Importance measures for inspections in binary networks

Chaochao Lin
Graduate Student, Dept. of Civil and Environmental Engineering, Carnegie Mellon University, Pittsburgh, USA

Matteo Pozzi
Associate professor, Dept. of Civil and Environmental Engineering, Carnegie Mellon University, Pittsburgh, USA

Junho Song
Professor, Dept. of Civil and Environmental Engineering, Seoul National University, Seoul, Korea

ABSTRACT: Many infrastructure systems can be modeled as networks of components with binary states (intact, damaged). Information about components' conditions is crucial for the maintenance process of the system. However, it is often impossible to collect information of all components due to budget constraints. Several metrics have been developed to assess the importance of the components in relation to maintenance actions: an important component is one that should receive high maintenance priority. Instead, in this paper we focus on the priority to be assigned for component inspections and information collection. We investigate metrics based on system level (global) and component level (local) decision making after inspection for networks with different topology, and compare these results with traditional ones. We then discuss the computational challenges of these metrics and provide possible approximation approaches.

1. Introduction

Many civil infrastructures consist of multiple binary components, arranged as a network to fulfill the function of the system. The binary states of the components, either intact or damaged, determine the system condition. The belief of the agent controlling the maintenance process can be described by a probabilistic distribution on the possible states of the components. Maintenance actions are selected to trade off the risk of system malfunctioning with the cost of maintenance (repair and retrofitting). Observations of the components' states can improve decision making and reduce the uncertainty and maintenance cost. However, due to budget constraints, it is often impossible to inspect all components. Therefore it is important to assign inspection priorities among components. Therefore it is important to assign inspection priorities among components.

Traditionally, importance measures have been developed for assigning priority to maintenance actions. Based on the needed input to determine the rank among the components, they can be categorized into structure, reliability and lifetime measures (Birnbaum, 1968). The structure importance measures take the topology structure of the network as input, such as the Fussell-Vesely structure importance (Vesely, 1970) and the permutation importance (Boland et al., 1989). The reliability importance measure also considers the failure probability of the system and the components, such as Birnbaum reliability importance (Birnbaum, 1968). The lifetime importance measures further take into account the varying distribution of failure probabilities over time, such as Birnbaum lifetime importance (Lambert, 1975).

Hwang (2005) summarized the dominant relations among importance measures of different forms. If importance measure A is dominant over B, then when component i has higher priority in A, it will also have higher priority in B.

2. IMPORTANCE MEASURES FOR BINARY SYSTEMS

When a binary system is composed of N components, $c_1, \cdots, c_N$, their conditions are described by vector $\mathbf{s} = (s_1, s_2, \cdots, s_N)$, where $s_i = 1$ if component c_i functions, and $s_i = 0$ if c_i fails. The system state $u = \phi(\mathbf{s})$ is also a binary variable, where $\phi : \mathbb{B}^N \to \mathbb{B}$ is the component-to-system function, and $\mathbb{B} = \{0, 1\}$.

Birnbaum structure importance measure (Birnbaum, 1968) quantifies whether component c_j is essential for the system as:

$$\delta_j(\mathbf{s}) = \phi_{j,1}(\mathbf{s}) - \phi_{j,0}(\mathbf{s}) \tag{1}$$

where: $\phi_{j,a} = \phi(s_1, \cdots, s_j = a, \cdots, s_N)$.

c_j is essential for the system in state $\mathbf{s}$ when $\delta_j(\mathbf{s}) = 1$: this means that the system functions if and only if that component does.

The reliability Birnbaum measure considers the possible failure of the components and the system:

$$\hat{\delta}_j = \mathbb{E}[\delta_j(\mathbf{s})] = P_{\omega|s_j=0} - P_{\omega|s_j=1} \tag{2}$$

where $P_{\omega|s_j=0}$ is the posterior system failure probability when component c_j is damaged, and $P_{\omega|s_j=1}$ is the posterior probability when c_j is functioning (properly, the first equality of (2) holds only for independent components, while the second is general).

