An Efficient Complex Modal Decomposition Method for Inelastic Stochastic Design Spectrum-based Analysis

Ioannis P. Mitseas
Postdoctoral Research Fellow, Institute for Risk and Reliability, Leibniz University Hannover, Hannover, Germany

Ioannis A. Kougioumtzoglou
Assistant Professor, Dept. of Civil Engineering & Engineering Mechanics, Columbia University, New York, USA

Michael Beer
Professor, Institute for Risk and Reliability, Leibniz University Hannover, Hannover, Germany

ABSTRACT: An efficient stochastic modal decomposition method for random vibration analysis of non-classically damped nonlinear multi-degree-of-freedom (MDOF) systems is proposed in accordance with contemporary aseismic code provisions (e.g., EC8). Specifically, relying on statistical linearization and state-variable formulation, the complex eigenvalue problem considering inelastic MDOF structural systems subject to stochastic seismic processes is formulated and solved. To this aim, equivalent linear modal properties (EMPs), i.e., natural frequencies and damping ratios, are appropriately defined and evaluated based on an iterative scheme involving the determination of the system response covariance matrix as well. Note that the stochastic excitations are characterized by power spectra compatible in a stochastic sense with a given elastic response uniform hazard spectrum (UHS) of specified modal damping ratio. Next, the system EMPs are utilized in conjunction with the response elastic UHS for determining peak nonlinear responses in modal coordinates. Further, modal participation factors are evaluated for the complex-valued mode shapes and the generalized complete-quadratic-combination (CQC) is employed as the modal combination rule for determining the peak total responses. The reliability of the proposed framework is assessed by considering a 3-storey nonlinear frame structure exposed to the Eurocode 8 elastic response spectrum. Nonlinear response time-history analysis (RHA) involving a large ensemble of Eurocode 8 spectrum compatible accelerograms is conducted to assess the accuracy of the proposed approach.

Realistic structural and mechanical systems exhibit various nonlinear behaviors, which usually become progressively more significant as the amplitude of vibration increases. Specifically, nonlinear/hysteretic modeling is widely used in the field of earthquake engineering (e.g., Soong and Dargush, 1997; Mitseas et al., 2014, 2016a, 2018). Further, the above system modeling is typically coupled with sophisticated stochastic representations of environmental excitations, such as earthquakes. From a mathematics perspective, this yields a coupled system of nonlinear stochastic differential equations to be solved for determining the stochastic response of the structure. In this regard, a computationally efficient methodology is developed herein for random vibration analysis of non-classically damped nonlinear multi-degree-of-freedom (MDOF) systems subject to excitations compatible with elastic response uniform hazard spectra (UHS).

1. STATEMENT OF THE PROBLEM

The dynamic response of a *n*-DOF structure excited by a base motion, the acceleration of which is $\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t})$, is governed by the system of differential equations of the form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(\mathrm{t}) = -\mathbf{M}\boldsymbol{\gamma}\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t}) \quad (1)$$

where $\mathbf{x}(t)$, $\dot{\mathbf{x}}(t)$, and $\ddot{\mathbf{x}}(t)$ are the response displacement, velocity, and acceleration vectors of the nodes relative to the base motion, respectively. The dot superscript denotes differentiation with respect to time; $\boldsymbol{\gamma}$ is a unit $(\mathrm{n} \times 1)$ column vector. Further, $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ denote the $(\mathrm{n} \times \mathrm{n})$ real-valued mass, damping, and stiffness matrices, respectively. The objective of this section is to elucidate the analysis process for the linear system response where no particular restrictions are imposed on the form of the damping matrix (e.g. Rayleigh damping assumption). In this setting, classical modal analysis has been a commonly employed dimension reduction and solution technique. In particular, the natural frequencies ω_r^o and mode shapes $\boldsymbol{\theta}_r$ are first determined for the undamped structure by considering a harmonic response solution of the form

