creative commons

COMMONS DEED

저작자표시-비영리-변경금지 2.0 대한민국

이용자는 아래의 조건을 따르는 경우에 한하여 자유롭게

- 이 저작물을 복제, 배포, 전송, 전시, 공연 및 방송할 수 있습니다.

다음과 같은 조건을 따라야 합니다:

저작자표시. 귀하는 원저작자를 표시하여야 합니다.

비영리. 귀하는 이 저작물을 영리 목적으로 이용할 수 없습니다.

변경금지. 귀하는 이 저작물을 개작, 변형 또는 가공할 수 없습니다.

- 귀하는, 이 저작물의 재이용이나 배포의 경우, 이 저작물에 적용된 이용허락조건을 명확하게 나타내어야 합니다.
- 저작권자로부터 별도의 허가를 받으면 이러한 조건들은 적용되지 않습니다.

저작권법에 따른 이용자의 권리는 위의 내용에 의하여 영향을 받지 않습니다.

이것은 이용허락규약(Legal Code)을 이해하기 쉽게 요약한 것입니다.

Disclaimer

dCollection

교육학 석사 학위논문

The trees which are m-step competition graphs of digraphs with a source

(내차수가 0인 점을 갖는 유향 그래프의 m-step 경쟁 그래프인 수형도)

2019년 8월

서울대학교 대학원

수학교육과

최명호

The trees which are m-step competition graphs of digraphs with a source

(내차수가 0인 점을 갖는 유향 그래프의 m-step 경쟁 그래프인 수형도)

지도교수 김서령

이 논문을 교육학 석사 학위논문으로 제출함

2019년 4월

서울대학교 대학원

수학교육과

최명호

최명호의 교육학 석사 학위논문을 인준함

2019년 6월

위 원 장 ____________________________ (인)

부 위 원 장 ____________________________ (인)

위 원 ____________________________ (인)

The trees which are m-step competition graphs of digraphs with a source

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Mathematics Education
to the faculty of the Graduate School of
Seoul National University

by

Myungho Choi

Dissertation Director : Professor Suh-Ryung Kim

Department of Mathematics Education
Seoul National University

August 2019

Abstract

Myungho Choi
Department of Mathematics Education
The Graduate School
Seoul National University

Cohen [1] introduced the notion of competition graph while studying predator-prey concepts in ecological food webs. Among the variants of competition graphs, the notion of m-step competition graph to be studied in this thesis, was introduced by Cho *et al.* [2]. In 2000, Cho *et al.* [2] posed the following question: For which values of m and n is P_n an m-step competition graph? Helleloid [4] and Kuhl *et al.* [5] partially answered the question in 2005 and 2010, respectively. In 2011, Belmont [6] presented a complete characterization of paths that are m-step competition graphs.

In this thesis, we study "tree-inducing digraphs" with a source. We call a digraph D with at least three vertices an m-step tree-inducing digraph if the m-step competition graph of D is a tree for some integer $m \geq 2$. We say that a digraph is a tree-inducing digraph if it is an m-step tree-inducing digraph for some integer $m \geq 2$. We first completely characterize a tree-inducing digraph with a source. Interestingly, it turns out that if a tree is the m-step competition graph of a digraph with a source, then it is a star graph. We also compute the number of tree-inducing digraphs with a source.

Key words: tree; star graph; m-step competition graph; tree-inducing digraph; idle vertex.
Student Number: 2016–22901

Contents

Chapter 1

Introduction

1.1 Basic graph terminology

We introduce some basic notions in graph theory. For undefined terms, readers may refer to [8].

Let G be a graph. Two vertices u and v in G are called *adjacent* if there is an edge e in G which connects u and v. Then we say u and v are the *end vertices* of e. Two distinct edges are also called *adjacent* if they have a common end vertex.

Two graphs G and H are said to be *isomorphic* if there exist bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that for every edge $e \in E(G)$, e connects vertices u and v in G if and only if $\phi(e)$ connects vertices $\theta(u)$ and $\theta(v)$ in H. If G and H are isomorphic, then we write $G \cong H$.

Let G (resp. D) be a graph (resp. digraph). A graph H (resp. digraph E) is a *subgraph* (resp. *subdigraph*) of G (resp. D) if $V(H) \subset V(G)$ (resp. $V(D) \subset V(E)$), $E(H) \subset E(G)$ (resp. $A(D) \subset A(E)$), and we write $H \subset G$ (resp. $E \subset D$). The subgraph resp. digraph of G (resp. D) whose vertex set is X and whose edge set (resp. arc set) consists of all edges (resp. arcs) of G (resp. D) which have both ends in X is called the *subgraph* (resp. *subdigraph*) of G (resp. D) *induced by* X and is denoted by $G[X]$ (resp. $D[X]$). The subgraph induced by $V(G) \setminus X$ (resp. $V(D) \setminus X$) is denoted by $G - X$ (resp. $D - x$). For notational convenience, we write notion $G - v$ (resp. $D - v$) instead of $G - \{v\}$ (resp. $D - \{v\}$) for a vertex v in G (resp. D).

For a vertex v in a digraph D, the *outdegree* of v is the number of vertices D to which v is adjacent, while the *indegree* of v is the number of vertices of D from which v is adjacent. In a digraph D, we call a vertex with indegree 0 and outdegree at least 1 a *source* of D.

A *walk* in a graph G is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_l \in V(G)$ such that $v_{i-1}v_i \in E(G)$ for each $2 \leq i \leq l$ and is denoted by $v_1 \to v_2 \to \cdots \to v_l$. If the vertices in a walk are distinct, then the walk is called a *path*. A *cycle* in G is a path $v_1 \to v_2 \to \cdots \to v_k$ together with the edge $v_k v_1$ where $k \geq 3$.

A *directed walk* in a digraph D is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots v_l \in V(D)$ such that $(v_{i-1}, v_i) \in A(D)$ for each $2 \leq i \leq l$ and is denoted by $v_1 \to v_2 \to \cdots \to v_l$. If the vertices in a walk are distinct, then the walk is called a *directed path*. A *directed cycle* is a directed walk formed by a directed path $v_1 \to v_2 \to \cdots \to v_k$ and the arc (v_k, v_1) where $k \geq 1$.

A graph is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 so that every edge has one end in V_1 and the other end in V_2; such a partition $(V1, V2)$ is called a *bipartition* of the graph, and V_1 and V_2 are called its *parts*. If a bipartite graph is simple and every vertex in one part is joined to every vertex in the other part, then the graph is called a *complete bipartite graph*. We denote by $K_{m,n}$ a complete bipartite graph with bipartition $(V1, V2)$ if $|V_1| = m$ and $|V_2| = n$. Especially, $K_{1,n}$ is called a *star* graph for some positive integer n. A graph that contains no cycles at all is called *acyclic* and a connected acyclic graph is called a *tree*. It is obvious that each star graph is a tree.

1.2 Competition graph and its variants

Cohen [1] introduced the notion of competition graph while studying predator-prey concepts in ecological food webs. The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph, which has the same vertex set as D and has an edge between two distinct vertices u and v if the arcs (u, x) and (v, x) are in D for some vertex $x \in V(D)$. Cohen's empirical observation that real-world competition graphs are usually interval graphs had led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs. For a comprehensive introduction to competition graphs, see [15, 25]. Competition graphs also have applications in coding, regulation of radio transmission, and modeling of complex economic systems (see [29] and [30] for a summary of these applications). For recent work on this topic, see [14, 28, 34, 36].

A variety of generalizations of the notion of competition graph have also been introduced, including the m-step competition graph in [2, 4], the common enemy graph (sometimes called the resource graph) in [26, 33], the competition-common enemy graph (sometimes called the competition-resource graph) in [10, 17, 19, 21, 24, 31, 32], the niche graph in [11, 12, 16] and the p-competition graph in [9, 22, 23].

Lundgren and Maybee [26] introduced the common enemy graph. The *common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exists a common in-neighbor of u and v in D. Their study led Scott [31] to introduce the competition-common enemy graph of D. The *competition-common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exists a common in-neighbor and a common outneighbor of u and v in D. This graph is fundamentally the intersection of the competition graph and the common enemy graph. That is, two vertices are adjacent if and only if they have both a common prey and a common enemy in D. On the other hand, the niche graph is the union of the competition graph and the common enemy graph. The *niche graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exists a common in-neighbor or a common outneighbor of u and v in D. For a digraph D, let $CE(D)$ be the common enemy graph, $CCE(D)$ the competition-common enemy graph, and $N(D)$ the niche graph. From the definition of those graphs, we might obtain the relationship among them: $CCE(D) \subset C(D) \subset N(D)$. Another variant of competition graph, the *p-competition graph*, denoted by $C_p(D)$, of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices u and v if and only if there exist p common out-neighbors of u and v in D for a positive integer p. If D happens to be a food web whose vertices are species in some ecosystem with an arc (x, y) if and only if x preys on y, then xy is an edge of $C_p(D)$ if and only if x and y have at least p common prey. Among other variants, the notion of m-step competition graph to be studied in this thesis, was introduced by Cho *et al.* [2]. Since its introduction, it has been extensively studied (see for example [4, 6, 13, 18, 20, 27, 35]).