Another intuitive approach for measuring importance, directly related to the cost of maintenance and system failure risk, is to compare the difference of the expected cost before and after maintenance. We define the system failure cost as C_F and replacement cost for component j as C_{R_j}. The value of maintenance for component j is defined as:

$$\xi_j = P_\pi C_F - (P_{\omega|s_j=1} C_F + C_{R_j}) \tag{3}$$

where P_π is the prior system failure probability.

Generally, the metrics can be formulated under the Critical Node Detection Problem (Lalou et al., 2018) for different applications, such as network robustness, security analysis, etc. However, as far as we know, few research has focused on the inspection process, i.e. how additional information collected at component level can affect the results of the importance measures. In this paper, we assess the value of information (VoI), i.e. the difference of expected cost before and after inspection (Malings and Pozzi, 2016) to assign inspection priorities among components, and compare the result with the extended Birnbaum measure of (2) and the value of maintenance of (3).

3. SYSTEM AND COMPONENT LEVEL ACTIONS FOR BINARY SYSTEMS

We assume that observations of the components are perfect and that any subset of components can be replaced. Given the topology of the system, the prior (joint) failure probability of the components, and the cost of system failure and component replacement, we are focusing on the following question: what is the most valuable component to inspect?

The VoI for inspecting component c_i can be formulated as:

$$\mathrm{VoI}(i) = C_\pi - \mathbb{E}C_{\omega,i} \tag{4}$$

where the expected posterior cost is:

$$\mathbb{E}C_{\omega,i} = P_i C_{\omega,s_i=0} + (1 - P_i) C_{\omega,s_i=1}$$

P_i is the failure probability for c_i, and C_π is the prior cost.

We define two alternative metrics: global metric for system level replacement and local metric for component level, to investigate the VoI.

3.1. *Global metric*

The global metric only allows two actions: doing nothing or replacing all components. The uncertainty on binary variable u, defining the system state, can be quantified by a penalty function f, that we can call *classification cost function*, and it depends on the system failure probability P. Examples of penalty functions are the misclassification rate, Shannon entropy and Gini index (Murphy, 2012).

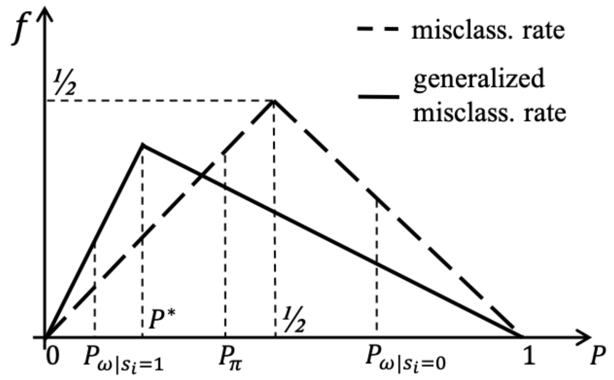

Figure 1: Misclassification and generalized misclassification rate

Here we assume that $f(P)$ can be any function that satisfies the following properties: (i) f is a concave function defined on $[0,1]$; (ii) $f(0) = f(1) = 0$. This is because when the system is known to fail ($P = 0$) or work ($P = 1$), there is no misclassification error.

Fig 1 shows two possible penalty functions. The misclassification rate is the probability that the optimal guess of the system state u is incorrect, which is $f = \min\{P, 1-P\}$. We define the cost to replace all components as $C_R = \sum_i C_{R_i}$. The generalized misclassification rate minimizes the expected cost of the corresponding action, as $f = \min\{(C_F - C_R)P, (1-P)C_R\}$.

3.1.1. Value of Information

The VoI is the expected reduction of penalty f. The global metric calculates VoI for each component and selects the one with the highest. From (4), the VoI of the global metric is:

$$\mathrm{VoI}_G(i) = f(P_\pi) - \mathbb{E}f_{\omega,i} \geq 0 \qquad (5)$$

where the expected misclassification cost after inspection is:

$$\mathbb{E}f_{\omega,i} = (1-P_i)f(P_{\omega|s_i=1}) + P_i f(P_{\omega|s_i=0}) \qquad (6)$$

Because the classification penalty f is concave, the VoI is always non-negative according to Jensen's Inequality.