$$\mathbf{x}(t) = \boldsymbol{\theta} \mathrm{e}^{i\omega t} \qquad (2)$$

to the system of Eq.(1). The mode shapes are normalized with respect to mass matrix so that $\boldsymbol{\theta}_r^{\boldsymbol{T}} \boldsymbol{M} \boldsymbol{\theta}_r = 1$. Subsequently, the damping ratios are provided by the diagonal elements of the modal damping matrix $\zeta_r^o = \boldsymbol{\theta}_r^{\boldsymbol{T}} \boldsymbol{C} \boldsymbol{\theta}_r / (2\omega_r^o)$. Note that the vibrational characteristics provide an insight into the dynamic character of the system. However, a more pertinent addressing of the non-classical damping feature would necessitate resorting to complex-valued modes stemming from the treatment of the damped eigenvalue problem (Igusa et al., 1984; Veletsos and Ventura, 1986). In this regard, it is possible to obtain a completely decoupled set of equations (e.g. Roberts and Spanos, 2003). Further, a more realistic and representative modelling of engineering systems subjected to severe excitations should thoroughly account for the presence of nonlinear mechanisms. In this setting, the development of an efficient analysis method for addressing the above challenges, while retaining at the same time the broad applicability of its linear counterpart among structural engineering practice is presented in the following.

2. MATHEMATICAL BACKBONE

This section reviews the mathematical details involved in undertaking the steps of (i) defining a power spectrum compatible in a stochastic sense with an assigned seismic response spectrum, of (ii) applying statistical linearization to a nonlinear *n*-DOF structural system using a state variable formulation, of (iii) decoupling the equivalent linear MDOF system by conducting complex modal analysis to derive the vibrational characteristics corresponding to each mode of vibration, of (iv) establishing an iterative algorithm based on the above three steps, and of (v) employing a modal combination rule and determining the modal participation factors for the complex-valued mode shapes.

2.1. Determination of compatible power spectra

An efficient numerical scheme is employed to statistically fit a stationary Gaussian acceleration process $\ddot{\mathrm{x}}_{\mathrm{g}}(\mathrm{t})$ of finite duration T_s, to an assigned elastic pseudo-acceleration response spectrum. In this regard, the following nonlinear equation consists the basis for relating a pseudo-acceleration response spectrum $S_a(\omega_i, \zeta_o)$ to an one-sided power spectrum corresponding to a Gaussian stationary stochastic process $X_i(t)$

$$S_a(\omega_i, \zeta_o) = \eta_{X_i} \omega_i^2 \sqrt{\lambda_{0,X_i}} \qquad (3)$$

where η_{X_i} and λ_{0,X_i} stand for the peak factor and the variance of the stationary stochastic response process $X_i(t)$ of an elastic oscillator of natural frequency ω_i and damping ratio ζ_o. The spectral moment of zeroth order of the stationary response process that appears in Eq.(3), reads for the general case of nth order

$$\lambda_{n,X_i} = \int_0^\infty \omega^n \frac{1}{(\omega_i^2 - \omega^2)^2 + (2\zeta_o \omega_i \omega)^2} G_{X_i}^{\zeta_o}(\omega) d\omega \qquad (4)$$

Note in passing that the evaluation of the stochastically compatible power spectrum $G_{X_i}^{\zeta_o}(\omega)$, which does not appear explicitly in Eq.(3), necessitates a careful handling of the inverse stochastic dynamics problem. The determination of the peak factor η_{X_i} which is related with first-passage problem (Vanmarcke, 1972; Mitseas et al., 2014; 2016b) passes from the computation of the mean zero crossing rate v_{X_i} and the spread factor δ_{X_i} of the stochastic response process $X_i(t)$. Pertinent information regarding their determination can be found in Cacciola et al. (2004). Next, the following direct scheme for the evaluation of the stochastically compatible power spectrum $G_{X_i}^{\zeta_o}(\omega)$ is derived

$$G_{X_i}^{\zeta_o}(\omega_i) = \begin{cases} \dfrac{4\zeta_o}{\omega_i \pi - 4\zeta_o \omega_{i-1}} \left(\dfrac{S_\alpha^2(\omega_i, \zeta_o)}{\eta_{X_i}^2} \cdots \right. \\ \left. -\Delta\omega \sum_{q=1}^{i-1} G_{X_i}^{\zeta_o}(\omega_q) \right), & \omega_i > \omega_b^l \\ 0, & \omega_i \leq \omega_b^l \end{cases} \tag{5}$$

where the discretization scheme $\omega_i = \omega_b^l + (i - 0.5)\Delta\omega$ is employed. The value of ω_b^l is associated with a lowest bound. Obviously, a preselection of an input power spectrum shape has to be preceded for deriving a stochastically compatible spectrum, according to the numerical scheme of Eq.(5). Note in passing that the time-limited stationary power spectrum is only used as a first step to represent the seismic input action, defined in terms of a pseudo-acceleration response spectrum.