1.3 m-step competition graphs

Given a digraph D and a positive integer m, we define the *m-step digraph D^m of D* as follows: $V(D^m) = V(D)$ and there exists an arc (u, v) in D^m if and only if there exists a directed walk of length m from a vertex u to a vertex v. If there is a directed walk of length m from a vertex x to a vertex y in D, we call y an *m-step prey* of x and x an *m-step predator* of y. Especially, a 1-step prey and a 1-step predator are just called a prey and a predator, respectively. If a vertex w is an m-step prey of both two distinct vertices u and v, then we say that w is an *m-step common prey* of u and v. If a vertex x is an m-step predator of both two distinct vertices u and v, then we say that

x is an *m-step common predator* of u and v. The *m-step competition graph* of D, denoted by $C^m(D)$, has the same vertex set as D and an edge between two distinct vertices x and y if and only if x and y have an m-step common prey in D. We call a graph an *m-step competition graph* for a positive integer m if it can be represented as the m-step competition graph of a digraph D. Note that $C^1(D)$ is the ordinary competition graph of D, and 'directed walk' in the definition of m-step prey can be replaced by 'directed path' for an acyclic digraph D.

In 2000, Cho *et al.*[2] posed the following question: For which values of m and n is P_n an m-step competition graph? In 2005, Helleloid *et al.*[4] partially answered the question. In 2010, Kuhl *et al.*[5] gave the sufficient condition for $C^m(D) = P_n$. In 2011, Belmont *et al.*[6] presented a complete characterization of paths that are m-step competition graphs. Helleloid *et al.*[4] characterized the trees which are m-step competition graphs of digrpahs with n vertices when $m \geq n$.

Proposition 1.1. *(Helleloid [4]). For all positive integers $m \geq n$, the only connected triangle-free m-step competition graph on n vertices is the star graph.*

Given a digraph D with n vertices whose m-step competition graph is a tree, unless otherwise stated, we assume that $m \geq 2$ and $m < n$. We call a digraph D with at least three vertices an *m-step tree-inducing digraph* if the m-step competition graph of D is a tree for some integer $m \geq 2$. A digraph is said to be a *tree-inducing digraph* if it is an m-step tree-inducing digraph for some integer $m \geq 2$.

Chapter 2

Tree-inducing digraphs

2.1 Some properties of tree-inducing digraphs

Given a digraph D and a vertex set W of D, $N^+_{D,m}(W)$ and $N^-_{D,m}(W)$ denote the set of all vertices reachable in m steps from some vertex $w \in W$ and the set of all vertices m steps behind some vertex $w \in W$, respectively. When no confusion is likely, we omit D in $N^+_{D,m}(W)$ and $N^-_{D,m}(W)$ to just write $N^+_m(W)$ and $N^-_m(W)$. We note that $N^+_1(W) = N^+(W)$ and $N^-_1(W) = N^-(W)$. Technically, we write $N^+_0(W) = N^-_0(W) = W$. Especially, we use the notation $N^+_m(x)$ and $N^-_m(x)$ for $N^+_m(\{x\})$ and $N^-_m(\{x\})$, respectively, for a vertex x of D. By definition,

$$N^+(N^+_i(W)) = N^+_{i+1}(W) \quad \text{and} \quad N^-(N^-_i(W)) = N^-_{i+1}(W)$$

for a vertex set W of D and a positive integer i. Thus, inductively, the following is true.

Proposition 2.1. *Let D be a digraph. Then $N^+_{j-i}(N^+_i(W)) = N^+_j(W)$ and $N^-_{j-i}(N^-_i(W)) = N^-_j(W)$ for a vertex set W of D, and any positive integers i and j satisfying $i \leq j$.*

Any vertex in a tree-inducing digraph has outdegree at least one since it does not have an isolated vertex. Therefore the following proposition is true.

Proposition 2.2. *Any vertex in an m-step tree-inducing digraph D has an i-step prey for any positive integer i.*

Proposition 2.3. *Let D be a tree-inducing digraph. If $N_i^+(u) \cap N_i^+(v) \neq \emptyset$ for some vertices u and v of D and a positive integer i, then $N_j^+(u) \cap N_j^+(v) \neq \emptyset$ for each integer $j \geq i$.*

Proof. Suppose $N_i^+(u) \cap N_i^+(v) \neq \emptyset$ for some vertices u and v of D and a positive integer i. Then u and v have an i-step common prey w. Let j be an integer grater than or equal to i. By Proposition 2.2, w has an $(j-i)$-step prey x. Then x is a j-step common prey of u and v. Therefore $x \in N_j^+(u) \cap N_j^+(v)$. □

Proposition 2.4. *Let D be an m-step tree-inducing digraph. Then, for any vertex $u \in D$, $|N_i^-(u)| \leq 2$ for any positive integer $i \leq m$.*

Proof. To reach a contradiction, suppose that $|N_i^-(u)| \geq 3$ for some vertex u of D and a positive integer $i \leq m$. Then there exist three distinct i-step predators x, y, and z of u. By Proposition 2.2, u has an $(m-i)$-step prey v. Then v is a m-step common prey of x, y, and z. Thus x, y, and z form a cycle in $C^m(D)$, which is a contradiction. □

Proposition 2.5. *Let D be an m-step tree-inducing digraph. Then, $N_i^+(u) \neq N_i^+(v)$ for any distinct vertices u and v in D and any positive integer $i \leq m$.*

Proof. Suppose, to the contrary, that $N_i^+(u) = N_i^+(v)$ for some distinct u and v in D and a positive integer $i \leq m$. Denote $C^m(D)$ by G. Since G is a tree, G has no isolate vertex. Then, by Proposition 2.2, u has an m-step prey. Take an m-step prey w of u. Then there exists a directed (u, w)-walk P of length m in D. Since $i \leq m$, there exists an i-step prey of u on P. Since $N_i^+(u) = N_i^+(v)$, it is an i-step prey of v and so w is an m-step prey of v. Thus $N_m^+(u) \subset N_m^+(v)$. Similarly, we can show that $N_m^+(v) \subset N_m^+(u)$, so $N_m^+(u) = N_m^+(v)$. Since G has at least three vertices, there exists a vertex x other than u and v in G which is adjacent to u or v. Without loss of generality, we may assume that x is adjacent to u in G. Then, u and x have an m-step common prey z in D. Since $N_m^+(u) = N_m^+(v)$, $\{u, v, x\} \subset N_m^-(z)$, which is a contradiction to Proposition 2.4. □

Lemma 2.6. *Let D be a tree-inducing digraph. For any nonempty proper subset U of $V(D)$, there exists a vertex $u \in N^+(U)$ such that $|N^-(u) \cap U| = 1$.*

Proof. To reach a contradiction, suppose that there exists a nonempty proper subset U^* of $V(D)$ such that $|N^-(v) \cap U^*| = 0$ or $|N^-(v) \cap U^*| \geq 2$ for each vertex v in $N^+(U^*)$. Since any vertex in $N^+(U^*)$ is an prey of a vertex in U^*, $|N^-(v) \cap U^*| \geq 1$ for each vertex v in $N^+(U^*)$ and so only the latter holds. Then, by Proposition 2.4, $|N^-(v) \cap U^*| = 2$ for each vertex $v \in N^+(U^*)$. Since U^* is a proper subset of $V(D)$, $V(D) \setminus U^* \neq \emptyset$. Since $U^* \neq \emptyset$ and $C^m(D)$ is connected, there exists a vertex x in $V(D) \setminus U^*$ which is adjacent to a vertex w in U^*.