3.1.2. Global metric on series systems

A series system works if and only if all components function properly. For this system, the global metric will always select the most vulnerable component, i.e. the component with highest failure probability. This conclusion also holds for systems with interdependent components. This can be proved as follows.

Because of the law of total expectation, the prior and posterior probabilities can be related as:

$$P_\pi = (1-P_i)P_{\omega|s_i=1} + P_i P_{\omega|s_i=0} \qquad (7)$$

In a series system, $P_{\omega|s_i=0} = 1$. As $f(P_{\omega|s_i=0}) = 0$, we derive:

$$\begin{aligned}\mathrm{VoI}_G(i) &= f(P_\pi) - (1-P_i)f(P_{\omega|s_i=1}) \\ &= f(P_\pi) - R_\pi \cdot \frac{f(P_{\omega|s_i=1})}{1-P_{\omega|s_i=1}}\end{aligned}$$

where

$$P_{\omega|s_i=1} = 1 - \frac{1-P_\pi}{1-P_i} \in [0,1]$$

is a non-increasing function of P_i, and prior reliability

$$R_\pi = 1 - P_\pi \geq 0$$

is independent of the component.

Ignoring the constant, because $f(x)/(1-x)$ is a non-increasing function of x (the proof is included in the appendix), the component with highest P_i will have the highest VoI. Notice that the proof does not require the statistical independence between components' states.

3.1.3. Global metric on parallel systems

A parallel system will function if and only if at least one component is intact. For such systems, the global metric will focus on the most reliable component instead, i.e. the component with lowest failure probability. This conclusion is also true for systems with interdependent components.

In a parallel system, $P_{\omega|s_i=1} = 0$ since any functioning component can prevent the system failure. Therefore $f(P_{\omega|s_i=1}) = 0$, and:

$$\mathrm{VoI}_G(i) = f(P_\pi) - P_{s_i=0} f(P_{\omega|s_i=0})$$

Again, (7) holds even when the failure probability of each component is dependent. Hence $\mathrm{VoI}(i)$ can be simplified as:

$$\mathrm{VoI}_G(i) = f(P_\pi) - R_\pi \cdot \frac{f(y_i)}{1-y_i}$$

where

$$y_i = 1 - \frac{R_\pi}{P_i} \in [0,1]$$

is a non-decreasing function of P_i.

Ignoring the constant, because $f(y)/(1-y)$ is a non-decreasing function of y, which is a non-decreasing function of P_i, the component with lowest P_i will have the highest VoI.

3.1.4. *Computational complexity for general systems*

The main complexity in applying the global metric is to compute the posterior probabilities $P_{\omega|s_i=0}$ and $P_{\omega|s_i=1}$. This can be tackled by numerical approaches, including Monte Carlo simulations.

3.1.5. *Comparison to Birnbaum reliability importance*

For independent components, the Birnbaum reliability importance can be measured by (2).

For series systems, we have: $\hat{\delta}_j = \dfrac{R_\pi}{1-P_i}$;

For parallel systems, we have: $\hat{\delta}_j = \dfrac{1-R_\pi}{P_i}$.

Obviously, we will select the most vulnerable component in series systems and the most reliable component in parallel systems, which is consistent with the global metric.

For general systems, such consistency may be violated. Consider a general system and its two components c_i and c_j, such that:

$$\begin{aligned}
&P_\pi = 0.4375 \\
&P_i = 0.875, P_{\omega|s_i=0} = 0.50, P_{\omega|s_i=1} = 0 \\
&P_j = 0.167, P_{\omega|s_j=0} = 0.75, P_{\omega|s_j=1} = 0.375
\end{aligned}$$

Since $\hat{\delta}_i = 0.5 > \hat{\delta}_j = 0.375$, the Birnbaum's measure will prioritize component c_i. Suppose:

$$f(P) = \min\{P, 1-P\}$$

VoI for component c_i and c_j is:

$$\begin{aligned}
&\text{VoI}_G(i) = 0 \\
&\text{VoI}_G(j) = f(P_\pi) - 0.354 = 0.083
\end{aligned}$$

thus the global metric prioritizes c_j for inspection.