2.2. Statistical linearization of MDOF systems

Consider a non-classically damped, nonlinear structural system with *n* number of DOFs base-excited by the Gaussian stationary acceleration stochastic process $\ddot{x}_g(t)$, characterized in the frequency domain by the power spectrum $G_{X_i}^{\zeta_o}(\omega)$. The response of the structure is governed by the system of differential equations written as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \mathbf{F}(t), \tag{6}$$

where $\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ is a nonlinear $n \times 1$ vector function of the variables $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$. For non-classically damped systems which do not satisfy the Caughey and O'Kelly identity (Caughey and O'Kelly, 1965) which states that

$$\mathbf{CM^{-1}K = KM^{-1}C} \tag{7}$$

the eigenvalues as well as the modal shapes are expected to be complex-valued. Note in passing that in case the damping matrix of a system satisfies the above relationship the natural modes are real-valued and equal to those of the associated undamped system. The eigenvalues of a classically damped system appear in complex conjugate pairs and the modulus of its pair is equal with the natural frequency ω_j^o of the associated undamped system. Further, $\mathbf{F}(t)$ can be expressed in the frequency domain as

$$\mathbf{S_{FF}}(\omega) = G_{X_i}^{\zeta_o}(\omega)\mathbf{M}\boldsymbol{\gamma}\boldsymbol{\gamma}^{\boldsymbol{T}}\mathbf{M}. \tag{8}$$

Relying on the standard assumption that the response processes are Gaussian, the standard spectral matrix solution procedure of the classical statistical linearization is employed to estimate the response power spectrum matrix of the nonlinear and non-classically damped MDOF structure. In this setting, a linearized version of Eq. (6) is considered in the form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + (\mathbf{C} + \mathbf{C_{eq}})\dot{\mathbf{x}}(t) + (\mathbf{K} + \mathbf{K_{eq}})\mathbf{x}(t) = \mathbf{F}(t) \tag{9}$$

Further, the coupled linearized equations can be written in the following normalized form by dividing the *j*-th equation with the corresponding m_j, i.e.,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} 2\zeta_{1e}\omega_{1e} & -2\zeta_{2e}\omega_{2e}\mu_{21} & \cdots & \cdots & 0 \\ 0 & 2\zeta_{2e}\omega_{2e} & \cdots & -2\zeta_{3e}\omega_{3e}\mu_{32} & 0 \\ 0 & 0 & \ddots & \ddots & -2\zeta_{ne}\omega_{ne}m_n\mu_{nn-1} \\ 0 & 0 & 0 & \cdots & 2\zeta_{ne}\omega_{ne}m_n \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{bmatrix} + \begin{bmatrix} \omega_{1e}^2 & -\omega_{2e}^2\mu_{21} & \cdots & 0 \\ 0 & \omega_{2e}^2 & -\omega_{3e}^2\mu_{32} & 0 \\ 0 & 0 & \ddots & -\omega_{ne}^2\mu_{nn-1} \\ 0 & 0 & 0 & \omega_{ne}^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = -\begin{bmatrix} \ddot{x}_g(t) \\ \ddot{x}_g(t) \\ \vdots \\ \ddot{x}_g(t) \end{bmatrix} \tag{10}$$

where $y_j(t)$ is the inter-story drift $y_j(t) = x_j - x_{j-1}$, x_j, $j = 0,1,2,\dots,n$ is the lateral floor displacement relative to the ground displacement with x_0=0. Next, $k_{je} = k_j + k_{jj}^{eq}$ and $c_{je} = c_j + c_{jj}^{eq}$. The $(d,l)^{th}$ element of the equivalent linear matrices $\mathbf{C_{eq}}$ and $\mathbf{K_{eq}}$ are given by the expressions

$$c_{d,l}^{eq} = E\left[\frac{\partial g_d}{\partial \dot{y}_l}\right], \tag{11}$$

and

$$k_{d,l}^{eq} = E\left[\frac{\partial g_d}{\partial y_l}\right], \tag{12}$$

in which E[·] is the mathematical expectation operator. Further, $\omega_{je}^2 = \frac{k_{je}}{m_j}$, $\zeta_{je} = \frac{c_{je}}{2\sqrt{k_{je}m_j}}$ and