Then, w and x have an m-step common prey a_m and so there exists a directed (w, a_m)-walk of length m in D. Let a_1 be a vertex outgoing from w on this walk. Then $a_1 \in N^+(w) \subset N^+(U^*)$. By the choice of U^*, each vertex of $N^+(U^*)$ has two one-step predators in U^*. Let y be the other one-step predator of a_1 in U^*. Then y and x are distinct. Furthermore, y is an m-step predator of a_m and so $\{w, x, y\} \subset N_m^-(a_m)$, which is a contradiction to Proposition 2.4. □

Proposition 2.7. *Let D be a tree-inducing digraph. Then $|N^+(U)| \geq |U|$ for any proper subset U of $V(D)$.*

Proof. We prove by induction on $|U|$. By Proposition 2.2, $|N^+(u)| \geq 1$ for each vertex u of D, so the inequality holds when $|U| = 1$. Now suppose that $|N^+(U)| \geq |U|$ for any proper vertex subset U of $V(D)$ such that $|U| \leq k$ for any positive integer $k \leq |V(D)| - 1$. Take a proper subset W of $V(D)$

with $k+1$ elements. Suppose, to the contrary, that $|N^+(W)| < |W|$. By Lemma 2.6, there exists a vertex $w \in N^+(W)$ such that $|N^-(w) \cap W| = 1$. Then $N^-(w) \cap W = \{x\}$ for some vertex $x \in W$. Now x is the only one-step predator of w in W, so $w \notin N^+(W - \{x\})$. Since $w \in N^+(W)$,

$$|N^+(W - \{x\})| \leq |N^+(W)| - 1.$$

By the assumption that $|N^+(W)| < |W|$,

$$|N^+(W - \{x\})| < |W| - 1. \tag{2.1}$$

On the other hand, since $W - \{x\}$ is a proper subset of $V(D)$ with $k-1$ elements, by induction hypothesis,

$$|N^+(W - \{x\})| \geq |W - \{x\}| = |W| - 1,$$

which contradicts (2.1). Therefore the proposition is true. □

The statement of Proposition 2.7 may be false for the vertex set of a digraph satisfying the hypothesis of the proposition.

Example 2.8. Let D be the digraph with $V(D) = \{v_1, v_2, \ldots, v_m, w\}$ and $A(D) = \{(v_i, v_{i+1}) : 1 \leq i < m\} \cup \{(v_m, v_1)\} \cup \{(w, v_i) : 1 \leq i \leq m\}$. Then $C^m(D)$ is a $K_{1,m}$ with bipartition $(\{w\}, \{v_1, v_2, \ldots, v_m\})$ for each positive integer m. Yet, $|N^+(V(D)| < |V(D)|$ since $w \notin N^+(V(D))$.

Theorem 2.9. *Let D be an m-step tree-inducing digraph. If $|N_i^+(v)| \geq l$ for a vertex v in D and some positive integers l and $i \leq m$, then $|N_j^+(v)| \geq l$ for any positive integer j such that $i \leq j \leq m$.*

Proof. Let v be a vertex of D such that $|N_i^+(v)| \geq l$ for some positive integers l and $i \leq m$. Let j be a positive integer at least i and at most m.

We first consider the case where $N_i^+(v) = V(D)$. Take a vertex x in D. If there exists a vertex u such that $N^-(u) = \emptyset$, then $u \notin N_i^+(v)$ and so

$N_i^+(v) \neq V(D)$. Therefore $N^-(w) \neq \emptyset$ for each vertex $w \in V(D)$. Thus there exists a directed (y, x)-walk W_1 of length $j - i$ for some vertex y in D. Since $N_i^+(v) = V(D)$, there exists a directed (v, y)-walk W_2 of length i. Now $W_2 \to W_1$ is a directed (v, x)-walk of length j. Therefore $x \in N_j^+(v)$. Thus $N_i^+(v) \subset N_j^+(v)$ and so $N_j^+(v) = V(D)$. Hence $|N_j^+(v)| = |V(D)| \geq l$.

Now we consider the case where $N_i^+(v) \subsetneq V(D)$. Denote $N_k^+(v)$ by U_k for each $i \leq k \leq j$. Then, by Proposition 2.7, $|N^+(U_i)| \geq |U_i|$. By (2.1), $N^+(U_i) = U_{i+1}$. If $U_{i+1} = V(D)$, then $|N_j^+(v)| \geq l$ by the argument in the previous case. If $U_{i+1} \subsetneq V(D)$, then $|N^+(U_{i+1})| \geq |U_{i+1}|$ by Proposition 2.7. We may repeat this process until we show that $|N_j^+(v)| \geq l$. □

By Theorem 2.9, we have the two following corollaries, which play a key role in determining $|N_m^+(v)|$ in $C^m(D)$.

Corollary 2.10. *Let D be an m-step tree-inducing digraph. If $|N^+(v)| \geq l$ for a vertex v in $V(D)$ and some positive integer l, then $|N_m^+(v)| \geq l$.*

Corollary 2.11. *Let D be an m-step tree-inducing digraph. Then $|N_m^+(v)| \geq \max_{u \in U} |N^+(u)|$ for a vertex v in D where $U = \left(\bigcup_{i=1}^{m-1} N_i^+(v)\right) \cup \{v\}$.*

Proof. Take a vertex v in D and let $U = (\bigcup_{i=1}^{m-1} N_i^+(v)) \cup \{v\}$. Let y be a vertex in U such that $|N^+(y)| = \max_{u \in U} |N^+(u)|$. If $y = v$, then $|N^+(v)| = |N^+(y)|$ and so, by Corollary 2.10, $|N_m^+(v)| \geq |N^+(y)|$. Suppose $y \neq v$. Then, by the definition of U, $y \in N_j^+(v)$ for some positive integer $j \in \{1, 2, \ldots, m-1\}$. Therefore $N^+(y) \subset N_{j+1}^+(v)$. Thus $|N_{j+1}^+(v)| \geq |N^+(y)|$. Then, by Theorem 2.9, $|N_m^+(v)| \geq |N^+(y)|$. □

2.2 An idle vertex of a tree-inducing digraph

Given a digraph D with at least three vertices whose m-step competition graph $C^m(D)$ is a tree and an edge $e = uv$ of $C^m(D)$, we denote the set of

m-step common prey of u and v by $P(e)$, that is, $P(e) = N_m(u) \cap N_m(v)$. Then

$$\bigcup_{e \in E(C^m(D))} P(e) \subset V(D). \tag{2.2}$$

Obviously

$$|P(e)| \geq 1 \tag{2.3}$$

for each edge $e \in E(C^m(D))$ and so

$$|E(C^m(D))| \leq \sum_{e \in E(C^m(D))} |P(e)| \tag{2.4}$$

Proposition 2.12. *Let D be a tree-inducing digraph. Then $P(e_1) \cap P(e_2) = \emptyset$ for distinct edges e_1 and e_2 in $E(C^m(D))$*

Proof. Suppose that $P(e_1) \cap P(e_2) \neq \emptyset$ for some two distinct edges e_1 and e_2 in $E(C^m(D))$. Without loss of generality, let v be a vertex in $P(e_1) \cap P(e_2)$. Since the edges e_1 and e_2 are distinct, there are at least three distinct points each of which is an end of e_1 or e_2. Let x, y, and z be such vertices. Then $v \in N_m^+(x) \cap N_m^+(y) \cap N_m^+(z)$ and so $|N_m^-(v)| \geq 3$, which is contradiction to Proposition 2.4. □

By (2.4) and Proposition 2.12,

$$|E(C^m(D))| \leq \sum_{e \in E(C^m(D))} |P(e)| = \left| \bigcup_{e \in E(C^m(D))} P(e) \right|.$$

Then, since $|\bigcup_{e \in E(C^m(D))} P(e)| \leq |V(D)|$ by (2.2),

$$|E(C^m(D))| \leq \sum_{e \in E(C^m(D))} |P(e)| \leq |V(D)|. \tag{2.5}$$

Since $C^m(D)$ is a tree, $|V(D)| = |E(C^m(D))| + 1$. Thus

$$\sum_{e \in E(C^m(D))} |P(e)| = |E(C^m(D))| \quad \text{or} \quad \sum_{e \in E(C^m(D))} |P(e)| = |E(C^m(D))| + 1.$$

Since at least $|V(D)|-1$ vertices are needed as m-step common prey by (2.3) and Proposition 2.12, the following is true.

Proposition 2.13. *Let D be an m-step tree-inducing digraph. Then*

(1) $|E(C^m(D))| = \sum_{e \in E(C^m(D))} |P(e)|$ *if and only if* $|P(e)| = 1$ *for each edge e in $C^m(D)$ if and only if there exists a unique vertex w in D such that* $|N_m^-(w)| = 0$ *or* 1.

(2) If $\sum_{e \in E(C^m(D))} |P(e)| = |E(C^m(D))| + 1$, *then* $|P(e^*)| = 2$ *for some edge e^* in $C^m(D)$ and* $|P(e)| = 1$ *for each edge e in* $E(C^m(D)) \setminus \{e^*\}$.

Given a tree-inducing digraph D, if $|E(C^m(D))|+1 = \sum_{e \in E(C^m(D))} |P(e)|$, then there exists a unique edge e^* with $|P(e^*)| = 2$ by Proposition 2.13(2) and we take one of the vertices in $P(e^*)$. Otherwise, we know that there exists a unique vertex w in D with $|N_m^-(w)| = 0$ or 1 by Proposition 2.13(1) and we take w. We call the vertex taken in each case an *idle vertex* of D and denote it by w.

