}_0^{1,2}(V,\mu)$ since

$$\begin{aligned}\left|\int_V (1-L^{0,V}u)u\,hd\mu\right| &= \left|\int_V \langle \rho\psi\mathbf{B}, \nabla u\rangle h\,dx\right| \\ &\le \|h\|_{L^\infty(V)}\|\rho\psi\mathbf{B}\|_{L^2(V,\mathbb{R}^d)}\|\nabla u\|_{L^2(V,\mathbb{R}^d)} \\ &\le \|h\|_{L^\infty(V)}\|\rho\psi\mathbf{B}\|_{L^2(V,\mathbb{R}^d)}\|u\|_{\widehat{H}_0^{1,2}(V,\mu)}.\end{aligned}$$

Thus, by the Riesz representation Theorem, there exists $v \in \widehat{H}_0^{1,2}(V,\mu)$ such that

$$\mathcal{E}_1^{0,V}(u,v) = \int_V (1-L^{0,V})u\cdot hd\mu \quad \text{for all } u \in D(L^{0,V})_b,$$

which implies that

$$\int_V (1-L^{0,V})u\cdot(h-v)d\mu = 0 \quad \text{for all } u \in D(L^{0,V})_b. \tag{8.8}$$

Since $(L^{0,V}, D(L^{0,V}))$ generates a sub-Markovian resolvent in $L^2(V,\mu)$,

$$L^1(V,\mu)\cap L^\infty(V,\mu) \subset (1-L^{0,V})(D(L^{0,V})_b),$$

hence $(1-L^{0,V})(D(L^{0,V})_b) \subset L^1(V,\mu)$ densely. Therefore (8.8) implies $h-v=0$. In particular, $h \in \widehat{H}_0^{1,2}(V,\mu)$ and

$$\begin{aligned}\mathcal{E}_1^{0,V}(h,h) &= \lim_{\alpha\to\infty} \mathcal{E}_1^{0,V}(\alpha G_\alpha^{0,V} h, h) \\ &= \lim_{\alpha\to\infty} \int_V (1-L^{0,V})(\alpha G_\alpha^{0,V} h)\, h d\mu \\ &= \lim_{\alpha\to\infty} \int_V \Big\langle \rho\psi\mathbf{B}, \nabla(\alpha G_\alpha h)\Big\rangle h dx \\ &= \int_V \Big\langle \rho\psi\mathbf{B}, \nabla h\Big\rangle h dx = \frac{1}{2}\int_V \Big\langle \mathbf{B}, \nabla h^2\Big\rangle d\mu = 0,\end{aligned}$$

therefore $h=0$. Then applying the Hahn-Banach Theorem [14, Proposition 1.9], Step 2 is proved. By the Lumer-Phillips Theorem [45, Theorem 3.1], the closure $(\overline{L}^V, D(\overline{L}^V))$ of $(L^V, D(L^{0,V})_b)$ generates a contraction C_0-semigroup $(\overline{T}_t^V)_{t>0}$ on $L^1(V,\mu)$.

Step 3: $(\overline{T}_t^V)_{t>0}$ is sub-Markovian.
Let $(\overline{G}_\alpha^V)_{\alpha>0}$ be the associated resolvent. It is enough to show that $(\overline{G}_\alpha^V)_{\alpha>0}$ is sub-Markovian since $T_t^V u = \lim_{\alpha\to\infty} \exp\Big(t\alpha(\alpha\overline{G}_\alpha^V u - u)\Big)$ in $L^1(V,\mu)$ by the proof of Hille-Yoshida (cf. [51, I. Theorem 1.12]). Observe that by construction

$$D(L^{0,V})_b \subset D(\overline{L}^V) \text{ densely with respect to the graph norm } \|\cdot\|_{D(\overline{L}^V)}.$$

Let $u \in D(\overline{L}^V)$ and take $u_n \in D(\overline{L}^{0,V})_b$ satisfying $\lim_{n\to\infty} u_n = u$ in $D(\overline{L}^V)$ and $\lim_{n\to\infty} u_n = u$, μ a.e. on V. Let $\varepsilon > 0$ and φ_ε be as in Step 1. Then by (8.7)

$$\int_V \overline{L}^V u\, 1_{\{u>1\}} d\mu = \lim_{\varepsilon\to 0+} \int_V \overline{L}^V u\, \varphi_\varepsilon'(u) d\mu \le 0.$$

Let $f \in L^1(V,\mu)$ and $u := \alpha\overline{G}_\alpha^V f \in D(\overline{L}^V)$. If $f \le 1$, then

$$\alpha\int_V u 1_{\{u>1\}} d\mu \le \int_V (\alpha u - \overline{L}^V u) 1_{\{u>1\}} d\mu = \alpha\int_V f 1_{\{u>1\}} d\mu \le \alpha\int_V 1_{\{u>1\}} d\mu.$$

Therefore, $\alpha\int_V (u-1)1_{\{u>1\}} d\mu \le 0$, which implies $u \le 1$. If $f \ge 0$, then $-nf \le 1$ for all $n \in \mathbb{N}$, hence $-nu \le 1$ for all $n \in \mathbb{N}$. Thus $u \ge 0$. Therefore $(\overline{G}_\alpha^V)_{\alpha>0}$ is sub-Markovian.

(ii) **Step 1:** It holds $D(\overline{L}^{0,V})_b \subset D(\overline{L}^{V})$ and $\overline{L}^{V} u = \overline{L}^{0,V} u + \langle \mathbf{B}, \nabla u \rangle$, $u \in D(\overline{L}^{0,V})_b$. Let $u \in D(\overline{L}^{0,V})_b$. Since $(T_t^{0,V})_{t>0}$ is an analytic semigroup, $T_t^{0,V} u \in D(L^{0,V})_b \subset D(\overline{L}^{V})$ and $\overline{L}^{V} T_t^{0,V} u = L^{0,V} T_t^{0,V} u + \langle \mathbf{B}, \nabla T_t^{0,V} u \rangle = \overline{T}_t^{0,V} \overline{L}^{0,V} u + \langle \mathbf{B}, \nabla T_t^{0,V} u \rangle$. By Lemma 8.2.1(ii), $\lim_{t \to 0+} T_t^{0,V} u = u$ in $\widehat{H}_0^{1,2}(V, \mu)$, which implies that

$$\lim_{t \to 0+} \overline{L}^{V} T_t^{0,V} u = \overline{L}^{0,V} u + \langle \mathbf{B}, \nabla u \rangle \quad \text{ in } \ L^1(V, \mu),$$

by (8.4). Since $\lim_{t \to 0+} T_t^{0,V} u = \lim_{t \to 0+} \overline{T}_t^{0,V} u = u$ in $L^1(V, \mu)$ and $(\overline{L}^{V}, D(\overline{L}^{V}))$ is a closed operator on $L^1(V, \mu)$, we obtain

$$u \in D(\overline{L}^{V}), \quad \overline{L}^{V} u = \overline{L}^{0,V} u + \langle \mathbf{B}, \nabla u \rangle.$$

Step 2: Let $u \in D(\overline{L}^{V})_b$ and take $u_n \in D(L^{0,V})_b$ as in Step 3 of the proof of Proposition 8.2.2(i). Let $M_1, M_2 > 0$ be such that $\|u\|_{L^\infty(V)} < M_1 < M_2$. Then

$$\lim_{n \to \infty} \int_{\{M_1 \le |u_n| \le M_2\}} \langle \widehat{A} \nabla u_n, \nabla u_n \rangle d\mu = 0. \tag{8.9}$$

Indeed, let $\varphi \in C^1(\mathbb{R})$ be such that $\varphi'(t) := (t - M_1)^+ \wedge (M_2 - M_1)$ with $\varphi(0) = 0$. Then by Lemma 8.2.1(i) (iv), we have $\varphi(u_n) \in \widehat{H}_0^{1,2}(V, \mu)$. Observe that $\varphi'(u) = 0$, μ-a.e. on V and

$$\begin{aligned}
&\int_{\{M_1 \le u_n \le M_2\}} \varphi''(u_n) \langle \widehat{A} \nabla u_n, \nabla u_n \rangle d\mu = \int_V \big\langle \widehat{A} \nabla u_n, \nabla \varphi'(u_n) \big\rangle d\mu \\
&= \mathcal{E}^{0,V}(u_n, \varphi'(u_n)) = -\int_V L^{0,V} u_n \, \varphi'(u_n) d\mu \\
&= -\int_V L^{0,V} u_n \, \varphi'(u_n) d\mu - \int_V \langle \mathbf{B}, \nabla \varphi(u_n) \rangle d\mu \\
&= -\int_V L^{V} u_n \, \varphi'(u_n) d\mu \longrightarrow -\int_V \overline{L}^{V} u \, \varphi'(u) d\mu = 0, \quad \text{ as } n \to \infty,
\end{aligned}$$

where the convergence of the last limit holds by Lebesgue's Theorem, since

$$\lim_{n\to\infty} \varphi'(u_n) = \varphi'(u) = 0, \quad \mu\text{-a.e. on } \mathbb{R}^d$$

and

$$\begin{aligned}
&\left| \int_V L^V u_n \cdot \varphi'(u_n) d\mu - \int_V \overline{L}^V u \cdot \varphi'(u) d\mu \right| \\
\le\ & \|\varphi'\|_{L^\infty(V)} \int_V \left| (L^V u_n - \overline{L}^V u) \right| d\mu + \int_V |\overline{L}^V u| \cdot |\varphi'(u_n) - \varphi'(u)| d\mu \\
& \longrightarrow 0 \ \text{ as } n \to \infty.
\end{aligned}$$

Similarly,

$$\int_{\{-M_2 \le u_n \le -M_1\}} \langle \widehat{A} \nabla u_n, \nabla u_n \rangle d\mu = \int_{\{M_1 \le -u_n \le M_2\}} \langle \widehat{A} \nabla(-u_n), \nabla(-u_n) \rangle d\mu = 0,$$

hence (8.9) is proved.

Step 3: Let u, u_n, $n \ge 1$ be as in Step 2. Let $\varphi \in C_0^2(\mathbb{R})$ be such that $\varphi(t) = t$ if $|t| < \|u\|_{L^\infty(V)} + 1$ and $\varphi(t) = 0$ if $|t| \ge \|u\|_{L^\infty(V)} + 2$. Using Step 2 and Lebesgue's Theorem

$$\overline{L}^V \varphi(u_n) = \varphi'(u_n) L^V u_n + \varphi''(u_n) \langle A \nabla u_n, \nabla u_n \rangle \longrightarrow \overline{L}^V u \quad \text{in } L^1(V, \mu) \ \text{ as } n \to \infty.$$

Therefore

$$\begin{aligned}
&\mathcal{E}^{0,V}(\varphi(u_n) - \varphi(u_m), \varphi(u_n) - \varphi(u_m)) \\
&\qquad = -\int_V \overline{L}^V \big(\varphi(u_n) - \varphi(u_m)\big) \cdot \big(\varphi(u_n) - \varphi(u_m)\big) d\mu \\
&\qquad \le 2 \|\varphi\|_{L^\infty(\mathbb{R}^d)} \|\overline{L}^V \varphi(u_n) - \overline{L}^V \varphi(u_m)\|_{L^1(V,\mu)} \longrightarrow 0 \quad \text{as } \ n, m \to \infty.
\end{aligned}$$

Thus $\lim_{n\to\infty} \varphi(u_n) = u$ in $\widehat{H}_0^{1,2}(V, \mu)$ by the completeness of $\widehat{H}_0^{1,2}(V, \mu)$. Then using

(8.4), for any $v \in \widehat{H}_0^{1,2}(V,\mu)_b$,

$$\begin{aligned}\mathcal{E}^{0,V}(u,v) - \int_V \langle \mathbf{B}, \nabla u\rangle\, v d\mu &= \lim_{n\to\infty} (\mathcal{E}^{0,V}\,(\varphi(u_n), v) - \int_V \langle \mathbf{B}, \nabla \varphi(u_n)\rangle d\mu) \\ &= -\lim_{n\to\infty} \int_V \overline{L}^V \varphi(u_n)\cdot v d\mu = -\int_V \overline{L}^V u \cdot v d\mu,\end{aligned}$$

which completes the proof of (ii).

□

Remark 8.2.3. *One can generalize the assumptions of Proposition 8.2.2 to more general positive functions ρ as follows. Consider ψ as in Section 8.1 and assume $\phi \in H^{1,2}_{loc}(\mathbb{R}^d)$ with $\phi > 0$ a.e. on $\mathbb{R}^d$ and let $\rho := \phi^2$, $\mu := \rho\psi dx$. Let $A = (a_{ij})_{1\le i,j\le d}$ be a symmetric matrix of functions that is locally uniformly strictly elliptic on $\mathbb{R}^d$ and $a_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d, \rho dx)$ for all $1 \le i,j \le d$. Assume $\mathbf{B}$ satisfies* (8.3) *and $\psi\mathbf{B} \in L^2_{loc}(\mathbb{R}^d, \rho dx)$. Let $(\mathcal{E}^0, D(\mathcal{E}^0))$, $(L^0, D(L^0))$ be defined in the same manner as in Section 8.1. For an open set U in $\mathbb{R}^d$, define $\widehat{H}^{1,2}_{0,\rho}(U,\mu)$ as the closure of $C_0^\infty(U)$ in $L^2(U,\mu)$ with respect to the norm $\left(\int_U \|\nabla u\|^2 \rho dx + \int_U u^2 d\mu\right)^{1/2}$. Then replacing $\widehat{H}_0^{1,2}(U,\mu)$ with $\widehat{H}^{1,2}_{0,\rho}(U,\mu)$, one may obtain the same results as in Lemma 8.2.1 and Propsotion 8.2.2. Especially, if $\psi \equiv 1$, it reduces to the framework of [69]. But considering a future goal in Theorem 8.3.1, we obtain $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{1-d/p}_{loc}(\mathbb{R}^d)$ with $\rho(x) > 0$ for all $x \in \mathbb{R}^d$, which in particular means that $\rho \in L^\infty_{loc}(\mathbb{R}^d)$ and $\frac{1}{\rho} \in L^\infty_{loc}(\mathbb{R}^d)$. In view of the latter, we maintain our present assumptions in Section 8.1 because it makes the arguments in the proofs simple.*

Remark 8.2.4. *Let V be a bounded open subset of $\mathbb{R}^d$. Define*

$$L^{*V} u := L^{0,V} + \langle -\mathbf{B}, \nabla u\rangle, \quad u \in D(L^{0,V})_b.$$

Note that $-\mathbf{B}$ has the same structural properties as $\mathbf{B}$ since (8.3) *and* (8.4) *hold. Thus Proposition 8.2.2 holds equally with $\mathbf{B}$ replaced by $-\mathbf{B}$. In particular, there exists a closed extension $(\overline{L}^{*V}, D(\overline{L}^{*V}))$ of $(L^{*V}, D(L^{0,V})_b)$ on $L^1(V,\mu)$, which generates a sub-Markovian C_0-resolvent of contractions $(\overline{G}^{*V}_\alpha)$ on $L^1(V,\mu)$ and*

$$\mathcal{E}^{0,V}(u,v) + \int_V \langle \mathbf{B}, \nabla u\rangle v d\mu = -\int_V \overline{L}^{*V} u\, v d\mu, \quad u \in D(\overline{L}^{*V})_b,\ v \in \widehat{H}_0^{1,2}(V,\mu).$$

Let $(L^{*V}, D(L^{*V}))$ be the part of $(\overline{L}^{*V}, D(\overline{L}^{*V}))$ on $L^2(V,\mu)$ and $(L^V, D(L^V))$ be the part of $(\overline{L}^V, D(\overline{L}^V))$ on $L^2(V,\mu)$. Then for any $u \in D(L^V)_b$, $v \in D(L^{*V})_b$

$$\begin{aligned} -\int_V L^V u \cdot v d\mu &= \mathcal{E}^{0,V}(u,v) - \int_V \langle \mathbf{B}, \nabla u \rangle v d\mu \\ &= \mathcal{E}^{0,V}(v,u) + \int_V \langle \mathbf{B}, \nabla v \rangle u d\mu \\ &= -\int_V L^{*V} v \cdot u d\mu \end{aligned} \tag{8.10}$$

Let $(G^V_\alpha)_{\alpha>0}$ and $(G^{*V}_\alpha)_{\alpha>0}$ be the resolvent associated to $(L^V, D(L^V))$, $(L^{*V}, D(L^{*V}))$ on $L^2(V,\mu)$, respectively. Then for any $f, g \in L^2(V,\mu) \cap L^\infty(V,\mu)$,

$$\begin{aligned} \int_V G^V_\alpha f \cdot g d\mu &= \int_V G^V_\alpha f \cdot (\alpha - L^{*V}) G^{*V}_\alpha g \, d\mu \\ &\underset{(8.10)}{=} \int_V (\alpha - L^V) G^V_\alpha f \cdot G^{*V}_\alpha g \, d\mu \\ &= \int_V f \cdot G^{*V}_\alpha g \, d\mu. \end{aligned} \tag{8.11}$$

By denseness of $L^2(V,\mu) \cap L^\infty(V,\mu)$ in $L^2(V,\mu)$, (8.11) extends to all $f, g \in L^2(V,\mu)$. Thus for each $\alpha > 0$, G^{*V}_α is the adjoint operator of G^V_α on $L^2(V,\mu)$.

Now let V be a bounded open subset of $\mathbb{R}^d$. Denote by $(\overline{G}^V_\alpha)_{\alpha>0}$ the resolvent associated with $(\overline{L}^V, D(\overline{L}^V))$ on $L^1(V,\mu)$. Then $(\overline{G}^V_\alpha)_{\alpha>0}$ can be extended on $L^1(\mathbb{R}^d,\mu)$ by

$$\overline{G}^V_\alpha f := \begin{cases} \overline{G}^V_\alpha (f 1_V) & \text{on } V \\ 0 & \text{on } \mathbb{R}^d \setminus V, \end{cases} \qquad f \in L^1(\mathbb{R}^d,\mu), \tag{8.12}$$

Let $g \in L^1(\mathbb{R}^d,\mu)_b$. Then $\overline{G}^V_\alpha(g 1_V) \in D(\overline{L}^V)_b \subset \widehat{H}^{1,2}_0(V,\mu)$, hence $\overline{G}^V_\alpha g \in \widehat{H}^{1,2}_0(V,\mu)$. Note that if $u \in D(\mathcal{E}^{0,V})$, then by definition it holds $u \in D(\mathcal{E}^0)$ and $\mathcal{E}^{0,V}(u,u) = \mathcal{E}^0(u,u)$. Therefore we obtain

$$\mathcal{E}^0(\overline{G}^{V_n}_\alpha g, \overline{G}^{V_n}_\alpha g) = \mathcal{E}^{0,V_n}\big(\overline{G}^{V_n}_\alpha (g 1_{V_n}), \overline{G}^{V_n}_\alpha (g 1_{V_n})\big). \tag{8.13}$$

Lemma 8.2.5. *Let V_1, V_2 be bounded open subsets of $\mathbb{R}^d$ and $\overline{V}_1 \subset V_2$. Let $u \in L^1(\mathbb{R}^d, \mu)$, $u \geq 0$, and $\alpha > 0$. Then $\overline{G}_\alpha^{V_1} u \leq \overline{G}_\alpha^{V_2} u$.*

Proof Using the denseness in $L^1(\mathbb{R}^d, \mu)$, we may assume $u \in L^1(\mathbb{R}^d, \mu)_b$. Let $w_\alpha := \overline{G}_\alpha^{V_1} u - \overline{G}_\alpha^{V_2} u$. Then clearly $w_\alpha \in \widehat{H}_0^{1,2}(V_2, \mu)$. Observe that $w_\alpha^+ \leq \overline{G}_\alpha^{V_1} u$ on $\mathbb{R}^d$, so that $w_\alpha^+ \in \widehat{H}_0^{1,2}(V_1, \mu)$ by Lemma 8.4.4. By [21, Theorem 4.4 (iii)], we obtain

$$\int_{V_2} \langle \mathbf{B}, \nabla w_\alpha \rangle w_\alpha^+ d\mu = \int_{V_2} \langle \mathbf{B}, \nabla w_\alpha^+ \rangle w_\alpha^+ d\mu = 0. \tag{8.14}$$

Since $\mathcal{E}^{0,V_2}$ is a symmetric Dirichlet form, $\mathcal{E}^{0,V_2}(w_\alpha^-, w_\alpha^+) = \mathcal{E}^{0,V_2}(w_\alpha^+, w_\alpha^-) \leq 0$. Therefore

$$\begin{aligned}
\mathcal{E}^{0,V_2}(w_\alpha^+, w_\alpha^+) &\leq \mathcal{E}_\alpha^{0,V_2}(w_\alpha, w_\alpha^+) - \int_{V_2} \langle \mathbf{B}, \nabla w_\alpha \rangle w_\alpha^+ d\mu \\
&\leq \Big(\mathcal{E}_\alpha^{0,V_1}(\overline{G}_\alpha^{V_1} u, w_\alpha^+) - \int_{V_1} \langle \mathbf{B}, \nabla \overline{G}_\alpha^{V_1} u \rangle w_\alpha^+ d\mu \Big) \\
&\qquad - \Big(\mathcal{E}_\alpha^{0,V_2}(\overline{G}_\alpha^{V_2} u, w_\alpha^+) - \int_{V_2} \langle \mathbf{B}, \nabla \overline{G}_\alpha^{V_2} u \rangle w_\alpha^+ d\mu \Big) \\
&\leq \int_{V_1} (\alpha - \overline{L}^{V_1}) \overline{G}_\alpha^{V_1} u \, w_\alpha^+ d\mu - \int_{V_2} (\alpha - \overline{L}^{V_2}) \overline{G}_\alpha^{V_2} u \, w_\alpha^+ d\mu \\
&= \int_{V_1} u w_\alpha^+ d\mu - \int_{V_2} u w_\alpha^+ d\mu = 0.
\end{aligned}$$

Therefore $w_\alpha^+ = 0$ in $\mathbb{R}^d$, hence $\overline{G}_\alpha^{V_1} u \leq \overline{G}_\alpha^{V_2} u$ on $\mathbb{R}^d$.

□

Remark 8.2.6. *Since $w_\alpha, w_\alpha^+ \in H^{1,2}(V_2)$ in Lemma 8.2.5, we could directly get* (8.14) *using [21, Theorem 4.4 (iii)]. However, in the general situation as in Remark 8.2.3, if ρ is not bounded below by a strictly positive constant, then w_α, w_α^+ may not be contained in $H^{1,2}(V_2)$. In that case by Lemma 8.4.2, we can take a sequence $(f_n)_{n \geq 1} \subset C_0^\infty(V_2)$ such that $\sup_{n \geq 1} \|f_n\|_{L^\infty(V_2)} \leq \|w_\alpha\|_{L^\infty(V_2)}$ and*

$$\lim_{n \to \infty} f_n = w_\alpha \ \text{ in } D(\mathcal{E}^{0,V_2}), \qquad \lim_{n \to \infty} f_n^+ = w_\alpha^+ \ \text{ weakly in } D(\mathcal{E}^{0,V_2}),$$
$$\lim_{n \to \infty} f_n^+ = w_\alpha^+ \quad \mu\text{-a.e. in } V_2.$$

By [21, Theorem 4.4 (iii)]

$$\int_{V_2} \langle \mathbf{B}, \nabla f_n \rangle f_n^+ d\mu = \int_{V_2} \langle \mathbf{B}, \nabla f_n^+ \rangle f_n^+ d\mu, \tag{8.15}$$

and

$$\begin{aligned} &\left| \int_{V_2} \langle \mathbf{B}, \nabla w_\alpha^+ \rangle w_\alpha^+ d\mu - \int_{V_2} \langle \mathbf{B}, \nabla f_n^+ \rangle f_n^+ d\mu \right| \\ \leq \ &\underbrace{\left| \int_{V_2} \langle \mathbf{B}, \nabla (w_\alpha^+ - f_n^+) \rangle w_\alpha^+ d\mu \right|}_{:=I_n} + \underbrace{\left| \int_{V_2} \langle \mathbf{B}, \nabla f_n^+ \rangle (w_\alpha^+ - f_n^+) d\mu \right|}_{=:J_n}. \end{aligned}$$

Since $\lim_{n\to\infty} f_n^+ = w_\alpha^+$ *weakly in* $D(\mathcal{E}^{0,V_2})$*, we have* $\lim_{n\to\infty} I_n = 0$*. Using the Cauchy-Schwarz inequality, it holds*

$$\begin{aligned} J_n &\leq \int_{V_2} \|\mathbf{B}\| \|\nabla f_n^+\| \, |w_\alpha^+ - f_n^+| d\mu \\ &\leq \left(\int_{V_2} \|\nabla f_n^+\|^2 \, |w_\alpha^+ - f_n^+| \, \rho dx \right)^{1/2} \left(\int_{V_2} \|\psi \mathbf{B}\|^2 |w_\alpha^+ - f_n^+| \, \rho dx \right)^{1/2} \\ &\leq \sqrt{2} \|w_\alpha\|_{L^\infty(V)}^{1/2} \sup_{n\geq 1} \|f_n^+\|_{\widehat{H}^{1,2}_{0,\rho}(V_2,\mu)} \left(\int_{V_2} \|\psi \mathbf{B}\|^2 |w_\alpha^+ - f_n^+| d\mu \right)^{1/2} \\ &\quad \longrightarrow 0 \quad \text{as } n \to \infty \end{aligned}$$

by Lebesgue's Theorem. Applying the same method for the left hand side of (8.15)*, we obtain*

$$\int_{V_2} \langle \mathbf{B}, \nabla w_\alpha \rangle w_\alpha^+ d\mu = \int_{V_2} \langle \mathbf{B}, \nabla w_\alpha^+ \rangle w_\alpha^+ d\mu.$$

By means of Proposition 8.2.2, we will derive the following Theorem 8.2.7.

Theorem 8.2.7. *There exists a closed extension* $(\overline{L}, D(\overline{L}))$ *of* $Lu := L^0 u + \langle \mathbf{B}, \nabla u \rangle$*,* $u \in D(L^0)_{0,b}$ *on* $L^1(\mathbb{R}^d, \mu)$ *satisfying the following properties:*

(a) $(\overline{L}, D(\overline{L}))$ *generates a sub-Markovian* C_0*-semigroup of contractions* $(\overline{T}_t)_{t>0}$ *on* $L^1(\mathbb{R}^d, \mu)$*.*

(b) *Let $(U_n)_{n\geq 1}$ is a family of bounded open subsets of $\mathbb{R}^d$ satisfying $\overline{U}_n \subset U_{n+1}$ and $\mathbb{R}^d = \bigcup_{n\geq 1} U_n$. Then $\lim_{n\to\infty} \overline{G}_\alpha^{U_n} f = (\alpha - \overline{L})^{-1} f$ in $L^1(\mathbb{R}^d, \mu)$, for all $f \in L^1(\mathbb{R}^d, \mu)$ and $\alpha > 0$.*

(c) *$D(\overline{L})_b \subset D(\mathcal{E}^0)$ and for all $u \in D(\overline{L})_b$, $v \in \widehat{H}_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$ it holds*

$$\begin{aligned} \mathcal{E}^0(u,u) &\leq -\int_{\mathbb{R}^d} \overline{L}u \cdot u d\mu, \\ \mathcal{E}^0(u,v) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u \rangle v d\mu &= -\int_{\mathbb{R}^d} \overline{L}u \cdot v d\mu. \end{aligned}$$

Proof Let $f \in L^1(\mathbb{R}^d, \mu)$ with $f \geq 0$. Let $(V_n)_{n\geq 1}$ be a family of bounded open subsets of $\mathbb{R}^d$ satisfying $\overline{V}_n \subset V_{n+1}$ for all $n \in \mathbb{N}$. Using Lemma 8.2.5, we can define for any $\alpha > 0$

$$\overline{G}_\alpha f := \lim_{n\to\infty} \overline{G}_\alpha^{V_n} f \quad \mu\text{-a.e. on } \mathbb{R}^d.$$

Using the L^1-contraction property, $\int_{\mathbb{R}^d} \alpha \overline{G}_\alpha^{V_n} f d\mu = \int_{V_n} \alpha \overline{G}_\alpha^{V_n} (f 1_{V_n}) d\mu \leq \int_{V_n} f d\mu \leq \int_{\mathbb{R}^d} f d\mu$. Thus by monotone integration, $\overline{G}_\alpha f \in L^1(\mathbb{R}^d, \mu)$ with

$$\int_{\mathbb{R}^d} \alpha \overline{G}_\alpha f d\mu \leq \int_{\mathbb{R}^d} f d\mu,$$

and by Lebesgue's Theorem, $\lim_{n\to\infty} \overline{G}_\alpha^{V_n} f = \overline{G}_\alpha f$ in $L^1(\mathbb{R}^d, \mu)$.
For any $f \in L^1(\mathbb{R}^d, \mu)$, define $\overline{G}_\alpha f := \overline{G}_\alpha f^+ - \overline{G}_\alpha f^-$. Then $\alpha \overline{G}_\alpha$ is a contraction on $L^1(\mathbb{R}^d, \mu)$, since

$$\int_{\mathbb{R}^d} |\alpha \overline{G}_\alpha f| d\mu \leq \int_{\mathbb{R}^d} \alpha \overline{G}_\alpha f^+ + \alpha \overline{G}_\alpha f^- d\mu \leq \int_{\mathbb{R}^d} f^+ d\mu + \int_{\mathbb{R}^d} f^- d\mu = \int_{\mathbb{R}^d} |f| d\mu.$$

Thus

$$\lim_{n\to\infty} \overline{G}_\alpha^{V_n} f = \overline{G}_\alpha f \text{ in } L^1(\mathbb{R}^d, \mu), \quad \lim_{n\to\infty} \overline{G}_\alpha^{V_n} f = \overline{G}_\alpha f \quad \mu\text{-a.e. on } \mathbb{R}^d.$$

Clearly, $(\overline{G}_\alpha)_{\alpha>0}$ is sub-Markovian, since $(\overline{G}_\alpha^{V_n})_{\alpha>0}$ is sub-Markovian on $L^1(V_n, \mu)$ for any $n \geq 1$. By the $L^1(\mathbb{R}^d,\mu)$-contraction property, for any $\alpha, \beta > 0$

$$\lim_{n\to\infty} \|\overline{G}_\alpha^{V_n}\overline{G}_\beta f - \overline{G}_\alpha^{V_n}\overline{G}_\beta^{V_n} f\|_{L^1(\mathbb{R}^d,\mu)} \leq \lim_{n\to\infty} \frac{1}{\alpha}\|\overline{G}_\beta f - \overline{G}_\beta^{V_n} f\|_{L^1(\mathbb{R}^d,\mu)} = 0. \tag{8.16}$$

Using (8.16) and the resolvent equation for $(\overline{G}_\alpha^{V_n})_{\alpha>0}$, we obtain for any $\alpha, \beta > 0$

$$\begin{aligned}(\beta-\alpha)\overline{G}_\alpha\overline{G}_\beta f &= \lim_{n\to\infty}(\beta-\alpha)\overline{G}_\alpha^{V_n}\overline{G}_\beta f = \lim_{n\to\infty}(\beta-\alpha)\overline{G}_\alpha^{V_n}\overline{G}_\beta^{V_n} f \\ &= \lim_{n\to\infty} \overline{G}_\alpha^{V_n} f - \overline{G}_\beta^{V_n} f = \overline{G}_\alpha f - \overline{G}_\beta f \quad \text{ in } L^1(\mathbb{R}^d,\mu).\end{aligned}$$

Let $f \in L^1(\mathbb{R}^d,\mu)_b$ and $\alpha > 0$. By (8.6), $\overline{G}_\alpha^{V_n}(f1_{V_n}) \in D(\overline{L}^V)_b \subset \widehat{H}_0^{1,2}(V_n,\mu)_b$. Using (8.13),

$$\begin{aligned}\mathcal{E}_\alpha^0(\overline{G}_\alpha^{V_n} f, \overline{G}_\alpha^{V_n} f) &= \mathcal{E}_\alpha^{0,V_n}\big(\overline{G}_\alpha^{V_n}(f1_{V_n}), \overline{G}_\alpha^{V_n}(f1_{V_n})\big) \\ &= -\int_{V_n} \overline{L}^{V_n}\overline{G}_\alpha^{V_n}(f1_{V_n})\cdot\overline{G}_\alpha^{V_n}(f1_{V_n})d\mu + \int_{V_n}\alpha\overline{G}_\alpha^{V_n}(f1_{V_n})\cdot\overline{G}_\alpha^{V_n}(f1_{V_n})d\mu \\ &= \int_{V_n}(f1_{V_n})\cdot\overline{G}_\alpha^{V_n}(f1_{V_n})d\mu \\ &\leq \int_{\mathbb{R}^d} f\cdot\overline{G}_\alpha f d\mu \qquad\qquad (8.17)\\ &\leq \frac{1}{\alpha}\|f\|_{L^\infty(\mathbb{R}^d,\mu)}\|f\|_{L^1(\mathbb{R}^d,\mu)}.\end{aligned}$$

Observe that $\lim_{n\to\infty}\overline{G}_\alpha^{V_n} f = \overline{G}_\alpha f$ in $L^2(\mathbb{R}^d,\mu)$ by Lebesgue's Theorem. Thus by the Banach-Alaoglu Theorem, $\overline{G}_\alpha f \in D(\mathcal{E}^0)$ and there exists subsequence of $(\overline{G}_\alpha^{V_n} f)_{n\geq 1}$, say again $(\overline{G}_\alpha^{V_n} f)_{n\geq1}$, such that

$$\lim_{n\to\infty}\overline{G}_\alpha^{V_n} f = \overline{G}_\alpha f \quad \text{ weakly in } D(\mathcal{E}^0). \tag{8.18}$$

Using the property of weak convergence and (8.17)

$$\mathcal{E}_\alpha^0(\overline{G}_\alpha f, \overline{G}_\alpha f) \leq \liminf_{n\to\infty}\mathcal{E}_\alpha^0(\overline{G}_\alpha^{V_n} f, \overline{G}_\alpha^{V_n} f) \leq \int_{\mathbb{R}^d} f\overline{G}_\alpha f d\mu. \tag{8.19}$$

Let $v \in \widehat{H}_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$. Then by Lemma 8.4.3, $v \in D(\mathcal{E}^0)$. Using (8.18),

$$\begin{aligned}
&\mathcal{E}^0_\alpha(\overline{G}_\alpha f, v) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla \overline{G}_\alpha f \rangle v \, d\mu \\
&= \lim_{n\to\infty} \Big(\mathcal{E}^0_\alpha(\overline{G}_\alpha^{V_n} f, v) - \int_{\mathbb{R}^d} \langle \rho\psi \mathbf{B}, \nabla \overline{G}_\alpha^{V_n} f \rangle v \, dx \Big) \\
&= \lim_{n\to\infty} \Big(\mathcal{E}^{0,V_n}_\alpha \big(\overline{G}_\alpha^{V_n}(f 1_{V_n}), v\big) - \int_{V_n} \big\langle \mathbf{B}, \nabla \overline{G}_\alpha^{V_n}(f 1_{V_n}) \big\rangle \, v \, d\mu \Big) \\
&= \lim_{n\to\infty} \int_{V_n} (\alpha - \overline{L}^{V_n}) \overline{G}_\alpha^{V_n}(f 1_{V_n}) \cdot v d\mu = \lim_{n\to\infty} \int_{V_n} f v d\mu = \int_{\mathbb{R}^d} f v d\mu. \qquad (8.20)
\end{aligned}$$

Let $u \in D(L^0)_{0,b}$ be given and take $j \in \mathbb{N}$ satisfying $\operatorname{supp} u \subset V_j$. Then by Lemma 8.4.3, $u \in \widehat{H}_0^{1,2}(V_j, \mu)$. Observe that $\operatorname{supp}(Lu) \subset V_j$ and for any $n \geq j$, $u 1_{V_n} \in D(L^{0,V_n})_b$, $L^{V_n}(u 1_{V_n}) = Lu$ on V_n, hence $\overline{G}_\alpha^{V_n}(\alpha - L)u = u$ on $\mathbb{R}^d$. Letting $n \to \infty$ we have

$$u = \overline{G}_\alpha(\alpha - L)u. \qquad (8.21)$$

Note that

$$\begin{aligned}
\|\alpha \overline{G}_\alpha u - u\|_{L^1(\mathbb{R}^d,\mu)} &= \big\|\alpha \overline{G}_\alpha u - \overline{G}_\alpha(\alpha - L)u\big\|_{L^1(\mathbb{R}^d,\mu)} \\
&= \big\|\overline{G}_\alpha L u\big\|_{L^1(\mathbb{R}^d,\mu)} \\
&\leq \frac{1}{\alpha}\|Lu\|_{L^1(\mathbb{R}^d,\mu)} \longrightarrow 0 \quad \text{as } \alpha \to \infty. \qquad (8.22)
\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^d) \subset D(L^0)_{0,b}$, (8.22) extends to all $u \in L^1(\mathbb{R}^d, \mu)$, which shows the strong continuity of $(\overline{G}_\alpha)_{\alpha>0}$ on $L^1(\mathbb{R}^d, \mu)$. Let $(\overline{L}, D(\overline{L}))$ be the generator of $(\overline{G}_\alpha)_{\alpha>0}$. Then (8.21) implies $\overline{L}u = Lu$ for all $u \in D(L^0)_{0,b}$. Thus $(\overline{L}, D(\overline{L}))$ is a closed extension of $(L, D(L^0)_{0,b})$ on $L^1(\mathbb{R}^d, \mu)$. By the Hille-Yosida Theorem, $(\overline{L}, D(\overline{L}))$ generates a C_0-semigroup of contractions $(\overline{T}_t)_{t>0}$ on $L^1(\mathbb{R}^d, \mu)$.
Since $\overline{T}_t u = \lim_{\alpha\to\infty} \exp\Big(t\alpha(\alpha \overline{G}_\alpha u - u)\Big)$ in $L^1(\mathbb{R}^d, \mu)$, $(\overline{T}_t)_{t>0}$ is also sub-Markovian, hence (a) is proved.

