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Inverse quantification of epistemic uncertainty under scarce data: Bayesian or Interval approach?

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ABSTRACT: This paper introduces a practical comparison of a newly introduced inverse method for the quantification of epistemically uncertain model parameters with the well-established probabilistic framework of Bayesian model updating via Transitional Markov Chain Monte Carlo. The paper gives a concise overview of both techniques, and both methods are applied to the quantification of a set of parameters in the well-known DLR Airmod test structure. Specifically, the case where only a very scarce set of experimentally obtained eigenfrequencies and eigenmodes are available is considered. It is shown that for such scarce data, the interval method provides more objective and robust bounds on the uncertain parameters than the Bayesian method, since no prior definition of the uncertainty is required, albeit at the cost that less information on parameter dependency or relative plausibility of different parameter values is obtained.

1. INTRODUCTION

In general engineering practice, the knowledge on a structure is usually incomplete, be it due to inherent variable model parameters or a lack of knowledge on the true parameter values Ferson and Ginzburg (1996). Hence, representing these model parameters as deterministic quantities might prove to be inadequate when a reliable and economic design is pursued, as a large degree of conservatism is needed to prevent premature failure and corresponding maintenance or insurance costs. This over-conservatism not only impairs the economic cost of producing the component; it also leads to unnecessary weight increase, which is impermissible in high-performance sectors such as machinery design, aerospace or automotive. In the last few decades, highly advanced techniques including probabilistic Stefanou (2009), possibilistic Moens and Hanss (2011) or imprecise probabilistic methods Beer et al. (2013) have been introduced to include non-determinism efficiently in these design models and mitigate these risks.

In order for these tools to deliver a realistic quantification of the non-determinism in the responses of the design model, the description of the non-deterministic parameters of the model should be made objectively and accurately. Since not all parameters (such as e.g. connection stiffness values or heterogeneous material properties) are trivial to measure directly, inverse uncertainty quantification (UQ) techniques have been introduced. Following inverse UQ, the responses of the structure are measured and used to infer knowledge on the non-determinism in the model parameters. As concerns inverse UQ in a probabilistic sense, the class of Bayesian methods is considered the standard approach Beck and Katafygiotis (1998); Katafygiotis and Beck (1998), even for random fields Soize (2011). However, in the context of limited, insufficient, vague or ambiguous data, the prior estimation of the joint probability density function of the non-deterministic parameter values is subjective. Moreover this estimate influences the quantified result to a large extent when insufficient independent measurement data are available.

Recently, a novel methodology for the identification of multivariate interval uncertainty was introduced by some of the authors in Faes et al. (2016, 2017), with an extension to interval fields in Faes and Moens (2017a) and complex interdependence structures Faes and Moens (2019). This method is based on the convex hull concept for the representation of dependent uncertain output quantities of an interval FE model, and iteratively minimizes the discrepancy between the convex hull of these uncertain output quantities with the convex hull over a set of replicated measurement data.

The literature on comparing forward UQ in a probabilistic and non-probabilistic context is abundant Elishakoff (2000); Vandepitte and Moens (2011); Beer and Kreinovich (2013) and both classes of techniques are considered complementary rather than competitive. However, a theoretical and practical comparison for inverse approaches is severely lacking in literature and limited to works of the authors Broggi et al. (2018); Faes et al. (2019). This paper therefore presents a practical comparison of Bayesian model updating with the inverse interval quantification technique under very scarce data via a case study approach. The paper starts by briefly introducing Bayesian model updating and interval quantification from a theoretical perspective, highlighting potential shortcomings of the techniques. Then, the results are compared using the well-known DLR Airmod test structure. Section 2 recalls the Bayesian approach to uncertainty quantification. Section 3 introduces the inverse method for interval quantification. Section 4 applies both methods to the DLR Airmod test structure for the case of limited data. Finally, section 5 lists the conclusions of the work.

2. BAYESIAN APPROACH

The use of Bayesian methods for inverse uncertainty quantification of is largely founded on the the pioneering work of Beck and Katafygiotis Beck and Katafygiotis (1998); Katafygiotis and Beck (1998) in the late 1990s. Following the Bayesian interpretation of probability, the probabilistic nature of an uncertain parameter is interpreted as the degree to which it is believed that each possible value of this parameter is consistent with the available information. Following Bayes' rule, this degree of belief is adjusted using independent information. As such, the Bayesian methods translate this prior knowledge on the uncertainty corresponding to the parameter values to an updated posterior knowledge, based on experimental data.

