이학 박사 학위논문

Study on structures of digraphs and graphs in the aspect of their holes

(Hole의 관점에서 그래프와 유향그래프의 구조에 관한 연구)

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이 논문을 이학 박사 학위논문으로 제출함

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Study on structures of digraphs and graphs in the aspect of their holes

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
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by

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Abstract

Study on structures of digraphs and graphs in the aspect of their holes

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This thesis aims at studying phylogeny graphs and graph completions in the aspect of holes of graphs or digraphs. A hole of a graph is an induced cycle of length at least four and a graph is chordal if it does not contain a hole. Specifically, we determine whether the phylogeny graphs of acyclic digraphs are chordal or not and find a way of chordalizing a graph without increasing the size of maximum clique not so much. In this vein, the thesis is divided into two parts.

In the first part, we completely characterize phylogeny graphs of $(1, i)$ digraphs and $(i, 1)$ digraphs, respectively, for a positive integer i. Then, we show that the phylogeny graph of a $(2, j)$ digraph D is chordal if the underlying graph of D is chordal for any positive integer j. In addition, we extend the existing theorems computing phylogeny numbers of connected graph with a small number of triangles to results computing phylogeny numbers of connected graphs with many triangles.

In the second part, we present a minimal chordal supergraph G^* of a graph G satisfying the inequality $\omega(G^*)-\omega(G) \leq i(G)$ for the non-chordality index $i(G)$ of G. Using the above chordal supergraph as a tool, we prove that the family of graphs satisfying the NC property satisfies the Hadwiger conjecture and the Erdős-Faber-Lovász Conjecture, and the family of graphs with bounded non-chordality indices is linearly χ-bounded.

Key words: hole, chordal graph, phylogeny graph, phylogeny number, non-chordality index, chromatic number
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Chapter 1

Introduction

This thesis aims at studying on structures of digraphs and graphs in the aspect of their holes. We first study the digraphs whose phylogeny graphs are chordal and phylogeny numbers of graphs in the aspect of the number of triangles and diamonds. Then we study a way of finding a chordal completion of a graph without increasing the size of maximum clique not so much.

Now we introduce the basic notions that will be used in the thesis.

1.1 Basic notions

Now we introduce basic notions in graph theory, which shall be frequently used in this thesis. For undefined terms, readers may refer to [7].

A *graph* G is defined as an ordered pair (V, E) where V is a set and E is a family of unordered pairs of elements in V. An element of V and an element of E are called a *vertex* and an *edge* of G, respectively. If $e = \{u, v\}$ is an edge, then we simply write it by uv for convenience when there is no confusion. The set of vertices and the set of edges of a graph G are called the *vertex set* and the *edge set* of G, respectively, and denoted by $V(G)$ and $E(G)$, respectively.

Let G be a graph and $e = uv$ be an edge of G. Then we say that e joins (or connects) u and v, u and v are the *end vertices* (or *ends*) of e, and each of u and v is *incident* to e. In addition, we write $u \sim_G v$ and say that u and v are *adjacent* in G.

Let G be a graph and uv be an edge of G. If $u = v$, then the edge uv is called a *loop*. If $u \neq v$ and there are more than one edge connecting u and v, then uv is called a *multiple edge* or a *parallel edge*. We call a graph *simple* if it does not have loops and no multiple edges.

Let G be a graph and u be a vertex of G. A vertex of G which is adjacent to u is called a *neighbor* of u. The set of all neighbors of u is called the *(open) neighborhood* of u and is denoted by $N_G(u)$. The *degree* of the vertex u is defined to be the number of edges incident to u and is denoted by $d_G(u)$ or $\deg_G(u)$. A vertex with degree 0 is called an isolated vertex. For a positive integer k, the set of k isolated vertices is denoted by I_k. When there is no confusion, we sometimes omit the subscript G in the notations defined above.

Two graphs G and H are said to be *isomorphic* if there exist bijections $f_V : V(G) \to V(H)$ and $f_E : E(G) \to E(H)$ such that for every edge $e \in E(G)$, e connects vertices u and v in G if and only if $f_E(e)$ connects vertices $f_V(u)$ and $f_V(v)$ in H. If G and H are isomorphic with bijections f_V and f_E described above, then we write $G \cong H$ and call (f_V, f_E) a *graph isomorphism* from G to H.

Let G be a graph. A graph H is called a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If H is a subgraph G, then G is called a *supergraph* of H. For a nonempty subset S of $V(G)$, the *subgraph of G induced by S*, denoted by $G[S]$, is the simple graph defined by $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) \mid u, v \in S\}$. For a nonempty proper subset S of $V(G)$, $G - S$ denotes the subgraph of G induced by $V(G) \setminus S$. For notational simplicity, we write $G - v$ instead of $G - \{v\}$ for a vertex v of G. An *induced subgraph* of G is a graph induced by a nonempty subset of $V(G)$. We say that G is *H-free* if G has no induced subgraph which is isomorphic to H.

An *edge contraction* is an operation which removes an edge from a graph while simultaneously merging the two vertices that it previously joined. A graph H is called a *minor* of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges. We say that G is H-*minor-free* if G has no minor which is isomorphic to H.

Given a simple graph G, the *complement* $\overline{G}$ of G is defined to be a simple graph obtained by reversing the adjacency of G, i.e., $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid uv \notin E(G)\}$.

A *complete graph* K_n is a graph with n vertices in which every pair of vertices are adjacent. A vertex subset S of $V(G)$ is called a *clique* if the induced subgraph $G[S]$ is complete. We sometimes call a complete subgraph a *clique*. The *clique number* of a graph G is defined to be the number of vertices in a maximum clique and denoted by $\omega(G)$.

For a clique K and an edge e of a graph G, we say that K *covers* e (or e *is covered by* K) if and only if K contains the two end points of e. An *edge clique cover* of a graph G is a collection of cliques that cover all the edges of G. The *edge clique cover number* of a graph G, denoted by $\theta_e(G)$, is the smallest number of cliques in an edge clique cover of G.

A *walk* from a vertex v_1 to a vertex v_{k+1} is an alternating sequence

$$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

of vertices and edges where each v_i $(i = 1, \ldots, k+1)$ is a vertex of G and each e_j $(j = 1, \ldots, k)$ is an edge of G joining v_i and v_{i+1}. The *length* $\ell(W)$ of a walk W is defined to be the number of edges belonging to it. If there exists a walk starting from a vertex v to another vertex w, then we say that v and w are *connected* by a walk. If any two vertices of a graph G are connected by a walk, then we say that G is *connected*. Otherwise, G is said to be *disconnected*. A maximally connected subgraph of G is called a *(connected) component* of G. It is easy to see that G is connected if and only if G has only one connected component.

A walk

$$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

is called a *path* if $v_1, \ldots, v_{k+1}$ are all distinct, and called a *cycle* if $v_1 = v_{k+1}$ and $v_1, \ldots, v_k$ are all distinct. We denote a path on n vertices by P_n, and a cycle on n vertices by C_n. If no subgraph of G is a cycle, then G is called *acyclic*. A connected acyclic graph is called a *tree*.

A *digraph* (or *directed graph*) D is defined as an ordered pair $(V(D), A(D))$ where $V(D)$ is a set and $A(D)$ is a family of ordered pairs of elements in $V(D)$. An element of $V(D)$ and an element of $A(D)$ are called a *vertex* and an *arc* (or *directed edge*) of D, respectively. The subdigraphs and induced subdigraphs of a digraph are similarly defined as the subgraphs and induced subgraphs of a graph. If $(u, v) \in A(D)$, then we say that u and v are the tail and the head of (u, v), respectively, so that the arc (u, v) goes from the tail u to the head v.

Let D be a digraph and u be a vertex of D. A vertex v is called an *out-neighbor* (resp. *in-neighbor*) of u if (u, v) (resp. (v, u)) is an arc in D. The set of all out-neighbors (resp. in-neighbors) of u is called the *out-neighborhood* (resp. *in-neighborhood*) of u in D and denoted by $N_D^+(u)$ (resp. $N_D^-(u)$). The *outdegree* $d_D^+(u)$ is the number of arcs with tail u and the *indegree* $d_D^-(u)$ is the number of arcs with head u.

A *directed walk* from a vertex v_1 to a vertex v_{k+1} is an alternating sequence

$$v_1, a_1, v_2, a_2, \ldots, v_k, a_k, v_{k+1}$$

of vertices and arcs where each v_i $(i = 1, \ldots, k+1)$ is a vertex and each a_j $(j = 1, \ldots, k)$ is an arc from v_i to v_{i+1}. The *length* $\ell(W)$ of a directed walk W is defined to be the number of arcs belonging to it. A directed walk

$$v_1, a_1, v_2, a_2, \ldots, v_k, a_k, v_{k+1}$$

is called a *directed path* if $v_1, \ldots, v_{k+1}$ are all distinct, and called a *directed* *cycle* if $v_1 = v_{k+1}$ and $v_1, \ldots, v_k$ are all distinct. If no subdigraph of D is a directed cycle, then G is said to be *acyclic*.

Let G be a digraph. If we assign an orientation to each edges of G, then the resulting digraph is called an *orientation* of G. An *oriented graph* is a graph with an orientation. If an orientation D of G satisfies the property that $(u, v), (v, w) \in A(D)$ imply $(u, w) \in A(D)$, then the orientation is said to be *transitive*.

A *hole* of a graph is an induced cycle of length at least four. A graph G is said to be *chordal* if every cycle in G of length greater than 3 has a chord, namely, an edge joining two nonconsecutive vertices on the cycle, that is, G does not contain a cycle of length at least 4 as an induced subgraph. A graph H is called a *chordal completion* (or *triangulation*) of a graph G, if H is a chordal spanning supergraph of G. See [28] for a survey on chordal completion.

If two sets A and B have a nonempty intersection, then we say that A and B *overlap* or *intersect*. A graph G is called the *intersection graph* of a family $\mathcal{F}$ of sets if there exists a bijection $\phi : V(G) \to \mathcal{F}$ such that two vertices x and y are adjacent in G if and only if $\phi(x) \cap \phi(y) \neq \emptyset$. A graph G is called an *interval graph* if we can assign to each vertex x of G a real interval $J(x)$ so that, whenever $x \neq y$, $xy \in E(G)$ if and only if $J(x) \cap J(y) \neq \emptyset$. Obviously, an interval graph is an intersection graph of a set of open intervals (or a set of closed intervals).

The notion of interval graph was introduced independently by G. Hajós [23] and S. Benzer [4]. Since the introduction of an interval graph, it has been extensively studied due to its important role in various fields such as scheduling theory, chemistry, biology, and genetics.

There are nice characterization for an interval graph.

The *asteroidal triple* (AT for short) is defined as a graph with the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $\{v_1v_2, v_3v_4, v_5v_6, v_1v_3, v_3v_5, v_5v_1\}$ (see Figure 1.1).

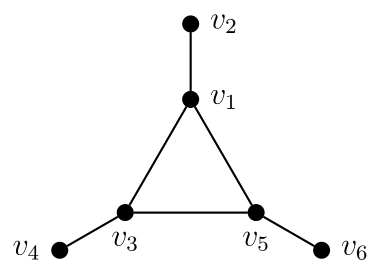


Figure 1.1: The asteroidal triple

Theorem 1.1.1 ([36]). *A graph is an interval graph if and only if it is chordal and AT-free.*

Theorem 1.1.2 ([20]). *A graph is an interval graph if and only if it is C_4-free and its complement has a transitive orientation.*

It is immediately true by Theorem 1.1.1 that an interval graph is chordal.

In this thesis, all the graphs and digraphs are assumed to be finite and simple unless otherwise stated.

Mathematical Notation

$\mathbb{N}$	:	The set of positive integers
$\mathbb{Z}_{\geq 0}$	:	The set of nonnegative integers
$\mathbb{Z}$	:	The set of integers
$\mathbb{R}$	:	The set of real numbers
$V(G)$	:	The vertex set of a graph (or a digraph) G
$E(G)$	:	The edge set of a graph G
$A(D)$	:	The arc set of a digraph D
uv in G	:	The edge between a vertex u and a vertex v in a graph G
(u, v) in D	:	The arc from a vertex u to a vertex v in a digraph D
$G[S]$	:	The subgraph of a graphs G induced by a vertex subset S
$G - S$	:	The subgraph of a graph G induced by $V(G) \setminus S$
$G - v$	:	The subgraph of a graph G induced by $V(G) \setminus \{v\}$
$\overline{G}$	:	The complement of a graph G
$d_G(u)$	:	The degree of a vertex u in a graph G
$d_D^-(u)$	:	The indegree of a vertex u in a digraph D
$d_D^+(u)$	:	The outdegree of a vertex u in a digraph D
$N_G(u)$	:	The neighborhood of a vertex u in a graph G
$N_D^-(u)$	:	The in-neighborhood of a vertex u in a digraph D
$N_D^+(u)$	:	The out-neighborhood of a vertex u in a digraph D
$\theta_e(G)$	:	The edge clique cover number of a graph G
I_k	:	The set of k isolated vertices
K_n	:	A complete graph of n vertices
P_n	:	A path of length n
C_n	:	A cycle of length n

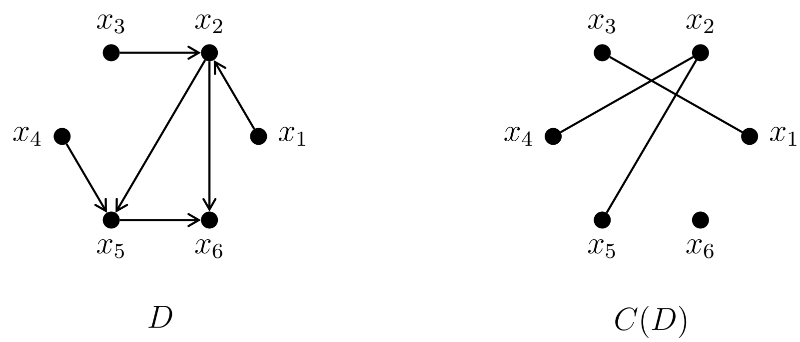


Figure 1.2: A digraph D and its competition graph $C(D)$

1.2 Preliminaries

1.2.1 Phylogeny graphs

Let D be a digraph which represents a food web in an ecosystem which is obtained by drawing an arc from a predator to a prey. The vertex set $V(D)$ represents the set of species in the ecosystem and an arc $(x, y) \in A(D)$ means that a species x preys on a species y. One important assumption in ecology is that two species compete if they have a common prey. Hence the rivalry between species in a food web, which is an important subject in ecology, can be represented by the competition graph of D. The competition graph of a digraph is defined as follows.

The *competition graph* of a digraph D, denoted by $C(D)$, is defined as a graph which has the same vertex set as D and has an edge xy between two distinct vertices x and y if and only if, for some vertex $z \in V(D)$, (x, z) and (y, z) are arcs in D (see Figure 1.2 for an example). Cohen [10] introduced the notion of competition graphs in the study on predator-prey concepts in ecological food webs. Competition graphs also have applications in areas such as modeling of complex economic systems, radio transmission, and coding. For a summary of these applications, the reader may refer to [44] and [51].

Cohen observed empirically that real-world competition graphs are usually interval graphs. Interval graphs have been widely studied and applied in many different area such as developmental psychology, scheduling theory, chemistry, biology, and genetics. Cohen's observation had led to a great deal of research on the structure of competition graphs and on the relationship between the structure of digraphs and their corresponding competition graphs. In the same vein, various variants of competition graphs have been introduced and studied. For recent work related to competition graphs, see [18, 31, 32, 37, 64].

There have been a large literature devoted to explaining Cohen's observation and to studying the properties of competition graphs. There are two different approaches in attempting to explain Cohen's observation. The first attempt is statistical, and develops models for randomly generated food webs from which one can show that the corresponding competition graphs are interval graphs. The second attempt is graph-theoretical. This involves the analysis of the properties of competition graphs that arise from different kinds of digraphs and attempts to characterize the acyclic digraphs whose competition graphs are interval.

Stief [57] showed that there is no forbidden subgraph characterization of acyclic digraphs whose competition graphs are interval. Unfortunately, this means that it is not easy to study the structural properties of competition graphs, which led researches to find another ways to explain Cohen's observation.

Dutton and Brigham [15] characterized the competition graphs arising from digraphs D which may have directed cycles and which also may have loops. They showed that if $|V(G)| = n$, then G is a competition graph of a digraph D (which may have loops) if and only if $\theta_e(G) \leq n$.

Roberts and Steif [50] obtain a similar characterization in the case that there are no loops. They showed that G is a competition graph of a digraph which has no loops if and only if there are cliques $C_1, C_2, \ldots, C_p$ which cover the edges of G and such that if $D_i = V(G) - C_i$, then $\{D_1, D_2, \ldots, D_p\}$ has a system of distinct representatives.

Roberts [45] observed that any graph G together with $|E(G)|$ additional isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* of a graph G to be the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph, and denoted it by $k(G)$. Acyclicity of a digraph is a natural assumption as it represents a food web in an ecosystem from which this subject is originated. However, the requirement of being acyclic is not necessary in general. In the literature, competition graphs of arbitrary digraphs are also widely studied.

Computing the competition number of a graph is one of the important problems in the field of competition graphs. Yet, computing the competition number of a graph is usually not easy as Opsut has shown that computation of the competition number in general is NP-hard in 1982. While an upper bound M of the competition number of a graph G may be obtained by constructing an acyclic digraph whose competition graph is G together with M isolated vertices, getting a good lower bound is a very difficult task because there are usually so many cases to consider. There has been much effort to compute the competition numbers of graphs.

The notion of phylogeny graphs was introduced by Roberts and Sheng [46] as a variant of competition graphs. (See also [26, 41, 47, 48, 49, 65] for study on phylogeny graphs.) Given an acyclic digraph D, the *underlying graph* of D, denoted by $U(D)$, is the graph with vertex set $V(D)$ and edge set $\{xy \mid (x,y) \in A(D) \text{ or } (y,x) \in A(D)\}$. The *phylogeny graph* of an acyclic digraph D, denoted by $P(D)$, is the graph with vertex set $V(D)$ and edge set $E(U(D)) \cup E(C(D))$.

"Moral graphs", having arisen from studying Bayesian networks, are the same as phylogeny graphs. One of the best-known problems, in the context of Bayesian networks, is related to the propagation of evidence. It consists

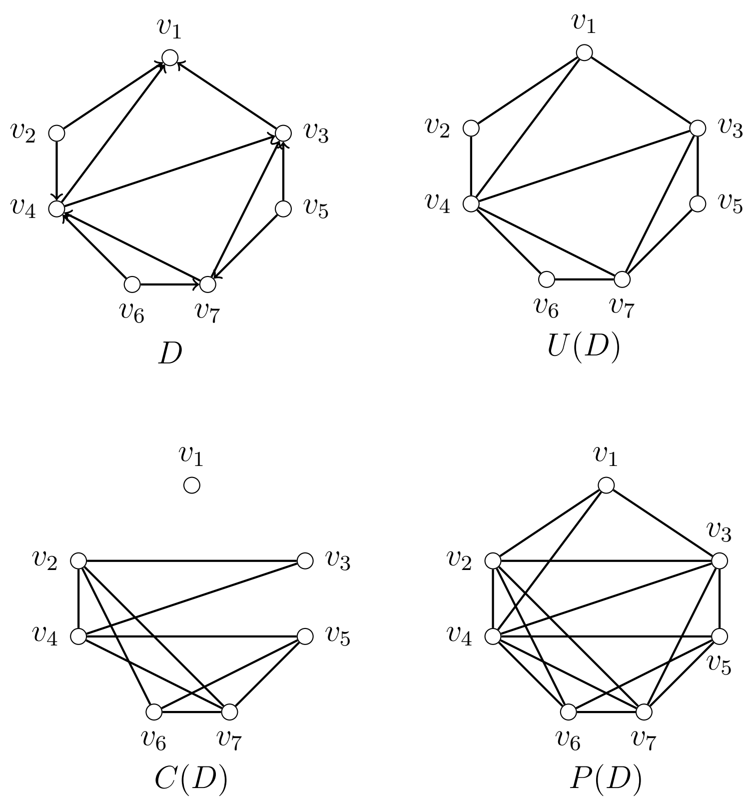


Figure 1.3: An acyclic digraph D, the underlying graph $U(D)$ of D, the competition graph $C(D)$ of D, and the phylogeny graph $P(D)$ of D.

of the assignment of probabilities to the values of the rest of the variables, once the values of some variables are known. Cooper [12] showed that this problem is NP-hard. Most noteworthy algorithms for this problem are given by Pearl [43], Shachter [55] and by Lauritzen and Spiegelhalter [33]. Those algorithms include a step of triangulating a moral graph, that is, adding edges to a moral graph to form a chordal graph.

As triangulations of moral graphs play an important role in algorithms for propagation of evidence in a Bayesian network, studying chordality of the phylogeny graphs of acyclic digraphs is meaningful. Yet, characterizing the acyclic digraphs whose phylogeny graphs are chordal seems to be more difficult than characterizing the acyclic digraphs whose competition graphs are interval. In this respect, hoping to provide insights for the further research, Lee *et al.* [34] studied the phylogeny graphs of $(2,2)$ digraphs and gave sufficient conditions and necessary conditions for $(2,2)$ digraphs having chordal phylogeny graphs. An acyclic digraph each vertex of which has indegree at most i and outdegree at most j is called an (i,j) digraph for some positive integers i and j. Hefner *et al.* [27] characterized $(2,2)$ digraphs whose competition graphs are interval. In the first section of Chapter 2, we extend their work to study phylogeny graphs of (i,j) digraphs.

Any acyclic digraph D for which G is an induced subgraph of $P(D)$ and such that D has no arcs from vertices outside of G to vertices in G is called a *phylogeny digraph* for G. The phylogeny number is defined analogously to the competition number. The *phylogeny number* $p(G)$ of G is the smallest r so that G has a phylogeny digraph D with $|V(D) \setminus V(G)| = r$. A phylogeny digraph D for a graph G for which $|V(D) \setminus V(G)| = p(G)$ is called an *optimal phylogeny digraph* for G. Given an optimal phylogeny digraph D for a graph G, we note that the digraph resulting from D by deleting the arcs outgoing from a vertex in $V(D) \setminus V(G)$ is still an optimal phylogeny digraph for G. In this vein, we may assume that outdegree of any vertex in $V(D) \setminus V(G)$ is zero for any optimal phylogeny digraph for a graph G [47].

Analogous to the competition number, the phylogeny number is closely related to the number of triangles as we may see from the following results.

Theorem 1.2.1 ([47]). *If G is a connected graph with no triangles, then*

$$p(G) = |E(G)| - |V(G)| + 1.$$

Given a graph G, we denote by G^- the graph obtained from G by deleting all the triangles edges of G where a triangle edge means an edge on a triangle.

Theorem 1.2.2 ([47]). *Let G be a connected graph with exactly one triangle. Then*

$$p(G) = \begin{cases} |E(G)| - |V(G)| & \textit{if } G^- \textit{ has three components;} \\ |E(G)| - |V(G)| - 1 & \textit{if } G^- \textit{ has one or two components.} \end{cases}$$

Theorem 1.2.3 ([49]). *Let G be a connected graph with exactly two triangles which share one of their edges. Let x, u, v, y be the vertices for these two triangles with the edge uv being their common edge. Then*

$$p(G) = \begin{cases} |E(G)| - |V(G)| - 1 & \textit{if } G^- \textit{ has four components or} \\ & \textit{if } G^- \textit{ has three components with } G_x = G_y; \\ |E(G)| - |V(G)| - 2 & \textit{otherwise.} \end{cases}$$

Theorem 1.2.4 ([49]). *Let G be a connected graph with exactly two triangles that are edge-disjoint. Then*

$$p(G) = \begin{cases} |E(G)| - |V(G)| - 1 & \textit{if } G^- \textit{ has five components;} \\ |E(G)| - |V(G)| - 2 & \textit{if } G^- \textit{ has four components;} \\ |E(G)| - |V(G)| - 2 & \textit{if } G^- \textit{ has three components, with each} \\ & \textit{component containing exactly two triangle} \\ & \textit{vertices, or with one component containing} \\ & \textit{a triangle of } G; \\ |E(G)| - |V(G)| - 3 & \textit{otherwise.} \end{cases}$$

As a matter of fact, Theorems 1.2.1-1.2.4 can be integrated into the following proposition. For a graph G containing at most two triangle,

$$|E(G)| - |V(G)| - 2t(G) + d(G) + 1 \leq p(G) \leq |E(G)| - |V(G)| - t(G) + 1 \tag{1.2.1}$$

where $t(G)$ and $d(G)$ denote by the number of triangles and the number of diamonds in G, respectively.

In the second section of Chapter 2, we extend the given inequalities in Theorems 1.2.1, 1.2.2, 1.2.3, and 1.2.4 to graphs with many triangles.

1.2.2 Graph colorings and chordal completions

Graph coloring problems are one of the most important, well-known and studied problems in graph theory. Graph coloring is a special case of graph labeling and actively studied in graph theory. It is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. A *vertex coloring* is a coloring the vertices of a graph. A vertex coloring is *proper* if no two adjacent vertices are of same color. For a given graph G, a function $f : V(G) \to \{1, 2, \ldots, k\}$ is called a *proper k-coloring* of G if $f(u) \neq f(v)$ for any adjacent vertices u and v. The *chromatic number* $\chi(G)$ of a graph G is defined to be the least positive integer k such that there exists a proper k-coloring of G. Similarly, an *edge coloring* assigns a color to each edge. An edge coloring is *proper* if no two adjacent edges of the same color. Other coloring problems can be transformed into a vertex coloring problems. For example, a face coloring of a plan graph is just a vertex coloring of its dual graph.

The convention of using colors originates from coloring of a map. In the middle of the nineteenth century, it was found that a map of England, with all counties, can be painted using only four colors in such a way that counties sharing a common border receive different colors. It became an interesting problem for many mathematicians whether it is possible to color any possible political map using only four colors. Appel and Haken [1] solved the problem in 1976 by using computer algorithm to check if the map can be four-colored in all possible cases. In 1943, Hadwiger [22] formulated the Hadwiger conjecture as follows.

Conjecture 1.2.5 ([22])**.** *If G is a K_n-minor-free graph, then $\chi(G) \leq n-1$.*

The Hadwiger conjecture is a far-reaching generalization of the four-color problem that still remains unsolved. There are many other graph coloring problems that have not been solved. (See [19], [29], [30], [39], and [40] for more problems on graph colorings.)

Graphs colorings have many practical applications as well as theoretical challenges. For example, vertex coloring of graphs can represent a mathematical model of various resource assignments. One of such a problem is to assign frequencies for stations of radio, or mobile phone network. Stations, which are in broadcasting range (and so their signals would interfere with each other) must be assigned different frequencies. To solve this problem, a mathematical model of the connection network is constructed, where vertices represent stations, and edges between them show conflicts (that is, pairs of stations, which need to be given different frequencies). The model itself is a graph with vertex coloring [19]. This application of vertex coloring has been widely studied in many papers. The reader may refer to [60], [56], [42], and [24].

Timetabling problems often involve restrictions in which pairs of activities cannot be performed simultaneously. For example, in scheduling courses at a school, two courses taught by the same individual cannot be scheduled at the same time. If the courses to be scheduled are represented by the vertices of a graph and every pair of courses that cannot be scheduled at the same time are connected by an edge, then a (proper) vertex coloring of this graph gives a feasible schedule of the courses. If the goal is to minimize the number of time slots needed, then the problem is that of finding the chromatic number of the graph (assuming each course take the same amount of time). An introduction to the timetabling problems can be found in the work of de Werra [13]. Timetabling problems have been studied extensively by many researchers including [11], [61], [35], and [14]. Schmidt and Ströhleim [54] provide an annotated bibliography for the timetabling problem [40].

Applications of graph coloring have also led to interesting generalizations of the graph coloring problem. Vising [59] and Erdős, Rubin, and Taylor [17] independently introduced the notion of list coloring to generalize that of ordinary graph coloring. Let G be a graph and C be a set of colors. A *list* for G is a mapping $L : V(G) \to \mathcal{P}(C)$ which assigns a set of colors to each vertex where $\mathcal{P}(C)$ denotes the power set of C. If $|L(x)| \geq k$ for all $x \in V(G)$, then L is called a *k-list*. A proper coloring $f : V(G) \to C$ is called an *L-coloring* of G if $f(x) \in L(x)$ for any $x \in V(G)$. The *list-chromatic number* of G, denoted by $\chi_l(G)$, is the smallest k such that G admits an L-coloring for every k-list L for G. A graph G is said to be *k-choosable* if $\chi_l(G) \leq k$. List colorings are important in the channel assignment problem when acceptable channels are specified. Brown *et al.* [8] and Mahadev and Roberts [38] have studied this class of coloring problem.

Dvořák and Postle [16] introduced the notion of DP-coloring, which is a generalization of list coloring. Let G be a graph and L be a list for G. The *auxiliary graph for G and L*, denoted by $H(G, L)$, is the graph with the vertex set $\{(v, c) \mid v \in V(G), c \in L(v)\}$ and the edge set $\{\{(v, c), (v', c')\} \mid$ either $c = c'$ and $vv' \in E(G)$ or $v = v'\}$. By construction, for every distinct vertices v and v' of G, the set of edges of $H(G, L)$ joining $\{(v, c) \mid c \in L(v)\}$ and $\{(v', c') \mid c' \in L(v')\}$ is empty if $vv' \notin E(G)$ and forms a matching (possibly empty) if $vv' \in E(G)$. Based on these properties of $H(G, L)$, Dvořák and Postle introduced the notion of DP-coloring. We follow the slightly modified version used by Bernshteyn *et al.* [5].

Definition 1.2.6. Let G be a graph. A *cover* of G is a pair (L, H), where L is a list for G and H is a graph with vertex set $\bigcup_{v \in V(G)} \{(v, c) \mid v \in V(G), c \in L(v)\}$, satisfying the following conditions.

