Polynomial Normal Transform Based on L-moments and Its Application to Structural Reliability

Ming-Na Tong
Graduate Student, School of Civil Engineering, Central South University, 22 Shaoshannan Road, Changsha 410075, China

Zhao-Hui Lu
Professor, Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing, China (corresponding author: luzhaohui@bjut.edu.cn)

Yan-Gang Zhao
Professor, Department of Architecture, Kanagawa University, Yokohama, Japan

ABSTRACT: Information on the distribution of the basic random variable is essential for the accurate analysis of structural reliability. The usual method for determining the distributions is to fit a candidate distribution to the histogram of available statistical data of the variable. Generally, such candidate distribution would have parameters that may be evaluated from the statistical moments of the statistical data. Probability distributions are usually determined using one or two parameters evaluated from the mean and standard deviation of statistical data. However, these distributions are not flexible enough to represent the skewness and the kurtosis of statistical data. Normal transformation is often used in probabilistic analysis especially when multivariate non-normal random variables are involved. This study proposes a probability distribution based on polynomial normal transform, of which parameters are determined using the first four L-moments (L-mean, L-standard deviation, L-skewness and L-kurtosis) of the available data. The simplicity, generality, flexibility and advantages of this distribution in statistical data analysis are discussed. The results are found to better than two- and three-parameter distributions, and similar to cubic normal distribution based on central moments (C-moments). With the aiming at illustrate the stability of polynomial normal transform based on L-moments, several extreme values are added to data. The proposed distribution is demonstrated to provide significant stability and flexibility. Then this method is applied to reliability index calculation, and its significance in structural reliability evaluation is discussed. The calculation results are compared with Monte Carlo calculations. Several numerical examples are further presented to demonstrate the accuracy and efficacy of the distribution for conducting reliability analyses.

1. INTRODUCTION

The reliability analysis of uncertain structural systems subjected to random loads is of great engineering interest and poses a challenging computational problem. A large number of reliability analysis methods have been developed during the last four decades, such as Monte Carlo simulation (MCS) (Melchers 1990; Gongkang Fu 1994), first-order reliability method (FORM) (Hasofer and Lind 1974; Rackwitz 1976), second-order reliability method (SORM) (Der Kiureghian et al. 1987; Der Kiureghian and De Stefano 1991; Cai and Elishakoff 1994), the moment methods and simulation methods for estimating the failure probability (Grigoriu 1982; Zhao and Ono 2001). These methods assume that their basic random variables have known PDFs or cumulative distribution functions (CDFs). The normal transformation can be applied using the Rosenblatt (Hohenbichler and Rackwitz 1981) or Nataf transformation (Liu and Kiureghian 1986) with known CDFs/PDFs. However, as we all known, the distributions of the random variables are often unknown in many practical engineering problems, so it is uneasy to realize the polynomial normal transformation by the conventional methods. To solve the problem, Fleishman suggested the third-order polynomial normal transformation, in which polynomial normal transformation can be realized by using the first four central moments (C-moments), which is generally formulated as follows

$$x = S_u\left(u\right) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 \qquad (1)$$

where x is the original non-normal random variable; u is the standard normal random variable; $S_u(u)$ is a third-order polynomial of u; and a_0,a_1, a_2, a_3 are the polynomial coefficients.

The linear moment (L-moment) is an alternative moment system used to characterize the shape of a probability distribution. This concept was proposed by Hosking (Hosking 1990), compared with C-moments, L-moments possess stronger robustness with respect to sample length and outliers as they are defined from expectations of linear combinations of order statistics (Pandey et al. 2001; MacKenzie and Winterstein 2011). Statistical uncertainties can be more efficiently eliminated by L-moments than by C-moments.

The objective of this paper is to measure the performance of the polynomial normal transform based on L-moments, and the polynomial normal transform based on L-moments is compared with the polynomial normal transform based on C-moments, then the polynomial normal transform based on L-moments is applied to structural reliability analysis.

