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Analysis of Structural Performance in the Framework of Imprecise Probabilities

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ABSTRACT: The analysis of structures with uncertain properties modeled as random variables with *imprecise Probability Density Function* (*PDF*) characterized by interval basic parameters (mean-value, variance, etc.) is addressed. A novel procedure able to provide approximate explicit expressions of the bounds of the interval mean-value and variance of the random stresses is proposed. The procedure stems from the joint application of the *Improved Interval Analysis* via *Extra Unitary Interval* and the *Rational Series Expansion*, introduced in the literature by the last two authors. The influence of imprecision of the *PDF* of the input parameters on structural performance is also investigated. For validation purposes, a 3D truss structure with uncertain Young's moduli is analyzed.

1. INTRODUCTION

It is now widely recognized that the results provided by the classical probabilistic structural analysis may be highly sensitive to the *basic* parameters (e.g., mean-value, variance, etc.) of the *Probability Density Function* (*PDF*) characterizing the uncertain properties. Indeed, a small change in the mean and variance values of the uncertain parameters may cause a large variation in the outcome of structural reliability assessment (Ben-Haim 1994, Elishakoff 1995).

To take into account the imprecise character of available information, over the last decades the theory of *imprecise probability* has been developed as a generalization of the classical probabilistic analysis. An *imprecise* probability arises when the probability for an event is bounded by a lower value and an upper value of the probability for the same event (see e.g., Walley 1991, Weichselberger 2000, Utkin and Kozine 2010, Beer et al. 2013).

The *interval analysis* (Moore et al. 2009) has proved to be an effective tool to evaluate the bounds of response statistical moments as well as to perform reliability assessment under uncertain parameters described by imprecise information (Jiang et al. 2011, Muscolino and Sofi 2017). Furthermore, it has been proved that, if the basic parameters of the *PDF* are modelled as intervals, the reliability belongs to an interval and the *reliability index* is also an interval quantity (Elishakoff 1995, Qiu et al. 2008).

The present study addresses the static analysis of discretized structures with uncertain parameters modeled as random variables characterized by *imprecise PDF*s. Recently, Muscolino and Sofi (2017) proposed an efficient procedure for estimating the bounds of interval statistics of the displacements of structures with *imprecise* random axial stiffness. This method is herein extended to evaluate the bounds of interval statistics of the stresses and the associated range of the *failure probability*. The main challenge is to reduce the overestimation which may significantly affect interval computations involving stress quantities due to the *dependency phenomenon* (Moore et al. 2009). To reduce conservatism, the proposed method relies on the joint application of the *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*) (Muscolino and Sofi 2012) and the so-called *Rational Series Expansion* (*RSE*) (Muscolino and Sofi 2013).

The developed procedure is applied to a 3D truss structure with random imprecise Young's moduli.

2. LINEAR STRUCTURES WITH UNCERTAIN AXIAL STIFFNESS

Let us consider a n-DOF discretized structural system subjected to deterministic static loads. Let $\rho_j = E_j A_j / L_j$ be the axial stiffness of the j-th element, where E_j, A_j and L_j are the Young's modulus, cross-sectional area and length of the element, respectively. Assume that r structural elements are characterized by uncertain axial stiffness, $\rho_i = \rho_{0,i}(1+X_i)$, $(i=1,2,\ldots,r)$, with dimensionless fluctuations X_i around the nominal value $\rho_{0,i}$ modelled as zero-mean random variables. To ensure always positive values of the uncertain properties, the random fluctuations satisfy the conditions $|X_i|<1$, $(i=1,2,\ldots,r)$, with the symbol $|\bullet|$ meaning absolute value.

The equilibrium equations of the structure with uncertain axial stiffness can be written as follows:

$$\mathbf{K}(\mathbf{X})\mathbf{U}(\mathbf{X}) = \mathbf{f} \tag{1}$$

where $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \cdots & X_r \end{bmatrix}^{\mathrm{T}}$ is the vector collecting the random fluctuations X_i, with T denoting the transpose operator; $\mathbf{K}(\mathbf{X})$ is the $n \times n$ stiffness matrix, which depends on the random fluctuations X_i; $\mathbf{U}(\mathbf{X})$ is the $n-$vector of random displacements; $\mathbf{f}$ is the $n-$ vector collecting the external nodal forces.