1. For each $v \in V(G)$, $H[\{v\} \times L(v)]$ is a complete graph.

2. For each $uv \in E(G)$, the edges between $\{u\} \times L(u)$ and $\{v\} \times L(v)$ form a matching (possibly empty).

3. For each distinct $u, v \in V(G)$ with $uv \notin E(G)$, no edges of H connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

We note that the auxiliary graph for the graph G and the list L in Definition 1.2.6 is a special type of H.

Definition 1.2.7. Suppose G is a graph and (L, H) is a cover of G. An (L, H)-*coloring* of G is an independent set $I \subset V(H)$ of size $|V(G)|$. In this context, we refer to the vertices of H as the *colors*. A graph G is said to be (L, H)-*colorable* if it admits an (L, H)-coloring.

Definition 1.2.8. The *DP-chromatic number*, denoted by $\chi_{DP}(G)$, is the minimum k such that G is (L, H)-colorable for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

It is well-known that, for a graph G,

$$\chi(G) \leq \chi_l(G) \leq \chi_{DP}(G) \tag{1.2.2}$$

where $\chi_l(G)$ and $\chi_{DP}(G)$ are the list-chromatic number and the DP-chromatic number of G, respectively.

For a given graph, we may obtain a chordal completion by adding edges. Many different chordal completions exist for a given graph in general. Most of the related graph problems that arise from practical applications seek to minimize various graph parameters of a chordal completion. For example, the *minimum* triangulation problem, also referred to as the *minimum fill-in* problem, asks to find a chordal completion with the smallest number of edges, and it has applications in sparse matrix computations [53], database management [58], [3], knowledge based systems [33], and computer vision [9].

The treewidth problem asks to find a chordal completion where the size of the largest clique is minimized, and many NP-complete problems are solvable in polynomial time when they are restricted to graphs of bounded treewidth [6] and [52]. Unfortunately, the minimum triangulation and the treewidth problems are NP-hard [2] and [63].

For a positive integer k, a graph G is *k-degenerate* if any subgraph of G contains a vertex having at most k neighbors in it. Dvořák and Postle [16] observed that if a graph G is k-degenerate, then $\chi_{DP}(G) \le k+1$. It is easy to check that every chordal graph G is $(\omega(G)-1)$-degenerate and so

$$\omega(G) \le \chi(G) \le \chi_l(G) \le \chi_{DP}(G) \le \omega(G).$$

Therefore,

(§) for a chordal graph G, $\chi(G) = \chi_l(G) = \chi_{DP}(G) = \omega(G)$.

The observation (§) directed our attention to the idea that, for a chordal completion G^* of a graph G, the chromatic number of G is bounded above by the clique number of G^*.

By (1.2.2), an upper bound of $\chi_{DP}(G)$ (resp. $\chi_l(G)$) is an upper bound of $\chi_l(G)$ (resp. $\chi(G)$). In this vein, it is interesting to check whether or not $\chi_{DP}(G) \le k$ (resp. $\chi_l(G) \le k$) when $\chi_l(G) \le k$ (resp. $\chi(G) \le k$) for a positive integer k.

In the Chapter 3, we introduce the notion of non-chordality index $i(G)$ of a graph G and present a minimal chordal completion G^* of a graph G satisfying the inequality $\omega(G^*) - \omega(G) \le i(G)$. Using the above chordal completion as a tool, we prove that the family of graphs with non-chordality indices at most one satisfies the Hadwiger conjecture and the Erdős-Faber-Lovász Conjecture, and the family of graphs with bounded non-chordality indices is a linearly χ-bounded class.

Chapter 2

Phylogeny graphs

In this chapter, we first study the digraphs whose phylogeny graphs are chordal. Then we study phylogeny numbers of graphs in the aspect of the number of triangles and diamonds.

2.1 Chordal phylogeny graphs

Lee *et al.* [34] studied the phylogeny graphs of $(2, 2)$ digraphs and gave sufficient conditions and necessary conditions for $(2, 2)$ digraphs having chordal phylogeny graphs.

In this section, we extend their work. First we completely characterize the phylogeny graphs of $(1, j)$ digraphs and those of $(i, 1)$ digraphs (Theorem 2.1.1 and 2.1.8). Then we study the phylogeny graphs of $(2, j)$ digraphs. We show that the phylogeny graph of any $(2, j)$ digraph whose underlying graph is chordal is chordal (Theorem 2.1.16). Finally, we show that the phylogeny graph of any $(2, 2)$ digraph whose underlying graph is chordal is chordal and planar (Theorem 2.1.27).

2.1.1 $(1, j)$ phylogeny graphs and $(i, 1)$ phylogeny graphs

In this section, we characterize the $(1, j)$ phylogeny graphs and the $(i, 1)$ phylogeny graphs.

A *component* of a digraph D is the subdigraph of D induced by the vertex of a component of its underlying graph. Given an acyclic digraph D, it is easy to check that D' is a component of D if and only if $P(D')$ is a component of $P(D)$. Thus,

($\star$) it is sufficient to consider only weakly connected digraphs (whose underlying graphs are connected) in studying phylogeny graphs of digraphs.

First we take care of $(1, j)$ phylogeny graphs.

A vertex of degree one is called a *pendant vertex.*

Given a graph G and a vertex v of G, we denote the set of neighbors of v in G by $N_G(v)$. We call $N_G(v) \cup \{v\}$ the *closed neighborhood* of v and denote it by $N_G[v]$. We call $\Delta(G) := \max\{|N_G(v)| \mid v \in V(G)\}$ the *maximum degree of* G.

Theorem 2.1.1. *For a positive integer j, a graph is a $(1, j)$ phylogeny graph if and only if it is a forest with the maximum degree at most $j + 1$.*

Proof. By ($\star$), it suffices to show that, for a positive integer j, a connected graph is a $(1, j)$ phylogeny graph if and only if it is a tree with the maximum degree at most $j + 1$. To show the "only if" part, suppose that a connected graph G is a $(1, j)$ phylogeny graph for some positive integer j. Then there is a $(1, j)$ digraph D such that $P(D)$ is isomorphic to G. Since every vertex of D has indegree at most one, $P(D) = U(D)$. Since $P(D)$ is connected, $U(D)$ is connected. Moreover, since D is a $(1, j)$ digraph, $U(D)$ has the maximum degree at most $j + 1$. If $U(D)$ contained a cycle C, then there would exist a vertex on C of indegree at least two by the acyclicity of D. Therefore $U(D)$ does not contain a cycle, and so $U(D)$ is tree.

Now we show the "if" part. If T is a tree with one or two vertices, then it is obviously a $(1, 1)$ phylogeny graph. We take a tree T with at least three vertices and let $j = \Delta(T) - 1$. Then there exist pendant vertices. We take one of them and denote it by u. We regard T as a rooted tree with the root u and define an oriented tree $\overrightarrow{T}$, which is acyclic, with $V(\overrightarrow{T}) = V(T)$ as follows. We take an edge xy in T. Then $d_T(u,x) = d_T(u,y)+1$ or $d_T(u,y) = d_T(u,x)+1$. If the former, $(y,x) \in A(\overrightarrow{T})$ and if the latter, $(x,y) \in A(\overrightarrow{T})$. By definition, $U(\overrightarrow{T}) = T$. Moreover, u has indegree zero and outdegree one, and each vertex in $\overrightarrow{T}$ except u has indegree one in $\overrightarrow{T}$. Then, since the degree of each vertex in T is at most $j+1$, the outdegree of each vertex in $\overrightarrow{T}$ is at most j. Therefore $\overrightarrow{T}$ is a $(1,j)$ digraph. Since each vertex in $\overrightarrow{T}$ has indegree at most one, $P(\overrightarrow{T}) = U(\overrightarrow{T}) = T$. $\square$

If $P(D)$ is triangle-free for an acyclic digraph D, then the indegree of each vertex is at most one in D, for otherwise, the vertex with indegree at least two form a triangle with two in-neighbors in $P(D)$. Thus, the following corollary immediately follows from Theorem 2.1.1.

Corollary 2.1.2. *For any positive integers i and j, if an (i,j) phylogeny graph is triangle-free, then it is a forest with the maximum degree at most $j+1$.*

Given a digraph D with n vertices, a one-to-one correspondence $f : V(D) \to [n]$ is called an *acyclic labeling* of D if $f(u) > f(v)$ for any arc (u,v) in D. It is well-known that D is acyclic if and only if there is an acyclic labeling of D.

Given a digraph D and a vertex v of D, we call $N_D^-(v) \cup \{v\}$ the *closed in-neighborhood* of v and denote it by $N_D^-[v]$.

Given a graph G, a vertex v of G is called a *simplicial vertex* if $N_G[v]$ forms a clique in G.

Now we consider phylogeny graphs of $(i,1)$ digraphs.

Lemma 2.1.3. *Let D be a nontrivial weakly connected $(i,1)$ digraph for some positive integer i and f be an acyclic labeling of D. Then every maximal clique* *in $P(D)$ is in the form of the closed in-neighborhood of the vertex with the least f-value among the vertices in the maximal clique.*

Proof. Let X be a maximal clique in $P(D)$ and x be the vertex having the least f-value among the vertices in X. Suppose $X \not\subset N^-_D[x]$. Then there is a vertex $y \in X$ such that $(y, x) \notin A(D)$. Since x has the least f-value among the vertices in X, $(x, y) \notin A(D)$. Yet, since x and y are adjacent in $P(D)$, they have a common out-neighbor, say z, in D. By the hypothesis that D is an $(i, 1)$ digraph, z is the only out-neighbor of x and y. Since x has the least f-value among the vertices in X, $z \notin X$. Since $N^-_D[z]$ forms a clique in $P(D)$, $X \not\subset N^-_D[z]$ by the maximality of X. That is, there exists a vertex w in X but not in $N^-_D[z]$. Then $w \neq z$. Since z is the unique out-neighbor of x and y in D, $(x, w) \notin A(D)$ and $(y, w) \notin A(D)$. Furthermore, since $w \notin N^-_D[z]$, neither w and x nor w and y have a common out-neighbor in D. However, w, x, and y belong to X, so (w, x) and (w, y) are arcs in D, which is a contradiction to the hypothesis that D is a $(i, 1)$ digraph. Hence $X \subset N^-_D[x]$. Since $N^-_D[x]$ is a clique in $P(D)$, $X = N^-_D[x]$ by the maximality of X. $\square$

Lemma 2.1.4. *Given a nontrivial weakly connected $(i, 1)$ digraph D for a positive integer i, the set of all the maximal cliques in $P(D)$ is exactly the set*

$$\{N^-_D[u] \mid u \in V(D) \text{ and } d^-_D(u) \geq 1\}.$$

Proof. Let f be an acyclic labeling of D. Take a maximal clique Y in $P(D)$. By Lemma 2.1.3, $Y = N^-_D[y]$ for the vertex y having the least f-value among the vertices in Y. Since D is nontrivial and weakly connected, $|Y| \geq 2$ and so y has an in-neighbor in D, i.e. $d^-_D(y) \geq 1$.

To prove a containment in the other direction, take a vertex u of indegree at least one in D. Let v be an in-neighbor of u in D. Suppose, to the contrary, $N^-_D[u]$ is not maximal. Then there is a maximal clique X properly containing $N^-_D[u]$. By Lemma 2.1.3, $X = N^-_D[x]$ for the vertex x with the least f-value among the vertices in X. Since $N^-_D[u]$ is properly contained in $N^-_D[x]$, $u \neq x$.

In addition, since $N_D^-[u]$ is included in $N_D^-[x]$, v is also an in-neighbor of x in D. Thus the outdegree of v is at least two, which contradicts the fact that D is an $(i,1)$ digraph. Therefore $N_D^-[u]$ forms a maximal clique in $P(D)$ and this completes the proof. $\square$

A *diamond* is a graph obtained from K_4 by deleting an edge. A graph is called *diamond-free* if it does not contain a diamond as an induced subgraph.

Lemma 2.1.5. *The phylogeny graph of a weakly connected* $(i,1)$ *digraph for a positive integer* i *is diamond-free and chordal.*

Proof. Let D be a weakly connected $(i,1)$ digraph for a positive integer i. We prove the lemma statement by induction on $|V(D)|$. If $|V(D)| = 1$ or 2, then the statement is trivially true. Suppose that $|V(D)| = n+1$ and the lemma statement is true for any weakly connected $(i,1)$ digraph with n vertices ($n \geq 2$). Since D is acyclic, there is a vertex u of indegree zero in D. Since D is a weakly connected $(i,1)$ digraph, $d_D^+(u) = 1$. Thus there is a unique out-neighbor v of u in D. Then, as u has indegree of zero in D, we may conclude that, for a vertex w in D, u is adjacent to w in $P(D)$ if and only if $w = v$ or w is an in-neighbor of v in D, i.e.

$$N_{P(D)}[u] = N_D^-[v]. \tag{2.1.1}$$

Since the indegree and the outdegree of u are zero and one, respectively, $D-u$ is weakly connected. Obviously $D-u$ is an $(i,1)$ digraph. Thus, by the induction hypothesis, $P(D-u)$ is diamond-free and chordal. Take two vertices x and y in $V(D) \setminus \{u\}$. Since u has indegree zero, u cannot be a common out-neighbor of x and y. Therefore, x and y are adjacent in $P(D) - u$ if and only if $(x,y) \in A(D)$ or $(y,x) \in A(D)$ or they have a common out-neighbor other than u in D if and only if x and y are adjacent in $P(D-u)$. Thus we have shown that $P(D) - u = P(D-u)$. By (2.1.1), u is simplicial in $P(D)$, so $P(D)$ is chordal. Now it remains to show that $P(D)$ is diamond-free.

Suppose that $P(D)$ has a diamond. Then, since $P(D)-u$ is diamond-free, every diamond of $P(D)$ contains u and a vertex which is not adjacent to u in $P(D)$. Let z be a vertex on a diamond which is not adjacent to u in $P(D)$. Then z is not contained in $N_D^-[v] \setminus \{u\}$ and is adjacent to two vertices y_1 and y_2 in $N_D^-[v] \setminus \{u\}$ by (2.1.1). Since $P(D) - u = P(D - u)$, z is adjacent to y_1 and y_2 in $P(D - u)$. Moreover, u is not a pendant vertex, so v has an in-neighbor distinct from u in D. Then v has indegree at least one in $D - u$, so $N_{D-u}^-[v]$ is a maximal clique in $P(D - u)$ by Lemma 2.1.4. Obviously $N_{D-u}^-[v] = N_D^-[v] \setminus \{u\}$, so $N_D^-[v] \setminus \{u\}$ is a maximal clique in $P(D - u)$. Then, since z belongs to $P(D-u)$ and is not contained in $N_D^-[v] \setminus \{u\}$, there exist a vertex w in $N_{D-u}^-[v]$ which is not adjacent to z in $P(D - u)$. Then the subgraph induced by z, w, y_1, and y_2 is a diamond in $P(D - u)$ and we have reached a contradiction. $\square$

Lemma 2.1.6. *Let D be an $(i, 1)$ digraph for a positive integer i and f be an acyclic labeling of D. Suppose that non-disjoint vertex sets X and Y form distinct maximal cliques in $P(D)$, respectively. Then X and Y have exactly one common vertex, namely v, and*

$$f(v) = \min\{f(w) \mid w \in X\} \text{ or } \min\{f(w) \mid w \in Y\}$$

whereas

$$f(v) > \min\{f(w) \mid w \in X \cup Y\}.$$

Proof. By $(\star)$, we may assume that $U(D)$ is connected. By Lemma 2.1.5, $P(D)$ is diamond-free, so $|X \cap Y| \leq 1$. Then, by the hypothesis that X and Y are non-disjoint vertex sets, $|X \cap Y| = 1$. Let v be the vertex common to X and Y. Since X and Y form maximal cliques, $X = N_D^-[x]$ and $Y = N_D^-[y]$ for the vertices x and y with the smallest f-values among the vertices in X and the vertices in Y, respectively, by Lemma 2.1.3. Since X and Y are distinct, $x \neq y$. If $v \notin \{x, y\}$, then x and y are two distinct out-neighbors of v, which is impossible. Thus $v \in \{x, y\}$. Without loss of generality, we may assume

$v = x$. Since x and y have the the smallest f-values among the vertices in X and the vertices in Y, respectively, v has the least f-value among the vertices in X but not among the vertices in Y, and the lemma statement is true. $\square$

We shall completely characterize the $(i, 1)$ phylogeny graphs in terms of "clique graph" which was introduced by Hamelink [25].

Definition 2.1.7. The *clique graph* of a graph G, denoted by $K(G)$, is a simple graph such that

- every vertex of $K(G)$ represents a maximal clique of G;
- two vertices of $K(G)$ are adjacent when they share at least one vertex in common in G.

Theorem 2.1.8. *For some positive integer i, a graph G is an $(i, 1)$ phylogeny graph if and only if it is a diamond-free chordal graph with $\omega(G) \leq i+1$ and its clique graph is a forest.*

Proof. By $(\star)$, it is sufficient to show that a connected graph G is an $(i, 1)$ phylogeny graph for some positive integer i if and only if it is a diamond-free chordal graph with $\omega(G) \leq i + 1$ and its clique graph is a tree. To show the "only if" part, suppose that a connected graph G is an $(i, 1)$ phylogeny graph for some positive integer i. Then $G = P(D)$ for some weakly connected $(i, 1)$ digraph D. By Lemma 2.1.3, $\omega(G) \leq i + 1$. In addition, by Lemma 2.1.5, $P(D)$ is diamond-free and chordal. Now we show that the clique graph $K(G)$ is a tree. As the clique graph of a connected graph is connected, it is sufficient to show that $K(G)$ is acyclic. Suppose, to the contrary, that $K(G)$ contains a cycle $C := X_1X_2 \cdots X_rX_1$ for an integer $r \geq 3$ and maximal cliques X_1, $\ldots$, X_r of G. Let f be an acyclic labeling of D. We denote by x_i the vertex which has the least f-value in X_i for each $i = 1, 2, \ldots, r$. By Lemma 2.1.6, $X_1 \cap X_2 = \{x_1\}$ or $X_1 \cap X_2 = \{x_2\}$. Without loss of generality, we may assume that $X_1 \cap X_2 = \{x_2\}$ so that $f(x_1) < f(x_2)$. By Lemma 2.1.6 again,

$X_2 \cap X_3 = \{x_2\}$ or $X_2 \cap X_3 = \{x_3\}$. Suppose that $X_2 \cap X_3 = \{x_2\}$. Then $f(x_3) < f(x_2)$ and $x_2 \in X_1 \cap X_3$. By Lemma 2.1.6, $X_1 \cap X_3 = \{x_2\}$, and either $f(x_2) = f(x_1)$ or $f(x_2) = f(x_3)$, which contradicts the fact that $f(x_1) < f(x_2)$ and $f(x_3) < f(x_2)$. Thus $X_2 \cap X_3 = \{x_3\}$. Continuing in this way, we may show that $X_i \cap X_{i+1} = \{x_{i+1}\}$ for each $i \in \{1, 2, \ldots, r-1\}$ and $X_r \cap X_1 = \{x_1\}$. By Lemma 2.1.3, $X_i = N_D^-[x_i]$ for each $i \in \{1, 2, \ldots, r\}$. Therefore $(x_1, x_r) \in A(D)$ and $(x_{i+1}, x_i) \in A(D)$ for each $i \in \{1, 2, \ldots, r-1\}$. Thus $x_1 \to x_r \to \cdots \to x_2 \to x_1$ is a directed cycle in D and we reach a contradiction to the acyclicity of D. Hence $K(G)$ does not contain a cycle and so the "only if" part is true.

To show the "if" part, suppose that a connected graph G is diamond-free and chordal with $\omega(G) \leq i + 1$ and that $K(G)$ is a tree for some positive integer i. If G is a complete graph, then it has at most $i + 1$ vertices and is obviously an $(i, 1)$ phylogeny graph. Thus we may assume that G is not a complete graph. Then $K(G)$ is not a trivial tree.

We show by induction on $|V(G)|$ that G is an $(i, 1)$ phylogeny graph. Since G is connected and not complete, $|V(G)| \geq 3$. If $|V(G)| = 3$, then G is a path of length two and, by Theorem 2.1.1, a $(1, 1)$ phylogeny graph. Assume that a connected non-complete graph is an $(i, 1)$ phylogeny graph if it is a diamond-free chordal graph with less than n vertices and the cliques of size at most $i + 1$ and its clique graph is a tree for $n \geq 4$. Suppose that $|V(G)| = n$. Since $K(G)$ is not a trivial tree, it contains a pendant vertex. Let X be a pendant vertex and Y be the neighbor of X in $K(G)$. Then $|X \cap Y| \geq 1$. Since G is a connected graph with at least four vertices, by the maximality of X and Y, $2 \leq |X|$ and $2 \leq |Y|$. By Lemma 2.1.6, $X \cap Y = \{u\}$ for some vertex u. Since $|X| \geq 2$, there exists a vertex v in $X \setminus \{u\}$. Since $K(G)$ does not contain a triangle, X and Y are the only maximal cliques that contain u in G and so

(†) $G - v$ does not have a maximal clique containing u other than $X \setminus \{v\}$ (not necessarily maximal) and Y.

Furthermore, since X is a pendant vertex in $K(G)$, every vertex in $X \setminus \{u\}$ is a simplicial vertex in G and therefore v is a simplicial vertex of G. Then the closed neighborhood of v in G is X. Moreover, it is obvious that $G-v$ is a connected diamond-free chordal graph with $\omega(G-v) \leq i+1$ and $K(G-v)$ is a tree. Therefore, by the induction hypothesis, $G-v$ is an $(i,1)$ phylogeny graph. Thus there is an $(i,1)$ digraph D^* such that $P(D^*) = G-v$. Let f^* be an acyclic labeling of D^*.

Case 1. The vertex u has the least f^*-value in Y. Then, by Lemma 2.1.3, $Y = N^-_{D^*}[u]$. Consider the case in which u has no out-neighbor in D^*. Then, by Lemma 2.1.6, $X \setminus \{v\} = \{u\}$. Adding the vertex v and the arc (u,v) to D^* results in an $(i,1)$ digraph whose phylogeny graph is G. Now consider the case in which u has an out-neighbor w in D^*. Then $f^*(w) < f^*(u)$ and $d^-_{D^*}(w) \geq 1$. Since $d^-_{D^*}(w) \geq 1$, $N^-_{D^*}[w]$ forms a maximal clique by Lemma 2.1.4. Since $f^*(w) < f^*(u)$ and $f^*(u)$ is the minimum in Y, $N^-_{D^*}[w]$ is distinct from Y. Since $N^-_{D^*}[w]$ contains u, $N^-_{D^*}[w] = X \setminus \{v\}$ by (†). Since $|X| \leq i+1$, $|X \setminus \{v\}| \leq i$ and so $d^-_{D^*}(w) \leq i-1$. Adding the vertex v and the arc (v,w) to D^* results in an $(i,1)$ digraph whose phylogeny graph is G.

Case 2. The vertex u does not have the least f^*-value in Y. Then u has the least f^*-value in $X \setminus \{v\}$ by Lemma 2.1.6. Thus, if u has no in-neighbor in D^*, then $X \setminus \{v\} = \{u\}$, and so adding the vertex v and the arc (v,u) to D^* results in an $(i,1)$ digraph whose phylogeny graph is G. Now consider the case in which u has an in-neighbor w in D^*. Then $d^-_{D^*}(u) \geq 1$, so $N^-_{D^*}[u]$ forms a maximal clique by Lemma 2.1.4. By Lemma 2.1.3, u has the least f^*-value in $N^-_{D^*}[u]$. Since u does not have the least f^*-value in Y, $N^-_{D^*}[u]$ is distinct from Y. Since $N^-_{D^*}[u]$ contains u, $N^-_{D^*}[u] = X \setminus \{v\}$ by (†). Since $|X| \leq i+1$, $|X \setminus \{v\}| \leq i$ and so $d^-_{D^*}(u) \leq i-1$. Adding the vertex v and the arc (v,u) to D^* results in an $(i,1)$ digraph whose phylogeny graph is G. $\square$

The *union* of two graphs G and H is the graph having its vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, we refer to their union as a *disjoint* union.

Proposition 2.1.9. *For a graph G, the following statements are equivalent.*

(i) G is a $(1, j)$ phylogeny graph and an $(i, 1)$ phylogeny graph for some positive integers i and j;

(ii) G is a disjoint union of paths;

(iii) G is a $(1, 1)$ phylogeny graph.

Proof. By Theorems 2.1.1 and 2.1.8, it is immediately true that (ii) is equivalent (iii). Obviously, (iii) implies (i). Now we show that (i) implies (ii). By Theorem 2.1.1, G is a forest. If G has a vertex of degree at least three, then $K(G)$ contains a triangle as each edge in G is a maximal clique, which contradicts Theorem 2.1.8. Therefore each vertex in G has degree at most two and so G is a disjoint union of paths. □

Remark 2.1.10. Theorems 2.1.1 and 2.1.8 tell us that an (i, j) phylogeny graph for positive integers i and j with $i = 1$ or $j = 1$ is diamond-free and chordal.

2.1.2 $(2, j)$ phylogeny graphs

In this section, we focus on phylogeny graphs of $(2, j)$ digraphs for a positive integer j. We thought that it is worth studying them in the context that a child has two biological parents in most species.

For an acyclic digraph D, an edge is called a *cared edge* in $P(D)$ if the edge belongs to the competition graph $C(D)$ but not to the $U(D)$. For a cared edge $xy \in P(D)$, there is a common out-neighbor v of x and y and it is said that *xy is taken care* of by v or that v *takes care* of xy. A vertex in D is called a *caring vertex* if an edge of $P(D)$ is taken care of by the vertex [34].

For example, the edges v_2v_3, v_2v_6, v_2v_7, v_4v_5, and v_5v_6 of $P(D)$ in Figure 1.3 are cared edges and the vertices v_1, v_4, v_4, v_3, and v_7 are vertices taking care of v_2v_3, v_2v_6, v_2v_7, v_4v_5, and v_5v_6, respectively.

Proposition 2.1.11. *Suppose that the phylogeny graph of a $(2, j)$ digraph D contains a hole H for a positive integer j. Then no vertex on H takes care of an edge on H.*

Proof. Suppose, to the contrary, that there exists a vertex v on H which takes care of an edge xy on H. Then $\{x, y, v\}$ forms a triangle in $P(D)$, so yv or vx is a chord of H in $P(D)$ and we reach a contradiction. □

Given a $(2, j)$ digraph D, suppose that $P(D)$ has a hole H and $e_1, e_2, \ldots, e_t$ are the cared edges on H. Let $w_1, w_2, \ldots, w_t$ be vertices taking care of $e_1, e_2, \ldots, e_t$, respectively. Since the indegree of w_i is at most two in D for $i = 1, \ldots, t$, $w_1, w_2, \ldots, w_t$ are distinct. We let $W = \{w_1, w_2, \ldots, w_t\}$ and call W a *set extending* H by extending the notion introduced in Lee *et al.* [34]. By Proposition 2.1.11,

$$W \subset V(D) \setminus V(H). \tag{2.1.2}$$

Therefore we may obtain a cycle in $U(D)$ from H by replacing each edge e_i with a path of length two from one end of e_i to the other end of e_i with the interior vertex w_i. We call such a cycle the *cycle obtained from* H *by* W. Let L be the subgraph of $U(D)$ induced by $V(H) \cup W$. We call L the *subgraph of* $U(D)$ *obtained from* H *by* W. These notions extend the ones introduced in Lee *et al.* [34].

Lee *et al.* [34] showed that, for a $(2, 2)$ digraph D such that the holes of $P(D)$ are mutually vertex-disjoint and no hole in $U(D)$ has length 4 or 6, the number of holes in $U(D)$ is greater than or equal to the number of holes in $P(D)$.

Theorem 2.1.12 ([34])**.** *Let H be a hole of the phylogeny graph $P(D)$ of a $(2, 2)$ digraph D. Then there is a hole $\phi(H)$ in the underlying graph $U(D)$ of D such that*

- *$\phi(H)$ equals H if H is a hole in $U(D)$;*
- *$\phi(H)$ is a hole in $U(D)$ only containing vertices in the subgraph obtained from H by a set extending H otherwise.*

Moreover, if the holes of $P(D)$ are mutually vertex-disjoint and no hole in $U(D)$ has length 4 or 6, then there exists an injective map from the set of holes in $P(D)$ to the set of holes in $U(D)$.

We shall devote the first part of this section to extending the above theorem given in [34]. To do so, we need the following lemmas.

Lemma 2.1.13. *Given a graph G and a cycle C of G with length at least four, suppose that a section Q of C forms an induced path of G and contains a path P with length at least two none of whose internal vertices is incident to a chord of C in G. Then P can be extended to a hole H in G so that $V(P) \subsetneq V(H) \subset V(C)$ and H contains a vertex on C not on Q.*

Proof. Let v_i and v_j be the origin and the terminal of P. Since P is an induced path of length at least two, v_i and v_j are nonadjacent. Now we take a shortest (v_j, v_i)-path P' with some vertices on the (v_i, v_j)-section of C other than P. Since v_i and v_j are nonadjacent, P' has length at least two. Therefore PP' is a cycle of length at least four. By the hypothesis, none of the internal vertices of P is incident to a chord of C. In addition, P and P' are induced paths, so $H := PP'$ is actually a hole in G. Note that $V(H) \subset V(C)$. If every vertex on H were on Q, then Q would have a chord as $V(H) \subset V(Q)$, which is impossible. Therefore H contains a vertex on C not on Q. □

Lemma 2.1.14. *Let D be a $(2, j)$ digraph and f be an acyclic labeling of D for a positive integer j. In addition, let H be a hole of $P(D)$, W be a set extending H, and w be a vertex with the least f-value in $V(H) \cup W$. Then $w \in W$. Moreover, there is a hole $\phi(H)$ in $U(D)$ such that $w \in V(\phi(H))$ and $V(\phi(H)) \subset V(H) \cup W$.*

Proof. Let $H = u_1u_2 \cdots u_lu_1$ for an integer $l \geq 4$. To reach a contradiction, we suppose that $w \in V(H)$. Without loss of generality, we may assume that $w = u_1$. Suppose that u_1u_2 and u_1u_l are edges of $U(D)$. Then, since u_1 has the least f-value in $V(H)$, $(u_2, u_1) \in A(D)$ and $(u_l, u_1) \in A(D)$ and so $\{u_1, u_2, u_l\}$ forms a triangle in $P(D)$, which is a contradiction to the supposition that H is a hole in $P(D)$. Therefore u_1u_2 or u_1u_l is a cared edge in $P(D)$. Without loss of generality, we may assume u_1u_2 is a cared edge in $P(D)$. Then u_1 and u_2 have a common out-neighbor, say v, in W, which implies that $f(v) < f(u_1)$. Thus we have reached a contradiction and so $w \in W$.

