이학박사 학위논문

# Gradient potential theory for nonlinear elliptic problems

## (비선형 편미분 방정식에 대한 그래디언트 퍼텐셜 이론)

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서울대학교 대학원

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# Gradient potential theory for nonlinear elliptic problems

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## Abstract

# Gradient potential theory for nonlinear elliptic problems

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The objective of this thesis is to provide a sharp gradient potential estimate for nonlinear elliptic problems under non-standard growth assumptions. The estimates have been found from the attempts to develop a unified method for the purpose of obtaining sharp pointwise bounds of the gradient of solutions.

First, we obtain gradient potential estimates, by using linearization techniques along with an exit time argument, for two non-autonomous elliptic measure data problems with superquadratic growth. One is variable exponent case and the other is mild phase transition case. In gradient potential theory for measure data problems, a unified method is still unknown, that covers both superquadratic and subquadratic cases, because of the difficulty stemming from the absence of energy solutions to such problems.

However, once we take energy solutions into account, we devise a new unified method to deal with superquadratic and subquadratic cases simultaneously. In particular, we show partial regularity of the gradient of solutions to subquadratic elliptic systems without the quasi-diagonal structure via Riesz potentials, when the given data belong to suitable Lebesgue spaces to ensure the existence of weak solutions.

In the process of a further research on developing a unified method for measure data problems, we establish global Calderón-Zygmund estimates for such problems with general growth via fractional maximal functions.

**Key words:** Measure data, Potential theory, Non-standard growth, Linearization technique, Harmonic approximation
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# Chapter 1

# Introduction

This thesis is devoted to gradient potential theory for non-autonomous elliptic equations with measure data and elliptic systems without the quasi-diagonal structure. It also aims at presenting global regularity results for measure data problems with general growth by using the fractional maximal function of order 1.

Gradient potential theory is a newborn area in the regularity theory for partial differential equations, and it has attracted much attention because of its difficulties and applications.

For examples, $C^1$-regularity criteria and gradient Hölder continuity can be described via potentials, and Calderón-Zygmund type results can be derived by applying embedding properties of the potentials to the gradient potential estimates.

One of the difficulties in gradient potential theory stems from a couple of facts that weak solutions are not suitable for measure data problems and that the problems lose their certain monotonicity property when both subquadratic and superquadratic growth are considered simultaneously, and so no applicable method has yet been developed to cover both cases simultaneously.

However, when the given data under consideration belong to suitable Lebesgue spaces to ensure the existence of weak solutions, we present a unified method in Chapter 4. Later on, we show Calderón-Zygmund type estimates for problems with general growth, which we discover in the process of trying to develop such a unified method for measure data problems.

## 1.1 Measure data problems with polynomial growth

Let us consider the following $p$-Laplace equation with measure data:

$$-\operatorname{div}(a(x)|Du|^{p-2}Du) = \mu \quad \text{in } \Omega \tag{1.1}$$

for $p \in (1,\infty)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$, $\mu$ is a Radon measure with finite total mass and $a : \Omega \to \mathbb{R}^+$ is a measurable function satisfying $0 < \nu \leq a(x) \leq L$ for some constants $\nu$ and $L$. If $p > n$ or $\mu \in L^{[p^*]'}(\Omega)$ for $p < n$ with $[p^*]' = \frac{np}{np-n+p}$, then there exists a unique weak solution to such an equation by the monotone operator theory, see [99]. On the other hand, the notion of weak solution is not appropriate if $p \leq n$ and $\mu$ merely belongs to $\mathcal{M}_b(\Omega)$ which is the space of bounded Radon measures in $\Omega$. For this reason, several concepts of solutions have been suggested to measure data problems, see for instance [18,19,37]. In [18], the authors considered $p$-Laplacian type equations for $p \in (2-1/n,\infty)$ and introduced a class of distributional solutions called Solutions Obtained by Limits of Approximations (SOLAs for short) which we are taking into account in Chapter 3 and Chapter 5.

There are several research areas regarding the regularity theory for (1.1). See for instance [8, 13, 88] for fractional differentiability, [53, 70, 71, 81, 90] for potential estimates and [26,94] for Calderón-Zygmund estimates. In [88], differentiability estimates for SOLAs to (1.1) were obtained in fractional Sobolev spaces when $\mu$ is a merely bounded Radon measure and $a(\cdot)$ is Lipschitz continuous. Moreover, if $\mu$ satisfies some density conditions, then there hold Morrey type regularity and BMO regularity for the gradient of SOLAs. In the recent paper [8], similar results were obtained in a completely linearized form by combining the difference quotient method and the technique used in [76], when $a(\cdot)$ is a positive constant.

This so-called linearization technique has an important role also in the gradient potential estimates. It is worth mentioning that pointwise estimates for SOLAs to (1.1) were first suggested in [70,71] by means of Wolff potentials and then developed in [9,20,85,100,101]. Wolff potentials are nonlinear potentials, and it is well known that Wolff potential estimates for SOLAs are optimal in the sense that there are no pointwise estimates via any other potentials that are sharper than Wolff potentials. Later in [90], potential estimates were upgraded to the gradient level for nonlinear equations with quadratic growth ($p = 2$). Surprisingly enough, even for the nonlinear equation (1.1) with Dini-continuous coefficient $a(\cdot)$ and $p \in (2-1/n, \infty)$, in light of the linearization technique, the gradient of SOLAs can be estimated by Riesz potentials that are originally designed for linear equations, see [53,76,77] and cf. [54]. For superquadratic growth case ($p \geq 2$), Riesz potential estimates given in [76, 77] are sharper than Wolff potential estimates given in [54] in the following sense:

$$\begin{aligned} I_1^{\mu}(x,R) &= \int_0^R \frac{|\mu|(B(x,\rho))}{\rho^{n-1}} \frac{d\rho}{\rho} \\ &\leq c(p) \left[ \int_0^{2R} \left( \frac{|\mu|(B(x,\rho))}{\rho^{n-1}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} \right]^{p-1} \\ &= c(p) \left[ W^{\mu}_{1/p,p}(x,2R) \right]^{p-1}, \end{aligned} \tag{1.2}$$

where $I_1^{\mu}$ is the (truncated) Riesz potential and $W^{\mu}_{1/p,p}$ is the nonlinear Wolff potential. Note that Dini-continuity on $a(\cdot)$ is known as the optimal assumption to obtain $C^1$-regularity of solutions, see (2.10) for Dini-continuity and see [52, 76, 78] for $C^1$-regularity results, respectively.

We would like to mention [77, Section 9] in which several regularity results were achieved by the gradient potential estimates. A local Calderón-Zygmund type result also can be obtained by using embedding properties of the Riesz potentials given in [6, 64]. As a matter of fact, even the global Calderón-Zygmund estimates still hold under weaker regularity assumptions on $a(\cdot)$ and $\Omega$ than those for potential estimates, see [26, 94]. We note that such regularity results via fractional maximal functions of $\mu$ were originally suggested in [89]. Moreover, their main results are written in terms of the fractional maximal function of order 1, $M_1(\mu)$, which satisfies

$$M_1(\mu)(x) = \sup_{r>0} r \frac{|\mu|(B_r(x))}{|B_r(x)|} \leq c(n) \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = c(n) I_1(\mu)(x), \tag{1.3}$$

where $I_1(\mu)$ is the Riesz potential of $\mu$. In this regard, the fractional maximal function estimates would be sharper than potential estimates.

Our main interests are to find optimal regularity assumptions on the structures and to develop a unified method to establish potential theory and the global Calderón-Zygmund theory to nonlinear elliptic problems.

## 1.2 Gradient potential theory for non-standard growth problems

We now consider the following model equation

$$-\operatorname{div}\left(\frac{g(x,|Du|)}{|Du|}\,Du\right)=\mu \quad \text{in } \Omega, \tag{1.4}$$

where $g(x,t)=\partial_t G(x,t)$ with a generalized $N$-function $G:\Omega\times\mathbb{R}^+\to\mathbb{R}^+$, see Section 2.2 for the generalized $N$-function. There are typical examples of $G(\cdot)$ which will be investigated later in the next chapters:

- Polynomial case:

$$G(x,t)=G_1(t)=(t^2+s^2)^{\frac{p-2}{2}}t^2 \quad \text{for } p\in(1,\infty), s\in[0,1]. \tag{P}$$

- Variable exponent case:

$$G(x,t)=G_2(x,t)=(t^2+s^2)^{\frac{p(x)-2}{2}}t^2, \tag{PX}$$

  where $1<\inf_{x\in\Omega}p(x)\le p(x)\le\sup_{x\in\Omega}p(x)<\infty$.

- Mild phase transition case:

$$G(x,t)=G_3(x,t)=t^p+a(x)\log(e+t)t^p \tag{PT}$$

  for $p\in(1,\infty)$ and $a\in C(\Omega;\mathbb{R}^+)$.

- Double phase case:

$$G(x,t)=G_4(x,t)=t^p+a(x)t^q \tag{DPT}$$

  for $1<p\le q<\infty$ and $a\in C^\alpha(\Omega;\mathbb{R}^+)$ for some $\alpha\in(0,1]$.

- $N$-function case: $G$ does not depends on the first variable. In other words, we can denote as follows: for any $x\in\Omega$

$$G(x,t)=G_5(t). \tag{O}$$

As we mentioned earlier, (1.4) with (P) is well known as $p$-Laplacian type equations with measure data. The case (O) is called general growth and is a natural generalization of (P). Later in this chapter, we will revisit (P) to discuss elliptic systems and (O) to study both subquadratic and superquadratic growth at the same time.

We now move on to non-autonomous cases (PX) and (PT), which has been studied extensively in the past 20 years. Non-autonomous equations were firstly investigated in [74,103–105] by Russian mathematicians, notably by Zhikov. These problems are of importance in that they naturally appear from the modeling of electrorheological fluids in [97,98] and image restoration in [1, 31]. To simplify our discussion, we always assume that $p(\cdot), a(\cdot)$ in (PX), (PT) are log-Hölder continuous. Here, we say that $p(\cdot)$ is log-Hölder continuous if there holds

$$\sup_{0<\rho\leq r} \omega(\rho) \log\left(\frac{1}{\rho}\right) < \infty \tag{1.5}$$

for some $r > 0$, where $\omega(\cdot)$ is a modulus of continuity of $p(\cdot)$ that means $|p(x) - p(y)| \leq \omega(|x-y|)$ for any $x, y \in \mathbb{R}^n$. Note that the log-Hölder continuity assumption ensures that $W^{1,\infty}(\Omega)$ is dense in Musielak-Orlicz-Sobolev spaces $W^{1,G}(\Omega)$ for each (PX) and (PT), see [3,105] and [45, Chapter 9].

In Chapter 3, we present gradient potential theory for (1.4) for (PT), which is announced in [30], and we also present a similar result for (PX), which is announced in [29]. In the same manner as in the case of (1.1), existence of weak solutions is not guaranteed for measure data problems in general. Hence, we consider SOLAs. In particular, we say that a distribution solution $u \in W^{1,1}(\Omega)$ to (1.4) is SOLA if there exists a sequence of weak solutions $\{u_k\}_{k\geq 1} \subset W^{1,G}(\Omega)$ to (1.4) with $\mu = \mu_k \in L^\infty(\Omega)$ and $u_k$ converges to $u$ in $W^{1,1}(\Omega)$. For (PX), Bögelein and Habermann [21] studied the existence of SOLAs for $p(\cdot)$ with $2 - 1/n < p(\cdot) < \infty$. On other hand, we refer to [30] for the proof of the existence of SOLAs for (PT) with $2 - 1/n < p < \infty$.

Gradient potential estimates for the case (PX) were first investigated by Bögelein and Habermann in [21] using non-standard Wolff potential for $2 \leq p(\cdot) < \infty$, and similar estimates were obtained by Baroni and Habermann to the whole range $2 - 1/n < p(\cdot) < \infty$ via a mixed potential in [14]. In

Chapter 3, for $2 \le p(\cdot) < \infty$, we show

$$g\left(x, |Du(x)|\right) \le cI_1^{\mu}(x,R) + cg\left(x, \fint_{B_R(x)} |Du|dy\right), \tag{1.6}$$

whenever $B_R(x) \Subset \Omega$ and $R > 0$ is sufficiently small. In light of (1.2), the estimate (1.6) covers the results given in [21]. We further show that (1.6) holds for (PT) with $2 \le p < \infty$. We refer to [29] and [30] for more details.

We would like to briefly mention that the associated ellipticity and growth conditions to (DPT) has significant changes depending on the point, and therefore, a stronger assumption than the one in (PT) should be made on the modulating coefficient $a(\cdot)$ in (DPT). For this reason, regularity results for double phase problems have been obtained only very recently in [11, 38–40].

We close this section by mentioning that in a small region, the energy functional for (PX) is very close to the one for (PT), and so the same regularity results hold under the same regularity assumptions on $p(\cdot)$ and $a(\cdot)$, respectively. For examples, whenever $\mu = 0$, if $p(\cdot)$ and $a(\cdot)$ are log-Hölder continuous, then a weak solution $u$ to (1.4) is Hölder continuous, and if $p(\cdot)$ and $a(\cdot)$ are Hölder continuous, then $Du$ is Hölder continuous, see [12]. Nevertheless, from the point of view of a perturbation argument, they should be treated differently because the associated reference problems are different each other.

## 1.3 Partial regularity via Riesz potentials

As previously stated, an increasing amount of attention has been given to gradient potential theory. In particular, the theory extended in the several directions. For instances, we refer to [76] for parabolic equations, [25, 72] for general nonlinearities, [10, 14, 29, 30] for non-standard growth problems, and [22, 52, 80, 81] for elliptic systems.

Let us focus on $p$-Laplacian type systems with the quasi-diagonal structure, in which one can obtain full regularity results for the systems, see for instance [102]. Gradient potential estimates for $p$-Laplace systems were first studied by Duzaar and Mingione in [52] when $p > n$. Later on, partial regularity for nonlinear elliptic systems without the quasi-diagonal structure was obtained, as a consequence of nonlinear potential estimates in [80], under the assumption that the associated data belong to suitable Lebesgue spaces to ensure the existence of weak solutions. On the other hand, the full regularity for the $p$-Laplace systems with measure data problems were achieved in [81]. In [80,81], superquadratic growth is considered, and harmonic approximation lemmas played an important role in their proofs. Meanwhile, sharp maximal function estimates were obtained in [22] for the nonlinear elliptic systems with data in divergence form.

In Chapter 4, we show gradient potential estimates for $p$-Laplacian type systems with subquadratic growth by using $\varepsilon$-regularity criteria, almost everywhere in $\Omega$. We assume that our data belong to suitable Lebesgue spaces. Note that if elliptic systems without the quasi-diagonal structure have data which do not belong to the dual space of the energy space, then no existence results are known so far. In addition, even for homogeneous elliptic systems without the quasi-diagonal structure, Hölder regularity of solutions holds almost everywhere in $\Omega$. In what follows, we call the systems without the quasi-diagonal structure to be general systems.

As we do not assume the quasi-diagonal structure, in which one can obtain full regularity results for the systems given in [102], there hold only partial regularity results for general systems, except for subtle higher integrability. Note that De Giorgi found discontinuous solutions to general systems in [62].

To deal with such general systems, we assume that our systems are asymptotically close to $p$-Laplace systems at the origin. Our approach is mainly based on harmonic approximation lemmas, which allow us to use perturbation arguments. In the process, we use shifted $N$-function techniques and higher integrability in order to apply harmonic approximation lemmas in a concise form. The main difficulty in our proof arises from the interaction between subquadratic growth and the associated data in the non-divergence form. Note that for the problems with subquadratic growth, in [53, Lemma 4.1 and Lemma 4.2], the local average of the gradient of solutions on the right-hand side of the comparison estimates can not be removed. Therefore, it does not hold for the problems with subquadratic growth that if the excess functional is small enough for some radius, then the excess functional is small enough for every small enough radii, which is in general true for the problems with superquadratic growth, see [80, Proposition 5.1]. To overcome this difficulty, we consider several alternatives in the proof of Lemma 4.5.1.

The argument that we use in Chapter 4 also can be applied to superquadratic systems. However, this argument only works for weak solutions, that is, any linearization technique covering both subquadratic and superquadratic growth for measure data problems is unknown. In the next section, we study elliptic equations with measure data covering both subquadratic and superquadratic growth.

## 1.4 Elliptic measure data problems with general growth

Turning back to (1.4) with (O), we first assume that $G_5 \in C^2(0,\infty) \cap C^1[0,\infty)$ is an $N$-function satisfying

$$0 < \gamma_1 - 1 \leq \frac{t g_5'(t)}{g_5(t)} \leq \gamma_2 - 1 < \infty \tag{1.7}$$

for some constants $\gamma_1, \gamma_2 > 1$, where $g_5(t) = G_5'(t)$. If (1.7) is satisfied, then $G_5$ and $\widetilde{G}_5$ satisfy $\Delta_2$-condition, see Chapter 2 for more details.

The goal of Chapter 5 is to develop a method to obtain the existence and regularity results for measure data problems with general growth, which is announced in [23]. As previously mentioned, the existence of weak solutions to measure data problems is not guaranteed, in general. Under (1.7), so-called approximable solutions, which are weaker than SOLAs, are introduced in the interesting paper [37], which is a natural extension of [17] to general growth, see Definition 5.2.1.

Note that every SOLAs are approximable solutions and both of the solutions are limits of weak solutions to regular problems. The only difference between SOLAs and approximable solutions is that an approximable solution only requires that the sequence of the gradient of regular solutions converges almost everywhere to the gradient of the approximable solution, while a SOLA requires that the sequence converges to the gradient of the SOLA in $L^1$. In Section 5.2, we show that approximable solutions are indeed SOLAs when $G_5(\cdot)$ satisfies (5.12). In fact, to establish Calderón-Zygmund theory, we take SOLAs into account instead of approximable solutions, because almost everywhere convergence of the sequence is insufficient, as far as we are concerned.

We refer to [10] where the measure data problem with general growth was first treated. To obtain Riesz potential estimates, Baroni considered superquadratic growth, that is, $2 \leq \gamma_1 \leq \gamma_2 < \infty$ in (1.7), because there is no unified method to obtain gradient potential estimates covering subquadratic and superquadratic growth simultaneously. To overcome the difficulties arising from dealing with SOLAs, several auxiliary $N$-functions and Sobolev-Poincaré type inequalities are introduced.

Note that, only in [37], a regularity result is obtained for measure data problems with general growth which covers subquadratic and superquadratic growth simultaneously. As a first step to develop a unified method for gradient potential estimates, we study Calderón-Zygmund type estimates via the fractional maximal functions of order 1, under the assumption (1.7) with $2 - 1/n < \gamma_1 \leq \gamma_2 < \infty$ which covers the whole region of $p$ for SOLAs to $p$-Laplacian type measure data problems. We further show Lorentz-Sobolev type estimates by employing the mapping properties of Riesz potentials and the inequality between fractional maximal functions and Riesz potentials. These Lorentz-Sobolev type estimates refine the classical result [19, Theorem 3].

The rest part of Chapter 5 is devoted to Calderón-Zygmund type result for spherical quasi-minimizers to the following functionals with variable exponent growth:

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x,Du) - |F|^{p(x)-2}F \cdot Du\, dx, \tag{1.8}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain and $F \in L^{p(\cdot)}(\Omega)$. This result is announced in [27].

Let us recall quasi-minimizers and spherical quasi-minimizers for the functional regarding (P). A quasi-minimizer is a weak type of the minimizer. For example, let $h : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfy $|\xi|^p \lesssim h(x,\xi) \lesssim |\xi|^p + 1$ and $U$ a subset of $\Omega$ to consider

$$\mathcal{H}(v,U) := \int_U h(x,Dv)\, dx.$$

Then we say $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is *a local quasi-minimizer* or a *$Q$-minimizer* of $\mathcal{H}$ provided

$$\mathcal{H}(u, \operatorname{supp}\varphi) \leq Q\mathcal{H}(u+\varphi, \operatorname{supp}\varphi)$$

for each ball $B \Subset \Omega$ and $\varphi \in W^{1,p}_0(B)$ and for some $Q \geq 1$. Of course, if $Q = 1$, then $u$ is a local minimizer of $\mathcal{H}$. On the other hand, we say

$u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *local spherical quasi-minimizer* of $\mathcal{H}$ provided

$$\mathcal{H}(u, B) = \int_B h(x, Du)\, dx \leq Q \int_B h(x, Du + D\varphi)\, dx = Q\mathcal{H}(u + \varphi, B)$$

for each $B \Subset \Omega$ and $\varphi \in W^{1,p}_0(B)$ and for some $Q \geq 1$.

The concept of quasi-minimizers was first introduced by Giaquinta and Giusti in [59,60]. The advantage of the use of quasi-minimizers is that they are Hölder continuous, see [59], as the De Giorgi argument still holds for the case. On the other hand, one can construct a functional and an associated spherical quasi-minimizer that is locally unbounded. However, it turns out that if $Q$ is sufficiently close to 1, then we can still obtain the Hölder continuity of a spherical quasi-minimizer, as follows from [63].

As mentioned before, we present a global Calderón-Zygmund type result for spherical quasi-minimizers to (1.8) under possibly the weakest assumptions on the domain $\Omega$ and the functional $f$.

# Chapter 2

# Preliminaries

## 2.1 Notations

The followings are standard notations, which will be used in what follows.

(1) $x = (x', x_n) \in \mathbb{R}^n$ for $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$.

(2) $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ is the ball centered at $x$ with radius $r > 0$ and $B_r^+(x)$t is the upper half ball. If there is no confusion, we write $B_r = B_r(x)$.

(3) $\Omega$ is a bounded domain of $\mathbb{R}^n, n \geq 2$, and $\partial\Omega$ is the boundary of $\Omega$.

(4) $\Omega_r(x) = \Omega \cap B_r(x)$ and $\Omega_r = \Omega \cap B_r$.

(5) $\mathrm{dist}(x, U) = \inf\{|x - y| : y \in U\}$ is the distance from $x$ to a set $U$.

(6) For each set $U \subset \mathbb{R}^n$, $|U|$ is the $n$-dimensional Lebesgue measure of $U$, and $\mathrm{diam}(U)$ is the diameter of $U$.

(7) For every $k \in \mathbb{N}$, we define truncation operators $T_k, \mathfrak{T}_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(t) = \min\{k, \max\{-k, t\}\} \quad \text{and} \quad \mathfrak{T}_k(t) = T_1(t - T_k(t)).$$

(8) For $f \in L^1_{loc}(\mathbb{R}^n)$, $(f)_U$ stands for the integral average of $f$ over a bounded open set $U \subset \mathbb{R}^n$ with positive measure, that is,

$$(f)_U = \fint_U f(x)\,dx = \frac{1}{|U|}\int_U f(x)\,dx.$$

(9) For each problems, we use the abbreviation `data` to specify the dependence of constants and denote $c \geq 1$ by a generic constant depending only on `data`, which may vary from line to line.

(10) We denote $A \lesssim B$ when there exists a generic constant $c$ such that $A \leq cB$. If $c$ depends also on $\chi$ which does not belong to `data`, then we write $A \lesssim_{\chi} B$ instead. Moreover, the notation $A \approx B$ shall mean $c^{-1}B \leq A \leq cB$ for some generic constant $c$.

(11) For any constant $p \geq 1$, $p' = \frac{p}{p-1}$ is the conjugate exponent of $p$. We call $p^* = \frac{np}{n-p}$ by Sobolev exponent for $p \in [1, n)$.

## 2.2 Musielak-Orlicz spaces

A real valued function $\Phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a generalized Young function, if $\Phi(x, \cdot)$ is a convex function that satisfies

$$\Phi(x, 0) = 0 \text{ for a.e. } x \in \Omega, \quad \lim_{t\to\infty} \Phi(x, t) = \infty \text{ for a.e. } x \in \Omega$$

and $\Phi(\cdot, t)$ is Lebesgue measurable for all $t > 0$. We define $\widetilde{\Phi} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\widetilde{\Phi}(x, t) := \sup_{t\geq 0}\{st - \Phi(x, t)\},$$

which is called the complementary function of $\Phi$. By the definition of complementary function, we see

$$st \leq \Phi(x, t) + \widetilde{\Phi}(x, s),$$

whenever $s, t \in \mathbb{R}^+$ and a.e. $x \in \Omega$. Moreover, for any $t \geq 0$ it holds that

$$\Phi^*\left(\frac{\Phi(t)}{t}\right) \leq \Phi(t) \leq \Phi^*\left(\frac{\Phi(2t)}{t}\right). \tag{2.1}$$

A generalized $N$-function $\Phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a generalized Young function such that $\Phi(x, t) > 0$ for all $t > 0$,

$$\lim_{t\to 0} \frac{\Phi(x, t)}{t} = 0 \quad \text{a.e. } x \in \Omega \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(x, t)}{t} = \infty \quad \text{a.e. } x \in \Omega.$$

Additionally, we assume that $\Phi(x,\cdot) \in C^2(0,\infty)$ for a.e. $x \in \Omega$ and satisfies

$$0 < \gamma_1 - 1 \le \frac{t\partial_t^2 \Phi(x,t)}{\partial_t \Phi(x,t)} \le \gamma_2 - 1 \tag{2.2}$$

for some positive constants $1 < \gamma_1 \le \gamma_2$, for $t > 0$ and a.e. $x \in \Omega$. In this section, we define $\texttt{data} = \{\gamma_1, \gamma_2, n\}$.

We readily check that (2.2) implies

$$\gamma_1 \le \frac{t\partial_t \Phi(x,t)}{\Phi(x,t)} \le \gamma_2, \quad \min\{\alpha^{\gamma_1}, \alpha^{\gamma_2}\} \le \frac{\Phi(x,\alpha t)}{\Phi(x,t)} \le \max\{\alpha^{\gamma_1}, \alpha^{\gamma_2}\} \tag{2.3}$$

for $t, \alpha \ge 0$ and a.e. $x \in \Omega$. If the last inequalities is satisfied, then we say that $\Phi$ and $\widetilde{\Phi}$ satisfy $\Delta_2$-condition. It is also well known that for any $\eta \in [0,1]$ and $s \ge 0$ (2.2) implies

$$\eta^{\gamma_2}\Phi(t) \le \Phi(\eta t) \le \eta^{\gamma_1}\Phi(t) \quad \text{and} \quad \eta^{\gamma_1'}\Phi^*(t) \le \Phi^*(\eta t) \le \eta^{\gamma_2'}\Phi^*(t). \tag{2.4}$$

For more details, we refer to [95] and [84, Lemma 1.1].

Since $\Phi(x,\cdot)$ is monotone increasing for a.e. $x \in \Omega$, we can define a function $\Phi^{-1} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\Phi^{-1}(x, \Phi(x,t)) = \Phi(x, \Phi^{-1}(x,t)) = t$ for all $t \in \mathbb{R}^+$. From (2.3), one can derive

$$\min\left\{\alpha^{\frac{1}{\gamma_1}}, \alpha^{\frac{1}{\gamma_2}}\right\} \le \frac{\Phi^{-1}(x,\alpha t)}{\Phi^{-1}(x,t)} \le \max\left\{\alpha^{\frac{1}{\gamma_1}}, \alpha^{\frac{1}{\gamma_2}}\right\} \tag{2.5}$$

for $t, \alpha \ge 0$ and a.e. $x \in \Omega$, see [10, Section 3]. Throughout this section, we assume that every generalized $N$-function satisfies (2.2). It is well known that

$$\widetilde{\Phi}\left(x, \frac{\Phi(x,t)}{t}\right) \approx \Phi(x,t) \tag{2.6}$$

for every $t > 0$ and a.e. $x \in \Omega$.

The Musielak-Orlicz space $L^{\Phi}(\Omega)$ is the set of Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \Phi(x, |f|)\, dx < \infty$$

with the following Luxemberg norm

$$\|f\|_{L^\Phi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f|}{\lambda}\right) dx \le 1\right\},$$

and the Musielak-Orlicz-Sobolev space is

$$W^{1,\Phi}(\Omega) = \left\{f \in L^\Phi(\Omega) : \int_\Omega \Phi(x, |Df|)\, dx < \infty\right\}$$

with the norm

$$\|f\|_{W^{1,\Phi}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f|}{\lambda}\right) dx + \int_\Omega \Phi\left(x, \frac{|Df|}{\lambda}\right) dx \le 1\right\}.$$

If (2.2) and $\inf_{x\in\Omega} \Phi(x, 1) > 0$ are satisfied, then they are Banach spaces, see [92, Theorem 10.2].

Similarly, we call a real valued function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ an $N$-function, if $\Psi$ is a nondecreasing convex function that satisfies $\Psi(0) = 0$, $\Psi(t) > 0$ for all $t > 0$,

$$\lim_{t\to 0} \frac{\Psi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Psi(t)}{t} = \infty.$$

Then we recall the Orlicz space

$$L^\Psi(\Omega) := \left\{f \in L^1(\Omega) : \int_\Omega \Psi(|Df|)\, dx < \infty\right\}$$

and the Orlicz-Sobolev space

$$W^{1,\Psi}(\Omega) := \left\{f \in L^\Psi(\Omega) : \int_\Omega \Psi(|Df|)\, dx < \infty\right\}.$$

One can find basic ingredients of Orlicz spaces and Musielak-Orlicz spaces in [43, 66, 92, 95] and references therein. Note that these spaces corresponds to the case (O).

Recall the examples given in Section 1.2. It is well known that the function spaces $W^{1,p}(\Omega)$ and $L^p(\Omega)$ regarding (P) are called simply Sobolev space and Lebesgue space, respectively. For (PX), we denote the Musielak-Orlicz space by $L^{p(\cdot)}(\Omega)$ and the Musielak-Orlicz-Sobolev space by $W^{1,p(\cdot)}(\Omega)$ and call them simply the variable exponent spaces. As mentioned in Chapter 1, these spaces are naturally appeared in the modeling of electrorheological fluids and image restoration. For further discussions about some important properties including reflexivity, separability, and Sobolev embeddings on variable exponent spaces, we refer to [42,45,55,57,58,105] and references therein.

We end this subsection with Sobolev type inequalities. Let $\Psi$ be an $N$-function satisfying (2.2). First, we introduce a condition on $\Psi$ corresponding to the case $1 < p \le n$ for (P). We say that $\Psi$ grows slowly if

$$\int_0 \left(\frac{s}{\Psi(s)}\right)^{\frac{1}{n-1}} ds < \infty \quad \text{and} \quad \int^\infty \left(\frac{s}{\Psi(s)}\right)^{\frac{1}{n-1}} ds = \infty. \tag{2.7}$$

We now define

$$H_n(t) := \left(\int_0^t \left[\frac{s}{\Psi(s)}\right]^{\frac{1}{n-1}} ds\right)^{\frac{n-1}{n}} \quad \text{and} \quad \Psi_n(t) := (\Psi \circ H_n^{-1})(t). \tag{2.8}$$

The Sobolev embedding theorem for $1 < p < n$ was extended to Orlicz spaces in [34, Theorem 3], which we now state.

**Lemma 2.2.1.** *Let $\Omega \subset \mathbb{R}^n, n \ge 2$, be a bounded domain and let $\Psi$ be an $N$-function satisfying* (2.7)*. Let $\Psi_n$ be the function defined by* (2.8)*. Then for every $u \in W_0^{1,\Psi}(\Omega)$ there holds*

$$\int_\Omega \Psi_n\left(\frac{|u|}{c_n(\int_\Omega \Psi(|Du|)\,dx)^{\frac{1}{n}}}\right) dx \le \int_\Omega \Psi(|Du|)\,dx,$$

*where $c_n$ is the measure of the $n$-dimensional unit ball.*

Next, we introduce an embedding theorem [33, Theorem 1a], corresponding to $\Psi$ growing fast, that is,

$$\int^\infty \left(\frac{s}{\Psi(s)}\right)^{\frac{1}{n-1}} ds < \infty. \tag{2.9}$$

**Lemma 2.2.2.** *Let $\Omega$ be as in Lemma 2.2.1 and let $\Psi$ satisfy* (2.2) *and* (2.9)*. Then there exists a constant $c$ depending only on $\gamma_2, |\Omega|, n$ such that for every $u \in W^{1,\Psi}(\Omega)$*

$$\|u\|_{L^\infty(\Omega)} \le c\|Du\|_{L^\Psi(\Omega)}.$$

We introduce another Sobolev embedding theorem, [10, Proposition 3.5].

**Lemma 2.2.3.** *Let $\Psi \in C^1(\mathbb{R}^+)$ be a positive $N$-function such that*

$$\gamma_1 \leq \frac{t\Psi'(t)}{\Psi(t)} \leq \gamma_2, \quad \text{for } t > 0, \quad \text{with } 1 \leq \gamma_1 \leq \gamma_2.$$

*Then there exists a constant $c$ depending only on $n, p_1$ such that*

$$\fint_{B_R} \Psi\left(\frac{|u|}{R}\right)^{\frac{n}{n-1}} dx \leq c\left(\fint_{B_R} \Psi(|Du|)\, dx\right)^{\frac{n}{n-1}}$$

*for every $u \in W_0^{1,\Psi}(B_R)$.*

## 2.3 Auxiliary results

### 2.3.1 log-Hölder continuity

The modulus of continuity of a continuous function $p : \Omega \to \mathbb{R}$ is the nondecreasing concave function $\omega_{p(\cdot)} : [0, \infty) \to [0, \infty)$ defined by

$$\omega_{p(\cdot)}(\rho) := \sup\{|p(x) - p(y)| : x, y \in \Omega, |x - y| \leq \rho\}.$$

We say that $p$ is Dini continuous if

$$\int_0 \omega_{p(\cdot)}(\rho) \frac{d\rho}{\rho} < \infty. \tag{2.10}$$

As we mentioned before, Dini-continuity assumption on the coefficient $a(\cdot)$ in (1.1) is the sharp one to obtain $C^1$-regularity of solutions.

Additionally, we say that $p$ is log-Dini continuous if

$$\int_0 \omega_{p(\cdot)}(\rho) \log\left(\frac{1}{\rho}\right) \frac{d\rho}{\rho} < \infty. \tag{2.11}$$

Note that the log-Dini continuity (2.11) on $p(\cdot)$ implies the strong log-Hölder continuity

$$\limsup_{\rho \searrow 0} \omega_{p(\cdot)}(\rho) \log\left(\frac{1}{\rho}\right) = 0,$$

which implies the log-Hölder continuity given in (1.5). Indeed, (1.5) implies that there exist constants $R_1, L > 0$ such that

$$\omega(\rho)\log\left(\frac{1}{\rho}\right) \leq L \tag{2.12}$$

for every $0 < \rho \leq R_1$. In the localization procedures of Chapter 3 and Chapter 5, for any function $p(\cdot)$ satisfying (2.12), the following estimates are often used: for any $0 < R \leq R_1$ and $0 \leq \tilde{\omega} \leq \omega_{p(\cdot)}(R)$ we have

$$R^{-\tilde{\omega}} \leq c(L) \tag{2.13}$$

whenever $0 < R \leq R_1$ and $0 \leq \tilde{\omega} \leq \omega_{p(\cdot)}(R)$, and

$$A^{\sigma} \leq c(L,\alpha)(A + R^{\alpha})^{\sigma+\tilde{\omega}} \tag{2.14}$$

for any $\sigma, \alpha > 0$, whenever $A \geq 0$.