The random stiffness matrix of the structure can be expressed as follows (Muscolino and Sofi 2017):

$$\mathbf{K}(\mathbf{X}) = \mathbf{C}^{\mathrm{T}}\mathbf{E}(\mathbf{X})\mathbf{C} \tag{2}$$

where $\mathbf{C}^{\mathrm{T}}$ is the $n \times m$ equilibrium matrix, m being the number of constituent unimodal components; $\mathbf{E}(\mathbf{X})$ is the $m \times m$ random diagonal internal stiffness matrix.

The random stiffness matrix can be rewritten as sum of its nominal value, $\mathbf{K}_0$, plus r rank-one random modifications, i.e.:

$$\mathbf{K}(\mathbf{X}) = \mathbf{K}_0 + \sum_{j=1}^{r} X_j\, \mathbf{w}_j\, \mathbf{w}_j^{\mathrm{T}} \tag{3}$$

where

$$\mathbf{K}_0 = \mathbf{C}^{\mathrm{T}}\, \mathbf{E}_0\, \mathbf{C}\,; \quad \mathbf{w}_j = \mathbf{C}^{\mathrm{T}}\, \mathbf{l}_{E,j} \tag{4a,b}$$

and $\mathbf{K}_j = \mathbf{w}_j\, \mathbf{w}_j^{\mathrm{T}}$ is a rank-one matrix. In the previous equations, $\mathbf{E}_0$ is a diagonal matrix listing the nominal axial stiffnesses $\rho_{0,j}$ $(j=1,2,\ldots,m)$; $\mathbf{l}_{E,j}$ is a $m-$vector having zero entries except the j-th which is equal to $\sqrt{\rho_{0,j}}$.

Recently, Muscolino and Sofi (2013) derived the so-called *Rational Series Expansion* (*RSE*) which provides an approximate explicit expression of the inverse of an invertible matrix with rank-r modifications. For small degrees of uncertainties, i.e. $|X_i| \ll 1$, $(i=1,2,\ldots,r)$, the *RSE* can be truncated to first-order terms, yielding:

$$\mathbf{U}(\mathbf{X}) = \mathbf{K}(\mathbf{X})^{-1}\mathbf{f} \approx \mathbf{K}_0^{-1}\mathbf{f} - \sum_{i=1}^{r} \frac{X_i}{1+X_i\, d_i}\, \mathbf{D}_i\, \mathbf{f} \tag{5}$$

where:

$$d_i = \mathbf{w}_i^{\mathrm{T}}\mathbf{K}_0^{-1}\mathbf{w}_i > 0\,; \;\; \mathbf{D}_i = \mathbf{K}_0^{-1}\mathbf{w}_i\, \mathbf{w}_i^{\mathrm{T}}\mathbf{K}_0^{-1}. \tag{6a,b}$$

Equation (5) represents an approximate explicit relationship between the displacement vector $\mathbf{U}(\mathbf{X})$ and the random variables X_j

which allows a straightforward evaluation of response statistics, as will be outlined in the next sections.

3. AXIAL STIFFNESS DESCRIBED BY IMPRECISE PROBABILITY DENSITY FUNCTION

Let us assume now that only imprecise information on the uncertain axial stiffness is available. Under this assumption, the random variables X_i are more appropriately described by a family of joint *imprecise Probability Density Function* (*PDF*s). Such a family is represented by the function $p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{\mathbf{X}}^I)$, depending on the vector $\mathbf{a}_{\mathbf{X}}^I$ $(i=1,2,\ldots,r)$ of basic parameters with the apex *I* characterizing interval variables.

From an engineering point of view, the r random variables X_i ($i=1,2,\ldots,r$) can be assumed to be independent so that the joint *imprecise PDF*, $p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{\mathbf{X}}^I)$, can be written as:

$$p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{\mathbf{X}}^I)=\prod_{i=1}^{r} p_{X_i}(x_i;\mathbf{a}_{X_i}^I) \tag{7}$$

where $p_{X_i}(x_i;\mathbf{a}_{X_i}^I)$ is the marginal *imprecise PDF* of the random variable X_i which depends on the interval vector $\mathbf{a}_{X_i}^I\in\mathbb{IR}^{s_i}$ where $\mathbb{IR}$ is the set of all closed real interval numbers. The *j*-th element of the interval vector $\mathbf{a}_{X_i}^I$ can be defined as $a_{X_i,j}^I\triangleq\left[\underline{a}_{X_i,j},\overline{a}_{X_i,j}\right]$, where $a_{X_i,j}^I\in\mathbb{IR}$, $\underline{a}_{X_i,j}$ and $\overline{a}_{X_i,j}$ are the *Lower Bound* (*LB*) and *Upper Bound* (*UB*) of the interval basic parameter $a_{X_i,j}^I$, respectively.