Next we will show (b). Let $(U_n)_{n\geq 1}$ be a family of bounded open subsets of $\mathbb{R}^d$ such that $\overline{U}_n \subset U_{n+1}$ for all $n \in \mathbb{N}$ and $\mathbb{R}^d = \bigcup_{n\geq 1} U_n$. Let $f \in L^1(\mathbb{R}^d, \mu)$ with $f \geq 0$. By the compactness of $\overline{V}_n$ in $\mathbb{R}^d$, there exists $n_0 \in \mathbb{N}$ such that $\overline{V}_n \subset U_{n_0}$, so that

$\overline{G}^{V_n} f \le \overline{G}^{U_{n_0}} f \le \lim_{n\to\infty} \overline{G}_\alpha^{U_n} f$. Letting $n \to \infty$, we obtain $\overline{G}_\alpha f \le \lim_{n\to\infty} \overline{G}_\alpha^{U_n} f$. Similarly we have $\lim_{n\to\infty} \overline{G}_\alpha^{U_n} f \le \overline{G}_\alpha f$, which shows (b).

Finally we will show (c). Let $u \in D(\overline{L})_b$ be given. Then by (8.18), $\alpha\overline{G}_\alpha u \in D(\mathcal{E}^0)$ and by (8.19)

$$\begin{aligned}
\mathcal{E}^0(\alpha\overline{G}_\alpha u, \alpha\overline{G}_\alpha u) &\le \int_{\mathbb{R}^d} \alpha u \cdot \alpha\overline{G}_\alpha u d\mu - \alpha \int_{\mathbb{R}^d} \alpha\overline{G}_\alpha u \cdot \alpha\overline{G}_\alpha u d\mu \\
&= \int_{\mathbb{R}^d} \alpha\big(u - \alpha\overline{G}_\alpha u\big) \cdot \alpha\overline{G}_\alpha u \, d\mu \\
&= \int_{\mathbb{R}^d} -\alpha\overline{L}\,\overline{G}_\alpha u \cdot \alpha\overline{G}_\alpha u \, d\mu \\
&= \int_{\mathbb{R}^d} -\alpha\overline{G}_\alpha\overline{L}u \cdot \alpha\overline{G}_\alpha u d\mu \qquad (8.23) \\
&\le \|\overline{L}u\|_{L^1(\mathbb{R}^d,\mu)} \|u\|_{L^\infty(\mathbb{R}^d,\mu)}.
\end{aligned}$$

Therefore $\sup_{\alpha>0} \mathcal{E}^0(\alpha\overline{G}_\alpha u, \alpha\overline{G}_\alpha u) < \infty$. By Banach-Alaoglu theorem, there exists a subsequence of $(\alpha\overline{G}_\alpha u)_{\alpha>0}$, say again $(\alpha\overline{G}_\alpha u)_{\alpha>0}$, such that $u \in D(\mathcal{E}^0)$ and $\lim_{\alpha\to\infty} \alpha\overline{G}_\alpha u = u$ weakly in $D(\mathcal{E}^0)$. Moreover by the property of weak convergence, (8.23) and Lebesgue's Theorem,

$$\begin{aligned}
\mathcal{E}^0(u,u) &\le \liminf_{\alpha\to\infty} \mathcal{E}^0(\alpha\overline{G}_\alpha u, \alpha\overline{G}_\alpha u) \le \liminf_{\alpha\to\infty} \left(-\int_{\mathbb{R}^d} \alpha\overline{G}_\alpha\overline{L}u \cdot \alpha\overline{G}_\alpha u d\mu \right) \\
&= -\int_{\mathbb{R}^d} \overline{L}u \, u d\mu.
\end{aligned}$$

If $v \in \widehat{H}^{1,2}(\mathbb{R}^d,\mu)_{0,b}$, then by (8.20)

$$\begin{aligned}
&\mathcal{E}^0(u,v) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u\rangle v d\mu = \lim_{\alpha\to\infty} \left(\mathcal{E}^0(\alpha\overline{G}_\alpha u, v) - \int_{\mathbb{R}^d} \langle \rho\psi\mathbf{B}, \nabla\alpha\overline{G}_\alpha u\rangle v dx \right) \\
&= \lim_{\alpha\to\infty} \left(\mathcal{E}^0_\alpha(\alpha\overline{G}_\alpha u, v) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla\alpha\overline{G}_\alpha u\rangle v d\mu - \alpha\int_{\mathbb{R}^d} \alpha\overline{G}_\alpha u \cdot v d\mu \right) \\
&= \lim_{\alpha\to\infty} \int_{\mathbb{R}^d} \alpha\Big(u - \alpha\overline{G}_\alpha u\Big) v d\mu = \lim_{\alpha\to\infty} \int_{\mathbb{R}^d} -\alpha\overline{G}_\alpha\overline{L}u \cdot v d\mu = -\int_{\mathbb{R}^d} \overline{L}u \cdot v d\mu,
\end{aligned}$$

as desired.

□

Remark 8.2.8. *In the same way as in Theorem 8.2.7, one can construct an* $L^1(\mathbb{R}^d,\mu)$ *closed extension* $(\overline{L}^*, D(\overline{L}^*))$ *of* $L^0u + \langle -\mathbf{B}, \nabla u\rangle$, $u \in D(L^0)_{0,b}$ *which generates a sub-Markovian* C_0*-resolvent of contractions* $(\overline{G}^*_\alpha)_{\alpha>0}$ *an* $L^1(\mathbb{R}^d,\mu)$*. Let* $(U_n)_{n\geq 1}$ *be as in Theorem 8.2.7(b). Observe that by Remark 8.2.4*

$$\int_{\mathbb{R}^d} \overline{G}_\alpha^{U_n} u \cdot v\, d\mu = \int_{\mathbb{R}^d} u \cdot \overline{G}_\alpha^{*\,U_n} v\, d\mu, \quad \text{for all } u, v \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu), \tag{8.24}$$

where $(\overline{G}_\alpha^{*\,U_n})_{\alpha>0}$ *is the resolvent associated to* $(\overline{L}^{*\,U_n}, D(\overline{L}^{*\,U_n}))$ *on* $L^1(U_n,\mu)$*, which is trivially extended to* $\mathbb{R}^d$ *as in* (8.12)*. Letting* $n \to \infty$ *in* (8.24),

$$\int_{\mathbb{R}^d} \overline{G}_\alpha u\, v\, d\mu = \int_{\mathbb{R}^d} u\, \overline{G}^*_\alpha v d\mu, \quad \text{for all } u, v \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu).$$

The following Theorem 8.2.9 which shows that $D(\overline{L})_b$ is an algebra is one of the ingredients to construct a Hunt process corresponding to the strict capacity (see, **SD3** in [78]). It will be used later. The proof of Theorem 8.2.9 is based on [69, Remark 1.7 (iii)], but we include its proof checking in detail some approximation arguments.

Theorem 8.2.9. $D(\overline{L})_b$ *is an algebra and* $\overline{L}u^2 = 2u\overline{L}u + \langle \widehat{A}\nabla u, \nabla u\rangle$ *for any* $u \in D(\overline{L})_b$.

Proof Let $u \in D(\overline{L})_b$. Since $D(\overline{L})_b$ is a linear space, it suffices to show $u^2 \in D(\overline{L})_b$. Let $(\overline{L}^*, D(\overline{L}^*))$, $(\overline{G}^*_\alpha)_{\alpha>0}$ be as in Remark 8.2.8 and set $g := 2u\overline{L}u + \langle \widehat{A}\nabla u, \nabla u\rangle$. If we can show

$$\int_{\mathbb{R}^d} (\overline{L}^*\, \overline{G}^*_1 h)\, u^2 d\mu = \int_{\mathbb{R}^d} g\, \overline{G}^*_1 h\, d\mu, \quad \text{for all } h \in L^1(\mathbb{R}^d,\mu)_b, \tag{8.25}$$

then

$$\begin{aligned} \int_{\mathbb{R}^d} \overline{G}_1(u^2 - g)\, h d\mu &= \int_{\mathbb{R}^d} (u^2 - g)\overline{G}^*_1 h\, d\mu \underset{(8.25)}{=} \int_{\mathbb{R}^d} u^2(\overline{G}^*_1 h - \overline{L}^*\, \overline{G}^*_1 h) d\mu \\ &= \int_{\mathbb{R}^d} u^2 h\, d\mu, \quad \text{for all } h \in L^1(\mathbb{R}^d,\mu)_b, \end{aligned}$$

hence $u^2 = \overline{G}_1(u^2 - g) \in D(\overline{L})_b$ and $\overline{L}u^2 = (1-\overline{L})\overline{G}_1(g-u^2) - \overline{G}_1(g-u^2) = g - u^2 + u^2 = g$, as desired.

Step 1: To prove (8.25), first assume $u = \overline{G}_1 f$ for some $f \in L^1(\mathbb{R}^d, \mu)_b$. Fix $v = \overline{G}_1^* h$ for some $h \in L^1(\mathbb{R}^d, \mu)_b$ with $h \geq 0$. Let $(U_n)_{n\geq 1}$ be as in Theorem 8.2.7(b) and $u_n := \overline{G}_1^{U_n} f$, $v_n := \overline{G}_1^{*\,U_n} h$. By Proposition 8.2.2 and Theorem 8.2.7

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\overline{L}^{*\,U_n} v_n)\, u u_n d\mu \\
=\;& -\mathcal{E}^0(v_n, u u_n) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla v_n \rangle u u_n d\mu, \qquad (\text{ since } v_n \in D(\mathcal{E}^0) \text{ and } u u_n \in D(\mathcal{E}^0)\ ) \\
=\;& -\frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla v_n, \nabla u \rangle u_n d\mu - \frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla v_n, \nabla u_n \rangle u d\mu + \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla (u u_n) \rangle v_n d\mu \\
=\;& -\frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla (v_n u_n), \nabla u \rangle d\mu + \frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla u_n, \nabla u \rangle v_n d\mu - \frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla v_n, \nabla u_n \rangle u d\mu \\
&\quad + \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u \rangle v_n u_n\, d\mu + \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u_n \rangle v_n u d\mu \\
=\;& -\frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla u, \nabla (v_n u_n) \rangle d\mu + \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u \rangle v_n u_n\, d\mu + \frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla u_n, \nabla u \rangle v_n d\mu \\
&\quad -\frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla u_n, \nabla (v_n u) \rangle d\mu + \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla u_n \rangle v_n u d\mu + \frac{1}{2}\int_{\mathbb{R}^d} \langle \widehat{A}\nabla u_n, \nabla u \rangle v_n d\mu \\
=\;& \int_{\mathbb{R}^d} \overline{L}u \cdot v_n u_n d\mu + \int_{\mathbb{R}^d} \overline{L}^{U_n} u_n \cdot v_n u d\mu + \int_{\mathbb{R}^d} \langle \widehat{A}\nabla u_n, \nabla u \rangle v_n d\mu. \qquad (8.26)
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \langle \widehat{A}\nabla u, \nabla u \rangle v d\mu - \int_{\mathbb{R}^d} \langle \widehat{A}\nabla u_n, \nabla u \rangle v_n d\mu \right| \\
&\leq \underbrace{\left| \int_{\mathbb{R}^d} \langle \widehat{A}\nabla (u - u_n), \nabla u \rangle v d\mu \right|}_{=:I_n} + \underbrace{\left| \int_{\mathbb{R}^d} \langle \widehat{A}\nabla u_n, \nabla u \rangle\, (v - v_n)\, d\mu \right|}_{=:J_n} \qquad (8.27)
\end{aligned}$$

Since $\lim_{n\to\infty} u_n = u$ weakly in $D(\mathcal{E}^0)$ and v is bounded on $\mathbb{R}^d$, $\lim_{n\to\infty} I_n = 0$. Note that $v_n = \overline{G}_1^{*\,U_n} h \leq \overline{G}_1^* h = v$, $\sup_{n\in\mathbb{N}} \mathcal{E}^0(u_n, u_n) < \infty$, $|v_n| \leq |v| \in L^\infty(\mathbb{R}^d, \mu)$ and

$$\lim_{n\to\infty} u_n = u \;\; \mu\text{-a.e. on } \mathbb{R}^d,$$

hence we obtain by the Cauchy–Schwarz inequality,

$$\begin{aligned} J_n &\leq \left(\int_{\mathbb{R}^d} \langle \widehat{A} \nabla u_n, \nabla u_n \rangle \, (v - v_n) \, d\mu \right)^{1/2} \left(\int_{\mathbb{R}^d} \langle \widehat{A} \nabla u, \nabla u \rangle \, (v - v_n) \, d\mu \right)^{1/2} \\ &\leq \sqrt{2} \|v\|_{L^\infty(\mathbb{R}^d,\mu)}^{1/2} \sup_{n \geq 1} \mathcal{E}^0(u_n, u_n)^{1/2} \left(\int_{\mathbb{R}^d} \langle \widehat{A} \nabla u, \nabla u \rangle \, (v - v_n) \, d\mu \right)^{1/2} \\ &\longrightarrow 0 \quad \text{ as } n \to \infty, \end{aligned}$$

where the latter convergence to zero followed by Lebesgue's Theorem for which we use

$$\left| \langle \widehat{A} \nabla u, \nabla u \rangle \, (v - v_n) \right| \leq 2 \|v\|_{L^\infty(\mathbb{R}^d,\mu)} \langle \widehat{A} \nabla u, \nabla u \rangle \in L^1(\mathbb{R}^d, \mu), \quad \mu\text{-a.e. on } \mathbb{R}^d$$

and

$$\lim_{n \to \infty} \langle \widehat{A} \nabla u, \nabla u \rangle \, (v - v_n) = 0, \quad \mu\text{-a.e. on } \mathbb{R}^d.$$

Therefore it follows by (8.27) that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \langle \widehat{A} \nabla u_n, \nabla u \rangle v_n d\mu = \int_{\mathbb{R}^d} \langle \widehat{A} \nabla u, \nabla u \rangle v d\mu. \tag{8.28}$$

By (8.26), (8.28) and Lebesgue's Theorem

$$\begin{aligned} & \int_{\mathbb{R}^d} \overline{L}^* v \cdot u^2 d\mu \\ = \;& \int_{\mathbb{R}^d} \left(\overline{G}_1^* h - h \right) u u_n d\mu \\ = \;& \lim_{n \to \infty} \int_{\mathbb{R}^d} \left(\overline{G}_1^{*\,U_n} h - h \right) u u_n \, d\mu \\ = \;& \lim_{n \to \infty} \int_{\mathbb{R}^d} (\overline{L}^{*\,U_n} v_n) u u_n d\mu \\ \underset{(8.26)}{=} \;& \lim_{n \to \infty} \int_{\mathbb{R}^d} \overline{L} u \cdot v_n u_n d\mu + \int_{\mathbb{R}^d} (\overline{G}_1^{U_n} f - f) \cdot v_n u d\mu + \int_{\mathbb{R}^d} \langle \widehat{A} \nabla u_n, \nabla u \rangle v_n d\mu \\ \underset{(8.28)}{=} \;& \int_{\mathbb{R}^d} \overline{L} u \cdot v u d\mu + \int_{\mathbb{R}^d} \overline{L} u \cdot v u d\mu + \int_{\mathbb{R}^d} \langle \widehat{A} \nabla u, \nabla u \rangle v d\mu \\ = \;& \int_{\mathbb{R}^d} g v d\mu. \end{aligned} \tag{8.29}$$

In the case of general $h \in L^1(\mathbb{R}^d, \mu)_b$, we also obtain (8.29) using $h = h^+ - h^-$ and linearity.

Step 2: Let $u \in D(\overline{L})_b$ be arbitrary. Set

$$g_\alpha := 2(\alpha\overline{G}_\alpha u)\overline{L}(\alpha\overline{G}_\alpha u) + \langle \widehat{A}\nabla\alpha\overline{G}_\alpha u, \nabla\alpha\overline{G}_\alpha u\rangle, \quad \alpha > 0.$$

By Theorem 8.2.7(c),

$$\begin{aligned}\mathcal{E}^0(\alpha\overline{G}_\alpha u - u, \alpha\overline{G}_\alpha u - u) &\le -\int_{\mathbb{R}^d} \overline{L}(\alpha\overline{G}_\alpha u - u)\cdot(\alpha\overline{G}_\alpha u - u)d\mu \\ &\le 2\|u\|_{L^\infty(\mathbb{R}^d,\mu)}\|\alpha\overline{G}_\alpha\overline{L}u - \overline{L}u\|_{L^1(\mathbb{R}^d,\mu)} \\ &\quad \longrightarrow 0 \quad \text{as } \alpha \to \infty,\end{aligned}$$

hence $\lim_{\alpha\to\infty} g_\alpha = g \quad \text{in } L^1(\mathbb{R}^d, \mu)$. Observe that by the resolvent equation

$$\overline{G}_\alpha u = \overline{G}_1\left((1-\alpha)\overline{G}_\alpha u + u\right)$$

and $(1-\alpha)\overline{G}_\alpha u + u \in L^1(\mathbb{R}^d, \mu)_b$. Replacing u in (8.29) with $\alpha\overline{G}_\alpha u$

$$\int_{\mathbb{R}^d} \overline{L}^* v\,\left(\alpha\overline{G}_\alpha u\right)^2 d\mu = \int_{\mathbb{R}^d} g_\alpha v d\mu.$$

Letting $\alpha \to \infty$, we finally obtain by Lebesgue's Theorem

$$\int_{\mathbb{R}^d} \overline{L}^* v \cdot u^2 d\mu = \int_{\mathbb{R}^d} g v d\mu,$$

so that our assertion holds.

□

8.3 Existence of a pre-invariant measure and general strong Feller properties

Here we state some conditions which will be used as our assumptions.

(A1) $p > d$ is fixed and $A = (a_{ij})_{1\le i,j\le d}$ is a symmetric matrix of functions which is locally uniformly strictly elliptic on $\mathbb{R}^d$ such that $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{0,1-d/p}_{loc}(\mathbb{R}^d)$ for all $1 \le i,j \le d$. $\psi \in L^1_{loc}(\mathbb{R}^d)$ is a positive function such that $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$ and $\mathbf{G}$ is a Borel measurable vector field on $\mathbb{R}^d$ satisfying $\psi\mathbf{G} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$.

(A2) $\psi \in L^q_{loc}(\mathbb{R}^d)$ with $q > \frac{d}{2}$. Fix $s > \frac{d}{2}$ such that $\frac{1}{q} + \frac{1}{s} < \frac{2}{d}$.

(A3) $q \ge \frac{p}{2} \vee 2$.

Theorem 8.3.1. *Under the assumption* **(A1)**, *there exists* $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{0,1-d/p}_{loc}(\mathbb{R}^d)$ *satisfying* $\rho(x) > 0$ *for all* $x \in \mathbb{R}^d$ *such that*

$$\int_{\mathbb{R}^d} \langle \mathbf{G} - \beta^{\rho,A,\psi}, \nabla\varphi\rangle \rho\psi dx = 0, \quad \textit{for all } \varphi \in C_0^\infty(\mathbb{R}^d). \tag{8.30}$$

Moreover $\rho\psi\,\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, *where* $\mathbf{B} := \mathbf{G} - \beta^{\rho,A,\psi}$.

Proof By Theorem 5.2.2, there exists $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{1-d/p}_{loc}(\mathbb{R}^d)$ satisfying $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \big\langle \frac{1}{2}A\nabla\rho + (\frac{1}{2}\nabla A - \psi\mathbf{G})\rho, \varphi\big\rangle dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d),$$

hence

$$\int_{\mathbb{R}^d} \big\langle \mathbf{G} - \frac{\nabla A}{2\psi} - \frac{A\nabla\rho}{2\rho\psi}, \nabla\varphi\big\rangle \rho\psi dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d),$$

and moreover

$$\rho\psi\,\mathbf{B} = \rho\psi\mathbf{G} - \frac{\rho}{2}\nabla A - \frac{A\nabla\rho}{2} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d).$$

□

From now on we assume that **(A1)** holds and fix A, ψ, ρ, $\mathbf{B}$ as in Theorem 8.3.1 and set as in Section 8.1 $\mu := \rho\psi\, dx$, $\widehat{A} := \frac{1}{\psi}A$, $\widehat{\rho} := \rho\psi$, $\widehat{a}_{ij} = \frac{1}{\psi}a_{ij}$ for all $1 \le i,j \le d$. Then A, ψ, ρ, $\mathbf{B}$ satisfy all assumptions of Section 8.1.

Remark 8.3.2. *If $\psi \in H^{1,2}(V) \cap L^\infty(V)$ for some bounded open set V in $\mathbb{R}^d$, then by the chain and product rules for weakly differentiable functions,*

$$\frac{1}{2}\nabla \widehat{A} = \frac{\nabla A}{2\psi} + \frac{-A\nabla\psi}{2\psi^2}, \qquad \frac{\widehat{A}\nabla\widehat{\rho}}{2\widehat{\rho}} = \frac{A\nabla\psi}{2\psi^2} + \frac{A\nabla\rho}{2\rho\psi} \quad \text{on } V.$$

Set $\beta^{\widehat{\rho},\widehat{A}} := \frac{1}{2}\nabla\widehat{A} + \frac{\widehat{A}\nabla\widehat{\rho}}{2\widehat{\rho}}$ on V. Then it holds $\beta^{\widehat{\rho},\widehat{A}} = \beta^{\rho.A,\psi}$ (a.e.) on V. If we assume $\psi \in H^{1,p}(V)$, then it holds

$$\widehat{\mathbf{F}} := \frac{1}{2}\nabla\widehat{A} + \mathbf{G} - 2\beta^{\widehat{\rho},\widehat{A}} \in L^p(V, \mathbb{R}^d).$$

By Theorem 8.2.7 there exists a closed extension $(\overline{L}, D(\overline{L}))$ of

$$Lf = L^0 f + \langle \mathbf{B}, \nabla f\rangle, \quad f \in D(L^0)_{0,b},$$

on $L^1(\mathbb{R}^d,\mu)$ which generates the sub-Markovian C_0-semigroup of contractions $(\overline{T}_t)_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$. Restricting $(\overline{T}_t)_{t>0}$ to $L^1(\mathbb{R}^d,\mu)_b$, it is well-known by Riesz-Thorin interpolation that $(\overline{T}_t)_{t>0}$ can be extended to a sub-Markovian C_0-semigroup of contractions $(T_t)_{t>0}$ on each $L^r(\mathbb{R}^d,\mu)$, $r \in [1,\infty)$. Denote by $(L_r, D(L_r))$ the corresponding closed generator with graph norm

$$\|f\|_{D(L_r)} := \|f\|_{L^r(\mathbb{R}^d,\mu)} + \|L_r f\|_{L^r(\mathbb{R}^d,\mu)},$$

and by $(G_\alpha)_{\alpha>0}$ the corresponding resolvent. Also $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ can be uniquely defined on $L^\infty(\mathbb{R}^d,\mu)$, but are no longer strongly continuous there.

For $f \in C_0^\infty(\mathbb{R}^d)$, we have

$$Lf = L^0 f + \langle \mathbf{B}, \nabla f\rangle = \frac{1}{2}\text{trace}(\widehat{A}\nabla^2 f) + \langle \mathbf{G}, \nabla f\rangle.$$

Define

$$L^* f : = L^0 f - \langle \mathbf{B}, \nabla f\rangle = \frac{1}{2}\text{trace}(\widehat{A}\nabla^2 f) + \langle \mathbf{G}^*, \nabla f\rangle,$$

with

$$\mathbf{G}^* := (g_1^*, \cdots, g_d^*) = 2\beta^{\rho,A,\psi} - \mathbf{G} = \beta^{\rho,A,\psi} - \mathbf{B} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu).$$

We see that L and L^* have the same structural properties, i.e. they are given as the sum of a symmetric second order elliptic differential operator L^0 and a divergence free first order perturbation $\langle \mathbf{B}, \nabla \cdot \rangle$ or $\langle -\mathbf{B}, \nabla \cdot \rangle$, respectively, with same integrability condition $\rho\psi\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Therefore all what will be derived below for L will hold analogously for L^*. Denote by $(L^*_r, D(L^*_r))$ the operators corresponding to L^* for the co-generator on $L^r(\mathbb{R}^d, \mu)$, $r \in [1, \infty)$, $(T^*_t)_{t>0}$ for the co-semigroup, $(G^*_\alpha)_{\alpha>0}$ for the co-resolvent. As in [69, Section 3], we obtain a corresponding bilinear form with domain $D(L_2) \times L^2(\mathbb{R}^d, \mu) \cup L^2(\mathbb{R}^d, \mu) \times D(\widehat{L}_2)$ by

$$\mathcal{E}(f, g) := \begin{cases} -\int_{\mathbb{R}^d} L_2 f \cdot g \, d\mu & \text{for } f \in D(L_2), \ g \in L^2(\mathbb{R}^d, \mu), \\ -\int_{\mathbb{R}^d} f \cdot L^*_2 g \, d\mu & \text{for } f \in L^2(\mathbb{R}^d, \mu), \ g \in D(L^*_2). \end{cases} \tag{8.31}$$

$\mathcal{E}$ is called the *generalized Dirichlet form associated with* $(L_2, D(L_2))$.

Theorem 8.3.3. *Assume* **(A1)**, **(A2)** *and let* $f \in \cup_{r\in[s,\infty]} L^r(\mathbb{R}^d, \mu)$. *Then* $G_\alpha f$ *has a locally Hölder continuous* μ*-version* $R_\alpha f$ *on* $\mathbb{R}^d$. *Furthermore for any open balls* B, B' *satisfying* $\overline{B} \subset B'$, *we have the following estimate*

$$\|R_\alpha f\|_{C^{0,\gamma}(\overline{B})} \le c_2 \left(\|f\|_{L^s(B',\mu)} + \|G_\alpha f\|_{L^1(B',\mu)} \right), \tag{8.32}$$

where $c_2 > 0$, $\gamma \in (0, 1)$ *are constants which are independent of* f.

Proof Let $f \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$. Then by Theorem 8.2.7, $G_\alpha f \in D(\overline{L})_b \subset D(\mathcal{E}^0)$ and

$$\begin{aligned} &\mathcal{E}^0(G_\alpha f, \varphi) - \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla G_\alpha f \rangle \varphi d\mu \\ &= -\int_{\mathbb{R}^d} \left(\overline{L}\,\overline{G}_\alpha f \right) \varphi \, d\mu \\ &= \int_{\mathbb{R}^d} (f - \alpha G_\alpha f) \varphi \, d\mu, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d). \end{aligned} \tag{8.33}$$

Thus (8.33) implies

$$\int_{\mathbb{R}^d} \langle \frac{1}{2}\rho A\nabla G_\alpha f, \nabla\varphi\rangle dx - \int_{\mathbb{R}^d} \langle \rho\psi\mathbf{B}, \nabla G_\alpha f\rangle \varphi\, dx + \int_{\mathbb{R}^d} (\alpha\rho\psi G_\alpha f)\,\varphi dx$$
$$= \int_{\mathbb{R}^d} (\rho\psi f)\,\varphi dx, \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d). \tag{8.34}$$

Note that ρ is locally bounded below and above on $\mathbb{R}^d$ and $\rho\psi\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $\alpha\rho\psi \in L^q_{loc}(\mathbb{R}^d)$. Let B, B' be open balls in $\mathbb{R}^d$ satisfying $\overline{B} \subset B'$. Since $\frac{1}{\psi} \in L^\infty(B')$, $G_\alpha f \in H^{1,2}(B')$. Thus by Theorem 7.2.2, there exists a Hölder continuous μ-version $R_\alpha f$ of $G_\alpha f$ on $\mathbb{R}^d$ and constants $\gamma \in (0,1)$, $c_1 > 0$, which are independent of f such that

$$\begin{aligned}\|R_\alpha f\|_{C^{0,\gamma}(\overline{B})} &\leq c_1 \left(\|G_\alpha f\|_{L^1(B')} + \|\rho\psi f\|_{L^{(\frac{1}{q}+\frac{1}{s})^{-1}}(B')} \right) \\ &\leq c_2 \left(\|G_\alpha f\|_{L^1(B',\mu)} + \|f\|_{L^s(B',\mu)} \right),\end{aligned} \tag{8.35}$$

where $c_2 := c_1 \left(\frac{1}{\inf_{B'} \rho\psi} \vee \frac{\|\rho\psi\|_{L^q(B')}}{(\inf_{B'} \rho\psi)^{1/s}} \right)$. Using the Hölder inequality and the contraction property, (8.35) extends to $f \in \cup_{r\in[s,\infty)} L^r(\mathbb{R}^d, \mu)$. In order to extend (8.35) to $f \in L^\infty(\mathbb{R}^d, m)$, let $f_n := 1_{B_n} \cdot f \in L^q(\mathbb{R}^d, \mu)_0$, $n \geq 1$. Then $\|f - f_n\|_{L^s(B',\mu)} + \|G_\alpha(f - f_n)\|_{L^1(B',m)} \to 0$ as $n \to \infty$ by Lebesgue's Theorem. Hence (8.35) also extends to $f \in L^\infty(\mathbb{R}^d, m)$.

□

Let $f \in D(L_r)$ for some $r \in [s,\infty)$. Then $f = G_1(1 - L_r)f$, hence by Theorem 8.3.3, f has a locally Hölder continuous μ-version on $\mathbb{R}^d$ and

$$\|f\|_{C^{0,\gamma}(\overline{B})} \leq c_3 \|f\|_{D(L_r)},$$

where $c_3 > 0$, $\gamma \in (0,1)$ are constants, independent of f. In particular, $T_t f \in D(L_r)$ and $T_t f$ has hence a continuous μ-version, say $P_t f$ with

$$\|P_t f\|_{C^{0,\gamma}(\overline{B})} \leq c_3 \|P_t f\|_{D(L_r)}. \tag{8.36}$$

Note that c_3 is independent of $t \geq 0$ as well as of f. The following Lemma will be quite important for later to show joint continuity of $P_\cdot g(\cdot)$ for $g \in \cup_{\nu\in[\frac{2p}{p-2},\infty]} L^\nu(\mathbb{R}^d, \mu)$.

