A New Online Bayesian Approach for the Joint Estimation of State and Input Forces using Response-only Measurements

Daniz Teymouri
MPhil Student, Dept. of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Hong Kong, China

Omid Sedehi
PhD Candidate, Dept. of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Hong Kong, China; Department of Civil Engineering, Sharif University of Technology, Tehran, Iran

Lambros S. Katafygiotis
Professor, Dept. of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Hong Kong, China

Costas Papadimitriou
Professor, Dept. of Mechanical Engineering, University of Thessaly, Volos, Greece

ABSTRACT: In this paper, a recursive Bayesian-filtering technique is presented for the joint estimation of the state and input forces. By introducing new prior distributions for the input forces, the direct transmission of the input into the state is eliminated, which allows removing low-frequency error components from the predictions and estimations. Eliminating such errors is of practical significance to the emerging fatigue monitoring methodologies. Furthermore, this new technique does not require a priori knowledge of the input covariance matrix and provides a powerful method to update the noise covariance matrices in a real-time manner. The performance of this algorithm is demonstrated using one numerical example and compared it with the state-of-the-art algorithms. Contrary to the present methods which often produce unreliable and inaccurate estimations, the proposed method provides remarkably accurate estimations for both the state and input.

1. INTRODUCTION

Real-time estimation of the state in linear time-invariant (LTI) dynamical systems is a relatively well-addressed topic. The Kalman filter (KF) provides optimal solutions for the estimation of the state when the process and observation noise are both Gaussian, and the input forces are known (Anderson and Moore 1979). In practice, however, it is often the case that the input forces are unknown as well. This requires estimating the state and input forces simultaneously. Gillijns and De Moor (2007) proposed an optimal filter in the unbiased minimum-variance sense for the joint estimation of the state and input adopting a two-stage algorithm. Lourens et al. (2012) developed it further for modally-reduced dynamical models. This filter is often referred to as GDF (Eftekhar Azam et al. 2015). Another approach is to use a first-order random walk model to describe the variation of input forces over time. Then, an augmented state vector comprising both the state and input can be constructed, to which the original formulation of the KF can be applied (Lourens et al. (2012b). Therefore, this filter is regarded to as augmented Kalman filter (AKF). However, both GDF and AKF are extremely error-prone working only under certain conditions. These conditions can be categorized into observability, controllability, direct inevitability, stability, and uniqueness of estimations discussed in Maes et al. (2015). However, it has been evidenced that the estimations of the state and input produced by both GDF and AKF are contaminated with low-frequency drift components, even if such conditions are met. Naets et al. (2015) have shown that the drift can be reduced by using dummy displacement/strain measurements in addition to acceleration responses. Eftekhar Azam et al. (2015) demonstrated that the drift problem originates from the double-integration of noisy acceleration response measurements and have proposed a dual Kalman filter (DKF) method for the simultaneous estimation of the state and input forces. They indicated that the drift problem could be diminished if the covariance matrix of the input forces is calibrated very well by using L-curve methods. However, such calibration methods cannot be applied in an online manner, and the performance of the DKF can be influenced when the characteristics of the input forces vary considerably.

In this paper, a recursive Bayesian filter is presented for the joint estimation of the state and input forces. This algorithm does not require knowing the covariance matrix of the input. It also updates the noise covariance matrices sequentially. The performance of this algorithm is demonstrated using one numerical example and compared with the AKF, GDF, and DKF.

2. STATE-SPACE STOCHASTIC MODEL

The discrete-time state-space representation of linear-time invariant dynamical systems can be expressed as (Anderson and Moore 1979):

$$\mathbf{z}_{k+1} = \mathbf{A}\mathbf{z}_k + \mathbf{B}\mathbf{p}_k + \mathbf{v}_k \tag{1}$$

where $\mathbf{z}_k$ denotes the state vector comprising the displacement and velocity of all degrees-of-freedom, $\mathbf{p}_k$ is the input force, $\mathbf{A}$ is the system matrix, $\mathbf{B}$ is the input-to-state matrix, and $\mathbf{v}_k$ is process noise described using a zero-mean Gaussian distribution having covariance matrix $\mathbf{Q}_k$, i.e. $N(\mathbf{v}_k \,|\, \mathbf{0}, \mathbf{Q}_k)$. Note that the sub-index k corresponds to discrete-time sample $t_k = k\Delta t$, $k = \{0,1,...,n\}$. The reader is referred to Chen (2003) for the definition of $\mathbf{A}$ and $\mathbf{B}$.