The statistics of the random variables X_i with *imprecise PDF* as well as those of the structural response are described by intervals. Such statistics can be defined by introducing the *interval stochastic average operator* $\mathrm{E}^I\langle\bullet\rangle$ (Muscolino and Sofi 2017).

By applying the *RSE* (Eq.(5)) in conjunction with the *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*), Muscolino and Sofi (2017) derived the bounds of the interval mean-value vector and covariance matrix of the nodal displacements $\mathbf{U}(\mathbf{X})$ in approximate explicit form. The aim of the present study is to extend this approach to evaluate the bounds of the normal stresses in the structural elements and then perform reliability assessment in the presence of imprecise random parameters.

3.1. Interval mean-value of the normal stress

The random normal stress in the $j-$th structural element can be evaluated in terms of the random displacement vector $\mathbf{U}(\mathbf{X})$ as follows:

$$S_j(\mathbf{X})=\frac{\rho_{0,j}\left(1+X_j\right)}{A_{0,j}}\mathbf{c}_j^{\mathrm{T}}\mathbf{U}(\mathbf{X}) \tag{8}$$

where $\mathbf{c}_j^{\mathrm{T}}$ is the $j-$ th row of the $m\times n$ compatibility matrix $\mathbf{C}$; $A_{0,j}$ is the cross-sectional area of the $j-$th structural element set equal to the nominal value.

Substituting Eq. (5) into Eq. (8), the following approximate explicit expression of the $j-$th random normal stress is obtained

$$S_j(\mathbf{X})=\rho_{0,j}(1+X_j)\left[b_{0,j}-\sum_{i=1}^{r}\chi_i b_{i,j}\right] \tag{9}$$

where

$$\chi_i(X_i)=\frac{X_i}{1+X_i\,d_i} \tag{10}$$

is an auxiliary random variable (Muscolino and Sofi 2017) and

$$b_{0,j}=\frac{1}{A_{0,j}}\mathbf{c}_j^{\mathrm{T}}\mathbf{K}_0^{-1}\mathbf{f};\quad b_{i,j}=\frac{1}{A_{0,j}}\mathbf{c}_j^{\mathrm{T}}\mathbf{D}_i\mathbf{f}. \tag{11a,b}$$

By applying the *interval stochastic average operator* $\mathrm{E}^I\langle\bullet\rangle$ to Eq. (9) and taking into account that the random variables X_i are independent, the interval mean-value of the $j-$th random normal stress, $S_j(\mathbf{X})$, can be evaluated as:

$$\mu_{S_j}^I = \mathrm{E}^I \left\langle S_j(X) \right\rangle = \rho_{0,j} \left[b_{0,j} - \sum_{i=1}^{r} \mu_{\chi_i}^I \, b_{i,j} - \mathrm{E}^I \left\langle X_j \chi_j \right\rangle b_{j,j} \right] \quad (12)$$

where $\mu_{\chi_i}^I = \mathrm{E}^I \left\langle \chi_i \right\rangle$ is the interval mean-value of the $i-$th auxiliary random variable.

Following the *IIA* via *EUI*, the previous equation can be rewritten as sum of the midpoint value plus an interval deviation as follows:

$$\mu_{S_j}^I = \mathrm{mid}\left\{\mu_{S_j}^I\right\} + \mathrm{dev}\left\{\mu_{S_j}^I\right\} \quad (13)$$

where

$$\mathrm{mid}\left\{\mu_{S_j}^I\right\} = \rho_{0,j} \left[b_{0,j} - \sum_{i=1}^{r} \mathrm{mid}\left\{\mu_{\chi_i}^I\right\} b_{i,j} - \mathrm{mid}\left\{\mathrm{E}^I \left\langle X_j \chi_j \right\rangle\right\} b_{j,j} \right];$$

$$\mathrm{dev}\left\{\mu_{S_j}^I\right\} = \rho_{0,j} \left[-\sum_{i=1}^{r} \Delta\mu_{\chi_i} \hat{e}_i^I \, b_{i,j} - \Delta \mathrm{E}\left\langle X_j \chi_j \right\rangle \hat{e}_j^I \, b_{j,j} \right]. \quad (14a,b)$$