We now introduce an estimate for $L \log L$ functions given in [4, (28)] and [68, Lemma 5.2]. For any $q, \beta > 1$ and $f \in L^q(\Omega)$, we have

$$\fint_{\Omega} f \log^{\beta}\left(e + \frac{|f|}{\|f\|_{L^1(\Omega)}}\right) dx \leq c(q,\beta)\left(\fint_{\Omega} |f|^q \, dx\right)^{\frac{1}{q}}. \tag{2.15}$$

In Chapter 3, we further use the following estimates: For any $t_1, t_2 \in [0, \infty)$, we see

$$\log(e + t_1 t_2) \leq \log(e + t_1) + \log(e + t_2). \tag{2.16}$$

Moreover, if $t_1 \geq 1$, then we have

$$\log(e + t_1 t_2) \leq t_1 \log(e + t_2).$$

### 2.3.2 Monotonicity of vector field $A(\cdot)$.

Let us consider the following continuous vector field $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^n$ with $\partial A = \partial_\xi A(\cdot)$ being Carathéodory regular and satisfying the following ellipticity and growth assumptions

$$\begin{cases} |A(x,\xi)| + |\partial A(x,\xi)|(|\xi|^2+s^2)^{\frac{1}{2}} \le Lg(x,|\xi|) \\ \nu \dfrac{g(x,|\xi|)}{|\xi|}|\eta|^2 \le \langle \partial_\xi A(x,\xi)\eta,\eta\rangle, \end{cases} \tag{2.17}$$

where $x \in \Omega$, $\xi,\eta \in \mathbb{R}^n$, $0 < \nu \le L$ and $g(x,t) = \partial_t G(x,t)$ for some generalized $N$-function $G$ satisfying (2.2) with $\Phi = G$.

We now define an auxiliary vector field $V : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$V(x,\xi) = \left(\frac{g(x,|\xi|)}{|\xi|}\right)^{\frac{1}{2}}\xi$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$. The monotonicity of $A(\cdot)$ can be written as follows

$$\begin{aligned} |V(x,z_1) - V(x,z_2)|^2 &\approx \frac{g(x,|\xi_1|+|\xi_2|)}{|\xi_1|+|\xi_2|}|\xi_1-\xi_2|^2 \\ &\lesssim \langle A(x,\xi_1) - A(x,\xi_2), \xi_1-\xi_2\rangle \end{aligned} \tag{2.18}$$

for every $z_1, z_2 \in \mathbb{R}^n$ and $x \in \Omega$, see [44, Lemma 3] and [65]. In case $t \mapsto g(x,t)/t$ is increasing for a.e. $x \in \Omega$, we further discover

$$G(x,|z_1-z_2|) \le \frac{g(x,|z_1|+|z_2|)}{|z_1|+|z_2|}|z_1-z_2|^2. \tag{2.19}$$

### 2.3.3 Regularity results for limiting equations

In this subsection, we present known decay estimate for limiting equations. For $A : \mathbb{R}^n \to \mathbb{R}^n$ satisfying (2.17) with some $N$-function $G$, let $v$ be the weak solution to the frozen equation:

$$\begin{cases} -\operatorname{div}(A(Dv)) = 0 & \text{in} \quad B_R, \\ v = w & \text{on} \quad \partial B_R, \end{cases} \tag{2.20}$$

where $w \in W^{1,G}(B_R)$. The following local Lipschitz regularity and excess decay estimates can be found in [10, Lemma 4.1] and [84, Lemma 5.1].

**Lemma 2.3.1.** *Let $w \in W^{1,G}(B_R)$ be the weak solution to* (2.20) *under* (2.17)*. Then the following estimate holds:*

$$\sup_{B_{R/2}} |Dv| \leq c_l \fint_{B_R} |Dv| \, dx.$$

*Moreover $v \in C^{1,\beta}_{loc}(B_R)$ for some $\beta \in (0,1)$ and the following excess decay estimate holds:*

$$\fint_{B_\rho} |Dv - (Dv)_{B_\rho}| \, dx \leq c_\beta \left(\frac{\rho}{r}\right)^\beta \fint_{B_r} |Dv - (Dv)_{B_r}| \, dx$$

*and*

$$\operatorname*{osc}_{B_\rho} Dv \leq c_\beta \left(\frac{\rho}{r}\right)^\beta \fint_{B_r} |Dv| \, dx$$

*for $0 < \rho < r \leq \widetilde{R}$, where the constants $c_l, c_\beta$ and the exponent $\beta$ depending only on $n, \gamma_1, \gamma_2, \nu, L$.*

Global Lipschitz regularity of solutions to equations with general growth has been actively investigated, see for instance [15, 35, 36, 84]. In particular, we mention [40, Theorem 2.2] and [32, Theorem 4.1] for boundary Lipschitz regularity for the weak solution $v$ to

$$\begin{cases} -\operatorname{div}(A(Dv)) = 0 & \text{in} \quad B_R^+, \\ v = w & \text{on} \quad \partial B_R^+, \end{cases} \tag{2.21}$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the same assumptions in (2.20) and $w \in W^{1,G}(B_R^+)$ satisfies $w = 0$ on $T_R = \{x \in B_R : x_n = 0\}$.

**Lemma 2.3.2.** *Let $v \in W^{1,G}(B_R^+)$ be the weak solution to* (2.21)*. Then there exists a constant $c_l = c_l(n, \gamma_1, \gamma_2, \nu, L) \geq 1$ such that*

$$\sup_{B_r^+} G(|Dv|) \leq c_l \fint_{B_{2r}^+} G(|Dv|) \, dx.$$

Recall that (P) is the typical example of (O) in Section 1.2. Therefore, Lemma 2.3.1 and Lemma 2.3.2 hold for the case (P) with constants $c_l, c_\beta$ and $\beta$ depending on $p$ instead of $\gamma_1$ and $\gamma_2$.

# Chapter 3

# Non-autonomous equations

## 3.1 Main results

We devote this chapter to gradient potential estimates for non-autonomous measure data problems regarding (PT) and (PX). Let us consider the following equation

$$-\mathrm{div}\,(\gamma(x)A(x,Du)) = \mu \quad \text{in } \Omega, \tag{3.1}$$

where $\mu$ is a finite Borel measure defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and the coefficient function $\gamma : \Omega \to \mathbb{R}$ satisfy the following boundedness

$$\nu \leq \gamma(x) \leq L < \infty. \tag{3.2}$$

We further assume Dini-continuity assumption on $\gamma$ as in (2.10) with modulus of continuity $\omega_{\gamma(\cdot)}$ to obtain pointwise estimates for the gradient of solution. Note that Dini-continuity is known as the optimal assumption on the coefficient to derive $C^1$-regularity for homogeneous elliptic equations as we mentioned in Section 1.1. Moreover, this continuity assumption has an important role in measuring the decay rate of oscillation of the gradient.

The mapping $A : \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ is assumed to be $C^1$-regular in the gradient variable $\xi$, with $\partial A(\cdot)$ being Carathéodory regular. In order to study elliptic equations generalized from (1.4) with (PT) or (PX), we are going to consider two types of assumptions, (GPT) and (GPX). It is readily checked that (1.4) satisfies (GPT) (resp. (GPX)) when (PT) (resp. (PX)) is considered.

We first consider the following growth, ellipticity and continuity on $A(\cdot)$:

$$\begin{cases} |A(x,\xi)| + |\partial A(x,\xi)||\xi| \leq L\left[|\xi|^{p-1} + a(x)\log(e+|\xi|)|\xi|^{p-1}\right] \\ \nu\left[|\xi|^{p-2} + a(x)\log(e+|\xi|)|\xi|^{p-2}\right]|\eta|^2 \leq \langle \partial A(x,\xi)\eta,\eta\rangle \\ |A(x,\xi) - A(x_0,\xi)| \leq L\omega_{a(\cdot)}(|x-x_0|)\log(e+|\xi|)|\xi|^{p-1} \end{cases} \tag{GPT}$$

for every $x, x_0 \in \Omega$ and $\xi$, $\eta \in \mathbb{R}^n$, where $0 < \nu \leq L$ are fixed and $2 \leq p$.

On the other hand, we also consider the following set of assumptions:

$$\begin{cases} |A(x,\xi)| + |\partial A(x,\xi)|(|\xi|^2+s^2)^{\frac{1}{2}} \leq L(|\xi|^2+s^2)^{\frac{p(x)-1}{2}} \\ \nu(|\xi|^2+s^2)^{\frac{p(x)-2}{2}}|\eta|^2 \leq \langle \partial A(x,\xi)\eta,\eta\rangle \\ |A(x,\xi) - A(x_0,\xi)| \leq L\omega_{p(\cdot)}(|x-x_0|)\left[(|\xi|^2+s^2)^{\frac{p(x)-1}{2}}\right. \\ \qquad\qquad \left. +(|\xi|^2+s^2)^{\frac{p(x_0)-1}{2}}\right]\left[1+|\log(|\xi|^2+s^2)|\right] \end{cases} \tag{GPX}$$

for every $x, x_0 \in \Omega$ and $\xi$, $\eta \in \mathbb{R}^n$, where $0 < \nu \leq L$ and $s \in [0,1]$ are fixed.

In addition, the variable exponent function $p : \Omega \to \mathbb{R}$ and the modulating coefficient $a : \Omega \to \mathbb{R}$ are assumed to be log-Dini continuous as in (2.11) and satisfy the following boundedness

$$2 \leq p(x) \leq \gamma_2 < \infty \quad \text{and} \quad 0 \leq a(x) \leq \|a\|_{L^\infty(\Omega)} < \infty. \tag{3.3}$$

In this chapter, we assume $2 \leq p$ and (3.3) to restrict our discussion to superquadratic growth case.

Remark that in the region $\{x \in \Omega : a(x) = 0\}$, (3.1) with (GPT) is reduced to be $p$-Laplace type equation, while in the remaining region, it is reduced to $L^p \log L$ type equation. For this reason, it is called a non-autonomous problem and so does the case (PX), see (1.2) for more details.

Throughout this chapter, `data` stands for the set of constants $\{n, p, \nu, L\}$ if (GPT) are considered, while it stands for $\{n, \gamma_2, \nu, L\}$ if (GPX) are considered. We define $g_3$ and $g_2$ by

$$g_3(x,t) = t^{p-1} + a(x)\log(e+t)t^{p-1} \approx \partial_t G_3(x,t)$$

and

$$g_2(x,t) = t^{p(x)-1} \approx \partial_t G_2(x,t),$$

where $G_3$ and $G_2$ are given in Section 1.2. If it there is no confusion, we omit the subscripts $_3$ and $_2$. We also write

$$\omega(t) = \omega_{\gamma(\cdot)}(t) + \omega_{a(\cdot)}(t)\log(e+t)$$
$$(resp.\, \omega(t) = \omega_{\gamma(\cdot)}(t) + \omega_{p(\cdot)}(t)\log(e+t))$$

if (GPT) (resp. (GPX)) are taken into account.

As mentioned in Section 1.2, we consider (3.1) under (GPT) or (GPX) with the right-hand side measure $\mu$ which does not necessarily belong to the dual space of $W^{1,G}(\Omega)$. Therefore, we consider the notion of SOLAs that is introduced by Boccardo and Gallouët in [18]. Indeed, for (3.1) with (GPX), Bögelein and Habermann proved the existence of SOLA in [21].

**Definition 3.1.1** (SOLA)**.** A function $u \in W_0^{1,1}(\Omega)$ is a SOLA to (3.1) under (GPX) if and only if there are a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ converges to $\mu$ weakly in measure and a corresponding sequence of weak solutions $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(\Omega)$ to the equation

$$\begin{cases} -\operatorname{div}(\gamma(x)A(x,Du_k)) = \mu_k & \text{in} \quad \Omega \\ \qquad\qquad\qquad\qquad u_k = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.4}$$

such that $u_k \to u$ in $W^{1,q}(\Omega_r)(y)$ for every $x \in \Omega$, $r > 0$ with

$$1 \le q < \min\left\{ \inf_{x\in\Omega_r} p(x), \frac{n}{n-1} \inf_{x\in\Omega_r}(p(x)-1)\right\}.$$

Note that weakly convergence in measure implies

$$\limsup_{k\to\infty} |\mu_k|(U) \le |\mu|(\bar{U}) \tag{3.5}$$

for every measurable set $U \subset \Omega$ and its closure $\bar{U}$, see [56, Theorem 1.9.1].

On the other hand, the proof of the existence of SOLAs to (3.1) under (GPT) in [30, Lemma 2.5] is quite similar to the one in [18] since the logarithmic perturbation does not affect the approximation procedure given in [18]. Now we define a SOLAs to (3.1) under (GPT):

**Definition 3.1.2.** We say that a function $u \in W_0^{1,1}(\Omega)$ is a local SOLA to (3.1) under (GPT) if there exists a sequence of local weak solutions $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,G}(\Omega)$ to (3.4) with $\{\mu_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ such that $\mu_k \rightharpoonup \mu$ weakly in measure, and $u_k \to u$ in $W^{1,q}(\Omega)$ for any $1 \le q < \min\{p, \frac{n(p-1)}{n-1}\}$.

We first obtain a priori estimates, Theorem 3.1.4 below, for weak solutions, and then justify that the estimates also holds true for the SOLA by an appropriate approximation procedure, which will be made under the a priori assumption that $L^1$-norm of $Du$ is uniformly bounded as

$$\int_\Omega |Du|\,dx =: M < \infty. \tag{3.6}$$

Henceforth, we assume $\mu \in L^\infty(\Omega)$ and $u \in W^{1,G}(\Omega)$, until Section 3.4.

We now state our first main result:

**Theorem 3.1.3.** *Let $u \in W^{1,G}(\Omega)$ be a weak solution to* (3.1) *under* (GPT). *Assume $a(\cdot)$ is* log-*Dini continuous and $\gamma(\cdot)$ is Dini continuous. Then there exist constants $c = c(\texttt{data}) > 1$ and $R_0 = R_0(\texttt{data}, \omega, M, |\mu|(\Omega))$ satisfying*

$$g(x_0, |Du(x_0)|) \le cI_1^\mu(x_0, R) + cg\left(x_0, \fint_{B_R(x_0)} |Du|\,dx\right) \tag{3.7}$$

*for every Lebesgue point $x_0$ of $Du$, whenever $B_R(x_0) \subset \Omega$ with $R \le R_0$ and the right-hand side is finite.*

Analogously, we have the following estimates for (GPX):

**Theorem 3.1.4.** *Let $u \in W^{1,G}(\Omega)$ be a weak solution to* (3.1) *under* (GPX). *Assume $p(\cdot)$ is* log-*Dini continuous and $\gamma(\cdot)$ is Dini continuous. Then there exist constants $c = c(\texttt{data}) > 1$ and $R_0 = R_0(\texttt{data}, \omega, M, |\mu|(\Omega))$ satisfying*

$$g(x_0, |Du(x_0)|) \le cI_1^\mu(x_0, R) + cg\left(x_0, \fint_{B_R(x_0)} (|Du| + s + R)\,dx\right) \tag{3.8}$$

*for every Lebesgue point $x_0$ of $Du$, whenever $B_R(x_0) \subset \Omega$ with $R \le R_0$ and the right-hand side is finite.*

**Remark 3.1.5.** Recalling (1.2) for the case that $p(x_0) \ge 2$, one can obtain the result of [21] as a consequence of Theorem 3.1.4, with the weaker continuity assumption (2.11) on the variable exponent function $p(\cdot)$ than the one used in [21].

**Remark 3.1.6.** Applying some embedding properties of Riesz potential to Theorem 3.1.4, one can prove Calderón-Zygmund type estimates. We refer to [6,64] for a classical result of Riesz potential embedding in Lebesgue spaces, [7,42] in variable exponent spaces, and [91, Theorem 1.2] in weighted variable exponent spaces, respectively. Calderón-Zygmund type estimates for (3.1) under (GPX) are recently obtained in [26]. Indeed, for any weak log-Hölder continuous function $q : \Omega \to \mathbb{R}$ such that

$$1 < \inf_{\Omega} q(x) \leq \sup_{\Omega} q(x) < n,$$

[7, Theorem 4.3] shows the implication:

$$\mu \in L^{q(\cdot)}(\Omega) \Rightarrow Du \in w\text{-}L^{\frac{nq(\cdot)}{(n-q(\cdot))(p(\cdot)-1)}}(\Omega),$$

where $w\text{-}L^{\frac{nq(\cdot)}{(n-q(\cdot))(p(\cdot)-1)}}(\Omega)$ is the variable exponent weak space defined in [7, Definition 3.1]. See (2.12) for the definition of weak log-Hölder continuity.

As another direct consequence of Theorem 3.1.3 and Theorem 3.1.4, we see that

$$I_1^{\mu}(\cdot, R) \in L^{\infty}_{loc}(\Omega) \text{ for some } R > 0 \quad \Rightarrow \quad Du \in L^{\infty}_{loc}(\Omega, R^n).$$

Then later in Proposition 3.5.1, we prove a local VMO-regularity of $Du$. Once we have Proposition 3.5.1, the gradient continuity criteria follow from the same spirit used as in the proof of [76, Theorem 1.5], as we now state without its proof.

**Theorem 3.1.7.** *Under the assumptions of Theorem 3.1.3, if*

$$\lim_{R \to 0} I_1^{\mu}(x, R) = 0 \quad \text{locally uniformly in } \Omega \text{ w.r.t } x, \tag{3.9}$$

*then $Du$ is continuous in $\Omega$. This $C^1$-regularity criteria also hold under the assumptions of Theorem 3.1.4.*

**Remark 3.1.8.** Recall the Lorentz space

$$L(p,q)(\Omega) := \left\{ f \in L^1(\Omega) : \left\| \rho | \{|f| > \rho\} |^{\frac{1}{p}} \right\|_{L^q(\frac{d\rho}{\rho})} < \infty \right\}$$

for $1 \leq p < \infty$ and $1 \leq q \leq \infty$, and [76, Corollary 1.6] to discover that if the right-hand side measure $\mu$ belongs to $L(n, 1)$, then (3.9) holds and $Du$ is continuous. Consequently, our result of Theorem 3.1.7 complements [78] for the $p$-Laplacian problems to the non-autonomous problems.

We now conclude this introduction with the following Riesz potential estimates for SOLA.

**Theorem 3.1.9.** *The statements of Theorem 3.1.3, Theorem 3.1.4 and Theorem 3.1.7 continue to hold for SOLAs to* (3.1) *under the same assumptions of each theorems.*

Before ending this section, we mention the very fine paper [75] where oscillation estimates are obtained in terms of the Riesz potentials of the measure $\mu$ for nonlinear degenerate elliptic equation of the $p$-Laplacian type. After the pointwise estimate have been obtained in Theorem 3.1.3, it seems possible to find a correct version of the oscillation estimate [75, Theorem 1.1].

# 3.2 Comparison estimates

From the assumption (2.11), when we consider (GPT), we can take a positive constant

$$R_1 = R_1(\omega, |\mu|(\Omega), \|Du\|_{L^1(\Omega)}) \leq \frac{1}{|\mu|(\Omega) + \|Du\|_{L^1(\Omega)} + 1}$$

such that

$$\omega_{\gamma(\cdot)}(r) + \omega_{a(\cdot)}(r) \log\left(\frac{1}{r}\right) \leq \frac{1}{100np}, \tag{3.10}$$

for every $0 < r \leq R_1$, which implies

$$r^{-\omega_{a(\cdot)}(r)} = e^{-\omega_{a(\cdot)}(r)\log(r)} \leq c \tag{3.11}$$

and

$$\left(|\mu|(\Omega) + \|Du\|_{L^1(\Omega)} + 1\right)^{\omega_{a(\cdot)}(r)} \leq r^{-\omega_{a(\cdot)}(r)} \leq c$$

whenever $0 < r \leq R_1$. We point out that the upper bound in (3.10) is chosen in order to handle some technical issues such as in (3.45) and (3.69).

On the other hand, in case of (GPX), we assume

$$\omega_{\gamma(\cdot)}(r) + \omega_{p(\cdot)}(r) \log\left(\frac{1}{r}\right) \leq \frac{1}{100n\gamma_2}. \tag{3.12}$$

These inequalities will be used often in next sections. Throughout this chapter, we always assume $0 < R \leq R_1$. To proceed further, we recall (2.13) and

(2.14) which will be used without mentioning them.

For any measurable set $U \subset \mathbb{R}^n$ with positive Lebesgue measure and an integrable function $f : U \to \mathbb{R}^k$ with some positive integer $k$, we denote the excess functional of $f$ by

$$E(f,U) := \fint_U |f - (f)_U|\, dx. \tag{3.13}$$

We now consider the following reference problems. As mentioned in after (3.6), we assume $\mu \in L^\infty(\Omega)$ and $u \in W^{1,G}(\Omega)$ till Section 3.4. Then there exists $w \in W^{1,G}(B_R(x_0))$ the unique weak solution to the homogeneous Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(\gamma(x)A(x,Dw)) = 0 & \text{in} \quad B_R(x_0) \\ \hfill w = u & \text{on} \quad \partial B_R(x_0), \end{cases} \tag{3.14}$$

where $B_R(x_0) \subset \Omega$. Indeed, there is a higher integrability result of $w$ in Lemma 3.2.7 for (GPT) and Lemma 3.2.10 for (GPX).

With a suitable assumption on $R$, we then let $v \in W^{1,G(x_1)}(B_{\tilde{R}/2}(x_1))$ be the unique weak solution to the homogeneous Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(A(x_1,Dv)) = 0 & \text{in} \quad B_{\tilde{R}/2}(x_1) \\ \hfill v = w & \text{on} \quad \partial B_{\tilde{R}/2}(x_1), \end{cases} \tag{3.15}$$

whenever $0 < \tilde{R} \le R$ and $B_{\tilde{R}}(x_1) \subset B_R(x_0)$, see Remark 3.2.12.

As we mentioned in Section 1.2, each energy functional regarding (GPT) and (GPX) are similar in local. Moreover, the proofs of Theorem 3.1.3 and Theorem 3.1.4 are parallel after certain comparison estimates. Therefore, in what follows, we focus on the proof of Theorem 3.1.3 to emphasize the difference between (GPT) and (GPX) by introducing only comparison estimates for Theorem 3.1.4.

### 3.2.1 Basic comparison estimates for (GPT)

Through this subsection, we assume that $A(\cdot)$ satisfies (GPT).

Before starting to discover comparison estimates, we introduce some auxiliary functions and their properties. For $B_R(x_0)$, we denote

$$a_{m,R,x_0} = \inf_{B_R(x_0)} a(x), \quad a_{M,R,x_0} = \sup_{B_R(x_0)} a(x) \quad \text{and} \quad a_0 = a(x_0).$$

For $\chi \geq -1$, $x \in \Omega$ and $t \in \mathbb{R}^+$, we define

$$\begin{aligned} & g_{m,R,x_0}(t) = \inf_{B_R(x_0)} g(x,t), \quad g_{M,R,x_0}(t) = \sup_{B_R(x_0)} g(x,t), \quad g_0(t) = g(x_0,t), \\ & g_{m,R,x_0,\chi}(t) := \left(\frac{g_m(t)}{t}\right)^{1+\chi} t, \quad g_{M,R,x_0,\chi}(t) := \left(\frac{g_M(t)}{t}\right)^{1+\chi} t, \\ & G_{m,R,x_0}(t) = \inf_{B_R(x_0)} G(x,t) \quad \text{and} \quad G_0(t) = G(x_0,t). \end{aligned} \tag{3.16}$$

If no confusion arises, then we omit the subscripts $R, x_0$ in the notations, that is, we simply write $a_m, g_m$ and so on.

A direct calculation yields

$$\frac{\partial G}{\partial t}(x,t) \approx g(x,t), \quad \frac{dG_m}{dt}(t) \approx g_m(t), \quad G(x,t) \approx \int_0^t g(x,s)\,ds$$

and

$$G_m(t) \approx \int_0^t g_m(s) ds$$

for every $x \in \Omega$ and $t \in \mathbb{R}^+$. Furthermore, $t \mapsto \frac{g(x,t)}{t}$ is increasing function for a.e $x \in \Omega$, i.e.,

$$\frac{g(x,t_1)}{t_1} \leq \frac{g(x,t_2)}{t_2}, \tag{3.17}$$

whenever $0 < t_1 < t_2 \in \mathbb{R}^+$.

Here, we provide some properties of the function $G(\cdot)$ for $p \geq 2$. Differentiating $G(\cdot)$ with respect to $t > 0$, we have

$$\partial_t G(x,t) = pt^{p-1}(1 + a(x)\log(e+t)) + a(x)\frac{t^p}{e+t}$$

and

$$\partial_t^2 G(x,t) = p(p-1)t^{p-2}(1 + a(x)\log(e+t)) + 2a(x)p\frac{t^{p-1}}{e+t} - a(x)\frac{t^p}{(e+t)^2}$$

for every $x \in \Omega$. To find the constant $\gamma_1$ and $\gamma_2$ in (2.2), we estimate

$$p-1 \le \frac{t\partial_t^2 G(t)}{\partial_t G(t)} \le p-1+\frac{a(x)(p+1)\frac{t}{e+t}-a(x)\frac{t^2}{(e+t)^2}}{p(1+a(x)\log(e+t))+a(x)\frac{t}{e+t}}. \tag{3.18}$$

If $t \ge e(e-1)$, then $p \ge 2$ implies

$$\begin{aligned} \frac{a(x)(p+1)\frac{t}{e+t}-a(x)\frac{t^2}{(e+t)^2}}{p(1+a(x)\log(e+t))+a(x)\frac{t}{e+t}} &\le \frac{a(x)(p+1)\frac{t}{e+t}}{p(1+2a(x))+a(x)\frac{t}{e+t}} \\ &\le \frac{a(x)(p+1)}{2a(x)p+a(x)+p} \le \frac{3}{5}. \end{aligned} \tag{3.19}$$

On the other hand, the concavity of log function implies

$$\log(e+t) \ge 1+\frac{t}{e(e-1)} \ge 1+\frac{t}{2e}$$

for $0 < t < e(e-1)$, and so we have

$$\begin{aligned} &3p(1+a(x)\log(e+t))+\tfrac{3a(x)t}{e+t}-4a(x)(p+1)\tfrac{t}{e+t}+4a(x)\tfrac{t^2}{(e+t)^2} \\ &\ge 3p\left(1+a(x)+\tfrac{a(x)t}{2e}\right)-(4a(x)p+a(x))\tfrac{t}{e+t} \\ &\ge 3p\left(1+\tfrac{3a(x)}{2}\right)\tfrac{t}{e+t}-(4a(x)p+\alpha)\tfrac{t}{e+t}>0. \end{aligned}$$

Therefore, for every $0 < t < e(e-1)$, we have

$$\frac{a(x)(p+1)\frac{t}{e+t}-a(x)\frac{t^2}{(e+t)^2}}{p(1+a(x)\log(e+t))+\frac{a(x)t}{e+t}} \le \frac{3}{4}. \tag{3.20}$$

Combining (3.18)-(3.20) and applying (2.3), we obtain

$$p-1 \le \frac{tG''(x,t)}{G'(x,t)} \le p-\frac{1}{4} \quad \text{and} \quad p \le \frac{tG'(x,t)}{G(x,t)} \le p+1 \tag{3.21}$$

for every $t > 0$ and $x \in \Omega$. Therefore, applying (2.5) with $\alpha \in (0,1)$, we have

$$\alpha^{\frac{1}{p}}\varphi^{-1}(x,t) \le \varphi^{-1}(x,\alpha t) \le \alpha^{\frac{1}{p+1}}\varphi^{-1}(x,t) \tag{3.22}$$

for every $x \in \Omega$ and $t \in \mathbb{R}^+$.

For the fixed ball $B_R$, $G_m(\cdot)$ also satisfies (3.21) and we have

$$\frac{d}{dt}\left(G_m(t)^{\frac{1}{8p}}\right) = \frac{1}{8p} G_m(t)^{\frac{1}{8p}-1} G_m'(t)$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\left(G_m(t)^{\frac{1}{8p}}\right) &= \frac{1}{8p} G_m(t)^{\frac{1}{8p}-2} G_m'(t)^2 \left(\frac{1}{8p} - 1 + \frac{G_m(t)G_m''(t)}{G_m'(t)^2}\right)\\ &\le \frac{1}{8p} G_m(t)^{\frac{1}{8p}-2} G_m'(t)^2 \left(\frac{1}{8p} - 1 + \frac{p-\frac{1}{4}}{p}\right) \le 0.\end{aligned}$$

Therefore, $G_m(\cdot)^{\frac{1}{8p}}$ is concave and then it follows from Jensen's inequality that

$$\left(\fint_{B_R} G_m(|f|)^{\frac{1}{8p}} dx\right)^{8p} \le G_m\left(\fint_{B_R} |f| dx\right) \tag{3.23}$$

for every $f \in L^1(B_R)$.

Recall the estimates in Section 2.3 which will be used frequently later in. For every $\alpha \in (0, \infty)$ and $t \ge 1$, $\log t \le \frac{1}{\alpha} t^\alpha$. Then we have

$$\begin{aligned} g_M(t) &\le g_m(t) + \omega_{a(\cdot)}(R) \log(e+t) t^{p-1}\\ &\le g_m(t) + (e+t)^{\omega_{a(\cdot)}(R)} t^{p-1} \lesssim g_m(t) + t^{p-1+\omega_{a(\cdot)}(R)}. \end{aligned} \tag{3.24}$$

We now set a constant

$$\mathcal{M}_{R,x_0} = g_0^{-1}\left(\frac{|\mu|(B_R(x_0))}{R^{n-1}}\right) \ge 0.$$

If there is nothing to be confused, we write $\mathcal{M} = \mathcal{M}_{R,x_0}$. By (3.24) and (3.11), we see

$$\begin{aligned} g_0\left(g_m^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right)\right) &\lesssim \frac{|\mu|(B_R)}{R^{n-1}} + g_m^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right)^{p-1+\omega(R)}\\ &\lesssim \frac{|\mu|(B_R)}{R^{n-1}} + \left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{1+\frac{\omega(R)}{p-1}} \lesssim \frac{|\mu|(B_R)}{R^{n-1}}. \end{aligned}$$

Since $g_0^{-1}(t) \leq g_m^{-1}(t)$ for $t \geq 0$, we see

$$\mathcal{M} \approx g_m^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right). \tag{3.25}$$

The next lemma shows a comparison estimate between (3.1) and (3.14) under (GPT). To do this, we reduce these equations to certain problems with general growth in the proof of Lemma 3.2.1 to apply known estimates in [10].

**Lemma 3.2.1.** *Let $u \in W^{1,G}(\Omega)$ be a weak solution to* (3.1) *and $w \in W^{1,G}(B_R)$ be the weak solution to* (3.14)*. Then for any*

$$\chi \in \left[-1, \min\left\{\frac{1}{p-1}, \frac{p}{(p-1)(n-1)}\right\}\right), \tag{3.26}$$

*there exists a constant $c$ depending only on* `data` *and $\chi$ such that*

$$\fint_{B_R} g_{m,\chi}(|Du - Dw|)\,dx \leq c g_{m,\chi}(\mathcal{M}).$$

*Proof.* The main idea is to rescale (3.1), (3.14) and reduce them to general growth problems. Without loss of generality, one can assume $\mathcal{M} > 0$. If not, then the uniqueness of weak solutions implies $u = w$ in $B_R$ and there is nothing to prove. We rescale (3.1) and (3.14) as follows:

$$\bar{u}(x) := \frac{u(x_0 + Rx)}{\mathcal{M}R}, \quad \bar{w}(x) := \frac{w(x_0 + Rx)}{\mathcal{M}R},$$
$$\bar{A}(x, z) := \frac{A(x_0 + Rx, \mathcal{M}z)}{g_m(\mathcal{M})}, \quad \bar{\mu}(x) = R\frac{\mu(x_0 + Rx)}{g_m(\mathcal{M})}$$

and

$$\bar{g}(x, t) := \frac{g(x_0 + Rx, \mathcal{M}t)}{g_m(\mathcal{M})},$$

for $x \in B_1(0)$ and $z \in \mathbb{R}^n$. It then follows from (3.25) that $|\bar{\mu}|(B_1(0)) \approx 1$. We further note that

$$\langle \partial_z \bar{A}(x, z)\lambda, \lambda\rangle \geq \nu \frac{\bar{g}(x, |z|)}{|z|}|\lambda|^2$$

and

$$(p-1)\bar{g}(x,t) \le t\frac{d}{dt}\bar{g}(x,t) \le p\bar{g}(x,t)$$

for all $x \in B_1(0)$ and $t > 0$. Subtracting (3.14) from (3.1), we discover

$$-\operatorname{div}\left[\bar{A}(x,D\bar{u}) - \bar{A}(x,D\bar{w})\right] = \bar{\mu}(x) \qquad \text{in } B_1(0). \tag{3.27}$$

Based on Lemma 2.2.1 and Lemma 2.2.2, we consider two cases $p \le n$ and $p > n$. Let us first consider the case $p \le n$ which implies the second condition in (2.7). The first condition in (2.7) holds for every $p < n$, excluding $p = n$. For this reason, we define

$$\mathfrak{f}(t) := \begin{cases} 0 & t = 0, \\ \bar{g}_{m,\chi}(1)t & \text{for } t \in (0,1), \\ \bar{g}_{m,\chi}(t) & \text{for } t \in [1,\infty). \end{cases}$$

By testing (3.27) with

$$\begin{aligned} \varphi &= T_k\left(\frac{\bar{u}-\bar{w}}{c_n\left(\displaystyle\int_{B_1}\mathfrak{f}(|D\bar{u}-D\bar{w}|)\,dx\right)^{\frac{1}{n}}}\right) \\ &=: T_k\left(\frac{\bar{u}-\bar{w}}{c_n\mathcal{F}}\right) \in W_0^{1,G}(B_1)\cap L^\infty(B_1) \end{aligned}$$

and using (2.18) and (2.19), we find

$$\begin{aligned} \frac{1}{c_n\mathcal{F}}\int_{C_k}\bar{G}_m(|D\bar{u}-D\bar{w}|)\,dx &\lesssim \int_{B_1}\langle \bar{A}(x,D\bar{u})-\bar{A}(x,D\bar{w}), D\varphi\rangle\,dx \\ &\le \int_{B_1} kd|\bar{\mu}| \lesssim k, \end{aligned} \tag{3.28}$$

where $c_n$ is the constant given in Lemma 2.2.1,

$$\bar{G}_m(t) := \int_0^t \bar{g}_m(s)ds \quad \text{and} \quad C_k := \left\{x \in B_1 : \frac{|\bar{u}-\bar{w}|}{c_n\mathcal{F}} \le k\right\}.$$

We have used $|\bar{\mu}|(B_1) \approx 1$ in (3.28). Now using

$$\varphi := \mathfrak{T}_k\left(\frac{\bar{u}-\bar{w}}{c_n\mathcal{F}}\right) \in W_0^{1,G}(B_1) \cap L^\infty(B_1),$$

as a test function to (3.27), we obtain

$$\int_{D_k} \bar{G}_m(|D\bar{u} - D\bar{w}|)\,dx \lesssim \mathcal{F}, \tag{3.29}$$

where

$$D_k := \left\{x \in B_1 : k < \frac{|\bar{u}-\bar{w}|}{c_n\mathcal{F}} \le k+1\right\}.$$

The estimates (3.28) and (3.29) corresponds to [10, (5.17) in the proof of Lemma 5.1]. Once (3.28) and (3.29) are obtained with $\bar{g}_m(1) = 1$, then we discover

$$\fint_{B_1} \bar{g}_{m,\chi}(|D\bar{u} - D\bar{w}|)\,dx \le c \tag{3.30}$$

by following the calculations after [10, (5.17) in *Step 2.1* of Lemma 5.1], where $c$ depends only on `data` and $\chi$.

