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A multiobjective optimization based approach for RBDO

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ABSTRACT: In this paper we present a novel algorithm in order to solve multiobjective design optimization problems of a sandwich plate when the objective functions are not smooth and when uncertainty is introduced into the material properties. The algorithm is based on the existence of a common descent vector for each sample of the random objective functions and on an extension of the stochastic gradient algorithm. It will be shown that a chance constraint optimization problem such as a RBDO problem can be written as a multiobjective optimization problem. Chance constraint optimization problems yields optimal designs for a fixed given level of probability for the constraint. However in real life problem it is not realistic to introduce a given probability because it is not known. It is more efficient to solve the problem for a whole range of probability in order to obtain an overview of the probability level appearing in the constraint effect on the solution. We show in this paper how to transform a chance constraint optimization problem into a multiobjective optimization problem and we give an illustration on simple examples.

Manufacturers are ever looking for designing products with better performance, higher reliability at lower cost and risk. One way to address these antagonistic objectives is to use multiobjective optimization approaches. But real world problems are rarely described through a collection of fixed parameters and uncertainty has to be taken into account, may it appear in the system description itself or in the environment and operational conditions. Indeed the system behavior can be very sensitive to modifications in some parameters Papadimitriou et al. (1997); Matthies et al. (1997); Arnaud and Poirion (2014). This is why uncertainty has to be introduced in the design process from the start. Optimization under uncertainty has known important advances since the second-half of the 20th century Dantzig (1955); Bellman and Zadeh (1970) and various approaches have been proposed including robust optimization, where only the bounds of the uncertain parameters are used, and stochastic optimization where uncertain parameters are modeled through random variables with a given distribution and where the probabilistic information is directly introduced in the numerical approaches. In that context the uncertain multi objective problems is written in terms of the expectation of each objective. In our paper we shall focus on this last interpretation of the optimization problem.Considering single objective stochastic optimization problems, a large variety of numerical approaches Sahinidis (2004); Roy et al. (2008) can be found in the literature. Two main distinct approaches exist, one based on stochastic approximations such as the Robbins Monro algorithm and the various stochastic gradient approaches Robbins and Monro (1951); Ermoliev (1983); Ermoliev and Wets (1988), the second one based on scenario approaches Shapiro (2003); Nemirovski and Shapiro (2006), the latter being more frequently applied for chance constrained problems. Again two directions can be found, a robust approach and a scenario based approach used to calculate an estimate of the mean objective function Fliege and Xu (2011); Bonnel and Collonge (2014); Mattson and Messac (2005). RBDO and more generally chance constraint problems are numerically difficult to solve and moreover their solution is obtained for a single given of chance level. An interesting situation would be to construct the solutions for a whole range of probability levels. Writing the probabilistic constraint as the expectation of particular random function, we show that if the original chance constraint problem is replaced by a stochastic multiobjective optimization problems, the Pareto solution set of the new problem contains the solutions for all levels of probability.

1. OPTIMIZATION OF UNCERTAIN OBJECTIVES

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be an abstract probabilistic space, and $W : \Omega \to \mathbb{R}^d$ a random vector. We denote μ the distribution of the random variable W and $\mathscr{W}$ its image space $W(\Omega)$. Let $W_1, ..., W_k, ...$ independent copies of the random variable W which will be used to generate independent random samples with distribution μ. Consider m convex functions $f_i : \mathbb{R}^n \times \mathscr{W} \to \mathbb{R}$, $i = 1, ...m$ depending on uncertain parameters modeled trough random vector $W(\omega)$. In this paper we shall consider the following optimization problem :

$$\min_{x \in \mathbb{R}^n} \left\{ \mathbb{E}[f_1(x, W(\omega))], ..., \mathbb{E}[f_m(x, W(\omega))] \right\}. \quad (1)$$

More precisely we want to construct the associated Pareto set: multiobjective optimization is based on the notion of *Pareto optimal* and *weak Pareto optimal* solutions. Consider m convex functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ...m$ and the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ f_1(x), ..., f_m(x) \right\}. \quad (2)$$