In the previous equations, $\mathrm{mid}\{\bullet\}$ and $\mathrm{dev}\{\bullet\}$ denote the midpoint and interval deviation of the quantity between curly brackets; $\Delta\mu_{\chi_i}$ and $\Delta \mathrm{E}\left\langle X_j \chi_j \right\rangle$ are the deviation amplitudes of $\mu_{\chi_i}^I$ and $\mathrm{E}^I\left\langle X_j \chi_j \right\rangle$, respectively; $\hat{e}_i^I$ is the $i-$th *EUI* (Muscolino and Sofi 2017).

Based on Eq. (13) and following the philosophy of the *IIA* via *EUI*, the *LB* and *UB* of the interval mean-value of the $j-$th random normal stress can be evaluated as:

$$\underline{\mu}_{S_j} = \mathrm{mid}\left\{\mu_{S_j}^I\right\} - \Delta\mu_{S_j}; \qquad \overline{\mu}_{S_j} = \mathrm{mid}\left\{\mu_{S_j}^I\right\} + \Delta\mu_{S_j} \quad (15a,b)$$

where $\Delta\mu_{S_j}$ is the deviation amplitude, defined as follows

$$\Delta\mu_{S_j} = \rho_{0,j} \left[\sum_{\substack{i=1 \\ i \neq j}}^{r} \left| \Delta\mu_{\chi_i} b_{i,j} \right| + \left| \Delta\mu_{\chi_j} b_{j,j} + \Delta \mathrm{E}\left\langle X_j \chi_j \right\rangle b_{j,j} \right| \right]. \quad (16)$$

3.2. Interval variance of the normal stress

Taking into account Eqs. (9) and (12), and neglecting terms involving powers of the random variables X_i of order higher than two, the following approximate explicit expression of the variance of the $j-$th random normal stress is obtained:

$$\sigma_{S_j}^{2\,I} = \mathrm{E}^I \left\langle S_j^2(\mathbf{X}) \right\rangle - \left(\mu_{S_j}^I\right)^2 = \rho_{0,j}^2 \left[b_{0,j}^2 \mathrm{E}^I \left\langle X_j^2 \right\rangle - 2 b_{0,j} b_{j,j} \mathrm{E}^I \left\langle X_j \chi_j \right\rangle + \sum_{i=1}^{r} \sigma_{\chi_i}^{2\,I} b_{i,j}^2 \right] \quad (17)$$

where $\sigma_{\chi_i}^{2I}$ is the interval variance of the auxiliary random variable χ_i.

Following the *IIA* via *EUI*, Eq. (17) can be rewritten as sum of the midpoint value plus an interval deviation as follows:

$$\sigma_{S_j}^{2\,I} = \mathrm{mid}\left\{\sigma_{S_j}^{2\,I}\right\} + \mathrm{dev}\left\{\sigma_{S_j}^{2\,I}\right\} \quad (18)$$

where

$$\mathrm{mid}\left\{\sigma_{S_j}^{2\,I}\right\} = \rho_{0,j}^2 \left[b_{0,j}^2 \mathrm{mid}\left\{\mathrm{E}^I\left\langle X_j^2 \right\rangle\right\} - 2 b_{0,j} b_{j,j} \mathrm{mid}\left\{\mathrm{E}^I\left\langle X_j \chi_j \right\rangle\right\} + \sum_{i=1}^{r} \mathrm{mid}\left\{\sigma_{\chi_i}^{2\,I}\right\} b_{i,j}^2 \right];$$

$$\mathrm{dev}\left\{\sigma_{S_j}^{2\,I}\right\} = \rho_{0,j}^2 \left[b_{0,j}^2 \Delta \mathrm{E}\left\langle X_j^2 \right\rangle \hat{e}_j^I - 2 b_{0,j} b_{j,j} \Delta \mathrm{E}\left\langle X_j \chi_j \right\rangle \hat{e}_j^I + \sum_{i=1}^{r} \Delta\sigma_{\chi_i}^2 \hat{e}_i^I b_{i,j}^2 \right]. \quad (19a,b)$$