As a first step in the inverse uncertainty quantification, a prior probability distribution $p(\theta|\mathscr{M})$, conditioned upon a chosen mathematical model

$\mathcal{M}$, is assigned to a set of uncertain parameters $\boldsymbol{\theta}$. These distributions represent prior information on the uncertain parameters. Then, independent experimental data $\mathcal{D}$ are used to update $p(\boldsymbol{\theta}|\mathcal{M})$ by means of Bayes' theorem to obtain the posterior distribution $p(\boldsymbol{\theta}|\mathcal{D},\mathcal{M})$:

$$p(\boldsymbol{\theta}|\mathcal{D},\mathcal{M}) = \frac{p(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})\, p(\boldsymbol{\theta}|\mathcal{M})}{p(\mathcal{D}|\mathcal{M})} \qquad (1)$$

where $p(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})$ is the likelihood of obtaining the data $\mathcal{D}$, given the value of the uncertain parameters $\boldsymbol{\theta}$ and a model of the structure $\mathcal{M}$. The denominator of eq. (1), also commonly referred to as evidence, ensures that the posterior distribution $p(\boldsymbol{\theta}|\mathcal{D},\mathcal{M})$ integrates to one.

In the context of structural dynamics, the data $\mathcal{D}$ usually consists of the residuals between experimental measurements and predictions of $\mathcal{M}$:

$$\varepsilon_i = z_i^e - z_i^m(\boldsymbol{\theta}), \quad i = 1,\ldots,d \qquad (2)$$

where z_i^e is the i-th measured eigenfrequency, $z_i^m(\boldsymbol{\theta})$ is the i-th predicted eigenfrequencies of a finite element model and d the number of considered responses.

In practice, the likelihood function is often chosen to be a zero-mean multivariate normal distribution:

$$p(\mathcal{D}|\boldsymbol{\theta},\mathcal{M}) = \prod_{i=1}^{N} \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\varepsilon_i^T(\boldsymbol{\theta})\Sigma^{-1}\varepsilon_i(\boldsymbol{\theta})\right) \qquad (3)$$

where N denotes the number of data points in $\mathcal{D}$. The solution to eq. (1) is commonly approximated by sampling from a Markov Chain that is ergodic and stationary with respect to $p(\mathcal{D}|\boldsymbol{\theta},\mathcal{M})$. In this paper, transitional Markov Chain Monte Carlo (TMCMC), as introduced by Ching and Chen Ching and Chen (2007), is applied for the computation of the posterior distribution.

3. INTERVAL APPROACH

An inverse method for the quantification of multivariate interval uncertainty, based on a set $\mathcal{D}$ of measured structural responses, was recently introduced by some of the authors Faes et al. (2016, 2017); Faes and Moens (2017a,b).

Following this method, the experimental data $\mathcal{D}$ are represented using their convex hull $\mathscr{C}^e$. Similarly, also the convex hull $\mathscr{C}^m$ of the uncertain realization set $\tilde{z}^m$ is constructed. Specifically, $\tilde{z}^m$ is obtained by propagating the multivariate interval uncertainty, captured in an interval vector $\boldsymbol{\theta}^I$, through $\mathcal{M}$:

$$\tilde{z}^m = \left\{ z_j^m \mid z_j^m = \mathcal{M}(\boldsymbol{\theta}_i); \boldsymbol{\theta}_i \in \boldsymbol{\theta}^I; i = 1,\ldots,q \right\} \qquad (4)$$

Since the computational complexity of computing a convex hull scales $\mathcal{O}(\lfloor v_c^{\frac{d}{2}} \rfloor / \lfloor \frac{d}{2} \rfloor !)$, with v_c the number of vertices of $\mathscr{C}^m$ Barber et al. (1996), both convex hulls are not computed in d dimensions. Therefore, both $\tilde{z}^m$ and $\mathcal{D}$ are projected onto d_r^+-dimensional sub-spaces prior to the computation of the convex hulls. The sub-spaces are defined by a lower-dimensional orthogonal basis $\mathcal{B}_i^+ \subset \mathcal{B}$, $i = 1,\ldots,\binom{d_r}{d_r^+}$, constructed as a subset of $\mathcal{B}$, with $d_r^+ << d_r$ and $\binom{d_r}{d_r^+}$ the binomial coefficient. This orthogonal basis $\mathcal{B}$ from which these sub-spaces are constructed is defined in $\mathbb{R}^{d_r}$, with $d_r << d$:

$$\mathcal{B} = \text{span}\{\phi_{e,d-d_r}, \phi_{e,d-d_r+1}, \ldots \phi_{e,d}\} \qquad (5)$$

with ϕ_e the d_r eigenvectors corresponding to the d_r largest eigenvalues of the covariance matrix Ξ_e of the measurement data set $\mathcal{D}$. Specifically, the i^{th} orthogonal subspace basis $\mathcal{B}_i^+ \subset \mathcal{B}$ is defined as:

$$\mathcal{B}_i^+ = \text{span}\{\phi_{m,\mathcal{I}_i(1)}, \phi_{m\mathcal{I}_i(2)}, \cdots \phi_{m,\mathcal{I}_i(d_r^+)}\} \qquad (6)$$

with $\mathcal{I}_i$ an index set containing the d_r^+ indices for the i^{th}, $i = 1,\ldots,\binom{d_r}{d_r^+}$ subspace of $\mathcal{B}$.

Based on the convex hulls of measurement and simulation data, the multivariate interval uncertainty in $\boldsymbol{\theta}^I$ is obtained by minimizing following objective function:

$$\delta(\boldsymbol{\theta}^I) = \sum_{i=1}^{\binom{d_r}{d_r^+}} \left(\Delta V_{m,i}^2 + w_o \Delta V_{o,i}^2 + \Delta c_i^2\right) \qquad (7)$$

with:

$$\Delta V_{m,i} = 1 - \frac{\mathscr{V}_{m,i}(\theta^I)}{\mathscr{V}_{e,i}} \tag{8a}$$

$$\Delta V_{o,i} = 1 - \frac{\mathscr{V}_{o,i}(\theta^I)}{\mathscr{V}_{e,i}} \tag{8b}$$

$$\Delta c_i = \left\| \mathbf{c}_{e,i} - \mathbf{c}_{m,i}(\theta^I) \right\|_2 \tag{8c}$$

where $\mathscr{V}_{m,i}$ and $\mathscr{V}_{e,i}$ are the d_r^+-dimensional volumes of respectively $\mathscr{C}^m_{\mathscr{B}_i^+}$ and $\mathscr{C}^e_{\mathscr{B}_i^+}$, $\mathscr{V}_{o,i}$ is the volume of the overlap of both convex hulls, and $\mathbf{c}_{e,i}$ and $\mathbf{c}_{m,i}$ are the centers of gravity of respectively $\mathscr{C}^e_{\mathscr{B}_i^+}$ and $\mathscr{C}^m_{\mathscr{B}_i^+}$. Note that for notation simplicity, the subscript $\mathscr{B}_i^+$ is simplified to i.

4. CASE STUDY: THE DLR AIRMOD

4.1. DLR AIRMOD model and dataset

The DLR AIRMOD structure, as shown in figure 1, is a scaled replica of the GARTEUR SM-AG19 benchmark airplane model Govers et al. (2014). The finite element model of this structure is explained in detail in Govers et al. (2014). A set of 18 parameters including support and joint stiffness values, as well as mass parameters are selected for the identification (see table 1), in correspondence with literature on the subject Govers et al. (2014). The locations of these parameters are indicated in figure 1.

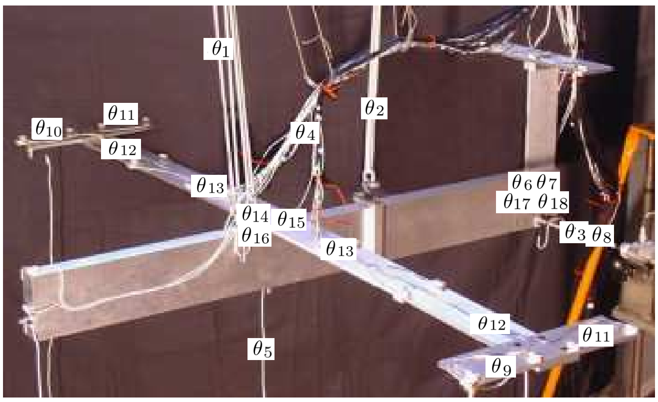

Figure 1: Illustration of the AIRMOD test structure (adapted after Govers et al. (2014)

