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Dimension-reduced FPK equation for structures excited by filtered noises

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ABSTRACT: Stochastic engineering dynamic actions such as earthquakes and strong wind can be regarded as colored noise with some certain power spectral density functions. By inserting filtering equations the response of the original system could be inverted to a Markov process because most colored noises can be generated through filtering white noises, thus making it possible to adopt the method of FPK equation and other related methods. However, the large dimension of the systems lead to great challenge in the solution of related high-dimensional FPK equations. For this purpose, the present paper proposed a simplified method for the extended system by integrating the high-dimensional FPK equation and establishing equivalent drift coefficients, thus resulting in a dimension-reduced FPK equation. The Kanai-Tajimi power spectral density model is used as an example. Inserting the estimated equivalent drift coefficients into the dimension-reduced FPK equation and solving it by the finite difference method leads to the PDF of response of the systems. Numerical examples are illustrated. The method established can be extended to multiplicative noises.

1. INTRODUCTION

The partial differential equations governing the evolution of PDF of response of systems excited by stochastic processes, e.g. the Fokker-Plank-Kolmogorov (FPK) equation for Gaussian white noises and the Kolmogorov-Feller (KF) equation (Rudenko et al, 2016) for Poissonian white noises, have been extensively studied. However, it is still almost impossible to apply these equations to practical nonlinear systems because of their high dimension. For the FPK equations of high-dimensional systems, usually no analytical solutions exist. Numerical solutions such as the finite difference method (Mohammadi & Borzì, 2015) and path integral solutions (PIS) have been investigated extensively. While only problems of modest dimensions can be solved by direct finite difference method or by PIS (Chai, et al, 2016, Kougioumtzoglou & Spano, 2012, Kougioumtzoglou et al, 2015). There is still huge gap between the available methods and the demanding of real-world applications.

Therefore, reducing the dimension of the FPK equation is essential (Er 2011). Er (2011) proposed a state space split approach combined with the statistical linearization method. Alternatively, the probability density evolution method (PDEM) provides a potential tool by establishing a state-decoupled generalized density evolution equation (Li & Chen 2009). Combining PDEM and FPK equation yields a flux-equivalent probability density evolution equation (Chen & Yuan, 2014), which is a one-dimensional partial differential equation. By introducing the concept of equivalent drift coefficient (Chen & Lin, 2014), an FPK-like equation for the marginal PDFs was obtained. Recently, the equivalent drift coefficient was reconstructed by the conditional mean (Chen & Rui, 2018) to solve the PDF of systems subjected to Gaussian white noise.

The practical engineering dynamic actions are not white-noise stochastic processes. To this end, in the present paper the dimension-reduction of FPK equation is extended by involving the filtered processes. Numerical examples demonstrate its effectiveness for structures subjected to earthquake ground motion.

2. METHODOLOGY

Consider a multi-degree-of-freedom structure subjected to earthquake ground motion. The equation of motion reads

$$\mathbf{M}\ddot{\mathbf{X}}+\mathbf{C}\dot{\mathbf{X}}+\mathbf{f}(\mathbf{X})=-\mathbf{M}\mathbf{I}a_g(t) \tag{1}$$

where $\mathbf{M}=[M_{ij}]_{n\times n}\in R^{n\times n}$ and $\mathbf{C}=[C_{ij}]_{n\times n}\in R^{n\times n}$ are the mass and damping matrices, $\mathbf{f}\in R^{n\times 1}$ is the vector of restoring force, $\mathbf{I}=(1,1,\cdots,1)^T\in R^{n\times 1}$ is the vector with all components being 1, and $a_g(t)$ is the earthquake ground acceleration process.

Generally, the earthquake ground accelerations are not white-noise processes. One of the most widely used power spectral density (PSD) function is the Kanai-Tajimi model (Kanai, 1957; Tajimi, 1960). In this case, the response of the structure is not a Markov process, and the FPK equation cannot be applied directly.