We now assume $p > n$. In this case, $u$ and $v$ are locally bounded in $B_R$ by Lemma 2.2.2. Thus we use $\bar{u} - \bar{w} \in W_0^{1,\bar{G}}(B_1) \cap L^\infty(B_1)$ as a test function in (3.27) to see

$$\begin{aligned}\int_{B_1} \bar{G}_m(|D\bar{u} - D\bar{w}|)\,dx &\le c\int_{B_1} (\bar{u}-\bar{w})d\mu \le c\|\bar{u}-\bar{w}\|_{L^\infty(B_1)}\\ &\le c\|D\bar{u} - D\bar{w}\|_{L^{\bar{G}_m}(B_1)}.\end{aligned}$$

Following the calculations [10, *Step 2.2* in Lemma 5.1], we obtain

$$\fint_{B_1} \bar{g}_{m,\chi}(|D\bar{u} - D\bar{w}|)\,dx \le c, \tag{3.31}$$

where $c$ depends only on `data` and $\chi$. By (3.30) and (3.31), we conclude that

$$\frac{1}{g_{m,\chi}(\mathcal{M})}\fint_{B_R} g_{m,\chi}(|Du - Dw|)\,dx = \fint_{B_1} \bar{g}_{m,\chi}(|D\bar{u} - D\bar{w}|)\,dx \le c.$$

□

Applying Lemma 3.2.1 with $\chi = -1$ and (3.25), we see

$$\fint_{B_R} |Du - Dw|\, dx \le c_1 g_0^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right), \tag{3.32}$$

where $c_1 = c_1(\texttt{data})$.

To proceed further, we define a function $h_{m,\chi} : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h_{m,\chi}(t) := \left[\frac{g_m(t)}{t}\right]^{1+\chi} = \frac{g_{m,\chi}(t)}{t}.$$

**Corollary 3.2.2.** *Let $u \in W^{1,G}(\Omega)$ be a weak solution to* (3.1) *and $w \in W^{1,G}(B_R)$ be the weak solution to* (3.14)*. Then for every $\chi$ satisfying (3.26), there exists a constant $c$ depending on* $\texttt{data}$ *and $\chi$ such that*

$$\fint_{B_R} h_{m,\chi}(|Du - Dw|)\, dx \le c h_{m,\chi}(\mathcal{M}).$$

*Proof.* We refer to the proof of [10, Corollary 5.2]. □

In the proof of next lemma, we follows similar procedures in that of Lemma 3.2.1.

**Lemma 3.2.3.** *Let $u \in W^{1,G}(\Omega)$ be a weak solution to* (3.1) *and $w \in W^{1,G}(B_R)$ be the weak solution to* (3.14)*. Then for every $\xi \in \left[1, \min\left\{\frac{p+1}{p}, \frac{n}{n-1}\right\}\right)$, there exists a constant $c$ depending on* $\texttt{data}$ *and $\xi$ such that*

$$\fint_{B_R} g_m(|Du - Dw|)^{\xi}\, dx \le c\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\xi}.$$

*Proof.* As in the proof of Lemma 3.2.1, we assume $\mathcal{M} > 0$ and we use the same scaling in there. We begin with the case $p \le n$. We define

$$\mathfrak{f}(t) := \begin{cases} 0 & t = 0, \\ g_m(1)^{\xi} t & \text{for } t \in (0,1), \\ g_m(t)^{\xi} & \text{for } t \in [1,\infty) \end{cases} \quad \text{and} \quad \mathcal{F} = \left(\int_{B_1} \mathfrak{f}(|D\bar{u} - D\bar{w}|)\, dx\right)^{\frac{1}{n}}.$$

We test (3.27) with

$$\varphi = T_k\left(\frac{\bar{u}-\bar{w}}{c_n\mathcal{F}}\right) \in W_0^{1,G}(B_1)\cap L^\infty(B_1)$$

to see

$$\int_{C_k} \bar{G}_m(|D\bar{u}-D\bar{w}|)\,dx \lesssim k\mathcal{F}, \quad \text{where} \quad C_k := \left\{x\in B_1 : \frac{|\bar{u}-\bar{w}|}{c_n\mathcal{F}} \le k\right\}.$$

Here, we have used (2.18) and (2.19). Similarly, take

$$\varphi = \mathfrak{T}_k\left(\frac{\bar{u}-\bar{w}}{c_n\mathcal{F}}\right) \in W_0^{1,G}(B_1)\cap L^\infty(B_1)$$

as a test function to (3.27) to see

$$\int_{D_k} \bar{G}_m(|D\bar{u}-D\bar{w}|)\,dx \lesssim \mathcal{F}, \quad \text{where} \quad D_k := \left\{x\in B_1 : k < \frac{|\bar{u}-\bar{w}|}{c_n\mathcal{F}} \le k+1\right\}.$$

Then by the same reasoning as in the proof of [10, Lemma 5.3], we discover

$$\fint_{B_1} \bar{g}_m(|D\bar{u}-D\bar{w}|)^\xi\,dx \le c(\texttt{data},\xi).$$

On the other hand for case $p>n$, we make the similar estimate as in Lemma 3.2.1 to discover

$$\fint_{B_1} \bar{g}_m(|D\bar{u}-D\bar{w}|)^\xi\,dx \le c(\texttt{data},\xi).$$

Consequently

$$\frac{1}{g_m(\mathcal{M})^\xi}\fint_{B_R} g_m(|Du-Dw|)^\xi dx = \fint_{B_1} \bar{g}_m(|D\bar{u}-D\bar{w}|)^\xi dx \le c.$$

□

We remark that Lemma 3.2.1, Lemma 3.2.3 and Corollary 3.2.2 are the natural extensions of [77, Lemma 2] to the setting of Musielak-Orlicz spaces. Next lemma shows a kind of weighted comparison estimate.

**Lemma 3.2.4.** *Let $u \in W^{1,G}(\Omega)$ be a weak solution to (3.1) and $w \in W^{1,G}(B_R)$ be the weak solution to (3.14). Then for any $\alpha > 0$ and $\xi > 1$, there exits a constant $c$ depending only on* `data` *such that*

$$\int_{B_R} \frac{|V(x,Du)-V(x,Dw)|^2}{(\alpha+|u-w|)^\xi}\,dx \le c\frac{\alpha^{1-\xi}}{\xi-1}|\mu|(B_R).$$

*Proof.* Testing $\eta^\pm := \alpha^{1-\xi} - (\alpha + (u-w)_\pm)^{1-\xi} \in W^{1,G}(B_R) \cap L^\infty(B_R)$ to (3.1) and (3.14), we discover

$$\begin{aligned} |I_\pm| &:= \left|(\xi-1)\int_{B_R} \gamma(x)\frac{\langle A(x,Du)-A(x,Dw), D(u-w)_\pm\rangle}{(\alpha+(u-w)_\pm)^\xi}\,dx\right| \\ &= \left|\int_{B_R}\gamma(x)\eta^\pm d\mu\right| \le L\alpha^{1-\xi}|\mu|(B_R). \end{aligned}$$

Then (2.18) imply the following inequality:

$$(\xi-1)\int_{B_R} \frac{|V(x,Du)-V(x,Dw)|^2}{(\alpha+|u-w|)^\xi}\,dx \le c(|I_+|+|I_-|) \le c\alpha^{1-\xi}|\mu|(B_R).$$

□

### 3.2.2 Basic comparison estimates for (GPX)

Through this subsection, we assume that $A(\cdot)$ satisfies (GPX).

In this subsection, we derive comparison estimates for (GPX), which are analogous results to lemmas in Subsection 3.2.1. However, instead of using scaling and normalization, we use Lemma 3.2.5 to prove Lemma 3.2.6.

For simplicity, we denote

$$p_0 = p(x_0), \quad p_1 := \inf_{B_R(x_0)} p(x) \quad \text{and} \quad p_2 := \sup_{B_R(x_0)} p(x). \tag{3.33}$$

As usual, all given balls are assumed to be centered at $x_0$, unless otherwise stated.

**Lemma 3.2.5.** *Given a weak solution $u \in W^{1,p(\cdot)}(\Omega)$ to (3.1), $w \in W^{1,p(\cdot)}(B_R)$*

*be the weak solution to* (3.14). *Then*

$$\int_{B_R} \frac{|V(x,Du)-V(x,Dw)|^2}{(h+|u-w|)^\xi}\,dx \le c\frac{h^{1-\xi}}{\xi-1}|\mu|(B_R) \tag{3.34}$$

*holds whenever* $h>0$ *and* $\xi>1$, *where* $c=c(\boldsymbol{data})\ge 1$.

*Proof.* Test $\eta^\pm := h^{1-\xi}-(h+(u-w)_\pm)^{1-\xi} \in W_0^{1,p(\cdot)}(B_R)\cap L^\infty(B_R)$ to (3.1) under (GPX). Then we have

$$\begin{aligned} |I_\pm| :=& (\xi-1)\left|\int_{B_R}\gamma(x)\frac{\langle a(x,Du)-a(x,Dw), D(u-w)_\pm\rangle}{(h+(u-w)_\pm)^\xi}\,dx\right| \\ =& \left|\int_{B_R}\gamma(x)\eta_\pm d\mu\right| \le Lh^{1-\xi}|\mu|(B_R). \end{aligned}$$

Now (2.18) and (2.18) imply the following inequality:

$$(\xi-1)\int_{B_R}\frac{|V(x,Du)-V(x,Dw)|^2}{(h+|u-w|)^\xi}\,dx \le c(|I_+|+|I_-|) \le cLh^{1-\xi}|\mu|(B_R).$$

□

From now on, we assume

$$0<R\le R_1, \tag{3.35}$$

where $R_1$ is the constant given in (3.10).

**Lemma 3.2.6.** *Let* $u\in W^{1,p(\cdot)}(\Omega)$ *be a weak solution to* (3.1) *under* (GPX) *and* $w\in W^{1,p(\cdot)}(B_R)$ *be the weak solution to* (3.14). *Then we have*

$$⨍_{B_R}|Du-Dw|^q\,dx \le c\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]+R^{p_0-1}\right)^{\frac{q}{p_0-1}}, \tag{3.36}$$

*whenever*

$$1\le q\le q_0 \quad \textit{for } q_0 := \min\left\{p_1-\frac{1}{4}, p_1-1+\frac{3(p_1-1)}{4(n-1)}\right\}. \tag{3.37}$$

*Proof.* Once we obtain (3.36) with $q=q_0$, it then follows from Hölder's inequality that

$$\fint_{B_R} |Du - Dw|^q \, dx \le \left( \fint_{B_R} |Du - Dw|^{q_0} dx \right)^{\frac{q}{q_0}}$$
$$\le c \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] + R^{p_0 - 1} \right)^{\frac{q}{p_0 - 1}}$$

for every $1 \le q \le q_0$. Therefore we shall prove for $q = q_0$. We first consider the case $p_1 \le n$. To apply Sobolev embedding theorem and Lemma 3.2.5, we choose $\xi = \frac{n(p_1 - q_0)}{n - q_0} \le n < \infty$, satisfying $\frac{\xi q_0}{(p_1 - q_0)} = q_0^*$ with the Sobolev conjugate $q_0^*$ of $q_0$, and $\xi \ge \frac{4n^2 - 7n}{4(n-1)^2 - 3} > 1$. We set

$$h = R^{\frac{\alpha + q_0}{q_0}} + R \left( \fint_{B_R} |Du - Dw|^{q_0} \, dx \right)^{\frac{1}{q_0}} > 0 \text{ with } \alpha = \frac{q_0(p_0 - 1)}{p_1 - 1}. \quad (3.38)$$

Applying (2.19), (2.14) and Hölder's inequality, we obtain

$$\begin{aligned}
&\fint_{B_R} |Du - Dw|^{q_0} \, dx \\
&= \fint_{B_R} \frac{|Du - Dw|^{q_0}}{(h + |u - w|)^{\frac{\xi q_0}{p(x)}}} (h + |u - w|)^{\frac{\xi q_0}{p(x)}} \, dx \\
&\le c \fint_{B_R} \frac{|V(x, Du) - V(x, Dw)|^{\frac{2q_0}{p(x)}}}{(h + |u - w|)^{\frac{\xi q_0}{p(x)}}} (h + |u - w|)^{\frac{\xi q_0}{p(x)}} \, dx \\
&\le c \left( \fint_{B_R} \frac{|V(x, Du) - V(x, Dw)|^{\frac{2p_1}{p(x)}}}{(h + |u - w|)^{\frac{\xi p_1}{p(x)}}} \, dx \right)^{\frac{q_0}{p_1}} \\
&\qquad \cdot \left( \fint_{B_R} (h + |u - w|)^{\frac{\xi q_0 p_1}{(p_1 - q_0) p(x)}} \, dx \right)^{\frac{p_1 - q_0}{p_1}} \\
&\le c \left( \fint_{B_R} \frac{|V(x, Du) - V(x, Dw)|^2}{(h + |u - w|)^{\xi}} \, dx + R^{\frac{\alpha(p_1 - \xi)}{q_0}} \right)^{\frac{q_0}{p_1}} \\
&\qquad \cdot \left( \fint_{B_R} \left( |u - w|^{\frac{\xi q_0}{p_1 - q_0}} + h^{\frac{\xi q_0}{p_1 - q_0}} + R^{\frac{\xi(\alpha + q_0)}{p_1 - q_0}} \right) dx \right)^{\frac{p_1 - q_0}{p_1}}. \qquad (3.39)
\end{aligned}$$

With the help of Sobolev embedding and (3.38), we have

$$\begin{aligned}
&\fint_{B_R}\left(|u-w|^{\frac{\xi q_0}{p_1-q_0}}+h^{\frac{\xi q_0}{p_1-q_0}}+R^{\frac{\xi(\alpha+q_0)}{p_1-q_0}}\right)dx\\
&\le c\left(R^{\frac{\xi q_0}{p_1}}\left(\fint_{B_R}|Du-Dw|^{q_0}\,dx\right)^{\frac{\xi}{p_1}}+h^{\frac{\xi q_0}{p_1}}+R^{\frac{\xi(\alpha+q_0)}{p_1}}\right)^{\frac{p_1}{p_1-q_0}}\\
&\le ch^{\frac{\xi q_0}{p_1-q_0}}, \qquad (3.40)
\end{aligned}$$

where $c$ depends only on $n, p_0, p_1$ and $q_0$. By enlarging the constant $c$, the dependence on $p_0, p_1$ and $q_0$ can be replaced by $\gamma_2$ and $n$, since $q_0 \le p_1 - \frac{1}{4} \le n - \frac{1}{4}$. Combining (3.39) with (3.40) and applying (3.34), then we find that

$$\begin{aligned}
\fint_{B_R}|Du-Dw|^{q_0}\,dx &\le ch^{\frac{\xi q_0}{p_1}}\left(\fint_{B_R}\frac{|V(x,Du)-V(x,Dw)|^2}{(h+|u-w|)^{\xi}}\,dx+R^{\frac{\alpha(p_1-\xi)}{q_0}}\right)^{\frac{q_0}{p_1}}\\
&\le ch^{\frac{\xi q_0}{p_1}}\left(h^{\frac{(1-\xi)q_0}{p_1}}\left[\frac{|\mu|(B_R)|}{R^n}\right]^{\frac{q_0}{p_1}}+R^{\frac{\alpha(p_1-\xi)}{p_1}}\right)\\
&\le c\left(\frac{h}{R}\right)^{\frac{q_0}{p_1}}\left[\frac{|\mu(B_R)|}{R^{n-1}}\right]^{\frac{q_0}{p_1}}+c\left(\frac{h}{R}\right)^{\frac{\xi q_0}{p_1}}R^{\frac{\alpha(p_1-\xi)}{p_1}}\\
&\le \frac{1}{2^{q_0}}\left(\frac{h}{R}\right)^{q_0}+c\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{q_0}{p_1-1}}+R^{\alpha}\right)\\
&\le \frac{1}{2}\fint_{B_R}|Du-Dw|^{q_0}\,dx+c\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]+R^{p_0-1}\right)^{\frac{q_0}{p_1-1}}.
\end{aligned}$$

On the other hand, we observe that for some constant $c$ depending only on $\gamma_2, n$ and $L$, it holds that

$$\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{q_0}{p_1-1}}=\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{q_0}{p_0-1}+\frac{q_0(p_0-p_1)}{(p_1-1)(p_0-1)}}\le c\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{q_0}{p_0-1}},$$

as $0 < R \le \min\{R_1, (M + |\mu|(\Omega) + 1)^{-1}\}$. This completes the proof for the case $p_1 < n$.

Next, we consider the case $n < p_1$, which implies $q_0 = p_1 - 1/4$. Let

$\xi = n\left(\frac{4p_1-2}{4p_1-1}\right)$, then we have

$$1 < \frac{6n}{7} \leq \xi \leq \min\left\{n\left(\frac{4\gamma_2-2}{4\gamma_2-1}\right), q_0\right\} \quad \text{and} \quad \frac{\xi q_0}{p_1-q_0} = \xi^*.$$

Then (3.40) holds true and the proof of Lemma 3.2.6 is finished. □

According to (3.10) and the choice of $q_0$, we have

$$q_0 = \min\left\{p_1 - \frac{1}{2}, p_1 - 1 + \frac{3(p_1-1)}{4(n-1)}\right\} \geq p_0 - 1.$$

Hence, we can always take $q = p_0 - 1$ to the Lemma 3.2.6.

### 3.2.3 Higher integrability and further comparison estimates for (GPT)

Through this subsection, we assume that $A(\cdot)$ satisfies (GPT).

**Lemma 3.2.7.** *Let $w \in W^{1,G}(B_R)$ be the weak solution to* (3.14)*, $0 < \rho \leq R$, $\theta \in (0,1)$ and $q \in (0,1]$. Then there exists a constant $\sigma = \sigma(\mathtt{data})$ such that*

$$\left(\fint_{B_{\theta\rho}(z)} G(x,|Dw|)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} \leq c\left(\fint_{B_\rho(z)} G(x,|Dw|)^q\,dx\right)^{\frac{1}{q}} \tag{3.41}$$

*for some constant $c$ depending only on $\mathtt{data}, \theta$ and $q$, whenever $B_\rho(z) \subset B_R$.*

*Proof.* One can find a higher integrability results for (3.14) in [12, Theorem 4.2], where the constants in the estimates depend on $L^p$ norm of $Dw$. To consider measure data problems, we need to eliminate such a dependence on the constants, otherwise no limiting process for *SOLA* can be made here. We start with employing [12, Lemma 4.1], a Caccioppoli type estimates:

$$\fint_{B_{r/2}(y)} G(x,|Dw|)\,dx \lesssim \fint_{B_r(y)} G\left(x, \frac{|w-(w)_{B_r(y)}|}{r}\right)dx, \tag{3.42}$$

where $c = c(\mathtt{data})$.

Applying (3.24) and the Poincaré inequality for Orlicz spaces in [44, Theorem 7 and Remark 8], there exists a constant $d_1 = d_1(p)$ such that

$$\begin{aligned}&\fint_{B_r(y)} G\left(x, \frac{|w-(w)_{B_r(y)}|}{r}\right) dx \\ &\lesssim \left(\fint_{B_r(y)} G_m\left(|Dw|\right)^{\frac{1}{d_1}} dx\right)^{d_1} + \fint_{B_r(y)} \left|\frac{w-(w)_{B_r(y)}}{r}\right|^{p+\omega_{a(\cdot)}(r)} dx. \qquad (3.43)\end{aligned}$$

We apply (3.32) and (3.11) to see

$$\begin{aligned}&\left(\fint_{B_r(y)} |Dw|\,dx\right)^{\omega_{a(\cdot)}(r)} \\ &\lesssim \left(\fint_{B_R} |Dw|\,dx\right)^{\omega_{a(\cdot)}(r)} \lesssim \left(\fint_{B_R} |Dw|\,dx\right)^{\omega_{a(\cdot)}(R)} + 1 \\ &\lesssim g_0^{-1}\left(\frac{|\mu|(B_R)}{R^{n-1}}\right)^{\omega_{a(\cdot)}(R)} + \left(\fint_{B_R} |Du|\,dx\right)^{\omega_{a(\cdot)}(R)} + 1 \le c. \qquad (3.44)\end{aligned}$$

From the assumption (3.10), there exist a constant $d_2 > 1$ depending only on $n, p$ such that

$$\frac{np}{n-(n-1)\omega_{a(\cdot)}(r)} \le \frac{np}{d_2 n - p}, \qquad (3.45)$$

when $p < n$. On the other hand, if $p \ge n$, then we take $d_2 = 2$. By the Sobolev-Poincaré inequality, we have

$$\begin{aligned}&\fint_{B_r(y)} \left|\frac{w-(w)_{B_r(y)}}{r}\right|^{p+\omega_{a(\cdot)}(r)} dx \\ &\le \left(\fint_{B_r(y)} \left|\frac{w-(w)_{B_r(y)}}{r}\right|^{\frac{n}{n-1}} dx\right)^{\frac{(n-1)\omega_{a(\cdot)}(r)}{n}} \\ &\quad\cdot \left(\fint_{B_r(y)} \left|\frac{w-(w)_{B_r(y)}}{r}\right|^{\frac{np}{n-(n-1)\omega_{a(\cdot)}(r)}} dx\right)^{\frac{n-(n-1)\omega_{a(\cdot)}(r)}{n}} \\ &\le \left(\fint_{B_r(y)} |Dw|\,dx\right)^{\omega_{a(\cdot)}(r)} \left(\fint_{B_r(y)} |Dw|^{\frac{p}{d_2}}\,dx\right)^{d_2}\end{aligned}$$

$$\lesssim \left( \fint_{B_r(y)} G(x, |Dw|)^{\frac{1}{d_2}} \, dx \right)^{d_2}. \tag{3.46}$$

Combining (3.42), (3.43) with (3.46), we have

$$\fint_{B_{r/2}(y)} G(x, Dw) \, dx \lesssim \left( \fint_{B_r(y)} G(x, |Dw|)^{\frac{1}{d}} dx \right)^{d},$$

where $d = \min\{d_1, d_2\}$. Applying Gehring's Lemma, [63, Section 6.4], we find the conclusion of the lemma. □

Note that the constant $c$ in Lemma 3.2.7 goes to infinity, when $\theta \to 1$. Theorem 3.2.7 implies that $v \in W^{1,p(1+\sigma)}(B_{3R/4}) \subset W^{1,G(x_1)}(B_{3R/4})$ for any $x_1 \in B_R$. Hence, we see that $v - w \in W_0^{1,G(x_1)}(B_{\tilde{R}/2}(x_1))$, where $w$ is the solution to (3.15). Thanks to the higher integrability result of $v$, one can obtain a comparison estimate between (3.14) and (3.15).

**Lemma 3.2.8.** *Let $w \in W^{1,G}(B_R)$ be the weak solution to* (3.14) *and $v \in W^{1,G(x_1)}(B_{\tilde{R}/2}(x_1))$ be the weak solution to* (3.15)*. Then we have*

$$\fint_{B_{\tilde{R}/2}(x_1)} |V(x_1, Dw) - V(x_1, Dv)|^2 \, dx \le c\omega(\tilde{R})^2 \fint_{B_{5\tilde{R}/8}(x_1)} G(x, |Dw|) \, dx.$$

*Proof.* In this proof, we denote $B = B_{\tilde{R}}(x_1)$ for simplicity. We test $v - w$ to both (3.14) and (3.15), to discover that

$$\begin{aligned}
&\fint_{\frac{1}{2}B} \frac{g(x_1, |Dw| + |Dv|)}{|Dw| + |Dv|} |Dw - Dv|^2 \, dx \\
&\lesssim \fint_{\frac{1}{2}B} \gamma(x_1) \langle A(x_1, Dw) - A(x_1, Dv), Dw - Dv \rangle \, dx \\
&= \fint_{\frac{1}{2}B} (\gamma(x_1) - \gamma(x)) \langle A(x_1, Dw), Dw - Dv \rangle \, dx \\
&\quad + \fint_{\frac{1}{2}B} \gamma(x) \langle A(x_1, Dw) - A(x, Dw), Dw - Dv \rangle \, dx =: I_1 + I_2.
\end{aligned} \tag{3.47}$$

Here, we have used (2.18). Using (GPT) and (3.17), we estimate $I_1$ as

$$
\begin{aligned}
I_1 &\le c \fint_{\frac{1}{2}B} \omega_{\gamma(\cdot)}(\tilde{R}) g(x_1, |Dw|)|Dw - Dv|\, dx \\
&= c \fint_{\frac{1}{2}B} \omega_{\gamma(\cdot)}(\tilde{R}) G(x_1, |Dw|)^{\frac{1}{2}} \left( \frac{g(x_1, |Dw|)}{|Dw|} \right)^{\frac{1}{2}} |Dw - Dv|\, dx \\
&\le \epsilon \fint_{\frac{1}{2}B} \frac{g(x_1, |Dw| + |Dv|)}{|Dw| + |Dv|} |Dw - Dv|^2\, dx \\
&\quad + c(\epsilon) \omega_{\gamma(\cdot)}(\tilde{R})^2 \fint_{\frac{1}{2}B} G(x_1, |Dw|)\, dx
\end{aligned} \tag{3.48}
$$

for some $\epsilon > 0$. By a direct calculation, we see

$$
\begin{aligned}
\fint_{\frac{1}{2}B} G(x_1, |Dw|)\, dx \le \fint_{\frac{1}{2}B} G(x, |Dw|)\, dx \\
+ \omega_{a(\cdot)}(\tilde{R}) \fint_{\frac{1}{2}B} |Dw|^p \log(e + |Dw|)\, dx
\end{aligned} \tag{3.49}
$$

Similarly, we estimate $I_2$ as

$$
\begin{aligned}
I_2 &\le c\omega_{a(\cdot)}(\tilde{R}) \fint_{\frac{1}{2}B} \log(e + |Dw|)|Dw|^{p-1} |Dw - Dv|\, dx \\
&\le \epsilon \fint_{\frac{1}{2}B} |Dw|^{p-2} |Dw - Dv|^2\, dx \\
&\quad + c(\epsilon) \omega_{a(\cdot)}(\tilde{R})^2 \fint_{\frac{1}{2}B} |Dw|^p \log^2(e + |Dw|)\, dx \\
&\le \epsilon \fint_{\frac{1}{2}B} \frac{g(x_1, |Dw| + |Dv|)}{|Dw| + |Dv|} |Dw - Dv|^2\, dx \\
&\quad + c(\epsilon) \omega_{a(\cdot)}(\tilde{R})^2 \fint_{\frac{1}{2}B} |Dw|^p \log^2(e + |Dw|)\, dx.
\end{aligned} \tag{3.50}
$$

Combining (3.47)-(3.50) and taking $\epsilon$ small enough, we see

$$
\fint_{\frac{1}{2}B} \frac{g(x_1, |Dw| + |Dv|)}{|Dw| + |Dv|} |Dw - Dv|^2\, dx
$$

$$\begin{aligned}
&\lesssim \omega_{\gamma(\cdot)}(\tilde{R})^2 \fint_{\frac{1}{2}B} G(x,|Dw|)\,dx \\
&\quad + \omega_{a(\cdot)}(\tilde{R})^2 \underbrace{\fint_{\frac{1}{2}B} |Dw|^p \log^2(e+|Dw|)\,dx}_{I_3} \\
&\quad + \omega_{\gamma(\cdot)}(\tilde{R})^2 \omega_{a(\cdot)}(\tilde{R}) \underbrace{\fint_{\frac{1}{2}B} |Dw|^p \log(e+|Dw|)\,dx}_{I_4}. \qquad (3.51)
\end{aligned}$$

By (2.16), we see

$$\begin{aligned}
I_3 &\lesssim \fint_{\frac{1}{2}B} |Dw|^p \log^2\left(e+|Dw|^p\right)dx \\
&\lesssim \fint_{\frac{1}{2}B} |Dw|^p \log^2\left(e+\frac{|Dw|^p}{\|Dw\|^p_{L^p(\frac{1}{2}B)}}\right)dx \\
&\quad + \log^2\left(e+\|Dw\|^p_{L^p(\frac{1}{2}B)}\right)\fint_{\frac{1}{2}B}|Dw|^p dx =: I_{3,1}+I_{3,2}.
\end{aligned} \qquad (3.52)$$

Applying (2.15) and Lemma 3.2.7, we discover

$$\begin{aligned}
I_{3,1} &\lesssim \left(\fint_{\frac{1}{2}B} |Dw|^{p(1+\sigma)}dx\right)^{\frac{1}{1+\sigma}} \\
&\lesssim \left(\fint_{\frac{1}{2}B} G(x,|Dw|)^{1+\sigma}dx\right)^{\frac{1}{1+\sigma}} \lesssim \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx,
\end{aligned} \qquad (3.53)$$

where $\sigma$ is the constant defined in Lemma 3.2.7. We apply Lemma 3.2.7, (3.32) and (3.11) to discover

$$\begin{aligned}
\fint_{\frac{1}{2}B} |Dw|^p dx \le \fint_{\frac{1}{2}B} G(x,|Dw|)\,dx &\le \left(\fint_B G(x,|Dw|)^{\frac{1}{2p}}dx\right)^{2p} \\
&\lesssim \left(\fint_B |Dw| dx\right)^{2p} \lesssim \left(\fint_B |Dw-Du|\,dx + \fint_B |Du|\,dx\right)^{2p} \\
&\lesssim \frac{1}{\tilde{R}^{2p(n+1)}}.
\end{aligned}$$

Now we estimate $I_{3,2}$ as follows:

$$I_{3,2} \lesssim \log^2\left(\frac{1}{\tilde{R}}\right) \fint_{\frac{1}{2}B} |Dw|^p dx \lesssim \log^2\left(\frac{1}{\tilde{R}}\right) \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx. \tag{3.54}$$

Combining (3.52)-(3.54), we have

$$I_3 \lesssim \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx + \log^2\left(\frac{1}{\tilde{R}}\right) \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx \tag{3.55}$$

Similarly, one can obtain

$$I_4 \lesssim \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx + \log\left(\frac{1}{\tilde{R}}\right) \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx \tag{3.56}$$

Combining (3.51), (3.55) and (3.56), it follows from (2.19) that

$$\begin{aligned}
&\fint_{\frac{1}{2}B} \frac{g(x_1,|Dw|+|Dv|)}{|Dw|+|Dv|}|Dw-Dv|^2 dx \\
&\lesssim \omega_{\gamma(\cdot)}(\tilde{R})^2 \fint_{\frac{1}{2}B} G(x,|Dw|)\,dx + \omega_{a(\cdot)}(\tilde{R})^2 \log^2\left(\frac{1}{\tilde{R}}\right) \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx \\
&\quad + \omega_{\gamma(\cdot)}(\tilde{R})^2 \omega_{a(\cdot)}(\tilde{R}) \log\left(\frac{1}{\tilde{R}}\right) \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx \\
&\lesssim \omega_{\gamma(\cdot)}(\tilde{R})^2 \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx + \omega_{a(\cdot)}(\tilde{R})^2 \log^2\left(\frac{1}{\tilde{R}}\right) \fint_{\frac{5}{8}B} G(x,|Dw|)\,dx,
\end{aligned}$$

where we have used (3.10) for the last estimate above. □

**Remark 3.2.9.** To obtain Riesz potential estimates for $u$, we need to obtain $L^1$ comparison estimates from Lemma 3.2.8. In this remark, we denote $G_{x_1}^{-1}$ as the inverse function of $G(x_1,\cdot)$. In the light of Jensen's inequality, (2.19) and Lemma 3.2.8, we discover that for $B = B_{\tilde{R}}(x_1)$

$$\begin{aligned}
\fint_{\frac{1}{2}B} |Dw - Dv|\,dx &\lesssim G_{x_1}^{-1}\left(\fint_{\frac{1}{2}B} G(x_1, |Dw - Dv|)\,dx\right) \\
&\lesssim G_{x_1}^{-1}\left(\omega(\tilde{R})^2 \fint_{\frac{5}{8}B} G(x, |Dw|)\,dx\right) \\
&\lesssim G_{x_1}^{-1}\left(\omega(\tilde{R})^2 \left(\fint_{\frac{3}{4}B} G(x, |Dw|)^{\frac{1}{8p}}\,dx\right)^{8p}\right) \\
&\lesssim G_{x_1}^{-1}\left(\omega(\tilde{R})^2 \left(\fint_{\frac{3}{4}B} G_m(|Dw|)^{\frac{1}{8p}} + |Dw|^{\frac{p+\omega(\tilde{R})}{8p}}\,dx\right)^{8p}\right).
\end{aligned}$$

We have used Lemma 3.2.7 in the third estimate. It then follows from (3.23) and (3.44) that

$$\begin{aligned}
&\left(\fint_{\frac{3}{4}B} G_m(|Dw|)^{\frac{1}{8p}} + |Dw|^{\frac{p+\omega(\tilde{R})}{8p}}\,dx\right)^{8p} \\
&\lesssim G_m\left(\fint_{\frac{3}{4}B} |Dw|\,dx\right) + \left(\fint_{\frac{3}{4}B} |Dw|\,dx\right)^{p+\omega(\tilde{R})} \lesssim G\left(x_1, \fint_{\frac{3}{4}B} |Dw|\,dx\right).
\end{aligned}$$

Therefore, there exists a constant $c_2 = c_2(\texttt{data})$ such that

$$\begin{aligned}
\fint_{\frac{1}{2}B} |Dw - Dv|\,dx &\le cG_{x_1}^{-1}\left(\omega(\tilde{R})^2 G\left(x_1, \fint_{\frac{3}{4}B} |Dw|\,dx\right)\right) \\
&\le c_2\omega(\tilde{R})^{\frac{2}{p+1}} \fint_B |Dw|\,dx. \qquad (3.57)
\end{aligned}$$

Here, we have used (2.5) and (3.21).