Based on Eq. (17) and following the philosophy of the *IIA via EUI*, the *LB* and *UB* of the interval variance of the $j-$th random normal stress can be evaluated as:

$$\underline{\sigma}_{S_j}^2 = \text{mid}\left\{\sigma_{S_j}^{2\,I}\right\} - \Delta\sigma_{S_j}^2;$$
$$\overline{\sigma}_{S_j}^2 = \text{mid}\left\{\sigma_{S_j}^{2\,I}\right\} + \Delta\sigma_{S_j}^2 \qquad (20a,b)$$

where

$$\Delta\sigma_{S_j}^2 = \rho_{0,j}^2 \left[\left| b_{0,j}^2 \Delta \mathrm{E}\left\langle X_j^2 \right\rangle - 2 b_{0,j} b_{j,j} \Delta \mathrm{E}\left\langle X_j \chi_j \right\rangle + \Delta\sigma_{\chi_j}^2 b_{j,j}^2 \right| + \sum_{\substack{i=1\\ i\neq j}}^{r} \left| \Delta\sigma_{\chi_i}^2 b_{i,j}^2 \right| \right] \qquad (21)$$

with $\Delta \mathrm{E}\left\langle X_j^2 \right\rangle$ and $\Delta\sigma_{\chi_i}^2$ denoting the deviation amplitudes of $\mathrm{E}^I\left\langle X_j^2 \right\rangle$ and $\sigma_{\chi_i}^{2\,I}$, respectively.

4. INTERVAL FAILURE PROBABILITY AND INTERVAL RELIABILITY INDEX

Once the bounds of the interval mean-value and variance of the most relevant stress are known, performance assessment of the structural system with *imprecise* random axial stiffness can be carried out by extending the classical probabilistic concept of reliability to the interval framework. As known, in reliability analysis, a measure of the risk is the *probability of failure*, $\mathcal{P}_{\mathcal{F}}$, while a measure of the success is the *probability of success* or *survival probability*, $\mathcal{P}_{\mathcal{S}} = 1 - \mathcal{P}_{\mathcal{F}}$, which can be defined in the following alternative ways:

$$\mathcal{P}_{\mathcal{F}} = \Pr\langle R \le S \rangle = \Pr\langle R/S \le 1 \rangle = \Pr\langle \ln R - \ln S \le 0 \rangle;$$
$$\mathcal{P}_{\mathcal{S}} = \Pr\langle S < R \rangle = \Pr\langle R/S > 1 \rangle = \Pr\langle \ln R - \ln S > 0 \rangle \qquad (22a,b)$$

where S is the most relevant structural response, caused by external loads and R is the corresponding resistance of materials.

The response and the resistance of materials are herein modelled as statistically independent random variables having lognormal distributions. In particular, the resistance of materials, R, is assumed to be characterized by a *precise* lognormal *PDF* with mean-value μ_R and standard deviation σ_R. On account of the imprecise character of the random axial stiffness of the structure, the response, S, is supposed to have an *imprecise* lognormal *PDF* with interval mean-value and standard deviation μ_S^I and σ_S^I.

Under these assumptions, the *probability of failure* turns out to be defined by an interval quantity. By using the well-known relationships for lognormal distributions (Haldar and Mahadevan 2000) and applying interval extension, the interval *probability of failure* reads as follows:

$$\mathcal{P}_{\mathcal{F}}^I = \left[\underline{\mathcal{P}}_{\mathcal{F}}, \overline{\mathcal{P}}_{\mathcal{F}}\right] = 1 - \Phi\left(\beta^I\right) \qquad (23)$$

where

$$\beta^I = \left[\underline{\beta}, \overline{\beta}\right] = \ln\left[\frac{\mu_R}{\mu_S^I}\sqrt{\frac{1+\delta_S^{2\,I}}{1+\delta_R^2}}\right] \times \frac{1}{\sqrt{\ln\left[(1+\delta_R^2)\left(1+\delta_S^{2\,I}\right)\right]}} \qquad (24)$$

is the *interval reliability index*. In the previous equation, δ_R and δ_S^I are the *coefficients of variation* of resistance and response, respectively:

$$\delta_R = \frac{\sigma_R}{\mu_R}; \quad \delta_S^I = \frac{\sigma_S^I}{\mu_S^I}. \qquad (25a,b)$$