Now we introduce a relation ψ from $E(C^m(D))$ to $V(D) \setminus \{w\}$ which relates each edge e to the element in $P(e)$ if $|P(e)| = 1$ and to the element in $P(e) \setminus \{w\}$ if $|P(e)| = 2$. Then it is easy to see that ψ is a one-to-one correspondence. We call ψ^{-1} an *m-step predator indicator of D*.

By the definition of idle vertex, the following is immediately true.

Proposition 2.14. *Let D be a tree-inducing digraph. Then D has exactly one idle vertex.*

The following propositions are immediate consequences of Proposition 2.13.

Proposition 2.15. *Let D be an m-step tree-inducing digraph. Then w is the idle vertex if and only if w in D has either at most one m-step predator in D or two m-step predators which have another vertex as m-step prey in D. Furthermore, if the latter of the "if part" is true, then w and the vertex is the only pair of vertices which shares two m-step common predators.*

Proposition 2.16. *Let D be an m-step tree-inducing digraph. Then every vertex v other than the idle vertex has exactly two m-step predators and shares at most one common m-step predator with any vertex that is distinct from v and the idle vertex.*

Proposition 2.17. *Let D be an m-step tree-inducing digraph and w be the idle vertex with no m-step predator in D. $N_j^-(w) = \emptyset$ for each $1 \leq j \leq m$.*

Proof. Since $N_m^-(w) = \emptyset$, $|P(e)| = 1$ for each edge e in $E(C^m(D))$ by Proposition 2.13(1). Therefore $N_m^-(v) \neq \emptyset$ for each vertex v in $V(D) \setminus \{w\}$. Fix $j \in \{1, \ldots, m\}$. If $N_j^-(x) \neq \emptyset$ for each vertex x in D, then it is easy to check that $N_m^-(w) \neq \emptyset$, which is a contradiction. Therefore there exists a vertex y such that $N_j^-(y) = \emptyset$. Then $N_m^-(y) = \emptyset$ and so, by Propositions 2.15 and 2.14, $y = w$. Thus $N_j^-(w) = \emptyset$. □

Corollary 2.18. *Let D be an m-step tree-inducing digraph without source. Then $1 \leq |N_i^-(v)| \leq 2$ for each $1 \leq i \leq m$ and each vertex v in D.*

Proof. By Proposition 2.4, $|N_i^-(v)| \leq 2$ for each $0 \leq i \leq m$ and each vertex v in D. If $N_j^-(u) = \emptyset$ for some $j \in \{1, \ldots, m\}$ and some vertex u in D, then $N_m^-(u) = \emptyset$ and so, by Propositions 2.15 and 2.17, $N^-(u) = \emptyset$, which contradicts the hypothesis. □

Now we obtain the following proposition.

Proposition 2.19. *Let D be an m-step tree-inducing digraph. Then $|N_m^+(x) \cap N_m^+(y)| \leq 2$ for any vertices x and y in D. Especially, if $|N_m^+(x) \cap N_m^+(y)| = 2$ for some x and y in D, then the following are true: the idle vertex is contained* *in* $N_m^+(x) \cap N_m^+(y)$*; there is no pair of vertices other than* $\{x, y\}$ *which shares two* m*-step common prey.*

Proof. There exists exactly one idle vertex in D by Proposition 2.14. Let w be the idle vertex in D. If $|N_m^+(x) \cap N_m^+(y)| \geq 3$, then, there are at least two vertices in $(N_m^+(x) \cap N_m^+(y)) \setminus \{w\}$ and we reach a contradiction to Proposition 2.16. To show the"especially" part, suppose that there exist two vertices x and y in D such that $|N_m^+(x) \cap N_m^+(y)| = 2$. Take two vertices u and v in $N_m^+(x) \cap N_m^+(y)$. Then $\{x, y\} \subset N_m^-(u) \cap N_m^-(v)$, so

$$\{x, y\} = N_m^-(u) = N_m^-(v)$$

by Proposition 2.4. If neither u nor v is the idle vertex, then it contradicts Proposition 2.16. Thus one of u and v is w and we may assume that $v = w$. Hence $w \in N_m^+(x) \cap N_m^+(y)$.

Suppose that there exists a pair of vertices a and b such that $|N_m^+(a) \cap N_m^+(b)| = 2$. Since $\{u, w\}$ is the only pair of vertices which shares two common m-step predators by furthermore part of Proposition 2.15, $N_m^+(a) \cap N_m^+(b) = \{u, w\}$. If $\{x, y\} \neq \{a, b\}$, then $|N_m^-(u)| \geq 3$, which contradicts Proposition 2.4. Therefore $\{x, y\} = \{a, b\}$. □

Proposition 2.20. *Let* D *be an* m*-step tree-inducing digraph. Suppose* $|N_m^+(u)| \geq l$ *for a vertex* u *in* D *and a positive integer* l*. Then the degree of* u *is at least* $l - 1$ *in* $C^m(D)$*. Especially, if the degree of* u *equals* $l - 1$ *in* $C^m(D)$*, then* $|N_m^+(u)| = l$ *and* $w \in N_m^+(u)$ *for the idle vertex* w *in* D*.*

Proof. Let ϕ be an m-step predator indicator from $V(D) \setminus \{w\}$ to $E(C^m(D))$ for the idle vertex w in D. We denote the degree of a vertex v in $C^m(D)$ by $d(v)$. By definition, for each vertex v in $N_m^+(u)$, $\phi(v)$ is an edge incident to u unless $v = w$. Then, since ϕ is one-to-one, there exists at least $|N_m^+(u)| - 1$ edges incident to u and so $d(u) \geq l - 1$. To show the "especially" part, suppose, to the contrary, that $d(u) = l - 1$ but $|N_m^+(u)| \neq l$. Then, by the hypothesis, $|N_m^+(u)| \geq l+1$. Then by the above argument, $d(u) \geq l$, which is a contradiction. Therefore $|N_m^+(u)| = l$. Since $d(u) = l-1$, $w \in N_m^+(u)$. □

Proposition 2.21. *Let D be an m-step tree-inducing digraph. Then the degree of each vertex u in $C^m(D)$ is $|N_m^+(u)| - 1$ or $|N_m^+(u)|$. Especially, the degree of u is $|N_m^+(u)| - 1$ if and only if $w \in N_m^+(u)$ for the idle vertex w in D.*

Proof. We denote the degree of a vertex u in $C^m(D)$ by $d(u)$. Since $d(u) = d$, there are d edges incident to u in $C^m(D)$. Let ϕ be an m-step predator indicator from $V(D) \setminus \{w\}$ to $E(C^m(D))$ for the idle vertex w in D. By definition, $\phi^{-1}(e)$ is an m-step common prey of the ends of e for each edge e incident to u, so there are d m-step prey of u. Therefore $|N_m^+(u)| \geq d$. If $|N_m^+(u)| \geq d+2$, $d(u) \geq d+1$ by Proposition 2.20, which is a contradiction. Thus $|N_m^+(u)| \leq d+1$, and so $d = |N_m^+(u)|$ or $|N_m^+(u)| - 1$. To show the "especially" part, suppose $|N_m^+(u)| = d+1$. Then one of vertices in $N_m^+(u)$ is not the image of ϕ^{-1} and so $w \in N_m^+(u)$. To show the converse, suppose $w \in N_m^+(v)$. Then, by definition, for each vertex v in $N_m^+(u) \setminus \{w\}$, $\phi(v)$ is an edge incident to u, which implies that there exists $|N_m^+(u)| - 1$ edges incident to u since ϕ is one-to-one. Thus $d(u) = |N_m^+(u)| - 1$. □

Proposition 2.22. *Let D be an m-step tree-inducing digraph. If $|N_i^+(u) \cap N_i^+(v)| \geq l$ for some vertices u and v of D and positive integers i and l, then $|N_j^+(u) \cap N_j^+(v)| \geq l$ for each integer j, $i \leq j \leq m$.*

Proof. Let u and v be vertices of D such that $|N_i^+(u) \cap N_i^+(v)| \geq l$ for some positive integers i and l. In addition, let j be a positive integer greater than or equal to i.