Lemma 8.3.4. *Assume* **(A1)**, **(A2)**. *For any* $f \in \bigcup_{r\in[s,\infty)} D(L_r)$ *the map*

$$(x,t) \mapsto P_t f(x)$$

is continuous on $\mathbb{R}^d \times [0,\infty)$.

Proof Let $f \in D(L_r)$ for some $r \geq s$ and $((x_n,t_n))_{n\geq 1}$ be a sequence in $\mathbb{R}^d \times [0,\infty)$ that converges to $(x_0,t_0) \in \mathbb{R}^d \times [0,\infty)$. Note that $P_{t_0} f \in C(\mathbb{R}^d)$. Then there exists an open ball B such that $x_n \in \overline{B}$ for all $n \geq 0$ and using (8.36)

$$\begin{aligned} |P_{t_n} f(x_n) - P_{t_0} f(x_0)| &\leq |P_{t_n} f(x_n) - P_{t_0} f(x_n)| + |P_{t_0} f(x_n) - P_{t_0} f(x_0)| \\ &\leq \|P_{t_n} f - P_{t_0} f\|_{C(\overline{B})} + |P_{t_0} f(x_n) - P_{t_0} f(x_0)| \\ &\leq c_3 \|P_{t_n} f - P_{t_0} f\|_{L^r(\mathbb{R}^d,m)} + c_3 \|P_{t_n} L_r f - P_{t_0} L_r f\|_{L^r(\mathbb{R}^d,m)} \\ &\quad + |P_{t_0} f(x_n) - P_{t_0} f(x_0)| \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

Remark 8.3.5. *If* $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ *satisfies the weak sector condition, then* $(T_t)_{t>0}$ *is an analytic semigroup on* $L^r(\mathbb{R}^d,\mu)$, $r \in [2,\infty)$ *by Stein interpolation. If* $f \in D(L_r)$ *with* $r \in [2,\infty)$, *then*

$$T_t f \in D(L_r), \quad \text{and} \quad \|L_r T_t f\|_{L^r(\mathbb{R}^d,\mu)} \leq \frac{c}{t} \|f\|_{L^r(\mathbb{R}^d,\mu)},$$

where $c > 0$ *is a constant whcih is indepencent of* f *and* $t > 0$. *Thus for any* $r \in [s \vee 2, \infty)$, $t > 0$, $f \in L^r(\mathbb{R}^d,\mu)$ *and any open ball* B

$$\begin{aligned} \|P_t f\|_{C^{0,\beta}(\overline{B})} &\leq c_3 \left(\|P_t f\|_{L^r(\mathbb{R}^d,\mu)} + \|L_r P_t f\|_{L^r(\mathbb{R}^d,\mu)} \right) \\ &\leq c_3 \left(1 + \frac{c}{t}\right) \|f\|_{L^r(\mathbb{R}^d,\mu)}. \end{aligned}$$

However, it is in general difficult to show a weak sector condition and moreover it does not need to hold. Thus we have to develop another way to show the joint continuity of $P_\cdot f(\cdot)$ *where* f *is in some suitable class.*

Theorem 8.3.6. *Assume* **(A1)**, **(A2)**, **(A3)** *and let* $f \in \bigcup_{\nu\in[\frac{2p}{p-2},\infty]} L^{\nu}(\mathbb{R}^d,\mu)$, $t>0$. *Then* $T_t f$ *has a continuous* μ*-version* $P_t f$ *on* $\mathbb{R}^d$ *and furthermore* $P_{\cdot} f(\cdot)$ *is continuous on* $\mathbb{R}^d \times (0,\infty)$. *For any bounded open sets* U, V *in* $\mathbb{R}^d$ *with* $\overline{U} \subset V$ *and* $0<\tau_3<\tau_1<\tau_2<\tau_4$, *i.e.* $[\tau_1,\tau_2]\subset(\tau_3,\tau_4)$, *we have the following estimate for all* $f \in \cup_{\nu\in[\frac{2p}{p-2},\infty]} L^{\nu}(\mathbb{R}^d,\mu)$

$$\|P_{\cdot} f(\cdot)\|_{C(\overline{U}\times[\tau_1,\tau_2])} \le C_1 \|P_{\cdot} f(\cdot)\|_{L^{\frac{2p}{p-2},2}(V\times(\tau_3,\tau_4))}, \tag{8.37}$$

where C_1 *is a constant that depend on* $\overline{U}\times[\tau_1,\tau_2], V\times(\tau_3,\tau_4)$, *but is independent of* f.

Proof First assume $f \in D(\overline{L})_b \cap D(L_s) \cap D(L_2)$. By means of Lemma 8.3.4, define $u \in C_b(\mathbb{R}^d \times [0,\infty))$ by $u(x,t) := P_t f(x)$. Note that for any bounded open set $O \subset \mathbb{R}^d$ and $T>0$, it holds $u \in H^{1,2}(O\times(0,T))$ by Theorem 9.3.4 below. Let $\varphi_1 \in C_0^{\infty}(\mathbb{R}^d)$, $\varphi_2 \in C_0^{\infty}((0,T))$. Observe that $T_t f \in D(\overline{L})_b$, hence

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times(0,T)} \Big\langle \frac{1}{2}\rho A\nabla u, \nabla(\varphi_1\varphi_2)\Big\rangle - \big\langle \rho\psi\mathbf{B}, \nabla(T_t f)\big\rangle \varphi_1\varphi_2 \, dxdt \\
&= \int_0^T \varphi_2 \left(\int_{\mathbb{R}^d} \Big\langle \frac{1}{2}\rho A\nabla(T_t f), \nabla\varphi_1\Big\rangle - \big\langle \rho\psi\mathbf{B}, \nabla(T_t f)\big\rangle \varphi_1 \, dx \right) dt \\
&= \int_0^T \varphi_2 \Big(\mathcal{E}^0(T_t f, \varphi_1) - \int_{\mathbb{R}^d} \big\langle \mathbf{B}, \nabla T_t f\big\rangle \varphi_1 \, d\mu \Big) dt \\
&= \int_0^T -\varphi_2 \Big(\int_{\mathbb{R}^d} \varphi_1 \overline{L}\, \overline{T}_t f \, d\mu \Big) dt \\
&= \int_0^T -\varphi_2 \left(\frac{d}{dt} \int_{\mathbb{R}^d} \varphi_1 T_t f \; \rho\psi dx \right) dt \\
&= \int_0^T \left(\frac{d}{dt}\varphi_2 \right) \left(\int_{\mathbb{R}^d} \varphi_1 T_t f \, \rho\psi dx \right) dt \\
&= \iint_{\mathbb{R}^d\times(0,T)} u \, \partial_t\big(\varphi_1\varphi_2\big)\rho\psi dxdt.
\end{aligned} \tag{8.38}$$

By Theorem 8.4.5, (8.38) extends to

$$\iint_{\mathbb{R}^d\times(0,T)} \Big\langle \frac{1}{2}\rho A\nabla u, \nabla\varphi \Big\rangle - \big\langle \rho\psi\mathbf{B}, \nabla\left(T_t f\right) \big\rangle \varphi \, dxdt$$
$$= \iint_{\mathbb{R}^d\times(0,T)} u \, \partial_t\varphi \cdot \rho\psi dxdt \quad \text{ for all } \varphi\in C_0^\infty(\mathbb{R}^d\times(0,T)). \tag{8.39}$$

Let $\tau_2^* := \frac{\tau_2+\tau_4}{2}$ and take $r>0$ so that

$$r < \frac{\sqrt{\tau_1-\tau_3}}{2} \quad \text{ and } \quad R_{\bar{x}}(2r)\subset V, \ \forall \bar{x}\in\overline{U}.$$

Then for all $(\bar{x},\bar{t})\in\overline{U}\times[\tau_1,\tau_2^*]$, we have $R_{\bar{x}}(2r)\times(\bar{t}-(2r)^2,\bar{t})\subset V\times(\tau_3,\tau_4)$. Using the compactness of $\overline{U}\times[\tau_1,\tau_2]$, there exist $(x_i,t_i)\in\overline{U}\times[\tau_1,\tau_2^*]$, $i=1,\dots,N$, such that

$$\overline{U}\times[\tau_1,\tau_2]\subset\bigcup_{i=1}^N R_{x_i}(r)\times(t_i-r^2,t_i).$$

Using Theorem 7.1.2,

$$\begin{aligned}
\|u\|_{C(\overline{U}\times[\tau_1,\tau_2])} &= \sup_{\overline{U}\times[\tau_1,\tau_2]}|u| \\
&\leq \max_{i=1,\dots,N}\ \sup_{R_{x_i}(r)\times(t_i-r^2,t_i)}|u| \\
&\leq \max_{i=1,\dots,N} c_i\|u\|_{L^{\frac{2p}{p-2},2}\left(R_{x_i}(2r)\times(t_i-(2r)^2,t_i)\right)} \\
&\leq \underbrace{\big(\max_{i=1,\dots,N} c_i\big)}_{=:C_1}\|u\|_{L^{\frac{2p}{p-2},2}\left(V\times(\tau_3,\tau_4)\right)},
\end{aligned}$$

where $c_i > 0$ $(1 \le i \le N)$ are constants which are independent of u. Thus for $\nu \ge \frac{2p}{p-2}$

$$\begin{aligned}
\|P_\cdot f\|_{C(\overline{U}\times[\tau_1,\tau_2])} &\le C_1 \|P_\cdot f\|_{L^{\frac{2p}{p-2},2}\left(V\times(\tau_3,\tau_4)\right)} && (8.40)\\
&= C_1 \left(\int_{\tau_3}^{\tau_4} \left(\int_V |T_t f|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} dt \right)^{1/2}\\
&\le C_1 \left(\frac{1}{\inf_V \rho\psi} \right)^{\frac{p-2}{2p}} \left(\int_{\tau_3}^{\tau_4} \left(\int_V |T_t f|^{\frac{2p}{p-2}} d\mu \right)^{\frac{p-2}{p}} dt \right)^{1/2}\\
&\le C_1 \left(\frac{1}{\inf_V \rho\psi} \right)^{\frac{p-2}{2p}} \left(\int_{\tau_3}^{\tau_4} \|T_t f\|^2_{L^{\frac{2p}{p-2}}(V,\mu)} dt \right)^{1/2}\\
&\le C_1 \underbrace{\left(\frac{1}{\inf_V \rho\psi} \right)^{\frac{p-2}{2p}} \mu(V)^{\frac{1}{2}-\frac{1}{p}-\frac{1}{\nu}}}_{=:C_2} \left(\int_{\tau_3}^{\tau_4} \|T_t f\|^2_{L^\nu(V,\mu)} dt \right)^{1/2}\\
&\le C_1 C_2 (\tau_4-\tau_3)^{1/2} \|f\|_{L^\nu(\mathbb{R}^d,\mu)}. && (8.41)
\end{aligned}$$

Now assume $f \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$. Then $nG_n f \in D(\overline{L})_b \cap D(L_s) \cap D(L_2)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} nG_n f = f$ in $L^\nu(\mathbb{R}^d,\mu)$. Thus (8.41) extends to all $f \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$. If $\nu \in [\frac{2p}{p-2},\infty)$, the above again extends to all $f \in L^\nu(\mathbb{R}^d,\mu)$ using the denseness of $L^1(\mathbb{R}^d,\mu)\cap L^\infty(\mathbb{R}^d,\mu)$ in $L^\nu(\mathbb{R}^d,\mu)$. Finally assume $f \in L^\infty(\mathbb{R}^d,\mu)$ and let $f_n := 1_{B_n}\cdot f$ for $n \ge 1$. Then $\lim_{n\to\infty} f_n = f$ μ-a.e. on $\mathbb{R}^d$ and

$$T_t f = \lim_{n\to\infty} T_t f_n = \lim_{n\to\infty} P_t f_n, \ \mu\text{-a.e. on } \mathbb{R}^d. \tag{8.42}$$

Thus using the sub-Markovian property and applying Lebesgue's Theorem in (8.40), $(P_\cdot f_n(\cdot))_{n\ge1}$ is a Cauchy sequence in $C(\overline{U}\times[\tau_1,\tau_2])$. Hence we can again define

$$P_\cdot f := \lim_{n\to\infty} P_\cdot f_n(\cdot) \ \text{ in } \ C(\overline{U}\times[\tau_1,\tau_2]).$$

For each $t>0$, $P_t f_n$ converges uniformly to $P_t f$ in U, hence in view of (8.42), $T_t f$ has continuous μ version $P_t f$ and $P_\cdot f \in C(\overline{U}\times[\tau_1,\tau_2])$. Therefore (8.41) extends to all $f \in L^\infty(\mathbb{R}^d,\mu)$. Since U and $[\tau_1,\tau_2]$ were arbitrary, it holds for any $f \in \cup_{\nu\in[\frac{2p}{p-2},\infty]} L^\nu(\mathbb{R}^d,m)$, $P_\cdot f(\cdot)$ is continuous on $\mathbb{R}^d\times(0,\infty)$ and for each $t>0$, $P_t f = T_t f$ μ-a.e. on $\mathbb{R}^d$. □

Remark 8.3.7. *(i) By Theorem 8.3.3, we get a resolvent kernel and a resolvent kernel density for any $x \in \mathbb{R}^d$. Indeed, for any $\alpha > 0$, $x \in \mathbb{R}^d$, (8.32) implies that*

$$R_\alpha(x, A) := \lim_{l\to\infty} R_\alpha(1_{B_l \cap A})(x), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

defines a sub-probability measure $\alpha R_\alpha(x, dy)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ that is absolutely continuous with respect to μ. Using the Radon-Nikodym derivative, the resolvent kernel density is defined by

$$r_\alpha(x, \cdot) := \frac{R_\alpha(x, dy)}{\mu(dy)}, \qquad x \in \mathbb{R}^d.$$

(ii) By Theorem 8.3.6, we also get a heat kernel and a heat kernel density for any $x \in \mathbb{R}^d$. Indeed, for any $t > 0$, $x \in \mathbb{R}^d$, (8.37) implies that

$$P_t(x, A) := \lim_{l\to\infty} P_t(1_{B_l \cap A})(x), \quad A \in \mathcal{B}(\mathbb{R}^d)$$

defines a sub-probability measure $P_t(x, dy)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ that is absolutely continuous with respect to μ. Using the Radon-Nikodym derivative, the heat kernel density is defined by

$$p_t(x, \cdot) := \frac{P_t(x, dy)}{\mu(dy)}, \qquad x \in \mathbb{R}^d.$$

Proposition 8.3.8. *Assume* **(A1)**, **(A2)**, **(A3)** *and let $t, \alpha > 0$. Then it holds:*

(i) $G_\alpha g$ has a locally Hölder continuous μ-version

$$R_\alpha g = \int_{\mathbb{R}^d} g(y) R_\alpha(\cdot, dy) = \int_{\mathbb{R}^d} g(y) r_\alpha(\cdot, y)\mu(dy), \quad \forall g \in \bigcup_{r\in[s,\infty]} L^r(\mathbb{R}^d, \mu). \tag{8.43}$$

In particular, (8.43) extends by linearity to all $g \in L^s(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$, i.e. $(R_\alpha)_{\alpha>0}$ is $L^{[s,\infty]}(\mathbb{R}^d, \mu)$-strong Feller.

(ii) $T_t f$ *has a continuous* μ*-version*

$$P_t f = \int_{\mathbb{R}^d} f(y) P_t(\cdot, dy) = \int_{\mathbb{R}^d} f(y) p_t(\cdot, y)\mu(dy), \quad \forall f \in \bigcup_{\nu \in [\frac{2p}{p-2}, \infty]} L^\nu(\mathbb{R}^d, \mu). \tag{8.44}$$

In particular, (8.44) *extends by linearity to all* $f \in L^{\frac{2p}{p-2}}(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$*, i.e.* $(P_t)_{t>0}$ *is* $L^{[\frac{2p}{p-2}, \infty]}(\mathbb{R}^d, \mu)$*-strong Feller.*

Finally, for any $\alpha > 0, x \in \mathbb{R}^d$*,* $g \in L^s(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$

$$R_\alpha g(x) = \int_0^\infty e^{-\alpha t} P_t g(x)\, dt.$$

8.4 Some auxiliary results

In this Section, we use all notations and assumptions from Section 8.2

Proposition 8.4.1. $(T_t^0)_{t>0}$ *restricted to* $L^1(\mathbb{R}^d, \mu) \cap L^\infty(\mathbb{R}^d, \mu)$ *can be extended to a sub-Markovian* C_0*-semigroup of contractions* $(\overline{T}_t^0)_{t>0}$ *with generator* $(\overline{L}^0, D(\overline{L}^0))$ *on* $L^1(\mathbb{R}^d, \mu)$*. If* $f \in D(L^0)$ *and* $f, L^0 f \in L^1(\mathbb{R}^d, \mu)$*, then* $f \in D(\overline{L}^0)$ *and* $\overline{L}^0 f = L^0 f$*. Set* $\mathcal{A} := \{u \in D(L^0) \cap L^1(\mathbb{R}^d, \mu) \mid L^0 u \in L^1(\mathbb{R}^d, \mu)\}$*. Then* $(\overline{L}^0, D(\overline{L}^0))$ *is the closure of* $(L^0, \mathcal{A})$ *on* $L^1(\mathbb{R}^d, \mu)$*.*

Similarly, for a bounded open subset V *of* $\mathbb{R}^d$*,* $(T_t^{0,V})_{t>0}$ *restricted to* $L^1(V, \mu) \cap L^\infty(V, \mu)$ *can be extended to a sub-Markovian* C_0*-semigroup of contractions* $(\overline{T}_t^{0,V})_{t>0}$ *on* $L^1(V, \mu)$*. Also if* $f \in D(L^{0,V})$ *and* f*,* $L^{0,V} f \in L^1(V, \mu)$*, then* $f \in D(\overline{L}^{0,V})$ *and* $\overline{L}^{0,V} f = L^{0,V} f$*. Finally* $(\overline{L}^{0,V}, D(\overline{L}^{0,V}))$ *is the closure of* $(L^{0,V}, D(L^{0,V}))$ *on* $L^1(V, \mu)$*.*

Proof Since the proof for the case of $(T_t^{0,V})_{t>0}$ is exactly same with the case of $(T_t^0)_{t>0}$, we will only prove the case of $(T_t^0)_{t>0}$. Since $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a regular Dirichlet from, there exists a Hunt process

$$\mathbb{M}^0 = (\Omega^0, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0}, (X_t^0)_{t\geq 0}, (\mathbb{P}_x^0)_{x\in\mathbb{R}^d\cup\Delta})$$

with life time $\zeta^0 = \inf\{t > 0 \mid X_t^0 = \Delta\}$ such that for any $g \in L^2(\mathbb{R}^d, \mu)$

$$x \mapsto \mathbb{E}_x^0 \left[g(X_t^0)\right] \quad \text{is a quasi-continuous } \mu\text{-version of } T_t^0 g.$$

Let $f \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$. Using Jensen inequality and sub-Markovian property of $(T_t^0)_{t>0}$

$$\begin{aligned}
\int_{\mathbb{R}^d} |T_t^0 f| d\mu &= \int_{\mathbb{R}^d} \left|\mathbb{E}^0_\cdot \left[f(X_t^0)\right]\right| d\mu \\
&\leq \int_{\mathbb{R}^d} \mathbb{E}^0_\cdot \left[\left|f(X_t^0)\right|\right] d\mu \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} T_t^0 |f| \cdot 1_{B_n} d\mu \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} |f| \cdot T_t^0 1_{B_n} d\mu \\
&\leq \int_{\mathbb{R}^d} |f| d\mu.
\end{aligned}$$

Since $L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$ is dense in $L^1(\mathbb{R}^d,\mu)$, $(T_t^0)_{t>0}$ restricted to $L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$ uniquely extend to the sub-Markovian contraction semigroup $(\overline{T}_t^0)_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$. Define

$$\mathcal{D} := L^\infty(\mathbb{R}^d,\mu) \cap \{g \mid g \geq 0 \text{ and there exists } A \in \mathcal{B}(\mathbb{R}^d) \\ \text{with } \mu(A) < \infty \text{ and } g = 0 \text{ on } \mathbb{R}^d \setminus A\}.$$

Since $\mathcal{D}$ is dense in $L^1(\mathbb{R}^d,\mu)^+$, $\mathcal{D}-\mathcal{D}$ is dense in $L^1(\mathbb{R}^d,\mu)$. Let $f \in \mathcal{D}-\mathcal{D}$. Then there exists $A \in \mathcal{B}(\mathbb{R}^d)$ with $\mu(A) < \infty$ such that $\text{supp}(f) \subset A$ and $f \in L^1(\mathbb{R}^d,\mu) \cap L^\infty(\mathbb{R}^d,\mu)$. By strong continuity of $(T_t^0)_{t>0}$ on $L^2(\mathbb{R}^d,\mu)$

$$\lim_{t\to 0+} \int_{\mathbb{R}^d} 1_A |T_t^0 f| d\mu = \int_{\mathbb{R}^d} 1_A |f| d\mu = \|f\|_{L^1(\mathbb{R}^d,\mu)},$$

hence using the contraction property on $L^1(\mathbb{R}^d,\mu)$,

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^d} 1_{\mathbb{R}^d\setminus A} |T_t^0 f| d\mu = \int_{\mathbb{R}^d} |T_t^0 f| d\mu - \int_{\mathbb{R}^d} 1_A |T_t^0 f| d\mu \\
&\leq \|f\|_{L^1(\mathbb{R}^d,\mu)} - \int_{\mathbb{R}^d} 1_A |T_t^0 f| d\mu \longrightarrow 0 \quad \text{as } t \to 0+.
\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{t\to 0+}\int_{\mathbb{R}^d}|T_t^0 f - f|d\mu &= \lim_{t\to 0+}\left(\int_{\mathbb{R}^d} 1_A|T_t^0 f - f|d\mu + \int_{\mathbb{R}^d} 1_{\mathbb{R}^d\setminus A}|T_t^0 f|d\mu\right)\\ &\le \mu(A)^{1/2}\lim_{t\to 0+}\|T_t f - f\|_{L^2(\mathbb{R}^d,\mu)} = 0.\end{aligned}$$

By the denseness of $\mathcal{D}-\mathcal{D}$ in $L^1(\mathbb{R}^d,\mu)$, we get the strong continuity of $(\overline{T}^0_t)_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$. Now let $f\in D(L^0)$ and $f,\ L^0 f\in L^1(\mathbb{R}^d,\mu)$. Then $f\in L^1(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$, $L^0 f\in L^1(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$, hence we get $\overline{T}^0_t f = T_t^0 f,\ \ \overline{T}^0_t L^0 f = T_t^0 L^0 f$ for every $t>0$. Using the 'Fundamental Theorem of Calculus on Banach Space' and strong continuity of $(\overline{T}^0_t)_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$

$$\begin{aligned}\frac{\overline{T}^0_t f - f}{t} &= \frac{T_t^0 f - f}{t} = \frac{1}{t}\int_0^t T_s^0 L^0 f\,ds\\ &= \frac{1}{t}\int_0^t \overline{T}^0_s L^0 f\,ds \longrightarrow L^0 f \quad \text{in } L^1(\mathbb{R}^d,\mu) \ \ \text{as } t\to 0+\,.\end{aligned}$$

Consequently, $f\in D(\overline{L}^0)$ and $\overline{L}^0 f = L^0 f$.

Let $(\overline{G}^0_\alpha)_{\alpha>0}$ be the resolvent generated by $(\overline{L}^0, D(\overline{L}^0))$. Set $\mathcal{C} := \left\{\overline{G}^0_1 g \mid g\in C_0^\infty(\mathbb{R}^d)\right\}$. Then $\mathcal{C}\subset\mathcal{A}$ and one can directly check that $\mathcal{C}\subset D(\overline{L}^0)$ is dense with respect to graph norm $\|\cdot\|_{D(\overline{L}^0)}$, hence it completes our proof.

□

Lemma 8.4.2. *Let V be a bounded open subset of $\mathbb{R}^d$ and $f\in\widehat{H}_0^{1,2}(V,\mu)_b$. Then there exists a sequence $(f_n)_{n\ge1}\subset C_0^\infty(V)$ and a constant $M>0$ such that $\|f_n\|_{L^\infty(V)}\le M$ for all $n\ge 1$ and*

$$\lim_{n\to\infty} f_n = f \quad \text{in } \ \widehat{H}_0^{1,2}(V,\mu), \qquad \lim_{n\to\infty} f_n = f \quad \mu\text{ -a.e. on } V.$$

Proof Take $(g_n)_{n\ge1}\subset C_0^\infty(V)$ such that

$$\lim_{n\to\infty} g_n = f \ \text{ in } \widehat{H}_0^{1,2}(V,\mu) \ \text{ and } \ \lim_{n\to\infty} g_n = f \quad \mu\text{ -a.e. on } V. \tag{8.45}$$

Define $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(t) = t$ if $|t| \le \|f\|_{L^\infty(\mathbb{R}^d)} + 1$ and $\varphi(t) = 0$ if $|t| \ge \|f\|_{L^\infty(\mathbb{R}^d)} + 2$. Let $M := \|\varphi\|_{L^\infty(\mathbb{R})}$ and $\widetilde{f}_n := \varphi(g_n)$. Then $\widetilde{f}_n \in C_0^\infty(V)$ and $\|\widetilde{f}_n\|_{L^\infty(V)} \le M$ for all $n \ge 1$. By Lebesgue's Theorem and (8.45),

$$\lim_{n\to\infty} \widetilde{f}_n = \lim_{n\to\infty} \varphi(g_n) = \varphi(f) = f \quad \text{in } L^2(V,\mu).$$

Using the chain rule and (8.45)

$$\begin{aligned}\sup_{n\ge1} \|\nabla \widetilde{f}_n\|_{L^2(V,\mathbb{R}^d)} &= \sup_{n\ge1} \|\nabla \varphi(g_n)\|_{L^2(V,\mathbb{R}^d)} \\ &\le \|\varphi'\|_{L^\infty(\mathbb{R})} \sup_{n\ge1} \|\nabla g_n\|_{L^2(V,\mathbb{R}^d)}. \\ &< \infty.\end{aligned}$$

Thus by the Banach-Alaoglu Theorem and the Banach-Saks Thoerem, there exists a subsequence of $(\widetilde{f}_n)_{n\ge1}$, say again $(\widetilde{f}_n)_{n\ge1}$, such that for the Cesaro mean

$$f_N := \frac{1}{N}\sum_{n=1}^{N} \widetilde{f}_n \longrightarrow f \quad \text{in } \widehat{H}_0^{1,2}(V,\mu) \quad \text{as } N \to \infty.$$

Note that $f_N \in C_0^\infty(V)$, $\|f_N\|_{L^\infty(V)} \le M$ for all $N \in \mathbb{N}$. Since the Cesaro mean of a convergent sequence in $\mathbb{R}$ is also converges, $(f_n)_{n\ge1}$ is the desired sequence.

□

Lemma 8.4.3. *Let $f \in \widehat{H}_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}$ and V be a bounded open subset of $\mathbb{R}^d$ with supp$(f) \subset V$. Then $f \in \widehat{H}_0^{1,2}(V,\mu)_b$. Moreover there exists $(f_n)_{n\ge1} \subset C_0^\infty(\mathbb{R}^d)$ and a constant $M > 0$ such that supp$(f_n) \subset V$, $\|f_n\|_{L^\infty(V)} \le M$ for all $n \ge 1$ and*

$$\lim_{n\to\infty} f_n = f \quad \text{in } \widehat{H}_0^{1,2}(\mathbb{R}^d,\mu), \qquad \lim_{n\to\infty} f_n = f \quad \mu\text{-a.e. on } \mathbb{R}^d.$$

Proof Let W be an open subset of $\mathbb{R}^d$ satisfying supp$(f) \subset W \subset \overline{W} \subset V$. Take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfying supp$(\chi) \subset V$ and $\chi \equiv 1$ on W. Since $f \in \widehat{H}_0^{1,2}(\mathbb{R}^d,\mu)$, there exists $\widetilde{g}_n \in C_0^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \widetilde{g}_n = f \quad \text{in } \widehat{H}_0^{1,2}(\mathbb{R}^d,\mu).$$

Thus $\chi\widetilde{g}_n \in C_0^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\chi\widetilde{g}_n) \subset V$ and

$$\begin{aligned}\|\chi\widetilde{g}_n - f\|_{L^2(\mathbb{R}^d,\mu)} &= \|\chi\widetilde{g}_n - \chi f\|_{L^2(\mathbb{R}^d,\mu)}\\ &\leq \|\chi\|_{L^\infty(\mathbb{R}^d)}\|\widetilde{g}_n - f\|_{L^2(\mathbb{R}^d,\mu)} \longrightarrow 0 \quad \text{as } n \to \infty.\end{aligned}$$

Note that $\chi\widetilde{g}_n \in C_0^\infty(\mathbb{R}^d) \subset \widehat{H}_0^{1,2}(V,\mu)$ and

$$\begin{aligned}\sup_{n\geq 1}\|\nabla(\chi\widetilde{g}_n)\|_{L^2(V,\mathbb{R}^d)} &= \sup_{n\geq 1}\left(\|\widetilde{g}_n\nabla\chi\|_{L^2(V,\mathbb{R}^d)} + \|\chi\nabla\widetilde{g}_n\|_{L^2(V,\mathbb{R}^d)}\right)\\ &\leq \sup_{n\geq 1}\left(\frac{\|\nabla\chi\|_{L^\infty(V,\mathbb{R}^d)}}{\inf(\rho\psi)}\|g_n\|_{L^2(\mathbb{R}^d,\mu)} + \|\chi\|_{L^\infty(\mathbb{R}^d)}\|\nabla\widetilde{g}_n\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}\right)\\ &< \infty.\end{aligned}$$

Since bounded sequences in Hilbert spaces have a weakly convergent subsequence, $f \in \widehat{H}_0^{1,2}(V,\mu)$. Taking $(f_n)_{n\geq 1} \subset C_0^\infty(V)$ as in Lemma 8.4.2 and extending it trivially to $C_0^\infty(\mathbb{R}^d)$, our assertion holds.

□

Lemma 8.4.4. *Let V_1, V_2 be bounded open subsets of $\mathbb{R}^d$ satisfying $\overline{V}_1 \subset V_2$. Assume $f \in \widehat{H}_0^{1,2}(V_2,\mu)$, $g \in \widehat{H}_0^{1,2}(V_1,\mu)$ with $g = 0$ on $V_2 \setminus V_1$. If $0 \leq f \leq g$, then $f \in \widehat{H}_0^{1,2}(V_1,\mu)$.*

Proof Take $(g_n)_{n\geq 1} \subset C_0^\infty(V_2)$ satisfying $\mathrm{supp}(g_n) \subset V_1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} g_n = g \quad \text{in } \widehat{H}_0^{1,2}(V_2,\mu).$$

Observe that for all $n \in \mathbb{N}$

$$\mathrm{supp}(f \wedge g_n) \subset V_1 \;\text{ and }\; f \wedge g_n = \frac{f+g_n}{2} - \frac{|f-g_n|}{2} \in \widehat{H}_0^{1,2}(V_2,\mu).$$

By Lemma 8.4.3, $f \wedge g_n \in \widehat{H}_0^{1,2}(V_1,\mu)$ for all $n \in \mathbb{N}$. Moreover

$$\lim_{n\to\infty} f\wedge g_n = \lim_{n\to\infty}\left(\frac{f+g_n}{2} - \frac{|f-g_n|}{2}\right) = \frac{f+g}{2} - \frac{|f-g|}{2} = f \wedge g = f \quad \text{in } L^2(V_1,\mu).$$

Since $\left(\langle\cdot,\cdot\rangle_{\widehat{H}_0^{1,2}(V_2,\mu)}, \widehat{H}_0^{1,2}(V_2,\mu)\right)$ is a Dirichlet form,

$$\begin{aligned}
&\sup_{n\geq 1}\|f\wedge g_n\|_{\widehat{H}_0^{1,2}(V_1,\mu)}\\
=\ &\sup_{n\geq 1}\|f\wedge g_n\|_{\widehat{H}_0^{1,2}(V_2,\mu)}\\
=\ &\sup_{n\geq 1}\left\|\frac{f+g_n}{2}-\frac{|f-g_n|}{2}\right\|_{\widehat{H}_0^{1,2}(V_2,\mu)}\\
\leq\ &\frac{1}{2}\sup_{n\geq 1}\left(\|f\|_{\widehat{H}_0^{1,2}(V_2,\mu)}+\|g_n\|_{\widehat{H}_0^{1,2}(V_2,\mu)}+\||f|\|_{\widehat{H}_0^{1,2}(V_2,\mu)}+\||g_n|\|_{\widehat{H}_0^{1,2}(V_2,\mu)}\right)\\
\leq\ &\sup_{n\geq 1}\left(\|f\|_{\widehat{H}_0^{1,2}(V_2,\mu)}+\|g_n\|_{\widehat{H}_0^{1,2}(V_2,\mu)}\right)<\infty.
\end{aligned}$$

Thus by the Banach-Alaoglu Theorem, $f\in\widehat{H}_0^{1,2}(V_1,\mu)$.

□

For a bounded open set U in $\mathbb{R}^d$ and $T>0$, $C^2(\overline{U}\times[0,T])$ denotes the space of all twice continuously differentiable functions on $\overline{U}\times[0,T]$ with the norm defined by

$$\|u\|_{C^2(\overline{U}\times[0,T])}:=\|u\|_{C(\overline{U}\times[0,T])}+\sum_{i=1}^{d+1}\|\partial_i u\|_{C^2(\overline{U}\times[0,T])}+\sum_{i,j=1}^{d+1}\|\partial_i\partial_j u\|_{C^2(\overline{U}\times[0,T])}.$$

Theorem 8.4.5. *Let U be a bounded open subset of $\mathbb{R}^d$ and $T>0$. Set*

$$\mathcal{S}:=\Big\{h\in C_0^\infty(U\times(0,T))\mid \text{there exists } N\in\mathbb{N} \text{ such that } h=\sum_{i=1}^N f_i g_i,\quad \text{where } f_i\in C_0^\infty(U),\ g_i\in C_0^\infty((0,T)) \text{ for all } i=1,\dots,N\Big\}.$$

Then $C_0^2(U\times(0,T))\subset\overline{\mathcal{S}}|_{C^2(\overline{U}\times[0,T])}$.