A solution x^* of problem (2) is Pareto optimal if no point x such that $f_i(x) \leq f_i(x^*)$ $\forall i = 1, ..., m$ and $f_j(x) < f_j(x^*)$ for an index $j \in \{1, ..., m\}$ exists. It is weakly Pareto optimal if no point x such that $f_i(x) < f_i(x^*)$ $\forall i = 1, ..., m$ exists. A complete review on multiobjective optimization can be found in Miettinen (1998). Before going on with the algorithm description that will be used to solve the previous problem we shall recall definitions of some notions appearing in the context of nonsmooth analysis and multiobjective optimization. Throughout the paper the standard inner product on $\mathbb{R}^n$ will be used and denoted $\langle \cdot, \cdot \rangle$, the norm being denoted $\| \cdot \|$.

2. COMMON DESCENT DIRECTION

The algorithm presented in the next section is based on the existence and construction of a descent direction. We first recall its definition.

Definition 1 *A vector d is called a descent direction if $\exists t_0 > 0$ such that $f(x + td) < f(x)$ for all $t \in [0, t_0]$.*

For smooth functions it is well known that the opposite direction of the gradient is a descent vector. In the nonsmooth convex or nonconvex context not all elements of the subdifferential is a descent vector. There exist several techniques to construct such a descent vector: proximal bundle methods Kiwiel (1985); Wilppu et al. (2014); Mäkelä et al. (2016), quasisecant methods Bagirov et al. (2013), or gradient sampling methods Burke et al. (2002, 2005). Considering now m functions $f_1, ..., f_m$ we show that there exists a vector d which is a descent direction for each function. Its construction is based on properties of the following convex set $\mathscr{C}$:

Lemma 1 *Let C be the convex hull of either*

1. *the gradients $\nabla f_i(x)$ of the objective functions when they are differentiable,*
2. *or the union of the subdifferentials $\partial f_i(x)$, $i = 1, ..., m$ when they are nondifferentiable but convex or*
3. *the union of the Clarke's subdifferentials $\partial f_i(x)$, $i = 1, ..., m$ if they are nonconvex.*

Then there exists a unique vector $p^* = \text{Argmin}_{p \in C} ||p||$ such that

$$\forall p \in C \; : \; p^T p^* \geq p^{*T} p^* = ||p^*||^2.$$

The existence of the common direction d and its construction is given by the next theorem:

Theorem 1 *Let C be the convex set defined in Lemma 1 and p^* its minimum norm element. Then either we have*

1. *$p^* = 0$ and the point x is Pareto stationary or*
2. *$p^* \neq 0$ and the vector $-p^*$ is a common descent direction for every objective function.*

We have now the sufficient materials to present the SMSGDA (Stochastic Multi Descent Algorithm) algorithm.

3. THE SMGDA ALGORITHM

As written problem (1) is a deterministic problem but the objective function expectations are seldom known. A classical approach, the sample average approximation (SAA) method, is to replace each expectancy by an estimator built using independent samples w_k of the random variable W, Bonnel and Collonge (2014); Fliege and Xu (2011). The algorithm we propose does not need to calculate the objective function expectancy and is based only on the construction of a common descent vector. Let ω be given in Ω and consider the deterministic multiobjective optimization problem:

$$\min_{x \in \mathbb{R}^n} \{f_1(x, W(\omega)), f_2(x, W(\omega)), ..., f_m(x, W(\omega))]\} \tag{3}$$

Following theorem 1 there exists a descent vector common to each objective function $f_k(x, W(\omega)), k = 1, ..., m$ at point x.

The common descent vector depends on x and ω and therefore will be considered as a random vector denoted $d(\omega)$ defined on the probability space $(\Omega, \mathscr{A}, \mathbb{P})$.

3.1. The algorithm

We give now the successive steps of the algorithm that we propose.

1. Choose an initial point x_0 in the design space, a number N of iterations and a σ-sequence t_k : $\sum t_k = \infty$; $\sum t_k^2 < \infty$,
2. at each step k draw a sample w_k of the random variable $W_k(\omega)$,
3. construct the common descent vector $d(w_k)$ using theorem 1 and the gradient sampling approximation method,
4. update the current point : $x_k = x_{k-1} + t_k d(w_k)$.