The *IIA* via *EUI* (Muscolino and Sofi 2012) yields the following expressions of the *LB* and *UB* of the *interval reliability index*:

$$\underline{\beta} = \min_{\mu_S \in \mu_S^I,\ \sigma_S \in \sigma_S^I} \{\beta(\mu_S, \sigma_S)\} = \beta(\overline{\mu}_S, \overline{\sigma}_S);$$
$$\overline{\beta} = \max_{\mu_S \in \mu_S^I,\ \sigma_S \in \sigma_S^I} \{\beta(\mu_S, \sigma_S)\} = \beta(\underline{\mu}_S, \underline{\sigma}_S) \qquad (26a,b)$$

where the bounds of the interval mean-value $\mu_S^I = [\underline{\mu}_S, \overline{\mu}_S]$ and standard deviation

$\sigma_S^I = [\underline{\sigma}_S, \overline{\sigma}_S]$ of the response can be evaluated by the proposed procedure.

Then, according to Eq.(23), the best possible value (or *LB*) and the worst possible value (or *UB*) of the *probability of failure* can be evaluated, respectively, as:

$$\underline{\mathcal{P}}_{\mathcal{F}} = 1 - \Phi\left(\overline{\beta}\right); \quad \overline{\mathcal{P}}_{\mathcal{F}} = 1 - \Phi\left(\underline{\beta}\right). \tag{27a,b}$$

Obviously, the *LB* and *UB* of the interval *survival probability* are given, respectively, by:

$$\underline{\mathcal{P}}_{\mathcal{S}} = 1 - \overline{\mathcal{P}}_{\mathcal{F}} = \Phi\left(\underline{\beta}\right); \quad \overline{\mathcal{P}}_{\mathcal{S}} = 1 - \underline{\mathcal{P}}_{\mathcal{F}} = \Phi\left(\overline{\beta}\right). \tag{28a,b}$$

The previous bounds allow us to compute the highest expected *failure probability*, $\overline{\mathcal{P}}_{\mathcal{F}}$, (see Eq. (27b)) which corresponds to the *LB* of the *survival probability* (see Eq. (28a)).

5. NUMERICAL APPLICATION

The 3D 26-bar truss structure under deterministic static loads shown in Figure 1 is selected as case study (Muscolino and Sofi 2017).

The following geometrical and mechanical properties are assumed: $A_{0,i} = 4.27 \times 10^{-4}\ \mathrm{m}^2$, $E_{0,i} = 2.1 \times 10^{8}\ \mathrm{kN/m}^2$, $i = 1,2,\ldots,26$, and $f = 200\ \mathrm{kN}$. Young's moduli of $r = 12$ bars are modeled as independent random variables, $E_i = E_0(1 + X_i)$, $i = 1,2,\ldots,12$, (see bar numbering in Figure 1) with fluctuations, X_i, around the nominal value modeled as zero-mean independent random variables with uniform *imprecise PDF*:

$$p_{X_i}(x_i; a_i^I) = \begin{cases} \dfrac{1}{2a_i^I}, & \text{for } -a_i^I \le x_i \le a_i^I \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

where $a_i^I = [\underline{a}_i, \overline{a}_i] = a_0(1 + \Delta\alpha\, \hat{e}_i^I)$ with $a_0 > 0$, $\Delta\alpha < 1$ and $\hat{e}_i^I = [-1, +1]$, $i = 1,2,\ldots,r = 12$.

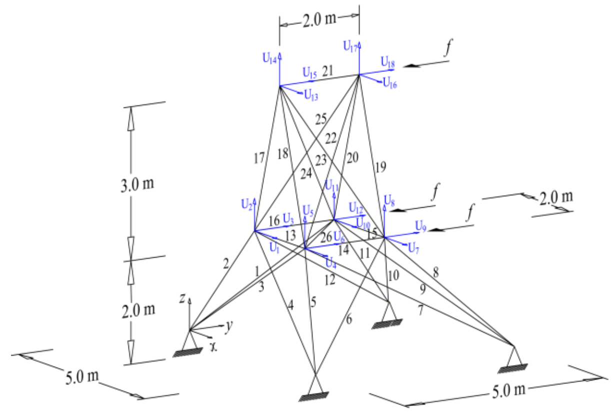

Figure 1: 3D truss structure with uncertain Young's moduli.