$\mu_{jj-1} = \frac{\mathrm{m_j}}{\mathrm{m_{j-1}}}$. Subsequently, the celebrated frequency domain relation is provided as

$$\mathbf{S_{yy}}(\omega) = \mathbf{H_y}(i\omega)\mathbf{S_{FF}}(\omega)\mathbf{H_y}^*(i\omega), \quad (13)$$

where the superscript (*) denotes Hermitian transposition and the frequency response function matrix is defined as

$$\mathbf{H_y}(i\omega) = \left[\left[\left(\mathbf{K} + \mathbf{K_{eq}}\right) + \mathbf{M}(i\omega)^2\right] + i\omega\left(\mathbf{C} + \mathbf{C_{eq}}\right)\right]^{-1}, \quad (14)$$

with i being the imaginary unit. Further, the cross–variance of the response due to a vector of stochastic excitation processes characterized by power spectra of the form $G_{X_i}^{\zeta_o}(\omega)$ is provided as

$$E[y_d(t)y_l(t)] = \int_{-\infty}^{\infty} S_{y_d y_l}(\omega)\, d\omega \quad (15)$$

where $S_{y_d y_l}(\omega)$ is the $(d,l)^{th}$ element of the response power spectrum matrix $\mathbf{S_{yy}}(\omega)$. It can be readily seen that Eqs.(10-15) constitute a coupled nonlinear system of algebraic equations to be solved iteratively for the system response covariance matrix. In fact, a simple iterative solution of Eqs. (10-15), until convergence of the elements of $\mathbf{C_{eq}}$ and $\mathbf{K_{eq}}$ matrices is achieved, is sufficient.

The decoupling step reviewed in the next section utilizes the ELPs in conjunction with a state variable formulation to define equivalent linear uncoupled oscillators in modal coordinates with effective damping and natural frequency properties.

2.3. System complex modal decomposition

Given that the system damping matrix has an arbitrary general form, the solution of the following complex eigenvalue problem

$$\left[\boldsymbol{\lambda}^2\mathbf{M} + \boldsymbol{\lambda}\left(\mathbf{C} + \mathbf{C_{eq}}\right) + \left(\mathbf{K} + \mathbf{K_{eq}}\right)\right]\boldsymbol{\psi} = \mathbf{0} \quad (16)$$

is pursued next. The eigenvalues $\boldsymbol{\lambda}$ and the associated eigenvectors $\boldsymbol{\psi}$ may be determined more conveniently by first reducing the system of n second order differential equations to a system of $2n$ first order differential equations. The general linearized equations of motion for a n-DOF system as expressed by Eq.(9), can be recast into the state variable form by defining a $2n$ state vector, $q(t)$, as follows

$$\boldsymbol{q}(t) = [\mathbf{x}(t) \quad \dot{\mathbf{x}}(t)]^{T} \quad (17)$$

A first-order matrix equation of motion may then be written as

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{G}\boldsymbol{q}(t) + \boldsymbol{f}(t) \quad (18)$$

where

$$\boldsymbol{G} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\mathbf{M}^{-1}\left(\mathbf{K} + \mathbf{K_{eq}}\right) & -\mathbf{M}^{-1}\left(\mathbf{C} + \mathbf{C_{eq}}\right) \end{bmatrix} \quad (19)$$

and

$$\boldsymbol{f}(t) = [\mathbf{0} \quad \mathbf{M}^{-1}\mathbf{F}(t)]^{T} \quad (20)$$

Next, the eigenvalues $\lambda_1, \lambda_2, \dots \lambda_{2n}$ of the $2\mathrm{n} \times 2\mathrm{n}$ matrix $\boldsymbol{G}$ are computed by solving

$$|\boldsymbol{G} - \boldsymbol{\lambda}\boldsymbol{I}| = \mathbf{0} \quad (21)$$

Provided the amount of damping in the system is not very high, the eigenvalues occur in complex conjugate pairs with negative real parts.

$$\begin{matrix} \lambda_r \\ \bar{\lambda}_r \end{matrix} = \begin{Bmatrix} -\zeta_{\mathrm{req}}\omega_{\mathrm{req}} + i\omega_{\mathrm{reqD}} \\ -\zeta_{\mathrm{req}}\omega_{\mathrm{req}} - i\omega_{\mathrm{reqD}} \end{Bmatrix}. \quad (22)$$