Fortunately, the earthquake ground acceleration processes generated by the Kanai model can be regarded as filtered white noises. As a matter of fact, in this case, the effect of soil site filtering can be characterized by the following equation

$$\ddot{X}_g+2\zeta_g\omega_g\dot{X}_g+\omega_g^2X_g=\xi(t) \tag{2}$$

where ζ_g is the damping ratio and ω_g is the effective frequency of the surface soil layer, X_g is the displacement relative to the bedrock, and $\xi(t)$ is the zero-mean white noise with intensity D on the bedrock. Then the earthquake ground acceleration on the structure is given by

$$a_g(t)=\ddot{X}_g(t)-\xi(t)=-2\zeta_g\omega_g\dot{X}_g-\omega_g^2X_g \tag{3}$$

Clearly, the PSD of a_g is

$$S(\omega)=\frac{1+4\zeta_g^2(\frac{\omega}{\omega_g})^2}{[1-(\frac{\omega}{\omega_g})^2]^2+4\zeta_g^2(\frac{\omega}{\omega_g})^2}D \tag{4}$$

which, as shown in Figure 1, is the Kanai-Tajimi spectrum. For engineering applications, the values of parameters take $\omega_g=5\pi,\zeta_g=0.6$ and the standard deviation of white noise is $0.2\mathrm{g}$, and thus $D=0.1207\mathrm{m}^2/\mathrm{s}^3$.

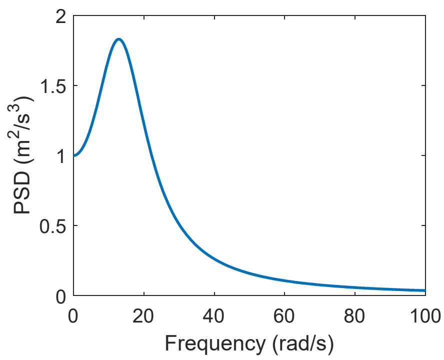


Figure 1: PSD of the Kanai-Tajimi filtered noise

Combining Eqs.(1), (2) and (3), and introducing the state vector $\mathbf{Z}=\left(\mathbf{X}^T,\dot{\mathbf{X}}^T,X_g,\dot{X}_g\right)^T=(\mathbf{X}^T,\mathbf{V}^T,X_g,V_g)^T$, where $\mathbf{V}=\dot{\mathbf{X}}$, $V_g=\dot{X}_g$, we have the following state equation

$$\dot{\mathbf{Z}}=\mathbf{A}(\mathbf{Z})+\mathbf{B}\xi(t) \tag{5}$$

where $\mathbf{A}\in R^{(2n+2)\times 1}$, $\mathbf{B}\in R^{(2n+2)\times 1}$. The expressions for $\mathbf{A}$ and $\mathbf{B}$ will not be detailed to avoid lengthiness of the paper.

This state equation can now be understood as an Itô stochastic differential equation. Therefore, a corresponding FPK equation exists for the joint PDF $p_{\mathbf{Z}}(\mathbf{z},t)$

$$\frac{\partial p_{\mathbf{Z}}(\mathbf{z},t)}{\partial t}=-\sum_{j=1}^{2n+2}\frac{\partial A_j(\mathbf{z},t)p_{\mathbf{Z}}(\mathbf{z},t)}{\partial z_j}+\frac{\sigma}{2}\frac{\partial^2 p_{\mathbf{Z}}(\mathbf{z},t)}{\partial v_g^2} \tag{6}$$

where A_j is the j-th component of $\mathbf{A}$, σ is the diffusion coefficient given by $\mathbf{B}D\mathbf{B}^{\mathrm{T}}$.

Though the solution of this high-dimensional partial differential equation is not readily available, the equation can be reduced to a lower-dimensional one if only some specified quantity is of interest (Chen & Rui, 2018). For instance, assume Z_l, the l-th component of $\mathbf{Z}$ is the quantity of interest. Denote the joint PDF of (Z_l, V_g) by by $p_{Z_lV_g}(z_l, v_g, t)$. Then Eq.(6) can be reduced, by adopting the dimension-reduction scheme in Chen & Rui (2018), to

$$\frac{\partial p_{Z_lV_g}(z_l,v_g,t)}{\partial t} = -\frac{\partial a_{Z_l}^{eq}(v_l,v_g,t)p_{Z_lV_g}(z_l,v_g,t)}{\partial v_l} - \frac{\partial a_{V_g}^{eq}(v_l,v_g,t)p_{Z_lV_g}(z_l,v_g,t)}{\partial v_g} + \frac{\sigma}{2}\frac{\partial^2 p_{Z_lV_g}(z_l,v_g,t)}{\partial v_g^2} \quad (7)$$

where $a_{Z_l}^{eq}(z_l, v_g, t), a_{V_g}^{eq}(z_l, v_g, t)$ are the equivalent drift coefficients given by the following conditional averaging

$$a_{V_l}^{eq} = E\left[A_l(\mathbf{Z},t)\middle|V_l = v_l, V_g = v_g\right] \quad (8)$$

$$a_{V_g}^{eq} = E\left[A_{2n+2}(\mathbf{Z},t)\middle|V_l = v_l, V_g = v_g\right] \quad (9)$$

which can be estimated by the embedded deterministic analyses.