Now we show that the "moreover" part of the lemma statement is true. Let C be the cycle in $U(D)$ obtained from H by W. Without loss of generality, we may assume that u_1u_l is taken care of by w. Then $(u_1, w) \in A(D)$ and $(u_l, w) \in A(D)$.

Suppose, to the contrary, that C has a chord which is incident to w in $U(D)$. Let xw be a chord of C in $U(D)$. Then $x \notin \{u_1, u_l\}$. Moreover, since w has the least f-value in $V(H) \cup W$, $(w, x) \notin A(D)$. Then u_1, u_l, and x are in-neighbors of w in D, which contradicts the hypothesis that D is a $(2, j)$ digraph. Hence there is no chord of C which is incident to w in $U(D)$. Since u_1u_l is a cared edge in $P(D)$, u_1wu_l is an induced path in $U(D)$. By applying Lemma 2.1.13 for $P = Q = u_1wu_l$, we may conclude that "moreover" part of the lemma statement is true. □

Now we are ready to extend Theorem 2.1.12 to not only make it valid for $(2, j)$ digraphs but also strengthen it.

Theorem 2.1.15. *For a positive integer j, let H be a hole of the phylogeny graph $P(D)$ of a $(2, j)$ digraph D. Then there is a hole in $U(D)$ which only contains vertices in the subgraph of $U(D)$ obtained from H by a set extending H. Moreover, if $P(D)$ has a hole and the holes of $P(D)$ are mutually edge-disjoint, then there exists an injective map from the set of holes in $P(D)$ to the set of holes in $U(D)$.*

Proof. The first part of this theorem is immediately true by Lemma 2.1.14.

To show the second part of the theorem statement, we assume that $P(D)$ has a hole and the holes in $P(D)$ are mutually edge-disjoint. Let f be an acyclic labeling of D, $\{H_1, \ldots, H_l\}$ be the set of holes in $P(D)$, and W_i be a set extending H_i for each $i = 1, \ldots, l$. Let w_i be the vertex with the least f-value in $V(H_i) \cup W_i$ for each $i = 1, \ldots, l$. Then, by Lemma 2.1.14, $w_i \in W_i$ and there exists a hole $\phi(H_i)$ such that $w_i \in V(\phi(H_i))$ and $V(\phi(H_i)) \subset V(H_i) \cup W_i$ for each $i = 1, \ldots, l$. At this point, we may regard ϕ as a map from the set of the holes in $P(D)$ to the set of holes in $U(D)$.

In the following, we show that ϕ is injective. Suppose, to the contrary, that $\phi(H_j) = \phi(H_k)$ for some j and k satisfying $1 \leq j < k \leq l$. Since w_i is the vertex with the least f-value in $V(H_i) \cup W_i$ and $V(\phi(H_i)) \subset V(H_i) \cup W_i$, w_i has the least f-value in $V(\phi(H_i))$ for each $i \in \{j, k\}$. Then, since $\phi(H_j) = \phi(H_k)$, $w_j = w_k$ and so $w_j \in W_j \cap W_k$. Thus w_j has two in-neighbors on H_j and two in-neighbors on H_k in D. Then, by the hypothesis that H_j and H_k are edge-disjoint, w_j has at least three distinct in-neighbors in D, which violates the indegree restriction on D. Hence $\phi(H_j) \neq \phi(H_k)$ for any j and k satisfying $1 \leq j < k \leq l$ and we have shown that ϕ is injective. $\square$

The underlying graph of an (i, j) digraph D being chordal does not guarantee that the phylogeny graph of D is chordal. For example, the underlying graph of the $(3, 2)$ digraph given in Figure 1.3 is chordal whereas its phylogeny graph has a hole $v_2v_3v_5v_6v_2$. However, if $i \leq 2$ or $j = 1$, then it does guarantee by the above theorem together with Theorems 2.1.1 and 2.1.8. As a matter of fact, we have shown the following theorem.

Theorem 2.1.16. *Let $\mathcal{D}^*_{i,j}$ be the set of (i, j) digraphs whose underlying graphs are chordal for positive integers i and j. Then the phylogeny graph of D is chordal for any $D \in \mathcal{D}^*_{i,j}$ if and only if $i \leq 2$ or $j = 1$.*

By Theorem 2.1.16, the phylogeny graph of a $(2, j)$ digraph D is chordal if the underlying graph of D is chordal for any positive integer j. By the way, if $j = 2$, then the underlying graph being chordal guarantees not only $P(D)$ being chordal but also $P(D)$ being planar, which will be to be shown later in this section. By the way, Lee *et al.* [34] showed that a $(2, 2)$ phylogeny graph is K_5-free.

Theorem 2.1.17 ([34]). *For any $(2,2)$ digraph D, the phylogeny graph of D is K_5-free.*

We shall extend this theorem in two aspects. On one hand, we find a sharp upper bound for the clique number of $(2,j)$ phylogeny graph for any positive integer j. On the other hand, we show that the phylogeny graph $P(D)$ of a $(2,2)$ digraph D is planar if the underlying graph of D is chordal by showing that $P(D)$ is K_5-minor-free and $K_{3,3}$-minor-free.

Lemma 2.1.18. *For a positive integer j, every $(2,j)$ phylogeny graph is $(j+2)$-degenerate.*

Proof. Let D be a $(2,j)$ digraph for a positive integer j and f be an acyclic labeling of D. We take a subgraph H of $P(D)$ and the vertex u which has the least f-value in $V(H)$. Then the out-neighbors of u in D cannot be in $V(H)$. Thus an edge incident to u in H is either a cared edge or the edge in $U(D)$ corresponding to an arc incoming toward u in D. Since u has at most j out-neighbors and each of the out-neighbors has at most one in-neighbor other than u in D, there are at most j cared edges which are incident to u in H. Moreover, since u has at most two in-neighbors in D, there are at most two edges incident to u in H which correspond to arcs incoming toward u in D. Thus u has degree at most $j+2$ in H. Since H was arbitrarily chosen, $P(D)$ is $(j+2)$-degenerate. $\square$

The following theorem gives a sharp upper bound for the clique number of $(2,j)$ phylogeny graph for any positive integer j to extend the Theorem 2.1.17.

Theorem 2.1.19. *Let D be a $(2,j)$ digraph for a positive integer j. Then*

$$\omega(P(D)) \leq \begin{cases} j+2 & \text{if } j \leq 2; \\ j+3 & \text{otherwise;} \end{cases}$$

and the inequalities are tight.

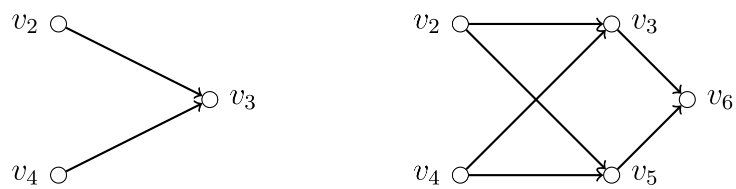


Figure 2.1: A $(2, 1)$ digraph and $(2, 2)$ digraph whose phylogeny graphs contain K_3 and K_4, respectively.

Proof. It is known that if a graph G is k-degenerate, then $\omega(G) \leq k + 1$. Thus, by Lemma 2.1.18, $\omega(P(D)) \leq j + 3$. By Theorems 2.1.8 and 2.1.17, $\omega(P(D)) \leq j + 2$ if $j \leq 2$.

The inequality is tight for $j \leq 2$ by the digraphs given in Figure 2.1. To show that the inequality is tight for $j \geq 3$, we construct a $(2, j)$ digraph in the following way. We start with an empty digraph D_0 with vertex set $\{v_1, \ldots v_{j+3}\}$. We add to D_0 the vertices $a_{1,2}, \ldots, a_{1,j+1}$ and the arcs $(v_1, a_{1,i})$, $(v_i, a_{1,i})$ for $i = 2, \ldots, j+1$ and arcs (v_{j+2}, v_1), (v_{j+3}, v_1) to obtain a digraph D_1. Then D_1 is a $(2, j)$-digraph with every vertex except v_1 having outdegree at most one and

$$E_1 := \{v_{j+2}v_{j+3}\} \cup \{v_1v_i \mid i = 2, \ldots, j+3\}$$

is an edge set of $P(D_1)$. We add to D_1 the vertices $a_{2,3}, \ldots, a_{2,j-1}, a_{2,j+1}, a_{2,j+2}$ and the arcs $(v_2, a_{2,i})$, $(v_i, a_{2,i})$ for each $i \in [j+2] \setminus \{1, 2, j\}$ and arcs (v_j, v_2), (v_{j+3}, v_2) to obtain a digraph D_2. Then D_2 is a $(2, j)$-digraph with every vertex except v_1 and v_2 having outdegree at most two and

$$E_2 := E_1 \cup \{v_jv_{j+3}\} \cup \{v_2v_i \mid i = 3, \ldots, j+3\}$$

is an edge set of $P(D_2)$.

For each $\ell \in [j-1] \setminus \{1, 2\}$, we add to $D_{\ell-1}$ the vertices $a_{\ell,\ell+1}, \ldots, a_{\ell,j+1}$ and the arcs $(v_\ell, a_{\ell,i})$, $(v_i, a_{\ell,i})$ for $i = \ell+1, \ldots, j+1$ and arcs (v_{j+2}, v_ℓ),

(v_{j+3}, v_ℓ) to obtain a digraph D_ℓ. Then, for each $\ell \in [j-1] \setminus \{1,2\}$ is a $(2, j)$-digraph with every vertex except v_1, …, and v_l having outdegree at most ℓ and

$$E_\ell := E_{\ell-1} \cup \{v_\ell v_i \mid i = \ell+1, \ldots, j+3\}$$

is an edge set of $P(D_\ell)$. Therefore v_i is adjacent to each of v_1, …, v_{j+3} except itself for $i = 1$, …, $j-1$. Now we add to D_{j-1} the arcs (v_{j+3}, v_{j+1}), (v_{j+2}, v_j), and (v_{j+1}, v_j) to obtain a $(2, j)$ digraph D_j. Clearly, v_j, …, v_{j+3} are mutually adjacent in $P(D_j)$ (recall that the edges $v_{j+2}v_{j+3}$ and v_jv_{j+3} are contained in E_1 and E_2, respectively). Thus v_1, …, v_{j+3} form a clique of size $j+3$ in $P(D_j)$. □

From Theorems 2.1.1 and 2.1.8, we know that the clique number of a $(1, j)$ phylogeny graph is at most two and the clique number of an $(i, 1)$ phylogeny graph is at most $i+1$ for any positive integers i and j.

In the rest of this section, we shall show that the phylogeny graph $P(D)$ of a $(2, 2)$ digraph D is planar if the underlying graph of D is chordal.

The following lemma is a known fact.

Lemma 2.1.20. *The class of chordal graphs is closed under contraction.*

We denote by $G \cdot e$ the graph obtained by contracting a graph G by an edge e in G.

Lemma 2.1.21. *For a graph G and two adjacent vertices u and v in G, let K be a clique with at least three vertices in $G \cdot uv$. If z is the vertex in K obtained by identifying u and v, then one of the following is true:*

- $K \setminus \{z\} \subset N_G(u)$;
- $K \setminus \{z\} \subset N_G(v)$;
- *the subgraph of G induced by $(K \setminus \{z\}) \cup \{u, v\}$ contains a hole in G.*

Proof. Suppose that $K \setminus \{z\} \not\subset N_G(u)$ and $K \setminus \{z\} \not\subset N_G(v)$. Then there is a vertex w and x in $K \setminus \{z\}$ such that w is not adjacent to u and x is not adjacent to v. Since K is a clique in $G \cdot uv$, w and x are adjacent to v and u, respectively, in G, and so $uxwvu$ is a hole in G. □

Lemma 2.1.22. *A chordal graph G is $K_{\omega(G)+1}$-minor-free.*

Proof. Denote $\omega(G)$ by ω for simplicity's sake. Suppose, to the contrary, that G contains $K_{\omega+1}$ as a minor. Then, since $K_{\omega+1}$ is complete, G contains an induced subgraph H such that $K_{\omega+1}$ is obtained from H by only contraction. Moreover, we may regard H as an induced subgraph of G for which the smallest number of contractions are required to obtain $K_{\omega+1}$. Then, since G is chordal, H is also chordal. Clearly H is $K_{\omega+1}$-free, so at least one edge of H is contracted to obtain $K_{\omega+1}$. Let uv be the last edge contracted to obtain $K_{\omega+1}$ from H. Let L be the second last graph obtained in the series of contractions to obtain $K_{\omega+1}$ from H, that is, $L \cdot uv = K_{\omega+1}$. Then, by Lemma 2.1.21, $V(L) \setminus \{u,v\} \subset N_L(u)$ or $V(L) \setminus \{u,v\} \subset N_L(v)$ or L contains a hole. If $V(L) \setminus \{u,v\} \subset N_L(u)$ or $V(L) \setminus \{u,v\} \subset N_L(v)$, then $L - v$ or $L - u$ is isomorphic to $K_{\omega+1}$, which contradicts the choice of H. Thus $V(L) \setminus \{u,v\} \not\subset N_L(u)$ and $V(L) \setminus \{u,v\} \not\subset N_L(v)$, and so L contains a hole. However, since H is chordal, by Lemma 2.1.20, L is chordal and we reach a contradiction. □

Theorem 2.1.23. *For a positive integer j and a $(2,j)$ digraph, if its underlying graph is chordal, then its phylogeny graph is K_{j+3}-minor-free if $j \leq 2$ and K_{j+4}-minor-free if $j \geq 3$.*

Proof. Let D be a $(2,j)$ digraph for a positive integer j whose underlying graph is chordal. Then, by Theorem 2.1.16, $P(D)$ is chordal. Moreover,

$$\omega(P(D)) \leq \begin{cases} j+2, & \text{if } j \leq 2 \\ j+3, & \text{otherwise.} \end{cases}$$

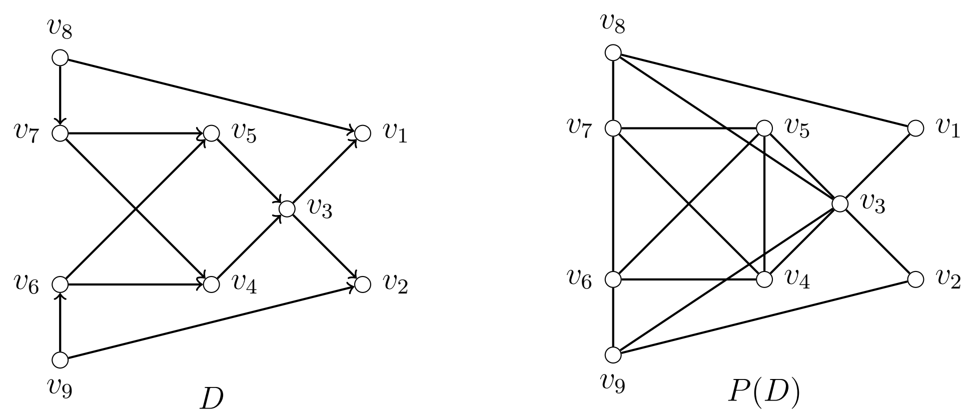


Figure 2.2: A $(2, 2)$ digraph D whose phylogeny graph contains K_5 as a minor.

by Theorem 2.1.19. Thus $P(D)$ is K_{j+3}-free (resp. K_{j+4}-free) if $j \leq 2$ (resp. $j \geq 3$). By Lemma 2.1.22, $P(D)$ is K_{j+3}-minor-free (resp. K_{j+4}-minor-free) if $j \leq 2$ (resp. $j \geq 3$). $\square$

The above theorem is false for a $(2, j)$ digraph whose underlying graph is non-chordal (see Figure 2.2).

Corollary 2.1.24. *If the underlying graph of a* $(2, 2)$ *digraph is chordal, then its phylogeny graph is* K_5*-minor-free.*

In the following, we show that the phylogeny graph of $(2, 2)$ digraph whose underlying graph is chordal is $K_{3,3}$-minor-free.

The *join* of two graphs G_1 and G_2 is denoted by $G_1 \vee G_2$ and has the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{xy \mid x \in G_1 \text{ and } y \in G_2\}$. Let I_n denote a set of n isolated vertices in a graph for a positive integer n.

Lemma 2.1.25. *For any* $(2, 2)$ *digraphs, if its underlying graph is chordal, then its phylogeny graph is* $K_3 \vee I_3$*-minor-free.*

Proof. Let G be the phylogeny graph of a $(2, 2)$ digraph D whose underlying graph is chordal. Then, by Theorem 2.1.16 and Corollary 2.1.24, G is chordal and K_5-minor-free.

Suppose, to the contrary, that $K_3 \vee I_3$ is a minor of G. Then G contains a subgraph H such that either $H = K_3 \vee I_3$ or $K_3 \vee I_3$ is obtained from H by using edge deletions or contractions. Let f be an acyclic labeling of D.

Suppose that $H = K_3 \vee I_3$. If H is not an induced subgraph of G, then two vertices of I_3 are adjacent in G, and so K_5 is a subgraph of G, which is impossible. Thus H is an induced subgraph of G. We denote the vertices of K_3 in H by x_1, x_2, x_3 and the vertices of I_3 in H by y_1, y_2, y_3. We may assume that $f(x_1) < f(x_2) < f(x_3)$ and $f(y_1) < f(y_2) < f(y_3)$.

If $f(x_1) < f(y_1)$, then the outdegree of x_1 in the subdigraph D_H of D induced by $V(H)$ is zero, which implies $d_H(x_1) \leq 4$ (recall that D is a $(2, 2)$ digraph), a contradiction. Thus

$$f(y_1) < f(x_1) < f(x_2) < f(x_3) \quad \text{and} \quad f(y_1) < f(y_2) < f(y_3). \tag{2.1.3}$$

If x_3 has two in-neighbors in D_H, then they must be y_2 and y_3, which implies their being adjacent in G, a contradiction. Therefore x_3 has at most one in-neighbor in D_H. Since D is a $(2, 2)$ digraph and $d_H(x_3) = 5$, x_3 has exactly one in-neighbor and two out-neighbors in D_H, and two cared edges in H are incident to x_3. The in-neighbor of x_3 in D_H is y_2 or y_3 by (2.1.3).

Let y be the in-neighbor of x_3 in D_H. Then $y \in \{y_2, y_3\}$ and $f(y) > f(x_3)$. Thus, by (2.1.3), none of x_1, x_2, and y_1 is an in-neighbor of y in D_H. Since D is a $(2, 2)$ digraph, y has at most one out-neighbor other than x_3 in D_H. Then, since $d_H(y) = 3$, by (2.1.3), one of x_2y and x_1y is a cared edge in G taken care of by x_1 or x_2. Since $f(x_1) < f(x_2)$, x_2y is a cared edge in G taken care of by x_1.

Let v be a vertex joined to x_3 by a cared edge in H. Then x_3 and v have a common out-neighbor in D. Since x_3 has all of its two out-neighbors in D_H, the common out-neighbors of x_3 and v should be in H. Since there are two cared edges incident to x_3 in H, the two out-neighbors of x_3 take care of those two cared edges incident to x_3. Since y_1 has the least f-value among the vertices in H, y_1 cannot be none of the other ends of two cared edges incident to x_3 in G. Hence y_1 must be one of the two out-neighbors of x_3 in D_H which takes care of a cared edge incident to x_3. Since $\{y_1, y_2, y_3\}$ is an independent set in G, neither y_2 nor y_3 can be an in-neighbor of y_1 in D_H. Thus x_1 or x_2 is the vertex joined to x_3 which is taken care of by y_1 in D_H.

If x_1 is an in-neighbor of y_1 in D_H, then $x_1y_1x_3yx_1$ is a hole in $U(D)$ since $\{y, y_1\} \subset I_3$ and x_1x_3 is a cared edge in G which is not an edge in $U(D)$. Thus x_2 is an in-neighbor of y_1 in D_H. In the following, we shall claim that $x_1x_2y_1x_3yx_1$ is a hole in $U(D)$ to reach a contradiction. Since $\{y, y_1\} \subset I_3$, y and y_1 are not adjacent in $U(D)$. Since x_2x_3 is a cared edge in G, x_2 and x_3 are not adjacent in $U(D)$. If x_1x_3 is an edge of $U(D)$, then there is an arc (x_3, x_1) since $f(x_1) < f(x_3)$, which contradicts the indegree condition on x_1. Therefore x_1 and x_3 are not adjacent in $U(D)$. By applying a similar argument, we may show that neither x_1 and y_1 nor y and x_2 are adjacent in $U(D)$.

Thus $H \neq K_3 \vee I_3$ and so $K_3 \vee I_3$ is obtained from H by using edge deletions or contractions. Then, $K_3 \vee I_3$ may be obtained from the subgraph of G induced by $V(H)$ by using edge deletions or contractions, so we may assume that H as an induced subgraph of G. Then H is chordal. If an edge deletion was required to obtain $K_3 \vee I_3$ from H, then it would mean that G contains K_5 as a minor, which is impossible. Thus, we may assume that $K_3 \vee I_3$ is obtained from H by only contractions.

Let H^* be a graph obtained from H by applying the smallest number of contractions to contain $K_3 \vee I_3$ as a subgraph. Since H is chordal, H^* is chordal by Lemma 2.1.20.

Let x_1, x_2, x_3 be the vertices of K_3 and y_1, y_2, y_3 be the vertices of I_3 for $K_3 \vee I_3$ contained in H^*. Let H' be the graph to which the last contraction is applied in the process of obtaining H^* and $e = uv$ be the edge contracted lastly. Then H' is chordal by Lemma 2.1.20. By the choice of H^*, u and v are identified to become a vertex in $\{x_1, x_2, x_3, y_1, y_2, y_3\}$.

Case 1. The vertices u and v are identified to become one of y_1, y_2, y_3. Without loss of generality, we may assume that u and v are identified to become the vertex y_3. By Lemma 2.1.21, $\{x_1, x_2, x_3\} \subset N_{H'}(u)$ or $\{x_1, x_2, x_3\} \subset N_{H'}(v)$ or $\{x_1, x_2, x_3, u, v\}$ contains a hole in H'. Since H' is chordal, $\{x_1, x_2, x_3\} \subset N_{H'}(u)$ or $\{x_1, x_2, x_3\} \subset N_{H'}(v)$. Then

$$\{x_1, x_2, x_3, y_1, y_2, u\} \quad \text{or} \quad \{x_1, x_2, x_3, y_1, y_2, v\}$$

forms $K_3 \vee I_3$ in H', which contradicts the choice of H^*.

Case 2. The vertices u and v are identified to become one of x_1, x_2, x_3. Then each of y_1, y_2, y_3 is adjacent to one of u, v in H'. Without loss of generality, we may assume that u and v are identified to become the vertex x_3. By Lemma 2.1.21, $\{x_1, x_2\} \subset N_{H'}(u)$ or $\{x_1, x_2\} \subset N_{H'}(v)$ or $\{x_1, x_2, u, v\}$ contains a hole in H'. Since H' is chordal, $\{x_1, x_2\} \subset N_{H'}(u)$ or $\{x_1, x_2\} \subset N_{H'}(v)$. Without loss of generality, we may assume that $\{x_1, x_2\} \subset N_{H'}(u)$. If u is adjacent to each of y_1, y_2, y_3, then $\{x_1, x_2, u, y_1, y_2, y_3\}$ forms $K_3 \vee I_3$ in H', a contradiction to the choice of H^*. Thus u is not adjacent to one of y_1, y_2, y_3 in H'. Without loss of generality, we may assume that u is not adjacent to y_3 in H'. Then v is adjacent to y_3 in H'. If v is not adjacent to one of x_1 and x_2, then $x_1y_3vux_1$ or $x_2y_3vux_2$ is a hole in H' and we reach a contradiction. Thus v is adjacent to x_1 and x_2. If one of y_1, y_2 is adjacent to both of u and v, then x_1, x_2, u, and v together with it form K_5 in H', a contradiction. Therefore $\{N_{H'}(u) \cap \{y_1, y_2, y_3\}, N_{H'}(v) \cap \{y_1, y_2, y_3\}\}$ is a partition of $\{y_1, y_2, y_3\}$. Thus $|N_{H'}(u) \cap \{y_1, y_2, y_3\}| + |N_{H'}(v) \cap \{y_1, y_2, y_3\}| = 3$. Without loss of generality, we may assume that $|N_{H'}(u) \cap \{y_1, y_2, y_3\}| = 1$. Then $\{x_1, x_2, v, y_1, y_2, y_3, u\} \setminus (N_{H'}(u) \cap \{y_1, y_2, y_3\})$ forms $K_3 \vee I_3$ in H' and we reach a contradiction. $\square$

Theorem 2.1.26. *For any $(2, 2)$ digraph D, if the underlying graph of D is* *chordal, then the phylogeny graph of D is $K_{3,3}$-minor-free.*

Proof. Suppose, to the contrary, that $K_{3,3}$ is a minor of $P(D)$. Then $K_{3,3}$ is obtained from $P(D)$ by edge deletions or vertex deletions or contractions. Let (X, Y) be a bipartition of $K_{3,3}$. Among the edge deletions, the vertex deletions, and the contractions to obtain $K_{3,3}$ from $P(D)$, we only take all the vertex deletions and all the contractions and apply them in the same order as the order in which vertex deletions and contractions applied to obtain $K_{3,3}$ from $P(D)$. Let H^* be a graph obtained from $P(D)$ in this way. Then H^* contains $K_{3,3}$ as a spanning subgraph. In addition, since $P(D)$ is chordal, H^* is chordal by Lemma 2.1.20 (it is clear that the chordality is preserved under vertex deletions). If there is a pair of nonadjacent vertices in H^* in each of X and Y, then those four vertices form a hole in H^* and we reach a contradiction. Thus X or Y forms a clique in H^* and so H^* contains $K_3 \vee I_3$ as a spanning subgraph. Then $K_3 \vee I_3$ is a minor of $P(D)$, which contradicts Lemma 2.1.25. Hence $P(D)$ is $K_{3,3}$-minor-free. □

Theorem 2.1.27. *For any $(2,2)$ digraphs, if its underlying graph is chordal, then its phylogeny graph is chordal and planar.*

Proof. Let D be a $(2,2)$ digraph whose underlying graph is chordal. Then, by Theorem 2.1.16, $P(D)$ is chordal. Furthermore, by Corollary 2.1.24 and Theorem 2.1.26, $P(D)$ is planar. □

Corollary 2.1.28. *A chordal graph one of whose orientations is a $(2,2)$ digraph is planar.*

Proof. Let G be a chordal graph one of whose orientations, namely D, is a $(2,2)$ digraph. Then $U(D)$ is G which is chordal. Thus, by Theorem 2.1.27, $P(D)$ is planar. Since $U(D)$ is a subgraph of $P(D)$, $U(D)$ is planar. □

2.2 The phylogeny number and the triangles and the diamonds of a graph

In this section, we deal with acyclic digraphs and their phylogeny graphs in the aspect of their holes, and phylogeny numbers of graphs.

We extend the given inequalities in Theorems 1.2.1, 1.2.2, 1.2.3, and 1.2.4 to graphs with many triangles (Theorem 2.2.12). In the process of doing so, we derive Theorem 2.2.2 which plays a key role in deducing various meaning results including Theorem 2.2.13 that answers a question given by Wu *et al.* [62] (Theorem 2.2.13). They showed that the difference between the phylogeny number and the competition number of a graph can be any integer greater than or equal to -1 and asked whether or not the same is true when limited to only connected graphs. We answer their question.

We begin with the following lemma.

Given a digraph D and two vertex sets U and V of D, we denote by $[U, V]_D$ the set of arcs in D having a tail in U and a head in V.

Lemma 2.2.1. *Let D be an acyclic digraph, G be an induced subgraph of $P(D)$, and H be a subgraph of G satisfying the following:*

(i) any maximal clique of H is also a maximal clique in G;

(ii) any maximal clique of G belonging to H and any maximal clique of G not belonging to H share at most one vertex.

In addition, we let D^* be the digraph with the vertex set

$$V(D^*) = V(H) \cup (V(D) \setminus V(G))$$

and the arc set

$$A(D^*) = \bigcup_{v \in X} [N_D^-[v] \cap V(H), \{v\}]$$

where

$$X = \{v \in V(H) \cup (V(D) \setminus V(G)) \mid N_D^-[v] \cap V(H) \text{ is a clique of size at least two in } H\}.$$

Then $P(D^*)$ contains H as an induced subgraph.

Proof. If H is an empty graph, then the statement is trivially true. Now suppose that H has an edge. Let $\mathcal{C}$ be the set of all maximal cliques of H. We first show that H is a subgraph of $P(D^*)$. By definition, $V(H) \subset V(D^*) = V(P(D^*))$. Take an edge $e := uv$ in H. Then $\{u, v\} \subset K$ for some $K \in \mathcal{C}$. By the condition (i), K is a maximal clique of G. Moreover, one of the following is true: either $(u, v) \in A(D)$ or $(v, u) \in A(D)$; $(u, w) \in A(D)$ and $(v, w) \in A(D)$ for some $w \in V(D)$.