This model is solved for the first 30 eigenmodes and corresponding -frequencies. Due to the low stiffness of the support, also rigid body modes are present in the model, which are also considered in the quantification procedure. From the set of 30 computed eigenmodes, the $1^{st} - 8^{th}$, $10^{th} - 12^{th}$, 14^{th}, 19^{th} and 20^{th} mode are selected for the identification (see Faes et al. (2019) for more details). These 14 modes are selected to be consistent with literature on the subject. A measurement data set containing 5 measurements of the 30 eigenmodes and -frequencies is applied for the inverse UQ. This is a small subset of the complete measurement data set that is available for the DLR Airmod test structure. For a complete explanation of the experimental campaign that was followed to construct this dataset, the reader is referred to Govers et al. (2014).

4.2. Inverse uncertainty quantification

For the propagation of the interval uncertainty θ^I, reduced transformation method Hanss (2002) is applied since the eigenfrequencies predicted by a linear numerical model are a strict monotonous function of the uncertain model parameters Adhikari (1999). However, the number of necessary function evaluations for the propagation scales exponentially with the number of uncertain model parameters. For the multivariate interval quantification method, it is therefore assumed that the masses at both wing-tips (i.e. θ_9 and θ_{10}) and the stiffness introduced by the cables at the top and the bottom of the structure (θ_4 and θ_5) are completely dependent, reducing the number of uncertain parameters to 16. Hence, the number of necessary function evaluations for a single interval computation reduces from 262144 to 65536. This assumption is not made for the Bayesian method, as the Monte Carlo sampling that underlies the applied TMCMC approach is dimension-independent.

The Bayesian uncertainty quantification was performed using 18 uncorrelated marginal uniform prior distributions. The range of the distribution of each parameter has been selected spanning an interval from 5% to 200% of the parameter nominal value. The parameters are selected as uncorrelated to make the base assumption as objective as possible. The likelihood, as introduced in eq. (3), is constructed under the assumption of independence of the data. The interval uncertainty quantification is performed by solving the optimization problem

Table 1: Parameters that are used in the identification

	Type	Description	Orientation	Deterministic value
θ_1	Stiffness	Support stiffness	y	$1.80 \cdot 10^{03}\ N/m$
θ_2	Stiffness	Support stiffness	y	$7.50 \cdot 10^{03}\ N/m$
θ_3	Stiffness	Cables	y	$1.30 \cdot 10^{02}\ N/m$
θ_4	Stiffness	Cables	y	$7.00 \cdot 10^{01}\ N/m$
θ_5	Stiffness	Cables	y	$7.00 \cdot 10^{01}\ N/m$
θ_6	Stiffness	Joint stiffness	x, y	$1.00 \cdot 10^{07}\ N/m$
θ_7	Stiffness	Joint stiffness	z	$1.00 \cdot 10^{09}\ N/m$
θ_8	Mass	Cables	/	$2.00 \cdot 10^{-01}\ kg$
θ_9	Mass	Screws	/	$1.86 \cdot 10^{-01}\ kg$
θ_{10}	Mass	Screws	/	$1.86 \cdot 10^{-01}\ kg$
θ_{11}	Mass	Cables	/	$1.50 \cdot 10^{-02}\ kg$
θ_{12}	Mass	Cables	/	$1.50 \cdot 10^{-02}\ kg$
θ_{13}	Mass	Cables	/	$1.50 \cdot 10^{-02}\ kg$
θ_{14}	Stiffness	Joint stiffness	x	$2.00 \cdot 10^{07}\ N/m$
θ_{15}	Stiffness	Joint stiffness	y	$2.00 \cdot 10^{07}\ N/m$
θ_{16}	Stiffness	Joint stiffness	z	$7.00 \cdot 10^{06}\ N/m$
θ_{17}	Stiffness	Joint stiffness	x	$5.00 \cdot 10^{07}\ N/m$
θ_{18}	Stiffness	Joint stiffness	y	$1.00 \cdot 10^{07}\ N/m$

introduced in equation (7) via Particle Swarm Optimization (PSO). A swarm size of 100 particles is used, and the optimization is considered to be converged when it reached 15 stalling iterations. These settings are found in a heuristic approach and based on prior experience with PSO. The datasets are, prior to solving eq. (7), projected on 2-dimensional sub-bases $\mathscr{B}_i^+$ of a 13-dimensional orthogonal basis $\mathscr{B}$. For a more in-depth discussion concerning the computational aspects and exact implementation procedure, the reader is referred to Faes et al. (2019).