The dimension-reduced FPK-like equation can then be solved, say by the path integral solution or the finite difference method. Finally, the PDF of Z_l can be obtained by

$$p_{Z_l}(z_l,t) = \int_{-\infty}^{\infty} p_{Z_lV_g}(z_l,v_g,t)dv_g \quad (10)$$

3. NUMERICAL EXAMPLES

The proposed method will be applied to the response analysis of structures subjected to earthquake ground accelerations.

3.1. A 10-story linear shear frame excited by earthquake ground acceleration with Kanai-Tajimi spectrum

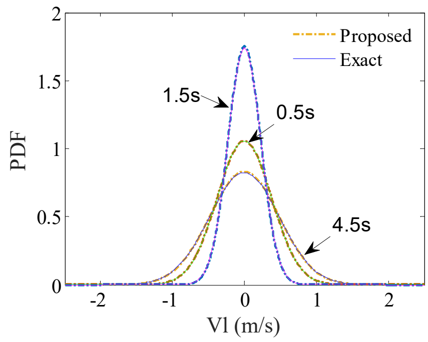
A 10-story linear story-shear frame subjected to earthquake excitation, modeled as filtered noise with Kanai-Tajimi spectrum, is analyzed. The lumped mass of each floor from bottom to top is [9.78, 9.78, 9.78, 9.78, 9.78, 9.78, 9.78, 9.78, 9.78, 9.78]×10^4kg. And the lateral inter-story stiffness of each story is [9.9, 9.9, 8.88, 8.88, 8.88, 8.88, 8.88, 8.88, 8.88, 8.88] ×10^7N/m. The natural periods of this building are calculated to be [1.3684, 0.4611, 0.2822, 0.2069, 0.1659, 0.1408, 0.1247, 0.1143, 0.1081, 0.1050] sec.

Suppose that the top displacement and the top velocity are quantities of interest. This linear case is analyzed so that numerical results can be compared with analytical ones.

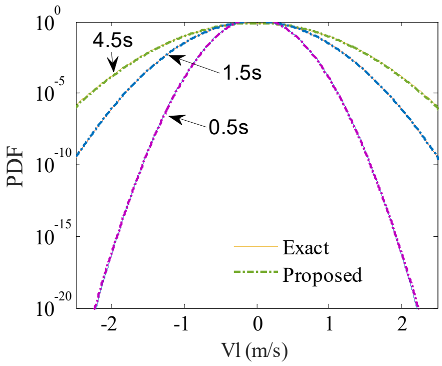
3.1.1. *Response of top velocity*

The conditional averaging in Eqs.(8) and (9) should be evaluated. Note that the conditional mean for linear systems is also linear (Bishop, 2006). 1000 embedded deterministic analyses are performed to yield data for the construction of the conditional mean. Then the reduced FPK-like equation (7) is solved by the explicit-implicit difference scheme with the time step 10^{-3}s. Afterwards, The marginal PDFs are obtained by integration in the direction of V_g according to Eq.(10).

Shown in Figures 2(a) and (b) are the PDFs of velocity at 3 typical time instants, compared with the analytical results, in linear and logarithmic coordinates, respectively. Figure 3 shows the joint PDF at 5s.



(a) PDFs in linear coordinates



(b) PDFs in logarithmic coordinates

Figure 2: PDFs of velocity at different time.

From the figures, perfect agreement between the results of the probability methods and the analytical counterparts is observed, in particular in the tail range which is usually highly concerned in reliability evaluation.

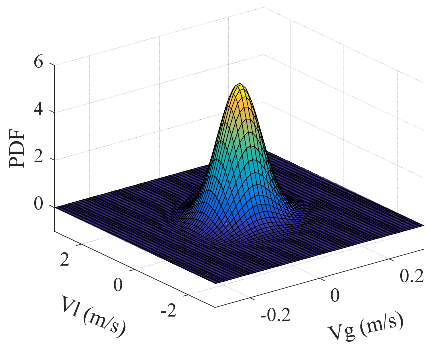
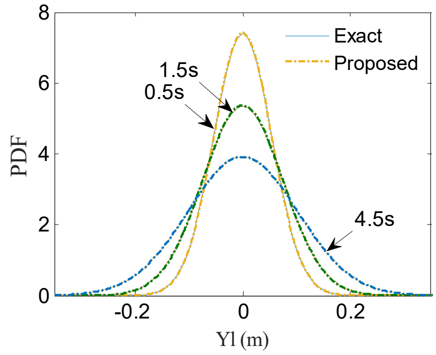