The equivalent modal properties (EMPs), namely ω_{req} and ζ_{req}, which correspond to the uncoupled equations of motion in modal coordinates are related to the eigenvalues as follows

$$\omega_{\mathrm{req}} = |\lambda_r|, \qquad \zeta_{\mathrm{req}} = -\frac{Re(\lambda_r)}{|\lambda_r|}. \quad (23)$$

In Eq.(22), ω_{reqD} is given by

$$\omega_{\mathrm{reqD}} = \omega_{\mathrm{req}}\left(1 - \zeta_{\mathrm{req}}^2\right)^{1/2}. \quad (24)$$

For a n-DOF system, there are n pairs of eigenvalues, and to each such pair corresponds a complex conjugate pair of eigenvectors

$$\begin{matrix} \boldsymbol{\psi}_r \\ \bar{\boldsymbol{\psi}}_r \end{matrix} = \begin{Bmatrix} \boldsymbol{\varphi}_r + i\boldsymbol{y}_r \\ \boldsymbol{\varphi}_r - i\boldsymbol{y}_r \end{Bmatrix} \quad (25)$$

The columns of the $2\mathrm{n} \times 2\mathrm{n}$ complex modal matrix $\boldsymbol{T_G}$ are formed from the eigenvectors or complex modes

$$\boldsymbol{T_G} = [\boldsymbol{\psi}_1, \bar{\boldsymbol{\psi}}_1, \boldsymbol{\psi}_2, \bar{\boldsymbol{\psi}}_2, \dots \boldsymbol{\psi}_n, \bar{\boldsymbol{\psi}}_n] \quad (26)$$

and can be used as an appropriate transformation matrix for introducing in matrix form the power spectrum of modal forces

$$\mathbf{S_{QQ}}(\omega) = \boldsymbol{T_G}^{-1}\mathbf{S_{FF}}(\omega)\, \boldsymbol{T_G}^{-1^{*\mathrm{T}}}. \quad (27)$$

The next section details the pertinent algorithm which utilizes the above three steps in a unified framework for conducting stochastic complex modal analysis of nonlinear MDOF systems without any restrictions on the nature of the damping matrix, in an iterative base.

2.4. Modal decoupling iterative algorithm

The proposed methodology incorporates an efficient iterative scheme which includes successive solution of an inverse stochastic dynamics problem for the determination of stochastically compatible power spectra with an assigned elastic design/response UHS of specified damping ratio. In the herein study, pseudo-acceleration design spectra prescribed by the European aseismic code provisions (EC8) are utilized for the determination of compatible design spectrum power spectra. At this point, it is deemed appropriate to note that the choice of EC8 is not binding and that the proposed methodology can readily be modified to account for provisions defined by various aseismic codes.

Relying on statistical linearization and utilizing the equivalent complex modal decomposition method, delineated in sections 2.2 and 2.3 respectively, the nonlinear *n*-DOF system is decoupled and cast into (*n*) uncoupled oscillators in modal coordinates with equivalent modal properties (EMPs). Next, the stochastically derived equivalent modal damping ratios ζ_{req} redefine the damping related to the updated input elastic response UHS which in turn define stochastically compatible design spectrum power spectra. The fully populated applied forces power spectrum required for the iterative algorithm, can be computed by considering the redetermined diagonal excitation response spectrum in modal coordinates as

$$\mathbf{S_{FF}}(\omega) = \boldsymbol{T_G} \mathbf{S_{QQ}}(\omega) \, \boldsymbol{T_G}^{*\mathrm{T}} \tag{28}$$

The aforementioned procedure establishes a cyclic relationship between the, output, stochastically equivalent damping coefficients of the decoupled modal oscillators and the input, damping ratios of the elastic response spectrum until input/output damping ratio consistency is achieved for all the modes. Lastly, once convergence between $\zeta_{\mathrm{req}}^{(k)}$ and $\zeta_{\mathrm{req}}^{(k-1)}$ is achieved after k iterations for a *r* mode, the equivalent linear vibrational modal properties, $\omega_{\mathrm{req}}^{(k_{end})}$ and $\zeta_{\mathrm{req}}^{(k_{end})}$ are defined. The idea of iteratively updating the nominal damping ratio of the input response spectrum corresponding to each mode ensures compliance with the basic definition of the response spectrum-based analysis which requires the considered decoupled linear/linearized oscillators and the imposed elastic UHS to share the same damping. This critical point only recently received the appropriate attention in the literature (Mitseas et al. 2017; 2018).