Figure 2 shows all combinations of the eigenfrequencies corresponding to the $1^{st} - 8^{th}$, $10^{th} - 12^{th}$, 14^{th}, 19^{th} and 20^{th} mode. These eigenfrequencies are obtained by propagating the intervals and posterior distributions, quantified using respectively the interval method and Bayesian model updating, through the FE model of the DLR Airmod structure.

Concerning the marginal eigenfrequencies, the bounds predicted by the quantified interval method circumscribe the measurement data set tighter as compared to the Bayesian samples, which are shown to be more over-conservative. This is e.g., the case for f_1, f_2, f_3, f_4, f_5, f_{11}, f_{14} and f_{19}. The only exception hereto is the 8^{th} eigenmode (i.e. the 1^{st} symmetric wing torsion mode), which is completely missed by the quantified interval method. Keeping in mind that the 7^{th}, 8^{th} and 12^{th} eigenmode correspond to respectively anti-symmetric torsion, symmetric torsion and wing fore-after bending modes, the inaccurate prediction of their dependence in the interval model is a direct cause of the assumption that θ_4 - θ_5 and θ_9 - θ_{10} are fully dependent. The main reason for this lack of overlap is explained by the assumed perfect dependence between parameters θ_4 - θ_5 and θ_9 - θ_{10}, combined with the low data availability.

Due to the scarcity of the data-set, none of the methods was able to give an accurate estimate on the actual dependence between the eigenfrequencies. As such, apart from the 8^{th} eigenmode, it is shown that the interval method outperforms the Bayesian model updating under the availability of these scarce data. Note that for the full data set, the Bayesian model updating was proven more accurate Broggi et al. (2018). This is a direct effect from the wide prior distribution that is applied for