Proof Step 1: Let V be an bounced open set in $\mathbb{R}^d$ and $T_1,T_2\in\mathbb{R}$ with $T_1<T_2$.

Define

$$\mathcal{R} := \Big\{ h \in C_0^\infty(V \times (T_1, T_2)) \mid \text{there exists } N \in \mathbb{N} \text{ such that } h = \sum_{i=1}^N f_i g_i, \\ \text{where } f_i \in C_0^\infty(V),\ g_i \in C_0^\infty((T_1, T_2)) \text{ for all } i = 1, \dots, N.$$

We claim that

$$C_0^2(V \times (T_1, T_2)) \subset \overline{\mathcal{R}}|_{C(\overline{V} \times [T_1, T_2])}. \tag{8.46}$$

Note that $V \times (T_1, T_2)$ is a locally compact space and $\overline{\mathcal{R}}|_{C(\overline{V} \times [T_1, T_2])}$ is a closed subalgebra of $C_\infty(V \times (T_1, T_2))$. We can easily check that for each $(x, t) \in V \times (T_1, T_2)$, there exists $\widetilde{h} \in \mathcal{R}$ such that $\widetilde{h}(x, t) \neq 0$. For $(x, t), (y, s) \in V \times (T_1, T_2)$ and $(x, t) \neq (y, s)$, there exists $\widehat{h} \in \mathcal{R}$ such that $\widehat{h}(x, t) = 1$ and $\widehat{h}(y, s) = 0$. Therefore by [15, Chapter V, 8.3 Corollary], we obtain $\overline{\mathcal{R}}|_{C(\overline{V} \times [T_1, T_2])} = C_\infty(V \times (T_1, T_2))$, so that our claim (8.46) holds.

Step 2: $C_0^2(U \times (0, T)) \subset \overline{\mathcal{S}}|_{C^2(\overline{U} \times [0, T])}$.
For $n \in \mathbb{N}$, let η_n be a standard mollifier on $\mathbb{R}^d$ and θ_n be a standard mollifier on $\mathbb{R}$. Then $\xi_n := \eta_n \theta_n$ is a standard mollifier on $\mathbb{R}^d \times \mathbb{R}$. Let $h \in C_0^2(U \times (0, T))$ be given. Then there exists a bounded open subset V of $\mathbb{R}^d$ and $T_1, T_2 \in \mathbb{R}$ with $0 < T_1 < T_2$ such that

$$\text{supp}(h) \subset V \times (T_1, T_2) \subset \overline{V} \times [T_1, T_2] \subset U \times (0, T).$$

Take $N \in \mathbb{N}$ such that $f * \xi_N \in C_0^\infty(U \times (0, T))$ for all $f \in C_0^\infty(V \times (T_1, T_2))$.
Note that by [14, Proposition 4.20], it holds

$$\partial_t (h * \xi_\varepsilon) = \partial_t h * \xi_\varepsilon,\ \partial_t^2 (h * \xi_\varepsilon) = \partial_t^2 h * \xi_\varepsilon,\ \partial_t \partial_i (h * \xi_\varepsilon) = \partial_t \partial_i h * \xi_\varepsilon, \\ \partial_i (h * \xi_\varepsilon) = \partial_i h * \xi_\varepsilon, \quad \partial_i \partial_j (h * \xi_\varepsilon) = \partial_i \partial_j h * \xi_\varepsilon \text{ for any } 1 \le i, j \le d.$$

Hence by [14, Proposition 4.21], $\lim_{n \to \infty} h * \xi_\varepsilon = h$ in $C^2(\overline{U} \times [0, T])$. Thus given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ with $n_\varepsilon \ge N$ such that

$$\| h - h * \xi_{n_\varepsilon} \|_{C^2(\overline{U} \times [0, T])} < \frac{\varepsilon}{2}.$$

Let $\mathcal{R}$ be as in Step 1. By (8.46), there exists $h_\varepsilon \in \mathcal{R} \subset C_0^\infty(V \times (T_1, T_2))$ such that

$$\|h - h_\varepsilon\|_{C(\overline{U}\times[0,T])} < \frac{\varepsilon}{2\|\xi_{n_\varepsilon}\|_{C^2(\overline{U}\times[0,T])}}.$$

Thus using [14, Propsotion 4.20] and Young's inequality,

$$\|h * \xi_{n_\varepsilon} - h_\varepsilon * \xi_{n_\varepsilon}\|_{C^2(\overline{U}\times[0,T])} \leq \|\xi_{n_\varepsilon}\|_{C^2(\overline{U}\times[0,T])}\|h - h_\varepsilon\|_{C(\overline{U}\times[0,T])} < \frac{\varepsilon}{2}.$$

Therefore

$$\|h - h_\varepsilon * \xi_{n_\varepsilon}\|_{C^2(\overline{U}\times[0,T])} < \varepsilon.$$

Since $h_\varepsilon * \xi_{n_\varepsilon} \in \mathcal{S}$, we have $h \in \overline{\mathcal{S}}|_{C^2(\overline{U}\times[0,T])}$, as desired.

$\square$

Chapter 9

Well-posedness and irreducibility for degenerate Itô-SDEs

9.1 Weak existence of degenerate Itô-SDEs with rough coefficients

The following assumption will in particular be necessary to obtain a Hunt process with transition function $(P_t)_{t\geq 0}$ (and consequently a weak solution to the corresponding SDE for every starting point). It will be first used in Theorem 9.1.3 below.

(A4) $\mathbf{G} \in L^s_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$, where s is as in **(A2)**

The condition **(A4)** is not necessary to get a Hunt processes (and consequently a weak solution to the corresponding SDE for merely quasi-every starting point) as in the following proposition.

Proposition 9.1.1. *There exists a Hunt process*

$$\tilde{\mathbb{M}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}})_{t\geq 0}, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$$

with life time $\tilde{\zeta} := \inf\{t \geq 0 \,|\, \tilde{X}_t = \Delta\}$ *and cemetery* Δ *such that* $\mathcal{E}$ *is (strictly properly) associated with* $\tilde{\mathbb{M}}$ *and for strictly* $\mathcal{E}$*-q.e.* $x \in \mathbb{R}^d$,

$$\tilde{\mathbb{P}}_x\Big(\Big\{\omega \in \tilde{\Omega} \mid \tilde{X}_\cdot(\omega) \in C\big([0,\infty), \mathbb{R}^d_\Delta\big),\ \tilde{X}_\cdot(\omega) = \Delta \ \forall t \geq \zeta(\omega)\Big\} \Big) = 1.$$

Proof First one shows the quasi-regularity of the generalized Dirichlet form $(\mathcal{E}, D(L_2))$ associated with $(L_2, D(L_2))$, and the existence of an μ-tight special standard process associated with $(\mathcal{E}, D(L_2))$. This can be done exactly as in [69, Theorem 3.5]. One only has to take care that the space $\mathcal{Y}$ as defined in the proof of [69, Theorem 3.5] is replaced because of a seemingly uncorrected version of the papaer by the following one

$$\mathcal{Y} := \{u \in D(\overline{L}))_b \,|\, \exists f, g \in L^1(\mathbb{R}^d, \mu)_b,\ f, g \geq 0, \text{ such that } u \leq G_1 f \text{ and } -u \leq G_1 g\}$$

in order to guarantee the convergence at the end of the proof. Then the assertion will follow exactly as in [78, Theorem 6], using for the proof instead $\mathcal{G}$ there the space $\mathcal{Y}$ defined above and defining $E_k \equiv \mathbb{R}^d$, $k \geq 1$.

□

Remark 9.1.2. *(i) Assume* **(A1)**, **(A2)**, **(A3)** *and* $\mathbf{G} \in L_{loc}^{\frac{sq}{q-1}}(\mathbb{R}^d)$. *Then for any bounded open subset* V *of* $\mathbb{R}^d$, *it holds*

$$\int_V \|\mathbf{G}\|^s d\mu \leq \|\mathbf{G}\|^s_{L^{\frac{sq}{q-1}}(V)} \|\rho\psi\|_{L^q(V)},$$

hence **(A4)** *is satisfied.*

(ii) Two simple examples where **(A1)**, **(A2)**, **(A3)**, **(A4)** *are satisfied are given as follows: for the first example let* A, ψ *satisfy the assumptions of* **(A1)**, $\psi \in L^p_{loc}(\mathbb{R}^d)$, $s = p$, *and* $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ *and for the second let* A, ψ *satisfy the assumptions of* **(A1)**, $\psi \in L^{2p}_{loc}(\mathbb{R}^d)$, $s = \frac{2p}{3}$ *and* $\mathbf{G} \in L^{2p}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$.

Analogously to [49, Theorem 3.12], we obtain:

Theorem 9.1.3. *Under the assumptions* **(A1)**, **(A2)**, **(A3)**, **(A4)**, *there exists a Hunt process*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$$

with state space $\mathbb{R}^d$ *and life time*

$$\zeta = \inf\{t \geq 0 \,:\, X_t = \Delta\} = \inf\{t \geq 0 \,:\, X_t \notin \mathbb{R}^d\},$$

having the transition function $(P_t)_{t\geq 0}$ as transition semigroup, such that $\mathbb{M}$ has continuous sample paths in the one point compactification $\mathbb{R}^d_\Delta$ of $\mathbb{R}^d$ with the cemetery Δ as point at infinity, i.e. for any $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\bigg(\big\{\omega \in \Omega \mid X_\cdot(\omega) \in C\big([0,\infty), \mathbb{R}^d_\Delta\big),\, X_\cdot(\omega) = \Delta \ \forall t \geq \zeta(\omega)\big\} \bigg) = 1.$$

Remark 9.1.4. *Note that the analogous reuslts to Lemma 3.2.3, Lemma 3.2.4, Proposition 3.2.5, Proposition 3.2.6, Theorem 3.2.8 of Part I hold in the situation of Part III. One of the main differences is that $q > \frac{d}{2}$ of Part I is replaced by $s > \frac{d}{2}$ of* **(A2)**. *Especially the Krylov type estimate for the Hunt process of Theorem 9.1.3 holds as stated in* (9.1) *right below. Let $g \in L^r(\mathbb{R}^d, \mu)$ for some $r \in [s, \infty]$ be given. Then for any ball B, there exists a constant $C_{B,r}$, depending in particular on B and r, such that for all $t \geq 0$,*

$$\sup_{x\in\overline{B}} \mathbb{E}_x\left[\int_0^t |g|(X_s)\, ds\right] < e^t C_{B,r} \|g\|_{L^r(\mathbb{R}^d,\mu)}. \tag{9.1}$$

Note that $C_{B,r}$ does not depend on the VMO condition of the diffusion matrix since we use the elliptic Hölder estimate of Theorem 7.2.2 which is different from the elliptic $H^{1,p}$-estimate of Part I, II. One can get the analogous conservativeness and moment inequalities to Theorem 4.1.2, Theorem 4.1.4 (i) in the situation of Part III. Since we have not derived a parabolic Harnack inequality related to (7.1), *irreducibility and strict irreducibility can not be directly obtained as in the proof of Lemma 4.2.2, Corollary 4.2.4. However, choosing a special ψ in Section 9.2, strict irreducibility can be derived and one can show the analogous recurrence and transience results to Proposition 4.2.5, Theorem 4.2.7, Lemma 4.2.8, Theorem 4.2.9 in the situation of Part III.*

The following theorem can be proved exactly as in Theorem 3.2.8 of Part I.

Theorem 9.1.5. *Consider the Hunt process $\mathbb{M}$ from Theorem 9.1.3 with coordinates $X_t = (X_t^1, ..., X_t^d)$. Let $(\widehat{\sigma}_{ij})_{1\leq i\leq d, 1\leq j\leq m}$, $m \in \mathbb{N}$ arbitrary but fixed, be any matrix consisting of locally bounded functions for all $1 \leq i \leq d$, $1 \leq j \leq m$, such that $\widehat{A} = \widehat{\sigma}\,\widehat{\sigma}^T$, i.e.*

$$\widehat{a}_{ij}(x) = \sum_{k=1}^m \widehat{\sigma}_{ik}(x)\widehat{\sigma}_{jk}(x), \quad \forall x \in \mathbb{R}^d,\ 1 \leq i,j \leq d.$$

Then on a standard extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, that we denote for notational convenience again by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$, $x \in \mathbb{R}^d$, there exists a standard m-dimensional Brownian motion $W = (W^1, \dots, W^m)$ starting from zero such that $\mathbb{P}_x$-a.s. for any $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $i = 1, \dots, d$

$$X_t^i = x_i + \sum_{j=1}^m \int_0^t \widehat{\sigma}_{ij}(X_s)\, dW_s^j + \int_0^t g_i(X_s)\, ds, \quad 0 \leq t < \zeta, \tag{9.2}$$

in short

$$X_t = x + \int_0^t \widehat{\sigma}(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad 0 \leq t < \zeta.$$

9.2 Strict irreducibility for special weight functions

Here we consider a special weight function $\psi(x) := \|x\|^{-\alpha}$ with $\alpha > 0$, $\alpha q < d$. Then ψ is smooth on $\mathbb{R}^d \setminus \overline{B}_\varepsilon$ for any $\varepsilon > 0$. In that case, we can also derive strict irreducibility, and irreducibility except $\mathbf{0}$.

Lemma 9.2.1. *Assume* **(A1)**, **(A2)**, **(A3)** *and $\psi(x) = \|x\|^{-\alpha}$ for some $\alpha > 0$ satisfying $\alpha q < d$. Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $P_{t_0} 1_A(x_0) = 0$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. Then $\mu(A) = 0$.*

Proof We use the proof by contradiction. Suppose $\mu(A) > 0$. Since $\mu(\{\mathbf{0}\}) = 0$, we have $\mu(A \setminus \{\mathbf{0}\}) = \mu(A) > 0$. For each $n \in \mathbb{N}$, let $E_n := \{x \in \mathbb{R}^d \mid \frac{1}{2n} < \|x\| < 2n\}$. Then $\mathbb{R}^d \setminus \{0\} = \bigcup_{n=1}^\infty E_n$, so that

$$A \setminus \{\mathbf{0}\} = \bigcup_{n=1}^\infty (A \cap E_n).$$

By the countable subadditivity of μ, there exists $n_0 \in \mathbb{N}$ such that $0 < \mu(A \cap E_{n_0}) < \infty$ and $x_0 \in E_{n_0}$. Using the compactness of $\overline{E}_{n_0}$ in $\mathbb{R}^d$, there exist $N_1 \in \mathbb{N}$ and a family of open balls $\{U_i\}_{i=1}^{N_1}$ in $\mathbb{R}^d$ such that

$$\overline{E}_{n_0} \subset \bigcup_{i=1}^{N_1} U_i \subset E_{n_0+1},$$

hence

$$A \cap E_{n_0} = \bigcup_{i=1}^{N_1} (A \cap E_{n_0} \cap U_i).$$

Therefore, there exists $i_0 \in \{1, \dots, N_1\}$ such that

$$0 < \mu(A \cap E_{n_0} \cap U_{i_0}) < \infty. \tag{9.3}$$

Let $y_0 \in E_{n_0+1}$ be the center of U_{i_0}. Since E_{n_0+1} is path-connected and $x_0, y_0 \in E_{n_0+1}$, there exists a continuous function $\gamma : [0,1] \to E_{n_0+1}$ such that $\gamma(0) = x_0$ and $\gamma(1) = y_0$. Set

$$\delta := \frac{1}{2} \cdot \inf\{\|a - b\| \mid a \in \gamma([0,1]), b \in \partial E_{n_0+1}\}.$$

Thus there exist $N_2 \in \mathbb{N}$ and distinct points $p_i \in \gamma((0,1))$, $i = 1, 2, \dots, N_2$ such that

$$B_\delta(x_0) \cup \left(\bigcup_{i=1}^{N_2} B_\delta(p_i) \right) \cup B_\delta(y_0) \subset E_{n_0+1},$$
$$B_\delta(x_0) \cap B_\delta(p_1) \neq \emptyset, \quad B_\delta(p_{N_2}) \cap B_\delta(y_0) \neq \emptyset,$$
$$B_\delta(p_i) \cap B_\delta(p_{i+1}) \neq \emptyset \ \text{ for all } i = 0, 1, \dots, N_2 - 1.$$

Now take $f_n := nG_n 1_{A \cap E_{n_0} \cap U_{i_0}}$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} f_n = 1_{A \cap E_{n_0} \cap U_{i_0}}$ in $L^{\frac{2p}{p-2}}(\mathbb{R}^d, \mu)$, hence by Theorem 8.3.6, $\lim_{n\to\infty} P_t f_n(x) = P_t 1_{A \cap E_{n_0} \cap U_{i_0}}(x)$ for any $(x,t) \in \mathbb{R}^d \times (0, \infty)$. For each $n \in \mathbb{N}$, let $u_n := \widehat{\rho} P_\cdot f_n$. Then by Remark 8.3.2 and as for (3.23) of Part I, we obtain for any $T > 0$.

$$\int_0^T \int_{E_{n_0+1}} \left(\frac{1}{2} \langle \widehat{A} \nabla u_n, \nabla \varphi \rangle + u_n \langle \widehat{\mathbf{F}}, \nabla \varphi \rangle - u_n \partial_t \varphi \right) dxdt = 0,$$
$$\text{for all } \varphi \in C_0^\infty(E_{n_0+1} \times (0,T)),$$

where $\widehat{\mathbf{F}} := \frac{1}{2} \nabla \widehat{A} + \mathbf{G} - 2\beta^{\widehat{\rho}, \widehat{A}} \in L^p(E_{n_0+1}, \mathbb{R}^d)$. Now take arbitrary but fixed $(x,t) \in B_\delta(x_0) \times (0, t_0)$. Then by [2, Theorem 5]

$$0 \le u_n(x,t) \le u_n(x_0, t_0) \ \exp\left(C\Big(\frac{\|x_0 - x\|^2}{t_0 - t} + \frac{t_0 - t}{\min(1,t)} + 1 \Big) \right),$$

where C is a constant which is independent of n. Letting $n \to \infty$

$$\begin{aligned}
0 &\leq \widehat{\rho}(x) P_t 1_{A\cap E_{n_0}\cap U_{i_0}}(x) \\
&\leq \widehat{\rho}(x_0) P_{t_0} 1_{A\cap E_{n_0}\cap U_{i_0}}(x_0) \cdot \exp\left(C\Big(\frac{\|x_0 - x\|^2}{t_0 - t} + \frac{t_0 - t}{\min(1,t)} + 1 \Big) \right) \\
&\leq \widehat{\rho}(x_0) P_{t_0} 1_{A}(x_0) \cdot \exp\left(C\Big(\frac{\|x_0 - x\|^2}{t_0 - t} + \frac{t_0 - t}{\min(1,t)} + 1 \Big) \right) \\
&= 0.
\end{aligned}$$

Therefore using Theorem 8.3.6, $P_t 1_{A\cap E_{n_0}\cap U_{i_0}}(x) = 0$ for any $(x,t) \in B_\delta(x_0) \times (0, t_0]$. Iterating this procedure $N_2 + 1$ times, we obtain

$$P_t 1_{A\cap E_{n_0}\cap U_{i_0}}(x) = 0 \ \text{ for any } (x,t) \in B_\delta(y_0) \times (0, t_0].$$

Without loss of generality, we may assume $B_\delta(y_0) \subset U_{i_0}$. Then similarly, applying [2, Theorem 5] to u_n on $U_{i_0} \times (0, t_0)$ and using the above similar procedure, $P_t 1_{A\cap E_{n_0}\cap U_{i_0}}(x) = 0$ for any $(x,t) \in U_{i_0} \times (0, t_0)$. Therefore

$$0 = \int_{\mathbb{R}^d} 1_{A\cap E_{n_0}\cap B_{i_0}} P_t 1_{A\cap E_{n_0}\cap U_{i_0}} d\mu \underset{t\to 0+}{\longrightarrow} \mu(A \cap E_{n_0} \cap U_{i_0}),$$

which contradicts (9.3), hence the assertion holds.

□

Corollary 9.2.2. *Assume* **(A1)**, **(A2)**, **(A3)** *and let* $\psi(x) := \|x\|^{-\alpha}$ *with* $\alpha > 0$, $\alpha q < d$. *Then*

(i) $(T_t)_{t>0}$ *is strictly irreducible.*

(ii) $(P_t)_{t>0}$ *is irreducible except in* $\mathbf{0}$, *i.e. given* $A \in \mathcal{B}(\mathbb{R}^d)$ *with* $\mu(A) > 0$, $P_t 1_A(x) > 0$ *for all* $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $t > 0$.

(iii) If additionally to **(A1)**, **(A2)**, **(A3)**, *we assume* **(A4)** *then* $\mathbb{M}$ *from Theorem 9.1.3 is irreducible except in* $\mathbf{0}$, *i.e. given* $A \in \mathcal{B}(\mathbb{R}^d)$ *with* $\mu(A) > 0$, $\mathbb{P}_x(X_t \in A) > 0$ *for all* $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $t > 0$.

Proof (i) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be a weakly invariant set with $\mu(\mathbb{R}^d \setminus A) \neq 0$. Then by monotone approximation with the L^2-functions $1_{B_n(\mathbf{0})} \nearrow 1_{\mathbb{R}^d}$ as $n \to \infty$, we get for any $t > 0$, $P_t 1_A(x) = 0$, for μ-a.e. $x \in \mathbb{R}^d \setminus A$. Fix $t > 0$. Since $\mu\Big((\mathbb{R}^d \setminus A) \setminus \{\mathbf{0}\}\Big) > 0$, there exists $x_0 \in (\mathbb{R}^d \setminus A) \setminus \{\mathbf{0}\}$ such that $P_t 1_A(x_0) = 0$. By Lemma 9.2.1, $\mu(A) = 0$ as desired.
(ii) By contraposition of Lemma 9.2.1, if $\mu(A) > 0$, then $P_t 1_A(x) > 0$, for all $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}, t > 0$.
(iii) Clear.

□

Example 9.2.3. *Given $p > d$, let $A = (a_{ij})_{1 \le i,j \le d}$ be a symmetric matrix of functions on $\mathbb{R}^d$ which is locally uniformly strictly elliptic and $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{0,1-d/p}_{loc}(\mathbb{R}^d)$ for all $1 \le i, j \le d$. Given $m \in \mathbb{N}$, let $\sigma = (\sigma_{ij})_{1 \le i \le d, 1 \le j \le m}$ be a matrix of functions satisfying $\sigma_{ij} \in C(\mathbb{R}^d)$ for all i, j, such that $A = \sigma\sigma^T$. Let $\phi \in L^\infty_{loc}(\mathbb{R}^d)$ be such that for any open ball B, there exist positive constants c_B, C_B such that*

$$c_B \le \phi \le C_B \quad a.e. \ on \ B.$$

Let $\psi(x) := \frac{1}{\|x\|^\alpha}\phi$ for some $\alpha > 0$ and consider following conditions.

(1) $\alpha p < d$, $\mathbf{G} \in L^\infty(B_\varepsilon(\mathbf{0})) \cap L^p(\mathbb{R}^d \setminus \overline{B}_\varepsilon(\mathbf{0}))$ for some $\varepsilon > 0$,

(2) $2\alpha p < d$, $\mathbf{G} \in L^{2p}(B_\varepsilon(\mathbf{0})) \cap L^p(\mathbb{R}^d \setminus \overline{B}_\varepsilon(\mathbf{0}))$ for some $\varepsilon > 0$,

(3) $\alpha \cdot (\frac{p}{2} \vee 2) < d$, $\mathbf{G} \equiv 0$ on $B_\varepsilon(\mathbf{0})$ and $\mathbf{G} \in L^s_{loc}(\mathbb{R}^d \setminus \overline{B_\varepsilon}(\mathbf{0}))$ for some $\varepsilon > 0$, where $s > d$ so that $(\frac{p}{2} \vee 2)^{-1} + \frac{1}{s} < \frac{2}{d}$.

Either of the conditions (1), (2), or (3) imply **(A1)**, **(A2)**, **(A3)**, **(A4)**. *Indeed, take $q := p$, $s := p$ in the case of (1), $q := 2p$, $s := \frac{2p}{3}$ in the case of (2), and $q := \frac{p}{2} \vee 2$, $s > d$ defined by (3) in the case of (3). Assuming (1), (2) or (3), the Hunt process $\mathbb{M}$ as in Theorem 9.1.5 solves weakly $\mathbb{P}_x$-a.s. for any $x \in \mathbb{R}^d$,*

$$X_t = x + \int_0^t \|X_s\|^{\alpha/2} \cdot \frac{1}{\sqrt{\phi}(X_s)} \cdot \sigma(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad 0 \le t < \zeta. \qquad (9.4)$$

Moreover, if we assume $\phi \equiv 1$, then by Corollary 9.2.2, the associated $L^2(\mathbb{R}^d, \mu)$-semigroup $(T_t)_{t>0}$ is strict irreducible and $\mathbb{M}$ is irreducible except in $\mathbf{0}$.

Remark 9.2.4. *Let $\psi(x) := \|x\|^{-\alpha}$ with $0 < \alpha < d$. Consider Cap that is the capaacity related to $(\mathcal{E}^0, D(\mathcal{E}^0))$ as defined in [25, Section 2.1]. Then by [25, Example 3.3.2],*

$$Cap(\{\mathbf{0}\}) > 0 \iff d-2 < \alpha < d. \tag{9.5}$$

Now define a generalized Dirichlet form $\mathcal{E}$ as in (8.31) and let $\mathrm{Cap}_{\mathcal{E}}$ be a strict capacity of $\mathcal{E}$ as defined in [78, Definition 1]. Then by [62, by Lemma 2.1] and (9.5), we obtain if $0 < \alpha \le d-2$ with $d \ge 3$, then

$$\mathrm{Cap}_{\mathcal{E}}(\{\mathbf{0}\}) = 0.$$

In that case, through the argument in [50, Theorem 3.8, Theorem 3.10] and [62, Lemma 2.2, Theorem 2.3], one may construct a Hunt process

$$\mathbb{M}^* = (\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t\ge 0}, (X^*_t)_{t\ge 0}, (\mathbb{P}^*_x)_{x\in(\mathbb{R}^d\setminus\{\mathbf{0}\})\cup\{\Delta\}})$$

with state space $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and life time

$$\zeta^* = \inf\{t \ge 0 \,:\, X^*_t = \Delta\} = \inf\{t \ge 0 \,:\, X^*_t \notin \mathbb{R}^d \setminus \{\mathbf{0}\}\},$$

having the transition function $(P_t^{\mathbb{R}^d\setminus\{\mathbf{0}\}})_{t\ge 0}$ as strong Feller transition semigroup, such that

$$T_t^{\mathbb{R}^d\setminus\{\mathbf{0}\}} f = P_t^{\mathbb{R}^d\setminus\{\mathbf{0}\}} f, \quad \mu\text{-a.e. on } \mathbb{R}^d \setminus \{\mathbf{0}\}, \quad t > 0, \quad f \in L^2(\mathbb{R}^d, \mu)_b$$

and $\mathbb{M}^*$ has continuous sample paths in the one point compactification $(\mathbb{R}^d \setminus \{\mathbf{0}\})_\Delta$ of $\mathbb{R}^d \setminus \{\mathbf{0}\}$ with the cemetery Δ as point at infinity. However if $\mathrm{Cap}_{\mathcal{E}}(\{\mathbf{0}\}) > 0$, then $\mathbb{M}^*$ as above would not be costructed by the arguments in [62].

Let $A = (a_{ij})_{1\le i,j\le d}$ be a symmetric matrix of function satisfying (8.2). Consider $d = 2$ and $\alpha = \frac{1}{4}$. Then by (9.5), $Cap(\{\mathbf{0}\}) > 0$. Now let $p := 3$, $q := 2p = 6$, $s := \frac{2p}{3} = 2$. Assume $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^2)$ for all $1 \le i,j \le 2$ and $\mathbf{G} \in L^{2p}_{loc}(B_\varepsilon, \mathbb{R}^2) \cap L^p(\mathbb{R}^2 \setminus \overline{B}_\varepsilon, \mathbb{R}^2)$ for some $\varepsilon > 0$. In that case, **(A1)**, **(A2)**, **(A3)**, **(A4)** *holds, hence we can construct* *a Hunt process* $\mathbb{M}$ *as in Theorem 9.1.3 which is a weak solution to Itô-SDE* (9.4) *and satisfies irreducibility.*

Consider $d = 3$ *and* $\alpha = 2$. *In that case, we also get* $Cap(\{\mathbf{0}\}) > 0$ *by* (9.5). *Let* $p = 4$, $q = 3$, $s = 4$. *Assume* $a_{ij} \in H^{1,p}_{loc}(\mathbb{R}^3)$ *for all* $1 \le i, j \le 3$ *and* $\mathbf{G} \in L^p_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ *and that there exists* $\varepsilon > 0$ *such that* $\mathbf{G} \equiv 0$ *on* $B_\varepsilon(\mathbf{0})$. *Then* **(A1)**, **(A2)**, **(A3)**, **(A4)** *holds, hence we could construct a Hunt process* $\mathbb{M}$ *as in Theorem 9.1.3 which is a weak solution to Itô-SDE* (9.4) *and satisfies irreducibility.*

9.3 Uniqueness in law for degenerate Itô-SDEs with discontinuous dispersion coefficient

Consider

(A4′): **(A1)** holds with $p := 2d + 2$, **(A2)** holds with $q > 2d + 2$, $s := d$, and $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$.

Note that if we assume **(A4′)**, then **(A3)** and **(A4)** hold.

Theorem 9.3.1 (**Local Krylov type estimate**). *Assume* **(A4′)**. *Let*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, (X_t)_{t \ge 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d})$$

be a canonical stochastic process, i.e.

$$\begin{aligned} \Omega = C([0, \infty), \mathbb{R}^d),\ \ \mathcal{F} = \mathcal{B}(\Omega),\ \ \mathcal{F}_t := \sigma(X_s : s \le t), \\ X_t(\omega) = \omega(t),\ \omega \in \Omega, \quad \mathbb{P}_x(X_0 = x) = 1,\ x \in \mathbb{R}^d, \end{aligned}$$

such that

$$X_t = x + \int_0^t \widehat{\sigma}(X_s) dW_s + \int_0^t \mathbf{G}(X_s) ds, \quad 0 \le t < \infty, \quad \mathbb{P}_x\text{-}a.s.\ \ \forall x \in \mathbb{R}^d, \tag{9.6}$$

where $\widehat{\sigma}$ is as in Theorem 9.1.5 and every term in (9.6) *is well-defined. In particular,*

$$(t,\omega) \mapsto \widehat{\sigma}(X_t(\omega)) \quad \text{and} \quad (t,\omega) \mapsto \mathbf{G}(X_t(\omega)) \text{ are progressively measurable.}$$

Let $x \in \mathbb{R}^d$, $T > 0$, $R > 0$ and assume $f \in L^{2d+2,d+1}(B_R \times (0,T))$. Then there exists a constant $C > 0$ which is independent of f such that

$$\mathbb{E}_x \left[\int_0^{T \wedge D_R} f(X_s, s) ds \right] \leq C \|f\|_{L^{2d+2,d+1}(B_R \times (0,T))},$$

where $D_R := D_{\mathbb{R}^d \setminus B_R} := \inf\{t \geq 0 \mid X_t \in \mathbb{R}^d \setminus B_R\}$. Moreover $\mathbb{P}_x$ is a solution to the time-homogeneous martingale problem in the sense of [37, Chapter 5, 4.15 Definition].

Proof Let $g \in L^{d+1}(B_R \times (0,T))$. (Note: all functions defined on $B_R \times (0,T)$ are trivially extended on $\mathbb{R}^d \times (0,\infty) \setminus B_R \times (0,T)$.) Using [38, 2. Theorem (2), p. 52], there exists a constant $C_1 > 0$ which is independent of g, such that

$$\begin{aligned}
& \mathbb{E}_x \left[\int_0^{T \wedge D_R} 2^{-\frac{d}{d+1}} \det(A)^{\frac{1}{d+1}} \cdot \psi^{-\frac{d}{d+1}} g(X_s, s) ds \right] \\
\leq \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}} \cdot \mathbb{E}_x \left[\int_0^{T \wedge D_R} e^{-\int_0^s \|\mathbf{G}(X_u)\| du} \cdot \det\left(\frac{1}{2}\widehat{A}\right)^{\frac{1}{d+1}} g(X_s, s) ds \right] \\
\leq \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}} \cdot C_1 \|g\|_{L^{d+1}(B_R \times (0,\infty))} \\
= \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}} \cdot C_1 \|g\|_{L^{d+1}(B_R \times (0,T))}.
\end{aligned}$$

Let $f \in L^{2d+2,d+1}_{loc}(B_R \times (0,T))$. Replacing g with $2^{\frac{d}{d+1}} \cdot \det(A)^{-\frac{1}{d+1}} \psi^{\frac{d}{d+1}} f$, we have

$$\begin{aligned}
& \mathbb{E}_x \left[\int_0^{T \wedge D_R} f(X_s, s) ds \right] \\
\leq \;& e^{T\|\mathbf{G}\|_{L^\infty(B_R)}} \cdot C_1 \|2^{\frac{d}{d+1}} \cdot \det(A)^{-\frac{1}{d+1}} \psi^{\frac{d}{d+1}} f\|_{L^{d+1}(B_R \times (0,T))} \\
\leq \;& \underbrace{2^{\frac{d}{d+1}} e^{T\|\mathbf{G}\|_{L^\infty(B_R)}} \cdot C_1 \|\det(A)^{-\frac{1}{d+1}}\|_{L^\infty(B_R)} \|\psi\|^{\frac{2d}{2d+2}}_{L^{2d}(B_R)}}_{=:C} \|f\|_{L^{2d+2,d+1}(B_R \times (0,T))}.
\end{aligned}$$

The last property follows from Itô's formula applied with $f \in C_0^2(\mathbb{R}^d)$, i.e.

$$\begin{aligned}&\mathbb{E}_x\left[f(X_t)-f(X_s)-\int_s^t\Big(\frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 f)+\langle \mathbf{G},\nabla f\rangle\Big)(X_u)du\Big|\,\mathcal{F}_s\right]\\ =\ &\mathbb{E}_x\left[\int_s^t \nabla f(X_u)\widehat{\sigma}(X_u)dW_u\Big|\,\mathcal{F}_s\right]=0,\quad 0\le s<t<\infty,\end{aligned}$$

since all coefficients are locally bounded.