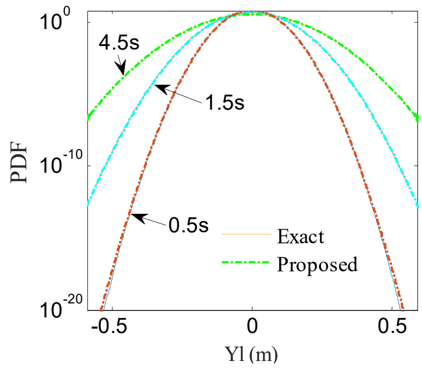
Figure 3: Joint PDF at 5s.

3.1.2. Top displacement

Applying the similar procedures, the PDF of top displacement can be obtained. The PDFs of top displacement at 3 typical time instants compared with the analytical results in linear and in logarithmic coordinate systems are shown in Figures 4(a) and (b), respectively. The cumulative distribution functions (CDFs) in linear and logarithmic coordinates are shown in Figures 5(a) and (b), respectively, and the complementary CDFs in logarithmic coordinate are plotted in Figure 5(c). Again high accuracy is observed, even in the tail range.

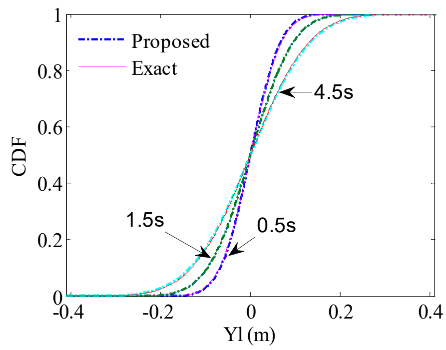


(a) PDFs in linear coordinates

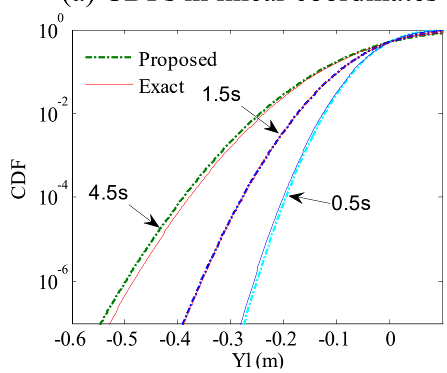


(b) PDFs in logarithmic coordinates

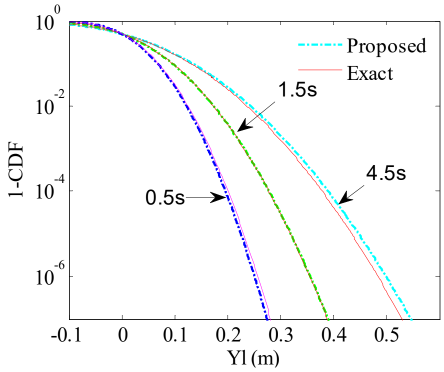
Figure 4: PDFs of top displacement at different time.



(a) CDFs in linear coordinates



(b) CDFs in logarithmic coordinates



(c) Complementary CDFs in logarithmic coordinate.

Figure 5: CDFs and complementary CDFs at different time.

3.2. *A Rayleigh oscillator excited by colored noise with Kanai-Tajimi PSD*

Consider a Rayleigh oscillator with the equation of motion

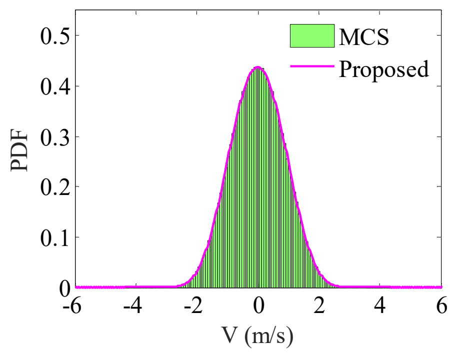
$$\ddot{X}+\omega_0^2 X+2\zeta_0\omega_0(-1+\varepsilon\dot{X}^2)\dot{X}=-a(t) \quad (11)$$

in which $a(t)$ is the colored noise with Kanai-Tajimi PSD, as described in Eq.(4). The parameter values are set to be $\omega_g=5\pi, \zeta_g=0.6, \omega_0=1.1, \zeta_0=0.1, \varepsilon=0.4$.