Consider the case $N_i^+(u) \cap N_i^+(v) = V(D)$. If there exists a vertex w such that $N^-(w) = \emptyset$, then $w \notin N_i^+(u) \cap N_i^+(v)$ and so $N_i^+(u) \cap N_i^+(v) \neq V(D)$. Therefore each vertex has at least one predator. Take a vertex x in D. Since each vertex has at least one predator, there exists a directed (y, x)-walk W_1 of length $m - i$ for some vertex y in D. Since $N_i^+(u) \cap N_i^+(v) =$

$V(D)$, there exists a directed (z, y)-walk W_2 of length i for some vertex z in $N_i^+(u) \cap N_i^+(v)$. Now $W_2 \to W_1$ is a directed (z, x)-walk of length m. Therefore $x \in N_m^+(u) \cap N_m^+(v)$. Thus $(N_i^+(u) \cap N_i^+(v)) \subset (N_m^+(u) \cap N_m^+(v))$ and so $N_m^+(u) \cap N_m^+(v) = V(D)$. Then $\{u, v\} = N_m^-(w)$ for each vertex $w \in V(D)$ by Proposition 2.4. Therefore $e = uv$ is the only one edge in $C^m(D)$, which is a contradiction to the hypothesis that $C^m(D)$ is a tree with at least three vertices.

Consider the case $N_i^+(u) \cap N_i^+(v) \subsetneq V(D)$. For notational convenience, let

$$U_k = N_k^+(u) \cap N_k^+(v)$$

for each $i \leq k \leq m$. To show that $N^+(U_i) \subset U_{i+1}$, take a vertex a in $N^+(U_i)$. Then $a \in N^+(b)$ for some vertex b of U_i, so there exists an arc (b, a) in D. Since $b \in U_i$, there exist a directed (u, b)-walk W_1 and a directed (v, b)-walk W_2 both of which have length i. Now $W_1 \to a$ is a directed (u, a)-walk of length $i + 1$ and $W_2 \to a$ is a directed (v, a)-walk of length $i + 1$. Therefore $a \in U_{i+1}$ and so

$$N^+(U_i) \subset U_{i+1}.$$

Thus $|N^+(U_i)| \leq |U_{i+1}|$. On the other hand, by Proposition 2.7, $|N^+(U_i)| \geq |U_i|$ and so $|U_i| \leq |U_{i+1}|$. Hence $l \leq |U_{i+1}|$. If $U_{i+1} = V(D)$, then it is a contradiction by the argument in the previous case. Therefore $U_{i+1} \subsetneq V(D)$. We may repeat this process until we show that $|N_j^+(u) \cap N_j^+(v)| \geq l$. □

Corollary 2.23. *Let D be an m-step tree-inducing digraph. If $|N_i^+(u) \cap N_i^+(v)| \geq l$ for some vertices u and v of D and positive integers i and l, $i \leq m$, then u and v are adjacent in $C^m(D)$.*

Theorem 2.24. *Let D be a tree-inducing digraph without sources. Then each vertex lies on a directed cycle in D.*

Proof. Suppose, to the contrary, there exists a vertex u which does not lie on any directed cycle in D. Let A, B, and C be subsets of $V(D)$ such that

$$A = \{v \in V(D) \mid v \in \bigcup_{i \geq 1} N_i^+(u)\};$$

$$B = \{v \in V(D) \mid v \in \bigcup_{i \geq 1} N_i^-(u)\};$$

$$C = V(D) \setminus (A \cup B).$$

By the hypothesis, $N^-(u) \neq \emptyset$, so $B \neq \emptyset$. By Proposition 2.2, $N^+(u) \neq \emptyset$, so $A \neq \emptyset$. Since there is no directed cycle containing u, $A \cap B = \emptyset$. If $u \in A$ or $u \in B$, then there exists a closed directed walk containing u and so there exists a directed cycle containing u, which contradicts our assumption. Thus $u \in C$ and so $C \neq \emptyset$. We will claim the following:

$$A \nrightarrow B, \quad A \nrightarrow C, \quad \text{and} \quad C \nrightarrow B \tag{2.6}$$

where $X \nrightarrow Y$ for vertices sets X and Y of D means that there is no arc from a vertex in X and to a vertex in Y. If there exists an arc (a, b) from a vertex $a \in A$ to a vertex $b \in B$, then, since there exists a closed directed walk W_1 (resp. W_2) from u (resp. b) to a (resp. u), the arc (a, b) together with W_1 and W_2 forms a closed directed walk containing u and we reach a contradiction. If there exists an arc (c, b) from a vertex $c \in C$ to a vertex $b \in B$, then since there exists a directed walk W_3 from b to u, the arc (c, b) together with W_3 forms a directed walk from c to u, which contradicts the choice of the subset C. If there exists an arc (a, c) from a vertex $a \in A$ to a vertex $c \in C$, then, since there exists a directed walk W_4 from u to a, the arc (a, c) together with W_4 forms a directed walk from u to c, which contradicts the choice of the subset C.

Now we show that

$$\{u\} \nrightarrow B, \quad \{u\} \nrightarrow C, \quad \text{and} \quad C \nrightarrow \{u\} \tag{2.7}$$

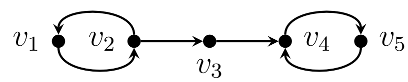


Figure 2.1: A digraph whose m-step competition graph is not tree

If there exists an arc (u, b) for some vertex b in B, then, since there is a directed walk W_5 from b to u, the arc (u, b) together with W_5 forms a closed directed walk containing u and we reach a contradiction. If there exists an arc (u, c) or (c, u) for a vertex c, then $c \in A$ or B, which contradicts the choice of the subset C.

Since D is a tree-inducing digraph, there exists an m-step predator indicator ϕ from $V(D) \setminus \{w\}$ to $E(C^m(D))$ for the idle vertex w in D. Since no vertices in A or C or $\{u\}$ can be m-step predators of vertices in $B \cup \{u\}$ by (2.6) and (2.7), any vertex in $B \cup \{u\}$ has an m-step predator only in B. By definition of the ϕ, at least $|B|$ vertices in $B \cup \{u\}$ are m-step common prey of the ends of edges in the image of ϕ. Therefore at least $|B|$ vertices in $B \cup \{u\}$ have two m-step predators only in B. Thus $C^m(D)[B]$ has at least $|B|$ edges which implies that there exists a cycle in $C^m(D)[B]$. Since $C^m(D)[B] \subset C^m(D)$, there exists a cycle in $C^m(D)$, which is a contradiction to the hypothesis that $C^m(D)$ is a tree. $\square$

Remark 2.25. It is likely that, for each vertex of a digraph without source, there is a directed cycle containing it. However, it is not true. For example, let D be a digraph given in Figure 2.1. Then D has no source and no directed cycle containing the vertex v_3.

Remark 2.26. For some tree-inducing digraph D with a source, Theorem 2.24 may be false. For example, the vertex w given in Example 2.8 does not lie on any directed cycle in D.

Chapter 3

Tree-inducing digraphs with a source

3.1 A characterization of tree-inducing digraphs with a source

Given a digraph D with vertex set $\{v_1, v_2, \ldots, v_n\}$, let A be the adjacency matrix of D such that

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an arc of } D, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. *Let D be a digraph with n vertices $(n \geq 1)$ satisfying the following conditions:*

(a) $|N^+(u)| = 1$ for each vertex u in D.

(b) $N^+(x) \cap N^+(y) = \emptyset$ for every pair of vertices x and y in D.

Then the following are true:

(1) $|N_i^+(u)| = 1$ for each vertex u in D and any positive integer i.

(2) $N_i^+(x) \cap N_i^+(y) = \emptyset$ *for every pair of vertices* x *and* y *in* D *and any positive integer* i.

Especially, each vertex in D *lies on exactly one directed cycle in* D.

Proof. Let A be the adjacency matrix of D. By the condition (a), each vertex in D has exactly one prey. Since the sum of indegrees of the vertices in D equals that of outdegrees of the vertices, each vertex in D has exactly one predator by the condition (b). Therefore each row and each column of A have exactly one 1 and 0s elsewhere. Thus A is a permutation matrix. It is well-known that the product of permutation matrices is a permutation matrix. Therefore A^i is a permutation matrix for any positive integer i. Thus the statement (1) is immediately true. If a vertex z belongs to $N_i^+(x) \cap N_i^+(y)$ for some vertices x and y in D and for some positive integer i, then the column of A^i corresponding to z contains at least two 1s. Hence the statement (2) is true.

To show the "especially" part, let $V(D) = \{v_1, v_2, \ldots, v_n\}$. Since we have shown that the adjacency matrix A of D is a permutation matrix, we may take a permutation α on $\{1, 2, \ldots, n\}$ such that $\alpha(i) = j$ for each arc (v_i, v_j) in D. It is well known that every permutation of finite set can be written as a cycle or as a product of disjoint cycles. Thus α can be written as

$$\alpha = (a_1, a_2, \ldots, a_p)(b_1, b_2, \ldots, b_q) \cdots (c_1, c_2, \ldots, c_r)$$

for each distinct a_i, b_j, and c_k $(1 \le i \le p, 0 \le j \le q, 0 \le k \le r)$ in $V(D)$. Now take a vertex v in $V(D)$. Without loss of generality, we may assume $v = a_1$. Then v lies on exactly one directed cycle $C = a_1 \to a_2 \to \cdots \to a_p \to a_1$ in D.