□

Theorem 9.3.2 (Local Itô's formula for weakly differentiable functions). *Let $R_0>0$, $T>0$. Assume $u\in W^{2,1}_{2d+2}(B_{R_0}\times(0,T))\cap C(\overline{B}_{R_0}\times[0,T])$ satisfying $\|\nabla u\|\in L^{4d+4}(B_{R_0}\times(0,T))$. Let $R>0$ with $R<R_0$. If $(X_t)_{t\ge 0}$ satisfies (9.6), then $\mathbb{P}_x$-a.s. for any $x\in\mathbb{R}^d$,*

$$u(X_{T\wedge D_R},T\wedge D_R)-u(x,0)=\int_0^{T\wedge D_R}\nabla u(X_s,s)\widehat{\sigma}(X_s)dW_s+\int_0^{T\wedge D_R}(\partial_t u+Lu)(X_s,s)ds.$$

where $Lu:=\frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 u)+\langle \mathbf{G},\nabla u\rangle$.

Proof Take $T_0>0$ satisfying $T_0>T$. Extend u to $\overline{B}_{R_0}\times[-T_0,T_0]$ by

$$u(x,t)=u(x,0)\ \text{ for }\ -T_0\le t<0,\quad u(x,t)=u(x,T)\ \text{ for } T<t\le T_0,\, x\in\overline{B}_{R_0}.$$

Then it holds

$$u\in W^{2,1}_{2d+2}(B_{R_0}\times(0,T))\cap C(\overline{B}_{R_0}\times[-T,T])\ \text{ and }\ \|\nabla u\|\in L^{4d+4}(B_{R_0}\times(-T_0,T_0)).$$

For sufficiently large $n\in\mathbb{N}$, let ζ_n be a standard mollifier on $\mathbb{R}^{d+1}$ and $u_n:=u*\zeta_n$. Then it holds $u_n\in C^\infty(\overline{B}_R\times[0,T])$, such that $\lim_{n\to\infty}\|u_n-u\|_{W^{2,1}_{2d+2}(B_R\times(0,T))}=0$ and $\lim_{n\to\infty}\|\nabla u_n-\nabla u\|_{L^{4d+4}(B_R\times(0,T))}=0$. By Itô's-formula, for $x\in\mathbb{R}^d$, it holds for any $n\ge 1$

$$\begin{aligned}&u_n(X_{T\wedge D_R},T\wedge D_R)-u_n(x,0)\\ =\ &\int_0^{T\wedge D_R}\nabla u_n(X_s,s)\,\widehat{\sigma}(X_s)dW_s+\int_0^{T\wedge D_R}(\partial_t u_n+Lu_n)(X_s,s)ds,\quad \mathbb{P}_x\text{-a.s.}\end{aligned}\tag{9.7}$$

By Sobolev embedding, there exists a constant $C > 0$, independent of u_n and u, such that

$$\sup_{\overline{B}_R \times [0,T]} |u_n - u| \le C\|u_n - u\|_{W^{1,2}_{2d+2}(B_R \times (0,T))}.$$

Thus $\lim_{n\to\infty} u_n(x,0) = u(x,0)$ and

$$u_n(X_{T\wedge D_R}, T \wedge D_R) \text{ converges } \mathbb{P}_x\text{-a.s. to } u(X_{T\wedge D_R}, T\wedge D_R) \text{ as } n \to \infty.$$

By Theorem 9.3.1,

$$\begin{aligned}
&\mathbb{E}_x\left[\left|\int_0^{T\wedge D_R} (\partial_t u_n + Lu_n)(X_s,s)ds - \int_0^{T\wedge D_R}(\partial_t u + Lu)(X_s,s)ds\right|\right] \\
&\le \mathbb{E}_x\left[\int_0^{T\wedge D_R} |\partial_t u - \partial_t u_n|(X_s,s)ds\right] + \mathbb{E}_x\left[\int_0^{T\wedge D_R}|Lu - Lu_n|ds\right] \\
&\le C\|\partial_t u_n - \partial_t u\|_{L^{2d+2,d+1}(B_R\times(0,T))} + C\|Lu - Lu_n\|_{L^{2d+2,d+1}(B_R\times(0,T))} \\
&\longrightarrow 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where $C > 0$ is a constant which is independent of u and u_n.

Using Jensen's inequality, Itô sometry, and Theorem 9.3.1, we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{T\wedge D_R} (\nabla u_n(X_s,s) - \nabla u(X_s,s))\, \widehat{\sigma}(X_s)dW_s\right] \\
&\le \mathbb{E}_x\left[\left|\int_0^{T\wedge D_R} (\nabla u_n(X_s,s) - \nabla u(X_s,s))\, \widehat{\sigma}(X_s)dW_s\right|^2\right]^{1/2} \\
&= \mathbb{E}_x\left[\int_0^{T\wedge D_R} \left\|\, (\nabla u_n(X_s,s) - \nabla u(X_s,s))\, \widehat{\sigma}(X_s)\right\|^2 ds\right]^{1/2} \\
&\le C\|(\nabla u_n - \nabla u)\widehat{\sigma}\|_{L^{4d+4,2d+2}(B_R\times(0,T))} \\
&\le CC'\|\widehat{\sigma}\|_{L^\infty(B_R)}\|\nabla u_n - \nabla u\|_{L^{4d+4,2d+2}(B_R\times(0,T))} \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Letting $n \to \infty$ in (9.7), our assertion holds.

□

Theorem 9.3.3. *Assume* **(A4′)** *and let* $q_0 > 2d+2$ *be such that* $\frac{1}{q_0} + \frac{1}{q} = \frac{1}{2d+2}$. *If* $u \in D(L_{q_0})$, *then* $u \in H^{2,2d+2}_{loc}(\mathbb{R}^d)$. *Moreover given an open ball* B *in* $\mathbb{R}^d$, *there exists a constnat* $C > 0$, *independent of* u, *such that*

$$\|u\|_{H^{2,2d+2}(B)} \le C\|u\|_{D(L_{q_0})}.$$

Proof By the assumption **(A4′)** and Theorem 8.3.1, $\rho \in H^{1,2d+2}_{loc}(\mathbb{R}^d) \cap C^{0,1-\frac{d}{2d+2}}_{loc}(\mathbb{R}^d)$ and $\rho\psi\mathbf{B} \in L^{2d+2}_{loc}(\mathbb{R}^d)$. Let $f \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$. Then by (8.34)

$$\begin{aligned}&\int_{\mathbb{R}^d} \langle \frac{1}{2}\rho A \nabla G_\alpha f, \nabla\varphi\rangle dx - \int_{\mathbb{R}^d} \langle \rho\psi\mathbf{B}, \nabla G_\alpha f\rangle \varphi\, dx + \int_{\mathbb{R}^d} (\alpha\rho\psi G_\alpha f)\,\varphi dx \\ &= \int_{\mathbb{R}^d} (\rho\psi f)\,\varphi dx, \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d).\end{aligned} \tag{9.8}$$

Let $\widetilde{q} := \left(\frac{1}{2d+2} + \frac{1}{d}\right)^{-1}$. Then $\alpha\rho\psi \in L^{2d+2}_{loc}(\mathbb{R}^d) \subset L^{\widetilde{q}}_{loc}(\mathbb{R}^d)$, $\rho\psi f \in L^{2d+2}_{loc}(\mathbb{R}^d) \subset L^{\widetilde{q}}_{loc}(\mathbb{R}^d)$, hence by [12, Theorem 1.8.3], $G_\alpha f \in H^{1,2d+2}_{loc}(\mathbb{R}^d)$. Moreover, using [12, Theorem 1.7.4] and the resolvent contraction property, for any open balls V, V' in $\mathbb{R}^d$ with $\overline{V} \subset V'$, there exists a constant $\widetilde{C} > 0$, independent of f, such that

$$\begin{aligned}&\|G_\alpha f\|_{H^{1,2d+2}(V)} \\ &\le \widetilde{C}(\|G_\alpha f\|_{L^1(V')} + \|\rho\psi f\|_{L^{\widetilde{q}}(V')}) \\ &\le \widetilde{C}(\|G_\alpha f\|_{L^1(V')} + \|\rho\psi\|_{L^{2d+2}(V')}\|f\|_{L^d(V')}) \\ &\le \widetilde{C} \cdot \left(\frac{1}{\inf_{V'}\rho\psi}\right)^{\frac{1}{q_0}} (\alpha^{-1}|V'|^{1-\frac{1}{q_0}} + \|\rho\psi\|_{L^{2d+2}(V')}|V'|^{\frac{1}{d}-\frac{1}{q_0}})\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}.\end{aligned} \tag{9.9}$$

Set $\widetilde{C}_1 := \left(\frac{1}{\inf \rho\psi}\right)^{\frac{1}{q_0}} (\alpha^{-1}|V'|^{1-\frac{1}{q_0}} + \|\rho\psi\|_{L^{2d+2}(V')}|V'|^{\frac{1}{d}-\frac{1}{q_0}})$. Using Morrey's inequality and (9.9), there exists a constant $\widetilde{C}_2 > 0$ which is independent of f such that

$$\|G_\alpha f\|_{L^\infty(V)} \le \widetilde{C}_2\widetilde{C}\widetilde{C}_1\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}. \tag{9.10}$$

Now set

$$h_1 := \langle \rho\psi\mathbf{B}, \nabla G_\alpha f\rangle - \alpha\rho\psi G_\alpha f + \rho\psi f \in L^{d+1}_{loc}(\mathbb{R}^d).$$

Then (9.8) implies

$$\int_{\mathbb{R}^d} \langle \frac{1}{2}\rho A \nabla G_\alpha f, \nabla \varphi \rangle dx = \int_{\mathbb{R}^d} h_1 \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d). \tag{9.11}$$

Let U_1, U_2 be open balls in $\mathbb{R}^d$ satisfying $\overline{B} \subset U_1 \subset \overline{U}_1 \subset U_2$.
Take $\zeta_1 \in C_0^\infty(U_2)$ such that $\zeta_1 \equiv 1$ on $\overline{U}_1$. Then using integration by parts, and (9.11)

$$\begin{aligned}
&\int_{U_2} \langle \frac{1}{2}\rho A \nabla(\zeta_1 G_\alpha f), \nabla \varphi \rangle dx = \int_{U_2} \langle \frac{1}{2}\rho A \nabla G_\alpha f, \zeta_1 \nabla \varphi \rangle dx + \int_{U_2} \frac{1}{2}\langle A\nabla \zeta_1, \nabla \varphi \rangle \rho G_\alpha f dx \\
&= \int_{U_2} \langle \frac{1}{2}\rho A \nabla G_\alpha f, \nabla(\zeta_1 \varphi) \rangle dx - \int_{U_2} \underbrace{\langle \frac{1}{2}\rho A \nabla G_\alpha f, \nabla \zeta_1 \rangle}_{=:h_2} \varphi dx \\
&\quad + \int_{U_2} \underbrace{-\frac{1}{2}\Big(\langle G_\alpha f \nabla \rho + \rho \nabla G_\alpha f, A \nabla \zeta_1 \rangle + \rho G_\alpha f \langle \nabla A, \nabla \zeta_1 \rangle + \rho G_\alpha f \operatorname{trace}(A \nabla^2 \zeta_1) \Big)}_{=:h_3} \varphi dx \\
&= \int_{U_2} (h_1 \zeta_1 - h_2 + h_3) \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(U_2).
\end{aligned} \tag{9.12}$$

Note that $h_2, h_3 \in L^{2d+2}_{loc}(\mathbb{R}^d)$. Let $h_4 := \langle \frac{1}{2}\nabla(\rho A), \nabla(\zeta_1 G_\alpha f) \rangle \in L^{d+1}_{loc}(\mathbb{R}^d)$. Using (9.12),

$$\begin{aligned}
&\int_{U_2} \langle \frac{1}{2}\rho A \nabla(\zeta_1 G_\alpha f), \nabla \varphi \rangle dx + \int_{U_2} \langle \frac{1}{2}\nabla(\rho A), \nabla(\zeta_1 G_\alpha f) \rangle \varphi dx \\
= &\int_{U_2} (h_1 \zeta_1 - h_2 + h_3 + h_4) \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(U_2).
\end{aligned} \tag{9.13}$$

We have $h := h_1 \zeta_1 - h_2 + h_3 + h_4 \in L^{d+1}_{loc}(\mathbb{R}^d)$ and

$$\|h\|_{L^{d+1}(U_2)} \le C_2 (\|G_\alpha f\|_{H^{1,2d+2}(U_2)} + \|\rho \psi f\|_{L^{d+1}(U_2)}), \tag{9.14}$$

where $C_2 > 0$ is a constant which is independent of f. By [27, Theorem 9.15], there exists $w \in H^{2,d+1}(U_2) \cap H_0^{1,d+1}(U_2)$ such that

$$-\frac{1}{2}\operatorname{trace}(\rho A \nabla^2 w) = h \quad \text{a.e. on } U_2. \tag{9.15}$$

Furthermore, using [27, Lemma 9.17], (9.14), (9.9), there exists a constant $C_1 > 0$ which is independent of f such that

$$\begin{aligned}\|w\|_{H^{2,d+1}(U_2)} &\leq C_1\|h\|_{L^{d+1}(U_2)} \\ &\leq C_1C_2\left(\|G_\alpha f\|_{H^{1,2d+2}(U_2)} + \|\rho\psi f\|_{L^{d+1}(U_2)}\right) \\ &\leq C_1C_2C_3\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)},\end{aligned}$$

where $C_3 := \widetilde{C}_1 + \|\rho\psi\|_{L^{2d+2}(U_2)}|U_2|^{\frac{1}{2d+2}-\frac{1}{q_0}}\left(\frac{1}{\inf_{V'}\rho\psi}\right)^{\frac{1}{q_0}}$. Note that (9.15) implies

$$\begin{aligned}&\int_{U_2}\langle\frac{1}{2}\rho A\nabla w, \nabla\varphi\rangle dx + \int_{U_2}\langle\frac{1}{2}\nabla(\rho A), \nabla w\rangle\varphi dx \\ =\ &\int_{U_2} h\varphi dx, \quad \text{for all } \varphi \in C_0^\infty(U_2).\end{aligned} \tag{9.16}$$

Using the maximum principle of [77, Theorem 1] and comparing (9.16) with (9.13), we obtain $\zeta G_\alpha f = w$ on U_2, hence $G_\alpha f = w$ on U_1, so that $G_\alpha f \in H^{2,d+1}(U_1)$. Therefore, by Morrey's inequality, we obtain $\partial_i G_\alpha f \in L^\infty(U_1),\ 1 \leq i \leq d$, and

$$\begin{aligned}\|\partial_i G_\alpha f\|_{L^\infty(U_1)} &\leq C_4\|G_\alpha f\|_{H^{2,d+1}(U_1)} \\ &\leq C_4\|w\|_{H^{2,d+1}(U_2)} \\ &\leq C_1C_2C_3C_4\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)},\end{aligned} \tag{9.17}$$

where $C_4 > 0$ is a constant which is independent of f. Thus we obtain $h \in L^{2d+2}(U_1)$. Now take $\zeta_2 \in C_0^\infty(U_1)$ such that $\zeta_2 \equiv 1$ on $\overline{B}$. Note that a.e. on U_1 it holds

$$\begin{aligned}&-\frac{1}{2}\text{trace}\left(\rho A\nabla^2(\zeta_2 G_\alpha f)\right) \\ &= -\frac{1}{2}\zeta_2\cdot\text{trace}(\rho A\nabla^2 G_\alpha f) - \frac{1}{2}G_\alpha f\cdot\text{trace}(\rho A\nabla^2\zeta_2) - \langle\rho A\nabla\zeta_2, \nabla G_\alpha f\rangle. \\ &= -\frac{1}{2}\zeta_2 h - \frac{1}{2}G_\alpha f\cdot\text{trace}(\rho A\nabla^2\zeta_2) - \langle\rho A\nabla\zeta_2, \nabla G_\alpha f\rangle =: \widetilde{h}.\end{aligned}$$

Since $\|\nabla G_\alpha f\| \in L^\infty(U_1)$, $\widetilde{h} \in L^{2d+2}(U_1)$, by [27, Theorem 9.15], we get $\zeta_2 G_\alpha f \in H^{2,2d+2}(U_1)$, hence $G_\alpha f \in H^{2,2d+2}(B)$. Using [27, Lemma 9.17], (9.10), (9.17), there exist positive constants C_5, C_6 which are independent of f such that

$$\begin{aligned}\|G_\alpha f\|_{H^{2,2d+2}(B)} &\le \|\zeta_2 G_\alpha\|_{H^{2,2d+2}(U_1)}\\ &\le C_5\|\widetilde{h}\|_{L^{2d+2}(U_1)}\\ &\le C_5C_6(\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}+\|\rho\psi f\|_{L^{2d+2}(U_1)})\\ &\le C_5C_6(\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}+\|\rho\psi\|_{L^q(U_1)}(\inf_U \rho\psi)^{-1/q_0}\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)})\\ &\le C\|f\|_{L^{q_0}(\mathbb{R}^d,\mu)}, \qquad (9.18)\end{aligned}$$

where $C := C_5C_6(1\vee \|\rho\psi\|_{L^q(U_1)}(\inf_U \rho\psi)^{-1/q_0})$. Using the denseness of $C_0^\infty(\mathbb{R}^d)$ in $L^{q_0}(\mathbb{R}^d,\mu)$, (9.18) extends to $f\in L^{q_0}(\mathbb{R}^d,\mu)$. Now let $u\in D(L_{q_0})$, Then $(1-L_{q_0})u\in L^{q_0}(\mathbb{R}^d,\mu)$, hence by (9.18), it holds $u=G_1(1-L_{q_0})u\in H^{2,2d+2}_{loc}(\mathbb{R}^d)$ and

$$\begin{aligned}\|u\|_{H^{2,2d+2}(B)} &= \|G_1(1-L_{q_0})u\|_{H^{2,2d+2}(B)}\\ &\le C\|(1-L_{q_0})u\|_{L^{q_0}(\mathbb{R}^d,\mu)}\\ &\le C\|u\|_{D(L_{q_0})}.\end{aligned}$$

□

Theorem 9.3.4. *Assume* **(A1)**, **(A2)**. *Let $f\in D(\overline{L})_b\cap D(L_s)\cap D(L_2)$ and define*

$$u_f := P_\cdot f \in C(\mathbb{R}^d\times[0,\infty))$$

as in Lemm 8.3.4. Then for any open set U in $\mathbb{R}^d$ and $T>0$,

$$\partial_t u_f,\, \partial_i u_f \in L^{2,\infty}(U\times(0,T)) \text{ for all } 1\le i\le d,$$

and for each $t\in(0,T)$, it holds

$$\partial_t u_f(\cdot,t) = T_tL_2 f\in L^2(U), \quad \text{and} \quad \partial_i u_f(\cdot,t)=\partial_i P_t f\in L^2(U).$$

Furthermore, if we additionally assume $f\in D(L_{q_0})$ and **(A4′)**, *then $\partial_i\partial_j u_f\in L^{2d+2,\infty}(U\times(0,T))$ for all $1\le i,j\le d$, and for each $t\in(0,T)$, it holds*

$$\partial_i\partial_j u_f(\cdot,t) = \partial_i\partial_j P_t f \in L^{2d+2}(U).$$

Proof Assume **(A1)**, **(A2)**. Let $f \in D(\overline{L})_b \cap D(L_s) \cap D(L_2)$ and $t > 0$, $t_0 \geq 0$. Then by Theorem 8.2.7(c),

$$P_{t_0} f = \overline{T}_{t_0} f \in D(\overline{L})_b \subset D(\mathcal{E}^0),$$

where $\overline{T}_0 := id$. Observe that by Theorem 8.2.7(c), for any open ball B in $\mathbb{R}^d$ with $\overline{U} \subset B$,

$$\begin{aligned}
&\|\nabla P_t f - \nabla P_{t_0} f\|^2_{L^2(B)} \\
&\leq (\lambda_B \inf_B \rho)^{-1} \int_B \langle A\nabla(P_t f - P_{t_0} f), \nabla(P_t f - P_{t_0} f)\rangle \rho dx \\
&\leq 2(\lambda_B \inf_B \rho)^{-1} \mathcal{E}^0(P_t f - P_{t_0} f, P_t f - P_{t_0} f) \\
&\leq 2(\lambda_B \inf_B \rho)^{-1} \int_{\mathbb{R}^d} -\overline{L}(\overline{T}_t f - \overline{T}_{t_0} f) \cdot (\overline{T}_t f - \overline{T}_{t_0} f) d\mu \\
&\leq 4(\lambda_B \inf_B \rho)^{-1} \|f\|_{L^\infty(\mathbb{R}^d,\mu)} \|\overline{T}_t \overline{L} f - \overline{T}_{t_0} \overline{L} f\|_{L^1(\mathbb{R}^d,\mu)}. \qquad (9.19)
\end{aligned}$$

Likewise,

$$\|\nabla P_t f\|^2_{L^2(B)} \leq 2(\lambda_B \inf_B \rho)^{-1} \|f\|_{L^\infty(\mathbb{R}^d,\mu)} \|\overline{T}_t \overline{L} f\|_{L^1(\mathbb{R}^d,\mu)}.$$

For each $i = 1, \ldots, d$, define a map

$$\partial_i P_\cdot f : [0, T] \to L^2(U), \quad t \mapsto \partial_i P_t f.$$

Then by (9.19) and the $L^1(\mathbb{R}^d, \mu)$-strong continuity of $(\overline{T}_t)_{t>0}$, the map $\partial_i P_\cdot f$ is continuous with respect to the $\|\cdot\|_{L^2(B)}$-norm, hence by [48, Theorem, p91](or [12, Exercise 1.8.15]), there exists a Borel measurable function u^i_f on $U \times (0, T)$ such that for each $t \in (0, T)$ it holds

$$u^i_f(\cdot, t) = \partial_i P_t f \in L^2(U).$$

Thus using (9.19) and the $L^1(\mathbb{R}^d, \mu)$-contraction property of $(\overline{T}_t)_{t>0}$, it holds $u^i_f \in$

$L^{2,\infty}(U \times (0,T))$ and

$$\begin{aligned} \|u_f^i\|_{L^{2,\infty}(U\times(0,T))} &= \sup_{t\in(0,T)} \|\partial_i P_t f\|_{L^2(U)} \\ &\le 2(\lambda_B \inf_B \rho)^{-1/2} \|f\|^{1/2}_{L^\infty(\mathbb{R}^d,\mu)} \|\overline{L} f\|^{1/2}_{L^1(\mathbb{R}^d,\mu)}. \end{aligned}$$

Now let $\varphi_1 \in C_0^\infty(U)$ and $\varphi_2 \in C_0^\infty((0,T))$. Then

$$\begin{aligned} \iint_{U\times(0,T)} u_f \cdot \partial_i(\varphi_1\varphi_2) dxdt &= \int_0^T \left(\int_U P_t f \cdot \partial_i \varphi_1 dx \right) \varphi_2 dt \\ &= \int_0^T - \left(\int_U \partial_i P_t f \cdot \varphi_1 dx \right) \varphi_2 dt \\ &= -\iint_U u_f^i \cdot \varphi_1 \varphi_2 dxdt. \end{aligned} \tag{9.20}$$

Using the approximation as in Theorem 8.4.5, $\partial_i u_f = u_f^i \in L^{2,\infty}(U \times (0,T))$.
Now define a map

$$T.L_2 f : [0,T] \to L^2(U), \quad t \mapsto T_t L_2 f,$$

where $T_0 := id$. Since

$$\|T_t L_2 f - T_{t_0} L_2 f\|_{L^2(U)} \le (\inf_U \rho\psi)^{-1/2} \|T_t L_2 f - T_{t_0} L_2 f\|_{L^2(\mathbb{R}^d,\mu)},$$

using the $L^2(\mathbb{R}^d,\mu)$-strong continuity of $(T_t)_{t>0}$ and [48, Theorem, p91](or [12, Exercise 1.8.15]), there exists a Borel measurable function u_f^0 on $U \times (0,T)$ such that for each $t \in (0,T)$ it holds

$$u_f^0(\cdot,t) = T_t L_2 f \in L^2(U).$$

Using the $L^2(\mathbb{R}^d,\mu)$-contraction property of $(T_t)_{t>0}$, it holds $u_f^0 \in L^{2,\infty}(U \times (0,T))$ and

$$\begin{aligned} \|u_f^0\|_{L^{2,\infty}(U\times(0,T))} &= \sup_{t\in(0,T)} \|T_t L_2 f\|_{L^2(U)} \\ &\le (\inf_U \rho\psi)^{-1/2} \|L_2 f\|_{L^2(\mathbb{R}^d,\mu)}. \end{aligned}$$

Observe that

$$\begin{aligned}\iint_{U\times(0,T)} u_f \cdot \partial_t(\varphi_1\varphi_2)dxdt &= \int_0^T \left(\int_U T_t f\cdot\varphi_1 dx\right)\partial_t\varphi_2 dt \\ &= \int_0^T -\left(\int_U T_t L_2 f\cdot \varphi_1 dx\right)\varphi_2 dt \\ &= -\iint_U u_f^0\cdot\varphi_1\varphi_2 dxdt.\end{aligned}$$

Using the approximation of Theorem 8.4.5, we obtain $\partial_t u_f = u_f^0 \in L^{2,\infty}(U\times(0,T))$. Now assume **(A4′)**. Then by Theorem 9.3.3, $P_{t_0}f \in D(L_{q_0}) \subset H^{2,2d+2}_{loc}(\mathbb{R}^d)$ and for each $1\le i,j\le d$, it holds

$$\begin{aligned}&\|\partial_i\partial_j P_t f - \partial_i\partial_j P_{t_0} f\|_{L^{2d+2}(U)} \\ &\le \|P_t f - P_{t_0} f\|_{H^{2,2d+2}(U)} \\ &\le \|T_t f - T_{t_0} f\|_{L^{q_0}(\mathbb{R}^d,\mu)} + \|T_t L_{q_0} f - T_{t_0} L_{q_0} f\|_{L^{q_0}(\mathbb{R}^d,\mu)}\end{aligned} \tag{9.21}$$

Define a map

$$\partial_i\partial_j P_\cdot f : [0,T] \to L^2(U), \quad t\mapsto \partial_i\partial_j P_t f.$$

By the $L^{q_0}(\mathbb{R}^d,\mu)$-strong continuity of $(T_t)_{t>0}$ and (9.21), the map $\partial_i\partial_j P_\cdot f$ is continuous with respect to the $\|\cdot\|_{L^{2d+2}(U)}$-norm. Hence by [48, Theorem, p91](or [12, Exercise 1.8.15]), there exists a Borel measurable function u_f^{ij} on $U\times(0,T)$ such that for each $t\in(0,T)$, it holds

$$u_f^{ij}(\cdot,t) = \partial_i\partial_j P_t f.$$

Using Theorem 9.3.3 and the $L^{q_0}(\mathbb{R}^d,\mu)$-contraction property of $(T_t)_{t>0}$, $u_f^{ij}\in L^{2d+2,\infty}(U\times(0,T))$ and

$$\begin{aligned}\|u_f^{ij}\|_{L^{2d+2,\infty}(U\times(0,T))} &\le \sup_{t\in(0,T)} \|\partial_i\partial_j P_t f\|_{L^{2d+2}(U)} \\ &\le \sup_{t\in(0,T)} \|P_t f\|_{H^{2,2d+2}(U)} \\ &\le \sup_{t\in(0,T)} C\left(\|T_t f\|_{L^{q_0}(\mathbb{R}^d,\mu)} + \|T_t L_{q_0} f\|_{L^{q_0}(\mathbb{R}^d,\mu)}\right) \\ &\le C\|f\|_{D(L_{q_0})},\end{aligned}$$

where $C > 0$ is a constant which is independent of f. Using the same line of arguments as in (9.20) and the approximation as in Theorem 8.4.5,

$$\partial_i \partial_j u_f = u_f^{ij} \in L^{2d+2,\infty}(U \times (0,T)).$$

$\square$

Theorem 9.3.5. *Assume* **(A4′)** *and* $f \in C_0^\infty(\mathbb{R}^d)$. *Then there exists*

$$u_f \in C_b\left(\mathbb{R}^d \times [0,\infty)\right) \cap \Big(\bigcap_{r>0} W^{2,1}_{2d+2,\infty}(B_r \times (0,\infty)) \Big)$$

satisfying $u_f(x,0) = f(x)$ *for all* $x \in \mathbb{R}^d$ *such that*

$$\partial_t u_f \in L^\infty(\mathbb{R}^d \times (0,\infty)),\ \partial_i u_f \in \bigcap_{r>0} L^\infty(B_r \times (0,\infty)) \ \ \textit{for all } 1 \le i \le d,$$

and

$$\partial_t u_f = \frac{1}{2}\mathrm{trace}(\widehat{A}\nabla^2 u_f) + \langle \mathbf{G}, \nabla u_f \rangle \quad \textit{a.e. on } \mathbb{R}^d \times (0,\infty).$$

Proof Let $f \in C_0^\infty(\mathbb{R}^d)$. Then $f \in D(L_s)$. Define $u_f := P_\cdot f(\cdot)$. Then by Lemma 8.3.4, $u_f \in C_b(\mathbb{R}^d \times [0,\infty))$ and $u_f(x,0) = f(x)$ for all $x \in \mathbb{R}^d$. In particular, since $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, it holds $f \in D(L_{q_0})$, so that $P_t f \in D(L_{q_0})$ for any $t \ge 0$. By Theorem 9.3.4, for each $t > 0$, it holds $\partial_t u_f(\cdot,t) = T_t L_s f = T_t L f$ μ-a.e. on $\mathbb{R}^d$. Note that for each $t > 0$, using the sub-Markovian property,

$$\begin{aligned} \|\partial_t u_f(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} &= \|T_t L f\|_{L^\infty(\mathbb{R}^d)} \\ &\le \|Lf\|_{L^\infty(\mathbb{R}^d,\mu)}, \end{aligned}$$

hence $\partial_t u_f \in L^\infty(\mathbb{R}^d \times (0,\infty))$. By Theorem 9.3.4, for $1 \le i,j \le d$, $t > 0$, $\partial_i u_f(\cdot,t) = \partial_i P_t f$, $\partial_i \partial_j u_f(\cdot,t) = \partial_i \partial_j P_t f$ μ-a.e. on $\mathbb{R}^d$. Using Theorem 9.3.3 and the $L^{q_0}(\mathbb{R}^d,\mu)$-

contraction property of $(T_t)_{t>0}$, for any $R>0$ and for each $1 \le i,j \le d$, $t>0$, it holds

$$\begin{aligned}\|\partial_i\partial_j u_f(\cdot,t)\|_{L^{2d+2}(B_R)} &\le \|P_t f\|_{H^{2,2d+2}(B_R)}\\ &\le C\left(\|T_t f\|_{L^{q_0}(\mathbb{R}^d,\mu)} + \|T_t L_{q_0} f\|_{L^{q_0}(\mathbb{R}^d,\mu)}\right)\\ &\le C\|f\|_{D(L_{q_0})},\end{aligned}$$

where $C>0$ is as in Theorem 9.3.3 and independent of f. By Morrey's inequality, there exists a constant $C_{R,d}>0$ such that for each $t>0$, $1\le i\le d$,

$$\begin{aligned}\|\partial_i u_f(\cdot,t)\|_{L^\infty(B_R)} &\le \|\partial_i P_t f\|_{L^\infty(B_R)}\\ &\le C_{R,d}\|P_t f\|_{H^{2,2d+2}(B_R)}\\ &\le C_{R,d}C\|f\|_{D(L_{q_0})}.\end{aligned}$$

Thus, $u_f \in W^{2,1}_{2d+2,\infty}(B_R\times(0,\infty))$ and $\partial_t u_f, \partial_i u_f \in L^\infty(B_R\times(0,\infty))$ for all $1\le i\le d$. By (8.39), it holds

$$\begin{aligned}&\iint_{\mathbb{R}^d\times(0,\infty)} \Big\langle \frac{1}{2}\rho A\nabla u_f, \nabla\varphi\Big\rangle - \big\langle \rho\psi\mathbf{B}, \nabla u_f\big\rangle\varphi \, dxdt\\ &\quad = \iint_{\mathbb{R}^d\times(0,\infty)} -\partial_t u_f\cdot\varphi\rho\psi dxdt \quad \text{ for all } \varphi\in C_0^\infty(\mathbb{R}^d\times(0,\infty)).\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}&-\iint_{\mathbb{R}^d\times(0,\infty)} \Big(\frac{1}{2}\text{trace}(\widehat{A}\nabla^2 u_f) + \big\langle \beta^{\rho,\psi,A}+\mathbf{B}, \nabla u_f\big\rangle\Big)\varphi\, d\mu dt\\ &\quad = \iint_{\mathbb{R}^d\times(0,\infty)} -\partial_t u_f\cdot\varphi\, d\mu dt \quad \text{ for all } \varphi\in C_0^\infty(\mathbb{R}^d\times(0,\infty)).\end{aligned}$$

Therefore,

$$\partial_t u_f = \frac{1}{2}\text{trace}(\widehat{A}\nabla^2 u_f) + \langle \mathbf{G}, \nabla u_f\rangle \quad \text{a.e. on } \mathbb{R}^d\times(0,\infty).$$

□

Theorem 9.3.6. *Assume* **(A4′)***. Then uniqueness in law for* (9.6) *holds.*

Proof Assume both $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d})$ and $\widetilde{\mathbb{M}} = (\widetilde{\Omega}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, (\widetilde{X}_t)_{t\geq 0}, (\widetilde{\mathbb{P}}_x)_{x\in\mathbb{R}^d})$ satisfy (9.6). Let $f \in C_0^\infty(\mathbb{R}^d)$. For $T > 0$, define $g(x,t) := u_f(x, T-t)$, $(x,t) \in \mathbb{R}^d \times [0,T]$, where u_f is defined as in Theorem 9.3.5. Then by Theorem 9.3.5,

$$g \in C_b\left(\mathbb{R}^d \times [0,T]\right) \cap \Big(\bigcap_{r>0} W^{1,2}_{2d+2,\infty}(B_r \times (0,T))\Big),$$
$$\partial_t g \in L^\infty(\mathbb{R}^d \times (0,T)), \quad \partial_i g \in \bigcap_{r>0} L^\infty(B_r \times (0,T)),\ 1 \leq i \leq d,$$

and it holds

$$\frac{\partial g}{\partial t} + Lg = 0 \quad \text{a.e. in } \mathbb{R}^d \times (0,T), \qquad g(x,T) = f(x) \ \text{ for all } x \in \mathbb{R}^d.$$

Applying Theorem 9.3.1 to $\mathbb{M}$, for $x \in \mathbb{R}^d$, $R > 0$, it holds

$$\mathbb{E}_x\left[\int_0^{T\wedge D_R} \left|\frac{\partial g}{\partial t} + Lg\right|(X_s, s)ds\right] = 0,$$

hence

$$\int_0^{T\wedge D_R} \left(\frac{\partial g}{\partial t} + Lg\right)(X_s, s)ds = 0, \quad \mathbb{P}_x\text{-a.s.},$$

hence by Theorem 9.3.2,

$$g(X_{T\wedge D_R}, T\wedge D_R) - g(x,0) = \int_0^{T\wedge D_R} \nabla g(X_s, s)\widehat{\sigma}(X_s)dW_s, \qquad \mathbb{P}_x\text{-a.s.}$$

Therefore

$$\mathbb{E}_x\left[g(X_{T\wedge D_R}, T\wedge D_R)\right] = g(x,0).$$

Letting $R \to \infty$ and using Lebesgue's Theorem, we obtain

$$\mathbb{E}_x[f(X_T)] = \mathbb{E}_x[g(X_T, T)] = g(x,0).$$

Analogously for $\widetilde{\mathbb{M}}$, we obtain $\widetilde{\mathbb{E}}_x[f(\widetilde{X}_T)] = g(x,0)$. Thus

$$\mathbb{E}_x[f(X_T)] = \widetilde{\mathbb{E}}_x[f(\widetilde{X}_T)].$$

Since $\widehat{\sigma}$ and $\mathbf{G}$ are locally bounded on $\mathbb{R}^d$, we can apply the Markov-like property obtained in [37, Chapter 5, 4.19 Lemma]. Thus using the same way of proof as in [37, Chapter 5, 4.27 Proposition], the assertion follows.