The last step of the algorithm defines a sequence of random variables on the probability space $(\Omega, \mathscr{A}, \mathbb{P})$ through the relation

$$X_k(\omega) = X_{k-1}(\omega) - t_k d(X_{k-1}(\omega), W_k(\omega)) \tag{4}$$

Theorem 2 (Mercier et al. (2018)) *Under a set of assumptions,*

1. *the sequence of random variables $X_k(\omega)$ defined by relation (4) converges in mean square towards a point X^* of the Pareto set:*
$$\lim_{k \to +\infty} \mathbb{E}[\|X_k(\omega) - X^\star\|^2] = 0.$$
2. *The sequence converges almost surely towards X^*.*
$$\mathbb{P}\left(\left\{\omega \in \Omega,\ \lim_{k \to \infty} X_k(\omega) = X^\star\right\}\right) = 1.$$

4. SOLVING RELIABILITY PROBLEM USING A STOCHASTIC MULTIOBJECTIVE FORMULATION

Introducing probabilistic constraints is a rather natural way to take into account the notion of risk in an optimization process. Let us consider the following chance constraint problem:

$$\underset{x \in X^{\text{ad}}}{\operatorname{argmin}} \{\mathbb{E}[f(x, \xi(\omega))] \,|\, \mathbb{P}[g(x, \xi(\omega)) \geq 0] \geq p_0\}. \tag{5}$$

Here, X^{ad} is a feasible closed convex set of the set of control variables X, $g : X \times \mathbb{R} \to \mathbb{R}$ represents a physical or structural quantity. In this formulation failure occurs when $g(x, \xi(\omega))$ becomes positive, and p_0 denotes the level of risk one is ready to accept. Such a problem is rather difficult to solve. The reason is twofold: first it is very difficult to check whether a given chance constraint is satisfied at a given point x or not. Typically Monte-Carlo simulation is the only way to estimate the probability of violating the constraint, but becomes too costly when

p_0 approaches unity. The second reason comes from the fact that the feasible set of problem (5) can be nonconvex even if the set X^{ad} is convex as well as function g. Several developments can be found in the literature in order to overcome those difficulties: transforming the problem into a combinatorial problem by discretizing the probability distribution Dentcheva et al. (2000), using convex approximation Nemirovski and Shapiro (2006) or sample average approximations Luedtke and Ahmed (2008); Pagnoncelli and Shapiro (2009).

The general formulation of a reliability based design optimization (RBDO) problem is the following

$$\begin{cases} \min_{\mathbf{x}\in\mathbb{R}^n} \{\mathbb{E}[f(\mathbf{x},W(\omega))]\} \\ s.t.\ \mathbb{P}\{g(\mathbf{x},W(\omega)) \geq 0\} \leq \alpha. \end{cases} \tag{6}$$

$\mathbb{P}\{g(\mathbf{x},W(\omega)) \geq 0\}$ represents the probability of failure of the design $\mathbf{x}$, $W(\omega)$ is a random variable modeling the uncertainties and α represents the threshold of failure probability authorized. In most cases, α is a very small positive number. Let us remark that $\mathbb{P}\{g(\mathbf{x},W(\omega)) \geq 0\} = \mathbb{E}[\mathbb{I}_{\mathbb{R}_+}(g(\mathbf{x},W(\omega)))] \stackrel{\text{def}}{=} \mathbb{E}[G(\mathbf{x},W(\omega))]$, where $\mathbb{I}_{\mathbb{R}_+}$ denotes the indicator function of $\mathbb{R}_+$: $\mathbb{I}_{\mathbb{R}_+}(x) = 1$ when x is positive and is equal to 0 otherwise. We replace the RBDO problem (6) by the following stochastic multiobjective optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \{\mathbb{E}[f(\mathbf{x},W(\omega))], \mathbb{E}[G(\mathbf{x},W(\omega))]\}. \tag{7}$$