Let $\mathbf{d}_k$ be the observation vector comprising discrete-time displacement, velocity, and acceleration measurements. This observation vector can be expressed as:

$$\mathbf{d}_k = \mathbf{C}\mathbf{z}_k + \mathbf{D}\mathbf{p}_k + \mathbf{w}_k \tag{2}$$

where matrices $\mathbf{C}$ and $\mathbf{D}$ are well-known in the context of joint input-state estimation methods can be found elsewhere (Eftekhar Azam et al. 2015), and $\mathbf{w}_k$ is observation noise described using a zero-mean Gaussian distribution having covariance matrix $\mathbf{R}_k$, i.e. $N(\mathbf{v}_k \,|\, \mathbf{0}, \mathbf{R}_k)$.

3. SEQUENTIAL BAYESIAN APPROACH

Given the characteristics of the process and observation models, we aim to estimate the state and input forces simultaneously using a sequential Bayesian approach. For this purpose, we use Gaussian conjugate distributions to establish a sequential relationship between the updating and estimating parameters such that the formulation can be implemented in a recursive non-iterative algorithm. In a recent study by the authors of this paper (Sedehi et al. 2018a,b), we have mathematically proved and proposed a new algorithm to handle this joint estimation problem. This novel Bayesian-filtering technique is outlined in Algorithm 1, offering several advantages over the existing algorithms summarized below:

- It does involve using and calibrating an ad-hoc covariance matrix for the input forces, contrary to the DKF and AKF, which require doing so. This allows it to eliminate the adverse effects caused by using random walk models for describing the variation of input forces over time.
- It offers to update the noise covariance matrices using a Bayesian updating paradigm, while the present methods

require tuning the covariance matrices at the very beginning.

- The correlation of the state and input forces is fully considered, whereas the DKF totally neglects it.
- The direct transmission of the input into the state required for estimating the input of the next step is entirely eliminated. This new formulation removes the adverse effects caused by the erroneous input estimations obtained at the preceding steps.

Algorithm 1: Bayesian estimation of the state and input in dynamical systems.

Set initial estimations for

$\mathbf{z}_{1|0}$, $\tilde{\mathbf{z}}_{1|0}$, $\mathbf{P}^{z}_{1|0}$, $\mathbf{P}^{p}_{1|0}$, $\mathbf{P}^{zP}_{1|0}$, $\mathbf{\Sigma}_0$, $\mathbf{\Omega}_0$, ρ_0, *and* ν_0

For time steps *k = 1 : n*

1- Kalman gain of the input

$$\mathbf{G}^{p}_{k} = \mathbf{P}^{p}_{k|k-1}\mathbf{J}^{T}(\mathbf{J}\mathbf{P}^{p}_{k|k-1}\mathbf{J}^{T} + \hat{\mathbf{R}}_{k-1})^{-1}$$

2-Input estimation

$$\mathbf{p}_{k|k} = \mathbf{G}^{p}_{k}(\mathbf{d}_k - \mathbf{G}\tilde{\mathbf{z}}_{k|k-1})$$

3-Estimating the input covariance matrix

$$\mathbf{P}^{p}_{k|k} = \mathbf{P}^{p}_{k|k-1} + \mathbf{G}^{p}_{k}\mathbf{G}\mathbf{P}^{z}_{k|k-1}\mathbf{G}^{T}\mathbf{G}^{p}_{k} - \mathbf{G}^{p}_{k}\mathbf{J}\mathbf{P}^{p}_{k|k-1}$$