2. THE FIRST FOUR L-MOMENTS OF A RANDOM VARIABLE

When the PDF/CDF of a random variable is known, the first four L-moments can be expressed as

$$\lambda_1 = \int_0^1 x dF(x) \qquad (2)$$

$$\lambda_2 = \int_0^1 x[2F(x)-1]dF(x) \qquad (3)$$

$$\lambda_3 = \int_0^1 x[6F^2(x)-6F(x)+1]dF(x) \qquad (4)$$

$$\lambda_4 = \int_0^1 x[20F^3(x)-30F^2(x)+12F(x)-1]dF(x) \qquad (5)$$

in which $F(\cdot)$ is the cumulative distribution function (CDF) of a random variable.

But when the CDF is unknown, and only the data is known, Eqs.(2-5) cannot be used to calculate the first four L-moments. If X is a discrete random variable with an ordered set of data of size n, ($x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$), its first four L-moments, are

$$l_1 = n^{-1}\sum_{j=1}^{n} x_{j:n} \qquad (6)$$

$$l_2 = 2n^{-1}\sum_{j=2}^{n}\frac{(j-1)}{(n-1)}x_{j:n} - n^{-1}\sum_{j=1}^{n} x_{j:n} \qquad (7)$$

$$l_3 = 6n^{-1}\sum_{j=3}^{n}\frac{(j-1)(j-2)}{(n-1)(n-2)}x_{j:n} - 6n^{-1}\sum_{j=2}^{n}\frac{(j-1)}{(n-1)}x_{j:n} + n^{-1}\sum_{j=1}^{n} x_{j:n} \qquad (8)$$

$$l_4 = 20n^{-1}\sum_{j=4}^{n}\frac{(j-1)(j-2)(j-3)}{(n-1)(n-2)(n-3)}x_{j:n} - 30n^{-1}\sum_{j=3}^{n}\frac{(j-1)(j-2)}{(n-1)(n-2)}x_{j:n} + 12n^{-1}\sum_{j=2}^{n}\frac{(j-1)}{(n-1)}x_{j:n} - n^{-1}\sum_{j=1}^{n} x_{j:n} \qquad (9)$$

3. POLYNOMIAL NORMAL TRANSFORM BASED ON L-MOMENTS

3.1 The u-x transformation based on L-moments

For a random variable, if the first four L-moments (L-mean λ_1, L-scale λ_2, L-skewness λ_3, L-kurtosis λ_4) are known, an explicit and simple solution of the polynomial coefficients in Eq.(1) can be obtained by using the numerical values of $C_{m,n}$ computed by Tung (1999)

$$a_0 = \lambda_1 - 1.81379937\lambda_3 \qquad (10a)$$

$$a_1 = 2.25518617\lambda_2 - 3.93740250\lambda_4 \qquad (10b)$$

$$a_2 = 1.81379937\lambda_3 \qquad (10c)$$

$$a_3 = -0.19309293\lambda_2 + 1.574961\lambda_4 \qquad (10d)$$

3.2 The x-u transformation based on L-moments

The key to find the x-u transformation is to find the root of Eq.(1), according to Cardano formula, the most common root can be expressed as

$$u = -\frac{p}{\sqrt[3]{-q/2+\sqrt{\Delta}}} + \sqrt[3]{-q/2+\sqrt{\Delta}} - \frac{a}{3} \quad (11)$$

in which $p = \frac{3b-a^2}{9}$, $q = \frac{2}{27}a^3 - \frac{ab}{3} + c - \frac{x}{a_3}$, $\Delta = p^3 + \frac{q^2}{4}$, $a = a_2/a_3$, $b = a_1/a_3$, $c=a_0/a_3$.

4. THE POLYNOMIAL NORMAL DISTRIBUTION

The distribution is defined on the base of the polynomial normal transformation, which is expressed as Eq.(1).