### 3.2.4 Higher integrability and further comparison estimates for (GPX)

Through this subsection, we assume that $A(\cdot)$ satisfies (GPX).

Higher integrability result for (3.14) under (GPX) is already obtained in [21, Lemma 3.2]. By minor modifications with the choice of $R$ in (3.10), one can see that the constant $c$ in the following lemma does not depend on $M$ and $|\mu|(\Omega)$.

**Lemma 3.2.10.** *Let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution to (3.1) under (GPX) and $w \in W^{1,p(\cdot)}(B_R)$ be the weak solution to (3.14). Then for any $\theta \in (0,1)$, there exist constants $\sigma_0 = \sigma_0(\boldsymbol{data}) \in (0,1]$ and $c = c(\boldsymbol{data}, \theta) \geq 1$ such that $|Dw|^{p(\cdot)} \in L^{1+\sigma_0}_{loc}(B_R)$ with the estimate:*

$$\left[\fint_{B_{\theta\rho}(y)} (|Dw| + s)^{(1+\sigma)p(x)}\,dx\right]^{\frac{1}{1+\sigma}} \leq c \fint_{B_\rho(y)} (|Dw| + s + \rho)^{p(x)}\,dx,$$

*whenever $B_\rho(y) \subset B_R$.*

**Remark 3.2.11.** With the help of Lemma 3.2.6 with $q = 1$, (2.14) and [63, Remark 6.12], for $0 < \theta_1 < \theta_2 \leq 1$, we can deduce

$$\begin{aligned}
&\left(\fint_{B_{\theta_1 R}(y)} (|Dw| + s)^{p(x)}\,dx\right)^{\omega(2R)} \\
&\leq c\left(\fint_{B_R(y)} (|Dw| + s + R)^{\frac{p(x)}{p_2}}\,dx\right)^{p_2\omega(2R)} \\
&\leq c\left(\fint_{B_R(y)} (|Dw| + s + R)\,dx\right)^{p_2\omega(2R)} \\
&\leq c\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{p_0-1}} + \frac{M}{R^n} + s + R\right)^{p_2\omega(2R)} \leq c,
\end{aligned} \tag{3.58}$$

and

$$\begin{aligned}
&\fint_{B_{\theta_1 R}(y)} (|Dw| + s)^{p(x)}\,dx \\
&\leq c\left[\fint_{B_{\theta_2 R}} (|Dw| + s + R)^{\frac{p(x)}{p_2}}\,dx\right]^{p_1} \\
&\qquad \cdot \left[\fint_{B_{\theta_1 R}(y)} (|Dw| + s + R)^{p(x)}\,dx\right]^{\frac{p_2-p_1}{p_2}}
\end{aligned}$$

$$\leq c \left[ \fint_{B_{\theta_2 R}(y)} (|Dw| + s + R)\, dx \right]^{p_1}, \tag{3.59}$$

where $c = c(\mathtt{data}, \theta_1, \theta_2)$ is increasing to infinity as $\frac{\theta_2}{\theta_1} \to 0$.

**Remark 3.2.12.** In what follows, we assume that the number $R_1$ given in (3.12) further satisfies

$$\omega_{p(\cdot)}(\rho) \log\left(\frac{1}{\rho}\right) \leq \frac{\sigma_0}{100 n \gamma_2}. \tag{3.60}$$

It then follows from Lemma 3.2.10 that

$$p_2 \leq p_1 + \omega_{p(\cdot)}(2R) \leq (1+\sigma_0) p(x) \quad \text{for } x \in B_R,$$

and so

$$w \in W^{1,(1+\sigma_0)p(\cdot)}(B_{\tilde{R}/2}(x_1)) \subset W^{1,p_2}(B_{\tilde{R}/2}(x_1)) \subset W^{1,p(x_1)}(B_{\tilde{R}/2}(x_1)),$$

which ensures the existence of the weak solution $v \in W^{1,p(x_1)}(B_{\tilde{R}/2}(x_1))$ to the homogeneous problem (3.15) with $p(x_1)$ growth.

**Lemma 3.2.13.** *Let $w \in W^{1,p(\cdot)}(B_R(x_0))$ be the weak solution to* (3.14) *and $v \in W^{1,p(x_1)}(B_{\tilde{R}/2}(x_1))$ be the weak solution to* (3.15)*. Then we have*

$$\begin{aligned} &\fint_{B_{\tilde{R}/2}(x_1)} |V(x_1, Dw) - V(x_1, Dv)|^2\, dx \\ &\qquad \leq c\omega(\tilde{R})^2 \left[ \fint_{B_{5\tilde{R}/8}(x_1)} (|Dw| + s)^{p(x)}\, dx + \tilde{R}^{p(x_1)} \right]. \end{aligned} \tag{3.61}$$

*and*

$$\fint_{B_{\tilde{R}/2}(x_1)} |Dw - Dv|\, dx \leq c\omega(\tilde{R})^{\frac{2}{p_0}} \fint_{B_{3\tilde{R}/4}(x_1)} (|Dw| + s + \tilde{R})\, dx. \tag{3.62}$$

*Proof.* From the monotonicity property of $a(\cdot)$, (2.18), we have

$$\fint_{B_{\tilde{R}/2}(x_1)} \left(|Dw|^2 + |Dv|^2 + s^2\right)^{\frac{p(x_1)-2}{2}} |Dw - Dv|^2\, dx$$

$$
\begin{aligned}
&\leq c \fint_{B_{\tilde{R}/2}(x_1)} \gamma(x_1)\langle a(x_1, Dw) - a(x_1, Dv), Dw - Dv\rangle\, dx \\
&= c \fint_{B_{\tilde{R}/2}(x_1)} \langle \gamma(x_1) a(x_1, Dw) - \gamma(x_1) a(x, Dw), Dw - Dv\rangle\, dx \\
&\quad + c \fint_{B_{\tilde{R}/2}(x_1)} (\gamma(x_1) - \gamma(x))\langle a(x, Dw), Dw - Dv\rangle\, dx \\
&=: I_1 + I_2.
\end{aligned}
$$

For the estimate of $I_1$, we refer to [21, Lemma 3.4]:

$$
|I_1| \leq c \left[\omega_{p(\cdot)}\left(\tilde{R}\right) \log\left(\frac{1}{\tilde{R}}\right)\right]^2 \left[\fint_{B_{5\tilde{R}/8}(x_1)} (|Dw| + s)^{p(x)}\, dx + \tilde{R}^{p_0}\right].
$$

On the other hand, by (GPX) and Hölder's inequality, we deduce

$$
\begin{aligned}
|I_2| &\leq c\omega_{\gamma(\cdot)}(\tilde{R}) \fint_{B_{\tilde{R}/2}(x_1)} (|Dw|^2 + s^2)^{\frac{p(x)}{4}} \\
&\qquad \cdot (|Dw|^2 + |Dv|^2 + s^2)^{\frac{p(x)-2}{4}} |Dw - Dv|\, dx \\
&\leq \epsilon \fint_{B_{\tilde{R}/2}(x_1)} (|Dw|^2 + |Dv|^2 + s)^{\frac{p(x)-2}{2}} |Dw - Dv|^2\, dx \\
&\qquad + c(\epsilon) \left[\omega_{\gamma(\cdot)}\left(\tilde{R}\right)\right]^2 \fint_{B_{\tilde{R}/2}(x_1)} (|Dw| + s)^{p(x)}\, dx. \qquad (3.63)
\end{aligned}
$$

Absorbing the first term in right-hand side of (3.63), we obtain (3.61). In view of (3.59) with $\theta_1 = \frac{5}{8}, \theta_2 = \frac{3}{4}$,(3.61) and (2.14), we estimate as follows

$$
\begin{aligned}
&\fint_{B_{\tilde{R}/2}(x_1)} |Dw - Dv|\, dx \\
&\leq \left(\fint_{B_{\tilde{R}/2}(x_1)} |Dw - Dv|^{p(x_1)}\, dx\right)^{\frac{1}{p(x_1)}} \\
&\leq c \left(\fint_{B_{\tilde{R}/2}(x_1)} \left(|Dw|^2 + |Dv|^2 + s^2\right)^{\frac{p(x_1)-2}{2}} |Dw - Dv|^2\, dx\right)^{\frac{1}{p(x_1)}}
\end{aligned}
$$

$$\le c\left(\omega(\tilde R)^2 \fint_{B_{5\tilde R/8}(x_1)} (|Dw|+s)^{p(x)}\,dx + \tilde R^{p_0}\right)^{\frac{1}{p(x_1)}}$$
$$\le c\omega(\tilde R)^{\frac{2}{p(x_1)}} \fint_{B_{3\tilde R/4}(x_1)} (|Dw|+s+\tilde R)\,dx.$$

This completes the proof. □

### 3.2.5 Sequence of comparison estimates for (GPT)

Through this subsection, we assume that $A(\cdot)$ satisfies (GPT) and set $u$ as a weak solution to (3.1).

For some $\delta \in \left(0, \frac{1}{16}\right)$, we define a sequence of shrinking balls

$$B_i = B_{r_i}(x_0) \quad \text{and} \quad r_i = \delta^i R \quad (i = 0, 1, \cdots). \tag{3.64}$$

Let $v_i \in W^{1,G}(B_i)$ be the weak solution to (3.14) with $B_R$ replaced by $B_i$. Moreover, let $w_i \in W^{1,G(x_1)}(\frac{1}{2}B_i)$ be the weak solution to the following equation:

$$\begin{cases} -\mathrm{div}\,(A(x_1, Dv_i)) = 0 & \text{in } \ \frac{1}{2}B_{r_i}(x_1), \\ w_i = v_i & \text{on } \ \partial\frac{1}{2}B_{r_i}(x_1). \end{cases} \tag{3.65}$$

For the functions defined in (3.16), we denote $g_{m,i} = g_{m,r_i}$, $g_{M,i} = g_{M,r_i}$, $g_{m,i,\chi} = g_{m,r_i,\chi}$ and so on. Recall (3.25). For any $\chi \in (-1, \infty)$, we define functions $h_0, h_{m,i}, h_{M,i}, h_{0,\chi}, h_{m,i,\chi}, h_{M,i,\chi}, g_{0,\chi} : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h_0(t) = \frac{g_0(t)}{t}, \quad h_{m,i}(t) = \frac{g_{m,i}(t)}{t}, \quad h_{M,i}(t) = \frac{g_{M,i}(t)}{t}, \quad h_{m,i,\chi}(t) = \frac{g_{m,i,\chi}(t)}{t},$$
$$h_{M,i,\chi}(t) = \frac{g_{M,i,\chi}(t)}{t} \quad \text{and} \quad g_{0,\chi}(t) = \left(\frac{g_0(t)}{t}\right)^{1+\chi} t.$$

In addition, we write $\mathcal{M}_i = \mathcal{M}_{r_i,x_0}$ for each $i \in \mathbb{N}$.

**Lemma 3.2.14.** *Assume*

$$\mathcal{M}_{i-1} \le \lambda \qquad \text{and} \qquad \frac{\lambda}{H} \le |Dw_{i-1}| \le H\lambda \quad \text{in } B_i \tag{3.66}$$

*for a constant $H \ge 1$. Then there exists a constant $c_3 = c_3(\texttt{data}, \delta, H)$ such* *that*

$$\fint_{B_i} |Du - Dw_i| dx \le c_3 \frac{\lambda}{g_0(\lambda)} \left[ \frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}} \right].$$

*Proof.* We set

$$\eta := \frac{1}{4(p-1)(n+1)} \qquad \text{and} \qquad \xi := 1 + 2\eta.$$

By the assumption (3.66),

$$\begin{aligned}
\fint_{B_i} |Du - Dw_i| dx &\lesssim_H \fint_{B_i} \frac{h_{M,i,\eta}(|Dw_{i-1}|)}{h_{M,i,\eta}(\lambda)} |Du - Dw_i|\, dx \\
&\lesssim \fint_{B_i} \frac{h_{M,i,\eta}(|Dw_i - Dw_{i-1}|)}{h_{M,i,\eta}(\lambda)} |Du - Dw_i|\, dx \\
&\quad + \fint_{B_i} \frac{h_{M,i,\eta}(|Dw_i|)}{h_{M,i,\eta}(\lambda)} |Du - Dw_i| dx =: A_1 + A_2. \qquad (3.67)
\end{aligned}$$

Since $g_{M,i,\eta}$ is a Young function, we apply Young's inequality and (2.6) to see that

$$\begin{aligned}
&h_{M,i,\eta}(\lambda) A_1 \\
&\lesssim \fint_{B_i} \tilde{g}_{M,i,\eta} \left( \frac{g_{M,i,\eta}(|Dw_i - Dw_{i-1}|)}{|Dw_i - Dw_{i-1}|} \right) dx + \fint_{B_i} g_{M,i,\eta}(|Du - Dw_i|)\, dx \\
&\lesssim \fint_{B_i} g_{M,i,\eta}(|Dw_i - Dw_{i-1}|)\, dx + \fint_{B_i} g_{M,i,\eta}(|Du - Dw_i|)\, dx \\
&\lesssim \fint_{B_i} g_{M,i,\eta}(|Du - Dw_{i-1}|)\, dx + \fint_{B_i} g_{M,i,\eta}(|Du - Dw_i|)\, dx. \qquad (3.68)
\end{aligned}$$

Note that (3.10) implies $\frac{(p-2+\omega_{a(\cdot)}(r_{i-1}))(1+\eta)+1}{p-1} \le \min\left\{\frac{p+1}{p}, \frac{n}{n-1}\right\}$. By (3.24), Lemma 3.2.1, Lemma 3.2.3 and (3.11), we discover

$$\begin{aligned}
&\fint_{B_i} g_{M,i,\eta}(|Du - Dw_{i-1}|)\, dx \\
&\lesssim_\delta \fint_{B_{i-1}} g_{m,i-1,\eta}(|Du - Dw_{i-1}|)\, dx \\
&\quad + \fint_{B_{i-1}} |Du - Dw_{i-1}|^{(p-2+\omega_{a(\cdot)}(r_{i-1}))(1+\eta)+1}\, dx
\end{aligned}$$

$$
\begin{aligned}
&\lesssim g_{m,i-1,\eta}(\mathcal{M}_{i-1}) + \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^{\frac{(p-2+\omega_{a(\cdot)}(r_{i-1}))(1+\eta)+1}{p-1}} \\
&\lesssim g_{0,\eta}(\mathcal{M}_{i-1}) + \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^{\frac{(p-2)(1+\eta)+1}{p-1}} \\
&= g_{0,\eta}(\mathcal{M}_{i-1}) + g_0(\mathcal{M}_{i-1})^{1+\frac{p-2}{p-1}\eta}. \qquad (3.69)
\end{aligned}
$$

In the same manner, we find

$$
\begin{aligned}
\fint_{B_i} g_{M,i,\eta}(|Du - Dw_i|)\,dx &\lesssim g_{0,\eta}(\mathcal{M}_i) + g_0(\mathcal{M}_i)^{1+\frac{p-2}{p-1}\eta} \\
&\lesssim_\delta g_{0,\eta}(\mathcal{M}_{i-1}) + g_0(\mathcal{M}_{i-1})^{1+\frac{p-2}{p-1}\eta}. \qquad (3.70)
\end{aligned}
$$

Combining (3.68), (3.69) and (3.70), we apply (3.66) to discover that

$$
\begin{aligned}
A_1 &\lesssim_\delta \frac{1}{h_{0,\eta}(\lambda)}\left(g_{0,\eta}(\mathcal{M}_{i-1}) + g_0(\mathcal{M}_{i-1})^{1+\frac{p-2}{p-1}\eta}\right) \\
&= \frac{\lambda g_{0,\eta}(\mathcal{M}_{i-1})}{g_{0,\eta}(\lambda)} + \frac{\lambda^{1+\eta}}{g_0(\lambda)^{1+\eta}} g_0(\mathcal{M}_{i-1})^{1+\frac{p-2}{p-1}\eta} \lesssim \frac{\lambda}{g_0(\lambda)}\left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]. \qquad (3.71)
\end{aligned}
$$

In the last inequality, we have used

$$
\frac{\lambda^\eta}{g_0(\lambda)^\eta} g_0(\mathcal{M}_{i-1})^{\frac{p-2}{p-1}\eta} = \frac{\lambda^\eta}{g_0(\lambda)^{\frac{\eta}{p-1}}}\left(\frac{g_0(\mathcal{M}_{i-1})}{g_0(\lambda)}\right)^{\frac{p-2}{p-1}\eta} \le 1.
$$

Applying (3.24), we have

$$
\begin{aligned}
h_{M,i,\eta}(\lambda)A_2 &\lesssim \fint_{B_i} h_{m,i,\eta}(|Dw_i|)|Du - Dw_i|\,dx \\
&\quad + \fint_{B_i} |Dw_i|^{(p-2+\omega_{a(\cdot)}(r_i))(1+\eta)}|Du - Dw_i|\,dx \\
&=: B_1 + B_2. \qquad (3.72)
\end{aligned}
$$

For any $\alpha > 0$ to be chosen, Lemma 3.2.4 and (2.18) yield

$$
B_1 \lesssim \fint_{B_i} \left[\frac{h(x, |Dw_i| + |Du|)}{(\alpha + |u - v_i|)^\xi}|Du - Dw_i|^2\right]^{\frac{1}{2}}
$$

$$
\begin{aligned}
&\qquad \cdot \left[h_{m,i}(|Dw_i|)(\alpha + |u - v_i|)\right]^{\frac{\xi}{2}} \, dx \\
&\lesssim \left( \fint_{B_i} \frac{|V(x, Du) - V(x, Dw_i)|^2}{(\alpha + |u - v_i|)^{\xi}} dx \right)^{\frac{1}{2}} \\
&\qquad \cdot \left( \fint_{B_i} \left[h_{m,i}(|Dw_i|)(\alpha + |u - v_i|)\right]^{\xi} \, dx \right)^{\frac{1}{2}} \\
&\lesssim \left( \left[ \frac{|\mu|(B_i)}{r_i^n} \right] \alpha^{1-\xi} \right)^{\frac{1}{2}} \left( \fint_{B_i} \left[h_{m,i}(|Dw_i|)(\alpha + |u - v_i|)\right]^{\xi} \, dx \right)^{\frac{1}{2}}. \qquad (3.73)
\end{aligned}
$$

By a similar calculation, we have

$$
\begin{aligned}
B_2 &\lesssim \fint_{B_i} \left[ \frac{h(x, |Dw_i| + |Du|)}{(\alpha + |u - v_i|)^{\xi}} |Du - Dw_i|^2 \right]^{\frac{1}{2}} \\
&\qquad \cdot \left[ |Dw_i|^{(p-2)\xi + \tilde{\omega}} (\alpha + |u - v_i|)^{\xi} \right]^{\frac{1}{2}} \, dx \\
&\lesssim \left( \left[ \frac{|\mu|(B_i)}{r_i^n} \right] \alpha^{1-\xi} \right)^{\frac{1}{2}} \left( \fint_{B_i} |Dw_i|^{\tilde{\omega}} \left[ |Dw_i|^{p-2} (\alpha + |u - v_i|) \right]^{\xi} \, dx \right)^{\frac{1}{2}} \\
&\lesssim \left( \left[ \frac{|\mu|(B_i)}{r_i^n} \right] \alpha^{1-\xi} \right)^{\frac{1}{2}} \left( \fint_{B_i} |Dw_i| \, dx \right)^{\frac{\tilde{\omega}}{2}} \\
&\qquad \cdot \left( \fint_{B_i} \left[h_{m,i}(|Dw_i|)(\alpha + |u - v_i|)\right]^{\frac{\xi}{1-\tilde{\omega}}} \, dx \right)^{\frac{1-\tilde{\omega}}{2}}, \qquad (3.74)
\end{aligned}
$$

where $\tilde{\omega} = 2\omega_{a(\cdot)}(r_i)(1 + \eta)$. Using (3.32) and (3.11), we see

$$
\begin{aligned}
\left( \fint_{B_i} |Dw_i| dx \right)^{\frac{\tilde{\omega}}{2}} &\lesssim \left( \fint_{B_i} |Dw_i - Du| \, dx \right)^{\frac{\tilde{\omega}}{2}} + \left( \fint_{B_i} |Du| \, dx \right)^{\frac{\tilde{\omega}}{2}} \\
&\lesssim \left[ \frac{|\mu|(B_i)}{r_i^{n-1}} \right]^{\frac{\tilde{\omega}}{2(p-1)}} + \left[ \frac{\|Du\|_{L^1(\Omega)}}{r_i^n} \right]^{\tilde{\omega}} \le c.
\end{aligned} \qquad (3.75)
$$

Combining (3.72)-(3.75) gives

$$
A_2 \lesssim_{\delta} \frac{1}{\sqrt{h_0(\lambda)}} \left( \left[ \frac{|\mu|(B_{i-1})}{r_{i-1}^n} \right] \alpha^{1-\xi} \right)^{\frac{1}{2}}
$$

$$\cdot\left(\fint_{B_i}\left[\frac{h_{m,i}(|Dw_i|)}{h_{M,i}(\lambda)}(\alpha+|u-v_i|)\right]^{\frac{\xi}{1-\tilde{\omega}}}\,dx\right)^{\frac{1-\tilde{\omega}}{2}}.$$

For any constant $\tau\in(0,1)$, we take

$$\alpha:=\left(\fint_{B_i}\left[\frac{h_{m,i}(|Dw_i|)}{h_{M,i}(\lambda)}|u-v_i|\right]^{\frac{\xi}{1-\tilde{\omega}}}\,dx\right)^{\frac{1-\tilde{\omega}}{\xi}}+\tau. \tag{3.76}$$

Then we find that

$$\begin{aligned}A_2 &\lesssim_\delta \frac{\alpha^{\frac{1}{2}}}{\sqrt{h_0(\lambda)}}\left[\frac{|\mu|(B_{i-1})}{r_{i-1}^n}\right]^{\frac{1}{2}}\left[1+\left(\fint_{B_i}\left[\frac{h_{m,i}(|Dw_i|)}{h_{M,i}(\lambda)}\right]^{\frac{\xi}{1-\tilde{\omega}}}dx\right)^{\frac{1-\tilde{\omega}}{2}}\right]\\ &=\left[\frac{\alpha}{r_{i-1}}\right]^{\frac{1}{2}}\left[\frac{|\mu|(B_{i-1})}{h_0(\lambda)r_{i-1}^{n-1}}\right]^{\frac{1}{2}}\\ &\quad\cdot\left[1+\frac{1}{h_{M,i,\frac{\xi}{2}-1}(\lambda)}\left(\fint_{B_i}h_{m,i,\frac{\xi}{1-\tilde{\omega}}-1}(|Dw_i|)\,dx\right)^{\frac{1-\tilde{\omega}}{2}}\right].\end{aligned} \tag{3.77}$$

We assert that the last term above can be bounded by some constant $c$ depending only on $\texttt{data}, H$ and $\delta$. To prove this, we apply (3.66) to have

$$\begin{aligned}&\left(\fint_{B_i}h_{m,i,\frac{\xi}{1-\tilde{\omega}}-1}(|Dw_i|)\,dx\right)^{\frac{1-\tilde{\omega}}{2}}\\ &\lesssim_H\left(\fint_{B_i}h_{m,i,\frac{\xi}{1-\tilde{\omega}}-1}(|Dw_i-Dw_{i-1}|)\,dx\right)^{\frac{1-\tilde{\omega}}{2}}+h_{m,i,\frac{\xi}{2}-1}(\lambda).\end{aligned}$$

In addition, (3.24) and Corollary 3.2.2 imply

$$\begin{aligned}&\left(\fint_{B_i}h_{m,i,\frac{\xi}{1-\tilde{\omega}}-1}(|Dw_i-Dw_{i-1}|)\,dx\right)^{\frac{1-\tilde{\omega}}{2}}\\ &\lesssim\left(\fint_{B_i}h_{m,i,\frac{\xi}{1-\tilde{\omega}}-1}(|Dw_i-Du|)\,dx\right)^{\frac{1-\tilde{\omega}}{2}}\\ &\quad+\left(\fint_{B_i}h_{m,i,\frac{\xi}{1-\tilde{\omega}}-1}(|Du-Dw_{i-1}|)\,dx\right)^{\frac{1-\tilde{\omega}}{2}}\end{aligned}$$

$$\begin{aligned}
&\lesssim_\delta h_{m,i,\frac{\xi}{2}-1}(\mathcal{M}_i) + \left(\fint_{B_{i-1}} h_{m,i-1,\frac{\xi}{1-\tilde{\omega}}-1}(|Du - Dw_{i-1}|)\,dx\right)^{\frac{1-\tilde{\omega}}{2}} \\
&\qquad + \left(\fint_{B_{i-1}} |Du - Dw_{i-1}|^{(p-2+\omega_{a(\cdot)}(r_{i-1}))\left(\frac{\xi}{1-\tilde{\omega}}-1\right)} dx\right)^{\frac{1-\tilde{\omega}}{2}} \\
&\lesssim_\delta h_{m,i,\frac{\xi}{2}-1}(\mathcal{M}_{i-1}) + \left(\fint_{B_{i-1}} |Du - Dw_{i-1}|^{(p-2+\omega_{a(\cdot)}(r_{i-1}))\frac{\xi}{1-\tilde{\omega}}} dx\right)^{\frac{1-\tilde{\omega}}{2}}
\end{aligned}$$

It then follows from Hölder's inequality, Corollary 3.2.2, (3.32) and (3.11) that

$$\begin{aligned}
&\left(\fint_{B_{i-1}} |Du - Dw_{i-1}|^{(p-2+\omega_{a(\cdot)}(r_{i-1}))\frac{\xi}{1-\tilde{\omega}}} dx\right)^{\frac{1-\tilde{\omega}}{2}} \\
&\le \left(\fint_{B_{i-1}} h_{m,i-1,\frac{\xi}{1-\tilde{\omega}-\omega_{a(\cdot)}(r_{i-1})\xi}-1}(|Du - Dw_{i-1}|)\,dx\right)^{\frac{1-\tilde{\omega}-\omega_{a(\cdot)}(r_{i-1})\xi}{2}} \\
&\qquad \cdot \left(\fint_{B_{i-1}} |Du - Dw_{i-1}|\,dx\right)^{\frac{\omega_{a(\cdot)}(r_{i-1})\xi}{2}} \\
&\lesssim h_{m,i-1,\frac{\xi}{2}-1}(\mathcal{M}_{i-1}) \lesssim h_{m,i,\frac{\xi}{2}-1}(\mathcal{M}_{i-1}),
\end{aligned}$$

and this is the assertion. Similarly, one can show

$$\fint_{B_i} g_{m,i}(|Dw_{i-1} - Dw_i|)^{\frac{\xi}{1-\tilde{\omega}}}\,dx \lesssim_{\delta,H} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^{\frac{\xi}{1-\tilde{\omega}}}. \tag{3.78}$$

Thus, for any $\epsilon \in (0,1)$, we further estimate (3.77) as

$$A_2 \lesssim_{\delta,H} \left[\frac{\alpha}{r_{i-1}}\right]^{\frac{1}{2}} \left[\frac{|\mu|(B_{i-1})}{h_0(\lambda) r_{i-1}^n}\right]^{\frac{1}{2}} \le \frac{c(\epsilon)\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^n}\right] + \frac{\epsilon\alpha}{r_{i-1}}. \tag{3.79}$$

Combining (3.67), (3.71), (3.79) gives

$$\fint_{B_i} |Du - Dw_i| dx \le \frac{c(\delta,H,\epsilon)\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^n}\right] + \frac{\epsilon\alpha}{r_{i-1}}. \tag{3.80}$$

From (3.76), we find

$$\begin{aligned} h_0(\lambda)\alpha &\lesssim \left(\fint_{B_i} [h_{m,i}(|Dw_{i-1} - Dw_i|)|u - v_i|]^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} \\ &\quad + \left(\fint_{B_i} [h_{m,i}(|Dw_{i-1}|)|u - v_i|]^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} + h_0(\lambda)\tau \\ &=: I_1 + I_2 + h_0(\lambda)\tau. \end{aligned} \tag{3.81}$$

We note $\frac{\xi}{1-\tilde{\omega}} \leq \frac{n}{n-1}$ from the assumption (3.10). We estimate $I_1$ as

$$\begin{aligned} \frac{I_1}{r_i} &= \left(\fint_{B_i} \left[\frac{g_{m,i}(|Dw_{i-1} - Dw_i|)}{|Dw_{i-1} - Dw_i|}\frac{|u - v_i|}{r_i}\right]^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} \\ &\lesssim \left(\fint_{B_i} \tilde{g}_{m,i}\left(\frac{g_{m,i}(|Dw_{i-1} - Dw_i|)}{|Dw_{i-1} - Dw_i|}\right)^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} \\ &\quad + \left(\fint_{B_i} g_{m,i}\left(\frac{|u - v_i|}{r_i}\right)^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} \\ &\lesssim \left(\fint_{B_i} g_{m,i}(|Dw_{i-1} - Dw_i|)^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} + \left(\fint_{B_i} g_{m,i}\left(\frac{|u - v_i|}{r_i}\right)^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \\ &\lesssim_{\delta,H} \left(\fint_{B_i} g_{m,i}\left(|Du - Dw_{i-1}|\right)^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} + \fint_{B_i} g_{m,i}\left(|Du - Dw_i|\right)\, dx \\ &\lesssim \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]. \end{aligned} \tag{3.82}$$

Here, we have used Young's inequality, (2.6), Lemma 2.2.3 and (3.78).

Applying (3.66) and Sobolev embedding theorem, we discover

$$\frac{I_2}{r_i h_0(\lambda)} \lesssim_H \left(\fint_{B_i} \left|\frac{u - v_i}{r_i}\right|^{\frac{\xi}{1-\tilde{\omega}}}\, dx\right)^{\frac{1-\tilde{\omega}}{\xi}} \lesssim \fint_{B_i} |Du - Dw_i| dx. \tag{3.83}$$

Combining (3.80)-(3.83), we have

$$\fint_{B_i} |Du - Dw_i| dx \le \frac{c(\delta, H, \epsilon)\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right] + c\epsilon \fint_{B_i} |Du - Dw_i| dx + \frac{\epsilon\tau}{r_{i-1}}.$$

We choose $\epsilon$ small enough and let $\tau \to 0$, to conclude that

$$\fint_{B_i} |Du - Dw_i| dx \le c_3 \frac{\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right].$$

□

Next, we obtain a sequence of comparison estimates between (3.1) and (3.65).

**Lemma 3.2.15.** *Assume*

$$\mathcal{M}_{i-1} \le \lambda, \qquad \sup_{\frac{3}{4}B_i} |Dw_i| \le H\lambda \quad \text{and} \quad \frac{\lambda}{H} \le |Dw_{i-1}| \le H\lambda \quad \text{in } B_i \tag{3.84}$$

*for a constant $H \ge 1$ and $\lambda > 0$. Then there exists a constant $c_4 = c_4(\textbf{data}, \delta, H)$ such that*

$$\fint_{\frac{1}{2}B_i} |Du - Dv_i|\, dx \le c_4 \frac{\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right] + c_4\omega(r_i)\lambda.$$

*Proof.* From Lemma 3.2.7 and (3.24), we have

$$\begin{aligned}
\fint_{\frac{5}{8}B_i} G(x, |Dw_i|)\, dx &\lesssim \left(\fint_{\frac{3}{4}B_i} G(x, |Dw_i|)^{\frac{1}{2p}}\, dx\right)^{2p} \\
&\lesssim \left(\fint_{\frac{3}{4}B_i} G_{m,i}(|Dw_i|)^{\frac{1}{2p}} + |Dw_i|^{\frac{p+\omega_{a(\cdot)}(r_i)}{2p}}\, dx\right)^{2p} \\
&\lesssim_H G_{m,i}(\lambda) + \left(\fint_{\frac{3}{4}B_i} |Dw_i|\, dx\right)^{p+\omega_{a(\cdot)}(r_i)}, \qquad (3.85)
\end{aligned}$$

where we have used (3.84) and Hölder's inequality for the last inequality. By

(3.32), (3.84) and (3.11), we find

$$\begin{aligned}\left(\fint_{\frac{3}{4}B_i}|Dw_i|\,dx\right)^{p+\omega_{a(\cdot)}(r_i)} &\lesssim \left(\left[\frac{|\mu|(B_i)}{r_i^{n-1}}\right]^{\frac{1}{p-1}}+\frac{M}{r_i^n}\right)^{\omega_{a(\cdot)}(r_i)}\lambda^p\\ &\lesssim \lambda^p \lesssim G_0(\lambda). \qquad (3.86)\end{aligned}$$

Combining Lemma 3.2.8, (3.85) and (3.86) gives

$$\fint_{\frac{1}{2}B_i} h_0(|Dv_i|+|Dw_i|)|Dw_i-Dv_i|^2dx \lesssim \omega(r_i)^2\fint_{\frac{5}{8}B_i} G(x,|Dw_i|)\,dx \lesssim \omega(r_i)^2G_0(\lambda). \qquad (3.87)$$

We define an auxiliary function $G_1(t) = G_0\big(t^{\frac{1}{p}}\big)$ for $t \in \mathbb{R}^+$. By a direct calculation, we discover

$$\frac{dG_1(t)}{dt} = 1 + a_0\log\left(e+t^{\frac{1}{p}}\right)+\frac{a_0t^{\frac{1}{p}}}{p(e+t^{\frac{1}{p}})} > 0,$$

and so the derivative of $G_1$ is increasing, i.e., $G_1^{-1}$ is concave. It then follows from Jensen's inequality, (2.19) and (3.87) that

$$\begin{aligned}\fint_{\frac{1}{2}B_i}|Dv_i-Dw_i|^p\,dx &\le G_1^{-1}\left(\fint_{\frac{1}{2}B_i}G_0\left(|Dv_i-Dw_i|\right)dx\right)\\ &\lesssim_H G_1^{-1}\left(\omega(r_i)^2G_0\left(\lambda\right)\right). \qquad (3.88)\end{aligned}$$

On the other hand, we apply (3.24) and Lemma 3.2.3 to discover

$$\begin{aligned}&\fint_{B_i} g_0(|Dw_i-Dw_{i-1}|)\,dx\\ &\lesssim \fint_{B_i} g_0(|Dw_i-Du|)\,dx + \fint_{B_i} g_0(|Du-Dw_{i-1}|)\,dx\\ &\lesssim_\delta \fint_{B_i} g_{m,i}(|Dw_i-Du|)+|Dw_i-Du|^{p-1+\omega_{a(\cdot)}(r_i)}\,dx\\ &+\fint_{B_{i-1}} g_{m,i-1}(|Du-Dw_{i-1}|)+|Du-Dw_{i-1}|^{p-1+\omega_{a(\cdot)}(r_{i-1})}\,dx\end{aligned}$$

$$\lesssim_\delta \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right] + \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]^{1+\frac{\omega_{a(\cdot)}(r_{i-1})}{p-1}} \lesssim \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]. \tag{3.89}$$

Applying (3.84), we see

$$\begin{aligned}
\fint_{\frac{1}{2}B_i} |Dv_i - Dw_i| dx &\lesssim_H \fint_{\frac{1}{2}B_i} \left(\frac{g_0(|Dw_{i-1}|)}{g_0(\lambda)}\right)^{\frac{1}{p'}} |Dv_i - Dw_i|\, dx \\
&\lesssim \fint_{\frac{1}{2}B_i} \left(\frac{g_0(|Dw_i|)}{g_0(\lambda)}\right)^{\frac{1}{p'}} |Dv_i - Dw_i|\, dx \\
&\quad + \fint_{\frac{1}{2}B_i} \left(\frac{g_0(|Dw_i - Dw_{i-1}|)}{g_0(\lambda)}\right)^{\frac{1}{p'}} |Dv_i - Dw_i|\, dx \\
&= I_1 + I_2.
\end{aligned} \tag{3.90}$$

Since $1 < p' \le 2$, we see $t^{\frac{1}{p'}} \le H t^{\frac{1}{2}}$ for every $t \in [0, H]$ and

$$\begin{aligned}
I_1 &\lesssim \fint_{\frac{1}{2}B_i} \left(\frac{g_0(|Dw_i|)}{g_0(\lambda)}\right)^{\frac{1}{2}} |Dv_i - Dw_i| dx \\
&\lesssim_H \frac{1}{h_0(\lambda)^{\frac{1}{2}}} \left(\fint_{\frac{1}{2}B_i} h_0(|Dv_i| + |Dw_i|)|Dw_i - Dv_i|^2 dx\right)^{\frac{1}{2}} \lesssim \omega(r_i)\lambda.
\end{aligned} \tag{3.91}$$

Here, we have used (3.84) and (3.87).