4- Kalman gain of the state

$$\mathbf{G}^{z}_{k} = \tilde{\mathbf{P}}^{z}_{k|k-1}\mathbf{G}^{T}(\mathbf{G}\tilde{\mathbf{P}}^{z}_{k|k-1}\mathbf{G}^{T} + \hat{\mathbf{R}}_{k-1})^{-1}$$

5-State estimation

$$\mathbf{z}_{k|k} = \mathbf{z}_{k|k-1} + \mathbf{G}^{z}_{k}(\mathbf{d}_k - \mathbf{G}\mathbf{z}_{k|k-1} - \mathbf{J}\mathbf{p}_{k|k})$$

6-Estimating the state covariance matrix

$$\mathbf{P}^{z}_{k|k} = \mathbf{P}^{z}_{k|k-1} + \mathbf{G}^{z}_{k}\mathbf{J}\mathbf{P}^{P}_{k|k-1}\mathbf{J}^{T}\mathbf{G}^{z}_{k} - \mathbf{G}^{p}_{k}\mathbf{G}\mathbf{P}^{z}_{k|k-1}$$

7-Estimating the correlation between the state and input

$$\mathbf{P}^{zp}_{k|k} = -\mathbf{G}^{z}_{k}\mathbf{J}\mathbf{P}^{p}_{k|k}$$

8-Updating the covariance matrix of observation noise

$$\mathbf{\Sigma}_k = \mathbf{\Sigma}_{k-1} + \left(\mathbf{d}_k - \mathbf{G}\mathbf{z}_{k|k} - \mathbf{J}\mathbf{p}_{k|k}\right)\times \left(\mathbf{d}_k - \mathbf{G}\mathbf{z}_{k|k} - \mathbf{J}\mathbf{p}_{k|k}\right)^{T}$$

$$\nu_k = \nu_{k-1} + 1$$

$$\hat{\mathbf{R}}_k = \mathbf{\Sigma}_k / \left(\nu_k + N_0 + 1\right)$$

9-Updating the covariance matrix of process noise

$$\mathbf{\Omega}_k = \mathbf{\Omega}_{k-1} + \left(\mathbf{z}_{k|k} - \mathbf{A}\mathbf{z}_{k-1|k-1} - \mathbf{B}\mathbf{p}_{k-1|k-1}\right)\times \left(\mathbf{z}_{k|k} - \mathbf{A}\mathbf{z}_{k-1|k-1} - \mathbf{B}\mathbf{p}_{k-1|k-1}\right)^{T}$$

$$\rho_k = \rho_{k-1} + 1$$

$$\hat{\mathbf{Q}}_k = \mathbf{\Omega}_k / \left(\rho_k + 2N_d + 1\right)$$

10-Prediction of the state without feedthrough of the input

$$\tilde{\mathbf{z}}_{k+1|k} = \mathbf{A}\mathbf{z}_{k|k}$$

11-Prediction of the state with feedthrough of the input

$$\mathbf{z}_{k+1|k} = \mathbf{A}\mathbf{z}_{k|k} + \mathbf{B}\mathbf{p}_{k|k}$$

12-Prediction of the input covariance matrix

$$\mathbf{P}^{p}_{k+1|k} = \mathbf{P}^{p}_{k|k}$$

13-Prediction of the state covariance matrix

$$\mathbf{P}^{z}_{k+1|k} = \mathbf{A}\mathbf{P}^{z}_{k|k}\mathbf{A}^{T} + \mathbf{B}\mathbf{P}^{p}_{k|k}\mathbf{B}^{T} + \mathbf{A}\mathbf{P}^{zP}_{k|k}\mathbf{B}^{T} + \mathbf{B}\mathbf{P}^{zPT}_{k|k}\mathbf{A}^{T} + \hat{\mathbf{Q}}_k$$

End For

In the next section, we use a numerical example to demonstrate the proposed method and compare it with the GDF, AKF, and DKF.