Therefore each vertex in D lies on exactly one directed cycle in D. □

The following is one of our main theorems.

Theorem 3.2. *Let* D *be an* m*-step tree inducing digraph with a source. Then the following statements are true.*

(1) $N^+(x) \cap N^+(y) = \emptyset$ *for every pair of vertices* x *and* y *in* $V(D) \setminus \{v\}$.

(2) $C^m(D)$ *is a star graph.*

(3) The vertex v *is a center of* $C^m(D)$ *and* $N^+(v) = V(D) \setminus \{v\}$.

(4) Each vertex u *in* $V(D) \setminus \{v\}$ *has outdegree* 1 *and lies on exactly one directed cycle in* D.

Proof. Let v be a source of D. Then $N_m^-(v) = \emptyset$, so v is the idle vertex by Proposition 2.15. Therefore

$$|N_m^-(u)| = 2 \tag{3.1}$$

for each vertex $u \in V(D) \setminus \{v\}$ by Proposition 2.16.

To show $|N^-(u)| = 2$ for each vertex $u \in V(D) \setminus \{v\}$, fix u in $V(D) \setminus \{v\}$. Then, since v is the only vertex with $N^-(v) = \emptyset$ by Proposition 2.14, $|N^-(u)| \geq 1$. By Proposition 2.4, $|N^-(u)| \leq 2$. Suppose, to the contrary, that $|N^-(u)| = 1$. Then $N^-(u) = \{x\}$ for some vertex x of D, so $u \notin N^+(V(D) \setminus \{x\})$. Since $N^-(v) = \emptyset$, $N^+(V(D) \setminus \{x\}) \subset (V(D) \setminus \{u, v\})$. Therefore

$$|N^+(V(D) \setminus \{x\})| \leq |V(D) \setminus \{u, v\}| < |V(D) \setminus \{x\}|.$$

Since $V(D) \setminus \{x\}$ is a proper subset of $V(D)$, we reach a contradiction to Proposition 2.7. Therefore

$$|N^-(u)| = 2 \tag{3.2}$$

for each vertex $u \in V(D) \setminus \{v\}$. To show $|N^+(u)| = 1$ for each vertex $u \in V(D) \setminus \{v\}$, fix x in $V(D) \setminus \{v\}$. By Proposition 2.2, $|N^+(x)| \geq 1$. Suppose, to the contrary, that $|N^+(x)| \geq 2$. Then, by Corollary 2.10, $|N_{m-1}^+(x)| \geq 2$. On the other hand, by (3.2), x is a common prey of two vertices y and y'. Then there exist arcs (y, x) and (y', x) in D. Take a vertex z in $N_{m-1}^+(x)$. Then there exists a directed (x, z)-walk W of length $m-1$. Therefore $y \to W$

is a directed (y, z)-walk and $y' \to W$ is a directed (y', z)-walk both of which have length m. Thus $z \in N_m^+(y) \cap N_m^+(y')$ and so $N_{m-1}^+(x) \subset N_m^+(y) \cap N_m^+(y')$. Then, since $|N_{m-1}^+(x)| \geq 2$, $|N_m^+(y) \cap N_m^+(y')| \geq 2$. Therefore we may take two vertices a and b in $N_m^+(y) \cap N_m^+(y')$. Since $N_m^-(v) = \emptyset$, a and b are distinct from v. Moreover, since $N_m^-(a) = N_m^-(b) = \{y, y'\}$ by Proposition 2.4, a or b is the idle vertex by Proposition 2.15, which contradicts Proposition 2.14. Therefore

$$|N^+(u)| = 1 \tag{3.3}$$

for each vertex $u \in V(D) \setminus \{v\}$.

To show the first statement, take two vertices a and b in $V(D) \setminus \{v\}$. By (3.3), $|N^+(a)| = |N^+(b)| = 1$. Therefore, if $N^+(a) \cap N^+(b) \neq \emptyset$, then $N^+(a) = N^+(b)$, which is a contradiction to Proposition 2.5. Thus

$$N^+(a) \cap N^+(b) = \emptyset \tag{3.4}$$

and hence the statement (1) is true.

Now we consider the subdigraph D' of D induced by $V(D) \setminus \{v\}$. The digraph D' satisfies the conditions (i) and (ii) of Proposition 3.1 by (3.3) and (3.4). Therefore

$$|N_{D',m}^+(u)| = 1$$

for each vertex u in $V(D) \setminus \{v\}$ and

$$N_{D',m}^+(u) \cap N_{D',m}^+(w) = \emptyset$$

for every pair of vertices u and w in $V(D) \setminus \{v\}$. Yet, $N_{D,1}^+(V(D) \setminus \{v\}) \subset (V(D) \setminus \{v\})$ since $N^-(v) = \emptyset$. Therefore $N_{D',m}^+(u) = N_m^+(u)$ and so

$$|N_m^+(u)| = 1 \tag{3.5}$$

for each vertex u in $V(D) \setminus \{v\}$. Thus

$$N_m^+(u) \cap N_m^+(w) = \emptyset$$

for every pair of vertices u and w in $V(D) \setminus \{v\}$. Hence u is not adjacent to w for every pair of vertices u and w in $V(D) \setminus \{v\}$. Moreover, we conclude that $|N_m^+(v)| = |V(D)| - 1$ by Proposition 2.16 and (3.5). Since $N^-(v) = \emptyset$, $N_m^+(v) = V(D) \setminus \{v\}$, that is, the vertex v is adjacent to u in $C^m(D)$ for each vertex u in $V(D) \setminus \{v\}$. Thus $C^m(D)$ is a star graph where v is a center of $C^m(D)$. Hence the statements (2) and (3) are true.

Each vertex in $V(D) \setminus \{v\}$ is contained in exactly one directed cycle in the subdiagraph of D induced by $V(D) - \{v\}$ by the "especially part" of Proposition 3.1. Since $N^-(v) = \emptyset$, the vertex v is not contained in any cycle in D. Thus each vertex in D is contained in exactly one directed cycle in D. Hence the statement (4) is true by (3.3). □

Corollary 3.3. *Let D be a digraph with at least three vertices whose m-step competition graph is a tree but not a star graph. Then each vertex in D lies on a directed cycle.*

Proof. If D has a source, then $C^m(D)$ is a star graph by Theorem 3.2, which contradicts the hypothesis. Therefore D has no source and so the statement is true by Theorem 2.24. □

Corollary 3.4. *Let D be an m-step tree-inducing digraph. If there exists a vertex v of D such that $N_m^-(v) = \emptyset$, then $C^m(D)$ is a star graph.*

Proof. Suppose that there exists a vertex v of D such that $N_m^-(v) = \emptyset$. Then $N^-(v) = \emptyset$ by Proposition 2.17. Therefore $C^m(D)$ is a star graph by Theorem 3.2. □

3.2 The number of tree-inducing digraphs with a source

Theorem 3.5. *Let D be a digraph with at least three vertices satisfying the following conditions:*

(a) There exists a vertex v in D such that $N^-(v) = \emptyset$ and $N^+(v) = V(D) \setminus \{v\}$.

(b) Each vertex u in $V(D) \setminus \{v\}$ has outdegree 1 and lies on exactly one directed cycle in D.

Then the following are true:

(1) Each of the components in the digraph $D - v$ is a directed cycle.

(2) $|N_i^+(u)| = 1$ for each vertex u in $V(D) \setminus \{v\}$ and any positive integer i.

(3) $N_i^+(x) \cap N_i^+(y) = \emptyset$ for every pair of vertices x and y in $V(D) \setminus \{v\}$ and any positive integer i.

(4) $C^m(D)$ is a star graph for any integer m.