It only remains to estimate $I_2$. Applying (3.89), (3.88) and (3.22), we obtain

$$\begin{aligned}
I_2 &\lesssim \left(\fint_{B_i} \frac{g_0(|Dw_i - Dw_{i-1}|)}{g_0(\lambda)} dx\right)^{\frac{1}{p'}} \left(\fint_{B_i} |Dv_i - Dw_i|^p\, dx\right)^{\frac{1}{p}} \\
&\lesssim_{\delta,H} \left(\frac{1}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]\right)^{\frac{1}{p'}} G_0^{-1}\left(\omega(r_i)^2 G_0(\lambda)\right) \\
&\lesssim \frac{\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right] + \lambda^{1-p} \left(G_0^{-1}\left(\omega(r_i)^2 G_0(\lambda)\right)\right)^p \\
&\lesssim \frac{\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right] + \omega(r_i)^{\frac{2p}{p+1}} \lambda.
\end{aligned} \tag{3.92}$$

Combining (3.90)-(3.92) yields

$$\fint_{\frac{1}{2}B_i} |Dv_i - Dw_i| dx \lesssim_{\delta,H} \frac{\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right] + \omega(r_i)\lambda.$$

This estimate and Lemma 3.2.14 complete the proof. □

### 3.2.6 Iterative comparison estimates for (GPX)

Through this subsection, we assume that $A(\cdot)$ satisfies (GPT) and set $u$ as a weak solution to (3.1).

Given a number $0 < \delta \leq \frac{1}{16}$, let $w_\delta \in W^{1,p(\cdot)}(B_{\delta R})$ be the weak solution to the following Dirichlet problem under the assumptions (GPX):

$$\begin{cases} -\mathrm{div}\,(\gamma(x)A(x, Dw_\delta)) = 0 & \text{in} \quad B_{\delta R}(x_0) \\ w_\delta = u & \text{on} \quad \partial B_{\delta R}(x_0). \end{cases} \tag{3.93}$$

**Lemma 3.2.16.** *Let $\lambda > 0$ and assume that*

$$\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right)^{\frac{1}{p_0-1}} \leq \lambda. \tag{3.94}$$

*We further assume that*

$$\frac{\lambda}{H} \leq |Dw| \leq H\lambda \quad \text{in} \quad B_{\delta R} \tag{3.95}$$

*holds for some constant $H \geq 1$. Then there exists a constant $c_3 = c_3(\boldsymbol{data}, \delta, H) \geq 1$ such that*

$$\fint_{B_{\delta R}} |Du - Dw_\delta|\, dx \leq c_3 \lambda^{2-p_0} \left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right). \tag{3.96}$$

*Proof.* We fix parameters $\eta$ and $\xi$ as

$$\eta := \frac{1}{4(n+1)(p_0-1)}, \quad \xi = 1 + 2\eta,$$

and introduce $\bar{w} = \frac{w}{\lambda}$ and $\bar{w}_\delta = \frac{w_\delta}{\lambda}$. We shall use exponents $q$ such that

$$1 \le q \le \frac{\xi(p_0-1)}{1-\omega_{p(\cdot)}(2\delta R)} = \left(p_0 - 1 + \frac{1}{2(n+1)}\right)\frac{1}{1-\omega_{p(\cdot)}(2\delta R)} < q_0, \quad (3.97)$$

where $q_0$ is given in (3.37). Note

$$\frac{\xi}{1-\omega_{p(\cdot)}(\delta R)} \le \frac{n}{n-1} = 1^*, \quad (3.98)$$

from (3.60). We start with estimating the lefr-hand side of (3.96), by applying (3.95) and (2.14):

$$\begin{aligned} \fint_{B_{\delta R}} |Du - Dw_\delta|\, dx &\le H^{(p_0-2)(1+\eta)} c \fint_{B_{\delta R}} |D\bar{w}|^{(p_0-2)(1+\eta)} |Du - Dw_\delta|\, dx \\ &\le c \fint_{B_{\delta R}} |D\bar{w}_\delta - D\bar{w}|^{(p_0-2)(1+\eta)} |Du - Dw_\delta|\, dx \\ &\quad + c \fint_{B_{\delta R}} |D\bar{w}_\delta|^{(p_0-2)(1+\eta)} |Du - Dw_\delta|\, dx. \end{aligned} \quad (3.99)$$

For any $q$ satisfying (3.97), we apply Lemma 3.2.6, to deduce that

$$\begin{aligned} &\fint_{B_{\delta R}} |D\bar{w}_\delta - D\bar{w}|^q\, dx \\ &\le c\lambda^{-q} \fint_{B_{\delta R}} |Du - Dw_\delta|^q\, dx + c\lambda^{-q} \fint_{B_{\delta R}} |Du - Dw|^q\, dx \\ &\le c\lambda^{-q} \left( \left[ \frac{|\mu|(B_{\delta R})}{(\delta R)^{n-1}} \right] + R^{p_0-1} \right)^{\frac{q}{p_0-1}} + c\lambda^{-q} \frac{|B_R|}{|B_{\delta R}|} \fint_{B_R} |Du - Dw|^q\, dx \\ &\le c\lambda^{-q} \left( \left[ \frac{|\mu|(B_{\delta R})}{(\delta R)^{n-1}} \right] + R^{p_0-1} \right)^{\frac{q}{p_0-1}} + c\lambda^{-q} \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] + R^{p_0-1} \right)^{\frac{q}{p_0-1}} \\ &\le c\lambda^{-q} \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] + R^{p_0-1} \right)^{\frac{q}{p_0-1}}, \end{aligned} \quad (3.100)$$

where $c = c(\mathtt{data}, \delta, q)$. Using Hölder's inequality, (3.36), (3.100) and (3.94), we have

$$
\begin{aligned}
&\fint_{B_{\delta R}} |D\bar{w}_\delta - D\bar{w}|^{(p_0-2)(1+\eta)}|Du - Dw_\delta|\,dx \\
&\le \left(\fint_{B_{\delta R}} |D\bar{w}_\delta - D\bar{w}|^{(p_0-1)(1+\eta)}\,dx\right)^{\frac{p_0-2}{p_0-1}} \\
&\qquad \cdot \left(\fint_{B_{\delta R}} |Du - Dw_\delta|^{p_0-1}\,dx\right)^{\frac{1}{p_0-1}} \\
&\le c\lambda^{1-(p_0-2)(1+\eta)-1}\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right)^{\frac{(p_0-2)(1+\eta)+1}{p_0-1}} \\
&\le c\lambda^{2-p_0}\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right),
\end{aligned}
\tag{3.101}
$$

with $c = c(\texttt{data}, \delta)$, where we have used the fact that

$$
\frac{(p_0-2)(1+\eta)+1}{p_0-1} \ge 1.
$$

We combine (3.99) with (3.101) to discover

$$
\begin{aligned}
&\fint_{B_{\delta R}} |Du - Dw_\delta|\,dx \\
&\le c\underbrace{\fint_{B_{\delta R}} |D\bar{w}_\delta|^{(p_0-2)(1+\eta)}|Du - Dw_\delta|\,dx}_{=:I_1} + c\lambda^{2-p_0}\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right)
\end{aligned}
\tag{3.102}
$$

with $c = c(\texttt{data}, \delta)$. We use (2.19) to estimate $I_1$ as follows: for any $h > 0$

$$
\begin{aligned}
I_1 = \lambda^{(2-p_0)(1+\eta)} &\fint_{B_{\delta R}} \left(\frac{|Dw_\delta|^{p(x)-2}}{(h+|u-w_\delta|)^\xi}|Du - Dw_\delta|^2\right)^{\frac{1}{2}} \\
&\cdot \left(|Dw_\delta|^{(p_0-2)\xi-(p(x)-p_0)}\,(h+|u-w_\delta|)^\xi\right)^{\frac{1}{2}}\,dx
\end{aligned}
$$

$$\leq c\lambda^{\frac{2-p_0}{2}} \fint_{B_{2\delta R}} \left( \frac{\left(|Du|^2+|Dw_\delta|^2+s^2\right)^{\frac{p(x)-2}{2}}}{(h+|u-w_\delta|)^\xi} |Du-Dw_\delta|^2 \right)^{\frac{1}{2}}$$

$$\cdot \left( \frac{|Dw_\delta|^{(p_0-2)\xi-(p(x)-p_0)}}{\lambda^{(p_0-2)\xi}} (h+|u-w_j|)^\xi \right)^{\frac{1}{2}} dx$$

$$\leq c\lambda^{\frac{2-p_0}{2}} \fint_{B_{\delta R}} \left( \frac{|V(x,Du)-V(x,Dw_\delta)|^2}{(h+|u-w_\delta|)^\xi} \right)^{\frac{1}{2}}$$

$$\cdot \left( \frac{(|Dw_\delta|+R)^{(p_0-2)\xi+\omega(2\delta R)}}{\lambda^{(p_0-2)\xi}} (h+|u-w_\delta|)^\xi \right)^{\frac{1}{2}} dx.$$

We use Hölder's inequality and (3.34) to deduce that

$$I_1 \leq c\lambda^{\frac{2-p_0}{2}} \left( \fint_{B_{\delta R}} \frac{|V(x,Du)-V(x,Dw_\delta)|^2}{(h+|u-w_\delta|)^\xi} dx \right)^{\frac{1}{2}}$$

$$\cdot \left( \fint_{B_{\delta R}} (|Dw_\delta|+R)\, dx \right)^{\frac{\omega_{p(\cdot)}(2\delta R)}{2}}$$

$$\cdot \left( \fint_{B_{\delta R}} \left( (|D\bar{w}_\delta|+1)^{(p_0-2)} (h+|u-w_\delta|) \right)^{\frac{\xi}{(1-\omega_{p(\cdot)}(2\delta R)}} dx \right)^{\frac{1-\omega_{p(\cdot)}(2\delta R))}{2}}$$

$$\leq c\lambda^{\frac{2-p_0}{2}} \left( h^{1-\xi} \left[ \frac{|\mu|(B_{\delta R})}{\delta R^n} \right] \right)^{\frac{1}{2}} \cdot \left( \fint_{B_{\delta R}} (|Dw_\delta|+R)\, dx \right)^{\frac{\omega_{p(\cdot)}(2\delta R)}{2}}$$

$$\cdot \left( \fint_{B_{\delta R}} \left( (|D\bar{w}_\delta|+1)^{(p_0-2)} (h+|u-w_\delta|) \right)^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx \right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{2}}.$$

We employ (3.58) to estimate $I_1$ as follows

$$I_1 \leq c\lambda^{\frac{2-p_0}{2}} \left( h^{1-\xi} \left[ \frac{|\mu|(B_R)}{R^n} \right] \right)^{\frac{1}{2}}$$

$$\cdot \left( \fint_{B_{\delta R}} \left( (|D\bar{w}_\delta|+1)^{(p_0-2)} (h+|u-w_\delta|) \right)^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx \right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{2}}$$

$$=: c\lambda^{\frac{2-p_0}{2}} \left( h^{1-\xi} \left[ \frac{|\mu|(B_R)}{R^n} \right] \right)^{\frac{1}{2}} I_2^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{2}}. \tag{3.103}$$

Now, for any small $\tilde{\eta} \in (0,1)$, we fix $h$ by

$$h := \left( \fint_{B_{\delta R}} \left( (|D\bar{w}_\delta| + 1)^{(p_0-2)} |u - w_\delta| \right)^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx \right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi}} + \tilde{\eta}. \tag{3.104}$$

Then it follows that

$$\begin{aligned} & I_2^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{2}} \\ & \leq ch^{\frac{\xi}{2}} \left( \fint_{B_{\delta R}} (|D\bar{w}_\delta| + 1)^{\frac{(p_0-2)\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx \right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{2}} + ch^{\frac{\xi}{2}}. \end{aligned} \tag{3.105}$$

Applying (3.100), (3.94) and (3.95), we find that

$$\begin{aligned} & \fint_{B_{\delta R}} (|D\bar{w}_\delta| + 1)^{\frac{(p_0-2)\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx \\ & \leq c \fint_{B_{\delta R}} |D\bar{w}_\delta - D\bar{w}|^{\frac{(p_0-2)\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx + c \fint_{B_{\delta R}} |D\bar{w}|^{\frac{(p_0-2)\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx + c \\ & \leq c. \end{aligned} \tag{3.106}$$

Combining (3.103), (3.105) with (3.106), for any $\epsilon \in (0,1)$, we conclude with

$$I_1 \leq \left( \lambda^{2-p_0} \frac{h}{R} \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] \right)^{\frac{1}{2}} \leq \frac{\epsilon h}{R} + c(\epsilon) \lambda^{2-p_0} \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right], \tag{3.107}$$

where $c = c(\texttt{data}, \delta, H)$. It finally remains to estimate $h$, which is defined in (3.104). According to (3.95), we have

$$\begin{aligned} h \leq & c \left( \fint_{B_{\delta R}} \left( |D\bar{w}_\delta - D\bar{w}|^{(p_0-2)} |u - w_\delta| \right)^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx \right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi}} \\ & + c \left( \fint_{B_{\delta R}} \left( |D\bar{w}|^{(p_0-2)} |u - w_\delta| \right)^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}} dx \right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi}} \end{aligned}$$

$$
\begin{aligned}
&+ c\left(\fint_{B_{\delta R}} |u - w_\delta|^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}}\, dx\right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi}} + \tilde{\eta} \\
&\le c\left(\fint_{B_{\delta R}} \left(|D\bar{w}_\delta - D\bar{w}|^{(p_0-2)}|u - w_\delta|\right)^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}}\, dx\right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi}} \\
&\quad + c\left(\fint_{B_{\delta R}} |u - w_\delta|^{\frac{\xi}{1-\omega_{p(\cdot)}(2\delta R)}}\, dx\right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi}} + \tilde{\eta} =: I_3 + I_4 + \tilde{\eta}. \qquad (3.108)
\end{aligned}
$$

We use Sobolev embedding, (3.100) and (3.36) to estimate $I_3$ as follows:

$$
\begin{aligned}
I_3 &\le c\left(\fint_{B_{\delta R}} |D\bar{w}_\delta - D\bar{w}|^{\frac{\xi(p_0-1)}{1-\omega_{p(\cdot)}(2\delta R)}}\, dx\right)^{\frac{(1-\omega_{p(\cdot)}(2\delta R))(p_0-2)}{\xi(p_0-1)}} \\
&\qquad \cdot \left(\fint_{B_{\delta R}} |u - w_\delta|^{\frac{\xi(p_0-1)}{1-\omega_{p(\cdot)}(2\delta R)}}\right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi(p_0-1)}} \\
&\le cR\lambda^{2-p_0}\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right)^{\frac{p_0-2}{p_0-1}} \\
&\qquad \cdot \left(\fint_{B_{\delta R}} |Du - Dw_\delta|^{\frac{\xi(p_0-1)}{1-\omega_{p(\cdot)}(2\delta R)}}\right)^{\frac{1-\omega_{p(\cdot)}(2\delta R)}{\xi(p_0-1)}} \\
&\le cR\lambda^{2-p_0}\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right). \qquad (3.109)
\end{aligned}
$$

According to (3.98), (3.95) and (3.101), we have

$$
\begin{aligned}
I_4 &\le c\left(\fint_{B_{\delta R}} |u - w_\delta|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \\
&\le cR\fint_{B_{\delta R}} |D\bar{w}|^{(p_0-2)(1+\eta)}|Du - Dw_\delta|\, dx \\
&\le cR\fint_{B_{\delta R}} |D\bar{w}_\delta|^{(p_0-2)(1+\eta)}|Du - Dw_\delta|\, dx \\
&\qquad + cR\fint_{B_{\delta R}} |D\bar{w}_\delta - D\bar{w}|^{(p_0-2)(1+\eta)}|Du - Dw_\delta|\, dx \\
&\le cR\fint_{B_{\delta R}} |D\bar{w}_\delta|^{(p_0-2)(1+\eta)}|Du - Dw_\delta|\, dx
\end{aligned}
$$

$$+ cR\lambda^{2-p_0}\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right). \tag{3.110}$$

We finally combine (3.108), (3.109) and (3.110) to discover

$$\begin{aligned}\frac{h}{R} &\le c \fint_{B_{\delta R}} |D\bar{w}|^{(p_0-2)(1+\eta)}|Du - Dw_\delta|\,dx \\ &\quad + c\lambda^{2-p_0}\left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right) + \frac{\tilde{\eta}}{R},\end{aligned}$$

where $c$ depends on $\texttt{data}, \delta$ and $H$. Plugging the inequality in (3.107) and choosing $\epsilon = \frac{1}{2c}$, we have

$$I_1 \le c\lambda^{2-p_0}\left(\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right] + R^{p_0-1}\right) + \frac{\tilde{\eta}}{R} \tag{3.111}$$

for any $\tilde{\eta} \in (0,1)$. Letting $\tilde{\eta} \to 0$ and combining (3.102) with (3.111) completes the proof.

□

We now compare $u \in W^{1,p(\cdot)}(\Omega)$ to $v_\delta \in W^{1,p_0}(B_{\delta R/2})$, the weak solution to the following reference problem:

$$\begin{cases} -\mathrm{div}\,(a(x_0, Dv_\delta)) = 0 & \text{in} \quad B_{\delta R/2} \\ v_\delta = w_\delta & \text{on} \quad \partial B_{\delta R/2}. \end{cases}$$

We present the last comparison estimates of this section.

**Lemma 3.2.17.** *Assume that*

$$\begin{cases} \left(\left[\frac{|\mu|(B_R)}{R^{n-1}}\right] + R^{p_0-1}\right)^{\frac{1}{p_0-1}} \le \lambda \\ \sup\limits_{B_{\frac{3\delta R}{4}}} (|Dw_\delta| + s + R) \le H\lambda \\ \frac{\lambda}{H} \le |Dw| \le H\lambda \quad \text{in } B_{\delta R} \end{cases} \tag{3.112}$$

*for some* $1 \le H$ *and every* $\lambda > 0$. *Then there exists a constant* $c_4 =$

$c_4(\boldsymbol{data}, \delta, H) \geq 1$ *such that*

$$\fint_{B_{\delta R/2}} |Du - Dv_\delta| \, dx \leq c_4 \omega(\delta R) \lambda + c_4 \lambda^{2-p_0} \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] + R^{p_0 - 1} \right).$$

*Proof.* Applying (3.61) with $x_1 = x_0$, and (3.112), we have

$$\begin{aligned}
&\fint_{B_{\delta R/2}} |Dw_\delta - Dv_\delta|^{p_0} \, dx \\
&\leq c \fint_{B_{\delta R/2}} \left( |Dw_\delta|^2 + |Dv_\delta|^2 + s^2 \right)^{\frac{p_0-2}{2}} |Dw_\delta - Dv_\delta|^2 \, dx \\
&\leq c\omega(\delta R)^2 \left[ \fint_{B_{\frac{5\delta R}{8}}} (|Dw_\delta| + s + R)^{p(x)} \, dx + R^{p_0} \right] \\
&\leq c\omega(\delta R)^2 \left[ \fint_{B_{\frac{3\delta R}{4}}} (|Dw_\delta| + s + R) \, dx \right]^{p_1} \leq c\omega(\delta R)^2 \lambda^{p_0}, \qquad (3.113)
\end{aligned}$$

where $c = c(\mathtt{data}, H, \delta)$. Here, we have used (3.59) with $\theta_1 = \frac{5}{8}$ and $\theta_2 = \frac{3}{4}$, and (2.14), in the last inequality. Now (3.112) implies

$$\begin{aligned}
\fint_{B_{\delta R/2}} |Dw_\delta - Dv_\delta| \, dx &\leq c \fint_{B_{\delta R/2}} |D\bar{w}|^{\frac{p_0-2}{p_0'}} |Dw_\delta - Dv_\delta| \, dx \\
&\leq c \fint_{B_{\delta R/2}} |D\bar{w}_\delta|^{\frac{p_0-2}{p_0'}} |Dw_\delta - Dv_\delta| \, dx \\
&\quad + c \fint_{B_{\delta R/2}} |D\bar{w}_\delta - D\bar{w}|^{\frac{p_0-2}{p_0'}} |Dw_\delta - Dv_\delta| \, dx.
\end{aligned} \qquad (3.114)$$

Since $p_0' \leq 2$, it follows from (3.112) and (3.113) that

$$\begin{aligned}
&\fint_{B_{\delta R/2}} |D\bar{w}_\delta|^{\frac{p_0-2}{p_0'}} |Dw_\delta - Dv_\delta| \, dx \\
&\leq c \fint_{B_{\delta R/2}} |D\bar{w}_\delta|^{\frac{p_0-2}{2}} |Dw_\delta - Dv_\delta| \, dx
\end{aligned}$$

$$\begin{aligned}
&\leq c\lambda^{\frac{2-p_0}{2}} \fint_{B_{\delta R/2}} \left(|Dw_\delta|^2 + |Dv_\delta|^2 + s^2\right)^{\frac{p_0-2}{4}} |Dw_\delta - Dv_\delta|\, dx \\
&\leq c\omega\,(\delta R)\,\lambda. \qquad (3.115)
\end{aligned}$$

Furthermore, we see from Lemma 3.2.6 with $q = p_0 - 2$, and (3.113) that

$$\begin{aligned}
&\fint_{B_{\delta R/2}} |D\bar{w}_\delta - D\bar{w}|^{\frac{p_0-2}{p_0'}} |Dw_\delta - Dv_\delta|\, dx \\
&\leq c\lambda^{\frac{2-p_0}{p_0'}} \left( \fint_{B_{\delta R/2}} |Dw_\delta - Dw|^{p_0-2}\, dx \right)^{\frac{1}{p_0'}} \\
&\quad \cdot \left( \fint_{B_{\delta R/2}} |Dw_\delta - Dv_\delta|^{p_0}\, dx \right)^{\frac{1}{p_0}} \\
&\leq c\lambda^{\frac{2-p_0}{p_0'}+1} \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] + R^{p_0-1} \right)^{\frac{p_0-2}{p_0}} \omega\,(\delta R)^{\frac{2}{p_0}} \\
&\leq c\omega\,(\delta R)\,\lambda + c\lambda^{2-p_0} \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] + R^{p_0-1} \right). \qquad (3.116)
\end{aligned}$$

Combining (3.114), (3.115) with (3.116), we find that

$$\fint_{B_{\delta R/2}} |Dw_\delta - Dv_\delta|\, dx \leq c\omega\,(\delta R)\,\lambda + c\lambda^{2-p_0} \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right] + R^{p_0-1} \right).$$

We recall Lemma 3.2.16 and combine with the last estimate to finish the proof.

□

## 3.3 Regularity results for homogeneous equation

The remaining parts of the proof of Theorem 3.1.3 are very similar to the one of Theorem 3.1.4. Therefore, from now on, we present only the proof of Theorem 3.1.3.

Our main purpose of this section is to show local $C^1$ regularity of (3.14). Let $x_1 \in B_{3R/4}$. For some $\delta \in \left(0, \frac{1}{16}\right)$ to be determined and for any $r \in$

$\left(0, \frac{1}{16}R\right]$, we define

$$\bar{B}_i = B_{\bar{r}_i}(x_1) \quad \text{and} \quad \bar{r}_i = \delta^i r \quad (i = 0, 1, \cdots). \tag{3.117}$$

Thus

$$\cdots \subset \frac{1}{2}\bar{B}_1 \subset \bar{B}_1 \subset \frac{1}{2}\bar{B}_0 \subset 2\bar{B}_0 \subset B_R = B_R(x_0).$$

Note that $\{\bar{B}_i\}_{i\in\mathbb{N}}$ is the sequence of balls centered at $x_1$, while $B_R$ is the ball centered at $x_0$. For simplicity, we denote $h_1(\cdot) = h(x_1, \cdot)$ and $G_1(\cdot) = G(x_1, \cdot)$ in this subsection. With the weak solution $v \in W^{1,G}(B_R)$ to (3.14), let $\bar{w}_i \in W^{1,G_1}(\frac{1}{2}\bar{B}_i)$ be the weak solution to (3.15) with $B_{\tilde{R}}(x_1)$ replaced by $\frac{1}{2}\bar{B}_i$.

**Lemma 3.3.1.** *Assume that*

$$\fint_{\bar{B}_{i-1}} |Dv|\, dx \le H\lambda \quad \textit{and} \quad \frac{\lambda}{H} \le |D\bar{w}_{i-1}| \le H\lambda \quad \textit{in } \bar{B}_i \tag{3.118}$$

*for a constant $H \ge 1$, a number $\lambda > 0$ and any index $i \ge 1$. Then there exists a constant $c_5 = c_5(\boldsymbol{data}, \delta, H) \ge 1$ such that*

$$\fint_{\bar{B}_i} |Dv - D\bar{w}_{i-1}|\, dx \le c_5 \omega\left(\bar{r}_{i-1}\right)\lambda.$$

*Proof.* By (3.118) and Lemma 3.2.8, we have

$$\begin{aligned}
\fint_{\bar{B}_i} |Dv - D\bar{w}_{i-1}|\, dx &\lesssim_H \fint_{\bar{B}_i} \left(\frac{h_1(|D\bar{w}_{i-1}|)}{h_1(\lambda)}\right)^{\frac{1}{2}} |Dv - D\bar{w}_{i-1}|\, dx \\
&\lesssim_\delta \frac{1}{h_1(\lambda)^{\frac{1}{2}}} \left(\fint_{\frac{1}{2}\bar{B}_i} h_1(|Dv| + |D\bar{w}_{i-1}|)\, |Dv - D\bar{w}_{i-1}|^2\, dx\right)^{\frac{1}{2}} \\
&\lesssim \frac{\omega\left(r_{i-1}\right)}{h_1(\lambda)^{\frac{1}{2}}} \left(\fint_{\frac{5}{8}\bar{B}_{i-1}} G(x, |Dv|)\, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

It then follows from (3.24), (3.23) and Lemma 3.2.7 that

$$\fint_{\frac{5}{8}\bar{B}_{i-1}} G(x, |Dv|)\, dx$$

$$
\begin{aligned}
&\lesssim \left( \fint_{\frac{5}{8}\bar{B}_{i-1}} G(x,|Dv|)^{\frac{1}{8p}}\,dx \right)^{8p} \\
&\lesssim \left( \fint_{\frac{3}{4}\bar{B}_{i-1}} G_{m,\bar{r}_{i-1}}(|Dv|)^{\frac{1}{8p}}dx \right)^{8p} + \left( \fint_{\frac{3}{4}\bar{B}_{i-1}} |Dv|^{\frac{p+\omega_{a(\cdot)}(\bar{r}_{i-1})}{8p}}dx \right)^{8p} \\
&\lesssim G_{m,\bar{r}_{i-1}} \left( \fint_{\frac{3}{4}\bar{B}_{i-1}} |Dv|\,dx \right) + \left( \fint_{\frac{3}{4}\bar{B}_{i-1}} |Dv|^{\frac{p+\omega_{a(\cdot)}(\bar{r}_{i-1})}{2p}}dx \right)^{2p} \\
&\lesssim_H G_{m,\bar{r}_i}(\lambda) + \left( \fint_{\frac{3}{4}\bar{B}_{i-1}} |Dv|\,dx \right)^{p+\omega_{a(\cdot)}(\bar{r}_{i-1})}
\end{aligned}
$$

By (3.32) and (3.11), we discover

$$
\begin{aligned}
&\left( \fint_{\frac{3}{4}\bar{B}_{i-1}} |Dv|\,dx \right)^{p+\omega_{a(\cdot)}(\bar{r}_{i-1})} \\
&\lesssim \left( \fint_{\bar{B}_{i-1}} |Dv|\,dx \right)^{p} \left( \fint_{B_R} |Dv|\,dx \right)^{\omega_{a(\cdot)}(\bar{r}_{i-1})} \\
&\le \left( \fint_{\bar{B}_{i-1}} |Dv|\,dx \right)^{p} \left( \fint_{B_R} |Du-Dv| + |Du|\,dx + 1 \right)^{\omega_{a(\cdot)}(R)} \\
&\lesssim_H \lambda^p \left( \left[ \frac{|\mu|(B_R)}{R^{n-1}} \right]^{\frac{1}{p-1}} + \frac{\|Du\|_{L^1(\Omega)}}{R^n} + 1 \right)^{\omega_{a(\cdot)}(R)} \lesssim G_1(\lambda).
\end{aligned}
$$

Merging all the estimates above, we complete the proof. □

We first need to establish local Lipschitz regularity for (3.14). We will use an exit time argument in the proof of next theorem.

**Theorem 3.3.2.** *Let $v \in W^{1,G}(B_R)$ be the weak solution to (3.14). Then there exist positive constants $R_2 = R_2(\boldsymbol{data}, \omega, |\mu|(\Omega), \|Du\|_{L^1(\Omega)})$ and $c_6 = c_6(\boldsymbol{data})$ such that*

$$
\|Dv\|_{L^\infty(B_{3R/4})} \le c_6 \fint_{B_R} |Dv|\,dx
$$

*whenever $0 < R \le R_2$.*

*Proof.* Our proof consists of three parts. At first, we define some significant constants and an exit time index. Next, we show some excess decay estimates for $v$. In the last part, we prove the Lipschitz regularity of $v$. We assume $x_1$ is a Lebesgue point of $Dv$ and take $r = \delta R$.