It is clear from the definition of the Pareto front that for a given value of α, the point $(\mathbf{x}^\star_\alpha, \alpha)$, where $\mathbf{x}^\star_\alpha$ is a solution of the RBDO problem for the given threshold, lies on the Pareto front of the problem (7). Therefore solving problem (7), one obtains directly the set of solutions of the RBDO problem (6) for all values of probability of failure $\alpha \in [0,1]$. However the *SMGDA* algorithm cannot be used directly to solve this last problem since the second objective function is not locally Lipschitz. We use the mollifier introduce in (Andrieu et al., 2011) in order to render this objective smooth. More precisely we introduce a smooth non-negative even function $h_r(x) = \frac{1}{r}h(x/r)$ such that it reaches its maximum for $x = 0$ and

$$\frac{1}{r}\int_{-\infty}^{+\infty} h\left(\frac{x}{r}\right) dx = 1$$

We have then:

$$\mathbb{I}^r_{\mathbb{R}^+}(\mathbf{z}) \approx \frac{1}{r}\int_0^{+\infty} h(\frac{\mathbf{z}-\mathbf{y}}{r}) d\mathbf{y},$$

which yields the following expression :

$$\mathbb{I}^r_{\mathbb{R}^+}(g(\mathbf{x},W(\omega))) = \frac{1}{r}\int_0^{+\infty} h\left(\frac{g(\mathbf{x},W(\omega))-\mathbf{y}}{r}\right) d\mathbf{y}.$$

This approximation can be differentiate with respect to $\mathbf{x}$ and it can be checked that :

$$\nabla\mathbb{I}^r_{\mathbb{R}^+}(g(\mathbf{x},W(\omega))) = -\frac{1}{r}h\left(\frac{g(\mathbf{x},W(\omega))}{r}\right)\nabla g(\mathbf{x},W(\omega)).$$

5. ILLUSTRATION

The above development is applied to a simple example used as a reference test cas in several publications Moustapha et al. (2016). The problem is to minimize the cross-sectional area $b \times h$ of a rectangular column submitted to a compression load F while avoiding buckling, which occurs when the compression load is higher than the critical Euler force. Failure will occur when:

$$F - \frac{\pi^2 Ebh^3}{12L^2} \geq 0, \tag{8}$$

where E is the Young's modulus of the column material and L its length. In their paper the authors considered the parameters E and L uncertain and modeled as lognormally distributed random variables. The critical Euler force is also considered uncertain through the appearance of a multiplicative lognormally distributed random variable k in its expression. The RBDO problem is then written

$$\text{Argmin}_{h,b}\, b \times h \tag{9}$$

under the constraints

$$h - b \leq 0 \text{ and } \mathbb{P}\left[F - \frac{k(\omega)\pi^2 E(\omega)bh^3}{12L(\omega)^2} \geq 0\right] \leq \alpha \tag{10}$$

The chosen parameters for the distributions are recalled in table 1. The value of the compression force is chosen as $F = 1.4622 \times 10^6$ N. With

Table 1: Probabilistic model for the column problem

parameter	distribution	mean : μ	covariance : δ
k	lognormal	0.6	10
E (MPA)	lognormal	10^4	5
L (mm)	lognormal	3	1

these specific probability distributions it is shown in Moustapha et al. (2016) that an analytic solution of the RBDO problem (9) and (10) exists and is given by:

$$b^* = h^* = \left(\frac{12F}{\pi^2 \exp(\lambda_k + \lambda_E - 2\lambda_L + \Phi^{-1}(\alpha) \sqrt{\zeta_k^2 + \zeta_E^2 + 4\zeta_L^2})} \right)^{\frac{1}{4}}, \quad (11)$$

where $\zeta_. = \sqrt{\ln(1+\delta_.^2)}$ and $\lambda_. = \ln(\mu_.) - 1/2\zeta_.^2$, μ and δ being the mean and covariance of the lognormal random variables, and Φ the cumulative distribution function of the standard normal distribution.

We consider now the stochastic multiobjective optimization problem

$$\min_{b,h} \{b \times h, \mathbb{E}[G(b,h,W(\omega))]\} \; ; \; h - b \leq 0, \quad (12)$$

where $W(\omega) = (k(\omega), E(\omega), L(\omega))$ and $G(b,h,W(\omega)) = \tilde{\mathbb{I}}_{\mathbb{R}^+}(F - \frac{k(\omega)\pi^2 E(\omega) bh^3}{12L(\omega)^2})$.