The proposed bounds of the statistics of normal stresses are compared with those provided by a procedure resulting from the joint application of classical *Monte Carlo Simulation* (*MCS*) with a combinatorial procedure known as *Vertex Method* (*VM*), herein referred to as *MCS-VM* (Muscolino and Sofi 2017). Both the proposed method and the *MCS-VM* have been implemented in MATLAB.

Figure 2 displays the *UB* and *LB* of the interval mean-value of the normal stress of the 26 bars for $a_0 = 0.2$ and $\Delta\alpha = 0.1$. An excellent agreement between the proposed estimates and those provided by the *MCS-VM* is observed. The proposed procedure yields also very accurate estimates of the bounds of the interval standard deviation of the normal stresses, as shown in Figure 3.

To assess the performance of the truss structure, attention is focused on bar 8 where the maximum normal stress S_8 is attained. The resistance of the material, R, is modeled as a lognormally distributed random variable with mean-value $\mu_R = 530\ \mathrm{MPa}$ and standard deviation $\sigma_R = 0.05\mu_R$, while the normal stress of bar 8, S_8, is assumed to be characterized by a lognormal *imprecise PDF* with interval mean-value and standard deviation $\mu_{S_8}^I$ and $\sigma_{S_8}^I$ (see Figures 2 and 3).

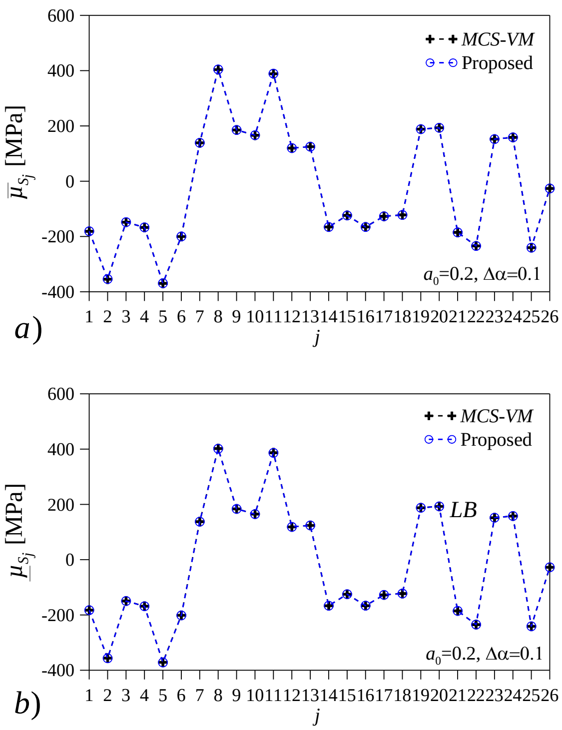

Figure 2: a) Upper bound and a) lower bound of the interval mean-value of the normal stresses of bars.

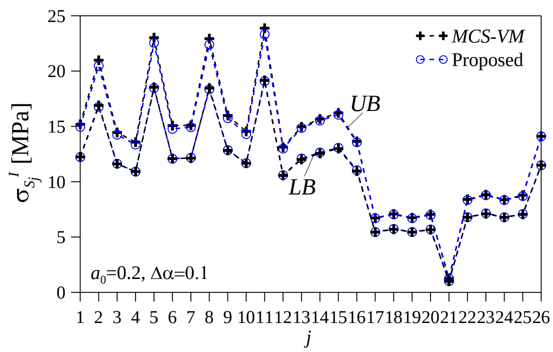

Figure 3: Bounds of the interval standard deviation of the normal stresses of bars.

Figure 4 shows the *PDF* of the resistance R along with three realizations of the *imprecise PDF* of the normal stress S_8 obtained setting the mean-value and standard deviation equal to $(\underline{\mu}_{S_8}, \underline{\sigma}_{S_8})$, $(\overline{\mu}_{S_8}, \overline{\sigma}_{S_8})$, and to the values $(\mu_{0,S_8}, \sigma_{0,S_8})$ pertaining to the uniform *PDF* (29) of the random variables X_i with nominal basic parameter a_0. Notice that the largest area of overlap between the *PDF*s of R and S_8, which gives a qualitative measure of the *probability of failure*, is obtained when the interval mean-value and standard deviation of S_8 are set to their *UB*, $\overline{\mu}_{S_8}$ and $\overline{\sigma}_{S_8}$. Indeed, in this case the *LB* of the *interval reliability index* and the *UB* of the *interval failure probability*, plotted in Figures 5 and 6, respectively, are achieved.