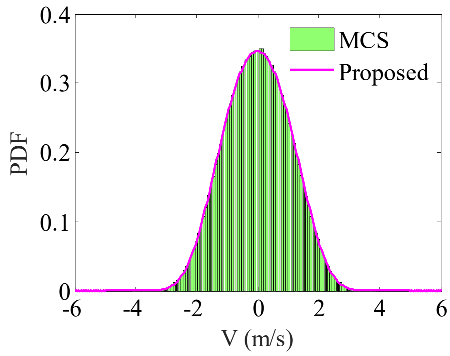
Likewise, to obtain the PDF of the system, now a 4-dimensional Itô stochastic differential equation is formed and then the corresponding FPK equation is reduced. In the numerical solution, 1000 embedded deterministic analyses via the 4th order stochastic Runge-Kutta algorithm are performed to estimate the equivalent drift coefficient, i.e., the conditional mean. Because of the cubic form of the damping force, the conditional means are estimated in a cubic form, at each time instant. Then the explicit-implicit difference scheme is employed to solve the reduced FPK-like equation. The time step is 0.005s.

In this example, no analytical results are available, therefore the proposed method is compared with Monte-Carlo simulation (MCS) of 10^6 times.

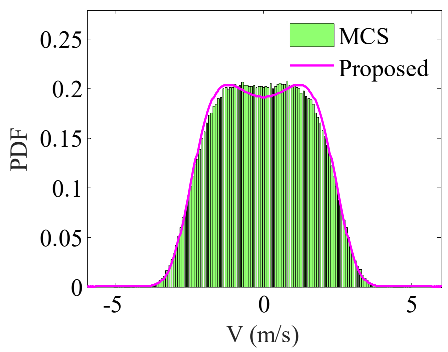
In Figures 6(a) through (c) shown are the PDFs of velocity at 3 time instants compared with MCS in linear coordinates. Shown in Figures 7 (a) through (c) are the CDFs in linear and logarithmic coordinates, and the complementary CDFs in logarithmic coordinate. Again, from the comparison with the Monte Carlo shown in these figures it is observed that proposed method is of high accuracy.



(a) PDF at 0.5s

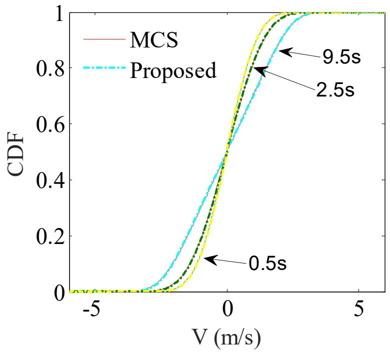


(b) PDF at 2.5s

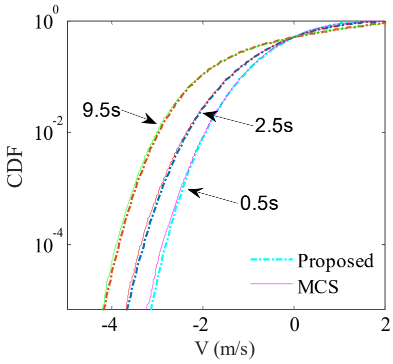


(c) PDF at 4.5s

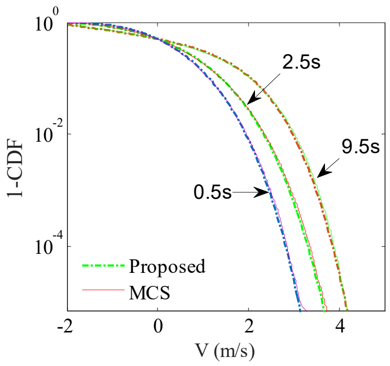
Figure 6: PDF at different time.



(a) CDFs in linear coordinates



(b) CDFs in logarithmic coordinates



(c) Complementary CDFs in logarithmic coordinate.

Figure 7: CDFs and complementary CDFs at different time.

4. CONCLUSIONS

The dimension-reduction method for FPK equations is extended to systems excited by non-white-noise stochastic processes. In particular, for the seismic response analysis of structures subjected to stochastic processes with Kanai-Tajimi spectrum is studied. Numerical examples are illustrated. The following conclusions can be drawn:

(1) The dimension-reduced FPK-like equation is not constrained by the degrees of freedom of the systems, therefore it is appropriate for general complex systems or structures subjected to non-white-noise modeled by filtered processes.

(2) The construction of the equivalent drift coefficients is implemented based on 1000 embedded deterministic analyses in this paper. The proposed method is of high accuracy.

More investigations on the rational construction of the equivalent drift coefficient should be done in the future. The method can be extended to multiplicative or Poissonian noises.

5. ACKNOWLEDGEMENTS

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