The CDF and PDF of x can be expressed as Eqs. (12a) and (12b), respectively:

$$F(x) = \Phi(u) \quad (12a)$$

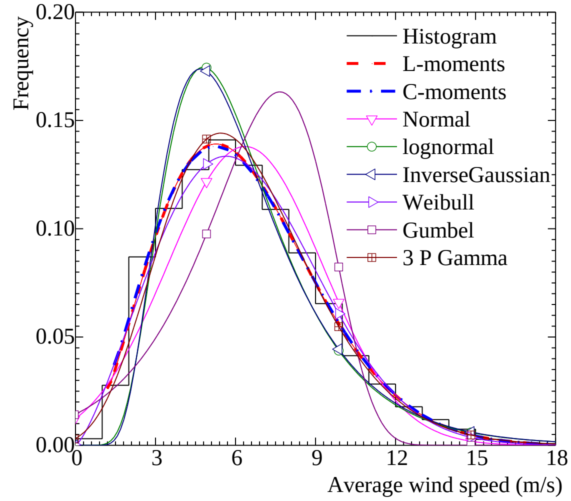
$$f(x) = \frac{\phi(u)}{a_1 + 2a_2u + 3a_3u^2} \quad (12b)$$

in which $F(\cdot)$ and $f(\cdot)$ are the CDF and PDF of x, respectively; $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and PDF of standard normal random variable u, respectively.

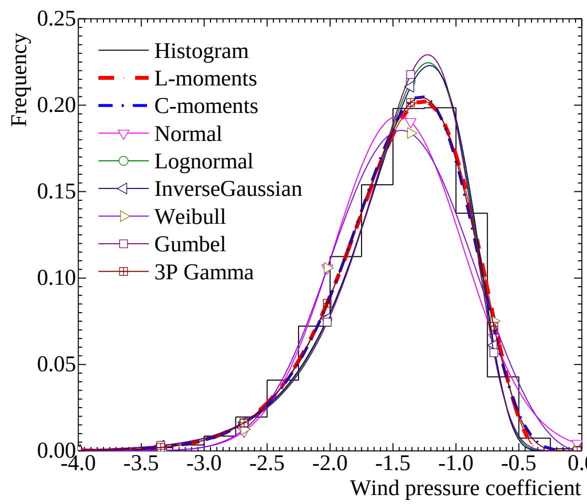
5. INVESTIGATION AND APPLICATION

5.1. Application to data fitting

To illustrate the flexibility of the polynomial normal distribution based on L-moments in fitting statistical data, two sets of data are investigated in the first example. Set A is observed data of the daily average wind speeds in the Republic of Ireland at a special station named as RPT (http://lib.stat.cmu.edu/datasets), set B is experimental data on the wind pressure coefficient of a wall-mounted finite-length square cylinder (Wang et al. 2015). The corresponding histograms are presented in Figure 1, which depicts the PDFs of the 2P distributions, i.e., Normal, Lognormal, Weibull, Gumbel and inverse Gaussian distributions whose mean values and standard deviations are equal to those of the data; the PDFs of the 3P gamma distributions whose first three moments are equal to those of the data; and the PDF of the cubic normal distribution based on C-moments and L-moments whose first four moments are equal to those of the data. Figure 1 reveal that the polynomial normal distribution based on C-moments and L-moments fits the histogram much better than the 2P distributions and the 3P gamma distribution for all cases, and the results of the polynomial normal distribution based on C-moments and L-moments are in close agreement with the histograms of the statistical data.