3.2. *Local metric*

The previous global metric refers to a single action at system level (e.g. replace all components). The local metric refers to actions at component level. The general decision process is to first select a component to inspect and then to select a set of components to replace based on the inspection outcome. The total cost is the sum of replacement cost and the system risk after replacement.

We define $\mathbf{r}$ as the binary decision vector for the components. If $r_i = 1$, component i is replaced; and $r_i = 0$ otherwise. $C_{T,\omega,s_i=0}$ is the expected cost when we discover that component i has failed, and $C_{T,\omega,s_i=1}$ is the cost when component i is working accordingly. $\mathbf{C}_R = (C_{R_1}, \cdots, C_{R_N})$ is a replacement cost vector for all components. The optimization searches for the replacement decision vector $\mathbf{r}$ to minimize the total cost. When the components are independent, the expected total cost for c_i can be calculated as follows:

$$\begin{aligned}
&C_{T,\omega,s_i=0} = \min_{\mathbf{r}}\{P_{\omega|s_j=1,\forall r_j=1}C_F + \mathbf{C}_R^T\mathbf{r}\} \\
&C_{T,\omega,s_i=1} = \min_{\mathbf{r}}\{P_{\omega|s_i=1,s_j=1,\forall r_j=1}C_F + \mathbf{C}_R^T\mathbf{r}\} \\
&\mathbb{E}[C_{T,\omega,i}] = P_i C_{T,\omega,s_i=0} + (1-P_i)C_{T,\omega,s_i=1}
\end{aligned} \quad (8)$$

The component with highest VoI can be found by:

$$\arg\min_i \mathbb{E}[C_{T,\omega,i}]$$

3.2.1. *Value of Information*

The VoI (4) according to the local metric is defined as the difference of the expected total cost before and after inspection on component i:

$$\text{VoI}_L(i) = C_{T,\pi} - \mathbb{E}[C_{T,\omega,i}]$$

Here $C_{T,\pi} = C_F P_\pi$ is the expected total cost before inspection.

3.2.2. *Local metric on series systems*

For series systems, if components' states are independent, (8) reads:

$$\begin{aligned}
&C_{T,\omega,s_i=0} = \min_{\mathbf{r}}\{(1 - \prod_{j:r_j=0}(1-P_j))C_F + \mathbf{C}_R^T\mathbf{r}\} \\
&C_{T,\omega,s_i=1} = \min_{\mathbf{r}}\{(1 - \prod_{j:r_j=0, j\neq i}(1-P_j))C_F + \mathbf{C}_R^T\mathbf{r}\}
\end{aligned} \quad (9)$$

For small probability of failures, the series system under local metric can be simplified as a cumulative system. A cumulative system has independent and isolated components, i.e. no connection. The system cost is the sum of the cost for each component (Malings and Pozzi, 2016).

The system failure risk for a series system is:

$$RISK_s = C_F[1 - \prod_i (1-P_i)^{1-r_i}]$$

For a cumulative system with component failure cost C_F:

$$RISK_c = C_F \sum_i (1-r_i)P_i$$

By neglecting higher order terms, the risk for series system can be written as:

$$RISK_s = C_F[\sum_i (1-r_i)P_i + o(P)] \approx RISK_c$$

The approximation reduces the computation complexity for series systems when searching for optimal replacement plans, since $r_i = 1$ if and only if $C_{R_i} \leq P_i C_F$. If we treat the series system as a cumulative system, the objective function is linear.

Other approximations for the local metric are provided in Section 4.

3.2.3. *Local metric for parallel systems*

The local metric will select the most reliable component in a parallel system, which is consistent with the global metric.