4. ILLUSTRATIVE EXAMPLE

Figure 1 shows the four degrees-of-freedom (DOF) dynamical system selected to demonstrate the proposed method. The mass matrix is a 4×4 identity matrix. The stiffness of springs is assumed to be 1kN/m. The damping mechanism is considered to be viscous having 1N.s/m damping coefficient. Given these assumptions, the state-space model can be constructed based on Eqs. (1) and (2).

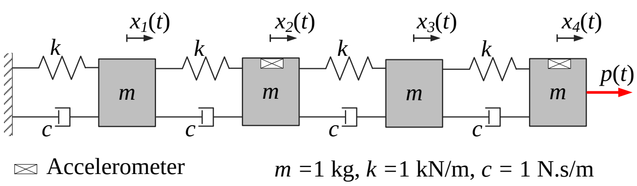

Figure 1: 4-DOF dynamical system considered for the joint input-state estimation.

External force $p(t)$ is applied to the fourth DOF. A Gaussian white noise (GWN) process and an impulse force are considered as the applied forces acting on the 4th DOF. The loadings are considered to be discrete-time functions sampled at intervals of 0.001s given by:

$$p(t) \underset{i.i.d.}{\sim} N(0,10) \quad \text{N} \tag{3}$$

$$p(t) = 5\big(u(t-1) - u(t-1.01)\big) \quad \text{N} \tag{4}$$

where $u(t)$ denotes unit step function; $N(0,10)$ is a Gaussian distribution with mean 0 and standard deviation 10N. Acceleration time-history responses of the 2nd and 4th DOF are considered as the measurements. This measurement selection falls within the category of collocated sensing owing to performing the acceleration measurement at the location of the applied loadings. Zero-mean GWN process having the standard deviation equal to 1% of the root-mean-square of the noise-free acceleration response is added to the response to account for the measurement noise.

In both cases of the input forces, the initial values $\mathbf{z}_{1|0}, \tilde{\mathbf{z}}_{1|0}$, $\mathbf{P}^{z}_{1|0}$, $\mathbf{P}^{p}_{1|0}$, ρ_0, ν_0, and $\mathbf{P}^{zP}_{1|0}$ are all set to zero. The parameters $\mathbf{\Sigma}_0$ and $\mathbf{\Omega}_0$ are considered to identity matrices of appropriate dimensions.

4.1. Gaussian input force

The GWN force expressed by Eq. (3) is applied to the 4th DOF. Figure 2(a) shows estimations of the input force compared with the actual values. As shown, the proposed method provides reasonable estimations of the applied force. Figure 2(b) compares the errors produced by the GDF, AKF, DKF and the proposed method when they are used to predict the input force. It can be seen that the proposed method outperforms the other three methods, produced the smallest errors. However, both the AKF and GDF diverge and the DKF is involved with large estimation errors.

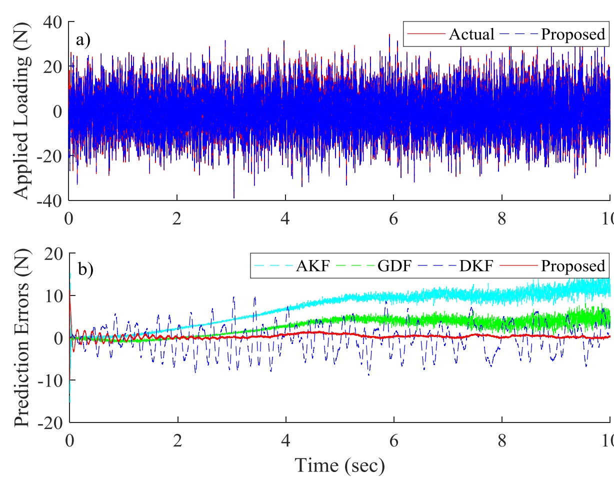

Figure 2: (a) estimation of the input force compared with the actual values (b) input prediction errors produced by the AKF, GDF, DKF, and the proposed method

Figure 3(a-b) shows the predictions of the displacement and velocity time-history responses corresponding to the 1st DOF obtained by using the proposed method. The accuracy of predictions is excellent, and the predicted response is free of low-frequency drift components. The response predictions corresponding to t<1s seem somewhat inaccurate, which is attributed to the convergence in estimating the noise covariance matrices.