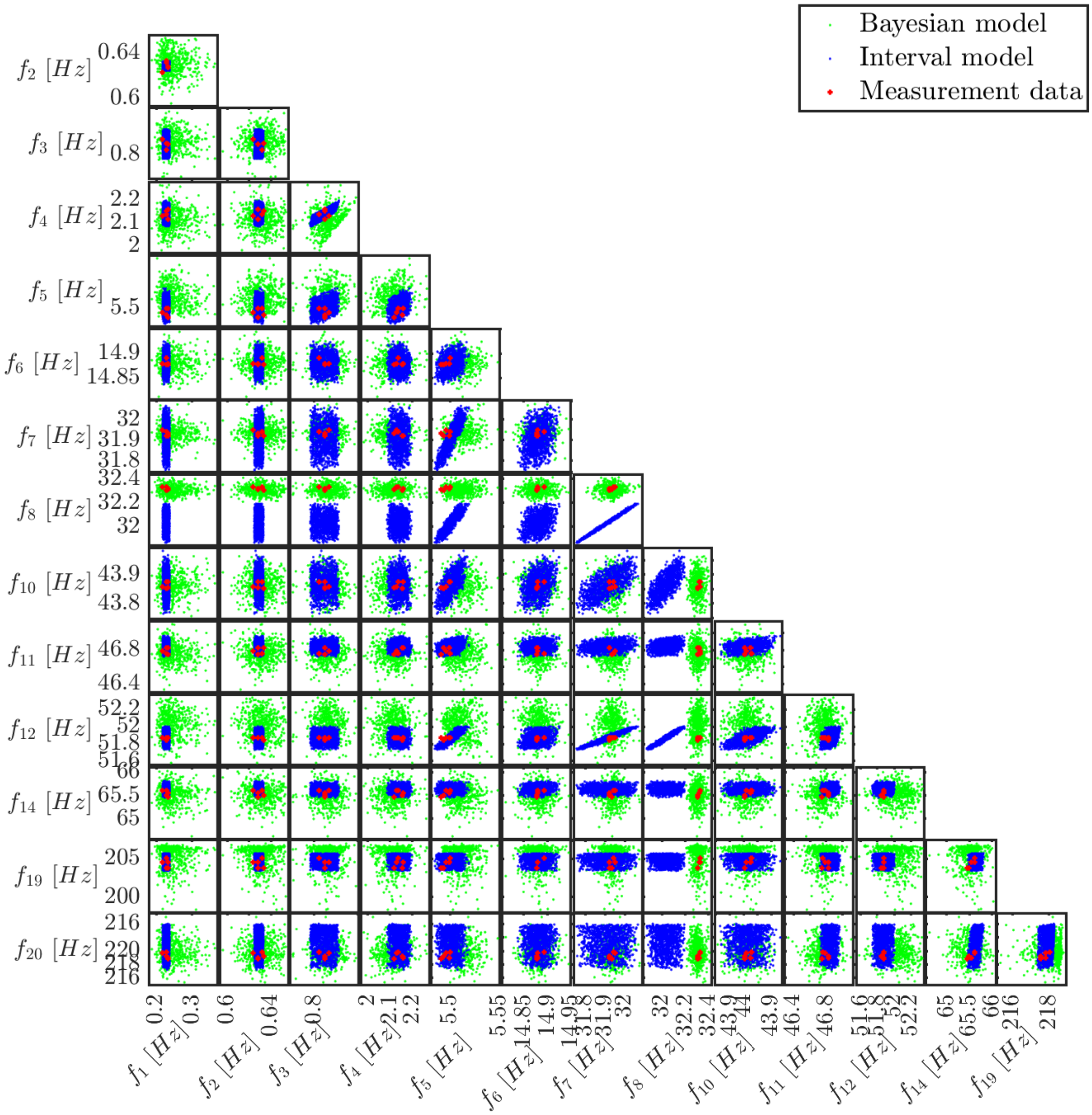

Figure 2: All combinations of considered eigenfrequencies, obtained by propagating the quantified intervals and posterior distributions through the AIRMOD FE model. The quantified results were obtained by using only 5 measured replica Faes et al. (2019).

the Bayesian model updating. In case there are insufficient or insufficiently informative experimental data, the results obtained via Bayesian methods indeed tend to be highly dependent on the defined prior information. Hence, when this prior is highly biased with respect to the actual parameter values, the obtained posterior distribution will show similar bias. The interval method on the other hand does not need an initial estimate of the parameter uncertainty, since the global optimization routine actively searches the space of input parameters for those intervals that best prescribe the available data.

4.3. *Reflection on the results and lessons learnt*

The most important observation that is made in this work is that the performance of the considered inverse UQ approaches depends largely on the data. This can be explained from a philosophical standpoint. Interval UQ methods approach the problem from the *outside*, as they bound possible values of the uncertain parameters between crisp bounds. This quantification is performed without making inference on the likelihood of each value within that interval. Consequently, only a worst-case inference is attainable based on the obtained information, but this inference is objective. Bayesian methods approach the uncertainty from the *inside*, as they assign a degree of plausibility to each possible value of the uncertain parameters within a range and employ independent data to infer the most plausible parameter values based on Bayes' theorem. Hence, more information on the uncertain parameters is obtained, however at the price that this might be subjective. As such, in case large data sets are available or the analyst has need for quantifying the relative likelihood of several parameter values being realized, including their (joint)-plausibility, correlation and multi-modal descriptors, Bayesian methods have the upper hand over interval approaches. However, the analyst should ensure that sufficient informative data are available to limit the effect of subjectivity or incorrectness of the prior distribution. On the other hand, when data are vague or scarce, interval methods are expected to provide a more objective and accurate quantification of the uncertainty, as less a priori assumptions on the underlying likelihood structure are needed. This however is achieved at the cost that only worst-case information is delivered to the analyst. As such, the selection of the most appropriate method must rely on the data that are available to the analyst and the information on the non-deterministic nature of the model quantities under consideration.

5. CONCLUSIONS

This paper presents a comparison of Bayesian model updating via Transitional Markov Chain Monte Carlo with a recently introduced inverse method for the quantification of multivariate interval uncertainty under scarce data. Hereto, both techniques are applied to the DLR AIRMOD case study using a small set of 5 measured eigenmodes and -frequencies. It is shown that the interval method provides, under these scarce data, tighter bounds on the uncertainty. The explanation for comparable large conservatism in the Bayesian estimates is the comparably large uncertainty in the defined prior distributions on the AIRMOD parameters combined with the lack of sufficient independent data to counteract this effect. The interval method on the other hand doesn't require initial estimates on the uncertain parameters and is hence more objective. This however comes at the cost that no statistical properties on the model parameters can be quantified.

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