**Step 1: Basic setup.**

We choose positive constants $H$ and $\delta < \frac{1}{16}$ such that

$$H = 10^5 2^{n+2} c_l \quad \text{and} \quad 2^{n+2} c_\beta \delta^\beta \le \frac{1}{4 \cdot 10^5}, \tag{3.119}$$

where $\beta, c_l, c_\beta$ are the constants given in Lemma 2.3.1. Then there exists an integer $k \ge 2$ such that

$$2c_\beta \delta^{k\beta} \le \delta^n. \tag{3.120}$$

Note that the constants $H, \delta, k$ depend only on `data`. Recalling (2.10) and (2.11), one can find a positive constant $R_2 \le R_1$ such that

$$\delta^{-kn} \omega(\tau)^{\frac{2}{p+1}} + \delta^{-1} \int_0^\tau \omega(\rho) \frac{d\rho}{\rho} \le \frac{\delta^{2n}}{2^{n+3} 10^6 c_\beta c_2 c_5} \tag{3.121}$$

for every $0 < \tau \le R_2$, where $c_2$ and $c_5$ are the constants given in (3.57) and Lemma 3.3.1, respectively. Assume $0 < R \le R_2$. Direct calculations give

$$\begin{aligned} \delta \omega_{a(\cdot)}(\bar{r}_{i+1}) \log\left(\frac{1}{\bar{r}_{i+1}}\right) &\le \delta \log\left(\frac{1}{\delta}\right) \omega_{a(\cdot)}(\bar{r}_{i+1}) \log\left(\frac{1}{\bar{r}_i}\right) \\ &\le \omega_{a(\cdot)}(\bar{r}_{i+1}) \log\left(\frac{1}{\bar{r}_i}\right) \end{aligned}$$

and

$$\begin{aligned} &\int_0^{2\bar{r}_0} \omega_{a(\cdot)}(\rho) \log\left(\frac{1}{\rho}\right) \frac{d\rho}{\rho} \\ &= \sum_{i=0}^{\infty} \int_{\bar{r}_{i+1}}^{\bar{r}_i} \omega_{a(\cdot)}(\rho) \log\left(\frac{1}{\rho}\right) \frac{d\rho}{\rho} + \int_{\bar{r}_0}^{2\bar{r}_0} \omega_{a(\cdot)}(\rho) \log\left(\frac{1}{\rho}\right) \frac{d\rho}{\rho} \\ &\ge \log\left(\frac{1}{\delta}\right) \sum_{i=0}^{\infty} \omega_{a(\cdot)}(\bar{r}_{i+1}) \log\left(\frac{1}{\bar{r}_i}\right) + (\log 2) \omega_{a(\cdot)}(\bar{r}_0) \log\left(\frac{1}{2\bar{r}_0}\right) \\ &\ge \delta \sum_{i=1}^{\infty} \omega_{a(\cdot)}(\bar{r}_i) \log\left(\frac{1}{\bar{r}_i}\right) + \frac{\log 2}{2} \omega_{a(\cdot)}(\bar{r}_0) \log\left(\frac{1}{\bar{r}_0}\right) \end{aligned}$$

$$\geq \delta \sum_{i=0}^{\infty} \omega_{a(\cdot)}(\bar{r}_i) \log\left(\frac{1}{\bar{r}_i}\right). \tag{3.122}$$

It then follows from (3.121) that

$$\sum_{i=0}^{\infty} \omega(\bar{r}_i) \leq \delta^{-1} \int_0^{2\bar{r}_0} \omega(\rho)\frac{d\rho}{\rho} \leq \frac{\delta^{2n}}{2^{n+3}10^6 c_\beta c_2 c_5}.$$

We now take

$$\lambda = H_1 \fint_{B_R} |Dv|\,dx \quad \text{with} \quad H_1 = 10^5\delta^{-3n}$$

and set

$$C_i = \sum_{m=-1}^{0} \fint_{\bar{B}_{i+m}} |Dv|\,dx + 2\delta^{-n} \fint_{\bar{B}_i} |Dv - (Dv)_{\bar{B}_i}|\,dx, \quad {}^{\forall} i \in \mathbb{N}. \tag{3.123}$$

Since $r = r_0 = \delta R$, we see

$$C_1 \leq 6\delta^{-3n} \fint_{B_R} |Dv|\,dx \leq \frac{6\delta^{-3n}}{H_1}\lambda \leq \frac{\lambda}{1000}.$$

If there exists an infinite sequence $\{i_j\}_{j\in N} \subset \mathbb{N}$ such that $C_{i_j} \leq \frac{\lambda}{1000}$ for every $j \in N$, then

$$|Dv(x_1)| = \lim_{j\to\infty} \fint_{\bar{B}_{i_j}} |Dv|\,dx \leq \frac{\lambda}{1000}.$$

Hence, it suffices to assume that there exits an exit time index $i_e > 1$ such that

$$C_{i_e} \leq \frac{\lambda}{1000} \quad \text{and} \quad C_i > \frac{\lambda}{1000}, \quad {}^{\forall} i > i_e.$$

Recalling (3.13), we denote

$$E_i := E(Dv, \bar{B}_i) \quad \text{and} \quad a_i := |(Dv)_{\bar{B}_i}|.$$

For $i \geq i_e$, we say that "Ind($i$) holds" if and only if

$$\fint_{\bar{B}_{i-1}} |Dv|\,dx \leq \lambda.$$

**Step 2: Excess decay estimate.**

First, we assume that Ind($i$) holds for some $i \geq i_e$. It then follows from (3.57) and (3.121) that

$$\fint_{\frac{1}{2}\bar{B}_{i-1}} |Dv - D\bar{w}_{i-1}| \, dx \leq c_2 \omega(\bar{r}_{i-1})^{\frac{2}{p+1}} \fint_{\bar{B}_{i-1}} |Dv| \, dx \leq c_2 \omega(\bar{r}_{i-1})^{\frac{2}{p+1}} \lambda \leq \frac{\delta^{(k+2)n}}{10^6} \lambda. \tag{3.124}$$

and

$$\fint_{\frac{1}{2}\bar{B}_{i-1}} |D\bar{w}_{i-1}| \, dx \leq \fint_{\frac{1}{2}\bar{B}_{i-1}} |Dv| \, dx + \fint_{\frac{1}{2}\bar{B}_{i-1}} |Dv - D\bar{w}_{i-1}| \, dx \leq 2^n \lambda + \frac{\delta^{(k+2)n}}{10^6} \lambda \leq 2^{n+1} \lambda.$$

Applying Lemma 2.3.1 and (3.119), we see

$$\|D\bar{w}_{i-1}\|_{L^\infty(\frac{3}{8}\bar{B}_{i-1})} \leq c_l \fint_{\frac{1}{2}\bar{B}_{i-1}} |D\bar{w}_{i-1}| \, dx \leq 2^{n+1} c_l \lambda \tag{3.125}$$

and

$$\operatorname*{osc}_{\bar{B}_i} |D\bar{w}_{i-1}| \leq 2c_\beta \delta^\beta \fint_{\frac{1}{2}\bar{B}_{i-1}} |D\bar{w}_{i-1}| \, dx \leq 2^{n+2} c_\beta \delta^\beta \lambda \leq \frac{\lambda}{10^5}. \tag{3.126}$$

Using Lemma 2.3.1, (3.124), (3.126) and (3.120), we discover

$$\begin{aligned}
\frac{2}{\delta^n} E(Dv, \bar{B}_{i+k}) &\leq \frac{2}{\delta^n} \fint_{\bar{B}_{i+k}} |Dv - (D\bar{w}_{i-1})_{\bar{B}_{i+k}}| \, dx \\
&\leq \frac{2}{\delta^n} E(D\bar{w}_{i-1}, \bar{B}_{i+k}) + \frac{2}{\delta^n} \fint_{\bar{B}_{i+k}} |Dv - D\bar{w}_{i-1}| \, dx \\
&\leq \frac{2c_\beta \delta^{k\beta}}{\delta^n} E(D\bar{w}_{i-1}, \bar{B}_i) + \frac{2}{\delta^{(k+2)n}} \fint_{\frac{1}{2}\bar{B}_{i-1}} |Dv - D\bar{w}_{i-1}| \, dx \\
&\leq \frac{2c_\beta \delta^{k\beta}}{10^5 \delta^n} \lambda + \frac{2}{10^5} \lambda \leq \frac{\lambda}{2000}.
\end{aligned}$$

Since $C_{i+k} > \dfrac{\lambda}{1000}$, we find from (3.123) that

$$\sum_{m=-1}^{0} \fint_{\bar{B}_{i+k+m}} |Dv|\,dx \geq \frac{\lambda}{2000}. \tag{3.127}$$

Combining (3.124) and (3.127), we discover

$$\begin{aligned}
\frac{\lambda}{2000} &\leq \sum_{m=-1}^{0} \fint_{\bar{B}_{i+m+k}} |Dv|\,dx \\
&\leq \frac{2}{\delta^{(k+1)n}} \fint_{\frac{1}{2}\bar{B}_{i-1}} |Dv - D\bar{w}_{i-1}|\ dx + \sum_{m=-1}^{0} \fint_{\bar{B}_{i+m+k}} |D\bar{w}_{i-1}|\ dx \\
&\leq \frac{2\lambda}{10^6} + 2\sup_{\bar{B}_i} |D\bar{w}_{i-1}|
\end{aligned}$$

and therefore

$$\frac{\lambda}{10^4} \leq \frac{\lambda}{4000} - \frac{\lambda}{10^6} \leq \sup_{\bar{B}_i} |D\bar{w}_{i-1}|. \tag{3.128}$$

From (3.125), (3.126), (3.128) and (3.119), we observe

$$\frac{\lambda}{H} \leq |D\bar{w}_{i-1}| \leq H\lambda \quad \text{in } \bar{B}_i.$$

Therefore, we are under the hypothesis of Lemma 3.3.1, which give us the following excess decay estimate:

$$\begin{aligned}
E(Dv, \bar{B}_{i+1}) &\leq E(D\bar{w}_{i-1}, \bar{B}_{i+1}) + \fint_{\bar{B}_{i+1}} |Dv - D\bar{w}_{i-1}|\,dx \\
&\leq c_\beta \delta^\beta E(D\bar{w}_{i-1}, \bar{B}_i) + \delta^{-n} \fint_{\bar{B}_i} |Dv - D\bar{w}_{i-1}|\,dx \\
&\leq c_\beta \delta^\beta E(Dv, \bar{B}_i) + 2c_\beta \delta^{-n} \fint_{\bar{B}_i} |Dv - D\bar{w}_{i-1}|\,dx \\
&\leq \frac{1}{4} E(Dv, \bar{B}_i) + 2c_\beta c_5 \delta^{-n} \omega(\bar{r}_{i-1})\lambda.
\end{aligned} \tag{3.129}$$

**Step 3: Final induction.**

We now assert

$$a_i + E_i \le \lambda, \qquad {}^{\forall} i \ge i_e. \tag{3.130}$$

Since $C_{i_e} \le \frac{\lambda}{1000}$, (3.130) holds for $i = i_e$. We now assume that (3.130) holds for every $j \in \{i_e, i_e+1, \cdots, i\}$ for some $i \ge i_e$. Consequently, Ind($i$) holds for every $j \in \{i_e, i_e+1, \cdots, i\}$. We apply (3.129) iteratively to discover

$$\begin{aligned}\sum_{j=i_e}^{i+1} E_j &\le E_{i_e} + \frac{1}{4}\sum_{j=i_e}^{i} E_j + \frac{2c_\beta c_5}{\delta^n}\sum_{j=i_e}^{i}\omega(\bar r_j)\lambda \\ &\le 2E_{i_e} + \frac{4c_\beta c_5}{\delta^n}\sum_{j=i_e}^{i}\omega(\bar r_j)\lambda \le \frac{\delta^n\lambda}{200}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}a_{i+1} &= a_{i_e} + \sum_{j=i_e}^{i}(a_{j+1} - a_j) \\ &\le a_{i_e} + \sum_{j=i_e}^{i} ⨍_{\bar B_{j+1}} |Dv - (Dv)_j|\,dx \le C_{i_e} + \frac{1}{\delta^n}\sum_{j=i_e}^{i} E_j \le \frac{\lambda}{100}.\end{aligned}$$

By induction, (3.130) holds for every $i \ge i_e$. Consequently, we obtain

$$|Dv(x_1)| = \lim_{i\to\infty} |a_i| \le \lambda.$$

□

**Theorem 3.3.3.** *Let $v \in W^{1,G}(B_R)$ be the weak solution to* (3.14)*. Assume that*

$$\sup_{B_{R/2}} |Dv| \le \mathcal{H}\lambda \tag{3.131}$$

*for a constant $\mathcal{H} \ge 1$ and a number $\lambda > 0$. Then for any $\sigma_1 \in (0,1)$, there exist constants $R_3 = R_3(\boldsymbol{data}, \mathcal{H}, \sigma_1, \omega, |\mu|(\Omega), \|Du\|_{L^1(\Omega)}) > 0$ and $\delta_1 = \delta_1(\boldsymbol{data}, \mathcal{H}, \sigma_1) \in \left(0, \frac{1}{16}\right)$ such that*

$$\operatorname*{osc}_{\delta_1 B_R} Dv \le \sigma_1\lambda,$$

*whenever* $0 < R \le R_3$.

*Proof.* **Step 1: Basic setup.**

Let $x_1 \in \frac{1}{4}B_R$ be any Lebesgue point of $Dv$. Let $\epsilon > 0$ be a small constant, which will be chosen later in this proof. We take positive constants $H_\epsilon$ and $\delta_\epsilon \le \frac{1}{16}$ such that

$$H_\epsilon := \frac{2^{n+7}c_l\mathcal{H}}{\epsilon} \quad \text{and} \quad 2c_\beta\delta_\epsilon^\beta\mathcal{H} \le \frac{\epsilon}{2^{n+7}}, \tag{3.132}$$

where $\beta, c_\beta, c_l$ are given in Lemma 2.3.1. We set $\delta = \delta_\epsilon$ in (3.117) and take a positive constant $R_{1,\epsilon} \le R_1$ such that

$$\omega(\tau)^{\frac{2}{p+1}} + \delta_\epsilon^{-2}\int_0^\tau \omega(\rho)\frac{d\rho}{\rho} \le \frac{\delta_\epsilon^{2n}\epsilon}{2^{n+10}c_2c_5\mathcal{H}}, \tag{3.133}$$

whenever $0 < \tau < R_{1,\epsilon}$. Here, $c_2 = c_2(n,\nu,L,p)$ and $c_5 = c_5(\texttt{data}, H_\epsilon, \delta_\epsilon)$ are the constants given in (3.57) and Lemma 3.3.1, respectively. Assume $0 < R \le R_{1,\epsilon}$. By (3.122), we see

$$\sum_{i=0}^{\infty}\omega(\bar{r}_i) \le \delta_\epsilon^{-2}\int_0^{2\bar{r}_0}\omega(\rho)\frac{d\rho}{\rho} \le \frac{\delta_\epsilon^{2n}\epsilon}{2^{n+10}c_2c_5\mathcal{H}}. \tag{3.134}$$

For $i \ge 1$, we set

$$C_i = \fint_{\bar{B}_i}|Dv|\,dx, \quad \mathcal{L} = \left\{i \in \mathbb{N} : C_i \le \frac{\epsilon\lambda}{2^{n+5}}\right\} \quad \text{and} \quad i_m = \min\mathcal{L}. \tag{3.135}$$

If $\mathcal{L}$ is empty, then we define $i_m = \infty$.

**Step 2: VMO estimate.**

First, we assume that $i \ge 1$ is an integer such that $i+1 \notin \mathcal{L}$. Using (3.57), (3.131) and (3.133), we have

$$\begin{aligned}\fint_{\bar{B}_{i+1}}|D\bar{w}_{i-1}|\,dx &\ge C_{i+1} - \delta_\epsilon^{-2n}\fint_{\frac{1}{2}\bar{B}_{i-1}}|Dv - D\bar{w}_{i-1}|\,dx \\ &\ge \frac{\epsilon\lambda}{2^{n+5}} - c_2\delta_\epsilon^{-2n}\omega(\bar{r}_{i-1})^{\frac{2}{p+1}}\mathcal{H}\lambda \ge \frac{\epsilon\lambda}{2^{n+6}},\end{aligned} \tag{3.136}$$

where we have used the assumption $i + 1 \notin \mathcal{L}$.

Likewise, we have

$$\fint_{\frac{1}{2}\bar{B}_{i-1}} |D\bar{w}_{i-1}|\,dx \le \fint_{\frac{1}{2}\bar{B}_{i-1}} |Dv|\,dx + \fint_{\frac{1}{2}\bar{B}_{i-1}} |D\bar{w}_{i-1} - Dv|\,dx$$
$$\le \mathcal{H}\lambda + c_2\delta_\epsilon^{-n}\omega(\bar{r}_{i-1})^{\frac{2}{p+1}}\mathcal{H}\lambda \le 2\mathcal{H}\lambda.$$

It then follows from Lemma 2.3.1 and (3.132) that

$$\|D\bar{w}_{i-1}\|_{L^\infty(\frac{1}{4}\bar{B}_{i-1})} \le H_\epsilon\lambda \quad \text{and} \quad \operatorname*{osc}_{\bar{B}_i} |D\bar{w}_{i-1}| \le \frac{\epsilon\lambda}{2^{n+7}}. \tag{3.137}$$

Combining (3.136) and (3.137), we find

$$\frac{\lambda}{H_\epsilon} \le \frac{\epsilon\lambda}{2^{n+7}} \le |D\bar{w}_{i-1}| \le H_\epsilon\lambda \quad \text{in } \bar{B}_i.$$

Therefore, we can apply Lemma 3.3.1 to discover

$$\fint_{\bar{B}_i} |Dv - D\bar{w}_{i-1}|\ dx \le c_5\omega\left(\bar{r}_{i-1}\right)\lambda.$$

Following the calculations as in (3.129) with (3.133) and (3.131), we have

$$E(Dv, \bar{B}_{i+1}) \le \frac{\epsilon}{10\mathcal{H}} E(Dv, \bar{B}_i) + 4c_5\delta_\epsilon^{-n}\omega(\bar{r}_{i-1})\lambda \le \epsilon\lambda. \tag{3.138}$$

On the other hand, if $i+1 \in \mathcal{L}$, then we have

$$E(Dv, \bar{B}_{i+1}) \le 2C_{i+1} \le \frac{\epsilon\lambda}{2^{n+4}}. \tag{3.139}$$

For any positive constant $\rho \le \delta_\epsilon^3 R$, there exist $m \ge 2$ and $r \in (\delta_\epsilon^2 R, \delta_\epsilon R]$ such that $\rho = \delta_\epsilon^m r$. It then follows from (3.138) and (3.139) that

$$\sup_{x_1 \in \frac{1}{4}B_R}\ \sup_{0 \le \rho \le \delta_\epsilon^3 R} E(Dv, B_\rho(x_1)) \le \epsilon\lambda. \tag{3.140}$$

**Step 3: Proof of Theorem 3.3.3.**
The estimate (3.140) shows that there exist positive constants $\tilde{\delta}$ and $R_3$

such that

$$\sup_{x_1 \in \frac{1}{4}B_R} \sup_{0 \le \rho \le \tilde{\delta} R} E(Dv, B_\rho(x_1)) \le \frac{\delta_{\sigma_1}^n \sigma_1 \lambda}{2^{n+8}}, \tag{3.141}$$

whenever $0 < R < R_3$. We now fix the constant $\epsilon = \sigma_1$ with $\delta = \delta_{\sigma_1}$ in **Step 1** and $r = \tilde{\delta} R$. Recall (3.135) in order to prove that

$$|(Dv)_{\bar{B}_k} - (Dv)_{\bar{B}_h}| \le \frac{\sigma_1 \lambda}{2^{n+3}} \quad \text{for every } 2 \le k \le h. \tag{3.142}$$

*Case* $1 : k < h \le i_m$. By the definition of $i_m$ in (3.135), $C_{i+1} \ge \frac{\sigma_1 \lambda}{2^{n+5}}$ holds for every $i \in \{k-1, k, \cdots, h-2\}$. Applying (3.138) iteratively, it follows from (3.141) and (3.134) that

$$\begin{aligned} \sum_{i=k}^{h-1} E(Dv, \bar{B}_i) &\le E(Dv, \bar{B}_k) + \frac{\sigma_1}{10} \sum_{i=k}^{h-2} E(Dv, \bar{B}_i) + \sum_{i=k}^{h-2} 4 c_5 \delta_{\sigma_1}^{-n} \omega(\bar{r}_{i-1}) \lambda \\ &\le 2E(Dv, \bar{B}_k) + 8 c_5 \delta_{\sigma_1}^{-n} \lambda \sum_{i=k-1}^{h-2} \omega(\bar{r}_{i-1}) \le \frac{\delta_{\sigma_1}^n \sigma_1 \lambda}{2^{n+6}}. \end{aligned} \tag{3.143}$$

Consequently, we obtain

$$\begin{aligned} |(Dv)_{\bar{B}_k} - (Dv)_{\bar{B}_h}| &\le \sum_{i=k}^{h-1} |(Dv)_{\bar{B}_i} - (Dv)_{\bar{B}_{i+1}}| \\ &\le \delta_{\sigma_1}^{-n} \sum_{i=k}^{h-1} E(Dv, \bar{B}_i) \le \frac{\sigma_1 \lambda}{2^{n+6}}. \end{aligned} \tag{3.144}$$

*Case* $2 : i_m \le k < h$. In this case, (3.142) is immediately obtained by the following estimates.

$$|(Dv)_{\bar{B}_h}| \le \frac{\sigma_1 \lambda}{2^{n+4}} \quad \text{and} \quad |(Dv)_{\bar{B}_k}| \le \frac{\sigma_1 \lambda}{2^{n+4}}. \tag{3.145}$$

If $h \in \mathcal{L}$, then the first inequality of (3.145) holds. We now assume $h \notin \mathcal{L}$. Then there exists $i_h \in \mathcal{L}$ such that $\{i_h + 1, i_h + 2, \cdots, h\} \cap \mathcal{L} = \varnothing$. The calculations as in (3.143) give

$$\sum_{i=i_h}^{h-1} E(Dv, \bar{B}_i) \le 2E(Dv, \bar{B}_{i_h}) + 8c_5\delta_{\sigma_1}^{-n}\lambda \sum_{i=k-1}^{h-2} \omega(\bar{r}_{i-1}) \le \frac{\delta_{\sigma_1}^n \sigma_1 \lambda}{2^{n+6}}.$$

Similarly to (3.144), we find

$$|(Dv)_{\bar{B}_h}| \le |(Dv)_{\bar{B}_{i_h}}| + \sum_{i=i_h}^{h-1} |(Dv)_{\bar{B}_i} - (Dv)_{\bar{B}_{i+1}}| \le \frac{\sigma_1\lambda}{2^{n+4}},$$

as $i_h \in \mathcal{L}$. One can obtain the second estimate of (3.145) by the same argument.

*Case* 3 : $k < i_m < h$. We assert that (3.145) also holds in this case. One can apply the calculations as in *Case* 2 to obtain the first estimate of (3.145). To prove the second one of (3.145), we recall (3.135) and (3.143) to estimate as follows:

$$|(Dv)_{\bar{B}_k}| \le |(Dv)_{\bar{B}_{i_m}}| + \sum_{i=k}^{i_m-1} |(Dv)_{\bar{B}_i} - (Dv)_{\bar{B}_{i+1}}| \le \frac{\sigma_1\lambda}{2^{n+4}}.$$

Consequently, these three cases show that (3.142) holds for every $2 \le k \le h$.

We now consider any $0 < \rho_1, \rho_2 \le \delta^2\tilde{\delta}R$. Then there exist two integers $k, h \ge 2$ such that $r_{k+1} < \rho_1 \le r_k$ and $r_{h+1} < \rho_2 \le r_h$. Applying (3.141), we have

$$\begin{aligned} |(Dv)_{\bar{B}_k} - (Dv)_{\bar{B}_{\rho_1}(x_1)}| &\le \fint_{B_{\rho_1}(x_1)} \left|Dv - (Dv)_{\bar{B}_k}\right| dx \\ &\le \delta_{\sigma_1}^{-n} E(Dv, \bar{B}_k) \le \frac{\sigma_1\lambda}{2^{n+8}} \end{aligned}$$

and

$$|(Dv)_{B_h} - (Dv)_{B_{\rho_2}(x_1)}| \le \frac{\sigma_1\lambda}{2^{n+8}}.$$

It then follows from (3.142) that

$$|(Dv)_{B_{\rho_1}(x_1)} - (Dv)_{B_{\rho_2}(x_2)}| \le \frac{\sigma_1\lambda}{2^{n+2}}.$$

Consequently, we conclude that

$$\begin{aligned}
&|Dv(y) - Dv(z)| \\
&\le |Dv(y) - (Dv)_{B_{2\delta_1 R}(y)}| + |(Dv)_{B_{2\delta_1 R}(y)} - (Dv)_{B_{4\delta_1 R}(z)}| \\
&\qquad + |(Dv)_{B_{4\delta_1 R}(z)} - Dv(z)| \\
&\le \frac{\sigma_1 \lambda}{2^n} + \fint_{B_{2\delta_1 R}(y)} |Dv - (Dv)_{B_{4\delta_1 R}(z)}|\, dx \\
&\le \frac{\sigma_1 \lambda}{2^n} + 2^n E(Dv, B_{4\delta_1 R}(z)) \le \sigma_1 \lambda.
\end{aligned}$$

for any $y, z \in B_{\delta_1 R}$ with $\delta_1 = \delta^2 \tilde{\delta}$. □

## 3.4 Proof of Theorem 3.1.3

We are now all set in proving Theorem 3.1.3. A main technique of our proof is based on a double step induction argument as in [79].

*Proof of Theorem 3.1.3.* Let $x_0 \in \Omega$ be a Lebesgue point of $Du$ and $B_{2R}(x_0) \subset \Omega$. We take the concentric balls given in (3.64) and the corresponding weak solutions $v_i$ and $w_i$.

**Step 1: Basic setup.**

Keeping Lemma 3.2.14, Lemma 3.2.15 and Theorem 3.3.3 in mind, we select

$$H := 1000^{4np+2} c_l c_6, \quad \sigma_1 := 10^{-5} \quad \text{and} \quad \delta_0 := \left( \frac{1}{4^{n+4} c_\beta} \right)^{\frac{1}{\beta}},$$

where $\beta, c_l, c_\beta$ are the constants given in Lemma 2.3.1 and $c_6$ is the constant given in Theorem 3.3.2. By replacing $H$ in Theorem 3.3.3 by $\mathcal{H}$, we can find a constant $\delta_1 = \delta_1(n, \nu, L, p)$ given there. We note that the constants in this section depend only on `data` and $\omega$. We further take $\delta \in (0, \frac{1}{16})$ and the smallest integer $k \ge 2$ satisfying

$$\delta := \min \left\{ \delta_0, \delta_1, H^{\frac{1}{n(k+6)}}, \left( \frac{1}{16^{n(p+1)} c_\beta} \right)^{\frac{1}{\beta}} \right\} \quad \text{and} \quad 2^{n+5} c_\beta \delta^{k\beta} \le \frac{\delta^n}{10^6}. \tag{3.146}$$

Recall the constants $c_1$ and $c_4$ from (3.32) and Lemma 3.2.15, respectively.

We then define a positive number $R_0 \leq \min\{R_1, R_2, R_3\}$ depending only on $\texttt{data}, \omega_{a(\cdot)}, \omega_{\gamma(\cdot)}, |\mu|(\Omega), \|Du\|_{L^1(\Omega)}$ such that

$$2c_2\delta^{-n(k+6)}\omega(\tau)^{\frac{2}{p+1}} + c_4\delta^{-(2n+1)}\int_0^{2\tau}\omega(\rho)\frac{d\rho}{\rho} \leq \frac{1}{2^n 10^6} \tag{3.147}$$

for every $0 < \tau \leq R_0$.

We now set

$$\lambda := H_1 \fint_{B_{2R}(x_0)} |Du|\,dx + H_2 g_0^{-1}\left(\int_0^{2R}\frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\frac{d\rho}{\rho}\right), \tag{3.148}$$

where

$$H_1 := 2^n 100^{\frac{n}{p-1}} 10^6 \delta^{-(k+6)n}, \quad H_2 := 2^n 10^6 H c_1 c_4 \delta^{-(k+6)n} \tag{3.149}$$

and $R = r_0 \in \left(0, \frac{R_0}{2}\right]$. By a direct calculation, we find

$$\begin{aligned}
\int_0^{2r_0}\frac{|\mu|(B_\rho)}{\rho^{n-1}}\frac{d\rho}{\rho} &= \sum_{i=0}^{\infty}\int_{r_{i+1}}^{r_i}\frac{|\mu|(B_\rho)}{\rho^{n-1}}\frac{d\rho}{\rho} + \int_{r_0}^{2r_0}\frac{|\mu|(B_\rho)}{\rho^{n-1}}\frac{d\rho}{\rho} \\
&\geq \delta^{n-1}\log\left(\frac{1}{\delta}\right)\sum_{i=0}^{\infty}\frac{|\mu|(\bar{B}_{i+1})}{r_{i+1}^{n-1}} + \frac{\log 2}{2^{n-1}}\left(\frac{|\mu|(\bar{B}_0)}{r_0^{n-1}}\right) \\
&\geq \delta^n \sum_{i=0}^{\infty}\frac{|\mu|(\bar{B}_i)}{r_i^{n-1}}.
\end{aligned}$$

Thus (3.148) and (2.5) implies

$$H_2\delta^{\frac{n}{p}} g_0^{-1}\left(\sum_{i=0}^{\infty}\frac{|\mu|(\bar{B}_i)}{r_i^{n-1}}\right) \leq \lambda \tag{3.150}$$

and

$$g_0^{-1}\left(\sum_{i=0}^{\infty}\frac{|\mu|(\bar{B}_i)}{r_i^{n-1}}\right) \leq \frac{\delta^{(k+5)n}}{2^n 10^6 H c_1 c_4}\lambda.$$

Note that (3.150) shows the first assumption in Lemma 3.2.15 for $i \geq 1$.

Recalling (3.122) and (3.147), we discover

$$2c_2\delta^{-n(k+4)}\omega(r_0)^{\frac{2}{p+1}} + c_4\sum_{i=0}^{\infty}\omega(r_i) \le \frac{\delta^{2n}}{2^n 10^6}. \tag{3.151}$$

For simplicity, we denote

$$E_i := E(Du, B_i) \quad \text{and} \quad a_i := |(Dv)_{\bar{B}_i}|. \tag{3.152}$$

**Step 2: Exit time argument and induction scheme.**
For each $i \ge 1$, we define

$$C_i := \sum_{m=-1}^{0} \fint_{B_{i+m}} |Du|\,dx + 2\delta^{-n}\fint_{B_i} |Du - (Du)_{B_i}|\,dx.$$

By (3.148) and (3.149), it follows that

$$C_1 \le 6\delta^{-2n}\fint_{B_0}|Du|\,dx \le \frac{\lambda}{1000}.$$

If there exists an infinite sequence $\{i_j\}_{j\in N} \subset \mathbb{N}$ such that $C_{i_j} \le \frac{\lambda}{1000}$ for every $j \in N$, then we see

$$|Du(x_0)| = \lim_{j\to\infty}\fint_{B_{i_j}}|Du|\,dx \le \frac{\lambda}{1000},$$

and we are done. Therefore, we only consider the case that there exists an exit time index $1 \le i_e$ satisfying

$$C_i > \frac{\lambda}{1000} \quad \text{for} \quad i \in \{i_e+1, i_e+2, \cdots\} \qquad \text{and} \qquad C_{i_e} \le \frac{\lambda}{1000}. \tag{3.153}$$

We say that "$\text{Ind}_1(i)$ holds", if

$$a_{i-1} + a_i = \fint_{B_{i-1}}|Du|\,dx + \fint_{B_i}|Du|\,dx \le \lambda \quad (i \ge i_e)$$

and say that "$\mathrm{Ind}_2(i)$ holds", if

$$\sum_{j=i_e+1}^{i} E_j \le \frac{1}{2}\sum_{j=i_e}^{i-1} E_j + \frac{2c_4}{\delta^n}\lambda\sum_{j=i_e}^{i-1}\omega(r_j) + \frac{2c_4}{\delta^n}\frac{\lambda}{g_0(\lambda)}\sum_{j=i_e-1}^{i-2}\left[\frac{|\mu|(B_j)}{r_j^{n-1}}\right] \quad (i > i_e).$$

Our aim is to show that $\mathrm{Ind}_1(i)$ holds for every $i \ge i_e$, which implies that

$$|Du(x_0)| = \lim_{i\to\infty} \fint_{B_i} |Du|\,dx \le \lambda. \tag{3.154}$$

Our proof will proceed as follows:

$$\begin{aligned} \mathrm{Ind}_1(i_e) \quad &\Rightarrow \quad \mathrm{Ind}_2(i_e+1), && (3.155)\\ \mathrm{Ind}_1(i) \quad \text{and} \quad \mathrm{Ind}_2(i) \quad &\Rightarrow \quad \mathrm{Ind}_2(i+1), \quad {}^\forall i > i_e, && (3.156)\\ \mathrm{Ind}_1(i) \quad \text{and} \quad \mathrm{Ind}_2(i+1) \quad &\Rightarrow \quad \mathrm{Ind}_1(i+1), \quad {}^\forall i > i_e. && (3.157)\end{aligned}$$

We note that $C_{i_e} \le \frac{\lambda}{1000}$ directly implies $\mathrm{Ind}_1(i_e)$.

**Step 3: Estimates obtained by $\mathrm{Ind}_1(i)$.**

Assuming that $\mathrm{Ind}_1(i)$ holds for $i \ge i_e$, we shall obtain the assumptions in Lemma 3.2.15. At first, we are going to find upper bounds of $|Dv_i|$ and $|Dv_{i-1}|$ in Lemma 3.2.15. Applying (3.32) and (3.150), we discover

$$\begin{aligned}\fint_{B_{i-1+l}} |Du - Dv_{i-1}|\,dx &\le \delta^{-nl}\fint_{B_{i-1}} |Du - Dv_{i-1}|\,dx\\ &\le c_1\delta^{-nl} g_0^{-1}\left(\frac{|\mu|(B_{i-1})}{r_{i-1}}\right) \le \frac{\delta^{4n}}{2^n 10^6}\lambda \end{aligned}\tag{3.158}$$

and similarly,

$$\fint_{B_{i+l}} |Du - Dv_i|\,dx \le \frac{\delta^{4n}}{2^n 10^6}\lambda, \tag{3.159}$$

whenever $l \in \{0, 1, ..., k+1\}$. $\mathrm{Ind}_1(i)$ and 3.158 with $l = 0$ implies

$$\fint_{B_{i-1}} |Dv_{i-1}|\,dx \le \frac{\delta^{4n}\lambda}{2^n 10^6} + \fint_{B_{i-1}} |Du|\,dx \le 2\lambda.$$

It then follows from Theorem 3.3.2 and Theorem 3.3.3 that

$$\sup_{\frac{3}{4}B_{i-1}} |Dv_{i-1}| \leq 2c_6 \fint_{B_{i-1}} |Dv_{i-1}|\, dx \leq H\lambda \text{ and } \operatorname*{osc}_{B_i} Dv_{i-1} \leq \frac{\lambda}{10^5}, \quad (3.160)$$

where we have used (3.146) and (3.160).

Similarly, one can obtain the following estimates for $Dv_i$:

$$\fint_{B_i} |Dv_i|\, dx \leq 2\lambda \quad \text{and} \quad \sup_{\frac{3}{4}B_i} |Dv_i| \leq H\lambda. \quad (3.161)$$

Next, we want to show a lower bound of $|Dv_i|$. Applying (3.57), (3.151) and (3.161), we discover

$$\begin{aligned}\fint_{\frac{1}{2}B_i} |Dv_i - Dw_i|\, dx &\leq c_2 \omega(r_i)^{\frac{2}{p+1}} \fint_{\frac{3}{4}B_i} |Dv_i|\, dx \\ &\leq 2^{n+1} c_2 \omega(r_i)^{\frac{2}{p+1}} \lambda \leq \frac{\delta^{n(k+6)}}{10^6}\lambda,\end{aligned}$$

and

$$\fint_{B_{i+l}} |Dv_i - Dw_i|\, dx \leq \frac{\delta^{5n}}{10^6}\lambda, \quad (3.162)$$

whenever $l \in \{1, ..., k+1\}$. For any $l \in \{1, ..., k+1\}$, we combine (3.159) and (3.162) to see

$$\fint_{B_{i+l}} |Du - Dw_i|\, dx \leq \frac{\delta^n}{10^6}\lambda. \quad (3.163)$$

For this reason, $\mathrm{Ind}_1(i)$ implies

$$\begin{aligned}\fint_{\frac{1}{2}B_i} |Dw_i|\, dx &\leq \fint_{\frac{1}{2}B_i} |Du|\, dx + \fint_{\frac{1}{2}B_i} |Du - Dw_i|\, dx \\ &\leq 2^n\lambda + \lambda \leq 2^{n+1}\lambda. \quad (3.164)\end{aligned}$$

It then follows from Lemma 2.3.1 that

$$\sup_{\frac{1}{4}B_i} |Dw_i| \leq c_l \fint_{\frac{1}{2}B_i} |Dw_i|\, dx \leq H\lambda.$$

At this time, we apply the second estimates of Lemma 2.3.1 to see

$$\begin{aligned}
2\delta^{-n}E(Du, B_{i+k}) &\leq 4\delta^{-n}E(Dw_i, B_{i+k}) + 4\delta^{-n} \fint_{B_{i+k}} |Du - Dw_i|\, dx \\
&\leq 8c_\beta \delta^{k\beta - n} E(Dw_i, \frac{1}{2}B_i) + \frac{4}{10^6}\lambda \\
&\leq 16 c_\beta \delta^{k\beta - n} \fint_{\frac{1}{2}B_i} |Dw_i|\, dx + \frac{4}{10^6}\lambda \\
&\leq 2^{n+5} c_\beta \delta^{k\beta - n}\lambda + \frac{2}{10^6}\lambda \leq \frac{1}{10^5}\lambda,
\end{aligned}$$

where we have used (3.163), (3.164) and (3.146).