Case 1. Either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Without loss of generality, we may assume $(u, v) \in A(D)$. Then $|N_D^-[v] \cap V(H)| \geq 2$. Suppose that there is no clique in $\mathcal{C}$ including $N_D^-[v] \cap V(H)$. Since $N_D^-[v] \cap V(H)$ is a clique of G, there is a maximal clique L of G containing $N_D^-[v] \cap V(H)$. By the assumption, L does not belong to H. Then $\{u, v\} \subset K \cap L$, which contradicts the condition (ii) given in the lemma statement. Therefore there is a maximal clique in $\mathcal{C}$ containing $N_D^-[v] \cap V(H)$. Thus $N_D^-[v] \cap V(H)$ is a clique in H and so $v \in X$. Hence, by the definition of D^*, $(u, v) \in A(D^*)$, which implies that e is an edge of $P(D^*)$.

Case 2. $(u, w) \in A(D)$ and $(v, w) \in A(D)$ for some $w \in V(D)$. Suppose that $w \notin V(H)$. Then $\{u, v, w\}$ be a clique in $P(D)$ while $\{u, v, w\}$ is not a clique in H. Thus, by the condition (ii), w does not belong to G. Hence $w \in V(H) \cup (V(D) \setminus V(G))$. Since $N_D^-[w] \cap V(H)$ forms a clique in G, there is a maximal clique Y in G including $N_D^-[w] \cap V(H)$, so $\{u, v\} \subset Y$. Since $\{u, v\} \subset Y \cap K$ and $K \in \mathcal{C}$, by the hypothesis (ii), $Y \in \mathcal{C}$. If $w \in V(G) \setminus V(H)$, $\{u, v, w\}$ forms a clique in G but not in H, which contradicts to the condition (ii) since $\{u, v\} \subset K$. Therefore $w \in V(H) \cup (V(D) \setminus V(G))$ and so $w \in X$.

By the definition of D^*, $(u,w) \in A(D^*)$ and $(v,w) \in A(D^*)$. Therefore e is an edge of $P(D^*)$. Thus we have shown that H is a subgraph of $P(D^*)$.

To show that H is an induced subgraph of $P(D^*)$, we take two vertices u and v in H which are adjacent in $P(D^*)$. Then either $(u,v) \in A(D^*)$ or $(v,u) \in A(D^*)$, or there is a vertex $w \in V(D^*)$ such that $(u,w) \in A(D^*)$ and $(v,w) \in A(D^*)$. We first assume that $(u,v) \in A(D^*)$. Then $v \in X$ and $u \in N_D^-[v] \cap V(H)$. By the definition of X, $N_D^-[v] \cap V(H)$ is a clique in H. Since v was taken from H, $\{u,v\} \subset N_D^-[v] \cap V(H)$ and so u and v are adjacent in H. By a similar argument, we may show that if $(v,u) \in A(D^*)$, then u and v are adjacent in H. Finally we assume that there is a vertex $w \in V(D^*)$ such that $(u,w) \in A(D^*)$ and $(v,w) \in A(D^*)$. Then $w \in X$, so $N_D^-[w] \cap V(H)$ is a clique in H. Since $\{u,v\} \subset N_D^-[w] \cap V(H)$, u and v are adjacent in H. Hence H is an induced subgraph of $P(D^*)$. $\square$

Theorem 2.2.2. *Let G be a graph and G_1, G_2, $\ldots$, G_k be subgraphs of G satisfying that*

(i) *$E(G_1)$, $E(G_2)$, $\ldots$, $E(G_k)$ are mutually disjoint;*

(ii) *any maximal clique of G_i is also a maximal clique in G for each $i = 1, \ldots, k$;*

(iii) *any maximal clique of G belonging to G_i and any maximal clique of G not belonging to G_i share at most one vertex for each $i = 1, \ldots, k$.*

Then $p(G) \geq \sum_{i=1}^{k} p(G_i)$.

Proof. By the definition of phylogeny number, there is an acyclic digraph D such that $p(G) = |V(D) \setminus V(G)|$ and $P(D)$ contains G as an induced subgraph. For each $i = 1, \ldots, k$, let D_i be a digraph with the vertex set

$$V(D_i) = V(G_i) \cup (V(D) \setminus V(G))$$

and the arc set

$$A(D_i) = \bigcup_{v \in X_i} \left[N_D^-[v] \cap V(G_i), \{v\} \right]_D$$

where

$$X_i = \{v \in V(G_i) \cup (V(D) \setminus V(G)) \mid N_D^-[v] \cap V(G_i) \text{ is a clique of size at least two in } G_i\}.$$

Then, by conditions (i) and (ii), we may apply Lemma 2.2.1 to conclude that $P(D_i)$ contains G_i as an induced subgraph. Since D_i is a subdigraph of D which is acyclic, D_i is acyclic for each $i = 1, \ldots, k$. Now, from D_i, we delete the vertices in $V(D_i) \setminus V(G_i)$ which have at most one in-neighbor in $V(G_i)$ and denote the resulting digraph by D_i^* for each $i = 1, \ldots, k$. Then

$$|N_{D_i^*}^-(w) \cap V(G_i)| \geq 2 \tag{2.2.1}$$

for each vertex $w \in V(D_i^*) \setminus V(G_i)$ and each $i = 1, \ldots, k$. It is easy to check that D_i^* is acyclic and $P(D_i^*)$ contains G_i as an induced subgraph. Thus $p(G_i) \leq |V(D_i^*) \setminus V(G_i)|$ for each $i = 1, \ldots, k$.

Now we show that $V(D_1^*) \setminus V(G_1)$, …, $V(D_k^*) \setminus V(G_k)$ are mutually disjoint. Suppose, to the contrary, that there are i and j with $1 \leq i < j \leq k$ such that $(V(D_i^*) \setminus V(G_i)) \cap (V(D_j^*) \setminus V(G_j)) \neq \emptyset$. Then there is a vertex $x \in (V(D_i^*) \setminus V(G_i)) \cap (V(D_j^*) \setminus V(G_j))$. By (2.2.1),

$$|N_{D_i^*}^-(x) \cap V(G_i)| \geq 2 \text{ and } |N_{D_j^*}^-(x) \cap V(G_j)| \geq 2. \tag{2.2.2}$$

Since D_i^* and D_j^* are subdigraphs of D and G_i and G_j are subgraphs of G,

$$(N_{D_i^*}^-(x) \cap V(G_i)) \cup (N_{D_j^*}^-(x) \cap V(G_j)) \subset N_D^-(x) \cap V(G). \tag{2.2.3}$$

Obviously, $N_{D_i^*}^-(x) \cap V(G_i)$ and $N_{D_j^*}^-(x) \cap V(G_j)$ form cliques in G_i and G_j, respectively. Then, since $E(G_i)$ and $E(G_j)$ are disjoint by the condition,

$N^-_{D^*_j}(x) \cap V(G_j)$ is not a clique of G_i. Thus $N^-_D(x) \cap V(G)$ is a clique in G which is not contained in G_i by (2.2.3). In addition, there exist a maximal clique K of G_i containing $N^-_{D^*_i}(x) \cap V(G_i)$. By the condition (ii), K is a maximal clique of G. By (2.2.3),

$$(N^-_D(x)\cap V(G))\cap K \supset (N^-_D(x)\cap V(G))\cap(N^-_{D^*_i}(x)\cap V(G_i)) = N^-_{D^*_i}(x)\cap V(G_i),$$

and, by (2.2.2), we reach a contradiction to the condition (iii). Thus $V(D^*_1)\setminus V(G_1)$, $\ldots$, $V(D^*_k)\setminus V(G_k)$ are mutually disjoint and so

$$\sum_{i=1}^{k} p(G_i) \leq \sum_{i=1}^{k} |V(D^*_i)\setminus V(G_i)| = |\bigcup_{i=1}^{k}(V(D^*_i)\setminus V(G_i))| \leq |\bigcup_{i=1}^{k}(V(D_i)\setminus V(G_i))|.$$

We note that $\bigcup_{i=1}^{k}(V(D_i)\setminus V(G_i)) = V(D)\setminus V(G)$. Hence

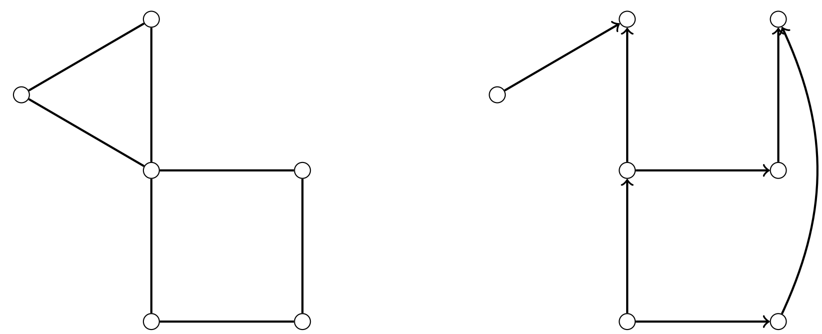
$$\sum_{i=1}^{k} p(G_i) \leq |V(D)\setminus V(G)| = p(G).$$

□

Corollary 2.2.3. *Let G be a graph and H be a triangle-free subgraph of G such that any maximal clique in H is a maximal clique in G. Then $p(G) \geq p(H)$.*

Proof. It is obvious that H satisfies the conditions (i) and (ii) in Theorem 2.2.2. Since H is triangle-free, any maximal clique of H consists of a vertex or two adjacent vertices. Furthermore, since any maximal clique of H is a maximal clique of G, any maximal clique of G not belonging to H shares at most one vertex with a maximal clique of H. Thus $p(G) \geq p(H)$ by Theorem 2.2.2. □

It is not easy to give a good lower bound for the phylogeny number of a graph. Corollary 2.2.3 is useful in a sense that there is a formula for



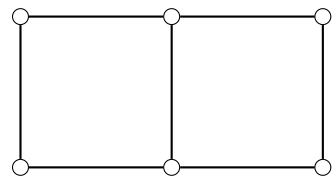
G D

Figure 2.3: A graph G whose phylogeny number can be computed by Corollary 2.2.3.

computing the phylogeny number of a triangle-free graph (see Theorem 1.2.1 and Lemma 2.2.4). For an example, we take the graph G given in Figure 2.3. Then the induced cycle of length 4 in G satisfies the condition for being H in Corollary 2.2.3. Thus $p(G) \geq 1$ by Theorem 1.2.1 and Corollary 2.2.3. The acyclic digraph D given in Figure 2.3 is a phylogeny digraph for G satisfying $|V(D) \setminus V(G)| = 1$. Hence $p(G) \leq 1$ and so $p(G) = 1$.

Lemma 2.2.4 ([47])**.** *Given a graph G, let $G_1, G_2, \ldots, G_m$ be the connected components of G and let D_i be an optimal phylogeny digraph for G_i for each $i = 1, 2, \ldots, m$. Then $D = D_1 \cup D_2 \cup \cdots \cup D_m$ is an optimal phylogeny digraph for G and $p(G) = p(G_1) + p(G_2) + \cdots + p(G_m)$.*

The inequality given in Theorem 2.2.2 may be strict if the number k of subgraphs satisfying the condition (i), (ii), and (iii) is at least two. By Theorem 1.2.1, $p(G) = 2$ for a graph G given in Figure 2.4. Yet, $p(G_1) + p(G_2) < 2$ for any two subgraphs G_1 and G_2 of G satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.2. To show it by contradiction, suppose that $p(G_1) + p(G_2) = 2$ for some two subgraphs G_1 and G_2 of G satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.2. Then one of the following



G

Figure 2.4: A graph G with $p(G) = 2$. Yet, $p(G_1) + p(G_2) < 2$ for any two subgraphs G_1 and G_2 of G satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.2.

is true: $p(G_1) = 2$ and $p(G_2) = 0$; $p(G_1) = 1$ and $p(G_2) = 1$; $p(G_1) = 0$ and $p(G_2) = 2$. A proper subgraph H of G contains at most one cycle, and, by Theorem 1.2.1 and Lemma 2.2.4, $p(H) = 1$ if H contains a cycle and $p(H) = 0$ otherwise. Therefore, if $p(G_1) = 2$ and $p(G_2) = 0$, then $G_1 = G$ and contradicts (i) or (ii) in Theorem 2.2.2. Similarly, the third case cannot happen. Now suppose that $p(G_1) = 1$ and $p(G_2) = 1$. Then each of G_1 and G_2 contains a cycle by the above observation, which contradicts (i) of Theorem 2.2.2.

In this vein, it is interesting to find properties of a graph G for which $p(G) = \sum_{i=1}^{k} p(G_k)$ for $k \geq 2$ and subgraphs $G_1, \ldots, G_k$ of G satisfying the conditions (i), (ii), and (iii) in Theorem 2.2.2. To do so, we need the following lemma.

A graph G is *separable* by a vertex w into two subgraphs G_1 and G_2 if $V(G_1) \cup V(G_2) = V(G)$, $E(G_1) \cup E(G_2) = E(G)$, and $V(G_1) \cap V(G_2) = \{w\}$.

Lemma 2.2.5 ([49]). *Let G be a graph separable by a vertex w into two graphs G_1 and G_2. If at least one of G_1 and G_2 has an optimal phylogeny digraph with no incoming arcs towards w, then $p(G) = p(G_1) + p(G_2)$.*

Theorem 2.2.6. *Let G be a graph and $G_1, G_2, \ldots, G_k$ be connected subgraphs of G satisfying that* *(i)* $\{E(G_1), E(G_2), \ldots, E(G_k)\}$ *is a partition of* $E(G)$*;*

(ii) every cycle of G *belongs to* G_i *for some* $i \in \{1, \ldots, k\}$*;*

(iii) at least $k-1$ *of* $G_1, \ldots, G_k$ *are vertex transitive.*

Then $p(G) = \sum_{i=1}^{k} p(G_i)$*.*

Proof. We show $p(G) = \sum_{i=1}^{k} p(G_i)$ by complete induction on k. If $k = 1$, then $G = G_1$ and so the inequality trivially holds. Suppose that $k \geq 2$ and the equality holds for any l subgraphs of G satisfying conditions (i), (ii), and (iii) for each $l \leq k-1$. Without loss of generality, we may assume that G_1 is not vertex transitive, if any. Since G is connected, G_1 must share a vertex with G_i for some $i \in \{2, \ldots, k\}$ by the condition (i). We may assume that $i = 2$.

Suppose that $|V(G_1) \cap V(G_2)| \geq 2$. Then we take two vertices $w_1, w_2 \in V(G_1) \cap V(G_2)$ the distance between which is the smallest in G_1. Let W_1 and W_2 be a shortest (w_1, w_2)-path in G_1 and a shortest (w_2, w_1)-path in G_2, respectively. Then the length of W_1 is the distance between w_1 and w_2 in G_1. Suppose that W_1 and W_2 have a common vertex w^* other than w_1 and w_2. Then $w^* \in V(G_1) \cap V(G_2)$. In addition, the (w_1, w^*)-section of W_1 is a path shorter than W_1 in G_1, so the distance between w_1 and w^* is smaller than the distance between w_1 and w_2 in G_1, which contradicts the choice of w_1 and w_2. Therefore W_1 and W_2 are internally vertex-disjoint and so W_1W_2 is a cycle in G. Then, by the condition (ii), G_r contains the cycle W_1W_2 for some $r \in [k]$. By the condition (i), W_1W_2 belongs to neither G_1 nor G_2, so $r \neq 1, 2$. Yet, G_1 and G_r share an edge, which contradicts the condition (i). Therefore $|V(G_1) \cap V(G_2)| = 1$.

Let $w \in V(G_1) \cap V(G_2)$. Then G is separable by w into two subgraphs G_1 and G_2. Let D_2 be an optimal phylogeny digraph for G_2. Since D_2 is acyclic, D_2 has a vertex v of indegree zero. If $v \notin V(G_2)$, $P(D_2 - v)$ contains G_2 as an induced subgraph, which contradicts the choice of D_2 to be optimal. Thus $v \in V(G_2)$. Since G_2 is vertex transitive, we may regard v as w. Thus, by Lemma 2.2.5, $p(G^*) = p(G_1) + p(G_2)$ where G^* is the union of G_1 and G_2. It is easy to check that the subgraphs G^*, G_3, …, G_k of G satisfy the conditions (i), (ii), and (iii). Hence, by the induction hypothesis,

$$p(G) = p(G^*) + \sum_{i=3}^{k} p(G_i) = \sum_{i=1}^{k} p(G_i)$$

and so $p(G) = \sum_{i=1}^{k} p(G_i)$. □

Corollary 2.2.7. *Let G be a graph and K be a clique of G that is a block in G and contains exactly one cut-vertex of G. Then G and the graph G_K obtained by deleting the vertices in K except the cut-vertex have the same phylogeny number.*

Proof. Let G_1, …, G_ω be the components of G for a positive integer ω. We may assume that G_1 contains K. Let H_1 be the graph obtained from G_1 by deleting the vertices in K except the cut-vertex. Obviously, H_1 and K satisfy the conditions (i), (ii), and (iii) of Theorem 2.2.6 as connected subgraphs of G_1. Thus, by the theorem, $p(G_1) = p(H_1)+p(K)$. Since the phylogeny number of a complete graph is zero, $p(K) = 0$ and so $p(G_1) = p(H_1)$. Therefore $p(G) = p(G_1) + \cdots p(G_\omega) = p(H_1) + \cdots + p(G_\omega)$ by Lemma 2.2.4. We note that replacing G_1 with H_1 among the components of G results in G_K. Thus the right hand side of the second equality above equals $p(G_K)$ by Lemma 2.2.4 and this completes the proof. □

Corollary 2.2.8. *Let G be a graph with a pendant vertex v. Then $p(G) = p(G - v)$.*

Now we are ready to extend the inequalities given in (1.2.1) to graphs with many triangles. To do so, we need the following lemmas.

Lemma 2.2.9 ([47])**.** *For any graph G, $p(G) \geq \theta_e(G) - |V(G)| + 1$.*

Lemma 2.2.10. *Let G be a graph and xy be an edge of G which is not an edge of any triangle in G. If a phylogeny digraph D for G contains the arc (x, y), then x is the only in-neighbor of y in D which belongs to $V(G)$.*

Proof. Suppose, to the contrary, that $z \in V(G)\setminus\{x\}$ is an in-neighbor of y in D. Then $\{x, y, z\}$ forms a triangle in $P(D)$. Since G is an induced subgraph of $P(D)$ and $\{x, y, z\} \subset V(G)$, $\{x, y, z\}$ forms a triangle in G and we reach a contradiction. □

Lemma 2.2.11. *Let G be a graph and xy be an edge of G which is not an edge of any triangle in G and D be a phylogeny digraph for G. If z is a common out-neighbor of x and y in D, then z does not belong to G and x and y are the only in-neighbors of z in D that belong to G.*

Proof. Suppose that z is a common out-neighbor of x and y in D. If z belongs to G, then $\{x, y, z\}$ forms a triangle in G and we reach a contradiction. Therefore z does not belong to G. If there is an in-neighbor w of z in D which belongs to $V(G) \setminus \{x, y\}$, then $\{x, y, w\}$ forms a triangle in G and we reach a contradiction. □

Theorem 2.2.12. *Let G be a connected K_4-free graph with mutually edge-disjoint diamonds. Then*

$$|E(G)| - |V(G)| - 2t(G) + d(G) + 1 \leq p(G) \leq |E(G)| - |V(G)| - t(G) + 1$$

where $t(G)$ and $d(G)$ denote the number of triangles and the number of diamonds in G, respectively. Especially, the first inequality becomes equality if G^- is connected and the second inequality becomes equality if G^- has exactly $2t(G) - d(G) + 1$ components.

Proof. It is easy to check that

$$\theta_e(G) = |E(G)| - 2(t(G) - 2d(G)) - 3d(G) = |E(G)| - 2t(G) + d(G).$$

By Lemma 2.2.9, $|E(G)| - |V(G)| - 2t(G) + d(G) + 1 \leq p(G)$. Now we show $p(G) \leq |E(G)| - |V(G)| - t(G) + 1$ by induction on $t(G)$. By Theorems 1.2.1, 1.2.2, 1.2.3, and 1.2.4, the inequalities hold for graphs having at most two triangles. Thus we may assume that G contains at least three triangles.

Case 1. There is no diamond in G. We take a triangle $uvwu$ in G. Then $E(G-uv) = E(G) \setminus \{uv\}$, $V(G-uv) = V(G)$ and $t(G-uv) = t(G)-1$. In addition, it is easy to check that $G-uv$ is connected, K_4-free, and diamond-free. Therefore, by the induction hypothesis,

$$\begin{aligned} p(G-uv) &\leq |E(G-uv)| - |V(G-uv)| - t(G-uv) + 1 \\ &= (|E(G)|-1) - |V(G)| - (t(G)-1) + 1 \\ &= |E(G)| - |V(G)| - t(G) + 1. \end{aligned} \tag{2.2.4}$$

Let D^* be an optimal phylogeny digraph for $G-uv$. Then, since uw and vw are edges of $G-uv$, one of the following is true: uw or vw is a cared edge of $P(D^*)$; none of uw and vw is a cared edge of $P(D^*)$.

Subcase 1-1. uw or vw is a cared edge of $P(D^*)$. Then u and w or v and w have a common out-neighbor in D^*. Without loss of generality, we may assume that u and w have a common out-neighbor z in D^*. Since G is diamond-free and K_4-free, uw is not an edge of any triangle in $G-uv$. Therefore $z \in V(D^*) \setminus V(G)$ and z has exactly two in-neighbors u and w which belong to $V(G-uv)$ by Lemma 2.2.11. Now we add an arc (v,z) to D^* and denote the resulting digraph by D. Then D is an acyclic digraph satisfying that $V(D) \setminus V(G) = V(D^*) \setminus V(G-uv)$ and $P(D)$ contains G as an induced subgraph.

Subcase 1-2. None of uw and vw is a cared edge of $P(D^*)$. Then one of (u,w) and (w,u) and one of (v,w) and (w,v) belong to $A(D^*)$. Since D^* is acyclic, we take an acyclic labeling ℓ of D^*. If w has the least ℓ-value among u, v, and w, then $(u,w) \in A(D^*)$ and $(v,w) \in A(D^*)$, and so uv is an edge of $G-uv$, which is a contradiction. Thus u or v has the least ℓ-value among u, v, and w. Without loss of generality, we may assume that u has the least ℓ-value among u, v, and w. Then $(w,u) \in A(D^*)$. Since uw is not an edge of any triangle of $G-uv$, w is the only in-neighbor of u in D^* that belongs to $V(G-uv)$ by Lemma 2.2.10. Now we add an arc

(v,u) to D^* to obtain an acyclic digraph D. Then it is easy to check that $V(D) \setminus V(G) = V(D^*) \setminus V(G - uv)$ and $P(D)$ contains G as an induced subgraph.

Since D^* is an optimal phylogeny digraph of $G-uv$, $|V(D^*) \setminus V(G-uv)| = p(G-uv)$. Then, since $V(D) \setminus V(G) = V(D^*) \setminus V(G-uv)$ in each subcase, $|V(D) \setminus V(G)| = p(G-uv)$. Therefore, by (2.2.4), $|V(D) \setminus V(G)| \leq |E(G)| - |V(G)| - t(G) + 1$ in each subcase and hence $p(G) \leq |E(G)| - |V(G)| - t(G) + 1$.

Case 2. There is a diamond in G. Let y and w be nonadjacent vertices and $\{x,y,z,w\}$ be a vertex set which forms a diamond Λ in G. Now let $G^* = G - \{xz, yz, wz\}$ and D^* be an optimal phylogeny digraph for G^*. Then G^* is still K_4-free graph and its diamonds are mutually edge-disjoint. Suppose that there exists an edge of Λ on a triangle T distinct from the triangles $xyzx$ and $xwzx$. Since G is K_4-free, T and $xyzx$ or T and $xwzx$ form a diamond. However, the resulting diamond shares an edge with Λ and we reach a contradiction. Therefore none of edges on Λ is on a triangle in G^*. Thus

$$|E(G^*)| = |E(G)| - 3, \quad |V(G^*)| = |V(G)|, \quad \text{and} \quad t(G^*) = t(G) - 2. \quad (2.2.5)$$

Furthermore, by Lemma 2.2.10,

(‡) u is the only in-neighbor of v that belongs to $V(G)$ if $(u,v) \in A(D^*)$ for $(u,v) \in \{(x,y),(y,x),(x,w),(w,x)\}$.

In addition, by Lemma 2.2.11, if xy (resp. xw) is a cared edge in $P(D^*)$, then

(♭) a caring vertex of xy (resp. xw) belongs to $V(D^*) \setminus V(G^*)$ (consequently $V(D^*) \setminus V(G)$) and x and y (resp. x and w) are the only in-neighbors in D^* of the caring vertex that belong to $V(G)$.

Subcase 2-1. G^* is disconnected. Then it has exactly two components G_1 and G_2 which contains z. Obviously G_i is connected and K_4-free, and the diamonds in G_i are mutually edge-disjoint for each $i = 1$, 2. Thus, by the induction hypothesis, $p(G_1) \leq |E(G_1)| - |V(G_1)| - t(G_1) + 1$ and $p(G_2) \leq |E(G_2)| - |V(G_2)| - t(G_2) + 1$. Then

$$\begin{aligned}
p(G^*) &= p(G_1) + p(G_2) \\
&\leq (|E(G_1)| - |V(G_1)| - t(G_1) + 1) + (|E(G_2)| - |V(G_2)| - t(G_2) + 1) \\
&= |E(G^*)| - |V(G^*)| - t(G^*) + 2 \\
&= (|E(G)| - 3) - |V(G)| - (t(G) - 2) + 2 \\
&= |E(G)| - |V(G)| - t(G) + 1. \qquad (2.2.6)
\end{aligned}$$

by (2.2.5) and Lemma 2.2.4.

Suppose that both of xy and xw are cared edges of $P(D^*)$. Then x and y (resp. x and w) have a common out-neighbor a (resp. b) in D^*. Now we add arcs (z, a) and (z, b) to D^* to obtain a digraph D.

Suppose that either xy or xw is cared edge of $P(D^*)$. Without loss of generality, we may assume that xy is a cared edge of $P(D^*)$. Then xw is not a cared edge of $P(D^*)$, and so either $(x, w) \in A(D^*)$ or $(w, x) \in A(D^*)$. Since xy is a cared edge, x and y have a common out-neighbor c in D^*. We construct a digraph D from D^* by adding the arcs (z, c), and (z, w) if $(x, w) \in A(D^*)$; (z, x) if $(w, x) \in A(D^*)$.

Now suppose that none of xy and xw is a cared edge of $P(D^*)$. Then either $(x, y) \in A(D^*)$ or $(y, x) \in A(D^*)$, and either $(x, w) \in A(D^*)$ or $(w, x) \in A(D^*)$. Since y and w are not adjacent in G^*, $(y, x) \notin A(D^*)$ or $(w, x) \notin A(D^*)$. We add the arcs to D^* as follows: (z, x) and (z, w) if $(y, x) \in A(D^*)$ and $(x, w) \in A(D^*)$; (z, y) and (z, x) if $(x, y) \in A(D^*)$ and $(w, x) \in A(D^*)$; (z, y) and (z, w) if $(x, y) \in A(D^*)$ and $(x, w) \in A(D^*)$; Let D be the resulting digraph.

We have constructed a digraph D from D^* in each of the three cases above. By (‡) and (♭), $P(D)$ contains G as an induced subgraph in each case. By (♭), the outdegree of a caring vertex is zero in D^* (we recall that we assumed that the outdegree of any vertex belonging to only optimal phylogeny digraph is zero). Moreover, since G_1 and G_2 are the components of G^*, there is no arc between a vertex in G_1 and a vertex in G_2 in D^*. Therefore D is acyclic in each case. Furthermore, D^* is an optimal phylogeny digraph for G^* and the added arcs have tails in $V(G)$. Thus we may conclude that D is a phylogeny digraph for G.

Since we did not add any new vertex to construct D from D^*, $V(D) \setminus V(G) = V(D^*) \setminus V(G^*)$. Since D^* was chosen as an optimal phylogeny digraph for G^*, $p(G^*) = |V(D^*) \setminus V(G^*)|$. Thus

$$p(G) \leq |V(D) \setminus V(G)| = |V(D^*) \setminus V(G^*)| = p(G^*) \leq |E(G)| - |V(G)| - t(G) + 1$$

by (2.2.6).

Subcase 2-2. G^* is connected. Clearly G^* is K_4-free and its diamonds are mutually edge-disjoint. Thus, by the induction hypothesis,

$$\begin{aligned} p(G^*) &\leq |E(G^*)| - |V(G^*)| - t(G^*) + 1 \\ &= (|E(G)| - 3) - |V(G)| - (t(G) - 2) + 1 \\ &= |E(G)| - |V(G)| - t(G) \end{aligned} \tag{2.2.7}$$

where the first equality holds by (2.2.5).

Suppose that one of xy and xw is a cared edges of $P(D^*)$. Without loss of generality, we may assume that xy is a cared edge of $P(D^*)$. Then x and y have a common out-neighbor a in D^*. We construct a digraph D from D^* by adding the vertex b and the arcs (z, a), (z, b), (x, b), and (w, b).