Figure 4(a-b) compares the prediction errors produced by the four methods used for capturing the actual responses. As indicated, the proposed method gives the smallest errors, while the AKF and GDF entirely diverge, and the DKF produces extremely large errors. Similar observations were made for all other DOF confirming the efficacy of the proposed method.

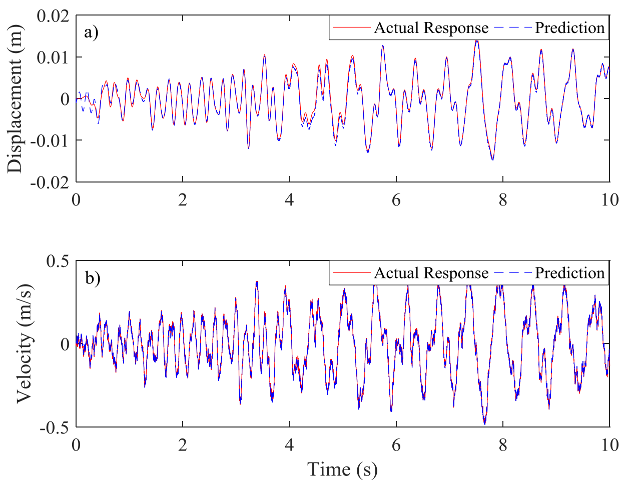

Figure 3: predictions of the 1st DOF time-history responses (a) displacement response (b) velocity response.

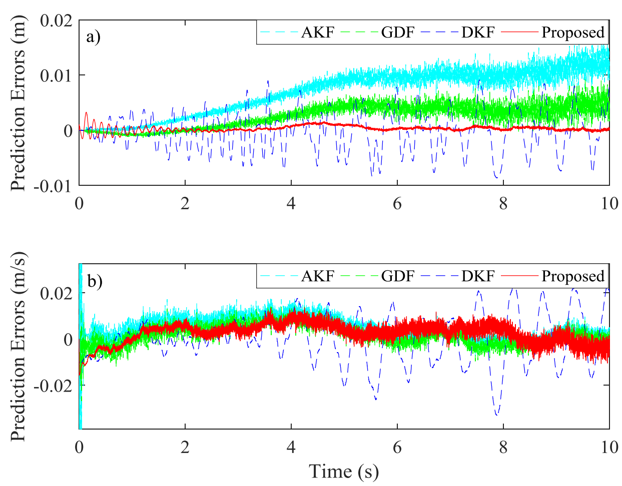

Figure 4: prediction errors produced by the AKF, GDF, DKF, and the proposed method in estimating (a) the displacement response of the 1st DOF (b) the velocity response of the 1st DOF.

Updating the noise covariance matrices is performed using the proposed algorithm. Figure 5 represents the convergence of the elements of covariance matrices to constant values. This remarkable stability and convergence achieved in estimating the covariance matrices broaden a new horizon to the state-input estimation problems.

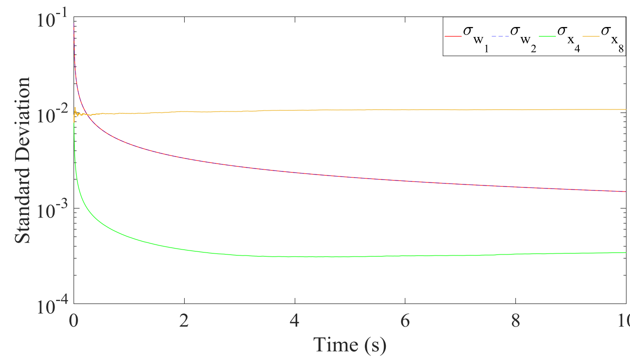

Figure 5: online update of the noise covariance matrices (σ_{w_1} and σ_{w_2} denote the measurement noise standard deviation corresponding to the 1st DOF and 2nd DOF acceleration responses, respectively; σ_{x_4} is the standard deviation of the 4th DOF displacement response; σ_{x_8} is the standard deviation of the 4th DOF velocity response).