The *SMGDA* algorithm is used in order to construct the Pareto front. At the same time the analytic solutions (h^*, b^*) of the RBDO problem (9) are constructed for a set of values of α from which the analytic Pareto front $(b^* \times h^*, \alpha)$ is obtained. In their paper Moustapha et al. (2016) used an adaptative Kriging surrogate model in order to solve the RBDO problem for the specific values $\alpha = .05$. Figure 1 shows the comparison of the two Pareto fronts, the solution found by Moustapha et al. (2016) being represented by a green triangle. Almost every solution proposed by *SMGDA* stick to the analytic Pareto front, and thus can be consider as good. During the optimization process *SMGDA* does not require the costly calculation of the failure probability and is still able to converge, even for very small values.

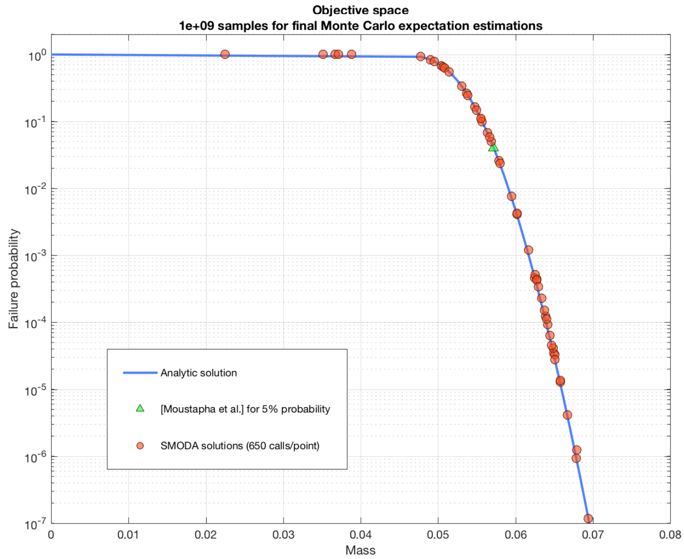

Figure 1: Pareto front of the reliability problem

6. CONCLUSION

Based on a new algorithm we have proposed to replace the study of a RBDO problem by a stochastic multiobjective optimization problem where the probabilistic constraint becomes a new objective. In this way we obtain the solutions of the original RBDO problem for all probability (or failure) levels, allowing to evaluate the impact of this level on the design parameter solutions. The *SMGDA* algorithm used is completely parallelizable and the numerical problem of evaluating the probability constraint is decoupled from the optimization procedure. It becomes a postprocessing procedure where any adequate method can be used.

7. REFERENCES

Andrieu, L., Cohen, G., and Vàzquez-Abad, F. (2011). "Gradient-based simulation optimization under probability constraints." *European Journal of Operational research*, 212, 345–351.

Arnaud, R. and Poirion, F. (2014). "Optimization of an uncertain aeroelastic system using stochastic gradient approaches." *Journal of Aircraft*, 51(3), 1061–1069.

Bagirov, A., Jin, L., Karmitsa, N., Nuimat, A. A., and Sultanova, N. (2013). "Subgradient method for nonconvex nonsmooth optimization." *Journal of Optimization Theory and Applications*, 157, 416–435.

Bellman, R. and Zadeh, L. (1970). "Decision-making in a fuzzy environment." *Management Science*, 17, 141–161.

Bonnel, H. and Collonge, J. (2014). "Stochastic optimization over a Pareto set associated with a stochastic multi-objective optimization problem." *J. Optim. Theory Appl.*, 162, 405–427.

Burke, J. V., Lewis, A. S., and Overton, M. L. (2002). "Approximating subdifferentials by random sampling of gradients." *Mathematics of Operation Research*, 27, 567–584.

Burke, J. V., Lewis, A. S., and Overton, M. L. (2005). "A robust gradient sampling algorithm for nonsmooth, nonconvex optimization." *SIAM J. Optim.*, 15, 751–779.