□

Combining Theorem 9.3.6, Remark 9.1.4 and Theorem 9.1.5, we directly obtain the following result.

Theorem 9.3.7. *Under the assumption* **(A4′)***, suppose there exists a constant* $M > 0$ *and some* $N_0 \in \mathbb{N}$ *such that*

$$-\frac{\langle \widehat{A}(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\mathrm{trace}\widehat{A}(x) + \big\langle \mathbf{G}(x), x\big\rangle \leq M\left(\|x\|^2+1\right)\left(\ln(\|x\|^2+1)+1\right)$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$*. Then* $\mathbb{M}$ *from Theorem 9.1.3 is non-explosive and a unique solution to* (9.6) *in a weak sense.*

Remark 9.3.8. *Consider the situation in Example 9.2.3 except the conditions (1), (2), (3). Let* $p := 2d+2$ *and assume* $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$*. Let* $\alpha \geq 0$ *be such that* $\alpha(2d+2) < d$*. Take* $q \in (2d+2, \frac{d}{\alpha})$*. Then* A*,* $\mathbf{G}$*,* ψ *satisfy* **(A4′)***. Therefore, the Hunt process* $\mathbb{M}$ *of Theorem 9.1.5 solves weakly* $\mathbb{P}_x$*-a.s. for any* $x \in \mathbb{R}^d$*,*

$$X_t = x + \int_0^t \|X_s\|^{\alpha/2} \cdot \frac{1}{\sqrt{\phi}(X_s)} \cdot \sigma(X_s)\,dW_s + \int_0^t \mathbf{G}(X_s)\,ds, \quad 0 \leq t < \zeta, \tag{9.22}$$

Assume that there exists a constant $M > 0$ *and some* $N_0 \in \mathbb{N}$*, such that*

$$\frac{\|x\|^\alpha}{\phi(x)}\left(-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\mathrm{trace}A(x)\right) + \big\langle \mathbf{G}(x), x\big\rangle \leq M\left(\|x\|^2+1\right)\left(\ln(\|x\|^2+1)+1\right)$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$*. Then by Remark 9.1.4,* $\mathbb{M}$ *is non-explosive, i.e.* $\mathbb{P}_x(\zeta = \infty) = 1$ *for all* $x \in \mathbb{R}^d$*. In that case, by Theorem 9.3.6,* $\mathbb{M}$ *is the unique solution to* (9.22) *in a weak sense.*

Part IV

Existence and regularity of transition functions with general pre-invariant measures and corresponding Itô-SDEs

Chapter 10

Regularity results for weighted parabolic PDEs

In this Chapter, we derive some regularity results including the parabolic Harnack inequality of solutions to linear parabolic equations in divergence form involving a weight function. We adapt some methods from [2] to derive a fundamental inequality, but some technical details are at times different to those of [2] since our parabolic PDEs involve weight functions in the time derivative term which are bounded below and above by some positive constants. To derive our regularity results, consider the following condition.

(I′) U is a bounded open subset of $\mathbb{R}^d$ and $T > 0$. $u \in H^{1,2}(U \times (0,T)) \cap L^\infty(U \times (0,T))$. $A = (a_{ij})_{1 \le i,j \le d}$ is a matrix of functions on U that is strictly elliptic and bounded, i.e. there exists constants $\lambda > 0$, $M > 0$ such that for any $\xi = (\xi_1 \ldots, \xi_d) \in \mathbb{R}^d$, $x \in U$,

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \lambda \|\xi\|^2, \qquad \max_{1 \le i,j \le d} |a_{ij}(x)| \le M.$$

$\mathbf{B} \in L^p(U, \mathbb{R}^d)$ for some $p > d$. ψ is a positive function on U satisfying locally bounded below and above, i.e. there exists constants $c_0, c_1 > 0$ such that
$c_0 \le \psi \le c_1$ on U.

Assume **(I′)** and consider the following divergence form of linear parabilic equation with a weight function in time derivative term.

$$\iint_{U\times(0,T)} (u\partial_t\varphi)\psi dxdt = \iint_{U\times(0,T)} \big\langle A\nabla u, \nabla\varphi\big\rangle + \langle \mathbf{B}, \nabla u\rangle \varphi \, dxdt,$$
$$\text{for all } \varphi \in C_0^\infty(U\times(0,T)). \tag{10.1}$$

Let $\eta \in C_0^\infty(U\times(0,T])$. Noting that assumption **(I′)** is surely stronger than assumption **(I)** in Part III, through the same procedure as in Section 7.1, we first get for $\beta \geq 1$ and a.e. $\tau \in (0,T)$

$$\frac{1}{\beta+1}\int_U \eta^2 (u^+)^{\beta+1} \,|_{t=\tau} \,\psi dx + \frac{\lambda\beta}{2}\int_0^\tau \int_U \eta^2 (u^+)^{\beta-1}\|\nabla u^+\|^2 dxdt$$
$$\leq \int_0^\tau \int_U \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4M^2}{\lambda}\|\nabla\eta\|^2\Big)(u^+)^{\beta+1}dxdt + \frac{2}{\beta+1}\int_0^\tau \int_U \eta|\partial_t\eta|(u^+)^{\beta+1}\,\psi dx. \tag{10.2}$$

Furthermore, if $\chi_{(0,\tau)}$ is replaced by $\chi_{(\tau_1,\tau_2)}$ for a.e. $\tau_1.\tau_2 \in (0,T)$, then for $\beta \geq 1$,

$$\frac{1}{\beta+1}\int_U \eta^2 (u^+)^{\beta+1} \,|_{t=\tau_1}^{t=\tau_2} \,\psi dx + \frac{\lambda\beta}{2}\int_{\tau_1}^{\tau_2} \int_U \eta^2 (u^+)^{\beta-1}\|\nabla u^+\|^2 dxdt$$
$$\leq \int_{\tau_1}^{\tau_2} \int_U \Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4M^2}{\lambda}\|\nabla\eta\|^2\Big)(u^+)^{\beta+1}dxdt + \frac{2}{\beta+1}\int_{\tau_1}^{\tau_2} \int_U \eta|\partial_t\eta|(u^+)^{\beta+1}\,\psi dx. \tag{10.3}$$

Finally we need another type of the fundamental inequality to derive a parabolic Harnack inequality. Given $\varepsilon > 0$, let $\bar{u} := u + \varepsilon$ and $G(u) := \bar{u}^\beta$, where $\beta \in \mathbb{R}$ is fixed. Define

$$H(u) := \begin{cases} \frac{1}{\beta+1}\bar{u}^{\beta+1} & \text{if } \beta \neq 1 \\ \log \bar{u} & \text{if } \beta = -1 \end{cases}$$

so that $H'(u) = G(u)$. Given $[\tau_1, \tau_2] \subset (0,T)$ define $\widetilde{\varphi} := \eta^2 G(u)\chi_{(\tau_1,\tau_1)}$. Then

$$\nabla\widetilde{\varphi} = \begin{cases} \eta^2 G'(u)\nabla u + 2\eta\nabla\eta\, G(u) & \tau_1 < t < \tau_2 \\ 0 & t \in (0,T)\setminus(\tau_1,\tau_2) \end{cases}$$

Using the same procedure as in Section 7.1, we obtain

$$\int_U \eta^2 H(u)\,|_{t=\tau_1}^{t=\tau_2}\,\psi dxdt + \int_{\tau_1}^{\tau_2}\int_U \left\langle A\nabla u, \nabla\widetilde{\varphi}\right\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi} dxdt$$
$$= \int_{\tau_1}^{\tau_2}\int_U 2\eta\,\partial_t\eta\,H(u)\,\psi dxdt, \qquad \text{for a.e. } \tau_1, \tau_2 \in (0,T). \tag{10.4}$$

Observe that

$$\begin{aligned}
&\operatorname{sign}\beta\Big(\left\langle A\nabla u, \nabla\widetilde{\varphi}\right\rangle + \langle \mathbf{B}, \nabla u\rangle\widetilde{\varphi}\Big)\\
=\;& \left\langle A\nabla u, \eta^2|G'(u)|\nabla u\right\rangle + \operatorname{sign}\beta\Big(\left\langle A\nabla u, 2\eta\nabla\eta\, G(u)\right\rangle + \langle \mathbf{B}, \nabla u\rangle\eta^2 G(u)\Big)\\
\geq\;& \lambda\eta^2|G'(u)|\|\nabla\bar{u}\|^2 - 2\eta|G(u)|dM\|\nabla\eta\|\|\nabla\bar{u}\| - \eta^2|G(u)|\|\mathbf{B}\|\|\nabla\bar{u}\|.
\end{aligned}$$

and

$$|\beta|(\bar{u})^{-\beta-1}\,G(u)^2 = |G'(u)|.$$

Thus using Cauchy inequality we obtain

$$\begin{aligned}
&2\eta G(u)dM\|\nabla\eta\|\|\nabla\bar{u}\|\\
&\leq 2\cdot\frac{1}{4}\frac{\Big(\sqrt{\lambda}\,\sqrt{|\beta|}(\bar{u})^{-\frac{\beta+1}{2}}G(u)\,\eta\,\|\nabla u\|\Big)^2}{2} + 2\cdot 4\frac{\Big(dM\sqrt{\lambda^{-1}}\sqrt{|\beta|^{-1}}\,(\bar{u})^{\frac{\beta+1}{2}}\,\|\nabla\eta\|\Big)^2}{2}\\
&= \frac{\lambda}{4}\eta^2|G'(u)|\|\nabla u\|^2 + \frac{4d^2M^2}{\lambda|\beta|}\|\nabla\eta\|^2\,(\bar{u})^{\beta+1},
\end{aligned}$$

and

$$\begin{aligned}
&\eta^2 G(u)\|\mathbf{B}\|\|\nabla u^+\|\\
\leq\;& \frac{1}{2}\cdot\frac{\Big(\sqrt{\lambda}\sqrt{|\beta|}\,(\bar{u})^{-\frac{\beta+1}{2}}\,G(u)\eta\|\nabla\bar{u}\|\Big)^2}{2} + 2\cdot\frac{\Big(\sqrt{\lambda^{-1}}\sqrt{|\beta|^{-1}}\,(u^+)^{\frac{\beta+1}{2}}\|\mathbf{B}\|\eta\Big)^2}{2}\\
\leq\;& \frac{\lambda}{4}\eta^2|G'(u)|\|\nabla\bar{u}\|^2 + \frac{1}{\lambda|\beta|}\|\mathbf{B}\|^2(\bar{u})^{\beta+1}\eta^2.
\end{aligned}$$

Therefore for a.e. $\tau_1, \tau_2 \in (0,T)$,

$$\begin{aligned}&\mathrm{sign}\beta\left(\int_U \eta^2 H(u)\,|_{t=\tau_1}^{t=\tau_2}\,\psi dxdt + \frac{\lambda\beta}{2}\int_{\tau_1}^{\tau_2}\int_U \eta^2 \bar{u}^{\beta-1}\|\nabla \bar{u}\|^2 dxdt\right)\\ &\le \int_{\tau_1}^{\tau_2}\int_U \Big(\frac{\|\mathbf{B}\|^2}{\lambda|\beta|}\eta^2 + \frac{4d^2M^2}{\lambda|\beta|}\|\nabla\eta\|^2\Big)(\bar{u})^{\beta+1}dxdt + \int_{\tau_1}^{\tau_2}\int_U 2\eta\,\partial_t\eta\, H(u)\,\psi dxdt\end{aligned} \tag{10.5}$$

The following Theorem 10.1.1 which presents an estimate of the L^∞-norm in terms of the L^2-norm improves Theorem 7.1.2 in which an estimate of the L^∞-norm in terms of the $L^{\frac{2p}{p-2}}$-norm is given. To prove the following Theorem 10.1.1, we use the fundamental inequalities (10.2) and (10.4). Given $r > 0$ and a fixed $(\bar{x}, \bar{t}) \in U \times (0,T)$, let $Q(r) := R_{\bar{x}}(r) \times (\bar{t} - r^2, \bar{t})$ and $R_{\bar{x}}(r) := R(r)$.

10.1 L^∞-estimate in terms of the L^2-norm

Theorem 10.1.1. *Assume* **(I′)** *and* $Q(3r) \subset U \times (0,T)$. *If* (10.1) *holds, then*

$$\|u\|_{L^\infty(Q(r))} \le C\|u\|_{L^2(Q(3r))},$$

where C is a constant depending only on r, λ, M and $\|\mathbf{B}\|_{L^p(R(3r))}$.

Proof Let $\eta \in C_0^\infty(R_{\bar{x}}(r) \times (\bar{t} - 9r^2, \bar{t}])$. Then (10.2), (10.3) hold with $U \times (0,T)$ replaced by $Q(3r)$. Using appropriate scaling arguments (cf. [2, proof of Theorem 2]), we may assume $r = \frac{1}{3}$. By Theorem 7.1.2,

$$\|u^+\|_{L^\infty(Q(1/3))} \le \sqrt{C_3}\|u\|_{L^{\frac{2p}{p-2},2}(Q(2/3))}. \tag{10.6}$$

where $C_3 > 0$ is a constant from Theorem 7.1.2. Now choose a smooth function η so that $\eta \equiv 1$ in $Q(2/3)$, $\eta \equiv 0$ outside $Q(1)$ and $0 \le \eta \le 1$, $|\partial_t \eta|, \|\nabla\eta\| \le 8d$. We will estimate $\|u\|_{L^{\frac{2p}{p-2},2}(Q(2/3))}$ in terms of $\|u\|_{L^2(Q(1))}$. By (10.3) with $\beta = 1$, for a.e.

$\tau_1, \tau_2 \in (\bar{t}-1, \bar{t})$, we get

$$\begin{aligned}
&\int_{R(1)} \eta^2 (u^+)^2 \big|_{t=\tau_1}^{t=\tau_2} \psi dx + \lambda \int_{\tau_1}^{\tau_2} \int_{R(1)} \eta^2 \|\nabla u^+\|^2 dx dt \\
&\quad \le 2 \int_{\tau_1}^{\tau_2} \int_{R(1)} \Big(\frac{\|\mathbf{B}\|^2}{\lambda} \eta^2 + \frac{4d^2M^2}{\lambda} \|\nabla \eta\|^2 + c_1 \eta |\partial_t \eta| \Big) (u^+)^2 dx dt =: \mathcal{I} \qquad (10.7)
\end{aligned}$$

Note that

$$2 \iint_{R(1)\times(\tau_1,\tau_2)} \frac{\|\mathbf{B}\|^2}{\lambda} \eta^2 (u^+)^2 dx dt \le \underbrace{\frac{2}{\lambda} \|\mathbf{B}\|^2_{L^p(R(1))}}_{=:C_4} \|\eta u^+\|^2_{L^{\frac{2p}{p-2},2}(R(1)\times(\tau_1,\tau_2))}$$

and

$$\begin{aligned}
& 2 \iint_{R(1)\times(\tau_1,\tau_2)} \Big(\frac{4d^2M^2}{\lambda} \|\nabla \eta\|^2 + c_1 \eta |\partial_t \eta| \Big) (u^+)^2 dx dt \\
\le\ & \underbrace{128 d^2 \left(\frac{4d^2M^2}{\lambda} \vee c_1 \right)}_{=:C_5} \|u^+\|^2_{L^2(R(1)\times(\tau_1,\tau_2))}.
\end{aligned}$$

By [2, Lemma 3] we have

$$\begin{aligned}
\mathcal{I} \ &\le\ C_4 \|\eta u^+\|^2_{L^{\frac{2p}{p-2},2}(R(1)\times(\tau_1,\tau_2))} + C_5 \|u^+\|^2_{L^2(R(1)\times(\tau_1,\tau_2))} \\
&\le\ K C_4 (\tau_1 - \tau_2)^\theta \left(\|\eta u^+\|^2_{L^{2,\infty}(R(1)\times(\tau_1,\tau_2))} + \|\nabla(\eta u^+)\|^2_{L^2(R(1)\times(\tau_1,\tau_2))} \right) \\
&\qquad + C_5 \|u^+\|^2_{L^2(R(1)\times(\tau_1,\tau_2))} \\
&\le\ \left(c_0^{-1} \vee 2\right) K C_4 (\tau_1 - \tau_2)^\theta \Bigg(\sup_{t\in(\tau_1,\tau_2)} \int_{R(1)} \eta^2 (u^+)^2 \psi dx \\
&\qquad + \|(\eta \nabla u^+)\|^2_{L^2(R(1)\times(\tau_1,\tau_2))} + 64 d^2 \|u^+\|^2_{L^2(R(1)\times(\tau_1,\tau_2))} \Bigg) + C_5 \|u^+\|^2_{L^2(R(1)\times(\tau_1,\tau_2))},
\end{aligned}$$

where $K > 0$ is a constant as in [2, Lemma 3] and $\theta := 1 - \frac{d}{p}$ if $d \ge 3$, $\theta := \frac{1}{2} - \frac{1}{p}$ if $d = 2$.

Now set $\varepsilon := \left(\frac{\lambda\wedge 1}{2K(c_0^{-1}\vee 2)}\right)^{\theta}$. Consider $\tau_2 = t$ as in $t\in(\tau_1,\tau_1+\varepsilon)$ and define

$$Z(t) := \int_{R(1)} \eta^2 (u^+)^2 \psi dx$$

Then by (10.7),

$$\begin{aligned} & Z(t) + \frac{\lambda}{2}\int_{\tau_1}^{t}\int_{R(1)} \eta^2 \|\nabla u^+\|^2 dxdt \\ \leq\ & \frac{\lambda\wedge 1}{2}\Big(\sup_{t\in(\tau_1,\tau_1+\varepsilon)} Z(t) + 64d^2\|u^+\|^2_{L^2(R(1)\times(\tau_1,\tau_1+\varepsilon))}\Big) \\ & +C_5\|u^+\|^2_{L^2(R(1)\times(\tau_1,\tau_1+\varepsilon))} + Z(\tau_1). \end{aligned} \tag{10.8}$$

Taking the supremum over $t\in(\tau_1,\tau_1+\varepsilon)$ on the left hand side of (10.8), we get

$$\begin{aligned} & \sup_{t\in(\tau_1,\tau_1+\varepsilon)} Z(t) + \lambda\int_{\tau_1}^{t+\varepsilon}\int_{R(1)} \eta^2\|\nabla u^+\|^2 dxdt \\ \leq\ & \underbrace{\big(64d^2+2C_5\big)\|u^+\|_{L^2(Q(1))}}_{=:\Theta} + 2Z(\tau_1). \end{aligned}$$

Similarly, we obtain

$$\sup_{t\in(\tau_1+\varepsilon,\tau_1+2\varepsilon)} Z(t) + \lambda\int_{\tau_1+\varepsilon}^{t+2\varepsilon}\int_{R(1)} \eta^2\|\nabla u^+\|^2 dxdt \leq \Theta + 2\sup_{t\in(\tau_1,\tau_1+\varepsilon)} Z(t).$$

Hence by iterating these procedures for $1+[\frac{1}{\varepsilon}]$ times with starting time $\tau_1\in(\bar{t}-\frac{2}{3},\bar{t})$, we get

$$\sup_{t\in(\bar{t}-1,\bar{t})} Z(t) \leq 2^{1+\frac{1}{\varepsilon}}\cdot\Theta$$

and

$$\int_{\bar{t}-1}^{\bar{t}}\int_{R(1)} \eta^2\|\nabla u^+\|^2 dxdt \leq \frac{\Theta}{\lambda}(1+2^{2+\frac{1}{\varepsilon}})\left(1+\frac{1}{\varepsilon}\right). \tag{10.9}$$

Hence,

$$\begin{aligned} \|\eta u^+\|^2_{L^{2,\infty}(Q(1))} &\leq c_0^{-1} \sup_{t\in(\bar{t}-1,\bar{t})} Z(t) \\ &\leq \underbrace{c_0^{-1} 2^{1+\frac{1}{\varepsilon}}\left(64d^2+2C_5\right)}_{=:C_6} \|u^+\|_{L^2(Q(1))} \end{aligned} \tag{10.10}$$

and

$$\|\eta\nabla u^+\|^2_{L^2(Q(1))} \leq \underbrace{\frac{1}{\lambda}(1+2^{2+\frac{1}{\varepsilon}})\left(1+\frac{1}{\varepsilon}\right)\left(64d^2+2C_5\right)}_{=:C_7} \|u^+\|_{L^2(Q(1))}.$$

Therefore by (10.9), (10.10) and the similar way as in the proof of [2, Lemma 3],

$$\begin{aligned} & \|u^+\|^2_{L^{\frac{2p}{p-2},2}(Q(2/3))} \\ \leq\ & \|\eta u^+\|^2_{L^{\frac{2p}{p-2},2}(Q(2/3))} \\ \leq\ & K\left(\|\eta u^+\|^2_{L^{2,\infty}(Q(1))} + \|\nabla(\eta u^+)\|^2_{L^2(Q(1))}\right) \\ \leq\ & 2K\left(\|\eta u^+\|^2_{L^{2,\infty}(Q(1))} + \left(\sup_{Q(1)}\|\nabla\eta\|^2\right)\|u^+\|^2_{L^2(Q(1))} + \|\eta\nabla u^+\|^2_{L^2(Q(1))}\right) \\ \leq\ & \underbrace{2K(C_6 \vee 64d^2 \vee C_7)}_{=:C_8} \|u^+\|^2_{L^2(Q(1))}. \end{aligned} \tag{10.11}$$

Combining (10.11) and (10.6), we obtain

$$\|u^+\|_{L^\infty(Q(1/3))} \leq \sqrt{C_3C_8}\|u^+\|_{L^2(Q(1))}.$$

Exactly in the same way, but with u replaced by $-u$, (7.9) holds with $C = 2\sqrt{C_3C_8}$.

□

10.2 Parabolic Harnack inequality

In this section, we prove a parabolic Harnack inequality, which is one of the most important results to derive the L^1-strong Feller property and irreducibility and strict irreducibility of the semigroup. Before proving the parabolic Harnack inequality, we prove the following technical lemma which generalizes [53, Lemma 7]. The generalization results from considering weight functions ψ, which then lead to a modification of the original proof.

Lemma 10.2.1. *Let U be a bounded open subset in $\mathbb{R}^d$. Let $p \in C_0(U)$ be positive on U and satisfy that there exists a constant $L > 0$ such that*

$$\sup\{\|x - y\| \mid x, y \in supp(p)\} \le L.$$

Moreover assume $\{x \in U \mid p(x) \ge c\}$ is convex for any constant $c \ge 0$. Let ψ be a Borel measurable function on U such that $c_0 \le \psi \le c_1$ for some positive constants c_0, c_1. Then for any $v \in H^{1,2}_{loc}(U)$, it holds

$$\int_U (v(x) - V)^2 \, p(x) \, dx \le \Lambda \int_U |\nabla v(x)|^2 p(x) \, dx,$$

where

$$V = \frac{\int_U v(x) p(x) \psi(x) dx}{\int_U p(x)\psi(x) dx}, \qquad \Lambda = \left(\frac{c_1}{c_0}\right)^2 \frac{\omega_d L^{d+2} \max_U p}{2 \int_U p(x) dx}.$$

Proof Since $\text{supp}(p) \subset U$, we may assume that U has Lipschitz boundary and $u \in H^{1,2}(U)$ by appropriately shrinking U so that $\text{supp}(p) \subset U$. Observe that a constant V satisfies that

$$\begin{aligned} 2 \int_U p(x)\psi(x) dx \int_U (v(x) - V)^2 \, p(x)\psi(x) dx \\ = \int_U \int_U (v(x) - v(y))^2 \, p(x)p(y)\psi(x)\psi(y) \, dx \, dy \qquad (10.12) \end{aligned}$$

if and only if

$$
\begin{aligned}
&2\left(\int_U p(x)\psi(x)dx\right)^2 \cdot V^2 - 4\left(\int_U p(x)\psi(x)dx\right)\left(\int_U v(x)p(x)\psi(x)dx\right)\cdot V \\
&\qquad +2\left(\int_U p(x)\psi(x)dx\right)\left(\int_U v^2(x)p(x)\psi(x)dx\right) \\
&= \left(\int_U v^2(x)p(x)\psi(x)dx\right)\left(\int_U p(y)\psi(y)dy\right) + \left(\int_U v^2(y)p(y)\psi(y)dy\right)\left(\int_U p(x)\psi(x)dx\right) \\
&\qquad -2\left(\int_U v(x)p(x)\psi(x)dx\right)\left(\int_U v(y)p(y)\psi(y)dy\right)
\end{aligned}
$$

if and only if

$$
\left(\int_U p(x)\psi(x)dx \cdot V - \int_U v(x)p(x)\psi(x)dx\right)^2 = 0.
$$

Therefore if we set $V := \dfrac{\int_U v(x)p(x)\psi(x)dx}{\int_U p(x)\psi(x)dx}$, then (10.12) holds. Now by [21, Theorem 4.7], extend $v \in H^{1,2}(U)$ on $\mathbb{R}^d$, say again $v \in H^{1,1}(\mathbb{R}^d)_0$ and extend p on $\mathbb{R}^d$ by zero extension. Let η_n be a standard mollifier on $\mathbb{R}^d$ and $v_n := v * \eta_n$.

Let $x, y \in U$ be given and we may assume $p(x) \le p(y)$. Let $l_{x,y}$ be the oriented straight line segement from x to y. Then by the 'Fundamental Theorem of Calculus' and Hölder inequality,

$$
\begin{aligned}
& (v_n(x) - v_n(y))^2\, p(x)p(y) \\
=\;& \left(\int_{l_{x,y}} \nabla v_n ds\right)^2 \cdot p(x)p(y) \\
=\;& \left(\int_{l_{x,y}} (\sqrt{p}\,\nabla v_n)\,\frac{1}{\sqrt{p}}\,ds\right)^2 \cdot p(x)p(y) \\
\le\;& \left(\int_{l_{x,y}} \|\nabla v_n\|^2 p\, ds\right)\left(\int_{l_{x,y}} \frac{1}{p} ds\right) p(x)p(y).
\end{aligned}
$$

Since $\{x \in U \mid p(x) \ge c\}$ is convex for any constant $c \ge 0,$ we have $\min\limits_{l_{x,y}} p = p(x)$, so that

$$\int_{l_{x,y}} \frac{p(x)}{p} ds \leq \int_{l_{x,y}} 1 ds \leq L.$$

Therefore we have

$$(v_n(x) - v_n(y))^2 \, p(x)p(y) \leq \left(\int_{l_{x,y}} \|\nabla v_n\|^2 p \, ds \right) \cdot L \cdot \max_U p. \tag{10.13}$$

Using the Fubini Theorem,

$$\begin{aligned}
& \int_U \int_U (v_n(x) - v_n(y))^2 \, p(x)p(y) dx \, dy \\
= \; & \int_U \int_{U-x} (v_n(x) - v_n(x+z))^2 \, p(x)p(x+z) dz \, dx \\
\leq \; & \int_U \int_{B_L(\mathbf{0})} (v_n(x) - v_n(x+z))^2 \, p(x)p(x+z) dz \, dx \\
= \; & \int_{B_L(\mathbf{0})} \int_U (v_n(x) - v_n(x+z))^2 \, p(x)p(x+z) dx \, dz. \qquad (10.14)
\end{aligned}$$

Let $z \in B_L(\mathbf{0})$ and $x \in U$. If $x + z \in U$, then by (10.13),

$$\begin{aligned}
& (v_n(x) - v_n(x+z))^2 \, p(x)p(x+z) \\
\leq \; & L \cdot \max_U p \cdot \left(\int_{l_{x,x+z}} \|\nabla v_n\|^2 p \, ds \right) \\
\leq \; & L^2 \cdot \max_U p \cdot \int_0^1 \|\nabla v_n(x+tz)\|^2 \, p(x+tz) \, dt. \qquad (10.15)
\end{aligned}$$

If $x + z \in \mathbb{R}^d \setminus U$, then $(v_n(x) - v_n(x+z))^2 \, p(x)p(x+z) = 0$, hence (10.15) holds. Thus for any $z \in B_L(\mathbf{0})$, using Fubini Theorem,

$$\begin{aligned}
& \int_U (v_n(x) - v_n(x+z))^2 \, p(x)p(x+z)dx \\
\leq \quad & L^2 \cdot \max_U p \int_0^1 \int_U \|\nabla v_n(x+tz)\|^2 \, p(x+tz)dx\,dt \\
= \quad & L^2 \cdot \max_U p \int_0^1 \int_{U+tz} \|\nabla v_n(y)\|^2 \, p(y)dy\,dt \\
\leq \quad & L^2 \cdot \max_U p \int_U \|\nabla v_n(x)\|^2 \, p(x)dx. \qquad (10.16)
\end{aligned}$$

Combining (10.14) and (10.16), we obtain

$$\begin{aligned}
& \int_U \int_U (v_n(x) - v_n(y))^2 \, p(x)p(y)dx\,dy \\
& \leq \omega_d L^{d+2} \max_U p \int_U \|\nabla v_n(x)\|^2 \, p(x)dx, \qquad (10.17)
\end{aligned}$$

where $\omega_d := \int_{B_1} 1dx$. Since $p \in L^\infty(U)$ and $\lim_{n\to\infty} v_n = v$ in $H^{1,2}(U)$, letting $n \to \infty$ in (10.17), we obtain

$$\begin{aligned}
& \int_U \int_U (v(x) - v(y))^2 \, p(x)p(y)dx\,dy \\
& \leq \omega_d L^{d+2} \max_U p \int_U \|\nabla v(x)\|^2 \, p(x)dx.
\end{aligned}$$

Finally by (10.12) and (10.17) we have

$$\begin{aligned}
& 2c_0^2 \int_U p(x)dx \int_U (v(x)-V)^2 p(x)dx \\
\leq\ & 2\int_U p(x)\psi(x)dx \int_U (v(x)-V)^2 p(x)\psi(x)dx \\
=\ & \int_U\int_U (v(x)-v(y))^2 p(x)p(y)\psi(x)\psi(y)\,dx\,dy \\
\leq\ & c_1^2 \int_U\int_U (v(x)-v(y))^2 p(x)p(y)\,dx\,dy \\
\leq\ & c_1^2\cdot\omega_d L^{d+2}\max_U p \int_U \|\nabla v(x)\|^2 p(x)dx,
\end{aligned}$$

as desired.

□

Gven $(\bar{x},\bar{t})\in U\times(0,T)$, set $Q^*(r) := R_{\bar{x}}(r)\times(\bar{t}-8r^2,\bar{t}-7r^2)$.

Theorem 10.2.2. *Assume* **(I)**$'$ *and* $Q(3r)\subset U\times(0,T)$. *If* (10.1) *holds, then*

$$\sup_{Q^*(r)} u \leq C\inf_{Q(r)} u,$$

where $C>0$ *is a constant which is independent of* u.