Proof. Since $N^-(v) = \emptyset$ by the condition (a), the statement (1) immediately follows from the condition (b). Now we consider the subdigraph D' of D induced by $V(D) \setminus \{v\}$. Since $N^-_{D,1}(v) = \emptyset$,

$$|N^+_{D',1}(u)| = 1 \tag{3.6}$$

for each vertex u in $V(D) \setminus \{v\}$ by the condition (b). Suppose, to the contrary, that $N^+_{D',1}(x) \cap N^+_{D',1}(y) \neq \emptyset$ for some vertices x and y in $V(D) \setminus \{v\}$. There exists a directed cycle C_x (resp. C_y) containing x (resp. y) in D by the condition (b). Since $N^-(v) = \emptyset$, v is not contained in the cycles C_x and C_y. Therefore, C_x and C_y are directed cycles in D'. Let z be a vertex in

$N^+_{D',1}(x) \cap N^+_{D',1}(y)$. Since $|N^+_{D',1}(x)| = |N^+_{D',1}(y)| = 1$ by the condition (b), z is contained in both cycles C_x and C_y in D'. Yet, $C_x \neq C_y$ since the arc (x, z) is distinct with the arc (y, z), so the condition (b) is violated. Therefore

$$N^+_{D',1}(x) \cap N^+_{D',1}(y) = \emptyset \tag{3.7}$$

for every pair of vertices x and y in $V(D) \setminus \{v\}$. On the other hand, since $N^-_{D,1}(v) = \emptyset$,

$$N^+_{D',i}(u) = N^+_{D,i}(u) \tag{3.8}$$

for each vertex u in $V(D) \setminus \{v\}$ and any positive integer i. Therefore

$$N^+_{D',i}(x) \cap N^+_{D',i}(y) = N^+_{D,i}(x) \cap N^+_{D,i}(y) \tag{3.9}$$

for every pair of vertices x and y in $V(D) \setminus \{v\}$ and any positive integer i. Since the digraph D' satisfies the conditions (a) and (b) of Proposition 3.1 by (3.6) and (3.7),

$$|N^+_{D',i}(u)| = 1 \quad \text{and} \quad N^+_{D',i}(x) \cap N^+_{D',i}(y) = \emptyset$$

for each vertex u in $V(D) \setminus \{v\}$ and any positive integer i. Then the statement (2) is true by (3.8) and the statement (3) is true by (3.9).

To show statement (4), take a vertex w in $V(D) \setminus \{v\}$. Since $|N^+_{D,1}(w)| = 1$ by the condition (2), $N^+_{D,1}(w) = \{x\}$ for some vertex x in D. Since $N^-_{D,1}(v) = \emptyset$ by the condition (a), $x \neq v$. Yet, $N^+_{D,1}(v) = V(D) \setminus \{v\}$ and so $x \in N^+_{D,1}(v) \cap N^+_{D,1}(w)$. Since each vertex in $V(D) \setminus \{v\}$ has outdegree at least 1 by the condition (b), $N^+_{D,m}(v) \cap N^+_{D,m}(w) \neq \emptyset$. Thus the vertex w and v are adjacent in $C^m(D)$. Since w is arbitrary chosen, the vertex v is adjacent to every vertex except itself in $C^m(D)$. On the other hand, since the statement (3) is true, every pair of vertices x and y in $V(D) \setminus \{v\}$ is not adjacent in $C^m(D)$. Thus each vertex in $V(D) \setminus \{v\}$ is only adjacent to v in $C^m(D)$. Hence $C^m(D)$ is a star graph. □

We call a digraph D with at least three vertices a *star-generating digraph with a source* satisfying the conditions given in Theorem 3.5. Now we are ready to give one of our main theorems.

Theorem 3.6. *Let D be a digraph with n vertices ($n \geq 3$) with a vertex of indegree 0. Then the m-step competition graph of D is a star graph if and only if D is star-generating digraph with a source. Especially, the number of star-generating digraph with a source with n vertices up to isomorphism equals the value $p(n-1)$ of the partition function, which is the number of distinct ways of representing $n-1$ as a sum of positive integers.*

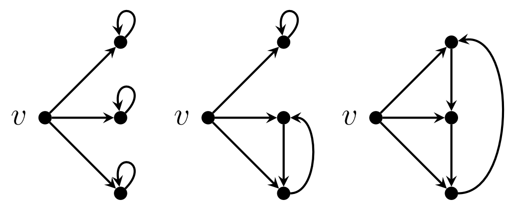
Proof. The "only if" part and "if" part immediately follow from Theorem 3.2 and Theorem 3.5 respectively.

To show the "especially" part, we take an integer n grater than or equal to three. Let A_k be the set of star-generating digraph with a source containing as many as k directed cycles with n vertices. Since A_i and A_j are disjoint for $i \neq j$,

$$\left| \bigcup_{k=1}^{n-1} A_k \right| = \sum_{k=1}^{n-1} |A_k|.$$

Moreover, since each star-generating digraph with a source with n vertices has a directed cycle of length less than n by definition, $\bigcup_{k=1}^{n-1} A_k$ is the set of the star-generating digraph with a source with n vertices. Therefore the number of star-generating digraph with a source with n vertices up to isomorphism equals $\sum_{k=1}^{n-1} |A_k|$. Any two distinct directed cycles in A_k are disjoint by the condition (b) given in Theorem 3.5. Therefore the sum of length of distinct directed cycles in a digraph belonging to A_k equals $n-1$. Thus each digraph in A_k gives rise to an integer partition with k parts of $n-1$. Two non-isomorphic digraphs in A_k give distinct integer partitions by the condition (a) and the statement (i) in Theorem 3.5. Thus the number of digraphs in A_k up to isomorphism equals the number of distinct ways of representing $n-1$ as a sum of k positive integers. Hence $\sum_{k=1}^{n-1} |A_k|$ equals $p(n-1)$. □

Example 3.7. The following three digraphs are all star-generating digraphs with a source with four vertices. Then the m-step ($m \geq 2$) competition graph of a digraph D with a source is a star graph if and only if D is one of the following three digraphs by Theorem 3.6. Especially, D_1, D_2, and D_3 correspond to the integer partitions of 3, $1+1+1$, $1+2$, and 3, respectively.



D_1 D_2 D_3

Bibliography

[1] J. E. Cohen: Interval graphs and food webs: a finding and a problem, *RAND Corporation Document 17696-PR*, Santa Monica, California, 1968.

[2] H. H. Cho, S. -R. Kim and Y. Nam: The m-step competition graph of a digraph, *Discrete Appl. Math.*, **105** (2000) 115–127.

[3] H. J. Greenberg, J. R. Lundgren, J. S. Maybee: Inverting graphs of rectangular matrices, *Discrete Appl. Math.*, **8** (1984) 255–265.

[4] Geir T.Helleloid: Connected triangle-free m-step competition graphs, *Discrete Appl. Math.*, **145** (2005) 376–383.

[5] Jaromy Kuhl, Brandon Christpoher Swan: Characterizing paths as m-step competition graphs *Discrete Appl. Math.*, **310** (2010) 2555–2559.

[6] Eva Belmont: A complete characterization of paths that are m-step competition graphs, *Discrete Appl. Math.*, **159** (2011) 1381–190.

[7] Joseph A.Gallian: Contemporary abstract algebra(seventh edition), *Brooks/cole cengage learning*.

[8] John Adrian Bondy, Uppaluri Siva Ramachandra Murty: Graph theory, *volume 244 of graduate texts in mathematics*, (2008).

[9] CA Anderson, L Langley, JR Lundgren, PA McKenna, and SK Merz. New classes of p-competition graphs and ϕ-tolerance competition graphs. *Congressus Numerantium*, pages 97–108, 1994.

[10] OB Bak and SR Kim. On the double competition number of a bipartite graph. *Congressus Numerantium*, pages 145–152, 1996.

[11] Stephen Bowser and Charles A Cable. Some recent results on niche graphs. *Discrete Applied Mathematics*, 30(2-3):101–108, 1991.

[12] Charles Cable, Kathryn F Jones, J Richard Lundgren, and Suzanne Seager. Niche graphs. *Discrete Applied Mathematics*, 23(3):231–241, 1989.

[13] Han Hyuk Cho and Hwa Kyung Kim. Competition indices of strongly connected digraphs. *Bull. Korean Math. Soc*, 48(3):637–646, 2011.

[14] Jihoon Choi, Suh-Ryung Kim, Jung Yeun Lee, and Yoshio Sano. On the partial order competition dimensions of chordal graphs. *Discrete Applied Mathematics*, 222:89–96, 2017.

[15] Ronald D Dutton and Robert C Brigham. A characterization of competition graphs. *Discrete Applied Mathematics*, 6(3):315–317, 1983.

[16] Peter C Fishburn and William V Gehrlein. Niche numbers. *Journal of Graph Theory*, 16(2):131–139, 1992.

[17] Zoltán Füredi. Competition graphs and clique dimensions. *Random Structures & Algorithms*, 1(2):183–189, 1990.

[18] Wei Ho. The m-step, same-step, and any-step competition graphs. *Discrete Applied Mathematics*, 152(1):159–175, 2005.

[19] Kathryn F Jones, J Richard Lundgren, FS Roberts, and S Seager. Some remarks on the double competition number of a graph. *Congr. Numer*, 60:17–24, 1987.

[20] Hwa Kyung Kim. Competition indices of tournaments. *Bull. Korean Math. Soc*, 45(2):385–396, 2008.

[21] Suh-Ryung Kim. On the inequality $dk(G) \leq k(G) + 1$. *Ars Combinatoria*, 51:173–182, 1999.