4.2. Impulse force

The impulse function expressed by Eq. (4) is applied to the 4th DOF. Figure 6(a) shows the estimated input forces by using the proposed algorithm compared with the actual values, and Figure 6(b) compares the prediction errors of the four methods. As can be seen, the proposed method produces the smallest errors in predicting the actual input force.

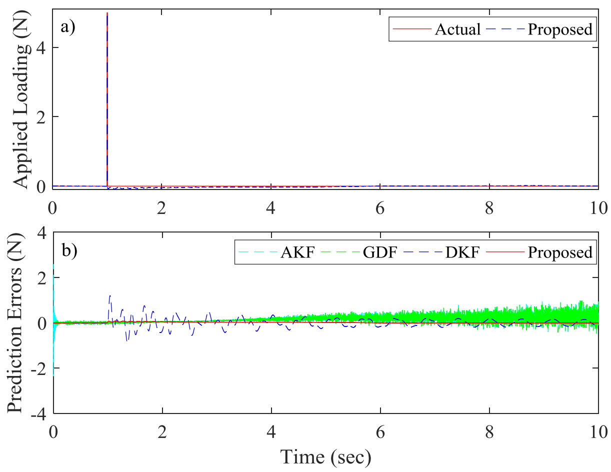

Figure 6: (a) estimation of the input force compared with the actual values (b) prediction errors produced by the AKF, GDF, DKF, and the proposed method.

Predictions of the displacement and velocity time-history responses corresponding to the 1st DOF are made by the proposed method. The results are shown in Figure 7(a-b) representing perfect accuracy in predictions of the state at an unmeasured DOF.

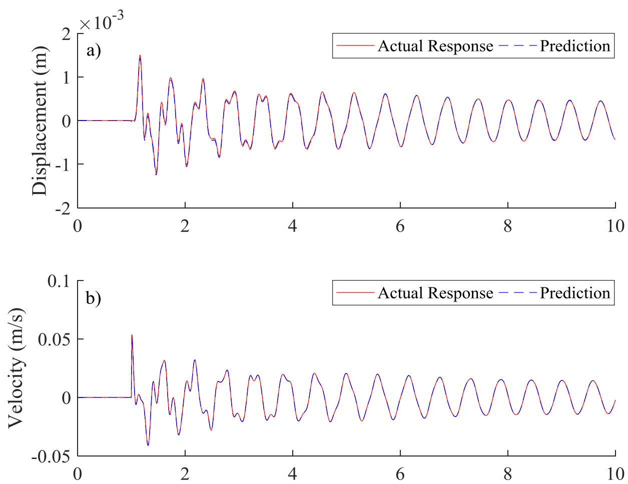

Figure 7: predictions of the 1st DOF time-history responses (a) displacement response (b) velocity response

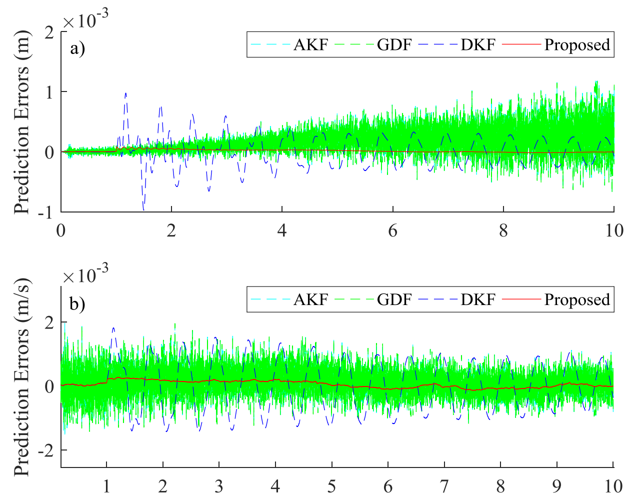

Figure 8: prediction errors produced by the AKF, GDF, DKF, and the proposed method in predicting (a) the displacement response of the 1st DOF (b) the velocity response of the 1st DOF

Prediction errors of the displacement and velocity responses are compared in Figure 8(a-b) to put the four methods into perspective. The results suggest that the proposed method greatly outperforms the other three methods. Moreover, estimations of the state and input are both free of low-frequency drift components. Resolving this drift problem shows promise for adopting this algorithm in the present fatigue prognosis methods in order to estimate fatigue damage accumulation and to predict the remaining fatigue life of critical system members.