Proof As in the proof of Theorem 10.1.1, we may assume $r=\frac{1}{3}$ and $U\times(0,T)\equiv Q(3r)$. Moreover considering a translation, we may assume $\bar{t}=1$. Given $\varepsilon>0$ define $\bar{u}:=u+\varepsilon$. For $\beta\in\mathbb{R}\setminus\{-1\}$, let $\gamma:=\frac{\beta+1}{2}$ and $v:=\bar{u}^\gamma$. Thus by (10.5), for a.e. $\tau_1,\tau_2\in(0,1)$, we have

$$\begin{aligned}
&\operatorname{sign}\beta\left(\frac{1}{\beta+1}\int_{R(1)}\eta^2v^2\,|_{t=\tau_1}^{t=\tau_2}\,\psi dxdt+\frac{\lambda\beta}{2\gamma^2}\int_{\tau_1}^{\tau_2}\int_{R(1)}\eta^2\|\nabla v\|^2dxdt\right)\\
&\leq\int_{\tau_1}^{\tau_2}\int_{R(1)}\Big(\frac{\|\mathbf{B}\|^2}{\lambda|\beta|}\eta^2+\frac{4d^2M^2}{\lambda|\beta|}\|\nabla\eta\|^2+\frac{2c_1}{|\beta+1|}\eta|\partial_t\eta|\Big)v^2=:\mathcal{I}
\end{aligned}\tag{10.18}$$

Step 1: Consider the case of $\beta>-1$. Given $s\in[1/3,1/2]$, set

$$S(s):=R(s)\times\Big(\frac{1}{6}(1-s),\frac{1}{6}(1+s)\Big).$$

Let $l, l' \in \mathbb{R}$ such that $\frac{1}{3} < l' < l \leq \frac{1}{2}$. Take $\eta \in C_0^\infty(S(l))$ so that $0 \leq \eta \leq 1$ on $S(l)$, $\eta \equiv 1$ on $S(l')$ and $\|\nabla \eta\| \leq \frac{2d}{l-l'}$, $|\partial_t \eta| \leq \frac{6}{l-l'}$ on $S(l)$. Then

$$\mathcal{I} \leq C_1 \left(|\beta|^{-1} \vee |\beta+1|^{-1}\right)(l-l')^{-2}\|v\|_{L^{\frac{2p}{p-2},2}(S(l))},$$

where $C_1 := 4d^2\left(\lambda^{-1}\|\mathbf{B}\|^2_{L^p(R(1))} + 4d^2M^2\lambda^{-1} + 2c_1\right)$.
For the case of $\beta > 0$ we set $\tau_1 := \frac{1}{12}$ and $\tau_2 := \tau$. Then we obtain

$$\begin{aligned}\lambda\|\eta\nabla v\|^2_{L^2(Q(1))} &\leq 2C_1\beta^{-2}(l-l')^{-2}\gamma^2\|v\|_{L^{\frac{2p}{p-2},2}(S(l))},\\ c_0\|\eta v\|^2_{L^{2,\infty}(Q(1))} &\leq 4C_1\beta^{-1}(l-l')^{-2}\gamma^2\|v\|_{L^{\frac{2p}{p-2},2}(S(l))}.\end{aligned}$$

For the case of $-1 < \beta < 0$ we set $\tau_1 = \tau$ and $\tau_2 = \frac{1}{4}$. Then we have

$$\begin{aligned}\lambda\|\eta\nabla v\|^2_{L^2(Q(1))} &\leq \frac{1}{2}C_1|\beta|^{-2}(l-l')^{-2}\|v\|_{L^{\frac{2p}{p-2},2}(S(l))},\\ c_0\|\eta v\|^2_{L^{2,\infty}(Q(1))} &\leq C_1|\beta|^{-1}(l-l')^{-2}\|v\|_{L^{\frac{2p}{p-2},2}(S(l))}.\end{aligned}$$

Therefore for any $\beta > -1$ with $\beta \neq 0$ we have

$$\begin{aligned}\|\eta\nabla v\|^2_{L^2(Q(1))} &\leq 2\lambda^{-1}C_1|\beta|^{-2}(l-l')^{-2}(1+\gamma^2)\|v\|_{L^{\frac{2p}{p-2},2}(S(l))},\\ \|\eta v\|^2_{L^{2,\infty}(Q(1))} &\leq 4c_0^{-1}C_1|\beta|^{-1}(l-l')^{-2}(1+\gamma^2)\|v\|_{L^{\frac{2p}{p-2},2}(S(l))}.\end{aligned}$$

Now set

$$\begin{aligned}\theta &:= 1 - \frac{d}{p}, \text{ and } \sigma := 1 + \frac{2\theta}{d} \text{ if } d \geq 3,\ \sigma := 1 + \frac{\theta}{2} \text{ if } d = 2,\\ p_\sigma &:= \left(\frac{\sigma p}{p-2}\right)' = \frac{\sigma p}{\sigma p - p + 2}, \qquad q_\sigma := \sigma' = \frac{\sigma}{\sigma - 1}.\end{aligned}$$

Then it holds

$$\frac{d}{2p_\sigma} + \frac{1}{q_\sigma} = 1 \text{ if } d \geq 3, \qquad \frac{d}{2p_\sigma} + \frac{1}{q_\sigma} < 1 \text{ if } d = 2.$$

By [2, Lemma 3],

$$\begin{aligned}
\|v^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2},2}(S(l'))} &\le \|(\eta v)^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2},2}(Q(1))} \\
&= \|\eta v\|^2_{L^{\frac{2\sigma p}{p-2},2\sigma}(Q(1))} \\
&= \|\eta v\|^2_{L^{2(p_\sigma)',2(q_\sigma)'}(Q(1))} \\
&\le K\Big(\|\eta v\|^2_{L^{\infty,2}(Q(1))} + \|\nabla(\eta v)\|^2_{L^2(Q(1))}\Big) \\
&\le K\Big(\|\eta v\|^2_{L^{\infty,2}(Q(1))} + 2\|\eta\nabla v\|^2_{L^2(Q(1))} + 8d^2(l-l')^{-2}\|v\|^2_{L^2(S(l))}\Big) \\
&\le C_2(1+|\beta|^{-2})(l-l')^{-2}(1+\gamma^2)\|v\|^2_{L^{\frac{2p}{p-2},2}(S(l))}, \qquad (10.19)
\end{aligned}$$

where $K > 0$ is a constant from [2, Lemma 3] and $C_2 := K(2\lambda^{-1}C_1 + 4c_0^{-1}C_1 + 8d^2)$. For the iteration method, choose a small number $\gamma_0 > 0$ and set $\gamma = \gamma_m = \sigma^m\gamma_0$, $m \in \mathbb{N}\cup\{0\}$. In order for iteration to work well, we have to get $\gamma_m \ne \frac{1}{2}$ for all $m \in \mathbb{N}\cup\{0\}$. To do this, we let γ_0 have the form

$$\gamma_0 := \frac{\sigma^{-N}}{1+\sigma} \quad \text{for some } N \in \mathbb{N} \qquad (10.20)$$

Then $\gamma_N < \frac{1}{2} < \gamma_{N+1}$ since $\sigma > 1$. Note that given $m \in \mathbb{N}\cup\{0\}$, $\beta = \beta_m = 2\gamma_m - 1 = \frac{2\sigma^{m-N}}{1+\sigma} - 1$. If $m \le N$, then $2\sigma^{m-N} \le 2 < 1+\sigma$, so that $|\beta_m| = 1 - \frac{2\sigma^{m-N}}{1+\sigma} \ge \frac{\sigma-1}{\sigma+1}$. If $m > N$, then $2\sigma^{m-N} \ge 2\sigma > 1+\sigma$, so that $|\beta_m| = 2\gamma_m - 1 = \frac{2\sigma^{m-N}}{1+\sigma} - 1 \ge \frac{2\sigma}{1+\sigma} - 1 = \frac{\sigma-1}{\sigma+1}$. Therefore,

$$|\beta|^{-1} = |\beta_m|^{-1} \le \frac{\sigma+1}{\sigma-1}.$$

Additionally we get

$$1+\gamma^2 = (1+\gamma_0^2)(\gamma/\gamma_0)^2 \le 2(\gamma/\gamma_0)^2.$$

Therefore from (10.19) we obtain

$$\|v^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2},2}(S(l'))} \le C_3(l-l')^{-2}(\gamma/\gamma_0)^2\|v\|^2_{L^{\frac{2p}{p-2},2}(S(l))},$$

where $C_3 = 2C_2\left(1 + \left(\frac{\sigma+1}{\sigma-1}\right)^2\right)$.

For $m = 0, 1, \dots,$ set $l = l_m := 3^{-1}(1 + 2^{-m-1})$, $l' = l'_m := 3^{-1}(1 + 2^{-m-2})$, $\varphi_m := \|\bar{u}^{r_0\sigma^m}\|^{2/\sigma^m}_{L^{\frac{2p}{p-2},2}(S(l_m))}$. Taking $r = r_0\sigma^m$ and $1/3 < l' = l'_m < l = l_m \leq 1/2$ for $m = 0, 1, 2, \dots$ we obtain

$$\varphi_{m+1} \leq (144C_3)^{\frac{1}{\sigma^m}}(2\sigma)^{\frac{2m}{\sigma^m}}\varphi_m. \tag{10.21}$$

Iterating (10.21) we have

$$\begin{aligned}\varphi_{m+1} &\leq (144C_3)^{\sum_{i=0}^m \frac{1}{\sigma^i}}(2\sigma)^{\sum_{i=0}^m \frac{2i}{\sigma^i}}\varphi_0 \\ &\leq \underbrace{(144C_3)^{\frac{\sigma}{\sigma-1}}(2\sigma)^{\frac{\sigma}{(\sigma-1)^2}}}_{=:C_4}\|u\|^2_{L^{\frac{2p}{p-2},2}\left(S(\frac{1}{2})\right)}.\end{aligned}$$

Letting $m \to \infty$ we have

$$\sup_{Q^*(1/3)} \bar{u}^{r_0} \leq \sqrt{C_4}\|\bar{u}^{r_0}\|_{L^{\frac{2p}{p-2},2}(S(1/2))}.$$

Step 2: Consider the case of $\beta < -1$. Let l and l' be real numbers satisfying $\frac{1}{3} < l' < l \leq \frac{1}{2}$ as in Step 1. Take a cut-off function $\eta \in C_0^\infty(R(l) \times (1 - l^2, l])$ satisfying $\eta \equiv 1$ in $Q(l')$ and $0 \leq \eta \leq 1$, $\|\nabla\eta\| \leq 2d(l-l)'$, $|\partial_t\eta| \leq 2(l-l')^{-1}$ in $Q(l)$. Choose $\tau_1 = \frac{1}{2}$ and $\tau_2 = \tau$ as in (10.18). Then by the same methods as in Step 1, we have

$$\lambda\|\eta\nabla v\|^2_{L^2(Q(1))} \leq \frac{1}{2}C_1(l-l')^{-2}\|v\|_{L^{\frac{2p}{p-2},2}(Q(l))},$$
$$c_0\|\eta v\|^2_{L^{2,\infty}(Q(1))} \leq C_1(l-l')^{-2}\|v\|_{L^{\frac{2p}{p-2},2}(Q(l))},$$

where C_1 is as in Step 1, hence using the same methods as in Step 1 and [2, Lemma 1],

$$\|v^\sigma\|^{2/\sigma}_{L^{\frac{2p}{p-2},2}(Q(l'))} \leq C_5(l-l')^{-2}\|v\|^2_{L^{\frac{2p}{p-2},2}(Q(l))},$$

where $C_5 := K(2^{-1}\lambda^{-1}C_1 + c_0^{-1}C_1 + 8d^2)$.

For the iteration, we let $l = l_m := 3^{-1}(1 + 2^{-m-1})$, $l' = l'_m := 3^{-1}(1 + 2^{-m-2})$, $\varphi_m := \|\bar{u}^{-r_0\sigma^m}\|^{2/\sigma^m}_{L^{\frac{2p}{p-2},2}(Q(l_m))}$. Considering $r = -r_0\sigma^m$ and $1/3 < l' = l'_m < l = l_m \leq 1/2$

for $m = 0, 1, 2, \ldots$, we get $\varphi_{m+1} \leq (144C_5)^{\frac{1}{\sigma^m}}\varphi_m$, so that

$$\varphi_{m+1} \leq (144C_5)^{\sum_{i=0}^{m}\frac{1}{\sigma^i}}\varphi_0 \leq \underbrace{(144C_5)^{\frac{\sigma}{\sigma-1}}(2\sigma)^{\frac{\sigma}{(\sigma-1)^2}}}_{=:C_6}\|\bar{u}^{-r_0}\|^2_{L^{\frac{2p}{p-2},2}(Q(\frac{1}{2}))}.$$

Letting $m \to \infty$ we have

$$\sup_{Q(\frac{1}{3})}\bar{u}^{-r_0} \leq \sqrt{C_6}\|\bar{u}^{-r_0}\|_{L^{\frac{2p}{p-2},2}(Q(\frac{1}{2}))},$$

hence

$$\|\bar{u}^{-r_0}\|^{-1}_{L^{\frac{2p}{p-2},2}(Q(\frac{1}{2}))} \leq \sqrt{C_6}\inf_{Q(\frac{1}{3})}\bar{u}^{r_0}.$$

Therefore if we show existence of a constant $\widetilde{C} > 0$ satisfying

$$\|\bar{u}^{r_0}\|_{L^{\frac{2p}{p-2},2}\left(S(\frac{1}{2})\right)}\|\bar{u}^{-r_0}\|_{L^{\frac{2p}{p-2},2}\left(Q(\frac{1}{2})\right)} \leq \widetilde{C}, \tag{10.22}$$

then the proof of Theorem 10.2.2 will be done.

Step 3: In order to show (10.22), consider the case of $\beta = -1$ as in (10.18). Set $v := -\log\bar{u}$. Then by (10.22), we obtain for a.e. $\tau_1, \tau_2 \in (0, 1)$

$$\begin{aligned}&\int_{R(1)}\eta^2 v\,|_{t=\tau_1}^{t=\tau_2}\,\psi dxdt + \frac{\lambda}{2}\int_{\tau_1}^{\tau_2}\int_{R(1)}\eta^2\|\nabla v\|^2 dxdt\\&\leq \int_{\tau_1}^{\tau_2}\int_{R(1)}\Big(\frac{\|\mathbf{B}\|^2}{\lambda}\eta^2 + \frac{4d^2M^2}{\lambda}\|\nabla\eta\|^2\Big)dxdt + \int_{\tau_1}^{\tau_2}\int_{R(1)}2\eta\,\partial_t\eta\,|v|\,\psi dxdt\end{aligned}$$

Choose a cut-off function η as in the form

$$\eta(x, t) = \zeta(x)\cdot\alpha(t),$$

where $\zeta \in C_0^\infty(R(1))$ satisfying $\zeta \equiv 1$ in $R(\frac{1}{2})$ and $\alpha \in C^\infty(\mathbb{R})$ satisfying $\alpha \equiv 1$ in

$[\tau_1, \infty)$, $\alpha \equiv 0$ in $(-\infty, \frac{\tau_1}{2})$. Moreover we can choose such functions ζ satisfying

$$0 \le \zeta \le 1,\ \|\nabla\zeta\| \le 6d,\ \text{and}\ \{\zeta \ge c\} \text{ is convex for any } c \in \mathbb{R}.$$

Then note that

$$\int_{\tau_1}^{\tau_2} \int_{R(1)} 2\eta\, \partial_t\eta\, |v|\, \psi dxdt = 0.$$

Define

$$V(t) := \frac{\int_{R(1)} \zeta^2(x) v(x,t)\, \psi(x) dx}{\int_{R(1)} \zeta^2(x)\, \psi(x) dx}, \qquad 0 < t < 1.$$

Applying Lemma 10.2.1 with $U \equiv R(1)$, $p = \zeta^2$, $L \equiv d^{1/d}$ and taking integration over $(0,1)$

$$\int_0^1 \int_{R(1)} \zeta^2 (v - V)^2 dxdt \le \left(\frac{c_1}{c_0}\right)^2 \frac{|\omega_d| d^{1+2/d}}{2\int_{R(1)} \zeta^2 dx} \int_0^1 \int_{R(1)} \zeta^2 \|\nabla v\|^2 dxdt. \tag{10.23}$$

Noting that $\int_{R(1)} \zeta^2 dx \ge |R(\frac{1}{2})| = 2^{-d}$ and $\alpha(\tau_1) = \alpha(\tau_2) = 1$, we obtain from (10.23),

$$\begin{aligned} & V(\tau_2) - V(\tau_1) + \frac{c_0^2 \lambda}{c_1^2 \omega_d d^{1+d/2}} \int_0^1 \int_{R(\frac{1}{2})} (v - V)^2 dxdt \\ \le\ & \left(\int_{R(1)} \zeta^2 dx\right)^{-1} \left(\int_{R(1)} \eta^2 v \,\Big|_{t=\tau_1}^{t=\tau_2} \psi dxdt + \frac{\lambda}{2} \int_{\tau_1}^{\tau_2} \int_{R(1)} \eta^2 \|\nabla v\|^2 dxdt\right) \\ \le\ & 2^d \int_{\tau_1}^{\tau_2} \int_{R(1)} \left(\frac{\|\mathbf{B}\|^2}{\lambda} + \frac{144 d^4 M^2}{\lambda}\right) dxdt. \end{aligned} \tag{10.24}$$

Since $\partial_t v(x, \cdot) \in L^2((0,1))$ for a.e. $x \in R(1)$, by 'Fundamental Theorem of Calculus', for a.e. $t \in (0,1)$,

$$v(x,t) = \int_0^t \partial_t v(x,s) ds, \qquad \text{for a.e. } x \in R(1).$$

Applying Fubini's Theorem

$$\int_{R(1)} \zeta^2(x)v(x,t)\,\psi(x)dx = \int_0^t \left(\int_{R(1)} t^2(x,t)\partial_t v(x,s)\psi(x)dx\right) ds,$$

hence $V(t)$ has a absolutely continuous dt-version on $(0,1)$, say again $V(t)$. Therefore $V(t)$ is a.e. differentiable on $(0,1)$, hence from (10.24) we get

$$\begin{aligned}\frac{dV}{dt} + A_0 \int_{R(\frac{1}{2})} (v-V)^2 dx &\leq 2^d \int_{R(1)} \left(\frac{\|\mathbf{B}\|^2}{\lambda} + \frac{144d^4M^2}{\lambda}\right) dx \\ &\leq \frac{2^d}{\lambda}\left(\|\mathbf{B}\|^2_{L^p(R(1))} + 144d^4M^2\right) =: C_7\end{aligned}$$

for a.e. in $(0,1)$, where $A_0 := \dfrac{c_0^2\lambda}{c_1^2\omega_d d^{1+d/2}}$. Let $Q^+ = R(\frac{1}{2}) \times (0,\frac{1}{2})$, and $Q^- = R(\frac{1}{2}) \times (\frac{1}{2},1)$. Define the function $\Psi : \mathbb{R} \to [0,\infty)$ by

$$\Psi(w) = \begin{cases} \sqrt{w} & \text{when } w > 0 \\ 0 & \text{when } w \leq 0 \end{cases}$$

By applying [2, Lemma 7] in the interval $[\frac{1}{2},1)$ with $\alpha = 2$, $\gamma = \frac{1}{2}$, we obtain

$$\iint_{Q^+} \Psi\Big(v(x,t) - V\,(1/2)\Big) dx\,dt \leq \frac{1}{A_0} + \frac{2+\sqrt{C_7/2}}{2^{d+1}},$$

Likewise, by [2, Corollary of Lemma 7] in the interval $[0,\frac{1}{2})$ with $\alpha = 2$, $\gamma = \frac{1}{2}$,

$$\iint_{Q^-} \Psi\Big(V\,(1/2) - v(y,s)\Big) dy\,ds \leq \frac{1}{A_0} + \frac{2+\sqrt{C_7/2}}{2^{d+1}}.$$

Therefore,

$$
\begin{aligned}
&\iint_{Q^-}\iint_{Q^+}\Psi\left(\log\frac{\bar{u}(y,s)}{\bar{u}(x,t)}\right)dx\,dt\,dy\,ds\\
&=\iint_{Q^-}\iint_{Q^+}\Psi\Big(v(x,t)-v(y,s)\Big)dx\,dt\,dy\,ds\\
&\le\iint_{Q^-}\iint_{Q^+}\Psi\Big(v(x,t)-V\,(1/2)\Big)+\Psi\Big(V\,(1/2)-v(y,s)\Big)dx\,dt\,dy\,ds\\
&\le 2^{-d-1}\left(\iint_{Q^+}\Psi\Big(v(x,t)-V\,(1/2)\Big)dx\,dt+\iint_{Q^-}\Psi\Big(V\,(1/2)-v(y,s)\Big)dy\,ds\right)\\
&\le 2^{-d}\left(\frac{1}{A_0}+\frac{2+\sqrt{C_7/2}}{2^{d+1}}\right)=:C_8. \qquad (10.25)
\end{aligned}
$$

Let $Q^-(l)$, $Q^+(l)$ be pairs of rectangles in $R(1/2)\times(0,1)$ obtained from the fixed pair Q^+, Q^-, respectively by the transformations

$$
x\mapsto lx+c_2,\quad t\mapsto l^2t+c_1,\qquad l\in(0,1],\ c_1,c_2>0.
$$

Now for $(x',t')\in R(1)\times(0,1)$ define

$$
\begin{aligned}
&u'(x',t'):=\bar{u}(lx'+c_1,l^2t'+c_2),\quad A'(x):=A(lx+c_1),\\
&\mathbf{B}'(x'):=l\cdot\mathbf{B}(lx'+c_1),\quad \psi'(x'):=\psi(lx+c_1).
\end{aligned}
$$

Then it holds

$$
\iint_{R(1)\times(0,1)}(u'\partial_t\varphi)\psi'dx'dt'=\iint_{R(1)\times(0,1)}\big\langle A'\nabla u',\nabla\varphi'\big\rangle+\langle\mathbf{B}',\nabla u'\rangle\varphi\,dx'dt',
$$
$$
\text{for all } \varphi'\in C_0^\infty(R(1)\times(0,1)).
$$

Now let $\overline{u'}:=u'+\varepsilon$. Then by (10.25),

$$
\iint_{Q^-}\iint_{Q^+}\Psi\left(\log\frac{\overline{u'}(y',s')}{\overline{u'}(x',t')}\right)dx'\,dt'\,dy'\,ds'\le C_8.
$$

Therefore

$$\begin{aligned}
&\frac{1}{l^{2n+4}}\iint_{Q^-(l)}\iint_{Q^+(l)}\Psi\left(-\frac{1}{C_8^2}\log\bar{u}(x,t)-\Big(-\frac{1}{C_8^2}\log\bar{u}(y,s)\Big)\right)dx\,dt\,dy\,ds\\
=&\ \frac{1}{l^{2n+4}C_8}\iint_{Q^-(l)}\iint_{Q^+(l)}\Psi\left(\log\frac{\bar{u}(y,s)}{\bar{u}(x,t)}\right)dx\,dt\,dy\,ds\\
=&\ \frac{1}{l^{2n+4}C_8}\iint_{Q^-(l)}\iint_{Q^+(l)}\Psi\left(\log\frac{\overline{u'}(\frac{y-c_1}{l},\frac{s-c_2}{l^2})}{\overline{u'}(\frac{x-c_1}{l},\frac{t-c_2}{l^2})}\right)dx\,dt\,dy\,ds\\
=&\ \frac{1}{C_8}\iint_{Q^-}\iint_{Q^+}\Psi\left(\log\frac{\overline{u'}(y',s')}{\overline{u'}(x',t')}\right)dx'\,dt'\,dy'\,ds'\le 1. \qquad (10.26)
\end{aligned}$$

Thus applying [53, Main Lemma, p. 106] to (10.26), there exist constants $c_2, c_3 > 0$ which only depends on d such that

$$\iint_{D^-}\bar{u}^{c_2/C_8^2}dy\,ds\cdot\iint_{D^+}(1/\bar{u})^{c_2/C_8^2}dx\,dt\le c_3, \qquad (10.27)$$

where $D^- := R(\frac{1}{2})\times(0,\frac{1}{4})$ and $D^+ := R(\frac{1}{2})\times(\frac{3}{4},1)$. Note that $S\left(\frac{1}{2}\right)\subset D^-$, $Q\left(\frac{1}{2}\right)\subset D^+$. Choose a small $\delta>0$ so that $\frac{p}{p-2}\le 1/\delta$ and take r_0 as in (10.20) satisfying

$$r_0\in\left[\frac{c_2\delta}{2\sigma C_8^2},\ \frac{c_2\delta}{2C_8^2}\right].$$

Hence

$$\begin{aligned}
&\|\bar{u}^{r_0}\|_{L^{\frac{2p}{p-2},2}(S(\frac{1}{2}))}\le\|\bar{u}^{r_0}\|_{L^{\frac{2}{\delta}}(S(\frac{1}{2}))}=\|\bar{u}\|^{r_0}_{L^{\frac{2r_0}{\delta}}(S(\frac{1}{2}))}\le\left(\iint_{D^-}\bar{u}^{c_2/C_8^2}dy\,ds\right)^{\frac{r_0C_8^2}{c_2}},\\
&\|\bar{u}^{r_0}\|_{L^{\frac{2p}{p-2},2}(Q(\frac{1}{2}))}\le\left(\iint_{D^+}(1/\bar{u})^{c_2/C_8^2}dx\,dt\right)^{\frac{r_0C_8^2}{c_2}}.
\end{aligned}$$

By (10.27), it holds

$$\|\bar{u}^{r_0}\|_{L^{\frac{2p}{p-2},2}(S(\frac{1}{2}))}\|\bar{u}^{r_0}\|_{L^{\frac{2p}{p-2},2}(Q(\frac{1}{2}))}\le c_3^{\frac{r_0C_8^2}{c_2}}.$$

Therefore

$$\begin{aligned}\sup_{Q^*(1/3)} \bar{u} &\leq \left(\sqrt{C_4}\sqrt{C_6}\cdot c_3^{\frac{r_0 C_8^2}{c_2}}\right)^{1/r_0} \inf_{Q(1/3)} \bar{u} \\ &\leq \left(1+\sqrt{C_4}\sqrt{C_6}\right)^{\frac{2\sigma C_8^2}{\delta c_2}} \cdot c_3^{\frac{C_8^2}{c_2}} \inf_{Q(1/3)} \bar{u},\end{aligned}$$

as desired.

□

In the same manner as in [2, Thoerem 4, Theorem 5], we obtain the following parabolic Hölder regularity, estimate and pointwise parabolic Harnack inequality as consequences of Theorem 10.2.2

Theorem 10.2.3. *Assume* **(I)**$'$ *and* $Q(3r) \subset U \times (0,T)$. *If* (10.1) *holds, then there exists a constant* $\gamma \in (0,1)$ *such that* $u \in C^{\gamma;\frac{\gamma}{2}}(\overline{Q}(r))$. *Furthermore for* $(x,t),(y,s) \in \overline{Q}(r)$ *, we have*

$$|u(x,t)-u(y,s)| \leq Cr^{-\gamma}\left(\|x-y\|+\sqrt{|t-s|}\right)^{\gamma} \sup_{Q(3r)} u,$$

where $C>0$ *is the constant which is independent of* u.

Theorem 10.2.4. *Assume* **(I)**$'$ *and* u *is non-negative. Suppose* U' *is convex with* $\overline{U'} \subset U$ *and let* $\delta := \inf_{x\in\overline{U'},y\in\overline{U}} \|x-y\|$, $T>0$. *If* (10.1) *holds, then for any* $x,y \in U'$ *and all* s,t *with* $0<s<t<T$, *we have*

$$u(y,s) \leq u(x,t)\cdot \exp C\left(\frac{\|x-y\|^2}{t-s}+\frac{t-s}{R}+1\right),$$

where $R := \min\{1,s,\delta^2\}$ *and* $C>0$ *is a constant which is independent of* u.

Chapter 11

Analytic and probabilistic results

11.1 Strong Feller property and irreducibility with general pre-invariant measures

Here we state a basic condition for our main results.

(C1) ψ is a positive Borel measruable function on $\mathbb{R}^d$. Given open ball B in $\mathbb{R}^d$, there exist positive constants c_B, C_B such that

$$c_B \le \psi \le C_B \quad \text{on } B. \tag{11.1}$$

$\rho \in H^{1,2}_{loc}(\mathbb{R}^d) \cap L^{\infty}_{loc}(\mathbb{R}^d)$ is a positive function and $\frac{1}{\rho} \in L^{\infty}_{loc}(\mathbb{R}^d)$. $A = (a_{ij})_{1\le i,j\le d}$ is a matrix of functions satisfying $a_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d)$ for all $1 \le i,j \le d$. Given open ball B in $\mathbb{R}^d$, there exist positive constants λ_B, M_B such that for any $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, $x \in B$, it holds

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \lambda_B \|\xi\|^2, \qquad \max_{1\le i,j\le d} |a_{ij}(x)| \le M_B. \tag{11.2}$$

Set $\widetilde{A} = (\widetilde{a}_{ij})_{1\le i,j\le d} := \frac{A+A^T}{2}$ and $\check{A} = (\check{a}_{ij})_{1\le i,j\le d} := \frac{A-A^T}{2}$. $\psi\mathbf{B} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d)$

satisfies

$$\int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla \varphi \rangle \rho\psi dx = 0, \qquad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^d). \tag{11.3}$$

From now on, we assume **(C1)** and let $\mu := \rho\psi dx$. For $f, g \in C_0^\infty(\mathbb{R}^d)$, define $(\mathcal{E}^0, C_0^\infty(\mathbb{R}^d))$ by

$$\mathcal{E}^0(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \Big\langle \frac{1}{\psi}\widetilde{A}\nabla f, \nabla g \Big\rangle \, d\mu.$$

Then by [51, Subsection II.2b)], $(\mathcal{E}^0, C_0^\infty(\mathbb{R}^d))$ is closable in $L^2(\mathbb{R}^d,\mu)$, hence denote its closure on $L^2(\mathbb{R}^d,\mu)$ by $(\mathcal{E}^0, D(\mathcal{E}^0))$ and its associated generator by $(L^0, D(L^0))$. Define

$$Lf := L^0 f + \langle \mathbf{B} + \beta^{\rho, \check{A}^T, \psi}, \nabla f \rangle, \quad f \in D(L^0)_{0,b}.$$

Note that

$$\int_{\mathbb{R}^d} \langle \mathbf{B} + \beta^{\rho, \check{A}^T, \psi}, \nabla \varphi \rangle d\mu = 0, \qquad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^d).$$

Moreover $C_0^\infty(\mathbb{R}^d) \subset D(L^0)_{0,b}$ and

$$Lf = \frac{1}{2\psi}\text{trace}(\widetilde{A}\nabla^2 f) + \langle \mathbf{B} + \beta^{\rho, A^T, \psi}, \nabla f \rangle, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Thus by Theorem 8.2.7, there exists an $L^1(\mathbb{R}^d,\mu)$-closed extension $(\overline{L}, D(\overline{L}))$ of $(L, D(L^0)_{0,b})$ in $L^1(\mathbb{R}^d,\mu)$ which generates a sub-Markovian C_0-semigroup of contractions $(T_t)_{t>0}$ on $L^1(\mathbb{R}^d,\mu)$. Restricting $(T_t)_{t>0}$ to $L^1(\mathbb{R}^d,\mu)_b$, by Riesz-Thorin interpolation, $(T_t)_{t>0}$ can be extended to a sub-Markovian C_0-semigroup of contractions on each $L^r(\mathbb{R}^d,\mu)$, $r \in [1,\infty)$. As in Part II, denote by $(L_r, D(L_r))$, $(G_\alpha)_{\alpha>0}$ the corresponding generator and resolvent in $L^r(\mathbb{R}^d,\mu)$, respectively. Denote by $(\widehat{L}_r, D(\widehat{L}_r))$ for the corresponding co-generator on $L^r(\mathbb{R}^d,\mu)$. Using sub-Markovian property, semigroup $(T_t)_{t>0}$ and resolvent $(G_\alpha)_{\alpha>0}$ can be extended on $L^\infty(\mathbb{R}^d,\mu)$ which satisfies contraction property, but no longer strongly continuous on $L^\infty(\mathbb{R}^d,\mu)$. Define $\mathcal{E}$ by

$$\mathcal{E}(f,g) := \begin{cases} -\int_{\mathbb{R}^d} L_2 f \cdot g\, d\mu & \text{for } f \in D(L_2),\ g \in L^2(\mathbb{R}^d,\mu), \\ -\int_{\mathbb{R}^d} f \cdot \widehat{L}_2 g\, d\mu & \text{for } f \in L^2(\mathbb{R}^d,\mu),\ g \in D(\widehat{L}_2). \end{cases}$$

Then $\mathcal{E}$ is called a *generalized Dirichlet form associated with* $(L_2, D(L_2))$.