If component c_i is found to be working, no action at system level is needed. Based on the optimal action we take after we discover the component is damaged, we can divide the components into two groups: for component c_t in the first group, replace it if $s_t = 0$, and we get:

$$C_{T,\omega,t} = P_i \min_j C_{R_j} \leq P_t P_{\omega|s_t=0} C_F = \prod_j P_j C_F$$

For c_k in the second group, do nothing if $s_k = 0$, and we get:

$$C_{T,\omega,k} = P_k P_{\omega,s_k=0} C_F = \prod_j P_j C_F \geq C_{T,\omega,i}$$

Thus the optimal component can only be in the first group. Then we prove that $P_k \geq P_t$ for all c_k in the second group and c_t in the first group. Suppose we find $c_{t'}$ and $c_{k'}$ such that $P_{t'} > P_k$, we have:

$$C_{T,\omega,k'} = \prod_j P_j C_F \geq P_{t'} \min_t C_{R_j} > P_{k'} \min_j C_{R_j}$$

which means that for $c_{k'}$, replacing it cost less than doing nothing, and it belongs to the first group. However, we have assumed $c_{k'}$ is in the second group. Hence, the component with lowest expected posterior cost must be in the first group:

$$\arg\min_i C_{T,\omega,i} = \arg\min_i P_i \tag{10}$$

3.2.4. *Computational complexity*

As we can see from (8), even for a series system, the computational difficulties are not only the assessment of the posterior probability for different combinations of replacement plans, but also the optimization of the maintenance actions, which is a non-linear integer programming with non-linear objective function and linear constraints. Generally the problem is NP-hard (Hemmecke et al., 2010).

4. LOCAL METRIC UNDER SPECIAL ASSUMPTIONS

For series systems, we have introduced one possible approximation in Section 3.2.2 related to cumulative systems. For general systems, we restrict our attention on cases where the following three assumptions hold:

i. Optimism

We assume that the agent's belief is that the system is in good condition. The prior decision is to do nothing, which requires:

$$P_\pi \cdot C_F = \min_Q \mathbb{E}[C_{\omega,Q}] \tag{11}$$

where

$$\mathbb{E}[C_{\omega,Q}] = \sum_{i \in Q} C_{R_i} + P_{\omega|s_i=1,c_i \in Q} \tag{12}$$

Also, if any observation decreases the system failure probability, the agent will still do nothing. For example, if the agent discovers that components in subset T are functioning and there is no negative correlation among components' states, the agent will do nothing:

$$P_{\omega|s_i=1,c_i\in T}C_F = \min_{Q,Q\cap T=\emptyset} \mathbb{E}[C_{\omega,Q,T}] \quad (13)$$

where

$$\mathbb{E}[C_{\omega,Q,T}] = \sum_{i\in Q} C_{R_i} + P_{\omega|s_i=1,c_i\in Q\cup T} \quad (14)$$

ii. Responsiveness

We assume that when the agent discovers a component is damaged, the expected cost of replacing the component is less than that of doing nothing:

$$P_{\omega|s_i=1}C_F + C_{R_i} \leq P_{\omega|s_i=0}C_F,\ \forall i$$

iii. Coherence

We extend the original concept of "coherent" systems (Xie, 1987). We assume that the system failure probability cannot increase after discovering that some components are functioning. For any set Q:

$$P_{\omega|s_i=1,c_i\in Q} \leq P_{s,\pi}$$

Based on the above three assumptions, the optimal actions under different inspection outcomes can be derived explicitly:

i. If a component is functioning, the agent will do nothing due to optimism assumption;

ii. If component i is discovered to be damaged, the agent will only replace the component.

First, we prove that the agent will not replace any other components. Compare the two cost:

$$C_{R_i} + P_{\omega|s_i=1}C_F$$
$$C_{R_i} + C_{R_j} + P_{\omega|s_i=1,s_j=1}C_F$$

Because of the "Coherence" assumption, replacing any other components is equivalent as doing nothing and replacing c_j when we discover c_i is working and the system failure probability reduces. From "Optimism" assumption we know the former is a better choice.