2.5. Generalized CQC method

The method uses the relative displacement response spectra and real-valued participation factors Γ_r which have been determined from the complex-valued mode shapes $\boldsymbol{\theta}_r$. The mode shapes, $\boldsymbol{\theta}_r$, are given by the upper half of the eigenvector $\boldsymbol{\psi}_r$. The real-valued coefficients a_r and c_r are defined as $a_r = -2Re(\eta_r \bar{\lambda}_r)$ and $c_r = 2Re(\eta_r)$, where

$$\eta_r = (\boldsymbol{w}^T \boldsymbol{\theta}_r)(\boldsymbol{\theta}_r^{\boldsymbol{T}} \boldsymbol{M\gamma})(-\lambda_r \boldsymbol{\theta}_r^{\boldsymbol{T}} \boldsymbol{M} \boldsymbol{\theta}_r + \lambda_r^{-1} \boldsymbol{\theta}_r^{\boldsymbol{T}} \mathbf{K} \boldsymbol{\theta}_r)^{-\mathbf{1}} \tag{29}$$

The transformation vector $\boldsymbol{w^T}$ for the case of the top floor relative displacement takes the values $[1 \ -1 \ \ 0 \dots 0]$. The modal responses can be combined by the generalized CQC rule to obtain (Sinha and Igusa, 1995)

$$max|y_j(t)| = \sqrt{\sum_{r=1}^{n} \sum_{p=1}^{n} \Gamma_r \Gamma_p \, \rho_{rp} S_d(\omega_r, \zeta_r) \, S_d(\omega_p, \zeta_p)} \tag{30}$$

where ρ_{rp} are correlation coefficients given by

$$\rho_{rp} = \left[\frac{8\sqrt{\zeta_r \zeta_p (\zeta_r + \varepsilon \zeta_p)} \varepsilon^{3/2}}{(1-\varepsilon^2)^2 + 4\zeta_r \zeta_p \varepsilon (1+\varepsilon^2) + 4(\zeta_r^2 + \zeta_p^2)\varepsilon^2} \right] \left[\frac{a_r a_p + \omega_r \omega_p c_r c_p}{\Gamma_r \Gamma_p} \right] \tag{31}$$

with $\varepsilon = \omega_p / \omega_r$ and Γ_r are the real-valued modal participation factors which are defined as

$$\Gamma_r = \sqrt{(a_r^2 + \omega_r^2 c_r^2)} \tag{32}$$

3. NUMERICAL IMPLEMENTATION

In this section the proposed stochastic dynamics technique is numerically exemplified by considering a nonlinear multi-storey frame structure subject to the Eurocode 8 elastic response spectrum (CEN, 2004). The degree of accuracy of the predicted peak mean inter-storey drifts is quantified by comparison with pertinent results derived from nonlinear RHA for a large ensemble of time-realizations compatible with the considered elastic response spectrum.

3.1. Nonlinear MDOF frame structure

The three-storey non-classically damped inelastic shear frame shown in Figure 1 is considered to illustrate the proposed approach. The lumped masses m_j, the stiffness and damping coefficients of the j-th story, k_j and c_j, respectively, are provided as $m_1 = m_2 = m_3 = 50\, ton$, $k_1 = 6.25\, MNm^{-1}$, $k_2 = 3\, MNm^{-1}$, $k_3 = 1.5\, MNm^{-1}$, $c_1 = 30\, kNsm^{-1}$, $c_2 = 20\, kNsm^{-1}$, and $c_3 = 10\, kNsm^{-1}$. The behavior of the shear frame is governed by nonlinear springs, of the linear-plus-cubic kind, connecting the structure to the ground and each storey with the above. Similarly, nonlinear dampers, of the linear-plus-cubic kind are employed in the same pattern.

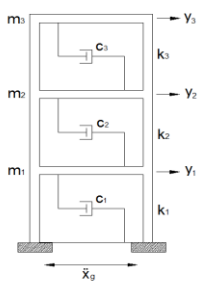

Figure 1. The three-storey nonlinear shear frame

Then the nonlinear vector function reads

$$\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \begin{Bmatrix} \varepsilon_1 k_1 \mathrm{x}_1^3(t) + \varepsilon_2 c_1 \dot{\mathrm{x}}_1^3(t) \\ \varepsilon_1 k_2 \mathrm{x}_2^3(t) + \varepsilon_2 c_2 \dot{\mathrm{x}}_2^3(t) \\ \varepsilon_1 k_3 \mathrm{x}_3^3(t) + \varepsilon_2 c_3 \dot{\mathrm{x}}_3^3(t) \end{Bmatrix} \quad (33)$$

where $\varepsilon_1 = 5$ and $\varepsilon_2 = 1 \times 10^{-5}$ have been assumed in the herein study.