Dantzig, B. (1955). "Linear programming under uncertainty." *Management Science*, 1, 197–206.

Dentcheva, D., Prékopa, A., and Ruszczynski, A. (2000). "Concavity and efficient points of discrete distributions in probabilistic programming." *Math. Programming Ser.A*, 89, 55–77.

Désidéri, J. (2012). "Multiple-gradient descent algorithm (MGDA) for multiobjective optimization." *CRAS Paris, Ser. I*, 350, 313–318.

Ermoliev, Y. (1983). "Stochastic quasigradient methods and their application to systems optimization.." *Stochastics*, 9, 1–36.

Ermoliev, Y. and Wets, R. (1988). *Numerical Techniques for Stochastic Optimization*. Springer verlag.

Fliege, J. and Xu, H. (2011). "Stochastic multiobjective optimization: Sample average approximation and applications." *Journal of Optimization Theory and Applications*, 151, 135–162.

Kiwiel, K. (1985). *Methods of Descent for Nondifferentiable Optimization*. Number 1133 in Lecture notes in mathematics. Berlin edition.

Luedtke, J. and Ahmed, S. (2008). "A sample approximation approach for optimization with probabilistic constraints." *SIAM Journal on Optimization*, 19, 674–699.

Mäkelä, M., Karmitsa, N., and Wilppu, O. (2016). *Mathematical Modeling and Optimization of Complex Structures*. Springer, Chapter Proximal Bundle Method for Nonsmooth and Nonconvex Multiobjective Optimization, 191–204.

Matthies, H., Brenner, C. E., Bucher, C., and Soares, C. (1997). "Uncertainties in probabilistic numerical analysis of structures and solids-stochastic finite elements." *Structural safety*, 3, 283–336.

Mattson, C. A. and Messac, A. (2005). "Pareto frontier based concept selection under uncertainty, with visualization." *Optimization and Engineering - Special Issue on Multidisciplinary Design Optimization*, Vol. 6, Kluwer Publishers, 85–115.

Mercier, Q., Poirion, F., and Désidéri, J. (2018). "A stochastic multi gradient descent algorithm." *European Journal of Operational Research*, 271,(3), 808–817.

Miettinen, K. (1998). *Nonlinear Multiobjective Optimization*, Vol. 12 of *International Series in Operations Research & Management Science*. Springer US.

Moustapha, M., Sudret, B., Bourinet, J., and Guillaume, B. (2016). "Quantile-based optimization under uncertainties using adaptive Kriging surrogate models." *Structural and Multidisciplinary Optimization*, 54, 1403–1421.

Nemirovski, A. and Shapiro, A. (2006). "Scenario approximations of chance constraints." *Probabilistic and Randomized Methods for Design under Uncertainty*, G. Calafiore and F. Dabbene, eds., Springer.

Pagnoncelli, B.K. and, S. A. and Shapiro, A. (2009). "Sample average approximation method for chance constrained programming: theory and applications." *Journal of Optimization Theory and Applications Vol. 142, No. 2, pp 399-416, July 2009.*, 142(2), 399–416.

Papadimitriou, C., Katafygiotis, L. S., and Au, S.-K. (1997). "Effects of structural uncertainties on tmd design : A reliability-based approach." *Journal of Structural Control*, 4, 65–88.

Robbins, H. and Monro, S. (1951). "A stochastic approximation method.." *Ann. Math. Statistics*, 22, 400–407.

Roy, R., Hinduja, S., and Teti, R. (2008). "Recent advances in engineering design optimisation : challenges and future trends." *manufacturing technology*, 57, 697–715.

Sahinidis, N. V. (2004). "Optimization under uncertainty: State-of-the-art and opportunities." *Computers & Chemical Engineering*, 28(6-7), 971–983.

Shapiro, A. (2003). *Handbooks in OR & MS, Vol. 10*. Elsevier Science B.V., Chapter 6. Monte Carlo Sampling Methods, 353–425.

Wilppu, O., Karmitsa, N., and Mäkelä, M. (2014). "New multiple subgradient descent bundle method for nonsmooth multiobjective optimization." *Report no.*, Turku Centre for Computer Science.