(a)Set A: Daily average wind speed

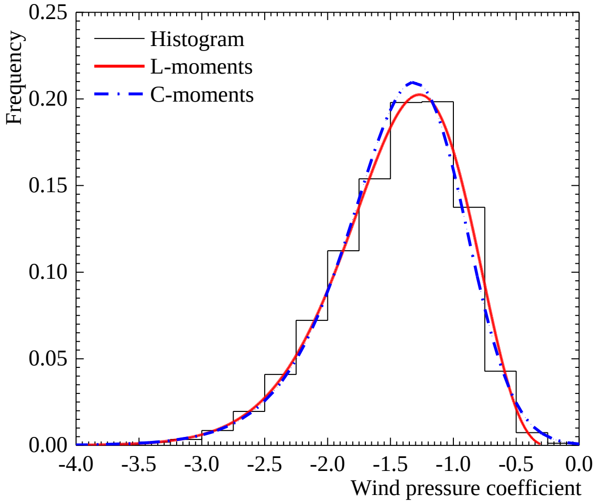


(b) Set B: Experimental wind pressure coefficient

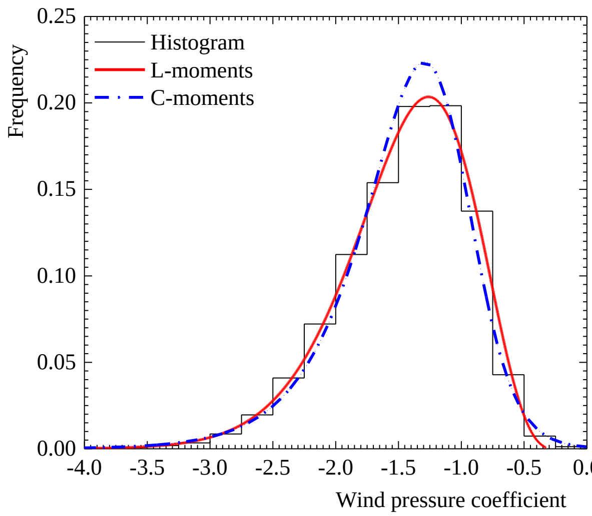
Figure 1: Comparison between some 2P, 3P, the polynomial normal distribution based on C-moments and L-moments in fitting actual data.

To verify the stability of L-moments when extreme values are existed, ten maximum value and ten minimum value are added to set B respectively (a total of 19999). The corresponding results are presented in Figure 2, which reveals that the polynomial normal distribution based on L-moments can be almost approach the data, and thus can fit the histogram much better than the polynomial normal distribution based on C-moments.

From the examples above, one can clearly see that L-moments are stable than C-moments in face of the extreme value when used to the polynomial normal distribution.



(a)Add maximum value



(b) Add minimum value

Figure 2: Comparison between the polynomial normal distribution based on L-moments and C-moments in facing of extreme value.

5.2 Application to first -order reliability analysis

Based on the proposed method, FORM for reliability analysis for independent variables can be readily realized. The computation procedure for FORM based on the proposed method is described as follows:

(1) Obtain the first four L-moments of each random variable for all variables and original correlation matrix for correlated variables by the probability information.

(2) Assume an initial checking point $\mathbf{x}_0$ (generally take the mean value).

(3) Obtain the corresponding checking point in the independent standard normal space, $\mathbf{u}_0$.

(4) Determine the initial reliability index β_0 .

(5) Determine the Jacobian matrix $\mathbf{J} = \dfrac{\partial \mathbf{X}}{\partial \mathbf{U}}$ evaluated at $\mathbf{u}_0$, where the Jacobian matrix is given by

$$\frac{\partial X_i}{\partial U_j} = l_{ij}\left[a_{1i} + 2a_{2i}\sum_{k=1}^{i} l_{ik}U_k + 3a_{3i}\left(\sum_{k=1}^{i} l_{ik}U_k\right)^2\right], \quad (i, j = 1, 2, \ldots, n) \tag{13}$$

(6) Determine the value of the gradient vector at u_0 and the value of the function in normal space:

$$G_u(\mathbf{u}_0) = G(\mathbf{x}_0) \tag{14}$$

$$\nabla G(\mathbf{u}_0) = \mathbf{J}^T \nabla G(\mathbf{x}_0) \tag{15}$$

in which $G(\mathbf{x}_0)$ can be computed directly by taking the derivative for explicit performance functions and can be computed by numerical differentiation such as central difference method for implicit functions. The central difference method is presented as follows:

$$\nabla G(x_0)=\begin{pmatrix} \dfrac{G(x_{01}+h,x_{02},...,x_{0n})-G(x_{01}-h,x_{02},...,x_{0n})}{2h_1} \\ . \\ . \\ . \\ \dfrac{G(x_{01},...,x_{0i}+h,...,x_{0n})-G(x_{01},...,x_{0i}-h,...,x_{0n})}{2h_i} \\ . \\ . \\ . \\ \dfrac{G(x_{01},x_{02},...,x_{0n}+h)-G(x_{01},x_{02},...,x_{0n}-h)}{2h_n} \end{pmatrix} \quad (16)$$