3.2. Eurocode 8 compatible power spectra

The Eurocode 8 (CEN, 2004) elastic response spectrum for soil conditions B, critical damping ratio $\zeta_o = 5\%$, and peak ground acceleration (PGA) equal to $0.36g$ is initially considered for exciting the structure in Figure 1. The employed spectrum is plotted in Figure 2 (black continuous curve) against the natural period T=2π/ω. The duration T_s is taken equal to $20\, s$, whereas the discretization step $\Delta\omega$ in Eq.(5) is set equal to $0.1\, rad/s$. A preselection of an input power spectrum shape has to be preceded for deriving a stochastically compatible spectrum, according to the numerical scheme presented in section 2.1. In the ensuing analysis, the Clough and Penzien (CP) spectrum is considered, i.e.,

$$D_{X_i}(\omega_i) = \frac{(\omega_i/\omega_f)^4}{\left(1 - (\omega_i/\omega_f)^2\right)^2 + 4\xi_f^2(\omega_i/\omega_f)^2} \frac{\omega_g^4 + 4\xi_g^2\omega_g^2\omega_i^2}{(\omega_g^2 - \omega_i^2)^2 + 4\xi_g^2\omega_g^2\omega_i^2} \quad (34)$$

where the requisite parameters are $\xi_g = 0.78$, $\omega_g = 10.78\, rad/s$, $\xi_f = 0.92$ and $\omega_f = 2.28\, rad/s$. The parameters ω_g and ξ_g describe the filtering effects of the geological formations on the propagation of the seismically induced waves, whereas ω_f and ξ_f control the incorporated CP high-pass filter.

The achieved level of compatibility between the derived power spectrum $G_{X_i}^{\zeta_o}(\omega)$ and the response spectrum S_a is presented in Figure 2 by comparing the assigned S_a with the response spectrum computed by Eq. (3) (broken line). A further comparison is shown in terms of a Monte Carlo analysis which involves an ensemble of 5000 stationary signals of 20s duration each compatible with the $G_{X_i}^{\zeta_o}(\omega)$ spectrum. The median response spectrum of these signals is plotted (dotted line) in Figure 2; a more detailed documentation accompanied by the appropriate commentary can be found in Mitseas et al. (2018).

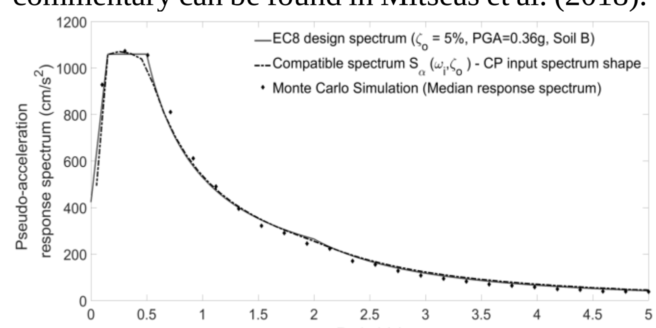

Figure 2. Eurocode 8 response spectrum for 5% damping ratio and compatibility assessment with the power spectrum using Eq.(3) and nonlinear RHA.

3.3. Derivation of equivalent modal properties

Following the efficient modal decoupling iterative algorithm delineated in sections 2.1-2.4, the nonlinear n-DOF system is decoupled and cast into (n) uncoupled modal oscillators. To this aim, a number of successive iterations is applied. The employed thresholds concerning the convergence checks are set to $\beta = 10^{-4}$. Sets of three $\omega_{\text{req}}^{(k)}$ and $\zeta_{\text{req}}^{(k)}$ EMPs $r = 0,1,2,3$ corresponding to the three mode shapes of the system are derived as by-products of the iterative algorithm as a function of the iteration index k. The iteratively repeated convergence process concerning each mode separately is terminated when successive values of the corresponding equivalent modal damping ratio $\zeta_{\text{req}}^{(k)}$ display difference lower than the threshold β. The values of the stochastically derived sets of EMPs attained at the last iteration correspond to the equivalent linear vibrational modal properties of the system. To illustrate the convergence rate, the derived EMPs $\omega_{\text{req}}^{(k)}$ and $\zeta_{\text{req}}^{(k)}$ are plotted in Figure 3(a-b), respectively, as a function of the iteration index k. It is readily seen that convergence is achieved after a small number of iterations for all the modes.