Therefore, from the "Responsiveness" assumption the optimal decision is to replace the damaged component other than doing nothing.

Hence, the expected posterior cost after inspecting c_i is:

$$\mathbb{E}[C_{T,\omega,i}] = P_{\omega|s_i=1}C_F + C_{R_i}P_i \quad (15)$$

4.1. *Comparison with the replacement metric*

The replacement metric compares the difference of the cost before and after maintenance, as in (3). Because the replacement metric does not consider the inspection process, the results can be different from the approximate local metric of (15) (though they have similar closed forms).

Suppose the replacement cost is identical for every component, and $C_F/C_{R_i} = \alpha$. Following (3), minimizing ξ_i is equivalent to minimizing $P_{\omega|s_i=1}$. Instead, following (15), we minimize $P_{\omega|s_i=1}\alpha + P_i$. Our question is whether (16) holds.

$$\arg\min_i\{P_{\omega|s_i=1}\} = \arg\min_i\{P_{\omega|s_i=1}\alpha + P_i\} \quad (16)$$

Suppose we have $k = \arg\min_i\{P_{s,\omega|s_i=1}\}$, and there exists a component t, such that

$$P_{\omega|s_k=1}\alpha + P_k > P_{\omega|s_t=1}\alpha + P_t$$

i.e. if we can find component t that satisfies the above inequality, the results from the replacement metric and approximate local metric are different. Such example can be easily found.

Consider a series system with two independent components c_1 and c_2. Assume that $\alpha = 2$, $P_1 = 0.8$ and $P_2 = 0.6$. Thus $k = 1$ and $t = 2$, i.e. the replacement metric selects component 1 while the approximate local metric gives higher priority to component 2.

However, consider a system which has high α and the components have low failure probabilities, which is true for many infrastructure systems. For an independent series system, such counter example cannot be found and the two metrics produce the same result. In this example, previous inequality becomes:

$$1/(1-P_2) > \alpha$$

which is impossible when $\alpha > 2$ and $P_i < 0.5$.

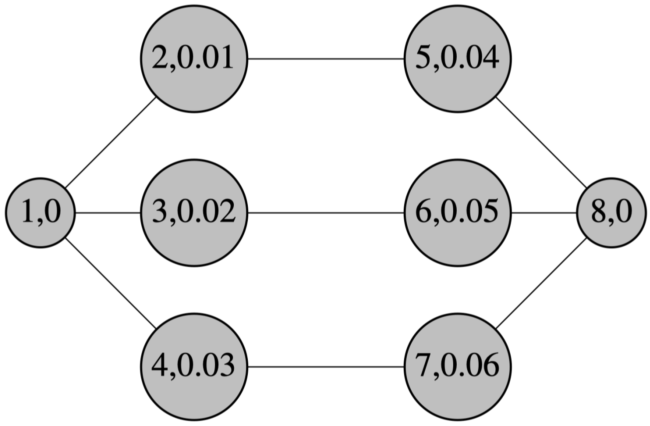

Figure 2: Examples of a general system. Inside each node, the first number is the node index, the second is the prior failure probability. Node 1 is the source and node 8 is the sink.

5. Examples of application to networks

In the following examples, we assume that the components have identical replacing costs of \$1K (the unit for all the following costs) and independent prior failure probabilities.

5.1. Inconsistency of prior action and optimal inspection

In Fig 2, we assume $\alpha = C_F/C_{R_i} = 16000$. The prior decision is to replace c_5, but the optimal inspection is c_2. Table 1 shows the action and cost under different inspections and outcomes. For example, if we inspect c_5 and discover it is intact, we do nothing with a cost of 0.97; otherwise, we replace c_5 with a cost of 1.97. Hence, the expected cost for inspecting c_5 is 1.01 (the cost unit is \$1K).

It is also obvious that the inspection outcome can change the prior replacing plans. For example, if we discover component 3 is intact, we decide to replace component 6 rather than c_5, which is the prior optimal.