[22] Suh-ryung Kim, Terry A McKee, FR McMorris, and Fred S Roberts. p-competition graphs. *Linear Algebra and its Applications*, 217:167–178, 1995.

[23] Suh-Ryung Kim, Terry A McKee, Fred R McMorris, and Fred S Roberts. p-competition numbers. *Discrete Applied Mathematics*, 46(1):87–92, 1993.

[24] Suh-Ryung Kim, Fred S Roberts, and Suzanne Seager. On 101-clear $(0, 1)$ matrices and the double competition number of bipartite graphs. *Journal of Combinatorics, Information & System Sciences*, 17:302–315, 1992.

[25] J Richard Lundgren. Food webs, competition graphs, competition-common enemy graphs, and niche graphs. *Applications of Combinatorics and Graph Theory in the Biological and Social Sciences, IMA Volumes in Mathematics and its Applications*, 17:221–243, 1989.

[26] J Richard Lundgren and John S Maybee. Food webs with interval competition graphs. In *Graphs and Applications: Proceedings of the First Colorado Symposium on Graph Theory. Wiley, New York*, 1984.

[27] Boram Park, Jung Yeun Lee, and Suh-Ryung Kim. The m-step competition graphs of doubly partial orders. *Applied Mathematics Letters*, 24(6):811–816, 2011.

[28] Tarasankar Pramanik, Sovan Samanta, Biswajit Sarkar, and Madhumangal Pal. Fuzzy ϕ-tolerance competition graphs. *Soft Computing*, 21(13):3723–3734, 2017.

[29] Arundhati Raychaudhuri and Fred S Roberts. Generalized competition graphs and their applications. *Methods of Operations Research*, 49:295–311, 1985.

[30] FS Roberts. Competition graphs and phylogeny graphs. *Graph Theory and Combinatorial Biology, Bolyai Mathematical Studies*, 7:333–362, 1996.

[31] Debra D Scott. The competition-common enemy graph of a digraph. *Discrete Applied Mathematics*, 17(3):269–280, 1987.

[32] Suzanne M Seager. The double competition number of some triangle-free graphs. *Discrete Applied Mathematics*, 28(3):265–269, 1990.

[33] George Sugihara. Graph theory, homology and food webs. In *Proc. Symp. App. Math.*, volume 30, pages 83–101. American Mathematical Society, 1984.

[34] Li-hua You, Fang Chen, Jian Shen, and Bo Zhou. Generalized competition index of primitive digraphs. *Acta Mathematicae Applicatae Sinica, English Series*, 33(2):475–484, 2017.

[35] Yongqiang Zhao and Gerard J Chang. Note on the m-step competition numbers of paths and cycles. *Discrete Applied Mathematics*, 157(8):1953–1958, 2009.

[36] Yongqiang Zhao, Zhiming Fang, Yonggang Cui, Guoyan Ye, and Zhijun Cao. Competition numbers of several kinds of triangulations of a sphere. *Open Journal of Discrete Mathematics*, 7(02):54, 2017.

국문초록

Cohen(1968)은 생태계의 먹이사슬에서 포식자-피식자 개념을 연구하면서 경쟁그래프의 개념을 고안하였다. Cho 외(2000)은 경쟁그래프의 많은 변형들 중의 하나로서 m-step 경쟁그래프라는 개념을 만들어 내었고 P_n이 m-step 경쟁그래프가 될 수 있는 m과 n에 대한 문제를 제기하였다. Helleloid(2005)와 Kuhl 외(2010)은 이 문제에 대한 부분적인 답을 제시하였다. Belmont(2011)는 m-step 경쟁그래프인 패스에 대하여 완벽하게 규명하였다.

이 논문에서는 내차수가 0인 점을 갖는 수형도 유발 유향그래프에 대하여 연구하였다. 점을 3개 이상을 갖는 유향그래프 D가 2이상의 어떤 정수 m에 대한 m-step 경쟁그래프가 수형도일 때, D를 m-step 수형도 유발 유향그래프라고 부른다. m-step 수형도 유발 유향그래프를 수형도 유발 유향그래프라고 부른다. 우선, m-step 경쟁그래프가 수형도인 내차수가 0인 점을 갖는 유향그래프의 구조를 완전하게 규명하였다. 흥미롭게도, 내차수가 0인 점을 갖는 유향 그래프의 m-step 경쟁그래프가 수형도일 때는 항상 별 그래프임을 보였다. 최종적으로는 m-step 경쟁그래프가 수형도인 내차수가 0인 점을 갖는 유향 그래프의 개수를 구하였다.

주요어휘: 수형도, 별그래프, m-step 경쟁 그래프, 수형도 유발 유향그래프, 한가한 꼭짓점

학번: 2016–22901

감사의 글

돌이켜보니, 석사과정은 저의 학문적 역량의 폭을 좀 더 넓힐 수 있었던 값진 시간이었습니다. 연구는 단순히 지식의 탐구뿐만이 아니라 저 자신을 돌아보게 만들었고, 인내와 끈기를 가르쳐주었으며, 나아가 겸손함을 느끼게 해주었습니다. 부족한 제가 석사과정을 무사히 마칠 수 있었던 것은 많은 분들께서 도움을 주신 덕분이라고 생각합니다. 이에 감사의 마음을 전하고자 글을 올립니다.

매 순간 최선을 다해 지도해주셨던 김서령 교수님께 깊이 감사드립니다. 여러 아이디어들을 간결하고 논리적으로 표현함에 있어서 어려움을 겪고 있었던 제가 교수님의 섬세한 지도와 아낌없는 조언들 덕분에 한 단계 더 발전할 수 있었습니다. 또한, 학문에 있어서는 냉철하고 진지하신 모습과 제자들에게는 많은 애정과 가르침을 주셨던 교수님을 보면서 연구자로서의 자세와 교육자로서의 면모를 배울 수 있었습니다. 언제나 진심이 담긴 격려와 긍정적인 기대를 견지해주셨던 교수님의 가르침을 잊지 않고, 훌륭한 교육자이자 연구자가 될 수 있도록 더욱 정진하겠습니다.

저희 팀 식구들에게도 정말 고마웠습니다. 대학원에서 생활하는 동안 많이 챙겨주시고 즐거운 팀 분위기를 만들어주셨던 민기 형님, 대학원의 입학부터 졸업까지 많은 도움을 줬던 고마운 마그, 출중한 연구 능력으로 연구자의 모범을 보여주셨던 수강이 형님, 여러 가지 팀 운영 전반에 애써주고 사소한 질문들에도 친절하게 답해준 홍군, 임용 동기이자 대학원에서는 같은 팀으로 만나게 된 다재다능한 호준이, 넘치는 에너지와 흥으로 연구실 생활에 활기를 더해준 성철이, 열심히 하는 자세로 저의 초심을 일깨워주신 임청 선생님, 파이선 활용을 도와준 정솔 선생님께 진심으로 감사합니다. 또한, 학부 때부터 대학원 생활까지 많은 조언을 주셨던 병주 형님, 그리고 교직 현장에서 훌륭한 교사로 학생들을 지도하고 계신 진환이 형님, 상은 누나, 그리고 예은이 모두 고마웠습니다. 그리고 파이선을 다루는 것에 많은 도움을 준 석희에게 진심으로 감사함을 전합니다.

교직 생활에서 잠시 벗어나 대학원 생활에 전념할 수 있도록 응원해주시

고 지지해주신 불암중학교의 선생님들, 그리고 학생들에게도 감사합니다. 일년 반의 휴직 기간 동안 교직 생활을 그리워했었고 그 소중함을 다시 느낄 수 있었습니다. 풍부한 경험에서 비롯된 조언들과 함께 교사의 전문성을 몸소 보여주신 불암중학교 김종복 부장님께 깊은 감사를 표합니다. 또한, 지난 학부시절의 삶에 대한 태도에서부터 학문적 기반의 토대까지 단단하게 다질 수 있도록 많은 가르침을 주셨던 고려대학교 수학교육과 권순희 교수님께 진심으로 감사합니다. 철없는 동생을 항상 챙겨주는 친구보다 가까운 우리 누나, 늘 긍정적인 믿음을 바탕으로 아낌없는 응원을 보내주시는 부모님께 마음 깊은 곳에서부터 감사와 사랑을 전합니다.

미국의 사상가 랠프 월도 에머슨은 “진정한 성공이란, 작은 정원을 가꾸든 사회 환경을 개선하든, 자기가 태어나기 전보다 세상을 조금이라도 더 살기 좋은 곳으로 만들어 놓고 떠나는 것”이라 하였습니다. 대학원에서 기른 역량을 바탕으로 세상에 긍정적인 영향을 줄 수 있는 사람으로 성장하기를 희망합니다. 이제 저는 새로운 출발선 위에 다시 서 있습니다. 열정 가득하게 살겠습니다. 감사합니다.