Since $C_{i+k} > \frac{\lambda}{1000}$ for $i_e \leq i$, we discover

$$\sum_{m=-1}^{0} \fint_{B_{i+m+k}} |Du|\, dx \geq \frac{\lambda}{1000} - \frac{\lambda}{10^5} \geq \frac{\lambda}{2000}. \tag{3.165}$$

In addition, (3.158) with $l = k, k+1$ gives

$$\begin{aligned}
&\sum_{m=-1}^{0} \fint_{B_{i+m+k}} |Du|\, dx \\
&\leq \sum_{m=-1}^{0} \left( \fint_{B_{i+m+k}} |Du - Dv_{i-1}|\, dx + \fint_{B_{i+m+k}} |Dv_{i-1}|\, dx \right) \\
&\leq \frac{\lambda}{10^6} + \sum_{m=-1}^{0} \fint_{B_{i+m+k}} |Dv_{i-1}|\, dx \leq \frac{\lambda}{10^6} + 2 \sup_{B_i} |Dv_{i-1}|.
\end{aligned} \tag{3.166}$$

Combining (3.165), (3.166) and (3.160), we have

$$\frac{\lambda}{H} \leq \frac{\lambda}{5000} - \frac{\lambda}{10^5} \leq \sup_{B_i} |Dv_{i-1}| - \operatorname*{osc}_{B_i} |Dv_{i-1}| = \inf_{B_i} |Dv_{i-1}|. \tag{3.167}$$

Hence, (3.150), (3.160), (3.161) and (3.167) allow us to apply Lemma 3.2.15.

**Step 4: Verification of** $\mathrm{Ind}_2(i_e + 1)$ **and** $\mathrm{Ind}_2(i + 1)$**.**

According to Lemma 2.3.1, Lemma 3.2.15 and the assumption (3.146), we find

$$\begin{aligned} E(Dw_i, B_{i+1}) &\le 2\delta^\beta c_\beta E(Dw_i, \frac{1}{2}B_i) \\ &\le 2^{-2n-5} \fint_{\frac{1}{2}B_i} |Du - (Du)_{B_i}|\, dx + 2^{-2n-5} \fint_{\frac{1}{2}B_i} |Du - Dw_i|\, dx \\ &\le \frac{1}{4}E_i + \frac{1}{4} \fint_{\frac{1}{2}B_i} |Du - Dw_i|\, dx \\ &\le \frac{1}{4}E_i + c_4\omega(r_i)\lambda + c_4 \frac{\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right], \end{aligned}$$

but then it follows from Lemma 3.2.15 that

$$\begin{aligned} E_{i+1} &\le 2 \fint_{B_{i+1}} |Du - Dw_i|\, dx + 2 \fint_{B_{i+1}} |Dw_i - (Dw_i)_{B_{i+1}}|\, dx \\ &\le \frac{1}{2}E_i + \frac{2c_4}{\delta^n}\omega(r_i)\lambda + \frac{2c_4}{\delta^n}\frac{\lambda}{g_0(\lambda)} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right]. \end{aligned} \tag{3.168}$$

This estimate (3.168) with $i = i_e$ shows that $\mathrm{Ind}_2(i_e + 1)$ holds.

To prove (3.156), we assume that $\mathrm{Ind}_1(i)$ and $\mathrm{Ind}_2(i)$ hold. Consequently, (3.168) and $\mathrm{Ind}_2(i)$ yields

$$\begin{aligned} \sum_{j=i_e+1}^{i+1} E_j &\le \frac{1}{2}\sum_{j=i_e}^{i-1} E_j + \frac{2c_4}{\delta^n}\lambda \sum_{j=i_e}^{i-1} \omega(r_j) + \frac{2c_4}{\delta^n}\frac{\lambda}{g_0(\lambda)} \sum_{j=i_e}^{i-1} \left[\frac{|\mu|(B_{i-1})}{r_{i-1}^{n-1}}\right] + E_{i+1} \\ &\le \frac{1}{2}\sum_{j=i_e}^{i} E_j + \frac{2c_4}{\delta^n}\lambda \sum_{j=i_e}^{i} \omega(r_j) + \frac{2c_4}{\delta^n}\frac{\lambda}{g_0(\lambda)} \sum_{j=i_e}^{i} \left[\frac{|\mu|(B_{j-1})}{r_{j-1}^{n-1}}\right]. \end{aligned} \tag{3.169}$$

**Step 5: Verification of** $\mathrm{Ind}_1(i+1)$**.**

It remains to show (3.157). We assume that $\mathrm{Ind}_1(i)$ and $\mathrm{Ind}_2(i+1)$ hold for some $i \ge i_e$. Then, for all $l \in \{i_e, ..., i\}$, we see

$$a_{l+1} - a_l \le \fint_{B_{l+1}} |Du - (Du)_{B_l}|\, dx \le \frac{1}{\delta^n} \fint_{B_l} |Du - (Du)_{B_l}|\, dx = \frac{E_l}{\delta^n}$$

and

$$a_{l+1} \leq a_{i_e} + \sum_{i=i_e}^{l+1} \fint_{B_{i+1}} |Du - (Du)_{B_i}|\, dx = a_{i_e} + \frac{1}{\delta^n} \sum_{i=i_e}^{l+1} E_i. \tag{3.170}$$

Continuously applying (3.169), (3.150) and (3.151) give

$$\begin{aligned}
\sum_{j=i_e}^{l+1} E_j &\leq E_{i_e} + \frac{1}{2} \sum_{j=i_e+1}^{l} E_j + \frac{2c_4}{\delta^n}\lambda \sum_{j=i_e}^{l} \omega(r_j) + \frac{2c_4}{\delta^n} \frac{\lambda}{g_0(\lambda)} \sum_{j=i_e}^{l} \left[\frac{|\mu|(B_{j-1})}{r_{j-1}^{n-1}}\right] \\
&\leq 2E_{i_e} + \frac{4c_4}{\delta^n}\lambda \sum_{j=i_e}^{l} \omega(r_j) + \frac{4c_4}{\delta^n} \frac{\lambda}{g_0(\lambda)} \sum_{j=i_e}^{l} \left[\frac{|\mu|(B_{j-1})}{r_{j-1}^{n-1}}\right] \\
&\leq 2E_{i_e} + \frac{\delta^n \lambda}{10^5} \leq \delta^n \left(C_{i_e} + \frac{\lambda}{10^5}\right)
\end{aligned} \tag{3.171}$$

for every $l \in \{i_e, ..., i\}$. Combining (3.153), (3.170) and (3.171),

$$a_{l+1} \leq 2C_{i_e} + \frac{\lambda}{10^5} \leq \frac{\lambda}{100}$$

for all $l \in \{i_e, ..., i\}$. The last inequality directly implies that $\mathrm{Ind}_1(i+1)$ holds. Therefore, (3.155), (3.156) and (3.157) hold, which implies the claim (3.154). This completes the proof. □

## 3.5 Gradient continuity via Riesz potentials

In this section, we prove Theorem 3.1.7 and Theorem 3.1.9. To this end, we assume that the nonhomogeneous term $\mu$ satisfies that

$$\lim_{\tau \to 0} \frac{|\mu|(B_\tau(x))}{\tau^{n-1}} = 0 \quad \text{locally uniformly in } \Omega \text{ w.r.t. } x. \tag{3.172}$$

Then according to Theorem 3.1.3, $Du$ is locally bounded. Therefore, for any open subsets $\Omega' \Subset \Omega'' \Subset \Omega$, we can define

$$\lambda := \|Du\|_{L^\infty(\Omega'')} \quad \text{and} \quad d := \mathrm{dist}(\Omega', \partial\Omega'') > 0. \tag{3.173}$$

Recall (3.64) and (3.152) with the corresponding solutions $v_i$ and $w_i$. Once we obtain the following proposition, then the proof of Theorem 3.1.7 is verbatim repetition of that of [76, Theorem 1.5].

**Proposition 3.5.1.** *Under the assumption* (3.172)*, $Du$ is locally VMO-regular in $\Omega$. More precisely, for any $\epsilon \in (0,1)$, there exists a small constant $r_\epsilon = r_\epsilon(\epsilon, n, p, \nu, L, \lambda, \omega_{a(\cdot)}, \omega_{\gamma(\cdot)}, |\mu|(\Omega), \|Du\|_{L^1(\Omega)}) < d$ such that*

$$\fint_{B_\rho(x_0)} \left|Du - (Du)_{B_\rho(x_0)}\right| \, dx < \epsilon\lambda,$$

*whenever $\rho \in (0, r_\epsilon]$ and $x_0 \in \Omega'$.*

*Proof.* Keeping Theorem 3.3.3 in mind, we take $H$ and $\sigma_1$ as follows:

$$H := \frac{400 c_1 c_6}{\epsilon} \quad \text{and} \quad \sigma_1 := \frac{\epsilon}{400},$$

where the constants $c_1$ is given in (3.32) and $c_6$ is given in Theorem 3.3.2, respectively. From this we can find a constant $\delta_1 = \delta_1(n, p, \nu, L, \omega)$ for which Theorem 3.3.3 holds.

We now choose the ratio of the shrinking balls in (3.64). Let $\delta \leq \min\left\{\frac{1}{16}, \delta_1\right\}$ be a small constant such that

$$2^{2n+10} c_\beta \delta^\beta \leq \frac{\epsilon}{2^4}, \tag{3.174}$$

where the constants $c_\beta$ and $\beta$ are given in Lemma 2.3.1. From the assumption (3.172), there exists $0 < R_\epsilon \leq \min\{R_1, R_2, R_3, d\}$ such that

$$\sup_{x \in \Omega'} \sup_{0 < \tau \leq R_\epsilon} g_0^{-1}\left(\frac{|\mu|(B_\tau(x))}{\tau^{n-1}}\right) \leq \frac{\delta^{2n}\epsilon}{2^{n+2} 400 c_\beta c_1 c_4}\lambda \tag{3.175}$$

and

$$\sup_{0 < \tau \leq R_\epsilon} \omega(\tau) \leq \frac{\delta^n}{80 c_\beta c_1},$$

where $c_4$ is the constant given in Lemma 3.2.15.

Take $R = r_0 \in (\delta R_\epsilon, R_\epsilon]$ in (3.64) and fix any $i \geq 1$, then we want to show

$$E_{i+2} = E(Du, B_{i+2}) < \epsilon\lambda. \tag{3.176}$$

If $a_{i+2} < \frac{\epsilon}{100}\lambda$, then (3.176) is trivial, and so we consider the case $a_{i+2} \geq \frac{\epsilon}{100}\lambda$. We now assert that

$$E_{i+2} \leq \frac{\epsilon}{2^4} E_{i+1} + \frac{8c_\beta c_4}{\delta^n}\omega(r_i)\lambda + \frac{8c_\beta c_4}{\delta^n}\frac{\lambda}{g_0(\lambda)}\left[\frac{|\mu|(B_i)}{r_i^{n-1}}\right]. \tag{3.177}$$

By (3.32) and (3.175), we discover that

$$\begin{aligned}\sup_{B_{i+1}} |Dv_i| &\geq \fint_{B_{i+2}} |Dv_i|\, dx \\ &\geq \fint_{B_{i+2}} |Du|\, dx - \fint_{B_{i+2}} |Du - Dv_i|\, dx \\ &\geq \frac{\epsilon}{100}\lambda - c_1\delta^{-2n} g_0^{-1}\left(\frac{|\mu|(B_i)}{r_i^{n-1}}\right) \geq \frac{\epsilon}{200}\lambda. \end{aligned} \tag{3.178}$$

Likewise, it follows from Theorem 3.3.2 and (3.32) that

$$\begin{aligned}\|Dv_i\|_{L^\infty(\frac{3}{4}B_i)} \leq c_6 \fint_{B_i} |Dv_i|\, dx &\leq c_6\lambda \\ &+ c_1 c_6 g_0^{-1}\left(\frac{|\mu|(B_i)}{r_i^{n-1}}\right) \leq 2c_6\lambda \end{aligned} \tag{3.179}$$

and

$$\|Dv_{i+1}\|_{L^\infty(\frac{3}{4}B_{i+1})} \leq 2c_6\lambda. \tag{3.180}$$

In addition, Theorem 3.3.3 and (3.179) imply

$$\operatorname*{osc}_{B_{i+1}} Dv_i \leq \frac{\epsilon}{400}\lambda. \tag{3.181}$$

We combine (3.178), (3.179) and (3.181) to discover

$$\frac{\lambda}{H} \leq \frac{\epsilon}{400}\lambda \leq |Dv_i| \leq 2c_6\lambda \leq H\lambda \quad \text{in } B_{i+1}. \tag{3.182}$$

Hence, (3.180) and (3.182) allow us to apply Lemma 3.2.15, so that

$$\fint_{\frac{1}{2}B_{i+1}} |Du - Dw_{i+1}|\, dx \leq c_4\omega(r_i)\lambda + c_4\frac{\lambda}{g_0(\lambda)}\left[\frac{|\mu|(B_i)}{r_i^{n-1}}\right]. \tag{3.183}$$

We apply Lemma 2.3.1 and (3.183) to conclude that

$$\begin{aligned} E_{i+2} &\leq 2E(Dw_{i+1}, B_{i+2}) + 2 \fint_{B_{i+2}} |Du - Dw_{i+1}|\, dx \\ &\leq 2c_\beta \delta^\beta E(Dw_{i+1}, B_{i+1}) + \frac{2}{\delta^n} \fint_{\frac{1}{2}B_{i+1}} |Du - Dw_{i+1}|\, dx \\ &\leq 4c_\beta \delta^\beta E(Du, B_{i+1}) + \frac{8c_\beta}{\delta^n} \fint_{\frac{1}{2}B_{i+1}} |Du - Dw_{i+1}|\, dx \\ &\leq \frac{\epsilon}{2^4} E_{i+1} + \frac{8c_\beta c_4}{\delta^n} \omega(r_i)\, \lambda + \frac{8c_\beta c_4}{\delta^n} \frac{\lambda}{g_0(\lambda)} \left[ \frac{|\mu|(B_i)}{r_i^{n-1}} \right]. \end{aligned}$$

This show the assertion (3.177).

Taking into account (3.173), (3.174), (3.175) and (3.177), we see that the claim (3.176) holds uniformly with respect to the point $x_0 \in \Omega'$ and to the initial radius $R \in (\delta R_\epsilon, R_\epsilon]$. We then take $r_\epsilon := \delta^3 R_\epsilon$ to observe that there exists a positive integer $m \geq 3$ such that $\delta^{m+1} R < \rho \leq \delta^m R$ for each $\rho \in (0, r_\epsilon]$. Consequently, (3.176) holds with $\rho = \delta^m r$ for some $r \in (\delta R, R]$. This completes the proof. □

Actually, we assumed $\mu \in L^\infty(\Omega)$, so that the solution $u$ to (3.1) belongs to $W^{1,G}(\Omega)$ in Theorem 3.1.3, Theorem 3.1.4 and Theorem 3.1.7. To complete Theorem 3.1.9, we need to consider a bounded Borel measure $\mu$ and a corresponding SOLA $u$.

*Proof of Theorem 3.1.9.* Let $\{u_k\}_{k\in\mathbb{N}} \in W^{1,G}(\Omega)$ be a sequence of weak solutions to (3.1) with right-hand side data $\mu_k \in L^\infty(\Omega)$ as in Definition 3.1.2. Let $v_k \in W^{1,G}(B_R)$ be the weak solution to

$$\begin{cases} -\mathrm{div}\,(\gamma(x) A(x, Dv_k)) = 0 & \text{in} \quad B_R \\ \qquad\qquad\qquad v_k = u_k & \text{on} \quad \partial B_R, \end{cases}$$

for each $k \in \mathbb{N}$. Then Lemma 3.2.1, Lemma 3.2.3, Lemma 3.2.4 and Corollary 3.2.2 holds for every $k \in \mathbb{N}$. Recall Lemma 3.2.3 and $\limsup_{i\to\infty} |\mu_i|(B_R) \leq |\mu|(\bar{B}_R)$. Then for any sufficiently large $k$, we have

$$\fint_{B_R} g_m\left(|Du_k - Dv_k|\right)^{1+\frac{1}{np}}\, dx \leq c \left[ \frac{|\mu_k|(B_R)}{R^{n-1}} \right]^{1+\frac{1}{np}}$$

$$\leq 2c \left[\frac{|\mu|(\bar{B}_R)}{R^{n-1}}\right]^{1+\frac{1}{np}}, \tag{3.184}$$

whenever $0 < R \leq R_1$. By the uniform boundedness of $\|u_k\|_{W^{1,g}(B_R)}$ and (3.24), we discover that $\|v_k\|_{W^{1,g}(B_R)}$ is uniformly bounded. Hence, there exists $v \in W^{1,g}(B_R)$ such that $v_k \rightharpoonup v$ in $W^{1,g}(B_R)$. Applying Theorem 3.3.2 and Theorem 3.3.3 along with a standard covering argument, we find that $\|Dv_k\|_{L^\infty(B_{\alpha R})} \leq c(\alpha)$ and $\{Dv_k\}$ is equicontinuous in $B_{\alpha R}$ for each $\alpha \in (0,1)$. We now apply Arzela-Ascoli theorem, so that $v \in C^1_{loc}(B_R)$ and $Dv_k(x)$ converges to $Dv(x)$ a.e. $x \in B_R$, up to a not relabeled subsequence. Consequently, (3.184) and the almost everywhere convergence imply $v_k \to v$ in $W^{1,g}(B_R)$. By Fatou's Lemma, $v$ solves

$$\begin{cases} -\operatorname{div}(\gamma(x)A(x,Dv)) = 0 & \text{in} \quad B_R \\ \qquad\qquad v = u & \text{on} \quad \partial B_R, \end{cases}$$

In addition, Lemma 3.2.1 holds for $u$ and $v$ with $\chi$ satisfying (3.26):

$$\begin{aligned} \fint_{B_R} g_{m,\chi}(|Du - Dv|)\,dx &= \fint_{B_R} \liminf_{k\to\infty} g_{m,\chi}(|Du_k - Dv_k|)\,dx \\ &\lesssim \liminf_{k\to\infty} g_{m,\chi}\left(\frac{|\mu_k|(B_R)}{R^{n-1}}\right) \lesssim g_{m,\chi}\left(\frac{|\mu|(\bar{B}_R)}{R^{n-1}}\right). \end{aligned}$$

Similarly, Lemma 3.2.3, Lemma 3.2.4 and Corollary 3.2.2 also holds for $u$ and $v$. We remark that Lemma 3.2.3, Lemma 3.2.4 and Corollary 3.2.2 we have used $|\mu|(B_R)$, while we use instead $|\mu|(\bar{B}_R)$ here. Since the remaining parts of the proof are still valid, we finish the proof. □

# Chapter 4

# Subquadratic systems without the quasi-diagonal structure

## 4.1 Main results

The goal of Chapter 4 is to obtain sharp potential bounds for the gradient of solutions to general nonlinear elliptic systems with subquadratic growth in terms of modified Riesz potentials, which provides a complimentary nonlinear potential theory to those with superquadratic growth in [80]. We refer to Section 1.3 for remarks about gradient potential theory for elliptic systems.

In this chapter, we consider general $p$-Laplace type systems of the form

$$-\operatorname{div}(A(x, Du)) = f \quad \text{in } \Omega \tag{4.1}$$

where $p \in (1, 2]$, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$ with $n, N \geq 2$.

The continuous vector field $A : \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is assumed to be $C^1$-regular in the second variable with $\partial A$ being Carathéodory regular, and to satisfy the following growth, ellipticity and continuity assumptions:

$$\begin{cases} |A(x,\xi)| + |\partial A(x,\xi)||\xi| \leq L|\xi|^{p-1} \\ \nu|\xi|^{p-2}|z|^2 \leq \langle \partial A(x,\xi)z, z\rangle \\ |A(x,\xi) - A(y,\xi)| \leq L\omega(|x-y|)|\xi|^{p-1} \\ |\partial A(x,\xi_2) - \partial A(x,\xi_1)| \leq L\mu\left(\frac{|\xi_2-\xi_1|}{|\xi_1|+|\xi_2|}\right)|\xi_1|^{p-2}|\xi_2|^{p-2}(|\xi_1|+|\xi_2|)^{2-p} \end{cases} \tag{4.2}$$

for every $\xi \in \mathbb{R}^{Nn} \setminus \{0\}$, $\xi_1, \xi_2 \in \mathbb{R}^{Nn}$ (except for $\xi_1 = \xi_2 = 0$), $x, y \in \Omega$

and $z \in \mathbb{R}^{Nn}$, where $0 < \nu \leq L$ are fixed constants and $\partial A$ stands for the derivative of $A$ with respect to the second variable. Two moduli of continuity $\omega, \mu : \mathbb{R}^+ \to [0,1]$ are concave, nondecreasing, bounded functions satisfying $\omega(0) = \mu(0) = 0$. Observe from $(4.2)_4$ and concavity of $\mu(\cdot)$ that we have the following locally Lipschitz continuity of $A$ away from the origin:

$$|\partial A(x, z_2) - \partial A(x, z_1)| \leq c(p, L)\mu\left(\frac{|z_2 - z_1|}{|z_1|}\right)|z_1|^{p-2} \tag{4.3}$$

for every $z_1, z_2 \in \mathbb{R}^{Nn}$ with $|z_1| > 2|z_2 - z_1|$.

We further assume Dini-continuity of the following partial map

$$x \mapsto \frac{A(x,\xi)}{|\xi|^{p-1}},$$

in the sense that for any $r > 0$

$$\int_0^r \omega(\rho)\,\frac{d\rho}{\rho} =: d(r) < \infty. \tag{4.4}$$

This partial map can be regarded as coefficients. When dealing with $C^1$-regularity of solutions to nonlinear $p$-Laplace equations with coefficients, Dini-continuity is known to be an optimal regularity assumption for the coefficients, see for instance [76, 78]. It is known in [69] that weak solutions to elliptic equations with continuous coefficient are not Lipschitz continuous in general. Under (4.4), we have partial $C^1$-regularity criteria in terms of Riesz potentials, see Theorem 4.1.3.

In this chapter, we do not assume quasi-diagonal structure, in which one can obtain full regularity results for the systems, see for instance [102]. On the other hand, for the systems without quasi-diagonal structure, only partial regularity results are available, except for subtle higher integrability. Indeed, De Giorgi constructed discontinuous solutions to general systems in [62]. To establish partial regularity, we assume that there exists $\eta : (0,\infty) \to (0,\infty)$ and $a : \Omega \to [\nu, L]$ such that

$$|\xi| \leq \eta(s) \Longrightarrow |A(x,\xi) - a(x)|\xi|^{p-2}\xi| \leq s|\xi|^{p-1} \tag{4.5}$$

for every $\xi \in \mathbb{R}^{Nn}$, $x \in \Omega$ and $s > 0$. In other words, $A(x,\cdot)$ is asymptotically close to $p$-Laplace operator with the coefficient $a(x)$ at the origin, uniformly with respect to $x \in \Omega$.

To guarantee the existence of weak solution to (4.1), we assume that $f \in L^q(\Omega)$, where $q \in (\bar{q}, \infty)$ with

$$\bar{q} := \begin{cases} [p^*]' = \dfrac{np}{np-(n-p)} & \text{if } p < n, \\ \dfrac{3}{2} & \text{if } p = n = 2. \end{cases} \tag{4.6}$$

One can easily see that $\bar{q} \in (1, n)$.

The approach that we will use through this chapter is the so-called $\varepsilon$-regularity criteria which mean that the point $x \in \Omega$ satisfying

$$E(Du, B_\rho(x)) := \left( ⨍_{B_\rho(x)} \left| V(Du) - V\big((Du)_{B_\rho(x)}\big) \right|^2 dy \right)^{\frac{1}{2}} < \varepsilon \tag{4.7}$$

is a regular point of $u$, where the bijection map $V : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is given by $V(\xi) = |\xi|^{\frac{p-2}{2}} \xi$. Here, we call $E(Du, B_\rho(x))$ as the excess functional of $Du$. Indeed, it is also reasonable to use $\varepsilon$-regularity criteria to the following modified Riesz potential

$$I^f_{1,q}(x_0, R) = \int_0^R \left( \rho^q ⨍_{B_\rho(x_0)} |f|^q \, dx \right)^{\frac{1}{q}} \frac{d\rho}{\rho}, \tag{4.8}$$

from the presence of $f \in L^q(\Omega)$ in (4.1). We refer to [2, 16, 47, 61, 86, 87] for further discussion about $\varepsilon$-regularity criteria regarding partial regularity results for the systems.

Through this chapter `data` stands for the set of constants $\{n, N, p, q, \nu, L\}$. We now state our main result.

**Theorem 4.1.1.** *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a weak solution to* (4.1) *under* (4.2), (4.4) *and* (4.5). *There exists a constant $\varepsilon_* = \varepsilon_*(\mathtt{data}, \mu(\cdot), \eta(\cdot)) > 0$ and a radius $R_* = R_*(\mathtt{data}, d(\cdot))$ such that if*

$$\big[E(Du, B_R(x_0))\big]^2 + \left[I^f_{1,q}(x_0, R)\right]^{\frac{p}{(p-1)}} \le \varepsilon_* \tag{4.9}$$

*holds for some $B_R(x_0) \subset \Omega$ with $R \in (0, R_*]$, then we have the limits*

$$\lim_{\rho \to 0} (Du)_{B_\rho(x_0)} = Du(x_0) \quad \text{and} \quad \lim_{\rho \to 0} \big(V(Du)\big)_{B_\rho(x_0)} = V(Du)(x_0), \tag{4.10}$$

*with the equality*

$$V(Du)(x_0) = V\big(Du(x_0)\big). \tag{4.11}$$

*Moreover, for any $\rho \in (0, R]$, we have*

$$\begin{aligned} &|V(Du)(x_0) - V\big((Du)_{B_\rho(x_0)}\big)| \\ &\le c\, E(Du, B_\rho(x_0)) + c\left[I^{\mu}_{1,q}(x_0,\rho)\right]^{\frac{p}{2(p-1)}} \\ &\quad + c\left(\fint_{B_\rho(x_0)} |Du|\,dx\right)^{\frac{2-p}{2}} I^{\mu}_{1,q}(x_0,\rho) + c\, d(\rho) \fint_{B_\rho(x_0)} |Du|\,dx, \end{aligned} \tag{4.12}$$

*where $c$ depends only on **data** and $\mu(\cdot)$.*

Note that if $I^f_{1,q}(x,R)$ is bounded for some radius $R > 0$, then one can take $I^f_{1,q}(x,\rho)$ as small as one want, by taking $\rho$ small enough. Hence, roughly speaking, (4.9) is not much more restrictive than (4.7) once Riesz potential is bounded.

The last term on the right hand side of (4.12) arises in the process of handling the coefficients. In addition, the second to the last term naturally arises from the interaction between lack of degeneracy for the problem with subquadratic growth and the data on the right hand side in non-divergence form, see [53] for such interaction and cf. [80] for the problem with superquadratic growth.

Our second main result is VMO-regularity.

**Theorem 4.1.2.** *Under the assumptions of Theorem 4.1.1, $Du$ is VMO-regular at $x_0 \in \Omega$, i.e.,*

$$\lim_{\rho\to 0} E(Du, B_\rho(x_0)) = 0. \tag{4.13}$$

*Moreover, if we replace the assumption* (4.9) *by*

$$\sup_{B_\rho \subset \Omega} \left[I^f_{1,q}(x_0,\rho)\right]^{\frac{p}{2(p-1)}} \le \frac{\varepsilon_*}{2}, \tag{4.14}$$

*and if*

$$\lim_{\rho\to 0}\left(\rho^q \fint_{B_\rho} |f|^q\,dx\right)^{\frac{1}{q}} = 0 \tag{4.15}$$

*holds locally uniformly in* $\Omega$, *then* $Du$ *is locally VMO-regular in the set*

$$\Omega_u = \Big\{ x \in \Omega : {}^{\exists} B_\rho(x) \subset\subset \Omega \text{ with } \rho \le R_* \text{ satisfying } E(Du, B_{\rho(x)}) < \frac{\varepsilon_*}{2} \Big\},$$

*which is an open subset of* $\Omega$, *satisfying* $|\Omega \setminus \Omega_u| = 0$.

One of the most important consequence of the gradient potential theory is $C^1$-regularity criteria in terms of potentials, see for instance [52, 76]. Since the proof of Theorem 4.1.3 is routine after obtaining Theorem 4.1.2, we now state $C^1$-regularity criteria without its proof.

**Theorem 4.1.3.** *Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be a weak solutions to* (4.1) *under the assumptions* (4.2), (4.4) *and* (4.5). *If*

$$\lim_{\rho \to 0} \Big[ I^f_{1,q}(x_0, \rho) \Big]^{\frac{p}{2(p-1)}} = 0 \tag{4.16}$$

*locally uniformly in* $\Omega$, *then* $Du$ *is continuous in the set* $\Omega_u$, *which is given in Theorem 4.1.2. Consequently,* $Du$ *is continuous in* $\Omega_u$, *whenever* $f \in L(n, 1)$ *that implies* (4.16).

## 4.2 Preliminaries

We refer to Section 2.2 for the basic ingredients regarding $N$-functions and Orlicz spaces.

We first recall a equivalent definition of excess functional given in 4.7. By denoting

$$\widetilde{E}(g, \mathcal{O}) = \bigg( \fint_{\mathcal{O}} \big| V(g(x)) - \big(V(g)\big)_{\mathcal{O}} \big|^2 \, dx \bigg)^{\frac{1}{2}}$$

for any $g \in L^p(\mathcal{O}; \mathbb{R}^{Nn})$, there exists a constant $c_1 = c_1(n, N, p)$ satisfying

$$\widetilde{E}(g, \mathcal{O}) \le E(g, \mathcal{O}) \le c_1 \widetilde{E}(g, \mathcal{O}), \tag{4.17}$$

see for instance [61, (2.6)] and see also [46, Lemma A.2] for more general cases.

Moreover, we have

$$\big| V\big((g)_{B_r}\big) - V\big((g)_{B_{\tilde{r}}}\big) \big|$$

$$
\begin{aligned}
&\leq \left|V\big((g)_{B_r}\big) - \big(V(g)\big)_{B_r}\right| + \left|\big(V(g)\big)_{B_r} - \big(V(g)\big)_{B_{\tilde r}}\right| + \left|\big(V(g)\big)_{B_{\tilde r}} - V\big((g)_{B_{\tilde r}}\big)\right| \\
&\leq \tau^{-\frac{n}{2}} \widetilde{E}(g, B_r) + 2\left( \fint_{B_{\tilde r}} \left|V(g) - V\big((g)_{B_r}\big)\right|^2 dx \right)^{\frac{1}{2}} \\
&\overset{(4.17)}{\leq} 3\tau^{-\frac{n}{2}} E(g, B_r),
\end{aligned} \tag{4.18}
$$

whenever $g \in L^p(B_r; \mathbb{R}^{Nn})$, $r > 0$, $\tau \in (0,1)$ and $\tilde r \in [\tau r, r]$. We use this estimate in Section 4.5, when $g = Du$ in order to deal with excess decay estimates for $Du$.

As we mentioned earlier, there are several types of Sobolev-Poincaré inequality according to $N$-functions, see for instance Lemma 2.2.1, Lemma 2.2.2 and Lemma 2.2.3. The next lemma shows another Sobolev-Poincaré type inequality for $N$-functions.

**Lemma 4.2.1.** *Let $\psi \in C^1[0,\infty)$ be a $N$-function satisfying (2.2) for any $1 < \gamma_1 \leq \gamma_2 < \infty$. Then for any $u \in W^{1,\psi}(B_r; \mathbb{R}^N)$ there exist constants $\gamma \geq 1$ and $c \geq 1$, both depending only on $n, N, \gamma_1, \gamma_2$, such that*

$$
\fint_{B_r} \psi\left( \left| \frac{u - (u)_{B_r}}{r} \right| \right) dx \leq c \left[ \fint_{B_r} \left[\psi\big(|Du|\big)\right]^{\frac{1}{\gamma}} dx \right]^{\gamma},
$$

*where $B_r$ is the ball with radius $r > 0$ in $\mathbb{R}^n$ .*

*Proof.* This lemma is a consequence of Lemma 2.2.3.

Set $\gamma = \min\left\{\frac{\gamma_1+1}{2}, \frac{n}{n-1}\right\} \in (1, \gamma_1)$, and define a $C^2(0,\infty)$ function

$$
\overline{\psi}(t) = \int_0^t \frac{\left[\psi(\tau)\right]^{\frac{1}{\gamma}}}{\tau} \, d\tau.
$$

Using (2.4), one can discover that $\overline{\psi}$ is also an $N$-function. By a straightforward calculation, we find that for every $t > 0$

$$
0 < \frac{\gamma_1 - \gamma}{\gamma} \leq \frac{t\overline{\psi}''(t)}{\overline{\psi}'(t)} = \frac{t\psi'(t)}{\gamma\psi(t)} - 1 \leq \frac{\gamma_2 - \gamma}{\gamma} < \infty
$$

and $\overline{\psi}$ is an $N$-function satisfying

$$
1 < \frac{\gamma_1}{\gamma} \leq \frac{t\overline{\psi}'(t)}{\overline{\psi}(t)} \leq \frac{\gamma_2}{\gamma} < \infty.
$$

Note that $\overline{\psi}(\cdot)$ is equivalent to $[\psi(\cdot)]^{\frac{1}{\gamma}}$, i.e., there exists $c = c(\gamma_2) > 1$ such that $c^{-1}[\psi(t)]^{\frac{1}{\gamma}} \le \overline{\psi}(t) \le c[\psi(t)]^{\frac{1}{\gamma}}$ for every $t > 0$. Indeed, by the definition of $\overline{\psi}$, we discover

$$\frac{\left[\psi(\frac{t}{2})\right]^{\frac{1}{\gamma}}}{2} \le \overline{\psi}(t) \le \sum_{i=0}^{\infty} \psi\left(\frac{t}{2^i}\right)^{\frac{1}{\gamma}} \le \left[\psi(t)\right]^{\frac{1}{\gamma}} \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{\frac{\gamma_1 i}{\gamma}} \le 2\left[\psi(t)\right]^{\frac{1}{\gamma}}.$$

Applying [10, Proposition 3.5] with $\overline{\psi}$, we find

$$\begin{aligned}
\fint_{B_r} \psi\left(\left|\frac{u-(u)_B}{r}\right|\right) dx &\le \left[\fint_{B_r} \left[\psi\left(\left|\frac{u-(u)_B}{r}\right|\right)\right]^{\frac{n}{\gamma(n-1)}} dx\right]^{\frac{\gamma(n-1)}{n}} \\
&\le c\left[\fint_{B_r} \left[\overline{\psi}\left(\left|\frac{u-(u)_B}{r}\right|\right)\right]^{\frac{n}{n-1}} dx\right]^{\frac{\gamma(n-1)}{n}} \\
&\le c\left[\fint_{B} \overline{\psi}(|Du|)\, dx\right]^{\gamma} \\
&\le c\left[\fint_{B} \psi(|Du|)^{\frac{1}{\gamma}}\, dx\right]^{\gamma}.
\end{aligned}$$

In the first line, we have used the fact that $\frac{n}{\gamma(n-1)} \ge 1$ to use Hölder's inequality. This completes the proof. $\square$

Let us consider a set of $N$-functions $\{\psi_s\}_{s\ge 0} \subset C^1[0,\infty) \cap C^2(0,\infty)$, where

$$\psi_s(t) = (t^2+s^2)^{\frac{p-2}{2}} t^2. \tag{4.19}$$

Then we have

$$2^{p-2}|\psi_{s_2}(t)|^2 \le |\psi_{s_1}(t)|^2 \le 2^{2-p}|\psi_{s_2}(t)|^2, \tag{4.20}$$

whenever $0 < \frac{1}{2}s_2 \le s_1 \le 2s_2$. We further see that there exists a constant $c = c(n, N, p) > 1$ such that

$$c^{-1}\psi_{|\xi_1|}(|\xi_1-\xi_2|) \le |V(\xi_1) - V(\xi_2)|^2 \le c\psi_{|\xi_1|}(|\xi_1-\xi_2|) \tag{4.21}$$

for any $\xi_1, \xi_2 \in \mathbb{R}^{Nn}$, see [44, Lemma 3]. Indeed, this set of $N$-functions are used in the context of partial differential equations with general growth, see [47, 48] and Section 5.