Now suppose that none of xy and xw is cared edge of $P(D^*)$. Then either $(x, y) \in A(D^*)$ or $(y, x) \in A(D^*)$, and either $(x, w) \in A(D^*)$ or $(w, x) \in A(D^*)$. Since y and w are not adjacent in G^*, $(y, x) \notin A(D^*)$ or $(w, x) \notin A(D^*)$. We construct a digraph D from D^* as follows: $V(D) = V(D^*) \cup \{c\}$; we alter the arcs incoming toward to z in D^* so that they go toward to c in D and add an arc (z, c); add arcs (z, x) and (z, w) if $(y, x) \in A(D^*)$ and $(x, w) \in A(D^*)$; (z, y) and (z, x) if $(x, y) \in A(D^*)$ and $(w, x) \in A(D^*)$; (z, y)

and (z, w) if $(x, y) \in A(D^*)$ and $(x, w) \in A(D^*)$, i.e.

$$A(D) = \begin{cases} A(D') \cup \{(z, c), (z, x), (z, w)\} & \text{if } (y, x), (x, w) \in A(D^*) \\ A(D') \cup \{(z, c), (z, x), (z, y)\} & \text{if } (x, y), (w, x) \in A(D^*) \\ A(D') \cup \{(z, c), (z, y), (z, w)\} & \text{if } (x, y), (x, w) \in A(D^*) \end{cases}$$

where D' is the digraph with $V(D') = V(D^*) \cup \{c\}$ and

$$\begin{aligned} A(D') = &(A(D^*) \setminus \{(u, z) \in A(D^*) \mid u \in V(D^*)\}) \\ &\cup \{(u, c) \mid u \in V(D^*) \text{ and } (u, z) \in A(D^*)\}. \end{aligned}$$

We have constructed a digraph D from D^* in each of the two cases above. By (‡) and (♭), $P(D)$ contains G as an induced subgraph in each case. By (♭), the outdegree of a caring vertex is zero in D^*. Therefore adding arcs (z, a), (z, b), (x, b), and (w, b) to D^* does not create a directed cycle in the first case. Since z has indegree zero in the second case, adding arcs with z as a tail does not create a directed cycle. Therefore D is acyclic in each case. Furthermore, D^* is an optimal phylogeny digraph for G^* and the added arcs have tails in $V(G)$. Thus we may conclude that D is a phylogeny digraph for G.

Since we added exactly one vertex to construct D from D^*, $|V(D) \setminus V(G)| = |V(D^*) \setminus V(G^*)| + 1$. Since D^* was chosen as an optimal phylogeny digraph for G^*, $p(G^*) = |V(D^*) \setminus V(G^*)|$. Thus

$$\begin{aligned} p(G) \leq |V(D) \setminus V(G)| &= |V(D^*) \setminus V(G^*)| + 1 \\ &= p(G^*) + 1 \leq |E(G)| - |V(G)| - t(G) + 1 \end{aligned}$$

where the last inequality holds by (2.2.7).

Now we prove the "especially" part. Clearly $V(G^-) = V(G)$. Since the diamonds in G are mutually edge-disjoint,

$$|E(G^-)| = |E(G)| - 3(t(G) - 2d(G)) - 5d(G) = |E(G)| - 3t(G) + d(G). \quad (2.2.8)$$

Suppose that G^- is connected. Since G^- is triangle-free,

$$p(G^-) = |E(G^-)| - |V(G^-)| + 1$$

by Theorem 1.2.1. Substituting $|V(G^-)| = |V(G)|$ and $|E(G^-)|$ given in (2.2.8) into the above equality results in

$$p(G^-) = |E(G)| - |V(G)| - 3t(G) + d(G) + 1. \quad (2.2.9)$$

Let D^- be an optimal phylogeny digraph for G^-. Now we add $t(G)$ vertices to D^- and arcs in such a way that each added vertex takes care of only the edges on a triangle and two triangle edges on distinct triangles are taken care of by distinct added vertices. Obviously the resulting digraph D is a phylogeny digraph for G and so

$$\begin{aligned} p(G) &\le |V(D) \setminus V(G)| = |V(D^-) \setminus V(G^-)| + t(G) \\ &= p(G^-) + t(G) \le |E(G)| - |V(G)| - 2t(G) + d(G) + 1 \end{aligned}$$

where the last inequality holds by (2.2.9). Consequently, we have shown that $p(G) = |E(G)| - |V(G)| - 2t(G) + d(G) + 1$ if G^- is connected.

Now suppose that G^- has exactly $r := 2t(G) - d(G) + 1$ components H_1, $\ldots$, H_r. For each component H_i of G^-, $p(H_i) = |E(H_i)| - |V(H_i)| + 1$ by Theorem 1.2.1. By Lemma 2.2.4,

$$p(G^-) = \sum_{i=1}^{r} p(H_i) = \sum_{i=1}^{r} (|E(H_i)| - |V(H_i)| + 1).$$

Since $|V(G^-)| = \sum_{i=1}^{r} |V(H_i)|$ and $|E(G^-)| = \sum_{i=1}^{r} |E(H_i)|$,

$$p(G^-) = |E(G^-)| - |V(G^-)| + r$$

or

$$p(G^-) = |E(G^-)| - |V(G^-)| + 2t(G) - d(G) + 1.$$

By (2.2.8),

$$p(G^-) = |E(G)| - |V(G)| - t(G) + 1. \tag{2.2.10}$$

We denote by L the graph obtained from G by attaching a new pendant vertex to each vertex of G. It is easy to see that the graph obtained from G^- by attaching a new pendant vertex to each vertex of G^- is L^-. Now

$$p(G) = p(L) \quad \text{and} \quad p(G^-) = p(L^-) \tag{2.2.11}$$

by Corollary 2.2.8. Moreover, a maximal clique of L^- is an edge which is an edge of G^- or a newly added edge incident to a pendant vertex. By the definition of G^-, each edge in G^- maximal clique of G. Therefore a maximal clique of L^- is a maximal clique of L. Thus $p(L) \geq p(L^-)$ by by Corollary 2.2.3. Then $p(G) \geq p(G^-)$ by (2.2.11). Therefore $p(G)) \geq |E(G)| - |V(G)| - t(G) + 1$ by (2.2.10). Accordingly, we have shown that $p(G) = |E(G)| - |V(G)| - t(G) + 1$ if G^- has exactly $2t(G) - d(G) + 1$ components. □

The graphs G_1 and G_2 given in Figure 2.5 are examples for G_1^- is connected and G_2^- has $2t(G_2) - d(G_2) + 1$ components, which implies that the lower bound and the upper bound both in Theorem 2.2.12 are achievable.

Wu *et al.* [62] showed that the difference between the phylogeny number and the competition number of a graph can be any integer greater than or equal to -1 and asked about the difference for a connected graph. We answer their question as follows.

The *Cartesian product* of two graphs G_1 and G_2 is denoted by $G_1 \times G_2$ and has the vertex set $V(G_1) \times V(G_2)$ and has an edge $(u_1, u_2)(v_1, v_2)$ if and

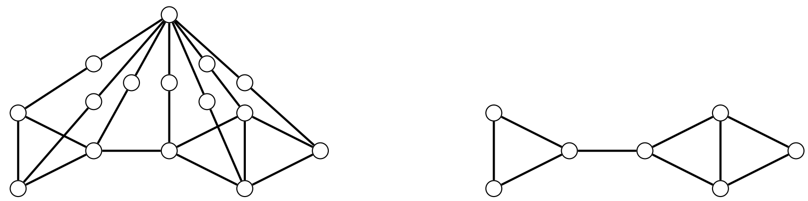


Figure 2.5: The graphs G_1 and G_2 showing that the lower bound and the upper bound given in Theorem 2.2.12, respectively, are sharp.

only if either $u_1 = v_1$ and u_2v_2 is an edge of G_2 or $u_2 = v_2$ and u_1v_1 is an edge of G_1.

Theorem 2.2.13. *For any nonnegative integer l, there is a connected graph G satisfying $p(G) - k(G) + 1 = l$.*

Proof. Let $G_0 = K_2$. Clearly $p(G_0) - k(G_0) + 1 = 0$. For each positive integer l, let G_l be the graph obtained by identifying a vertex on a complete graph K_{l+2} and a vertex on a Cartesian product of P_{l+1} and P_2 denoted by $P_{l+1} \times P_2$ (See Figure 2.6). We call the identified vertex in G_l v_l.

Fix a positive integer l. Obviously $P_{l+1} \times P_2$ is triangle-free and so the competition number is $|E(P_{l+1} \times P_2)| - |V(P_{l+1} \times P_2)| + 2 = l + 1$ by a well-known theorem that $k(G) = |E(G)| - |V(G)| + 2$ for a connected graph G. Then there is an acyclic digraph D'_l whose competition graph is $P_{l+1} \times P_2$ with newly added isolated vertices $b_{1,l}, b_{2,l}, \ldots, b_{l+1,l}$.

Now we define a digraph D_l as follows. We let

$$V(D_l) = V(D'_l) \cup \{a_l\} \quad \text{and} \quad A(D_l) = A(D'_l) \cup \{(v_l, a_l)\} \cup \bigcup_{i=1}^{l+1} \{(b_{i,l}, a_l)\}.$$

Then it is easy to check that D_l is acyclic and the competition graph of D_l is isomorphic to G_l with one isolated vertex. Thus $k(G_l) \leq 1$. It is known that the competition number of a connected graph is at least one. Since G_l

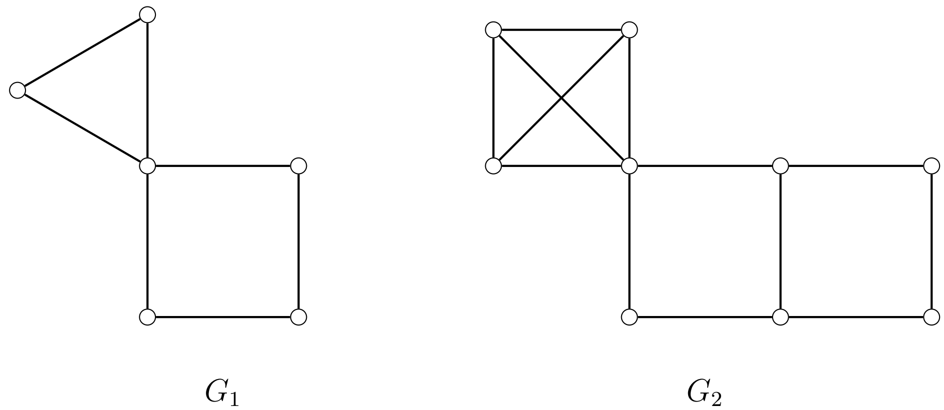


Figure 2.6: The graphs G_1 and G_2 defined in the proof of Theorem 2.2.13.

is connected, $k(G_l) \geq 1$ and so $k(G_l) = 1$.

It is easy to see that K_{l+2} and $P_{l+1} \times P_2$ satisfy (i) and (ii) of Theorem 2.2.6 as subgraphs of G_l. Obviously K_{l+2} is vertex transitive. Thus, by Theorem 2.2.6, $p(G_l) = p(K_{l+2}) + p(P_{l+1} \times P_2)$. It is known that the phylogeny number of a chordal graph is zero, so $p(K_{l+2}) = 0$. By Theorem 1.2.1, $p(P_{l+1} \times P_2) = l$. Therefore $p(G_l) = l$. Hence $p(G_l) - k(G_l) + 1 = l$ for each positive integer l. $\square$

Chapter 3

A new minimal chordal completion

We need the following notions.

A class of graphs is said to be *hereditary* if it is closed under isomorphism and induced subgraphs.

We say that a hole H *contains* a vertex v (resp. an edge e) if v (resp. e) is a vertex (resp. an edge) on H. We denote the set of holes in a graph G by $\mathcal{H}(G)$ and the set of holes in G containing u by $\mathcal{H}(G, u)$.

A nonempty subset X of $V(G)$ is called a *hole cover* of G provided that every hole in G contains at least one vertex of X. Note that, if G has no hole, that is, G is a chordal graph, then any nonempty vertex set is a hole cover of G.

For a vertex u of a graph G, we say that u satisfies the *non-consecutive property* (*NC property* for short) if any hole in $\mathcal{H}(G, u)$ and any hole not in $\mathcal{H}(G, u)$ do not share consecutive edges. A vertex subset $\mathcal{C}$ of G is said to satisfy the *NC property in G* if every vertex in $\mathcal{C}$ satisfies the NC property and every hole in G contains at most one vertex in $\mathcal{C}$. We say that a graph satisfies the *NC property* if it has a hole cover satisfying the NC property. It is easy to see that

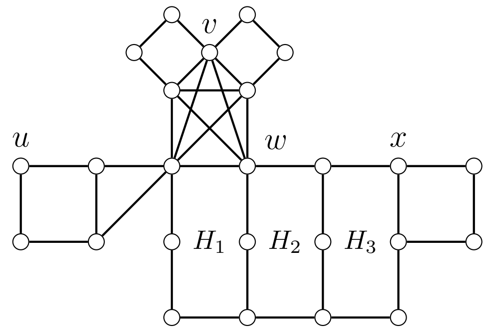


Figure 3.1: A graph G not satisfying the NC property

(♮) If a hole cover $\mathcal{C}$ of G satisfies the NC property in G, then a nonempty set $\mathcal{C} \setminus \mathcal{A}$ is a hole cover satisfying the NC property in $G - \mathcal{A}$ for any proper subset $\mathcal{A}$ of $V(G)$ not including $\mathcal{C}$.

Then it is immediately true that the family of graphs satisfying the NC property is hereditary. See Figure 3.1 for a graph not satisfying the NC property. To see why, suppose to the contrary that there exists a hole cover $\mathcal{C}$ of G satisfying the NC property. To cover the hole H_2, $\mathcal{C}$ must contain a vertex on H_2. Suppose that a vertex in $V(H_1) \cap V(H_2)$ is contained in $\mathcal{C}$. Since $\mathcal{C}$ is a hole cover satisfying the NC property, a vertex in $V(H_3) \setminus V(H_2)$ must be contained in $\mathcal{C}$ to cover H_3. Then, however, those two vertices are on the hole of length 8 surrounding H_2 and H_3, which contradicts the assumption that $\mathcal{C}$ satisfies the NC property. Even if a vertex in $V(H_2) \cap V(H_3)$ is contained in $\mathcal{C}$, we may reach a contradiction by applying a similar argument to the holes H_1 and H_2. Therefore we may conclude that there is no hole cover of G satisfying the NC property.

Let G be a graph with a hole H. For a vertex u on H, we *locally chordalize* the hole H by u in the following manner: we join u and each vertex on H nonadjacent to u.

In this chapter, we present a chordal completion of a graph G which is efficient in the following sense: Only edges joining two vertices on holes of G

are added to obtain our chordal completion (Theorem 3.3.8). Furthermore, we show that any minimal chordal completion of a graph can be obtained by joining two vertices on holes of G by edges (Proposition 3.3.12). As a matter of fact, for a nonnegative integer k, we give a sufficient condition for a graph G which has a chordal completion G^* satisfying the inequality $\omega(G^*)-\omega(G) \leq k$ (Theorem 3.3.8). This is a strong point of our chordal completion which differentiate it from other chordal completions. For example, it is shown that a graph G has treewidth at most k if and only if it has a chordal completion G^* satisfying $\omega(G^*) \leq k+1$. Yet, this characterization gives no information on $\omega(G)$, accordingly no significant information on $\omega(G^*) - \omega(G)$.

For a graph G and a vertex u satisfying the NC property, locally chordalizing all the holes in $\mathcal{H}(G, u)$ does not create any new hole (Theorem 3.1.5). Based on this observation, we found that, a hole cover $\mathcal{C}$ of a graph G can be partitioned into $\mathcal{C}_1, \dots, \mathcal{C}_k$ for some positive integer k so that

(i) $\mathcal{C}_i$ is a hole cover of the graph G_i satisfying the NC property,

(ii) G_i^* is chordal,

where $G_0 = G_0^* = G - \mathcal{C}$, G_i is the graph defined by $V(G_i) = V(G_{i-1}^*) \cup \mathcal{C}_i$ and

$$E(G_i) = E(G_{i-1}^*) \cup E\left(G - \bigcup_{j=i+1}^{k} \mathcal{C}_j\right),$$

and G_i^* is a chordal completion of G_i obtained by applying local chordalizations recursively by the vertices in $\mathcal{C}_i$ for each $i = 1, \dots, k$ (Theorem 3.3.1). Our chordal completion is G_k^* obtained for a hole cover with the smallest number k of partitions in Theorem 3.3.1. The smallest number k is called the non-chordality index of G and denoted by $i(G)$ (see Definition 3.3.3).

Then we obtain sharp upper bounds for the chromatic number, the list chromatic number, and the DP chromatic number of a graph in terms of non-chordality index and prove that the family of graphs with bounded non-chordality indices satisfies the Hadwiger conjecture and the Erdős-Faber-

Lovász Conjecture (Theorems 3.1.1, 3.3.6, 3.3.11, and 3.2.2). Other than obtaining sharp upper bounds for chromatic numbers, we prove that the family of graphs with bounded non-chordality is a linearly χ-bounded class (Theorem 3.4.1).

3.1 Graphs with the NC property

In this section, we devote ourselves to proving the following theorem.

Theorem 3.1.1. *Let G be a graph with the NC property. Then $\chi_{DP}(G) \leq \omega(G) + 1$. If G is K_n-minor-free, then $\chi_{DP}(G) \leq n - 1$.*

As a corollary of Theorem 3.1.1, we can prove a special case of Four Color Theorem.

Corollary 3.1.2. *For a planar graph G with the NC property, $\chi_{DP}(G) \leq 4$.*

Given a graph G and nonempty vertex sets S_1 and S_2, we denote the set of edges joining vertices of S_1 and vertices of S_2 by $[S_1, S_2]$. For simplicity, we use $[v, S]$ instead of $[\{v\}, S]$ for a vertex v and a nonempty vertex set S of a graph G.

Lemma 3.1.3. *Given a graph G, suppose that there exist a hole H, an induced path P, and two nonadjacent vertices u and v on H not on P satisfying the properties that*

(i) v is nonadjacent to any vertex on P in G;

(ii) there exist an internal vertex on a (u, v)-section of H and an internal vertex on the other (u, v)-section of H such that each of them is adjacent to a vertex on P.

Then there is a hole not containing u but containing two consecutive edges on H incident to v and containing a vertex on P but not on H.

Proof. Let $P = z_1z_2 \cdots z_r$ ($r \geq 1$). By the hypothesis that u and v are not on P, $z_i \neq u, v$ for each $i = 1, \ldots, r$. Since u and v are nonconsecutive vertices on H, we may give a sequence of H as follows:

$$H = vx_1x_2 \cdots x_p uy_qy_{q-1} \cdots y_1v \ (p, q \geq 1).$$

For notational convenience, we let $S_x = \{x_1, \ldots, x_p\}$ and $S_y = \{y_1, \ldots, y_q\}$. Let $\alpha = \min\{i \in \{1, \ldots, p\} \mid [x_i, V(P)] \neq \emptyset\}$ and $\beta = \min\{j \in \{1, \ldots, q\} \mid [y_j, V(P)] \neq \emptyset\}$. By the property (ii), $[S_x, V(P)] \neq \emptyset$ and $[S_y, V(P)] \neq \emptyset$ and so α and β exist. Among the vertices on P which are adjacent to x_α and among the vertices on P which are adjacent to y_β, we take z_γ and z_δ from them, respectively, with the smallest distance on P. Let P^* be the (z_γ, z_δ)-section of P. Then $C := vx_1x_2 \cdots x_\alpha P^* y_\beta y_{\beta-1} \cdots y_1 v$ is a cycle not containing u. We also note that C contains x_1v and y_1v, which are consecutive edges on H incident to v. It is easy to check that C has length at least four. No two vertices in $V(C) \setminus V(P^*)$ or in $V(P^*)$ can form a chord of C since the vertices in $V(C) \setminus V(P^*)$ are on the hole H and P^* is an induced path. Moreover, a vertex in $V(C) \setminus V(P^*)$ and a vertex in $V(P^*)$ cannot form a chord of C by the choice of α, β, z_γ, and z_δ. Therefore we can conclude that C is a hole in G. Since u is not on C, C is distinct from H. We note that C and H both are holes and the vertices on C other than the ones on P^* lie on H. Therefore there must be a vertex on P^* not on H. Since P^* is a section of P, C contains a vertex on P but not on H. $\square$

Let G be a graph with a hole H. For a vertex u on H, we recall that locally chordalizing the hole H by u means the following procedure: we join u and each vertex on H nonadjacent to u. We call an edge added in the process of a local chordalization of a hole a *newly added edge*.

Remark 3.1.4. Note that, for a graph G, locally chordalizing the holes in $\mathcal{H}(G, u)$ by a vertex u will destroy all the holes in $\mathcal{H}(G, u)$. That is, if $H \in \mathcal{H}(G, u)$, then $H \notin \mathcal{H}(G^*, u)$ where G^* is the graph resulting from the local chordalization by u.

Theorem 3.1.5. *Let G be a graph and u be a vertex of G satisfying the NC property. Then locally chordalizing all the holes in $\mathcal{H}(G,u)$ by u does not create any new hole.*

Proof. Let G^* be the graph obtained by locally chordalizing all the holes in $\mathcal{H}(G,u)$ by u. Suppose to the contrary that G^* has a hole, say H^*, not in G. Obviously H^* contains u and at least one newly added edge. Then, since u is adjacent to exactly two vertices on H^*, H^* contains one newly added edge or two newly added edges.

Let uv be a newly added edge and

$$H^* = uu_1u_2\cdots u_pvu \quad (p \geq 2).$$

Next, we define a cycle C by considering two cases.

Case 1. H^* contains uv as the only newly added edge. By the definition of local chordalization, there exists a hole H_1 in $\mathcal{H}(G,u)$ containing v on which u and v are not consecutive. Then u is adjacent to all the vertices on H_1 in G^*. However, u is not adjacent to u_k $(k = 2,\ldots,p)$ in G^*, so we can conclude that u_k $(k = 2,\ldots,p)$ is not on H_1. If u_1 is on H_1, then u_1 is adjacent to u in H_1. Thus, if u_1 is on H_1, then $uu_1u_2\cdots u_pP$ is a cycle in G for the (v,u)-section, denoted by P, of H_1 not containing u_1.

If u_1 is not on H_1 and u_1 is not adjacent to any vertex on one of the (v,u)-sections of H_1 except u, then we denote such a section by P'.

Now we define the cycle C as follows:

$$C = \begin{cases} uu_1u_2\cdots u_pP & \text{if } u_1 \text{ is on } H_1; \\ uu_1u_2\cdots u_pP' & \text{if } u_1 \text{ is not on } H_1 \text{ and } u_1 \text{ is not adjacent to any vertex on one of the } (v,u)\text{-sections of } H_1 \text{ except } u. \end{cases}$$

See (a) and (b) of Figure 3.2 for an illustration.

Case 2. H^* contains another newly added edge uw. Then $u_1 = w$. Assume that there is a hole H_2 in G which contains u, v, w. Then no two of u, v, w are consecutive on H_2. Let Q be the (v, w)-section of H_2 containing u. Since u is adjacent to all the vertices on H_2 but is not adjacent to u_i in G^*, we may conclude that u_i is not on H_2 for each $i = 2, \dots, p$. Therefore $wu_2u_3 \cdots u_pQ$ is a cycle in G. Now we let

$$C = wu_2u_3 \cdots u_pQ.$$

See Figure 3.2(c) for an illustration.

It is obvious that the cycle C defined in each case has length at least four and Pu_1, $P'u_1$, and Q are induced paths of G including u and the two vertices right next to u on C. Moreover, u is not adjacent to any vertex on C except the two vertices right next to u, and the two vertices right next to u on C are not adjacent in G. Thus, by Lemma 2.1.13, the path U composed of u and the two vertices right next to it can be extended to a hole H in G so that $V(U) \subsetneq V(H) \subset V(C)$ and H contains a vertex among $u_2, u_3, \dots, u_p$. Then u is adjacent to u_i for some $i \in \{2, 3, \dots, p\}$ in G^* by the definition of local chordalization, which contradicts the choice of H^*.

Now it remains to consider the following cases:

(i) the edge uv is the only newly added edge contained in H^*, u_1 is not on H_1, and there is a vertex on each (u, v)-section of H_1 which is adjacent to u_1 in G;

(ii) a newly added edge uw other than uv exists in H^* and there is no hole in G which contains all of u, v, and w.

We assume the case (i). The hypothesis of Lemma 3.1.3 is satisfied by H_1 for H, u_1 for P, u, and v. Therefore there exists a hole not containing u but containing consecutive edges on H_1 incident to v. This contradicts the hypothesis that u satisfies the NC property. Therefore the case (i) cannot happen.

Now we assume the case (ii). Since v and w are not consecutive vertices on H^*, w is not adjacent to v in G. Since uv and uw are newly added edges, there exist a hole H_3 containing u and v, and a hole H_4 containing u and w in G. By the case (ii) assumption, w is not on H_3 and v is not on H_4. Let $H_3 = vx_1x_2 \cdots x_q u y_r y_{r-1} \cdots y_1 v$ and $H_4 = wz_1z_2 \cdots z_s u w_t w_{t-1} \cdots w_1 w$ $(q, r, s, t \geq 1)$. See Figure 3.2(d) for an illustration. Since u is adjacent to all the vertices on H_3 (resp. H_4) and is not adjacent to u_i in G^*, we may conclude that u_i is not on H_3 (resp. H_4) for each $i = 2, \ldots, p$. For notational convenience, we let $S_x = \{x_1, \ldots, x_q\}$, $S_y = \{y_1, \ldots, y_r\}$, $S_z = \{z_1, \ldots, z_s\}$, and $S_w = \{w_1, \ldots, w_t\}$.

Suppose that, in G, $[w, S_x] \neq \emptyset$ and $[w, S_y] \neq \emptyset$. We apply Lemma 3.1.3 with H_3 for H, w for P, u, and v to reach a contradiction as before. Therefore $[w, S_x] = \emptyset$ or $[w, S_y] = \emptyset$. Without loss of generality, we may assume $[w, S_x] = \emptyset$. In addition, w is not adjacent to v in G. Thus $[w, S_x \cup \{v\}] = \emptyset$.

Suppose that $[S_x \cup \{v\}, S_z] \neq \emptyset$ and $[S_x \cup \{v\}, S_w] \neq \emptyset$. Then we apply Lemma 3.1.3 with H_4 for H, $vx_1x_2 \cdots x_q$ for P, u, and w for v to reach a contradiction as before. Therefore $[S_x \cup \{v\}, S_z] = \emptyset$ or $[S_x \cup \{v\}, S_w] = \emptyset$. Without loss of generality, we may assume $[S_x \cup \{v\}, S_z] = \emptyset$. Then $[S_x \cup \{v\}, S_z \cup \{w\}] = \emptyset$.

Now we consider the sequence $Q := vx_1x_2 \cdots x_q u z_s z_{s-1} \cdots z_1 w$. As being sections of H_3 and H_4, respectively, the two subsequences $vx_1x_2 \cdots x_q u$ and $uz_s z_{s-1} \cdots z_1 w$ of Q are induced paths in G. In addition, since $[S_x \cup \{v\}, S_z \cup \{w\}] = \emptyset$, Q is an induced path in G. Consider the cycle $C := Qu_2u_3 \cdots u_p v$. Since u is on H_3, H_4, and H^*, u is not incident to any chord of C in G. Then we apply Lemma 2.1.13 with C, Q, and $x_q u z_s$ for P to reach a contradiction as before. □

Corollary 3.1.6. *Suppose that a graph G has a hole cover $\mathcal{C} = \{u_1, \ldots, u_k\}$ satisfying the NC property and that $G_0 = G$ and, for $i = 1, \ldots, k$, G_i is the*

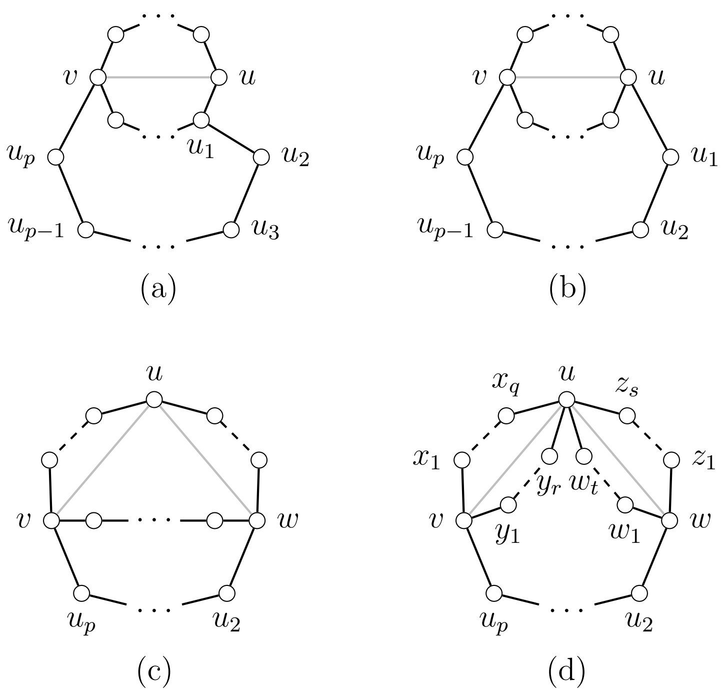


Figure 3.2: The cycle C defined in the proof of Theorem 3.1.5. The gray colored edges represent the newly edges on the hole H^* in G^* and w in (c) and (d) turns out to be u_1.

graph obtained by locally chordalizing the holes in $\mathcal{H}(G_{i-1}, u_i)$ *by* u_i. *Then* G_k *is chordal. Moreover, the resulting chordal graph is independent of the order of* $u_1, \ldots, u_k$ *by which the local chordalizations are performed.*

Proof. By induction on the size k of a hole cover satisfying the NC property. If $k = 1$, then G_k is chordal by Theorem 3.1.5. Suppose that the statement is true for any graph with a hole cover with size $k - 1$ satisfying the NC property. Now we locally chordalize the holes in $\mathcal{H}(G, u_1)$ by u_1 to obtain G_1. By Theorem 3.1.5, $\mathcal{C} \setminus \{u_1\}$ is a hole cover of G_1. By ($\natural$), $\mathcal{C} \setminus \{u_1\}$ still satisfies the NC property in G_1. Therefore, by the induction hypothesis, G_k is chordal.