Remark 11.1.1. *Let* $\mathbb{R}^d_+ := \{x=(x_1,\dots,x_d)\in\mathbb{R}^d \mid x_d \geq 0\}$*. Given* $\alpha\in(0,1)$*, define* $\phi := 2(\alpha 1_{\mathbb{R}^d_+} + (1-\alpha)1_{\mathbb{R}^d\setminus\mathbb{R}^d_+})$*. Let* $\rho \in H^{1,2}_{loc}(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ *be positive and define* $\widetilde{\rho} := \phi\rho$*,* $m := \widetilde{\rho}\,dx$*. Let* $A=(a_{ij})_{1\leq i,j\leq d}$ *be a matrix of functions satisfying* (11.2) *and assume* $a_{ij}\in H^{1,2}_{loc}(\mathbb{R}^d)$ *for all* $1\leq i,j\leq d$*. Let* $\breve{A} = (\breve{a}_{ij})_{1\leq i,j\leq d} := \frac{A-A^T}{2}$ *and assume there exists a positive constant* Λ *such that for any open ball* B *in* $\mathbb{R}^d$*, it holds*

$$\max_{1\leq i,j\leq d} |\breve{a}_{ij}(x)| \leq \Lambda\cdot\lambda_B, \qquad \textit{for all } x\in B.$$

Let

$$\mathcal{E}^0(f,g) := \int_{\mathbb{R}^d} \langle A\nabla f, \nabla g\rangle dm, \quad f,g\in C_0^\infty(\mathbb{R}^d).$$

Then $(\mathcal{E}^0, C_0^\infty(\mathbb{R}^d))$ *satisfies the strong sector condition and we can hence define* $(\mathcal{E}^0, D(\mathcal{E}^0))$ *as the closure of* $(\mathcal{E}^0, C_0^\infty(\mathbb{R}^d))$ *on* $L^2(\mathbb{R}^d,m)$*. Denote by* $(L^0, D(L^0))$ *the associated generator on* $L^2(\mathbb{R}^d,m)$*. Let* $\mathbf{B}\in L^2(\mathbb{R}^d,m)$ *be such that*

$$\int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla\varphi\rangle dm = 0, \qquad \textit{for any } \varphi\in C_0^\infty(\mathbb{R}^d).$$

Define

$$Lf := L^0 f + \langle \mathbf{B}, \nabla f\rangle, \quad f\in D(L^0)_{0,b}.$$

Using integration by parts, for any $f \in C_0^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+)$, $g \in C_0^\infty(\mathbb{R}^d)$ *we have*

$$\begin{aligned}
\mathcal{E}^0(f,g) &= \frac{1}{2}\int_{\mathbb{R}^d} \langle A\nabla f, \nabla g\rangle \, dm \\
&= \alpha \int_{\mathbb{R}^d_+} \langle A\nabla f, \nabla g\rangle \rho \, dx + (1-\alpha)\int_{\mathbb{R}^d\setminus\mathbb{R}^d_+} \langle A\nabla f, \nabla g\rangle \rho \, dx \\
&= \int_{\mathbb{R}^d_+} \left(\frac{1}{2}\mathrm{trace}(A\nabla^2 f) + \langle \beta^{\rho, A^T}, \nabla f\rangle\right) g \cdot 2\alpha\rho \, dx \\
&\qquad + \int_{\mathbb{R}^d\setminus\mathbb{R}^d_+} \left(\frac{1}{2}\mathrm{trace}(A\nabla^2 f) + \langle \beta^{\rho, A^T}, \nabla f\rangle\right) g \cdot 2(1-\alpha)\rho \, dx \\
&= \int_{\mathbb{R}^d} \left(\frac{1}{2}\mathrm{trace}(A\nabla^2 f) + \langle \beta^{\rho, A^T}, \nabla f\rangle\right) g \, dm.
\end{aligned}$$

Hence $f \in D(L^0)$ *and* $L^0 f = \frac{1}{2}\mathrm{trace}(A\nabla^2 f) + \langle \beta^{\widetilde{\rho}, A^T}, \nabla f\rangle$. *Note that* $C_0^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+)$ *is dense in* $L^1(\mathbb{R}^d, m)$ *and* $C_0^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+) \subset D(L^0)_{0,b}$. *Hence by [29, Lemma 13] there exists an* $L^1(\mathbb{R}^d, m)$ *closed extension* $(\overline{L}, D(\overline{L}))$ *of* $(L, D(L^0)_{0,b})$ *on* $L^1(\mathbb{R}^d, \mu)$ *which generates a sub-Markovian* C_0*-semigroup of contractions* $(T_t)_{t>0}$ *on* $L^1(\mathbb{R}^d, m)$. *Like above, we obtain correspondingly the sub-Markovian semigroup of contractions* $(T_t)_{t>0}$ *and the sub-Markovian resolvent of contractions* $(G_\alpha)_{\alpha>0}$ *on* $L^r(\mathbb{R}^d, m)$, $r \in [1,\infty)$. *And we also obtain the corresponding generator* $(L_r, D(L_r))$, *co-generator* $(\widehat{L}_r, D(\widehat{L}_r))$ *on* $L^r(\mathbb{R}^d, m)$, $r \in [1,\infty)$ *and a generalized Dirichlet form* $\mathcal{E}$ *associated with* $(L_2, D(L_2))$. $\mathcal{E}$ *is associated with a Hunt process with skew-reflection on* $\partial\mathbb{R}^d_+$.

From now on, we fix $p > d$ and let $q := \frac{pd}{p+d}$.

Theorem 11.1.2. *Assume* **(C1)** *and* $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. *Let* $f \in \cup_{r\in[q,\infty]} L^r(\mathbb{R}^d, \mu)$. *Then* $G_\alpha f$ *has a locally Hölder continuous* μ*-version* $R_\alpha f$ *on* $\mathbb{R}^d$. *Furthermore for any open balls* B, B' *with* $\overline{B} \subset B'$, *we have the following estimate*

$$\|R_\alpha f\|_{C^{0,\gamma}(\overline{B})} \le c_2 \left(\|f\|_{L^q(B',\mu)} + \|G_\alpha f\|_{L^1(B',\mu)}\right),$$

where $c_2 > 0$, $\gamma \in (0,1)$ *are constants which are independent of* f.

Proof Let $f \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$. Then by Theorem 8.2.7 (c),

$$G_\alpha f \in D(\overline{L})_b \subset D(\mathcal{E}^0) \subset H^{1,2}_{loc}(\mathbb{R}^d)$$

and

$$\begin{aligned}
&\mathcal{E}^0(G_\alpha f, \varphi) - \int_{\mathbb{R}^d} \langle \mathbf{B} + \beta^{\rho,\psi,\check{A}^T}, \nabla G_\alpha f\rangle \varphi \, d\mu \\
&= - \int_{\mathbb{R}^d} \left(\overline{L}\, G_\alpha f\right) \varphi d\mu \\
&= \int_{\mathbb{R}^d} (f - \alpha G_\alpha f)\varphi \, d\mu, \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^d} \langle \frac{1}{2}\rho A \nabla G_\alpha f, \nabla \varphi\rangle dx - \int_{\mathbb{R}^d} \langle \rho\psi\mathbf{B}, \nabla G_\alpha f\rangle \varphi \, dx + \int_{\mathbb{R}^d} (\alpha\rho\psi G_\alpha f)\, \varphi \, dx \\
&= \int_{\mathbb{R}^d} (\rho\psi f)\, \varphi\, dx, \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d).
\end{aligned}$$

Note that $\rho\psi\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $\alpha\rho\psi \in L^\infty_{loc}(\mathbb{R}^d)$. Let B, B' be open balls in $\mathbb{R}^d$ satisfying $\overline{B} \subset B'$. By Theorem 7.2.2, there exists a locally Hölder continuou μ-version $R_\alpha f$ of $G_\alpha f$ on $\mathbb{R}^d$ and there exist positive constants $\gamma \in (0,1)$, c_1, c_2, independent of f, such that

$$\begin{aligned}
\|R_\alpha f\|_{C^\gamma(\overline{B})} &\leq c_1 \left(\|G_\alpha f\|_{L^1(B')} + \|\rho\psi f\|_{L^q(B')}\right) \\
&\leq c_2 \left(\|G_\alpha f\|_{L^1(B',\mu)} + \|f\|_{L^q(B',\mu)}\right).
\end{aligned}$$

The remained part is analogous to Theorem 5.3.1. For $f \in \bigcup_{r\in[q,\infty)} L^r(\mathbb{R}^d,\mu)$, we use the denseness of $C_0^\infty(\mathbb{R}^d)$ and contraction properties. And for $f \in L^\infty(\mathbb{R}^d,\mu)$, we use pointwise approximation by $L^1(\mathbb{R}^d,\mu)_b$ and Lebesgue's Theorem which is analogous to Theorem 5.3.1.

□

Analogously to Lemma 8.3.4, we obtain

Lemma 11.1.3. *Assume* **(C1)** *and* $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$*. For any* $t > 0$*,* $f \in \bigcup_{r\in[q,\infty)} D(L_r)$*,*

$T_t f$ has a locally Hölder continuous version $P_t f$ on $\mathbb{R}^d$. Moreover the map

$$(x, t) \mapsto P_t f(x)$$

is continuous on $\mathbb{R}^d \times [0, \infty)$.

Theorem 11.1.4. *Assume* **(C1)** *and* $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. *Let $f \in \cup_{s\in[1,\infty]} L^s(\mathbb{R}^d, \mu)$ and $t > 0$. Then $T_t f$ has a locally Hölder continuous μ-version $P_t f$ on $\mathbb{R}^d$ and $P_\cdot f(\cdot)$ is locally parabolic Hölder continuous on $\mathbb{R}^d \times (0, \infty)$. Furthermore, for any bounded open sets U, V in $\mathbb{R}^d$ with $\overline{U} \subset V$ and $0 < \tau_3 < \tau_1 < \tau_2 < \tau_4$, i.e. $[\tau_1, \tau_2] \subset (\tau_3, \tau_4)$, we have the following estimate for all $f \in \cup_{s\in[1,\infty]} L^s(\mathbb{R}^d, \mu)$ with $f \geq 0$,*

$$\|P_\cdot f(\cdot)\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} \leq C_6 \|P_\cdot f(\cdot)\|_{L^1(V\times(\tau_3,\tau_4),\mu\otimes dt)}, \qquad (11.4)$$

where C_6, γ are positive constants that depend on $\overline{U} \times [\tau_1, \tau_2], V \times (\tau_3, \tau_4)$, but are independent of f.

Proof First assume $f \in D(\overline{L})_b \cap D(L_2) \cap D(L_q)$ with $f \geq 0$. Using Lemma 11.1.3, define $u \in C(\mathbb{R}^d \times [0, \infty))$ by $u(x, t) := P_t f(x)$. Then for any bounded open set $O \subset \mathbb{R}^d$ and $T > 0$, we have $u \in H^{1,2}(O \times (0, T))$ by the same way as in Theorem 9.3.4. Using the same argument as (8.38), it holds

$$\begin{aligned}\iint_{\mathbb{R}^d\times(0,T)} &\Big\langle \frac{1}{2}\rho A\nabla u, \nabla\varphi \Big\rangle - \langle \rho\psi\mathbf{B}, \nabla u\rangle\varphi \, dxdt \\ &= \iint_{\mathbb{R}^d\times(0,T)} u\,\partial_t\varphi \cdot \rho\psi dxdt \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d \times (0, T)). \qquad (11.5)\end{aligned}$$

Then by Theorem 10.2.3, Theorem 10.1.1 and Theorem 10.2.2 and using the same method as in the proof of Theorem 3.1.8, we obtain $u \in C^{\gamma;\frac{\gamma}{2}}(\overline{U} \times [\tau_1, \tau_2])$ and there exists a constant $\gamma \in (0, 1)$ and $C > 0$ which is independent of u such that

$$\|u\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} \leq C\|u\|_{L^1(V\times(\tau_3,\tau_4),\mu\otimes dt)}.$$

Given $s \in [1,\infty)$, using $L^s(\mathbb{R}^d,\mu)$-contraction property of $(P_t)_{t>0}$ we have

$$\begin{aligned} \|P_\cdot f\|_{C^{\gamma;\frac{\gamma}{2}}(\overline{U}\times[\tau_1,\tau_2])} &\leq C\|P_\cdot f\|_{L^1(V\times(\tau_3,\tau_4),\mu\otimes dt)}. \\ &\leq C(\tau_4-\tau_3)\|\rho\psi\|_{L^1(V)}^{\frac{s-1}{s}}\|f\|_{L^s(\mathbb{R}^d,\mu)}, \quad s\in[1,\infty], \end{aligned}$$

For $f \in L^1(\mathbb{R}^d,\mu)\cap L^\infty(\mathbb{R}^d,\mu)$ with $f \geq 0$ let $f_n := nG_n f$. Then $f_n \in D(\overline{L})_b \cap D(L_2) \cap D(L_q)$ with $f_n \geq 0$ and $f_n \to f$ in $L^s(\mathbb{R}^d,\mu)$ for any $s \in [1,\infty)$. Thus (11.4) extend to $f \in L^1(\mathbb{R}^d,\mu)\cap L^\infty(\mathbb{R}^d,\mu)$ with $f \geq 0$. If $f \in L^s(\mathbb{R}^d,\mu)$, $f \geq 0$ and $s \in [1,\infty)$, let $f_n := 1_{B_n}\cdot(f\wedge n)$. Then $f_n \in L^1(\mathbb{R}^d,\mu)\cap L^\infty(\mathbb{R}^d,\mu)$ with $f_n \geq 0$ and $f_n \to f$ in $L^s(\mathbb{R}^d,\mu)$. Thus (11.4) extend to $f \in L^s(\mathbb{R}^d,\mu)$ with $f \geq 0$. For $f \in L^\infty(\mathbb{R}^d,\mu)$, the result follows exactly as in Theorem 3.1.8.

□

The following Lemma is a key intermediate step to show irreducible and strict irreducible of $(P_t)_{t>0}$

Lemma 11.1.5. *Assume* **(C1)** *and* $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d,\mathbb{R}^d)$.

(i) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $P_{t_0}1_A(x_0) = 0$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$. Then $\mu(A) = 0$.

(ii) Let $A \in \mathcal{B}(\mathbb{R}^d)$ be such that $P_{t_0}1_A(x_0) = 1$ for some $t_0 > 0$ and $x_0 \in \mathbb{R}^d$. Then $P_t 1_A(x) = 1$ for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$.

Proof The proof of (ii) is almost analogous with (i) noting the proof of Lemm 4.2.2 (ii), hence we will only prove (i). Suppose $\mu(A) > 0$. Choose an open ball $B_r(x_0) \subset \mathbb{R}^d$ such that

$$0 < \mu\left(A \cap B_r(x_0)\right) < \infty.$$

Let $u := P_\cdot 1_{A\cap B_r(x_0)}$. Then $0 = u(x_0,t_0) \leq P_{t_0}1_A(x_0) = 0$. Set $f_n := nG_n 1_{A\cap B_r}(x_0)$. Then $f_n \in D(\overline{L})_b \cap D(L_2)\cap D(L_q)$ with $f_n \geq 0$ such that $f_n \to 1_{A\cap B_r(x_0)}$ in $L^1(\mathbb{R}^d,\mu)$. Let $u_n := P_\cdot f_n$. Fix $T > t_0$ and $U \supset \overline{B}_r(x_0)$. Note that by (11.5), $u_n \in H^{1,2}(U\times(0,T))$

satisfies

$$\iint_{U\times(0,T)} \Big\langle \frac{1}{2}\rho A\nabla u_n, \nabla\varphi\Big\rangle - \langle \rho\psi\mathbf{B}, \nabla u_n\rangle\varphi \, dxdt$$
$$= \iint_{U\times(0,T)} u_n \, \partial_t\varphi \cdot \rho\psi dxdt \quad \text{for all } \varphi \in C_0^\infty(U\times(0,T)).$$

Now take arbitrary but fixed $(x,t) \in B_r(x_0)\times(0,t_0)$. By Theorem 10.2.4,

$$0 \le u_n(x,t) \le u_n(x_0,t_0)\ \exp\left(C\Big(\frac{\|x_0-x\|^2}{t_0-t} + \frac{t_0-t}{\min(1,t)} + 1\Big)\right).$$

Applying (11.4) with $U \supset \overline{B}_r(x_0)$, $[\tau_1,\tau_2] \supset [t,t_0]$, it holds

$$0 \le u(x,t) \le u(x_0,t_0)\ \exp\left(C\Big(\frac{\|x_0-x\|^2}{t_0-t} + \frac{t_0-t}{\min(1,t)} + 1\Big)\right) = 0.$$

Thus, $P_t 1_{A\cap B_r(x_0)}(x) = 0$ for all $x \in B_r(x_0)$ and $0 < t < t_0$, so that

$$0 = \int_{\mathbb{R}^d} 1_{A\cap B_r(x_0)} P_t 1_{A\cap B_r(x_0)} d\mu \underset{t\to 0+}{\longrightarrow} \mu(B_r(x_0)\cap A) > 0,$$

which is contradiction. Therefore, we must have $\mu(A) = 0$.

□

Directly using Lemma 11.1.5 and proof of Theorem 4.2.4, we obtain the following result.

Corollary 11.1.6. *Assume* **(C1)** *and* $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d,\mathbb{R}^d)$.

(i) $(T_t)_{t>0}$ *is strictly irreducible.*

(ii) Let $A \in \mathcal{B}(\mathbb{R}^d)$ *with* $m(A) > 0$*. Then* $\mathbb{P}_x(X_t \in A) > 0$ *for all* $x \in \mathbb{R}^d, t > 0$*, i.e.* $(P_t)_{t>0}$ *is irreducible.*

From Theorem 11.1.2, for any $\alpha > 0$, $x \in \mathbb{R}^d$, we define

$$R_\alpha(x,A) := \lim_{l\to\infty} R_\alpha(1_{B_l\cap A})(x), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Then $\alpha R_\alpha(\cdot, A)$ is a sub-probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ that is absolutely continuous with respect to μ. Using the Radon-Nikodym derivative, the resolvent kernel density is defined by

$$r_\alpha(x, \cdot) := \frac{R_\alpha(x, dy)}{\mu(dy)}, \qquad x \in \mathbb{R}^d.$$

Similarly, from Theorem 11.1.4, for any $t > 0$, $x \in \mathbb{R}^d$ we define

$$P_t(x, A) := \lim_{l \to \infty} P_t(1_{B_l \cap A})(x), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Then $P_t(\cdot, A)$ is a sub-probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ that is absolutely continuous with respect to μ. Using the Radon-Nikodym derivative, the resolvent kernel density is defined by

$$p_t(x, \cdot) := \frac{P_t(x, dy)}{\mu(dy)}, \qquad x \in \mathbb{R}^d.$$

Therefore using the exacly same method as in Proposition 8.3.8, we derive the following result.

Proposition 11.1.7. *Assume* **(C1)** *and* $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. *Let* $\alpha > 0$, $t > 0$. *Then it holds:*

(i) $G_\alpha g$ *has a locally Hölder continuous* μ*-version and*

$$R_\alpha g = \int_{\mathbb{R}^d} g(y) R_\alpha(\cdot, dy) = \int_{\mathbb{R}^d} g(y) r_\alpha(\cdot, y)\mu(dy), \quad \forall g \in \bigcup_{r \in [q,\infty]} L^r(\mathbb{R}^d, \mu). \qquad (11.6)$$

In particular, (11.6) *extends by linearity to all* $g \in L^q(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$, *i.e.* $(R_\alpha)_{\alpha>0}$ *is* $L^{[q,\infty]}(\mathbb{R}^d, \mu)$*-strong Feller.*

(ii) $T_t f$ *has a continuous* μ*-version* $\forall f \in \bigcup_{s \in [1,\infty]} L^s(\mathbb{R}^d, \mu)$ *and*

$$P_t f = \int_{\mathbb{R}^d} f(y) P_t(\cdot, dy) = \int_{\mathbb{R}^d} f(y) p_t(\cdot, y)\mu(dy), \qquad (11.7)$$

In particular, (11.7) *extends by linearity to all* $f \in L^1(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$, *i.e.* $(P_t)_{t>0}$ *is* $L^{[1,\infty]}(\mathbb{R}^d, \mu)$*-strong Feller.*

Finally, for any $\alpha > 0, x \in \mathbb{R}^d$, $g \in L^q(\mathbb{R}^d, \mu) + L^\infty(\mathbb{R}^d, \mu)$, *we have*

$$R_\alpha g(x) = \int_0^\infty e^{-\alpha t} P_t g(x)\, dt.$$

Remark 11.1.8. *Assume the situation of Remark 11.1.1. Then we obtain*

$$\int_{\mathbb{R}^d} \langle \frac{1}{2}\rho\phi A \nabla G_\alpha f, \nabla \varphi \rangle dx - \int_{\mathbb{R}^d} \langle \rho\phi \mathbf{B}, \nabla G_\alpha f \rangle \varphi\, dx + \int_{\mathbb{R}^d} (\alpha \rho\phi G_\alpha f)\, \varphi\, dx$$
$$= \int_{\mathbb{R}^d} (\rho\phi f)\, \varphi\, dx, \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d)$$

and given $T > 0$, *it holds*

$$\iint_{\mathbb{R}^d \times (0,T)} \Big\langle \frac{1}{2}\rho\phi A \nabla u, \nabla \varphi \Big\rangle - \big\langle \rho\phi \mathbf{B}, \nabla u \big\rangle \varphi\; dxdt$$
$$= \iint_{\mathbb{R}^d \times (0,T)} u\, \partial_t \varphi \cdot \rho\phi dxdt \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d \times (0,T)).$$

Thus using analogous methods to the above, we obtain the analogue of Theorem 11.1.2, Lemma 11.1.3, Theorem 11.1.4, Lemma 11.1.5, Corollary 11.1.6.

11.2 Application to weak existence of Itô-SDEs

In order to construct a Hunt process associated with $(P_t)_{t>0}$ which is identified to a weak solution to the corresponding Itô-SDE, we present a final condition.

(C2): Fix $p > d$ and $q := \frac{pd}{p+d}$. $\mathbf{B} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$, $\nabla A^T \in L^q_{loc}(\mathbb{R}^d)$ and $\rho \in H^{1,q}_{loc}(\mathbb{R}^d)$.

If we assume **(C1)** and **(C2)**, then one can directly check that $(\mathbf{H2})'$ of Part I holds. Thus, using Proposition 9.1.1 and the analogous method to Theorem 3.2.1, we arrive at the following theorem.

Theorem 11.2.1. *Assume* **(C1)**, **(C2)**. *Then there exists a Hunt process*

$$\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d\cup\{\Delta\}})$$

with state space $\mathbb{R}^d$ *and life time*

$$\zeta = \inf\{t \geq 0 \,:\, X_t = \Delta\} = \inf\{t \geq 0 \,:\, X_t \notin \mathbb{R}^d\},$$

having the transition function $(P_t)_{t\geq 0}$ *as transition semigroup, such that* $\mathbb{M}$ *has continuous sample paths in the one point compactification* $\mathbb{R}^d_\Delta$ *of* $\mathbb{R}^d$ *with the cemetery* Δ *as point at infinity, i.e. for all* $x \in \mathbb{R}^d$

$$\mathbb{P}_x\bigg(\big\{\omega \in \Omega \mid X_\cdot(\omega) \in C\big([0,\infty), \mathbb{R}^d_\Delta\big),\, X_\cdot(\omega) = \Delta\ \forall t \geq \zeta(\omega)\big\} \bigg) = 1.$$

Remark 11.2.2. *Consider the situation of Remark 11.1.1 and assume* **(C2)**. *Then one can check that* (***H2***)$'$ *of Part I also holds since*

$$C_0^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+) \subset D(L^0)_{0,b} \subset D(L_1) \cap C_0(\mathbb{R}^d)$$

and $L_1 f \in L^q(\mathbb{R}^d)_0$ *for all* $f \in C_0^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+)$. *Hence using the analogous way to Theorem 3.2.1, there exists a Hunt process* $\mathbb{M}$ *which has skew reflection on the hyperplane* $\partial\mathbb{R}^d_+$. *Moreover the transition function* $(P_t)_{t>0}$ *of* $\mathbb{M}$ *satisfies general strong Feller properties, irreduciblity and strict irreducibility.*

Using Theorem 11.1.2 and the analogous method to Theorem 3.2.3 (ii), we obtain the following Krylov type estimate.

Proposition 11.2.3. *Assume* **(C1)**, **(C2)**. *Let* $g \in L^r(\mathbb{R}^d, \mu)$ *for some* $r \in [q, \infty]$. *Then for any open ball* B *there exists a constant* $C_{B,r}$ *which depends on* B *and* r *and does not depends on the* VMO *condition of* A, *such that for all* $t \geq 0$

$$\sup_{x\in\overline{B}} \mathbb{E}_x \left[\int_0^t |g|(X_s)\, ds \right] < e^t C_{B,r} \|g\|_{L^r(\mathbb{R}^d,\mu)}. \tag{11.8}$$

Using the analogous method to proof of Theorem 3.2.8, we obtain the following result.

Theorem 11.2.4. *Assume* **(C1)**, **(C2)**. *Consider the Hunt process* $\mathbb{M}$ *from Theorem 11.2.1. Let* $\sigma = (\sigma_{ij})_{1\le i\le d, 1\le j\le m}$, $m \in \mathbb{N}$ *arbitrary but fixed, be any matrix of functions* $\sigma_{ij} \in L^\infty_{loc}(\mathbb{R}^d)$ *for all* $1 \le i \le d$, $1 \le j \le m$, *such that* $\widetilde{A} = \sigma\sigma^T$, *i.e.*

$$\widetilde{a}_{ij}(x) = \sum_{k=1}^{m} \sigma_{ik}(x)\sigma_{jk}(x), \quad \forall x \in \mathbb{R}^d,\ 1 \le i,j \le d,$$

Then on a standard extension of $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}_x)$ *with life time* ζ, $x \in \mathbb{R}^d$, *that we denote for notational convenience again by* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}_x)$ *with life time* ζ, $x \in \mathbb{R}^d$, *there exists a standard m-dimensional Brownian motion* $W = (W^1, \dots, W^m)$ *starting from zero such that* $\mathbb{P}_x$*-a.s. for any* $x \in \mathbb{R}^d$, *it holds*

$$X_t = x + \int_0^t \frac{1}{\sqrt{\psi}(X_s)}\sigma(X_s)dW_s + \int_0^t \left(\frac{1}{2\psi}\nabla A^T + \frac{A^T\nabla\rho}{2\rho\psi} + \mathbf{B}\right)(X_s)ds, \quad 0 \le t < \zeta.$$

The corresponding resolvent $(G_\alpha)_{\alpha>0}$ *and semigroup* $(T_t)_{t>0}$ *satisfy general strong Feller properties as in Theorem 11.1.2 and Theorem 11.1.4, respectively. Furthermore,* $\mathbb{M}$ *satisfies irreducibility and strict irreducibility as in Corollary 11.1.6. Various properties of Part I, II, such as conservativeness in Theorem 4.1.2, moment inequality in Theorem 4.1.4 (i), Theorem 6.1.4, recurrence and transience in Proposition 4.2.5, Theorem 4.2.7, Lemma 4.2.8, Theorem 4.2.9, ergodic properties in Theorem 4.2.11 hold in the situation of Part IV.*

11.3 Explicit conditions for global well-posedness and ergodic properties

The finial section is devoted to present some conditions to derive our previous results in the case where diffusion and drift coefficients are explicitly given. By a direct application of Theorem 8.3.1, we show existence of a pre-invariant measure for a large class of second order partial differential operators.

Theorem 11.3.1. *Let* $A = (a_{ij})_{1\le i,j\le d}$ *be a matrix of functions satisfying* (11.2) *and* $a_{ij} \in H^{1,2}_{loc}(\mathbb{R}^d)$ *for all* $1 \le i,j \le d$. *Let* ψ *be a positive function satisfying* (11.1). *Let*

$\mathbf{G} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ be such that

$$\frac{1}{2}\nabla A^T - \psi \mathbf{G} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d).$$

Then there exists $\rho \in H^{1,p}_{loc}(\mathbb{R}^d) \cap C^{0,1-d/p}_{loc}(\mathbb{R}^d)$ satisfying $\rho(x) > 0$ for all $x \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \langle \mathbf{G} - \beta^{\rho, A^T, \psi}, \nabla \varphi \rangle \rho \psi dx = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d),$$

i.e.

$$\int_{\mathbb{R}^d} \left(\frac{1}{2\psi} \mathrm{div}\,(A\nabla\varphi) + \langle \mathbf{G} - \frac{1}{2\psi}\nabla A^T, \nabla\varphi \rangle \right) \rho\psi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d).$$

Moreover $\mathbf{G} - \beta^{\rho, A^T, \psi} \in L^p_{loc}(\mathbb{R}^d, \mathbb{R}^d)$.

Now let $\mathbf{B} := \mathbf{G} - \beta^{\rho, A^T, \psi}$ and consider all situations of Section 11.1. Then all results of Section 11.1 automatically hold under the assumption of Theorem 11.3.1. Using Theorem 11.2.4, we obtain the following result which presents global well-posedness and ergodic properties in the case where diffusion and drift coefficients that are possibly discontinuous are explicitly given.

Theorem 11.3.2. *Under the assumption of Theorem 11.3.1, suppose $\nabla A^T \in L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Let $\widetilde{A} := \frac{A+A^T}{2}$ and $(\sigma_{ij})_{1 \le i \le d, 1 \le j \le m}$, $m \in \mathbb{N}$ arbitrary but fixed, be a matrix consisting of functions $\sigma_{ij} \in L^\infty_{loc}(\mathbb{R}^d)$ for all $1 \le i \le d$, $1 \le j \le m$, such that $\widetilde{A} = \sigma\sigma^T$, i.e.*

$$\widetilde{a}_{ij}(x) = \sum_{k=1}^m \sigma_{ik}(x)\sigma_{jk}(x), \quad \forall x \in \mathbb{R}^d,\ 1 \le i, j \le d.$$

Then there exists a standard extension of a Hunt process $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P}_x)$ with life time ζ, $x \in \mathbb{R}^d$, that we denote for notational convenience again by $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P}_x)$ with life time ζ, $x \in \mathbb{R}^d$, and there exists a standard m-dimensional Brownian motion $W = (W^1, \ldots, W^m)$ starting from zero such that for any $x \in \mathbb{R}^d$, it *weakly solves* $\mathbb{P}_x$*-a.s*

$$X_t = x + \int_0^t \frac{1}{\sqrt{\psi}(X_s)}\sigma(X_s)dW_s + \int_0^t \mathbf{G}(X_s)ds, \quad 0 \le t < \zeta. \tag{11.9}$$

If there exists a constant $M > 0$ *and some* $N_0 \in \mathbb{N}$*, such that*

$$\frac{1}{\psi(x)}\left(-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\text{trace}A(x)\right) + \big\langle \mathbf{G}(x), x\big\rangle \le M\left(\|x\|^2+1\right)\left(\ln(\|x\|^2+1)+1\right)$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$*, then* $\mathbb{M}$ *is non-explosive, i.e.* $\mathbb{P}_x(\zeta = \infty) = 1$ *for all* $x \in \mathbb{R}^d$*. Moreover* $\mathbb{M}$ *is irreducible and strict irreducible, hence satisfies the result as in Proposition 4.2.5 in the situation of Part IV. If there exists a constant* $M > 0$ *and some* $N_0 \in \mathbb{N}$*, such that*

$$\frac{1}{\psi(x)}\left(-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\text{trace}A(x)\right) + \big\langle \mathbf{G}(x), x\big\rangle \le 0$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$*, then* $\mathbb{M}$ *is recurrent in the probabilistic sense as in* (4.16)*. If there exists a constant* $C > 0$ *and some* $N_0 \in \mathbb{N}$*, such that*

$$\frac{1}{\psi(x)}\left(-\frac{\langle A(x)x, x\rangle}{\|x\|^2+1} + \frac{1}{2}\text{trace}A(x)\right) + \big\langle \mathbf{G}(x), x\big\rangle \le -C(\|x\|^2+1)$$

for a.e. $x \in \mathbb{R}^d \setminus B_{N_0}$*, then* $\rho\psi dx$ *is a probability invariant measure of* $\mathbb{M}$ *and ergodic properties as in Theorem 4.2.11 holds in the situation of Part IV. Finally if* $\mathbf{G} \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ *and* $\mathbb{M}$ *is non-explosive, then* $\mathbb{M}$ *is a unique solution to* (11.9) *in a weak sense.*

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국문초록

이 논문에서 우리는 거친 계수를 갖는 이토 확률미분방정식의 약한 해들의 대역적인 존재 및 유일성, 장시간의 행동에 대한 해석학적 접근을 연구한다. 타원형 및 포물형 정칙성 이론과 일반화된 디리클레 형식 이론을 사용함으로써, 우리는 넓은 유형의 타원형 2계 편미분 작용소의 예비 불변측도의 존재성을 보이고, 그 작용소는 사실은 어느 헌트 과정의 극소 생성자가 됨을 보인다. 그 후, 이 헌트 과정은 $\mathbb{R}^d$ 위의 모든 점들을 시작점으로 갖는 이토 확률미분방정식의 폭발 시간 안에서 약한 해로 동일시된다. 그 헌트 프로세스는 $\mathbb{R}^d$의 한점 컴팩트화된 공간에서 연속인 샘플 경로를 갖고 알려진 존재 및 유일성 정리에 의해 그것은 폭발 시간 안에서 경로마다 유일한 강한 해가 된다. 해석학적, 확률론적 방법을 사용하여 우리는 고전적인 강한 펠러 성질을 포함하는 일반화된 강한 펠러 성질들, 크릴로프 유형의 가늠, 모먼트 부등식, 다양한 비폭발 판정법을 유도한다. 포물형 하르낙 부등식을 이용하여 우리는 프로세스의 기약성과 강한 기약성을 보이고 재귀성과 에르고딕 행동들에 대한 명확한 조건들을 이끌어 낸다. 더 나아가서 우리는 퇴화된 정도의 점들이 르벡 측도 0을 만족하는 퇴화된 거친 확산 계수에 관한 이토 확률미분방정식의 약한 해의 존재성과 유일성을 조사한다. 마지막으로 우리는 예비 불변측도의 밀도함수가 명확히 주어졌을 때를 고려한다. 단순히 그 예비측도의 존재성과 어떤 정칙성만 알았던 이전의 경우와는 달리, 우리는 퇴화되지 않은 비대칭, 불연속 확산 행렬인 경우에 얼마나 우리의 방법들이 확장되고 적용될 수 있는지 조사한다. 이를 위해 우리는 시간 텀에 무게를 갖는 발산 형식 선형 포물형 편미분방정식의 정칙이론에 대한 변분적 접근을 발전시킨다.

주요어휘: 일반화된 디리클레 형식, 불변 측도, 헌트 과정, 이토 확률미분방정식, 타원형 및 포물형 정칙성, 강한 펠러 성질, 비폭발성, 보존성, 기약성, 강한 기약성, 재귀성, 일시적임, 에르고딕성, 약한 유일성, 크릴로프 타입 추정

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