Calculate the new check point:

$$\mathbf{u}^{(k+1)}=\frac{\nabla G(\mathbf{u}^{(k)})}{\nabla G^T(\mathbf{u}^{(k)})\nabla G(\mathbf{u}^{(k)})}[\nabla G^T(\mathbf{u}^{(k)})\mathbf{u}^{(k)}-\nabla G(\mathbf{u}^{(k)})] \quad (17)$$

$$\mathbf{x}^{(k+1)}=\mathbf{x}^{(k)}+\mathbf{J}(\mathbf{u}^{(k+1)}-\mathbf{u}^{(k)}) \quad (18)$$

The corresponding reliability index can be calculated as $\beta=(\mathbf{u}_1^T\cdot\mathbf{u}_1)^{1/2}$.

(7) Calculate the absolute difference between β and β_0 until $|\beta-\beta_0|\le\varepsilon$, where ε is the permissible error (generally ε=10^{-6}).

Otherwise, repeat the step 3 through step 8 until convergence is achieved.

The second example considers a simplified bridge model formed by two girders and two continuous spans as shown in Figure 3 (Ghosn and Frangopol 1999). Assuming plastic behavior, one collapse mechanism for this bridge is shown in Figure 4. The collapse mechanism can be represented by a limit state function (LSF), Z_1, which can be written as:

$$Z_1=2(M_1-D_1)+(M_2-D_2)-\frac{P\times L_1}{2} \quad (19)$$

where M_i is the moment capacity at section i, D_i is the dead load moment at section i, P is the applied maximum lifetime truck load, and L_j is the length of span j. The concentrated load P is used to model the weight of an applied truck. Table 1 gives the properties of the random variables. The applied load is represented as a function of the HS-20 AASHTO design truck.

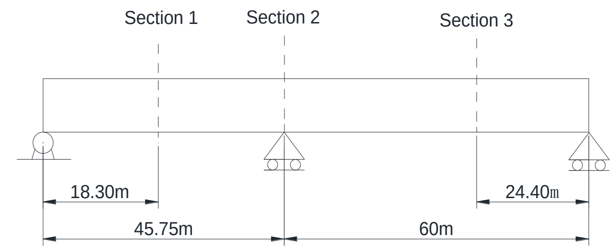


Figure 3: Two-girder continuous bridge configuration.

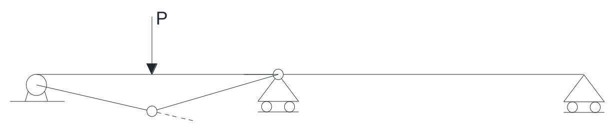


Figure 4: Two span bridge collapse mechanism.

Table 1: Random variables for Example 2.

Section	Variable	Distribution	Mean	Bias COV(%)
Section 1	Moment capacity	Lognormal	8190	1.12 10
	Dead load	Normal	3640	1.05 9
	Live load	Extreme I	HS-20	2.07 19
Section 2	Moment capacity	Lognormal	23400	1.12 10
	Dead load	Normal	13755	1.05 9
	Live load	Extreme I	HS-20	2.07 19

Note: COV denotes coefficient of variation

Using the proposed method, the reliability index can be obtained as 3.56. Using the MCS with 4×10^{-6}, the reliability index can be obtained as 3.51. Apparently, the results obtained by the proposed method agree well with the results obtained by MCS.

6. CONCLUSION

In this paper, the third-order polynomial normal transform based on L-moments is investigated, four-parameter distribution based on this proposed transformation is proposed, and a first-order reliability analysis method based on the transformation is developed. Numerical examples show that L-moments are more stable than C-moments, and L-moments are convenient for structural reliability analysis, especially when the probability distribution is unknown.

7. ACKNOWLEDGEMENTS

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