It is sufficient to show the uniqueness for the case $k = 2$. Let G' and G''

be the graphs obtained by locally chordalizing the holes in $\mathcal{H}(G, u_2)$ by u_2 and the holes in $\mathcal{H}(G', u_1)$ by u_1, respectively.

Since $\mathcal{C}$ satisfies the NC property, no hole in G contains two vertices in $\mathcal{C}$. Therefore, by Theorem 3.1.5, $\mathcal{H}(G, u_1) = \mathcal{H}(G', u_1)$ and $\mathcal{H}(G_1, u_2) = \mathcal{H}(G, u_2)$, which implies $G_2 = G''$. □

Let G be a graph with a hole cover $\mathcal{C}$ satisfying the NC property. Corollary 3.1.6 says that a chordal graph can be obtained by applying local chordalizations recursively by the vertices in $\mathcal{C}$ and the resulting chordal graph is the same no matter which order of the vertices is taken. The uniqueness of the resulting chordal graph allows us to denote it by a notation, say $\widehat{G}(\mathcal{C})$. In the rest of this paper, we derive some noteworthy theorems by utilizing $\widehat{G}(\mathcal{C})$ for graphs G having hole covers $\mathcal{C}$ satisfying the NC property.

Lemma 3.1.7. *Let G be a graph with a hole cover $\mathcal{C}$ satisfying the NC property. Suppose that vertices u and w in $\mathcal{C}$ are adjacent in G. Then any newly added edge incident to u and any newly added edge incident to w are not adjacent in $\widehat{G}(\mathcal{C})$.*

Proof. Suppose to the contrary that there exist a newly added edge incident to u and a newly added edge incident to w which are adjacent in $\widehat{G}(\mathcal{C})$. Let uv and wv be such edges for some $v \in V(G)$. Then, by the definition of local chordalization, neither uv nor wv is an edge in G and there exist $H_u \in \mathcal{H}(G, u)$ and $H_w \in \mathcal{H}(G, w)$ sharing the vertex v.

To reach a contradiction, suppose that there exist an internal vertex on a (u, v)-section of H_u and an internal vertex on the other (u, v)-section of H_u each of which is adjacent to w. Then, by Lemma 3.1.3 with $P = w$, there is a hole in G not containing u but containing two consecutive edges on H_u incident to v, which contradicts the hypothesis that $\mathcal{C}$ satisfies the NC property. Therefore there exists one of the (u, v)-sections of H_u such that w is not adjacent to any internal vertex on it. Let Q be such a section. By symmetry, we may conclude that there exists one of the (v, w)-sections of

H_w such that u is not adjacent to any internal vertex on it. Let R be such a section.

Let W be the concatenation of Q and R at v. Then W is a (u, w)-walk in $G - uw$. Now W contains a (u, w)-path S as an induced subgraph in $G - uw$. By the previous argument, the vertex immediately following u on S cannot be on R while the vertex immediately preceding w on S cannot be on Q. Therefore we may conclude that the length of S is at least three. Thus S and the edge uw form a hole in G. However, this hole contains both u and w, which is impossible as $\mathcal{C}$ satisfies the NC property. $\square$

Theorem 3.1.8. *Let G be a graph with a hole cover $\mathcal{C}$ satisfying the NC property. Suppose that a vertex set K forms a clique in $\widehat{G}(\mathcal{C})$ but not in G. Then there exists a vertex $u \in K \cap \mathcal{C}$ such that $K \setminus \{u\}$ is a clique in G.*

Proof. Since K is a clique in $\widehat{G}(\mathcal{C})$ but is not a clique in G, $K \cap \mathcal{C} \neq \emptyset$. Suppose that $K \cap \mathcal{C}$ is not a clique in G. Then there exist two vertices x and y in $K \cap \mathcal{C}$ such that $xy \notin E(G)$. This implies that there exists a hole in G containing both x and y, which is impossible by the hypothesis that $\mathcal{C}$ satisfies the NC property. Therefore $K \cap \mathcal{C}$ is a clique in G. However, K is not a clique in G, so there exist vertices $u \in K \cap \mathcal{C}$ and $v \in K \setminus \mathcal{C}$ such that uv is a newly added edge. We claim that every newly added edge whose end vertices belong to K is incident with u by contradiction. Suppose that there exists a newly added edge zw such that $\{z, w\} \subset K \setminus \{u\}$. By the definition of $\widehat{G}(\mathcal{C})$, we may assume $z \in \mathcal{C}$ and $w \notin \mathcal{C}$. Since $K \cap \mathcal{C}$ is a clique in G, $zu \in E(G)$. Then Lemma 3.1.7 implies that $v \neq w$, and uw and zv are edges in G. If vw is a newly added edge, then either v or w belongs to $\mathcal{C}$, which is not the case. Therefore $vw \in E(G)$. Then the cycle $uzvwu$ is obviously a hole in G containing u and z, which contradicts the hypothesis that $\mathcal{C}$ satisfies the NC property. Thus we have shown that every newly added edge in K is incident with u. Hence $K \setminus \{u\}$ is a clique in G. $\square$

Corollary 3.1.9. *Let G be a graph with a hole cover $\mathcal{C}$ satisfying the NC* *property. Then* $\omega(\widehat{G}(\mathcal{C})) \leq \omega(G) + 1$. *Furthermore the equality holds if and only if*

(†) *There exists a vertex* $u \in \mathcal{C}$ *such that the set*

$$\left\{ \left(\bigcup_{H \in \mathcal{H}(G,u)} V(H) \right) \cup N_G(u) \right\} \setminus \{u\}$$

contains a maximum clique K *of* G.

Proof. By Theorem 3.1.8, $\omega(\widehat{G}(\mathcal{C})) \leq \omega(G) + 1$. Furthermore, by the same theorem, $\omega(\widehat{G}(\mathcal{C})) = \omega(G) + 1$ if and only if there is a clique K in $\widehat{G}(\mathcal{C})$ of size $\omega(G) + 1$ and there is a vertex $u \in K \cap \mathcal{C}$ such that $K \setminus \{u\}$ forms a clique in G, which is equivalent to (†). □

Theorem 3.1.10. *Let* G *be a graph with a hole cover* $\mathcal{C}$ *satisfying the NC property. Then every clique of* $\widehat{G}(\mathcal{C})$ *is a minor of* G.

Proof. Let K be a clique in $\widehat{G}(\mathcal{C})$ of size n. If K is a clique in G, then we are done. Suppose that K is not a clique in G. By Theorem 3.1.8, there exists a vertex $u \in K \cap \mathcal{C}$ such that $K \setminus \{u\}$ is a clique in G. Therefore $|V(H) \cap (K \setminus \{u\})| \leq 2$ for every $H \in \mathcal{H}(G,u)$. Furthermore, every newly added edge whose end vertices are in K is incident with u.

Let $uv_1, \ldots, uv_l$ be the newly added edges whose end vertices are in K and $X = \{v_1, \ldots, v_l\}$. Take a vertex $v_i \in X$. Then there exists $H \in \mathcal{H}(G,u)$ containing v_i. Since uv_i is a newly added edge, u and v_i are not consecutive on H. Then each of the (u, v_i)-sections of H contains at least one internal vertex. In addition, $V(H) \cap X \subset V(H) \cap (K \setminus \{u\})$. Since we have shown that $|V(H) \cap (K \setminus \{u\})| \leq 2$, $|V(H) \cap X| \leq 2$. Since $v_i \in V(H) \cap X$, $V(H)$ contains at most one vertex in X other than v_i. Thus one of the (u, v_i)-sections of H does not contain any vertex in K as an internal vertex. Let P_i be such a section. In G, we contract the edges on P_1 except the edge incident to v_1 to obtain the edge e_1 joining u and v_1. Then P_2 is transformed to a (u, v_2)-walk

W_2 in the graph G_1 resulting from the contractions and still does not contain any vertex in K other than u and v_2 by the way of contractions and by the choice of P_i. In G_1, we contract the edges on W_2 except the edge incident to v_2 to obtain the graph G_2 and the edge e_2 joining u and v_2 in G_2. We may repeat this process until we obtain the graph G_l from G_{l-1} and the edge e_l joining u and v_l in G_l. Now, G_l contains the vertices of K and the edges $uv_1, \dots, uv_l$ so that K is clique of size n in G_l. $\square$

Now we have the following corollary.

Corollary 3.1.11. *Let G be a graph with a hole cover $\mathcal{C}$ satisfying the NC property. If G is K_n-minor-free, then $\widehat{G}(\mathcal{C})$ is K_n-free.*

Now we are ready to give a proof of Theorem 3.1.1.

A proof of Theorem 3.1.1. Since $\widehat{G}(C)$ is a chordal completion,

$$\chi_{DP}(G) \le \chi_{DP}(\widehat{G}(C)) = \omega(\widehat{G}(C)).$$

By Corollary 3.1.9, $\omega(\widehat{G}(C)) \le \omega(G) + 1$, so $\chi_{DP}(G) \le \omega(G) + 1$. Moreover, by Corollary 3.1.11, if G is K_n-minor-free, then $\omega(\widehat{G}(C)) \le n - 1$ and so $\chi_{DP}(G) \le n - 1$. $\square$

3.2 The Erdős-Faber-Lovász Conjecture

The following is one of the versions equivalent to the conjecture given by Erdős, Faber, and Lovász in 1972.

Conjecture 3.2.1. *If G is the union of k edge-disjoint copies of K_k for a positive integer k, then $\chi(G) = k$.*

In this section, we show that the above conjecture is true for the graphs satisfying the NC property by deriving the following theorem.

Theorem 3.2.2. *If a graph G satisfying the NC property is the union of k edge-disjoint copies of K_k for a positive integer k, then $\chi_{DP}(G) = k$.*

We start by showing the following lemmas.

Lemma 3.2.3. *Let G be a graph and L be a maximal clique of G. Suppose that every vertex in $G - L$ is a simplicial vertex in G. Then G is chordal.*

Proof. It suffices to prove the lemma when G is connected. Suppose to the contrary that G has a hole H. Since L is complete and H is a hole in G, $|V(H) \cap L| \leq 2$. Then $V(H) \setminus L$ forms an induced path in G and, by the hypothesis that any vertex in $G - L$ is a simplicial vertex in G, $|V(H) \setminus L| \leq 2$. Since H is a hole, $4 \leq |V(H)| = |V(H) \cap L| + |V(H) \setminus L| \leq 4$ and so $|V(H) \cap L| = 2$ and $|V(H) \setminus L| = 2$. Since $V(H) \setminus L := \{u, v\}$ and $V(H) \cap L := \{x, y\}$ are cliques in G, uv and xy are edges in G. Since H is a hole, u cannot be a simplicial vertex in G and we reach a contradiction. □

Lemma 3.2.4. *Let G be a union of k edge-disjoint copies of K_k and $\mathcal{L}$ be the set of those k copies of K_k for a positive integer k. Then $\omega(G) = k$. Furthermore, if a maximal clique of G with size k does not belong to $\mathcal{L}$, then G is chordal.*

Proof. Since G contains K_k, $\omega(G) \geq k$. We prove that any maximal clique of G not belonging to $\mathcal{L}$ has size at most k to show $\omega(G) \leq k$. Let L be a maximal clique of G with size l which does not belong to $\mathcal{L}$. For each vertex u in L, let n_u be the minimum number of cliques in $\mathcal{L}$ needed to cover the edges in the edge cut $[u, L \setminus \{u\}]$. Since each edge of G is covered by a unique maximal clique in $\mathcal{L}$, n_u is the number of cliques in $\mathcal{L}$ which share an edge with L. Since L is a maximal clique of G and does not belong to $\mathcal{L}$, the edges on L are covered by at least two cliques in $\mathcal{L}$ and so $n_u \geq 2$ for each $u \in L$. Now let u^* be a vertex in L with the minimum $p := n_{u^*}$. By the observation that $n_u \geq 2$ for each $u \in L$, $p \geq 2$. Let $L_1, \ldots, L_p$ be the cliques in $\mathcal{L}$ which cover the edges in $[u^*, L \setminus \{u^*\}]$. Let $l_i = |L \cap L_i| - 1$ for each $i = 1, \dots, p$. Without loss of generality, we may assume

$$l_1 \geq l_2 \geq \cdots \geq l_p \geq 1. \tag{3.2.1}$$

Suppose that there exist distinct vertices u_1 and u_2 in $L \cap L_i$ for some $i \in \{1, \dots, p\}$ such that an edge in $[u_1, L \setminus L_i]$ and an edge in $[u_2, L \setminus L_i]$ are covered by the same clique K in $\mathcal{L}$. Then $K \neq L_i$. However, since K is a clique, u_1u_2 is covered by K, a contradiction to the hypothesis. Therefore

($\sharp$) two edges in $[L \cap L_i, L \setminus L_i]$ are covered by distinct cliques in $\mathcal{L}$ for $i = 1, \dots, p$ unless they have a common end in $L \cap L_i$.

Since $L_1, \dots, L_p$ are mutually edge-disjoint,

$$\begin{aligned} l = \left| L \cap \bigcup_{i=1}^{p} L_i \right| &= \left| \left(\bigcup_{i=1}^{p} (L \cap L_i) \setminus \{u^*\} \right) \cup \{u^*\} \right| \\ &= \sum_{i=1}^{p} |(L \cap L_i) \setminus \{u^*\}| + 1 = \sum_{i=1}^{p} l_i + 1. \end{aligned} \tag{3.2.2}$$

Since $p \geq 2$, L_1 and L_2 exist. Each edge in $[(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ is covered by exactly one clique in $\mathcal{L}$ by the hypothesis. Since any edge in $[(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ is not incident to u^*, any clique in $\mathcal{L}$ covering an edge in $[(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ cannot be L_i for any $i = 1, \dots, p$. Therefore we need at least $p + l_1 l_2$ cliques in $\mathcal{L}$ to cover the edges in $[u^*, L \setminus \{u^*\}] \cup [(L \cap L_1) \setminus \{u^*\}, (L \cap L_2) \setminus \{u^*\}]$ and so

$$p + l_1 l_2 \leq |\mathcal{L}| = k. \tag{3.2.3}$$

For each vertex u in $L \cap L_1$, $n_u \geq p$ and so there are at least p cliques in $\mathcal{L}$ needed to cover the edges in $[u, L \setminus \{u\}]$. By ($\sharp$), we need at least $p + l_1(p-1)$ distinct cliques in $\mathcal{L}$ to cover the edges in $[u^*, L \setminus \{u^*\}] \cup [(L \cap L_1) \setminus \{u^*\}, L \setminus L_1]$ and so

$$p + l_1(p-1) \leq k. \tag{3.2.4}$$

If $l_2 \geq p$, then

$$\begin{aligned}
l &= \sum_{i=1}^{p} l_i + 1 && \text{(by (3.2.2))} \\
&\leq l_1 p + 1 && \text{(by (3.2.1))} \\
&< l_1 l_2 + p && \text{(by the case assumption and the fact that } p \geq 2) \\
&\leq k. && \text{(by (3.2.3))}
\end{aligned}$$

Therefore we have shown that $l < k$ if $l_2 \geq p$ and so the "furthermore" part is vacuously true.

Now assume $l_2 \leq p-1$. Then

$$\begin{aligned}
l &= \sum_{i=1}^{p} l_i + 1 && \text{(by (3.2.2))} \\
&\leq (p-1)l_1 + l_2 + 1 && \text{(by (3.2.1))} \\
&\leq (p-1)l_1 + p && \text{(by the assumption that } l_2 \leq p-1) \\
&\leq k && \text{(by (3.2.4))}
\end{aligned}$$

To show the "furthermore" part, suppose $l = k$. Then each of the three inequalities above becomes the equality. Now, if $p = 2$, then $l_2 = 1$ and $l = l_1 + l_2 + 1 = l_1 + 2 = k$, which implies $l_1 = k - 2$. If $p \geq 3$, then, by (3.2.1), $l_1 = \cdots = l_p = p - 1$ and $k = p^2 - p + 1$.

Case 1. $p = 2$. Let $L \cap L_1 = \{u^*, u_1, u_2, \ldots, u_{k-2}\}$ and $L \cap L_2 = \{u^*, v\}$. Since u_i and v belong to L, $u_i v$ is an edge in G for each $i = 1, \ldots, k-2$. Since $\mathcal{L}$ is an edge clique cover of G, there is a clique in $\mathcal{L}$ covering the edge $u_i v$ for each $i = 1, \ldots, k-2$. By ($\sharp$), no clique in $\mathcal{L}$ contains u_i, u_j, v for $1 \leq i < j \leq k-2$. Therefore, by relabelling the cliques in $\mathcal{L}$ if necessary, we may assume L_{i+2} is a clique covering $u_i v$ for each $i = 1, \ldots, k-2$. Then $(L_1 \cap L_2) \cap L = \{u^*\}$,

$(L_1 \cap L_i) \cap L = \{u_{i-2}\}$ for $i = 3, \ldots, k$, and $(L_i \cap L_j) \cap L = \{v\}$ for $2 \le i < j \le k$. Therefore L_i and L_j share exactly one vertex in L for distinct i, j in $\{1, \ldots, k\}$

Case 2. $p \ge 3$. Then $l_1 = \cdots = l_p = p - 1$ and $k = p^2 - p + 1$. Let $L \cap L_1 = \{u^*, v_1, \ldots, v_{p-1}\}$ and $L \cap L_2 = \{u^*, w_1, \ldots, w_{p-1}\}$. Since L is a clique in G, v_i and w_j are adjacent in G and the edge $v_i w_j$ must be covered by a clique in the edge clique cover $\mathcal{L}$ for any $i, j \in \{1, \ldots, p-1\}$. Let $K_{i,j} \in \mathcal{L}$ be a clique which covers the edge $v_i w_j$ for $i, j \in \{1, \ldots, p-1\}$ and let $\mathcal{K} = \{K_{i,j} \mid i, j \in \{1, \ldots, p-1\}\}$. Suppose $K_{i,j} = L_t$ for some $i, j \in \{1, \ldots, p-1\}$ and $t \in \{1, \ldots, p\}$. Then the edges $u^* w_j \in [L \cap L_1, L \setminus L_1]$ and $v_i w_j \in [L \cap L_1, L \setminus L_1]$ are covered by $K_{i,j}$, which is impossible by ($\sharp$). Therefore $K_{i,j}$ cannot be any of $L_1, \ldots, L_p$. By ($\sharp$), $K_{i,j} \neq K_{i',j'}$ if $(i,j) \neq (i',j')$. Therefore $|\mathcal{K}| = (p-1)^2$ and

$$|\{L_1, \ldots, L_p\} \cup \mathcal{K}| = p + (p-1)^2 = p^2 - p + 1.$$

Since $|\mathcal{L}| = k = p^2 - p + 1$, $\mathcal{L} = \{L_1, \ldots, L_p\} \cup \mathcal{K}$.

To apply Lemma 3.2.3, we first claim that $M \cap N \subset L$ for any distinct cliques $M, N \in \mathcal{L}$. Take two distinct cliques M and N in $\mathcal{L}$. If M and N belong to $\{L_1, \ldots, L_p\}$, then $M \cap N = \{u^*\} \subset L$. Suppose that one of M and N is in $\{L_1, \ldots, L_p\}$ and the other is in $\mathcal{K}$. Without loss of generality, we may assume $M = L_t := \{u^*, x_1, \ldots, x_{p-1}\}$ and $N = K_{i,j}$ for some $t \in \{1, \ldots, p\}$ and $i, j \in \{1, \ldots, p-1\}$. By the hypothesis that the cliques in $\mathcal{L}$ are mutually edge-disjoint,

$$L_t \cap K_{i,j} = \begin{cases} \{v_i\} & \text{if } t = 1 \\ \{w_j\} & \text{if } t = 2. \end{cases}$$

Therefore $M \cap N = L_t \cap K_{i,j} \subset L$ for $t = 1, 2$. Assume $3 \le t \le p$. Note that

$$E_{1t} := [\{v_1, \ldots, v_{p-1}\}, \{x_1, \ldots, x_{p-1}\}] \subset [L \cap L_1, L \setminus L_1] \cap [L \cap L_t, L \setminus L_t]. \tag{3.2.5}$$

Suppose that an edge $v_r x_s$ is covered by L_a for some $a \in \{1, \ldots, p\}$. Then the edges u^*x_s and u^*v_r are covered by L_a. However, u^* and v_r belong to $L \cap L_1$, $\{u^*x_s, v_rx_s\} \in [L \cap L_1, L \setminus L_1]$, and we reach a contradiction to ($\sharp$). Therefore each edge in E_{1t} should be covered by a clique in $\mathcal{K}$. Since $\mathcal{K} \subset \mathcal{L}$, it follows from ($\sharp$) that each clique in $\mathcal{K}$ covers at most one edge in $E_{1t} \subset [L \cap L_1, L \setminus L_1] \cap [L \cap L_t, L \setminus L_t]$. Since $|\mathcal{K}| = (p-1)^2 = |E_{1t}|$, each clique in $\mathcal{K}$ covers exactly one edge in E_{1t}. Therefore $K_{i,j}$ covers v_rx_s for some $r, s \in \{1, \ldots, p-1\}$. Thus $L_t \cap K_{i,j}$ contains the vertex x_s. By the hypothesis that the cliques in $\mathcal{L}$ are mutually edge-disjoint, $L_t \cap K_{i,j} = \{x_s\} \subset L$. Hence $M \cap N \subset L$ for $M = L_t$ and $N = K_{i,j}$. Finally we suppose that M and N belong to $\mathcal{K}$. Then $M = K_{i,j}$ and $N = K_{i',j'}$ for some $i, i', j, j' \in \{1, \ldots, p-1\}$ with $(i, j) \neq (i', j')$. If $i = i'$, then $M \cap N = \{v_i\} \subset L$ by the hypothesis. Suppose $i \neq i'$. Take a vertex $y \in L \setminus L_1$. Since L is a clique and $\{v_i, v_{i'}, y\} \subset L$, v_iy and $v_{i'}y$ are edges of G and should be covered by cliques in $\mathcal{L}$. We note that L_b covers u^*y if L_b covers v_iy or $v_{i'}y$ for any $b \in \{1, \ldots, p\}$. Therefore, by the hypothesis that the cliques in $\mathcal{L}$ are mutually edge-disjoint, v_iy and $v_{i'}y$ are covered by cliques in $\mathcal{K}$. Let $K_{c,d}$ be a clique in $\mathcal{K}$ covering v_iy. Then v_c, v_i, y belong to $K_{c,d}$. Since $K_{c,d}$ is a clique, v_c and y are adjacent. Then v_cy and v_iy belong to $[L \cap L_1, L \setminus L_1]$ and are covered by $K_{c,d}$. Thus, by ($\sharp$), $v_i = v_c$ and so $i = c$. Similarly, $v_{i'}y$ is covered by $K_{i',d'}$ for some $d' \in \{1, \ldots, p-1\}$. By the hypothesis on $\mathcal{L}$, $K_{i,d}$ and $K_{i',d'}$ are the unique cliques in $\mathcal{L}$ covering v_iy and $v_{i'}y$, respectively. As $K_{i,d}$ and $K_{i',d'}$ are uniquely determined by y, we may denote $K_{i,d}$ and $K_{i',d'}$ by $A(y)$ and $B(y)$, respectively. Now we define a function $F : L \setminus L_1 \to \{(K_{i,q}, K_{i',q'}) \mid 1 \leq q, q' \leq p-1\}$ by $F(y) = (A(y), B(y))$ for $y \in L \setminus L_1$. Then F is well-defined. By the hypothesis on $\mathcal{L}$ again, $A(y) \cap B(y) = \{y\}$ for each $y \in L \setminus L_1$ and so F is injective. Since the domain and the codomain of F have the same cardinality $(p-1)^2$, F is bijective. Since M and N belong to $\mathcal{K}$, (M, N) is contained in the codomain of F and so there exists a vertex $z \in L \setminus L_1$ such that $F(z) = (M, N)$. Then $M = A(z)$ and $N = B(z)$, so $M \cap N = A(z) \cap B(z) = \{z\} \subset L$. Hence we have shown that $M \cap N \subset L$ for any distinct cliques M and N in $\mathcal{L}$.

In both cases, we have shown that $M \cap N \subset L$ for any distinct cliques M and N in $\mathcal{L}$. Now we will show that every vertex in $G - L$ is simplicial in G. Take a vertex v in $G - L$. Suppose to the contrary that v is not a simplicial vertex in G. Then v has two neighbors z_1 and z_2 which are nonadjacent in G. Since $\mathcal{L}$ is an edge clique cover of G, $\mathcal{L}$ contains a clique covering vz_1 and a clique covering vz_2. Since z_1 and z_2 are nonadjacent, these two cliques are distinct. However, they share a vertex v which is not in L. This contradicts our claim that the intersection of any two cliques in $\mathcal{L}$ is a subset of L. Therefore every vertex in $G - L$ is a simplicial vertex in G. Thus, by Lemma 3.2.3, G is chordal. $\square$

A proof of Theorem 3.2.2. Let G be a graph satisfying the NC property which is the union of k edge-disjoint copies $L_1, \ldots, L_k$ of K_k. Obviously $\chi_{DP}(G) \geq k$. By Lemma 3.2.4, $\omega(G) = k$. Let $\mathcal{L} = \{L_1, \ldots, L_k\}$. Then $\mathcal{L}$ is an edge clique cover consisting of cliques of size k.

Fix $i \in \{1, \ldots, k\}$. Then $|L_i \cap L_j| \leq 1$ for any $j \in \{1, \ldots, k\} \setminus \{i\}$. Since L_i has k vertices, L_i has a vertex v not contained in L_j for any $j \in \{1, \ldots, k\} \setminus \{i\}$. Then v is a simplicial vertex of G. Since i is arbitrarily chosen, L_i has a simplicial vertex for any $i = 1, \ldots, k$.

If G is chordal, then $\chi_{DP}(G) = \omega(G) = k$ by (§). Now we suppose that G is non-chordal. Then, by the "furthermore part" of Lemma 3.2.4, any clique not belonging to $\mathcal{L}$ has size less than k. Since L_i has a simplicial vertex of G, we may take a simplicial vertex from L_i and denote it by v_i for each $i = 1, \ldots, k$. Let $G' = G - \{v_1, \ldots, v_k\}$. Then G' still satisfies the NC property. Since any clique not belonging to $\mathcal{L}$ has size less than k, $\omega(G') = k - 1$. Let $\mathcal{C}$ be a hole cover of G' satisfying the NC property. Then $\widehat{G'}(\mathcal{C})$ is chordal by definition and, by Corollary 3.1.9, $\omega(\widehat{G'}(\mathcal{C})) \leq \omega(G') + 1 = k$. Let G^* be the graph obtained from $\widehat{G'}(\mathcal{C})$ by adding the vertices $v_1, \ldots, v_k$ and the edges which were incident to $v_1, \ldots, v_k$ in G. Then G is a spanning subgraph of G^*. Since $v_1, \ldots, v_k$ are simplicial vertices of G, they are still simplicial vertices of G^*. Therefore, the fact that $\widehat{G'}(\mathcal{C})$ is chordal implies that G^* is chordal. Moreover, we note that exactly $k-1$ edges are added for v_i for each $i = 1, \ldots, k$ to obtain G^* from $\widehat{G'}(\mathcal{C})$. Then, since $\omega(\widehat{G'}(\mathcal{C})) \leq k$,

$$k \leq \chi_{DP}(G) \leq \chi_{DP}(G^*) = \omega(G^*) \leq k$$

and so $\chi_{DP}(G) = k$. □

3.3 A minimal chordal completion of a graph

3.3.1 Non-chordality indices of graphs

Given a graph G, we apply a sequence of local chordalizations to obtain a chordal completion G^* of G as follows: Let $\mathcal{C} = \{v_1, \ldots, v_l\}$ be a hole cover of G and $G_0 = G_0^* = G - \mathcal{C}$. By the definition of hole cover, G_0^* is chordal. Let G_1 be the graph with

$$V(G_1) = V(G_0^*) \cup \{v_1\} \quad \text{and} \quad E(G_1) = E(G_0^*) \cup E\left(G - \bigcup_{j=2}^{l} \{v_j\}\right).$$

Obviously $\{v_1\}$ is a hole cover of G_1 satisfying the NC property. By Corollary 3.1.6, we obtain the chordal graph $G_1^* = \widehat{G_1}(\{v_1\})$. Let G_2 be the graph with

$$V(G_2) = V(G_1^*) \cup \{v_2\} \quad \text{and} \quad E(G_2) = E(G_1^*) \cup E\left(G - \bigcup_{j=3}^{l} \{v_j\}\right).$$

Again, $\{v_2\}$ is a hole cover of G_2 satisfying the NC property. Let $G_2^* = \widehat{G_2}(\{v_2\})$ and we repeat this process until we obtain the chordal graph $G_l^* = \widehat{G_l}(\{v_l\})$ as a desired graph G^*. Then G_l^* is a chordal completion of G. We note that if G is chordal, then $G = G_l^*$. Now we have shown the following theorem.

In the rest of this chapter, for the notation $\bigcup_{j=p}^{q} S_j$ of a finite union of sets, we assume that it refers to an empty set if $p > q$.

Theorem 3.3.1. *Let G be a graph with a hole cover $\mathcal{C}$. Then $\mathcal{C}$ can be partitioned into $\mathcal{C}_1, \ldots, \mathcal{C}_k$ for some positive integer k so that*

(i) $\mathcal{C}_i$ is a hole cover of the graph G_i satisfying the NC property,

(ii) G_i^* is chordal,

where $G_0 = G_0^* = G - \mathcal{C}$; G_i is the graph defined by $V(G_i) = V(G_{i-1}^*) \cup \mathcal{C}_i$,

$$E(G_i) = E(G_{i-1}^*) \cup E\left(G - \bigcup_{j=i+1}^{k} \mathcal{C}_j\right),$$

and $G_i^* = \widehat{G_i}(\mathcal{C}_i)$ for each $i = 1, \ldots, k$.