5. CONCLUSIONS

A new Bayesian-filtering technique is presented for estimating the state and input, as well as updating the noise covariance matrices. A numerical example is selected to demonstrate the method and to compare it with three state-of-the-art methods. It is observed that the proposed method outperforms them in terms of the stability of estimations made for the state and input. While the three methods give unreliable estimations contaminated with large low-frequency error components, the offered method provides great accuracy without any drift in results. Eliminating this significant drawback appearing in the outcome of the present joint input-state estimation methods has significance in the fatigue monitoring methodologies.

6. ACKNOWLEDGEMENTS

Financial support from the Hong Kong research grants councils under grant numbers 16234816 and 16212918 is gratefully appreciated. The last author gratefully acknowledges the European Commission for its support of the Marie Sklodowska Curie program through the ETN DyVirt project (GA 764547).

This paper is completed as a part of the second author's PhD dissertation conducted jointly at Sharif University of Technology and the Hong Kong University of Science and Technology. The second author would like to gratefully appreciate kind support and supervision of Professor Fayaz R. Rofooei at Sharif University of Technology.

7. REFERENCES

Anderson, B. D. O., and Moore, J. B. (1979). "Optimal filtering." *Englewood Cliffs*, 21, 22–95.

Chen, Z. H. E. (2003). "Bayesian Filtering : From Kalman Filters to Particle Filters , and Beyond VI Sequential Monte Carlo Estimation : Particle Filters." *Statistics*, 182(1), 1–69.

Eftekhar Azam, S., Chatzi, E., and Papadimitriou, C. (2015). "A dual Kalman filter approach for state estimation via output-only acceleration measurements." *Mechanical Systems and Signal Processing*, Elsevier, 60, 866–886.

Eftekhar, S., Politecnico, A., and Azam, S. E. (2015). *Online Damage Detection in Structural Systems Applications of Proper Orthogonal Decomposition , and Kalman and Particle Filters*.

Gillijns, S., and De Moor, B. (2007). "Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feedthrough." *Automatica*, 43(5), 934–937.

Lourens, E., Papadimitriou, C., Gillijns, S., Reynders, E., De Roeck, G., and Lombaert, G. (2012a). "Joint input-response estimation for structural systems based on reduced-order models and vibration data from a limited number of sensors." *Mechanical Systems and Signal Processing*, Elsevier, 29, 310–327.

Lourens, E., Reynders, E., De Roeck, G., Degrande, G., and Lombaert, G. (2012b). "An augmented Kalman filter for force identification in structural dynamics." *Mechanical Systems and Signal Processing*, Elsevier, 27(1), 446–460.

Maes, K., Lourens, E., Van Nimmen, K., Reynders, E., De Roeck, G., and Lombaert, G. (2015). "Design of sensor networks for instantaneous inversion of modally reduced order models in structural dynamics." *Mechanical Systems and Signal Processing*, Elsevier, 52–53(1), 628–644.

Naets, F., Cuadrado, J., and Desmet, W. (2015). "Stable force identification in structural dynamics using Kalman filtering and dummy-measurements." *Mechanical Systems and Signal Processing*, Elsevier, 50–51, 235–248.

Sedehi, O., Teymouri, D., Papadimitriou, C., and Katafygiotis, L. S. (2019). "Online Bayesian Estimation of State and Input of Dynamical Systems using Acceleration-only Measurements." *Mechanical Systems and Signal Processing*, Elsevior, 123, 648-673.