5.2. Approximate local metric

When $\alpha = 4000$, Fig 2 becomes a system that satisfies the three assumptions in Section 4. Table 2 shows the optimal action under different inspection outcomes and its cost.

The optimal decision is to inspect c_5. The intuition is that the system is a parallel system made of series systems. We first focus on c_2 and c_5 since

Insp.	Intact	Damaged	$\mathbb{E}C$
2	5(1.00)	2,5(2.00)	1.0100
5	DN(0.9737)	5(1.9737)	1.0137
3	6(1.00)	3,6(2.00)	1.0200
6	3(1.00)	3,6(2.00)	1.0500
4	7(1.00)	4,7(2.00)	1.0300
7	4(1.00)	4,7(2.00)	1.0600

Table 1: Optimal repair plan and the cost for each inspection scenario, the cost unit is \$1K

Insp.	Intact	Damaged	$\mathbb{E}C$
2	DN(0.9737)	2(1.9737)	0.9837
5	DN(0.2434)	5(1.2434)	0.2834
3	DN(0.8749)	3(1.8749)	0.8949
6	DN(0.3500)	6(1.3500)	0.4000
4	DN(0.8214)	4(1.8214)	0.8514
7	DN(0.4107)	7(1.4107)	0.4707

Table 2: Optimal repair plan and the cost for each inspection scenario: approximate local metric, the cost unit is \$1K

they form the most reliable link in the parallel system; then we narrow the attention to c_5 as it is the most vulnerable in the series system of c_2 and c_5.

In this case, (16) holds, i.e. if we have to choose a component to replace without inspection, c_5 will also be the optimal choice.

5.3. General system example

Fig 3 shows a more complicated general network. We assume that the failure probability for each component is 0.05, except for components

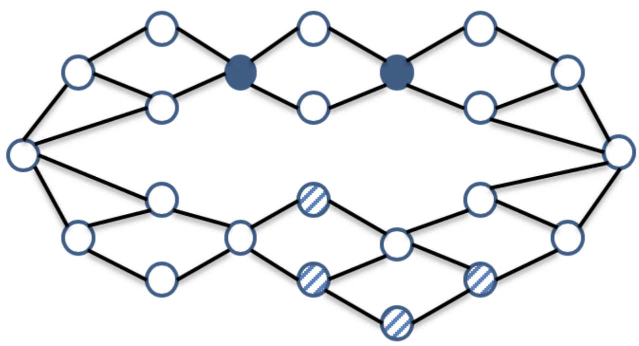

Figure 3: General system example

filled with diagonal stripes is 0.2, and $\alpha = 100$. The generalized global, local and approximate local metric select either one of the components with solid fill. When the prior decision is to do nothing, the posterior probability is dominant compared to component prior failure probability, as shown in (16), which causes the consistency.

6. DISCUSSIONS AND FUTURE WORK

This paper presents an overview of assigning priority for component inspections in binary systems, and develops global and local metrics based on the possible actions (system or component level). These metrics are based on the value of information. We have provided simple rules for assigning priorities among components in series and parallel systems. We have developed approximations for local metric in general networks. We have also compared these metrics with previous criterion on network examples. For parallel systems, global metric, local metric, and Birnbaum reliability importance provide the same results. For series systems, global metric and Birnbaum reliability importance prioritize the same component. For general systems, we have shown that the Birnbaum measure is not always consistent with the more general VoI metric.

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APPENDIX

Proof that $f(x)/(1-x)$ is non-decreasing

Suppose $x_1 < x_2$. Let $\alpha = \frac{1-x_2}{1-x_1} \in (0,1)$. Because $f(x)$ is concave, we know:

$$
\begin{aligned}
f(x_2) &= f(\alpha x_1 + (1-\alpha)\cdot 1) \\
&\geq \alpha f(x_1) + (1-\alpha) f(1) = \frac{1-x_2}{1-x_1} f(x_1) \\
\Leftrightarrow & \frac{f(x_2)}{1-x_2} \geq \frac{f(x_1)}{1-x_1}
\end{aligned}
$$