Let G be a graph with a hole cover $\mathcal{C}$. We call an ordered partition $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ of a hole cover $\mathcal{C}$ satisfying the conditions (i) and (ii) in Theorem 3.3.1 a *local chordalization partition* of $\mathcal{C}$. Then the graphs G_i, G_i^* are uniquely determined by the given local chordalization partition $\tilde{\mathcal{C}} := (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ of $\mathcal{C}$. We call the process of obtaining G_i and G_i^* the *chordalization chain* corresponding to $\tilde{\mathcal{C}}$. Especially, we write the process of obtaining G_i from G_{i-1}^* as $G_{i-1}^* <_{\mathcal{C}_i} G_i$ (in the context that G_{i-1}^* is a proper subgraph of G_i, we use "strictly less" notation) for $i = 1, \ldots, k$. Then the chordalization chain corresponding to $\tilde{\mathcal{C}}$ may be represented as

$$G_0 = G_0^* <_{\mathcal{C}_1} G_1 \leq G_1^* <_{\mathcal{C}_2} G_2 \leq G_2^* < \cdots <_{\mathcal{C}_k} G_k \leq G_k^*.$$

We note that G_k^* is a chordal completion of G. By the way, the last chordal completion in the chordalization chain corresponding to $\tilde{\mathcal{C}}$ is a minimal chordal spanning supergraph of G.

Proposition 3.3.2. *Let G be a graph, $\tilde{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_\ell)$ be a local chordalization partition of a hole cover $\mathcal{C}$ of G, and G^* be the last graph in the* *chordalization chain corresponding to $\tilde{\mathcal{C}}$. Then G^* is a minimal chordal completion of G.*

Proof. Let H be a graph that is a spanning supergraph of G and a proper subgraph of G^*. Then $E(G^*) \setminus E(H) \neq \emptyset$. By definition, each edge of $E(G^*) \setminus E(H)$ is incident to one of vertices in $\mathcal{C}$. Let s be the smallest index such that some vertices in $\mathcal{C}_s$ are incident to edges in $E(G^*) \setminus E(H)$. Now let B be the set of edges in $E(G^*) \setminus E(H)$ which are incident to vertices in $\mathcal{C}_s$. By the definition of local chordalization, $G_s^* - B$ is not chordal. Thus there exists a hole C in $G_s^* - B$. By the choice of s, the edges in $E(G^*) \setminus E(G_s^*)$ are incident to vertices in $\bigcup_{j=s+1}^{\ell} \mathcal{C}_j$. By definition, $(\bigcup_{j=s+1}^{\ell} \mathcal{C}_j) \cap V(G_s^*) = \emptyset$. Since $V(G_s^*) = V(G_s^* - B)$, the edges in $E(G^*) \setminus E(G_s^*)$ cannot be chords of C. Since $E(H) \subset E(G^*)$, the edges in $E(H) \setminus E(G_s^*)$ cannot be chords of C. Therefore C is a hole in H and so H is not chordal. Hence we have shown that G^* is a minimal chordal completion of G. $\square$

Now we are ready to introduce a parameter of a graph which measures the number of steps of adding new edges to reach one of its chordal completion.

Definition 3.3.3. The *non-chordality index* of a graph G, denoted by $i(G)$, is defined as follows: If G is chordal, $i(G) = 0$. If G is not chordal, then $i(G)$ is defined to be the smallest k over all the hole covers of G in Theorem 3.3.1.

Remark 3.3.4. A graph G satisfies the NC property if and only if G satisfies $i(G) \leq 1$.

Example 3.3.5. We consider the graph G given in Figure 3.1. Since G does not satisfy the NC property, $i(G) \geq 2$ by Remark 3.3.4. It is easy to check that $\mathcal{C} = \{u, v, w, x\}$ is a hole cover of G. See Figure 3.3 for an illustration. Since G_2^* is a chordal completion of G, $i(G) \leq 2$. Thus $i(G) = 2$.

In this section, we prove the following statement.

Theorem 3.3.6. *For any graph G, $\chi_{DP}(G) \leq \omega(G) + i(G)$. Especially, if G is non-chordal and K_n-minor-free, then $\chi_{DP}(G) \leq n - 2 + i(G)$.*

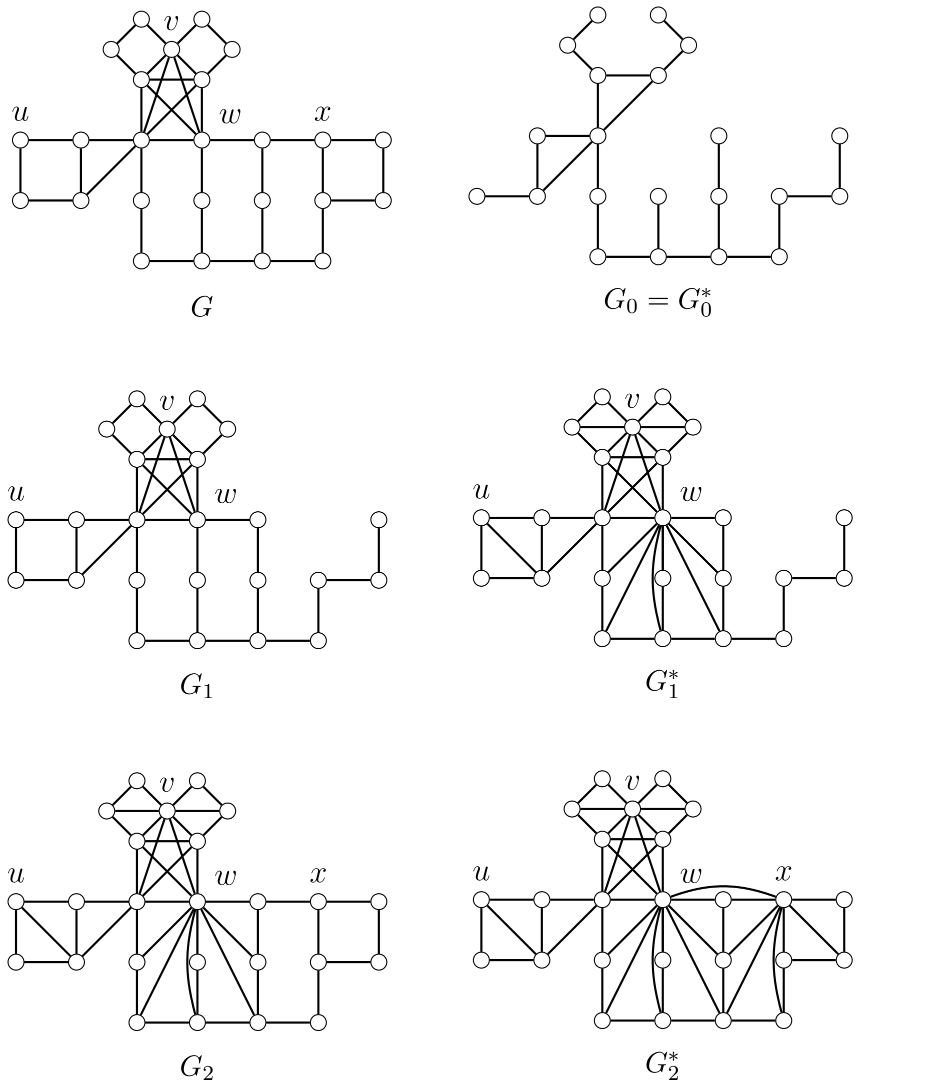


Figure 3.3: A chordalization chain $G_0 = G_0^* <_{\{u,v,w\}} G_1 \leq G_1^* <_{\{x\}} G_2 \leq G_2^*$ for a local chordalization partition $\tilde{\mathcal{C}} = (\{u, v, w\}, \{x\})$ of G

In order to do that, we show the following theorem first.

Theorem 3.3.7. *Let G be a graph, $\tilde{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_\ell)$ be a local chordalization partition of a hole cover $\mathcal{C}$ of G, and G^* be the last graph in the chordalization chain corresponding to $\tilde{\mathcal{C}}$. If a vertex set K of G forms a clique in G^*, then there exists a subset $\mathcal{C}^*$ of $K \cap \mathcal{C}$ such that $K \setminus \mathcal{C}^*$ is a clique in G and $|\mathcal{C}^* \cap \mathcal{C}_i| \leq 1$ for each $i = 1, \ldots, \ell$.*

Proof. Let

$$G_0 = G_0^* <_{\mathcal{C}_1} G_1 \leq G_1^* <_{\mathcal{C}_2} G_2 \leq G_2^* < \cdots <_{\mathcal{C}_\ell} G_\ell \leq G_\ell^* = G^*$$

be the chordalization chain corresponding to $\tilde{\mathcal{C}}$ for graphs G_i and chordal graphs G_i^*. Then $\mathcal{C}_i$ is a hole cover of the graph G_i satisfying the NC property for each $i = 1, \ldots, \ell$. For each $i = 0, 1, \ldots, \ell$, we add the vertices in $\bigcup_{j=i+1}^{\ell} \mathcal{C}_j$ to G_i^* and then restore the edges in G to obtain H_i, that is, H_i is the spanning supergraph of G with the edge set $E(G) \cup E(G_i^*)$. Then, by the definitions of G_i and G_i^*, $H_\ell = G_\ell^*$ and, for each $i = 0, \ldots, \ell - 1$,

$$H_i - \bigcup_{j=i+1}^{\ell} \mathcal{C}_j = G_i^*, \quad H_i - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j = G_{i+1},$$

and, since $G_{i+1}^* = \widehat{G_{i+1}}(\mathcal{C}_{i+1})$,

$$H_{i+1} - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j = \widehat{\left(H_i - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j\right)}(\mathcal{C}_{i+1}). \tag{3.3.1}$$

We claim that if L is a clique in H_{i+1} but is not a clique in H_i, then $L \setminus \{u\}$ is a clique in H_i for some vertex $u \in L \cap \mathcal{C}_{i+1}$ for each $i = 0, 1, \ldots, \ell - 1$. Suppose L is a clique in H_{i+1} but not a clique in H_i for some $i \in \{0, 1, \ldots, \ell - 1\}$. Then $L^* := L \setminus \bigcup_{j=i+2}^{\ell} \mathcal{C}_j$ is a clique in $H_{i+1} - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j$. Since two vertices joined by an edge in H_{i+1} but not in H_i belong to $V(G_{i+1}^*) = V(G) \setminus \bigcup_{j=i+2}^{\ell} \mathcal{C}_j$,

L^* is not a clique in $H_i - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j$. We note that (3.3.1) holds and $\mathcal{C}_{i+1}$ is a hole cover of $G_{i+1} = H_i - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j$ satisfying the NC property. Thus, by Theorem 3.1.8, there exists a vertex u in $L^* \cap \mathcal{C}_{i+1}$ such that $L^* \setminus \{u\}$ is a clique in $H_i - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j$. For the same reason why L^* is not a clique in $H_i - \bigcup_{j=i+2}^{\ell} \mathcal{C}_j$, $L \setminus \{u\}$ is still a clique in H_i.

Now we take a clique $L_0 := K$ in H_ℓ. For $i = 0, \ldots, \ell - 1$, we sequentially obtain a clique L_{i+1} in $H_{\ell-i-1}$ in the following way. If L_i is a clique in $H_{\ell-i-1}$, then we let $L_{i+1} = L_i$. If L_i is not a clique in $H_{\ell-i-1}$, then, by the claim which has been proven above, there exists a vertex $u \in L_i \cap \mathcal{C}_{\ell-i}$ such that $L_i \setminus \{u\}$ is a clique in $H_{\ell-i-1}$ and we let $L_{i+1} = L_i \setminus \{u\}$. Let $\mathcal{C}^* = K \setminus L_\ell$. Then $K \setminus \mathcal{C}^*$ equals L_ℓ and so is a clique as L_ℓ is a clique in $H_0 = G$. Moreover, since at most one vertex in $\mathcal{C}_{\ell-i}$ was deleted to obtain L_{i+1} from L_i, we have $\mathcal{C}^* \subset \mathcal{C}$ and $|\mathcal{C}^* \cap \mathcal{C}_i| \leq 1$ for each $i = 1, \ldots, \ell$, which completes the proof. $\square$

Theorem 3.3.8. *Let G be a graph, $\tilde{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_{i(G)})$ be a local chordalization partition of a hole cover $\mathcal{C}$ of G, and G^* be the last graph in the chordalization chain corresponding to $\tilde{\mathcal{C}}$. Then, for an induced subgraph H of G, $\omega(H^*) \leq \omega(H) + i(G)$ where H^* is the subgraph of G^* induced by $V(H)$. Especially, if G is non-chordal and K_n-minor-free, then $\omega(G^*) \leq n - 2 + i(G)$.*

Proof. If G is chordal, then the first part of the statement is immediately true as we may take G as G^* and the second statement is vacuously true. Thus we may assume G is non-chordal. Then $\ell := i(G) \geq 1$. Let

$$G_0 = G_0^* <_{\mathcal{C}_1} G_1 \leq G_1^* <_{\mathcal{C}_2} G_2 \leq G_2^* < \cdots <_{\mathcal{C}_\ell} G_\ell \leq G_\ell^* = G^*.$$

be the chordalization chain corresponding to $\tilde{\mathcal{C}}$. Clearly H^* is a chordal completion of H. Let K be a maximum clique of H^*. Then K is a clique in G^*. By Theorem 3.3.7, there exists a subset $\mathcal{C}^*$ of $K \cap \mathcal{C}$ such that $K \setminus \mathcal{C}^*$ is a clique in G and $|\mathcal{C}^* \cap \mathcal{C}_i| \leq 1$ for each $i = 1, \ldots, \ell$. Then

$$|\mathcal{C}^*| = \left|\mathcal{C}^* \cap \bigcup_{j=1}^{\ell} \mathcal{C}_j\right| \leq \sum_{j=1}^{\ell} |\mathcal{C}^* \cap \mathcal{C}_j| \leq \ell.$$

Now we note that $K \setminus \mathcal{C}^*$ is a clique in G, $K \subset V(H)$, and H is an induced subgraph of G. Thus $K \setminus \mathcal{C}^*$ is a clique in H and so $|K \setminus \mathcal{C}^*| \leq \omega(H)$. Therefore $\omega(H^*) = |K| \leq |K \setminus \mathcal{C}^*| + |\mathcal{C}^*| \leq \omega(H) + \ell$ and so the first statement is true.

To show the "especially" part, assume that G is K_n-minor-free. Let Z be the graph with the vertex set $V(G)$ and the edge set $E(G) \cup E(G_1^*)$. Then $\bigcup_{j=2}^{\ell} \mathcal{C}_j$ is a hole cover of Z and $(\mathcal{C}_2, \ldots, \mathcal{C}_\ell)$ is a local chordalization partition of $\bigcup_{j=2}^{\ell} \mathcal{C}_j$. By the definition of non-chordality index, $i(Z) \leq \ell - 1$. Let

$$Z_0 = Z_0^* <_{\mathcal{C}_2} Z_1 \leq Z_1^* <_{\mathcal{C}_3} Z_2 \leq Z_2^* < \cdots <_{\mathcal{C}_\ell} Z_{\ell-1} \leq Z_{\ell-1}^*$$

be the chordalization chain corresponding to $(\mathcal{C}_2, \ldots, \mathcal{C}_\ell)$. By the way, $Z_0 = Z_0^* = G_1^*$, $Z_i = G_{i+1}$ and $Z_i^* = G_{i+1}^*$ for $i = 1, \ldots, \ell - 1$. To reach a contradiction, suppose that Z has a clique L of size n. Then $L^* := L \setminus \bigcup_{j=2}^{\ell} \mathcal{C}_j$ is a clique in G_1^*. By the definition of Z, the edges in L but not in L^* belong to G. Since $\mathcal{C}_1$ is a hole cover of G_1 satisfying the NC property, by Theorem 3.1.10, L^* is a minor of G_1 as $G_1^* = \widehat{G_1}(\mathcal{C}_1)$. As G_1 is a subgraph of G and the edges in L but not in L^* belong to G, we may conclude that L is a minor of G with size n, which is a contradiction. Therefore Z is K_n-free and so $\omega(Z) \leq n - 1$. Take a maximum clique K of G^*. If K is a clique of Z, then $\omega(G^*) = |K| \leq \omega(Z) \leq n - 1 \leq n - 2 + i(G)$ and so the inequality holds. Suppose that K is not a clique of Z. By Theorem 3.3.7, there exists a subset $\mathcal{C}^{**}$ of $K \cap \left(\bigcup_{j=2}^{\ell} \mathcal{C}_j\right)$ such that $K \setminus \mathcal{C}^{**}$ is a clique in Z and $|\mathcal{C}^{**} \cap \mathcal{C}_i| \leq 1$ for each $i = 2, \ldots, \ell$. Then

$$|\mathcal{C}^{**}| = \left|\mathcal{C}^{**} \cap \bigcup_{j=2}^{\ell} \mathcal{C}_j\right| \leq \sum_{j=2}^{\ell} |\mathcal{C}^{**} \cap \mathcal{C}_j| \leq \ell - 1.$$

Thus

$$n - 1 \geq \omega(Z) \geq |K \setminus \mathcal{C}^{**}| \geq |K| - |\mathcal{C}^{**}| \geq \omega(G^*) - (\ell - 1)$$

and the "especially" part is true. $\square$

A proof of Theorem 3.3.6. Take a graph G and let G^* be a chordal completion of G given in Theorem 3.3.8. Then, since G^* is chordal, $\chi_{DP}(G^*) = \omega(G^*)$ by (§). Thus, by Theorem 3.3.8,

$$\chi_{DP}(G) \leq \chi_{DP}(G^*) = \omega(G^*) \leq \omega(G) + i(G)$$

and, if G is non-chordal and K_n-minor-free, then the right hand side of the second inequality above may be replaced with $n - 2 + i(G)$. $\square$

By (1.2.2), Theorem 3.3.6 gives $\chi_l(G) \leq \omega(G) + i(G)$ for a graph G and $\chi_l(G) \leq n-2+i(G)$ if G is non-chordal and K_n-minor-free. Actually, the inequality $\chi_l(G) \leq \omega(G)+i(G)$ is sharp and accordingly so is the first inequality given in Theorem 3.3.6. To show it, we need the following proposition.

Given a graph G, we denote the independence number and the vertex cover number of G by $\alpha(G)$ and $\beta(G)$, respectively. It is well known that $\alpha(G) + \beta(G) = |V(G)|$.

Proposition 3.3.9. *Every graph G is $\beta(G)$-degenerate.*

Proof. Take a graph G. Let I be an independent set of G with size $\alpha(G)$. Take a subgraph H of G. Suppose $V(H) \cap I \neq \emptyset$. Then, as I is an independent set of G, $V(H) \cap I$ is an independent set of H. Thus any vertex in $V(H) \cap I$ has degree at most $|V(H) \setminus I| \leq |V(G) \setminus I| = \beta(G)$. If $V(H) \cap I = \emptyset$, then $|V(H)| \leq |V(G) \setminus I| = \beta(G)$, and so any vertex of H has degree at most $\beta(G) - 1$. Hence G is $\beta(G)$-degenerate. $\square$

We recall that if a graph G is k-degenerate, then $\chi_{DP}(G) \leq k+1$, from which the corollary below is immediately true. As a matter of fact, the corollary enhances the known inequality $\chi(G) \leq \beta(G) + 1$ for a graph G.

Corollary 3.3.10. *For a graph G, $\chi_{DP}(G) \leq \beta(G) + 1$.*

Consider a complete graph K_n with $n \geq 2$. Then $\alpha(K_n) = 1$, $\beta(K_n) = |V(K_n)| - 1$, and $\chi(K_n) = \chi_l(K_n) = \chi_{DP}(K_n) = |V(K_n)| = \beta(K_n)+1$. Hence the upper bound for $\chi_{DP}(K_n)$ in Corollary 3.3.10 is sharp.

For a complete graph K_n with $n \geq 2$, the inequality given in Corollary 3.3.10 is sharp even for $\chi(K_n)$ and $\chi_l(K_n)$ as we have seen above. Yet, it is not necessarily in that way as it is known that $\beta(C_4) = 2$, $\chi(C_4) = \chi_l(C_4) = 2 < \beta(C_4) + 1$, and $\chi_{DP}(C_4) = 3 = \beta(C_4) + 1$.
Now we are ready to present the following theorem, which implies that the inequality $\chi_l(G) \leq \omega(G) + i(G)$ is sharp (and so $\chi_{DP}(G) \leq \omega(G) + i(G)$ is sharp).

Theorem 3.3.11. *For a positive integer s and a nonnegative integer t, there is a graph G with $\chi(G) = \omega(G) = s + 1$, $i(G) = t$, and $\chi_l(G) = s + t + 1$.*

Proof. If $t = 0$, then we let $G = K_{s+1}$. Suppose $t \geq 1$. We may represent t as the sum of s nonnegative integers, that is, $t = \sum_{i=1}^{s} m_i$ for nonnegative integers $m_1, m_2, \ldots, m_s$. Let G be a graph isomorphic to $K_{1+m_1,1+m_2,\ldots,1+m_s,m}$ where $m = (s+t)^{(s+t)}$. Let V_1, V_2, …, V_s, and V_{s+1} be the partite sets of G with $|V_i| = m_i + 1$ for $i = 1$, …, s and $|V_{s+1}| = m$. Now we take a vertex v_i from V_i for $i = 1$, …, s. Then $\mathcal{C} := \bigcup_{i=1}^{s} (V_i \setminus \{v_i\})$ is a hole cover of G with size $\sum_{i=1}^{s} m_i = t$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t$ be all the singleton subsets of $\mathcal{C}$. Then it is easy to check that $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t)$ is a local chordalization partition of $\mathcal{C}$. Thus $i(G) \leq t$.

On the other hand, it is obvious that $\omega(G) = s + 1$. Then, as it is easy to check that a complete multipartite graph is perfect,

$$\chi(G) = \omega(G) = s + 1.$$

Since $|V_{s+1}| = m$ and $|V(G) \setminus V_{s+1}| = \sum_{i=1}^{s}(1 + m_i) = s + t$,

$$\chi_{DP}(G) \leq s + t + 1 \tag{3.3.2}$$

by Corollary 3.3.10. In addition, $\bigcup_{i=1}^{s} V_i$ and V_{s+1} form two disjoint vertex sets of G with sizes $s+t$ and $(s+t)^{(s+t)}$, respectively, so G contains $K_{s+t,(s+t)^{s+t}}$ as a subgraph. Then, from the observation made by Gravier [21] that $\chi_l(K_{k,k^k}) > k$ for any positive integer k, we obtain

$$\chi_l(G) \geq s + t + 1. \tag{3.3.3}$$

Thus, by (1.2.2), (3.3.2), and (3.3.3), $s+t+1 \leq \chi_l(G) = \chi_{DP}(G) \leq s+t+1$ and so $\chi_l(G) = s+t+1$. Since $\omega(G) = s+1$, $i(G) \geq t$ by Theorem 3.3.6. As we have shown that $i(G) \leq t$, we complete the proof. $\square$

It is worthy of attention that Theorem 3.3.11 guarantees the existence of a graph G with $i(G) = t$ for any nonnegative integer t.

We recall that $\omega(G) \leq \chi(G) \leq \chi_l(G) \leq \chi_{DP}(G)$ for a graph G and that the gaps between $\omega(G)$ and $\chi(G)$, between $\chi(G)$ and $\chi_l(G)$, and between $\chi_l(G)$ and $\chi_{DP}(G)$ can be arbitrarily large. Yet, Theorem 3.3.6 tells us that the sum of those gaps cannot exceed $i(G)$. Especially, if G satisfies the NC property, then those gaps cannot exceed one and at most one of them can be one.

3.3.2 Making a local chordalization really local

In this section, we devote ourselves to convincing readers that the "local" in our terminology "local chordalization" makes a sense.

Let G be a non-chordal graph and $\Omega(G) = \bigcup_{H \in \mathcal{H}(G)} V(H)$. We define a relation $\sim_G$ on $\Omega(G)$ so that, for $u, v \in \Omega(G)$,

$u \sim_G v \Leftrightarrow$ either u and v are on the same hole or there exists a sequence $H_1, \ldots, H_t$ of distinct holes in $\mathcal{H}(G)$ such that $u \in H_1$, $v \in H_t$, and H_i and H_{i+1} share a vertex for each $i = 1, \ldots, t-1$.

It is easy to see that $\sim_G$ is an equivalence relation and that, for each hole in G, the vertices on the hole belong to the same equivalence class.

Proposition 3.3.12. *Let G be a non-chordal graph, H be a hole in G, and S be the equivalence class under $\sim_G$ containing $V(H)$. If adding a chord of H to G yields a new hole H^*, then $V(H^*) \subset S$.*

Proof. Since H is a hole, there are two nonadjacent vertices u and v on H. Suppose that adding the edge joining u and v to G creates a new hole H^*. Obviously uv is a chord of H in $G + uv$. Let x be a vertex in H^* other than u and v. It suffices to show $x \in S$ to complete the proof. If x is on H, then we are done. Thus we may assume that x is not on H.
Case 1. x is adjacent to an internal vertex of each of the two (u, v)-sections of H. Since u, v, and x are on the hole H^* with u and v consecutive on H^*, x is nonadjacent to one of u and v in $G + uv$. Without loss of generality, we may assume that x is nonadjacent to v in $G+uv$. Obviously x is nonadjacent to v in G. By applying Lemma 3.1.3 for $P = \{x\}$, there exists a hole in G containing x and v. Therefore $x \sim_G v$. Since $v \in S$, $x \in S$.
Case 2. One of the two (u, v)-sections of H has no internal vertex that is adjacent to x. Let R be such a (u, v)-section. Then none of x and its neighbors on H^* is an internal vertex on R. While traversing along the (x, v)-section (resp. (x, u)-section) of H^* not containing u (resp. v), let y (resp. z) be the first vertex at which we meet R. Let Q_1 be the (y, z)-section of H^* containing x, Q_2 be the (y, z)-section of R, and $Q = Q_1Q_2$. By the choices of y and z, Q is an induced cycle of G containing x and a vertex on H. Since two neighbors

of x on H^* are nonadjacent in G, Q is a hole in G. Since Q contains x and a vertex on H, $x \in S$. $\square$

Remark 3.3.13. Let G be a non-chordal graph and $\Omega(G)/{\sim_G}$ be the set of equivalence classes under $\sim_G$. Take an equivalence class $S \in \Omega(G)/{\sim_G}$, a hole H with $V(H) \subset S$, and vertices u and v on H which are not consecutive. Proposition 3.3.12 implies that the equivalence classes in $\Omega(G)/{\sim_G}$ except S are still equivalence classes under $\sim_{G+uv}$, and if there are other equivalence classes under $\sim_{G+uv}$, they are disjoint subsets of S. Therefore $\Omega(G+uv) \subset \Omega(G)$.

Remark 3.3.14. Let G be a non-chordal graph and $\ell = i(G)$. By the definition of $i(G)$, there exist a hole cover $\mathcal{C}$ of G and a local chordalization partition $\tilde{\mathcal{C}} = (\mathcal{C}_1, \dots, \mathcal{C}_\ell)$ of $\mathcal{C}$. Let

$$G_0 = G_0^* <_{\mathcal{C}_1} G_1 \leq G_1^* <_{\mathcal{C}_2} G_2 \leq G_2^* < \cdots <_{\mathcal{C}_\ell} G_\ell \leq G_\ell^* =: G^* \qquad (3.3.4)$$

be the chordalization chain corresponding to $\tilde{\mathcal{C}}$. Let H be the subgraph of G induced by $\Omega(G)$. Then, by the definition of induced subgraph, all the holes in H are contained in G. By the definition of $\Omega(G)$, all the holes in G are contained in H. Therefore $\mathcal{H}(G) = \mathcal{H}(H)$, $\Omega(G) = \Omega(H)$, and $\mathcal{C}$ is a hole cover of H. Thus the equivalence classes under $\sim_G$ are the equivalence classes under $\sim_H$. We recall that

$$G_0 = G_0^* = G - \mathcal{C}; \qquad (3.3.5)$$

$$V(G_i) = V(G_{i-1}^*) \cup \mathcal{C}_i, \ E(G_i) = E(G_{i-1}^*) \cup E\left(G - \bigcup_{j=i+1}^{\ell} \mathcal{C}_j\right); \qquad (3.3.6)$$

$$G_i^* = \widehat{G_i}(\mathcal{C}_i).$$

for each $i = 1, \dots, \ell$. Let $H_0 = H_0^* = H - \mathcal{C}$. Since H is an induced subgraph of G, H_0 is an induced subgraph of G_0 by (3.3.5). Furthermore, G_0, G_0^*, H_0, and H_0^* are chordal and so $\mathcal{H}(G_0^*) = \mathcal{H}(G_0) = \mathcal{H}(H_0) = \mathcal{H}(H_0^*) = \emptyset$ and

$\Omega(G_0^*) = \Omega(G_0) = \Omega(H_0) = \Omega(H_0^*) = \emptyset$. Let H_1 be the graph defined by $V(H_1) = V(H_0^*) \cup \mathcal{C}_1$ and

$$E(H_1) = E(H_0^*) \cup E\left(H - \bigcup_{j=2}^{\ell} \mathcal{C}_j\right).$$

Since H and H_0^* are induced subgraphs of G and G_0^*, respectively, H_1 is an induced subgraph of G_1 and $\mathcal{H}(H_1) \subset \mathcal{H}(G_1)$ by (3.3.6). Take a hole Ω_1 in G_1. Since G_1 is an induced subgraph of G, $V(\Omega_1) \subset \Omega(G) \setminus \bigcup_{i=2}^{\ell} \mathcal{C}_i$. Since $\Omega(G) = V(H)$ and $V(H) \setminus \bigcup_{i=2}^{\ell} \mathcal{C}_i = V(H_1)$, $V(\Omega_1) \subset V(H_1)$. Since H_1 is an induced subgraph of G, Ω_1 is a hole in H_1. Thus we have shown that $\mathcal{H}(H_1) = \mathcal{H}(G_1)$. Hence, since $\mathcal{C}_1$ is a hole cover of G_1 satisfying the NC property, it is a hole cover of H_1 satisfying the NC property and so we obtain $\widehat{H_1}(\mathcal{C}_1) =: H_1^*$. Since $\mathcal{H}(H_1) = \mathcal{H}(G_1)$ and H_1 is an induced subgraph of G_1, H_1^* is an induced subgraph of G_1^*. Let H_2 be the graph defined by $V(H_2) = V(H_1^*) \cup \mathcal{C}_2$ and

$$E(H_2) = E(H_1^*) \cup E\left(H - \bigcup_{j=3}^{\ell} \mathcal{C}_j\right).$$

Then $\Omega(G) \setminus \bigcup_{i=3}^{\ell} \mathcal{C}_i = V(H_2)$. Since H and H_1^* are induced subgraphs of G and G_1^*, respectively, H_2 is an induced subgraph of G_2 and $\mathcal{H}(H_2) \subset \mathcal{H}(G_2)$ by (3.3.6). Take a hole Ω_2 in G_2. Since G_1^* is chordal, Ω_2 must contain a vertex v in $\mathcal{C}_2$. By the way, since $\mathcal{C}_2$ is a hole cover of G_2 satisfying the NC property, Ω_2 contains exactly one vertex in $\mathcal{C}_2$ and so v is the only vertex on Ω_2 that is contained in $\mathcal{C}_2$.