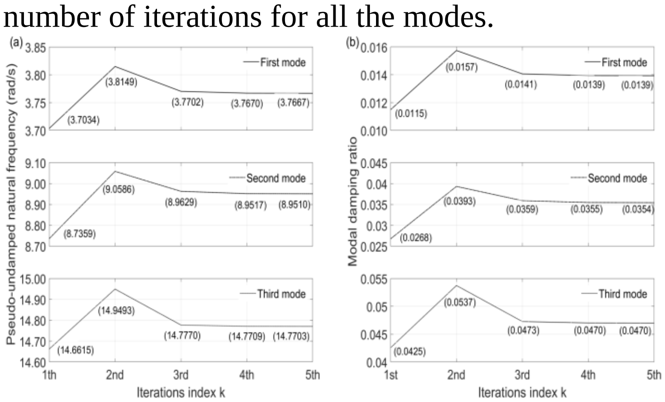

Figure 3. Equivalent pseudo-undamped natural frequency and modal damping ratio coefficients from successive iterations.

The herein defined equivalent linear vibrational modal properties can be used directly in conjunction with the corresponding appropriately updated response UHS to estimate the peak inelastic modal responses of the system. Subsequently, the generalized CQC rule is employed for nonlinear response estimates in physical coordinates.

Proposed methodology-based data are compared with nonlinear RHA within a Monte-Carlo simulation context utilizing an ensemble of 5000 artificial acceleration time-histories compatible in the mean sense with the Eurocode 8 elastic spectrum for 5% damping. For comparison reasons, results pertaining to the derived EMPs after the first iteration of the proposed framework are included, hence, the corresponding estimates for the peak inter-storey drifts (in cm) are 8.07, 11.95, and 15.76, whereas the associated estimation error is gauged as 12.76%, 9.95%, and 15.31%. Harnessing the potential of the equivalent linear vibrational modal properties $\omega_{\text{req}}^{(k_{end})}$ and $\zeta_{\text{req}}^{(k_{end})}$, comprehensive numerical data are presented in Table 1.

Table 1: Comparison of peak drift estimates.

Floor level	Peak inter-storey drifts (cm)	Nonlinear RHA (cm)	Percentage error
1	8.51	9.25	8.68%
2	12.32	13.27	7.69%
3	17.05	18.61	9.14%

Based on the presented results, the proposed methodology evinces that the iterative scheme contributes substantially in the enhancement of the accuracy through identifying efficiently equivalent vibrational modal properties based on the degree of the exhibiting nonlinearity. Note that the method leads to substantial reduction of computational effort as compared with nonlinear RHA within a MCS framework. Indicatively, to provide with an indicative order of magnitude for the computational cost involved, utilizing a laptop computer with standard configurations, the proposed technique requires 1-2 min, whereas the MCS based system peak response estimation (5,000 time histories) requires 4–5 h. Further, the bilinear hysteretic model which constitutes one of the more common practices in aseismic engineering, is recognized as a potential extension.

4. CONCLUSIONS

A novel inelastic modal decomposition method has been proposed for conducting dynamic response analysis of nonlinear MDOF structural systems excited by an elastic response UHS (e.g., Eurocode 8), while circumventing the need of undertaking computationally demanding nonlinear RHA. Specifically, the proposed methodology is provided in the rather advantageous response-spectrum variant rather than in a time-history version, in an attempt to increase its attractiveness among the practicing engineers. This feature hopefully renders the herein proposed approach a potent analysis tool for seismic design of nolinear/yielding structures, without any restrictions on the nature of the damping matrices (e.g. such problems fairly often arise in equipment-structure-type systems).

Particular attention has been given to identify and elucidate the physical significance of the equivalent linear vibrational modal properties. It has been shown that the displacements of a non-classically damped, nonlinear MDOF system may be expressed as a linear combination of the displacements of a number of excited modal oscillators by response spectra appropriately adjusted to the equivalent linear vibrational modal characteristics corresponding to each mode of vibration.

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