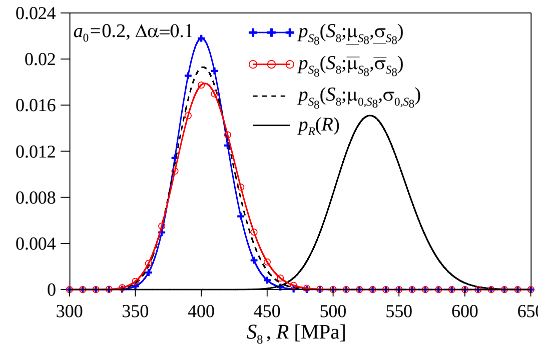

Figure 4: Probability density function of the normal stress of bar 8 and of the resistance of material.

In Figures 5 and 6, the *LB* of the *interval reliability index*, $\underline{\beta}_{S_8}$, and the corresponding *UB* of the *failure probability*, $\overline{\mathcal{P}}_{\mathcal{F},S_8}$, for the normal stress S_8 versus the nominal basic parameter a_0 of the *PDF* of the uncertain Young's moduli are plotted. The proposed estimates are very close to those provided by the *MCS-VM*. Indeed, the maximum absolute percentage errors affecting $\underline{\beta}_{S_8}$ and $\overline{\mathcal{P}}_{\mathcal{F},S_8}$, which are achieved for $a_0 = 0.2$, are equal to 0.077% and 1.083%, respectively. Though the *LB* of the *interval reliability index*, $\underline{\beta}_{S_8}$, and the *UB* of the *failure probability*, $\overline{\mathcal{P}}_{\mathcal{F},S_8}$, are slightly overestimated and underestimated, respectively, the percentage errors are very small. The nominal values pertaining to a uniform *precise PDF* of the uncertain Young's moduli with basic parameter a_0 are also reported. Notice that the performance of the structure is significantly affected by the imprecision of the *PDF* of the random Young's moduli. This implies that classical probabilistic reliability analysis, based on the nominal value a_0 of the basic parameters, may lead to serious underestimation of the *failure probability*.

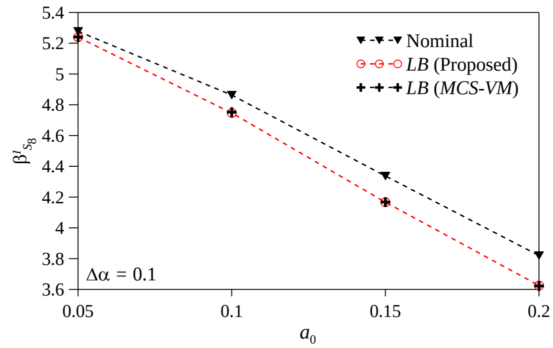

Figure 5: Lower bound and nominal value of the reliability index for the normal stress of bar 8.

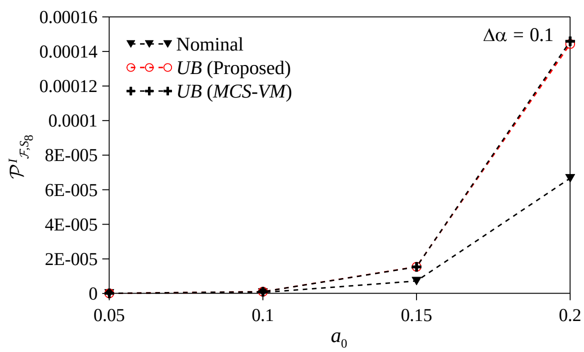

Figure 6: Upper bound and nominal value of the probability of failure for the normal stress of bar 8.

6. CONCLUSIONS

The analysis of discretized structures with uncertainties described by *imprecise Probability Density Functions* (*PDF*s) with interval basic parameters has been addressed. By extending a procedure recently proposed by Muscolino and Sofi (2017), approximate explicit expressions of the bounds of the interval mean-value and variance of stresses have been derived. Furthermore, analytical expressions of the bounds of the *interval reliability index* and the associated *interval failure probability* have been obtained. A notable feature of the developed method is the capability to limit the overestimation affecting interval computations involving stress quantities. Numerical results have demonstrated the accuracy of the proposed method as well as the remarkable influence of imprecision of the *PDF* of the uncertain parameters on the performance of structural systems.

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