Since G is non-chordal, there exist a hole in G. The chain given in (3.3.4) is the shortest, one of the holes in G must be in G_1. Thus there exists an edge in $E(G_1^*) \setminus E(G_1)$. Take an edge e in $E(G_1^*) \setminus E(G_1)$. Then there is a hole in G such that e is its chord in $G+e$. By Proposition 3.3.12, $\Omega(G+e) \subset \Omega(G)$.

If $E(G_1^*) \setminus E(G_1) = \{e\}$, then, by Proposition 3.3.12, $V(\Omega_2) \subset \Omega(G_2) \subset$

$\Omega(G+e) \subset \Omega(G)$ and so $V(\Omega_2) \subset \Omega(G)$. Suppose that $E(G_1^*)\setminus(E(G_1)\cup\{e\}) \neq \emptyset$ and take an edge e' in $E(G_1^*) \setminus (E(G_1) \cup \{e\})$. Then there is a hole C in G such that e' is its chord in $G + e'$. Now there is a hole in $G + e$ such that e' is its chord in $G \cup \{e, e'\}$. For, if C is a hole in $G + e$, then it is such a hole. Otherwise, by the definition of local chordalization, e is a chord of C and e' is a chord of a hole from $C + e$.

By applying Proposition 3.3.12 for $G + e$ and an edge e', $\Omega(G \cup \{e, e'\}) \subset \Omega(G)$. We may repeat this argument to conclude that $\Omega(G\cup(E(G_1^*)\setminus E(G_1))) \subset \Omega(G)$. Since G_2 is an induced subgraph of $G \cup (E(G_1^*) \setminus E(G_1))$ and Ω_2 is a hole in G_2,

$$V(\Omega_2) \subset \Omega(G_2) \subset \Omega(G \cup (E(G_1^*) \setminus E(G_1))) \subset \Omega(G),$$

and so $V(\Omega_2) \subset \Omega(G)$. Therefore we have shown that $V(\Omega_2) \subset \Omega(G)$ whether or not $E(G_1^*) \setminus (E(G_1) \cup \{e\}) \neq \emptyset$. Thus the vertices on Ω_2 belong to $\Omega(G) \setminus \bigcup_{i=3}^{\ell} \mathcal{C}_i$. Since $\Omega(G) \setminus \bigcup_{i=3}^{\ell} \mathcal{C}_i = V(H_2)$ and H_2 is an induced subgraph of G_2, Ω_2 is a hole in H_2 and so $\mathcal{H}(G_2) \subset \mathcal{H}(H_2)$. Thus $\mathcal{H}(G_2) = \mathcal{H}(H_2)$. Hence, since $\mathcal{C}_2$ is a hole cover of G_2 satisfying the NC property, it is a hole cover of H_2 satisfying the NC property and so we obtain $\widehat{H_2}(\mathcal{C}_2) =: H_2^*$. We may repeat this process to obtain $H_3, H_3^*, \ldots, H_\ell, H_\ell^*$ such that

$$\begin{aligned} &V(H_i) = V(H_{i-1}^*) \cup \mathcal{C}_i, \ E(H_i) = E(H_{i-1}^*) \cup E\left(H - \bigcup_{j=i+1}^{\ell} \mathcal{C}_j\right), \\ &H_i^* = \widehat{H}_i(\mathcal{C}_i), \end{aligned}$$

and $\mathcal{H}(G_i) = \mathcal{H}(H_i)$ for $i = 3, \ldots, \ell$. Noting that $\mathcal{H}(G) = \mathcal{H}(H)$ and G_ℓ^* (resp. H_ℓ^*) is a chordal completion of G (resp. H), we may conclude that $i(H) \leq \ell = i(G)$.

To show that $i(G) \leq i(H)$, we need to introduce the chordalization chain corresponding to a local chordalization partition $\tilde{\mathcal{C}}'$ of a hole cover $\mathcal{C}'$ of H terminating at $H_{i(H)}^*$. By mimicking the previous argument constructing

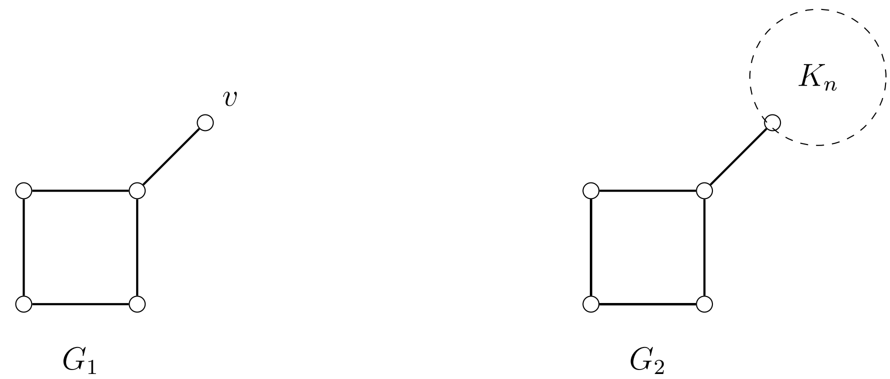


Figure 3.4: $\Omega(G_1) = \Omega(G_2)$, so $i(G_1) = i(G_2)$ by the argument given in Remark 3.3.14.

the chordalization chain corresponding to $\tilde{\mathcal{C}}$ for H, we may construct the chordalization chain corresponding to $\tilde{\mathcal{C}}'$ for G to conclude $i(G) \leq i(H)$. Thus $i(G) = i(H)$ and it is sufficient to apply local chordalization process to the induced subgraph H of G, which is a *local* structure, to obtain a desired chordal completion of G. In this vein, we may claim that the "local" in our terminology "local chordalization" is meaningful in another respect.

Example 3.3.15. The graph G_2 in Figure 3.4 is obtained from G_1 by replacing the vertex v of G_1 by the complete graph K_n. Then $\Omega(G_1) = \Omega(G_2)$. By the argument given in Remark 3.3.14, $i(G_1) = i(G_2)$. Yet, the treewidths of G_1 and G_2 are 2 and $n - 1$, respectively.

By the argument given in Remark 3.3.13, the following proposition is true.

Proposition 3.3.16. *For a non-chordal graph G,*

$$i(G) = \max\{i(G[S_1]), \ldots, i(G[S_r])\}$$

where $S_1, \ldots, S_r$ are the equivalence classes under $\sim_G$.

Proof. Let G be a graph and $\tilde{\mathcal{C}} = (\mathcal{C}_1, \ldots, \mathcal{C}_{i(G)})$ be a local chordalization partition of a hole cover $\mathcal{C}$ of G. Then, for each $j = 1, \ldots, i(G)$, $\mathcal{C} \cap S_j$ is a hole cover of $G[S_j]$. In addition, by the argument given in Remark 3.3.13, a subset of $\{\mathcal{C}_1 \cap S_j, \ldots, \mathcal{C}_{i(G)} \cap S_j\}$ forms a local chordalization partition of $\mathcal{C} \cap S_j$. Thus $i(G[S_j]) \leq i(G)$ for each $j = 1, \ldots, i(G)$ and so $i(G) \geq \max\{i(G[S_1]), \ldots, i(G[S_r])\}$.

Now let $\tilde{\mathcal{C}^j} = (\mathcal{C}^j_1, \ldots, \mathcal{C}^j_{i(G[S_j])})$ be a local chordalization partition of a hole cover $\mathcal{C}^j$ of $G[S_j]$ for each $j = 1, \ldots, r$. Clearly $\bigcup_{j=1}^r \mathcal{C}^j$ is a hole cover of G. In addition, by the argument given in Remark 3.3.13,

$$(\bigcup_{j=1}^r \mathcal{C}^j_1, \bigcup_{j=1}^r \mathcal{C}^j_2, \ldots, \bigcup_{j=1}^r \mathcal{C}^j_{\max\{i(G[S_1]),\ldots,i(G[S_r])\}})$$

is a local chordalization partition of $\bigcup_{j=1}^r \mathcal{C}^j$ where $C^j_p = \emptyset$ for any $j = 1, \ldots, r$ and any p, $i(G[S_j]) < p \leq \max\{i(G[S_1]), \ldots, i(G[S_r])\}$. Hence

$$\max\{i(G[S_1]), \ldots, i(G[S_r])\} \geq i(G).$$

□

The *join*, denoted by $G_1 \vee G_2$, of two graphs G_1 and G_2 is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$. We denote by I_m an empty graph with m vertices.

Theorem 3.3.17. *Suppose that a non-chordal graph G does not contain $I_m \vee K_n$ for positive integers $m \geq n$ as a subgraph and $\omega(G[\Omega(G)]) + i(G) \leq m$. Then there is a chordal completion G^* of G with $\omega(G^*) < m + n$.*

Proof. Since G is non-chordal, $i(G) \geq 1$. Let H be the subgraph of G induced by $\Omega(G)$. By the argument in Remark 3.3.14, $i(G) = i(H)$. Let H^* be the subgraph of G^* induced by $\Omega(G)$ where G^* is a chordal completion of G obtained in Remark 3.3.14. Then H^* is a chordal completion of H.

Suppose to the contrary that $\omega(G^*) \geq m+n$. Then there is a clique K of size $m+n$ in G^*. Clearly $K \cap \Omega(G)$ forms a clique in G^*. Since H^* is an induced subgraph of G^*, $K \cap \Omega(G)$ forms a clique in H^*. By Theorem 3.3.8, $|K \cap \Omega(G)| \leq \omega(H) + i(G)$. By the hypothesis, $|K \cap \Omega(G)| \leq m$. Since $|K| = m+n$, $|K \setminus \Omega(G)| \geq n$. By the definition of local chordalization and Remark 3.3.13, the end vertices of each of the edges newly added to obtain G^* belong to $\Omega(G)$, so $K \setminus \Omega(G)$ still forms a clique in G and each vertex in $K \cap \Omega(G)$ is adjacent to each vertex in $K \setminus \Omega(G)$ in G. By moving $m - |K \cap \Omega(G)|$ vertices in $K \setminus \Omega(G)$ into $K \cap \Omega(G)$ if $|K \cap \Omega(G)| < m$, we may claim that G contains $I_m \vee K_n$ as a subgraph. This contradicts the hypothesis, so we conclude that $\omega(G^*) < m+n$. □

The following corollary is an immediate consequence of Theorem 3.3.17.

Corollary 3.3.18. *Suppose a graph G does not contain $I_m \vee K_n$ for positive integers $m \geq n$ as a subgraph and $\omega(G[\Omega(G)]) + i(G) \leq m$. Then $\chi_{DP}(G) < m+n$.*

Remark 3.3.19. Since $K_{2,4}$ is non-chordal and has a hole cover which is a singleton, $i(K_{2,4}) = 1$. Then, by Theorem 3.3.6, $\chi_{DP}(K_{2,4}) \leq 3$. Yet, $\chi_{DP}(K_{2,4}) \leq 5$ by Corollary 3.3.18. Thus, for $\chi_{DP}(K_{2,4})$, Theorem 3.3.6 gives a better upper bound than Corollary 3.3.18.

On the other hand, for a certain graph G, Corollary 3.3.18 gives a better upper bound of $\chi_{DP}(G)$ than Theorem 3.3.6. To see why, consider the graph G given in Figure 3.5. If G contained a subgraph isomorphic to $I_8 \vee K_4$, then G would have at least four vertices with degree at least 11, which does not happen in G as the two vertices common to K_6 and K_{11} are the only vertices with degree at least 11. Hence G does not contain $I_8 \vee K_4$ as a subgraph.

It is easy to check that $\omega(G) = 11$ and $\omega(G[\Omega(G)]) = 5$. The graph $G[\Omega(G)]$ is represented by using bold edges in Figure 3.5 and happens to be the graph given in Figure 3.3. Therefore $i(G) = 2$. Then Theorem 3.3.6 gives rise to $\chi_{DP}(G) \leq 11 + 2$ while Corollary 3.3.18 gives rise to $\chi_{DP}(G) \leq 11$. Furthermore, since $\omega(G) = 11$, $\chi_{DP}(G)$ is actually equal to 11.

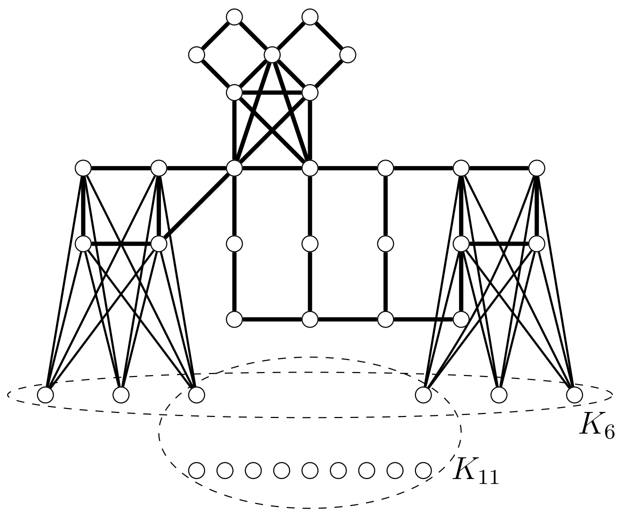


Figure 3.5: A graph G which shows that Theorem 3.3.17 may be regarded as an improvement of Theorem 3.3.8. The vertices enclosed by a dotted ellipse form a clique.

3.4 New χ-bounded classes

A class $\mathcal{F}$ of graphs is said to be *χ-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{F}$ and every induced subgraph H of G, $\chi(H) \leq f(\omega(H))$.

We may extend the notion of χ-boundedness as follows. A class $\mathcal{F}$ of graphs is said to be *χ_l-bounded* (resp. *χ_{DP}-bounded*) if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{F}$ and every induced subgraph H of G, $\chi_l(H) \leq f(\omega(H))$ (resp. $\chi_{DP}(H) \leq f(\omega(H))$).

A graph G is called *perfect graph* if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

We may also extend the notion of perfect graph as follows. We say that a graph G is *list-perfect* (resp. *DP-perfect*) if $\chi_l(H) = \omega(H)$ (resp. $\chi_{DP}(H) = \omega(H)$) for every induced subgraph H of G.

We denote the class of perfect graphs, the class of list-perfect graphs, and the class of DP-perfect graphs by $\mathcal{P}$, $\mathcal{P}_l$, and $\mathcal{P}_{DP}$, respectively.

By (1.2.2), a χ_{DP}-bounded graph class is χ_l-bounded and a χ_l-bounded graph class is χ-bounded. In the proof of Theorem 3.3.11, we have shown that for any positive integer s and any nonnegative integer t, there exist a complete multipartite graph G with $\omega(G) = s+1$ and $\chi_l(G) = s+t+1$, which implies that the class of complete multipartite graphs is not χ_l-bounded. Any complete multipartite graph is, however, perfect, which implies that the class of complete multipartite graphs is χ-bounded. Accordingly, a χ-bounded class is not necessarily χ_l-bounded. Furthermore, $\mathcal{P}_{DP} \subset \mathcal{P}_l \subset \mathcal{P}$ by (1.2.2). Yet, $\mathcal{P}_{DP} \subsetneq \mathcal{P}_l \subsetneq \mathcal{P}$ as $K_{2,4}$ is perfect but not list-perfect and C_4 is list-perfect but not DP-perfect.

Note that $\omega(C_n) = 2$ and $\chi_{DP}(C_n) = 3$ for even integer $n \geq 4$. Thus no graph in $\mathcal{P}_{DP}$ contains a hole of even length. Since a graph containing a hole of odd length is not perfect, no graph in $\mathcal{P}_{DP}$ contains a hole of odd length. Therefore $\mathcal{P}_{DP}$ is included in the class of chordal graphs. Thus, by (§), $\mathcal{P}_{DP}$ is the class of chordal graphs.

Now we present new χ-bounded classes.

Theorem 3.4.1. *A family of graphs the non-chordality index of each of which does not exceed k for some nonnegative integer k is linearly χ_{DP}-bounded.*

Proof. Take a family $\mathcal{F}$ of graphs the non-chordality index of each of which does not exceed k for a nonnegative integer k. Let $f : \mathbb{N} \to \mathbb{R}$ be a function defined by $f(x) = x + k$. Take a graph G in $\mathcal{F}$. Then $i(G) \leq k$. Let H be an induced subgraph of G. By the first part of Theorem 3.3.8, there exists a chordal completion H^* of H such that $\omega(H^*) \leq \omega(H) + i(G)$. Thus $\chi_{DP}(H) \leq \chi_{DP}(H^*) = \omega(H^*) \leq \omega(H)+i(G) \leq f(\omega(H))$. Hence the theorem is true. $\square$

By Remark 3.3.4, the following corollary is immediately true.

Corollary 3.4.2. *The class of graphs with the NC property is* χ_{DP}*-bounded.*

Bibliography

[1] Kenneth Appel and Wolfgang Haken. Every planar map is four colorable. *Bulletin of the American mathematical Society*, 82(5):711–712, 1976.

[2] Stefan Arnborg, Derek G Corneil, and Andrzej Proskurowski. Complexity of finding embeddings in ak-tree. *SIAM Journal on Algebraic Discrete Methods*, 8(2):277–284, 1987.

[3] Catriel Beeri, Ronald Fagin, David Maier, and Mihalis Yannakakis. On the desirability of acyclic database schemes. *Journal of the ACM (JACM)*, 30(3):479–513, 1983.

[4] Seymour Benzer. On the topology of the genetic fine structure. *Proceedings of the National Academy of Sciences*, 45(11):1607–1620, 1959.

[5] A Yu Bernshteyn, AV Kostochka, and SP Pron. On DP-coloring of graphs and multigraphs. *Siberian Mathematical Journal*, 58(1):28–36, 2017.

[6] Hans L Bodlaender. A tourist guide through treewidth. *Acta cybernetica*, 11(1-2):1, 1994.

[7] John Adrian Bondy and Uppaluri Siva Ramachandra Murty. *Graph theory, volume 244 of Graduate Texts in Mathematics.* Springer, New York, 2008.

[8] Jason I Brown, David Kelly, Johanan Schönheim, and Robert E Woodrow. Graph coloring satisfying restraints. *Discrete Mathematics*, 80(2):123–143, 1990.

[9] Fan RK Chung and David Mumford. Chordal completions of planar graphs. *Journal of Combinatorial Theory, Series B*, 62(1):96–106, 1994.

[10] Joel E Cohen. Interval graphs and food webs: a finding and a problem. *RAND Corporation Document*, 17696, 1968.

[11] AJ Cole. The preparation of examination time-tables using a small-store computer. *The Computer Journal*, 7(2):117–121, 1964.

[12] Gregory F Cooper. The computational complexity of probabilistic inference using bayesian belief networks. *Artificial intelligence*, 42(2-3):393–405, 1990.

[13] Dominique de Werra. An introduction to timetabling. *European journal of operational research*, 19(2):151–162, 1985.

[14] Kathryn A Dowsland. A timetabling problem in which clashes are inevitable. *Journal of the Operational Research Society*, 41(10):907–918, 1990.

[15] Ronald D Dutton and Robert C Brigham. A characterization of competition graphs. *Discrete Applied Mathematics*, 6(3):315–317, 1983.

[16] Zdeněk Dvořák and Luke Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *Journal of Combinatorial Theory, Series B*, 2017.

[17] Paul Erdos, Arthur L Rubin, and Herbert Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979.

[18] Kim A S Factor and Sarah K Merz. The $(1, 2)$-step competition graph of a tournament. *Discrete Applied Mathematics*, 159(2):100–103, 2011.

[19] Piotr Formanowicz and Krzysztof Tanaś. A survey of graph coloring-its types, methods and applications. *Foundations of Computing and Decision Sciences*, 37(3):223–238, 2012.

[20] Paul C Gilmore and Alan J Hoffman. A characterization of comparability graphs and of interval graphs. In *Selected Papers Of Alan J Hoffman: With Commentary*, pages 65–74. World Scientific, 2003.

[21] Sylvain Gravier. A hajós-like theorem for list coloring. *Discrete Mathematics*, 152(1-3):299–302, 1996.

[22] Hugo Hadwiger. Über eine klassifikation der streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88(2):133–142, 1943.

[23] György Hajós. Uber eine art von graphen. *Internationale mathematische nachrichten*, 1957.

[24] William K Hale. Frequency assignment: Theory and applications. *Proceedings of the IEEE*, 68(12):1497–1514, 1980.

[25] Ronald C Hamelink. A partial characterization of clique graphs. *Journal of Combinatorial Theory*, 5(2):192–197, 1968.

[26] Stephen Hartke. The elimination procedure for the phylogeny number. *Ars Combinatoria*, 75:297–312, 2005.

[27] Kim A S Hefner, Kathryn F Jones, Suh-ryung Kim, J Richard Lundgren, and Fred S Roberts. (i, j) competition graphs. *Discrete Applied Mathematics*, 32:241–262, 1991.

[28] Pinar Heggernes. Minimal triangulations of graphs: A survey. *Discrete Mathematics*, 306(3):297–317, 2006.

[29] Tommy R Jensen and Bjarne Toft. 25 pretty graph colouring problems. *Discrete Mathematics*, 229(1-3):167–169, 2001.

[30] Tommy R Jensen and Bjarne Toft. *Graph coloring problems*, volume 39. John Wiley & Sons, 2011.

[31] Akira Kamibeppu. A sufficient condition for kim's conjecture on the competition numbers of graphs. *Discrete Mathematics*, 312(6):1123–1127, 2012.

[32] Jaromy Kuhl. Transversals and competition numbers of complete multipartite graphs. *Discrete Applied Mathematics*, 161(3):435–440, 2013.

[33] Steffen L Lauritzen and David J Spiegelhalter. Local computations with probabilities on graphical structures and their application to expert systems. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 157–224, 1988.

[34] Seung Chul Lee, Jihoon Choi, Suh-Ryung Kim, and Yoshio Sano. On the phylogeny graphs of degree-bounded digraphs. *Discrete Applied Mathematics*, 233:83–93, 2017.

[35] Frank Thomson Leighton. A graph coloring algorithm for large scheduling problems. *Journal of research of the national bureau of standards*, 84(6):489–506, 1979.

[36] C Lekkeikerker and J Boland. Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae*, 51(1):45–64, 1962.

[37] Bo-Jr Li and Gerard J Chang. Competition numbers of complete r-partite graphs. *Discrete Applied Mathematics*, 160(15):2271–2276, 2012.

[38] NVR Mahadev and Fred S Roberts. *Amenable Colorings*. Rutgers University. Rutgers Center for Operations Research [RUTCOR], 1992.

[39] Enrico Malaguti and Paolo Toth. A survey on vertex coloring problems. *International transactions in operational research*, 17(1):1–34, 2010.

[40] Panos M Pardalos, Thelma Mavridou, and Jue Xue. The graph coloring problem: A bibliographic survey. In *Handbook of combinatorial optimization*, pages 1077–1141. Springer, 1998.

[41] Boram Park and Yoshio Sano. The phylogeny graphs of doubly partial orders. *Discussiones Mathematicae Graph Theory*, 33(4):657–664, 2013.

[42] Taehoon Park and Chae Y Lee. Application of the graph coloring algorithm to the frequency assignment problem. *Journal of the Operations Research society of Japan*, 39(2):258–265, 1996.

[43] Judea Pearl. Fusion, propagation, and structuring in belief networks. *Artificial Intelligence*, 29(3):241–288, 1986.

[44] Arundhati Raychaudhuri and Fred S Roberts. Generalized competition graphs and their applications. *Methods of Operations Research*, 49:295–311, 1985.

[45] Fred S Roberts. Food webs, competition graphs, and the boxicity of ecological phase space. In *Theory and Applications of Graphs*, pages 477–490. Springer, 1978.

[46] Fred S Roberts and Li Sheng. Phylogeny graphs of arbitrary digraphs. *Mathematical Hierarchies in Biology*, pages 233–238, 1997.

[47] Fred S Roberts and Li Sheng. Phylogeny numbers. *Discrete Applied Mathematics*, 87(1-3):213–228, 1998.

[48] Fred S Roberts and Li Sheng. Extremal phylogeny numbers. *Journal of Combinatorics, Information & System Sciences*, 24:143–149, 1999.

[49] Fred S Roberts and Li Sheng. Phylogeny numbers for graphs with two triangles. *Discrete Applied Mathematics*, 103(1):191–207, 2000.

[50] Fred S Roberts and Jeffrey E Steif. A characterization of competition graphs of arbitrary digraphs. *Discrete Applied Mathematics*, 6(3):323–326, 1983.

[51] FS Roberts. Competition graphs and phylogeny graphs. *Graph Theory and Combinatorial Biology, Bolyai Society Mathematical Studies*, 7:333–362, 1999.

[52] Neil Robertson and Paul D. Seymour. Graph minors. ii. algorithmic aspects of tree-width. *Journal of algorithms*, 7(3):309–322, 1986.

[53] Donald J Rose. A graph-theoretic study of the numerical solution of sparse positive definite systems of linear equations. In *Graph theory and computing*, pages 183–217. Elsevier, 1972.

[54] Gunther Schmidt and Thomas Ströhlein. Timetable construction–an annotated bibliography. *The Computer Journal*, 23(4):307–316, 1980.

[55] Ross D Shachter. Probabilistic inference and influence diagrams. *Operations Research*, 36(4):589–604, 1988.

[56] DH Smith. Graph colouring and frequency assignment. *Ars Combin.*, 25:205–212, 1988.

[57] Jeffrey E Steif. Frame dimension, generalized competition graphs, and forbidden sublist characterizations. *Henry Rutgers Thesis, Department of Mathematics, Rutgers University, New Brunswick, NJ*, 1982.

[58] Robert E Tarjan and Mihalis Yannakakis. Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM Journal on computing*, 13(3):566–579, 1984.

[59] Vadim G Vizing. Vertex colorings with given colors. *Diskret. Analiz*, 29:3–10, 1976.

[60] Robert James Waters. *Graph colouring and frequency assignment*. PhD thesis, University of Nottingham, 2005.

[61] De Werra. On a particular conference scheduling problem. *INFOR: Information Systems and Operational Research*, 13(3):308–315, 1975.

[62] Yaokun Wu, Yanzhen Xiong, and Soesoe Zaw. Competition numbers and phylogeny numbers. *To appear in Graphs and combinatorics.*

[63] Mihalis Yannakakis. Computing the minimum fill-in is np-complete. *SIAM Journal on Algebraic Discrete Methods*, 2(1):77–79, 1981.

[64] Xinhong Zhang and Ruijuan Li. The $(1, 2)$-step competition graph of a pure local tournament that is not round decomposable. *Discrete Applied Mathematics*, 205:180–190, 2016.

[65] Yongqiang Zhao and Wenjie He. Note on competition and phylogeny numbers. *Australasian Journal of Combinatorics*, 34:239–246, 2006.

국문초록

이 논문에서는 유향그래프와 그래프의 홀의 관점에서 계통발생 그래프와 그래프의 삼각화에 대하여 연구한다. 길이 4 이상인 유도된 싸이클을 홀이라 하고 홀이 없는 그래프를 삼각화된 그래프라 한다. 구체적으로, 싸이클을 갖지 않는 유향그래프의 계통발생 그래프가 삼각화된 그래프인지 판정하고, 주어진 그래프를 삼각화하여 클릭수가 크게 차이 나지 않는 그래프를 만드는 방법을 찾고자 한다. 이 논문은 연구 내용에 따라 두 부분으로 나뉜다.

먼저 $(1,i)$ 유향그래프와 $(i,1)$ 유향그래프의 계통발생 그래프를 완전하게 특징화하고, $(2,j)$ 유향그래프 D의 모든 유향변에서 방향을 제거한 그래프가 삼각화된 그래프이면, D의 계통발생 그래프 역시 삼각화된 그래프임을 보였다. 또한 적은 수의 삼각형을 갖는 연결된 그래프의 계통발생수를 계산한 정리를 확장하여 많은 수의 삼각형을 포함한 연결된 그래프의 계통발생수를 계산하였다.

다른 한 편 그래프 G의 비삼각화 지수 $i(G)$에 대하여 $\omega(G^*)-\omega(G) \le i(G)$를 만족하는 G의 삼각화된 그래프 G^*가 존재함을 보였다. 그리고 이를 도구로 이용하여 NC property를 만족하는 그래프가 Hadwiger 추측과 Erdős-Faber-Lovász 추측을 만족함을 증명하고, 비삼각화 지수가 유계인 그래프들이 linearly χ-bounded임을 증명하였다.

주요어휘: 홀, 삼각화된 그래프, 계통발생 그래프, 계통발생수, 비삼각화 지수, 채색수

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