We remark that (2.2) holds with $\gamma_1 = p$ and $\gamma_2 = 2$, for every $\psi_s$ uniformly with respect to $s \geq 0$. Furthermore, constant $c$ in Lemma 4.2.1 does not depend on $s$, when applying $\psi_s$ to the lemma, later in this chapter.

### 4.2.1 Approximation lemmas.

Our analysis relies on $\mathcal{A}$-harmonic approximation lemma, and $p$-harmonic approximation lemma. We first look at $\mathcal{A}$-harmonic approximation lemma, which will be used later in Section 4.4.1 to discuss the non-singular case. Let $\mathcal{A} : \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \to \mathbb{R}$ be a bilinear form satisfying

$$|\mathcal{A}| \leq L \quad \text{and} \quad \nu|\xi|^2 \leq \mathcal{A}(\xi, \xi) \quad \text{for every } \xi \in \mathbb{R}^{Nn}. \tag{4.22}$$

We say that $h \in W^{1,p}(B_r; \mathbb{R}^N)$ is $\mathcal{A}$-harmonic if and only if

$$\int_{B_r} \mathcal{A}(Dh, D\varphi)\, dx = 0$$

holds for every $\varphi \in W_0^{1,p}(B_r; \mathbb{R}^N)$.

**Lemma 4.2.2** ($\mathcal{A}$-harmonic approximation lemma, see [47], Theorem 14)**.** *Let $\mathcal{A}$ be a bilinear form on $\mathbb{R}^{Nn}$ satisfying (4.22) and $\Psi$ be an $N$-function satisfying (2.2). For any $\varepsilon, \sigma > 0$, there exists $\delta = \delta(n, N, \gamma_1, \gamma_2, L, \nu, \sigma, \varepsilon) > 0$ such that the following statement holds: Assume that $v \in W^{1,\Psi}(B_r; \mathbb{R}^N)$ is approximately $\mathcal{A}$-harmonic i.e., $v$ satisfies*

$$\fint_{B_{r/2}} \mathcal{A}(Dv, D\varphi)\, dx \leq \delta \fint_{B_r} |Dv|\, dx\, \|D\varphi\|_{L^\infty(B_r)}$$

*for all $\varphi \in C_0^\infty(B_r; \mathbb{R}^N)$. Then there exists a unique $\mathcal{A}$-harmonic map $h \in v + W_0^{1,\Psi}(B_r; \mathbb{R}^N)$ satisfying*

$$\fint_{B_{r/2}} \Psi(|Dv - Dh|)\, dx \leq \varepsilon \left( \left[ \fint_{B_{r/2}} \big[\Psi(|Dv|)\big]^{1+\sigma} dx \right]^{\frac{1}{1+\sigma}} + \fint_{B_r} \Psi(|Dv|)\, dx \right),$$

*whenever the right-hand side is finite.*

From the classical theory of elliptic partial differential equations, $\mathcal{A}$-harmonic function is locally smooth. Recall the following excess decay type estimate of $\mathcal{A}$-harmonic map: Let $h \in W^{1,2}(B_r;\mathbb{R}^N)$ be $\mathcal{A}$-harmonic function and $0 < \tilde{r} \leq r/2$. Then we have

$$\fint_{B_{\tilde{r}}} |Dh - (Dh)_{B_{\tilde{r}}}|\,dx \leq \frac{\tilde{r}}{r} \fint_{B_r} |Dh - (Dh)_{B_r}|\,dx.$$

For any $N$-function $\Psi$ satisfying (2.2), Lipschitz regularity for $Dh$ further yields

$$\fint_{B_{\tilde{r}}} \Psi\big(|Dh - (Dh)_{B_{\tilde{r}}}|\big)\,dx \leq \left(\frac{\tilde{r}}{r}\right)^{\gamma_1} \fint_{B_r} \Psi\big(|Dh - (Dh)_{B_r}|\big)\,dx, \tag{4.23}$$

as follows from [47, Proposition 27].

Now, turn our attention to $p$-harmonic approximation lemma which is first introduced by Duzaar and Mingione in [50, Lemma 1]. If $h \in W^{1,p}(B_r;\mathbb{R}^N)$ satisfies

$$\int_{B_r} |Dh|^{p-2} Dh \cdot D\varphi\,dx = 0$$

for every $\varphi \in C_0^\infty(B_r;\mathbb{R}^N)$, then we call $h$ a $p$-harmonic map. We now present a modified version of $p$-harmonic approximation lemma, see [49, 50].

**Lemma 4.2.3** ($p$-harmonic approximation lemma)**.** *For any $\varepsilon > 0$ and $p_1 \in (0,2]$, there exists $\delta = \delta(n,N,p,p_1,\varepsilon) > 0$ such that the following statement holds: Assume that $v \in W^{1,p}(B_r;\mathbb{R}^N)$ is approximately $p$-harmonic i.e., $v$ satisfies*

$$\fint_{B_r} |Dv|^{p-2}\langle Dv, D\varphi\rangle\,dx \leq \delta\left(\fint_{B_r} |Dv|^p\,dx\right)^{\frac{p-1}{p}} \|D\varphi\|_{L^\infty(B_r)}$$

*for all $\varphi \in C_0^\infty(B_r;\mathbb{R}^N)$. Then there exists a unique $p$-harmonic map $h \in v + W_0^{1,p}(B_r;\mathbb{R}^N)$ satisfying*

$$\left(\fint_{B_r} |V(Dv) - V(Dh)|^{p_1}\,dx\right)^{\frac{2}{p_1}} \leq \varepsilon \fint_{B_r} |Dv|^p\,dx.$$

Excess decay estimates for $p$-harmonic maps were shown by Giaquinta and Modica for $p \geq 2$ in [61], and by Acerbi and Fusco for $1 < p < 2$ in [2, Proposition 2.11]. By virtue of (4.17), for any $p$-harmonic function

$h \in W^{1,p}(B_r)$ and any $0 < \tilde{r} \leq r$, there are two constants $\alpha \in (0,1)$ and $c \geq 1$ both depending only on $n, N, p$ such that

$$E(Dh, B_{\tilde{r}}) \leq c\left(\frac{\tilde{r}}{r}\right)^{\alpha} E(Dh, B_r). \tag{4.24}$$

It is enough to assume $\alpha \leq p/2$ to simplify the notations in later sections.

As mentioned earlier, our proof relies on Lemma 4.2.2 and Lemma 4.2.3 to deal with the condition (4.5), instead of Uhlenbeck condition on the vector field $A(\cdot)$. For interested readers, we refer to [51], which summarized affluent results on harmonic approximation lemmas.

We are going to show some higher integrability results in Section 4.3, which will play a key role in approximation lemmas. We will apply Lemma 4.2.2 to obtain excess decay estimates for the non-singular case in Section 4.4.1. On the other hand, we apply Lemma 4.2.3 to establish excess decay estimates for the singular case in Section 4.4.4.

## 4.3 higher integrability

We start this section with the following higher integrability result.

**Lemma 4.3.1.** *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (4.1) *with* (4.2)*. There exist two constants $c = c(\boldsymbol{data}) > 1$ and $\sigma_g = \sigma_g(\boldsymbol{data}) \in (0,1]$, such that*

$$\left(\fint_{B_{r/2}} |Du|^{p(1+\sigma_g)}\,dx\right)^{\frac{1}{1+\sigma_g}} \leq c\fint_{B_r} |Du|^p\,dx + c\left(r^q \fint_{B_r} |f|^q\,dx\right)^{\frac{p}{q(p-1)}} \tag{4.25}$$

*for every ball $B_r \subset \Omega$, with $\bar{q}(1+\sigma_g) \leq q < n$.*

*Proof.* For the sake of completeness, we sketch its proof. We refer to [63, Section 6] and [87, Section 4.1] for the detailed proof.

Fix a ball $B_{\tilde{r}} := B_{\tilde{r}}(y) \subset B_r$ and test $\varphi = \zeta^2(u - (u)_{B_{\tilde{r}}})$ to (4.1), where $\zeta \in C_c^{\infty}(B_{\tilde{r}})$ is a cutoff function satisfying $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on $B_{\tilde{r}/2}$ and $|D\zeta| \leq 4/\tilde{r}$. Using (4.2) and Hölder's inequality, we have the following Caccioppoli type estimate:

$$\fint_{B_{\tilde{r}/2}} |Du|^p\,dx \leq c\fint_{B_{\tilde{r}}} \left(\frac{|u - (u)_{B_{\tilde{r}}}|}{\tilde{r}}\right)^p dx$$

$$+ c\left(\tilde{r}^{\bar{q}} \fint_{B_{\tilde{r}}} |f|^{\bar{q}}\,dx\right)^{\frac{1}{\bar{q}}} \left(\fint_{B_{\tilde{r}}} \left(\frac{|u-(u)_{B_{\tilde{r}}}|}{\tilde{r}}\right)^{p^*} dx\right)^{\frac{1}{p^*}},$$

where $\bar{q} = [p^*]'$. For any $\varepsilon > 0$, we now apply Sobolev-Poincaré inequality to discover

$$\fint_{B_{\tilde{r}/2}} |Du|^p\,dx \le \varepsilon \fint_{B_{\tilde{r}}} |Du|^p\,dx + c\left(\fint_{B_{\tilde{r}}} |Du|^{p_*}\,dx\right)^{\frac{p}{p_*}} + c(\varepsilon)K \fint_{B_{\tilde{r}}} |f|^{\bar{q}}\,dx,$$

where

$$p_* = \frac{np}{n+p} < p \quad \text{and} \quad K = \left(\int_{B_r} |f|^{\bar{q}}\,dx\right)^{\frac{p}{\bar{q}(p-1)}-1}.$$

Applying Gehring's lemma (see [63, Corollary 6.1]) for sufficiently small $\varepsilon > 0$, we complete the proof. □

We next establish a modified version of the above higher integrability result. It will play the central role in the proof of excess decay estimates for the non-singular case, (4.32). We recall the notation (4.19) and simply denote $\psi := \psi_0$. Note that $x$ dependence of the vector field $A$ with $(4.2)_3$ is a natural generalization of $p$-Laplace systems.

**Lemma 4.3.2.** *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (4.1) *with* (4.2). *Then there exist constants $\sigma = \sigma(\texttt{data}) \in (0, \sigma_g)$ and $c = c(\texttt{data})$ such that the estimate*

$$\left(\fint_{B_{r/2}} |V(Du) - V(\xi)|^{2(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}}$$
$$\le c \fint_{B_r} |V(Du) - V(\xi)|^2\,dx + c\omega(r)^2 \fint_{B_r} (|Du| + |\xi|)^p\,dx$$
$$+ c\left(\fint_{B_r} (|Du| + |\xi|)^p\,dx\right)^{\frac{2-p}{p}} \left(r^q \fint_{B_r} |f|^q\,dx\right)^{\frac{2}{q}} + c\left(r^q \fint_{B_r} |f|^q\,dx\right)^{\frac{p}{q(p-1)}}$$

*holds for any $\xi \in \mathbb{R}^{Nn}$ and ball $B_r \subset \Omega$, where $\sigma_g > 0$ is the constant determined in Lemma 4.3.1.*

*Proof.* Take any ball $B_{\tilde{r}} = B_{\tilde{r}}(y) \subset B_r$, and set

$$l := (u)_{B_{\tilde{r}}} + \xi(x - \bar{x}).$$

Define a test function $\varphi = \zeta^2(u-l)$, where $\zeta \in C_0^\infty(B_{\tilde r})$ is a cutoff function such that $0 \le \zeta \le 1$, $\zeta = 1$ on $B_{\tilde r/2}$ and $|D\zeta| \le 4/\tilde r$. Applying $\varphi$ to (4.1), we see

$$\begin{aligned} I_1 &:= \fint_{B_{\tilde r}} \int_0^1 \langle \partial A\big(x, \xi + (Du-\xi)t\big)(Du-\xi), D\varphi\rangle \,dt\,dx \\ &= \fint_{B_{\tilde r}} \langle A(x,Du) - A(x,\xi), D\varphi\rangle\,dx \\ &= \fint_{B_{\tilde r}} \langle A(x_0,\xi) - A(x,\xi), D\varphi\rangle\,dx + \fint_{B_{\tilde r}} f\varphi\,dx =: I_2 + I_3. \end{aligned}$$

We first estimate $I_1$ as

$$\begin{aligned} I_1 &\overset{(4.2)}{\ge} \nu \fint_{B_{\tilde r}} \int_0^1 |\xi + (Du-\xi)t|^{p-2}\,dt\ |Du-\xi|^2\zeta^2\,dx \\ &\qquad - 2L \fint_{B_{\tilde r}} \int_0^1 |\xi + (Du-\xi)t|^{p-2}\,dt\ |Du-\xi||u-l||D\zeta|\zeta\,dx \\ &\ge c^{-1} \fint_{B_{\tilde r}} |V(Du) - V(\xi)|^2\zeta^2\,dx \\ &\qquad - c\fint_{B_{\tilde r}} (|\xi| + |Du-\xi|)^{p-2}|Du-\xi||u-l||D\zeta|\zeta\,dx. \end{aligned}$$

We now have

$$\begin{aligned} &\fint_{B_{\tilde r}} |V(Du) - V(\xi)|^2\zeta^2\,dx \\ &\le c\fint_{B_{\tilde r}} \frac{\psi_{|\xi|}(|Du-\xi|)}{|Du-\xi|}|u-l||D\zeta|\zeta\,dx + c\,I_2 + c\,I_3. \end{aligned} \tag{4.26}$$

Using Young's inequality with $\bar\varepsilon \in (0,1)$, (2.1), (2.4) and Lemma 4.2.1 with $\psi_{|\xi|}$, we discover

$$\begin{aligned} &\fint_{B_{\tilde r}} \zeta \frac{\psi_{|\xi|}(|Du-\xi|)}{|Du-\xi|}\Big|\frac{u-l}{\tilde r}\Big|\,dx \\ &\le \bar\varepsilon \fint_{B_{\tilde r}} \psi^*_{|\xi|}\Big(\zeta\frac{\psi_{|\xi|}(|Du-\xi|)}{|Du-\xi|}\Big)dx + c(\bar\varepsilon)\fint_{B_{\tilde r}} \psi_{|\xi|}\Big(\Big|\frac{u-l}{\tilde r}\Big|\Big)\,dx \end{aligned}$$

$$
\begin{aligned}
&\leq c(p)\bar{\varepsilon} \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \zeta^2 \, dx \\
&\qquad + c(\bar{\varepsilon}) \left( \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^{\frac{2}{\gamma}} \, dx \right)^{\gamma}.
\end{aligned} \tag{4.27}
$$

Here, we have used (4.21) in the last line, and $\gamma = \gamma(\texttt{data}) \in (0, 1)$ is the constant determined in Lemma 4.2.1.

We now estimate $I_2$ and $I_3$. By $(4.2)_3$ and Sobolev's inequality, we discover

$$
\begin{aligned}
|I_2| &\leq c\omega(\tilde{r})|\xi|^{p-1} \fint_{B_{\tilde{r}}} |Du - \xi| \, dx \\
&\leq c\omega(\tilde{r}) \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|(|Du| + |\xi|)^{\frac{p}{2}} \, dx \\
&\leq \bar{\varepsilon} \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \, dx + c(\bar{\varepsilon})\omega(\tilde{r})^2 \fint_{B_{\tilde{r}}} (|Du| + |\xi|)^p \, dx.
\end{aligned} \tag{4.28}
$$

Recalling (4.6) and performing some standard manipulations leads to

$$
|I_3| \leq c\tilde{r} \left( \fint_{B_{\tilde{r}}} |f|^{\bar{q}} \, dx \right)^{\frac{1}{\bar{q}}} \left( \fint_{B_{\tilde{r}}} |Du - \xi|^p \, dx \right)^{\frac{1}{p}}. \tag{4.29}
$$

We now estimate second term in (4.29) as

$$
\begin{aligned}
&\fint_{B_{\tilde{r}}} |Du - \xi|^p \, dx \\
&\leq c \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^p (|Du| + |\xi|)^{\frac{p(2-p)}{2}} \, dx \\
&\leq c \left( \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \, dx \right)^{\frac{p}{2}} \left( \fint_{B_{\tilde{r}}} (|Du| + |\xi|)^p \, dx \right)^{\frac{2-p}{2}},
\end{aligned}
$$

in order to discover

$$
\begin{aligned}
|I_3| &\leq \bar{\varepsilon} \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \, dx \\
&\qquad + c(\bar{\varepsilon}) \left( \fint_{B_{\tilde{r}}} (|Du| + |\xi|)^p \, dx \right)^{\frac{2-p}{p}} \left( \tilde{r}^{\bar{q}} \fint_{B_{\tilde{r}}} |f|^{\bar{q}} \, dx \right)^{\frac{2}{\bar{q}}}.
\end{aligned} \tag{4.30}
$$

Combining (4.26)-(4.28) and (4.30), we obtain

$$\begin{aligned}
&\fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \, dx \\
&\leq \bar{\varepsilon} \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \, dx + c(\bar{\varepsilon}) \left( \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^{\frac{2}{\gamma}} \, dx \right)^{\gamma} \\
&\quad + c(\bar{\varepsilon}) \left( \fint_{B_{\tilde{r}}} (|Du| + |\xi|)^p \, dx \right)^{\frac{2-p}{p}} \left( \tilde{r}^{\bar{q}} \fint_{B_{\tilde{r}}} |f|^{\bar{q}} \, dx \right)^{\frac{2}{\bar{q}}} \\
&\quad + c(\bar{\varepsilon}) \omega(\tilde{r})^2 \fint_{B_{\tilde{r}}} (|Du| + |\xi|)^p \, dx.
\end{aligned} \tag{4.31}$$

Recalling the definition of $\bar{q}$ given in (4.6) and writing

$$K := \left( \int_{B_r} (|Du| + |\xi|)^p \, dx \right)^{\frac{2-p}{p}} \left( \int_{B_r} |f|^{\bar{q}} \, dx \right)^{\frac{2}{\bar{q}} - 1},$$

we further estimate (4.31) as

$$\begin{aligned}
&\fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \, dx \\
&\leq \bar{\varepsilon} \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^2 \, dx + c(\bar{\varepsilon}) \left( \fint_{B_{\tilde{r}}} |V(Du) - V(\xi)|^{\frac{2}{\gamma}} \, dx \right)^{\gamma} \\
&\quad + c(\bar{\varepsilon}) K \fint_{B_r} |f|^{\bar{q}} \, dx + c(\bar{\varepsilon}) \omega(r)^2 \fint_{B_{\tilde{r}}} (|Du| + |\xi|)^p \, dx.
\end{aligned}$$

From Lemma 4.3.1, we know that $|Du| \in L^{1+\sigma_g}(B_r)$. Therefore, we apply Gehring's lemma and use (4.25) to deduce the desired result. □

We remark that if there is no $x$ dependence on $A$, then one can obtain Lemma 4.3.2 without Lemma 4.3.1.

## 4.4 Excess decay estimates

Throughout the rest of this chapter, we denote by $u \in W^{1,p}(\Omega)$ to mean a weak solution to (4.1) satisfying (4.2) and (4.5). In this section, we study two cases, the non-singular case and the singular case, via $\mathcal{A}$-harmonic approximation lemma, Lemma 4.2.2, and $p$-harmonic approximation lemma, Lemma 4.2.3, respectively.

### 4.4.1 The non-singular case

Here we deal with the non-singular case which is characterized by the assumption that

$$|(Du)_{B_r}|^p > \frac{1}{\theta_1}\big[E(Du, B_r)\big]^2 \tag{4.32}$$

for some fixed $B_r \Subset \Omega$, while the constant $\theta_1 \in (0,1)$ will be selected later in Lemma 4.4.3. For the sake of readability, we use the short notation

$$\xi_1 := (Du)_{B_r} = \fint_{B_r} Du\, dx \in \mathbb{R}^{Nn}.$$

We start with the following useful lemma.

**Lemma 4.4.1.** *If* (4.32) *holds, then there exists a constant* $c = c(\texttt{data}) \geq 1$ *such that*

$$\fint_{B_r} |Du|^p\, dx \leq c\,|\xi_1|^p. \tag{4.33}$$

*Proof.* By a direct calculation, we see

$$\fint_{B_r} |Du|^p\, dx \leq 2^{p-1}\bigg[\fint_{B_r} |Du - \xi_1|^p\, dx + |\xi_1|^p\bigg].$$

To estimate the first term on the right-hand side, we divide $B_r$ into

$$B^+ = \Big\{x \in B_r : |Du(x) - \xi_1| \geq \frac{1}{2}\Big[|Du(x)| + |\xi_1|\Big]\Big\}$$

and $B^- := B_r \setminus B^+$. In $B^-$, it holds that $|Du(x) - \xi_1| < 2|\xi_1|$, so

$$\frac{1}{|B_r|}\int_{B^-} |Du - \xi_1|^p\, dx < 2^p|\xi_1|^p.$$

On the other hand in $B^+$, we estimate

$$\frac{1}{|B_r|}\int_{B^+} |Du - \xi_1|^p\, dx \leq \frac{2^{2-p}}{|B_r|}\int_{B^+} (|Du| + |\xi_1|)^{p-2}|Du - \xi_1|^2\, dx$$

$$\leq c[E(Du, B_r)]^2.$$

Finally, (4.33) follows from (4.32). □

### 4.4.2 Large measure or oscillatory coefficient

For any constant $\theta_2 \in (0,1)$ to be chosen, we call

$$\left(r^q \fint_{B_r} |f|^q \, dx\right)^{\frac{1}{q}} > \theta_2 |\xi_1|^{\frac{p-2}{2}} E(Du, B_r) \tag{4.34}$$

as *large measure condition*. In this case, we find from (4.17), (4.32) and (4.34) that for every $\tau \in (0,1)$

$$\begin{aligned}
\big[E(Du, B_{\tau r})\big]^2 &\leq c \fint_{B_{\tau r}} \big|V(Du) - \big(V(Du)\big)_{B_{\tau r}}\big|^2 \, dx \\
&\leq c \fint_{B_{\tau r}} \big|V(Du) - V\big((Du)_{B_r}\big)\big|^2 \, dx \\
&\leq \frac{c}{\tau^n} \big[E(Du, B_r)\big]^2 \\
&\leq \frac{c\sqrt{\theta_1}}{\tau^n} |\xi_1|^{\frac{p}{2}} E(Du, B_r) \\
&\leq \frac{c_2}{\theta_2 \, \tau^n} |\xi_1| \left(R^q \fint_{B_r} |f|^q \, dx\right)^{\frac{1}{q}},
\end{aligned} \tag{4.35}$$

where $c_2$ depends only on `data`. In a similar way, if $|\xi_1| \leq \lambda$ for any $\lambda > 0$, then we have

$$\begin{aligned}
\big[E(Du, B_{\tau r})\big]^2 &\leq \frac{c}{\tau^n} \big[E(Du, B_r)\big]^2 \\
&\leq \frac{c}{\tau^n} \lambda^{2-p} |\xi_1|^{p-2} \big[E(Du, B_r)\big]^2 \\
&\leq \frac{c_2}{\theta_2^2 \tau^n} \lambda^{2-p} \left(R^q \fint_{B_r} |f|^q \, dx\right)^{\frac{2}{q}}.
\end{aligned} \tag{4.36}$$

We next consider *oscillatory coefficient condition*

$$\omega(r)^2 |\xi_1|^p > \theta_3 \big[E(Du, B_r)\big]^2$$

for another constant $\theta_3 \in (0,1)$ to be chosen. Then we have

$$\big[E(Du, B_{\tau r})\big]^2 \le \frac{c_2}{\theta_3 \tau^n}\omega(r)^2|\xi_1|^p. \tag{4.37}$$

Here, we have abused the notation $c_2$, since the dependence of the constants is the same as in (4.35), (4.36) and (4.37).

### 4.4.3 Small measure and stable coefficient

We now deal with the case that (4.32),

$$\left(r^q \fint_{B_r} |f|^q\,dx\right)^{\frac{1}{q}} \le \theta_2 E(Du, B_r)|\xi_1|^{\frac{p-2}{2}}$$

and

$$\omega(r)^2|\xi_1|^p \le \theta_3\big[E(Du, B_r)\big]^2 \tag{4.38}$$

hold. As a direct consequence of (4.32) and $(4.38)_1$, we have

$$\left(R^q \fint_{B_R} |f|^q\,dx\right)^{\frac{p}{q(p-1)}} \le \theta_1^{\frac{2-p}{2(p-1)}}\theta_2^{\frac{p}{p-1}}\big[E(Du, B_R)\big]^2. \tag{4.39}$$

To establish excess decay estimate of $Du$, we are going to use $\mathcal{A}$-harmonic approximation lemma with the following bilinear form

$$\mathcal{A} := \frac{\partial A(x_0, \xi_1)}{|\xi_1|^{p-2}} \in \mathbb{R}^{N^2n^2},$$

which is strongly elliptic and has linear growth:

$$\nu|\lambda|^2 \le \mathcal{A}(\lambda,\lambda) \qquad \text{and} \qquad |\mathcal{A}| \le L$$

for any $\lambda \in \mathbb{R}^{Nn}$ by (4.2). We define a normalized function $v \in W^{1,p}(\Omega;\mathbb{R}^N)$ by

$$v(x) = \frac{|\xi_1|^{\frac{p-2}{2}}}{E(Du, B_r)}\big[u(x) - (u)_{B_r} - \xi_1 x\big], \tag{4.40}$$

which is indeed approximately $\mathcal{A}$-harmonic, as we now have

**Lemma 4.4.2.** *Assume* (4.32) *and* (4.38). *For any $\delta > 0$ there exist $\bar\theta_1 =$*

$\bar{\theta}_1(\boldsymbol{data}, \mu(\cdot), \delta)$, *and* $\theta_2, \theta_3$ *depending on* $\boldsymbol{data}$ *and* $\delta$ *such that for every* $\varphi \in C_c^\infty(B_r; \mathbb{R}^N)$

$$\fint_{B_r} \mathcal{A}(Dv, D\varphi)\, dx \le \delta \|D\varphi\|_{L^\infty(B_r)}$$

*whenever* $\theta_1 \le \bar{\theta}_1$.

*Proof.* Fix a test function $\varphi \in C_c^\infty(B_r; \mathbb{R}^N)$. Taking into account (4.1), we have

$$\begin{aligned}
&\fint_{B_r} \big\langle \partial A(x_0, \xi_1)(Du - \xi_1), D\varphi \big\rangle\, dx \\
&\quad = \fint_{B_r} \int_0^1 \big\langle \big[\partial A(x_0, \xi_1) - \partial A(x_0, \xi_1 + t(Du - \xi_1))\big](Du - \xi_1), D\varphi \big\rangle\, dt\, dx \\
&\qquad + \fint_{B_r} \big\langle A(x_0, Du) - A(x, Du), D\varphi \big\rangle\, dx - \fint_{B_r} f \cdot \varphi\, dx \\
&\quad =: I + II - \fint_{B_r} f \cdot \varphi\, dx. 
\end{aligned} \tag{4.41}$$

Denoting $B^+ := \{x \in B_r : |\xi_1| > 2|Du(x) - \xi_1|\} \subset B_r$ and $B^- = B_r \setminus B^+$, we estimate $I$ as

$$\begin{aligned}
I \le \frac{\|D\varphi\|_{L^\infty(B_r)}}{|B_r|} \Bigg[ &\underbrace{\int_{B^+} \int_0^1 \Big| \big[\partial A(x_0, \xi_1) - \partial A(x_0, \xi_{1,t})\big](Du - \xi_1) \Big|\, dt\, dx}_{=:I_1} \\
&+ \underbrace{\int_{B^-} \int_0^1 \Big| \big[\partial A(x_0, \xi_1) - \partial A(x_0, \xi_{1,t})\big](Du - \xi_1) \Big|\, dt\, dx \Bigg]}_{=:I_2},
\end{aligned} \tag{4.42}$$

where we have used the abbreviation $\xi_{1,t} := \xi_1 + t(Du - \xi_1)$ for every $t \in [0,1]$.

Note that $|Du(x)| + |\xi_1| \le 3|\xi_1|$ and $t|Du(x) - \xi_1| \le |\xi_1|/2$ for all $t \in [0,1]$ in $B^+$. On the other hand, concavity of $\mu(\cdot)$ implies that we have $\mu(ct) \le c\mu(t)$ for any $c \ge 1$ and $t \ge 0$. Taking these into account, we discover

$$I_1 \overset{(4.3)}{\le} c|\xi_1|^{p-2} \int_{B^+} \mu\bigg(\frac{|Du - \xi_1|}{|\xi_1|}\bigg) |Du - \xi_1|\, dx$$

$$
\begin{aligned}
&\leq c|\xi_1|^{\frac{p-2}{2}} \int_{B^+} \mu\left(\frac{|V(Du)-V(\xi_1)|}{|\xi_1|^{\frac{p}{2}}}\right)|V(Du)-V(\xi_1)|\,dx \\
&\leq c|\xi_1|^{\frac{p-2}{2}} \left(\int_{B^+} \mu\left(\frac{|V(Du)-V(\xi_1)|}{|\xi_1|^{\frac{p}{2}}}\right)^2 dx\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{B_r} |V(Du)-V(\xi_1)|^2\,dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.43}
$$

By Jensen's inequality and (4.32), we have

$$
\begin{aligned}
\frac{1}{|B_r|}\int_{B^+} \mu\left(\frac{|V(Du)-V(\xi_1)|}{|\xi_1|^{\frac{p}{2}}}\right)^2 dx &\leq \frac{c}{|B_r|}\int_{B^+} \mu\left(\frac{|V(Du)-V(\xi_1)|}{|\xi_1|^{\frac{p}{2}}}\right)dx \\
&\leq c\mu\left(\frac{E(Du,B_r)}{|\xi_1|^{\frac{p}{2}}}\right) \leq c\mu(\theta_1^{\frac{1}{2}}).
\end{aligned} \tag{4.44}
$$

We combine (4.43) and (4.44) to obtain

$$
\frac{I_1}{|B_r|} \leq c\mu(\theta_1^{\frac{1}{2}})|\xi_1|^{\frac{p-2}{2}} E(Du,B_r). \tag{4.45}
$$

On the other hand in $B^-$, $|Du(x)-\xi_1|+|\xi_1| \leq 3|Du(x)-\xi_1|$ holds, and we estimate $I_2$ as follows:

$$
\begin{aligned}
I_2 &\overset{(4.2)}{\leq} c\int_{B^-} \left[|\xi_1|^{p-2} + \int_0^1 |\xi_1 + t(Du-\xi_1)|^{p-2}\,dt\right]|Du-\xi_1|\,dx \\
&\leq c\int_{B^-} \left[|\xi_1|^{p-2} + (|\xi_1|+|Du-\xi_1|)^{p-2}\right]|Du-\xi_1|\,dx \\
&\leq \frac{c}{|\xi_1|}\int_{B^-} \left[|Du-\xi_1|^{p-1} + \frac{|Du-\xi_1|^{p-1}}{|\xi_1|+|Du-\xi_1|}|\xi_1|\right]|Du-\xi_1|\,dx \\
&\leq \frac{c}{|\xi_1|}\int_{B_r} \left|V(Du)-V(\xi_1)\right|^2 dx.
\end{aligned} \tag{4.46}
$$

Combining (4.42), (4.45) and (4.46), we obtain

$$
I \leq \frac{c\|D\varphi\|_{L^\infty(B_r)}}{|B_r|}\left[\mu(\theta_1^{\frac{1}{2}})|\xi_1|^{\frac{p-2}{2}} E(Du,B_r) + \frac{[E(Du,B_r)]^2}{|\xi_1|}\right]. \tag{4.47}
$$

We now use Lemma 4.4.1 and $(4.38)_2$ to estimate $II$ as

$$\begin{aligned} II &\overset{(4.2)}{\leq} c\omega(R)\|D\varphi\|_{L^\infty(B_r)}\left(\fint_{B_r}|Du|^p\,dx\right)^{\frac{p-1}{p}} \\ &\leq c\theta_3^{\frac{1}{2}}\|D\varphi\|_{L^\infty(B_r)}|\xi_1|^{\frac{p-2}{2}}E(Du,B_r). \end{aligned} \tag{4.48}$$

Merging (4.41), (4.47) and (4.48), and dividing the resulting estimate by $|\xi_1|^{p-2}$, we find

$$\begin{aligned} &\fint_{B_r}\langle \mathcal{A}(Du-\xi_1), D\varphi\rangle\,dx \\ &\overset{(4.40)}{\leq} c\Bigg[\mu(\theta_1^{\frac{1}{2}}) + \frac{\big[E(Du,B_r)\big]}{|\xi_1|^{\frac{p}{2}}} + \theta_3^{\frac{1}{2}} \\ &\qquad + \frac{1}{|\xi_1|^{\frac{p-2}{2}}E(Du,B_r)}\left(R^q\fint_{B_r}|f|^q\,dx\right)^{\frac{1}{q}}\Bigg]\|D\varphi\|_{L^\infty(B_r)} \\ &\leq c_*\big(\mu(\theta_1^{\frac{1}{2}}) + \theta_1^{\frac{1}{2}} + \theta_2 + \theta_3^{\frac{1}{2}}\big)\|D\varphi\|_{L^\infty(B_r)}, \end{aligned}$$

for some $c_* = c_*(\mathtt{data}) \geq 1$. In the last line, we also have used (4.32) and (4.38). Taking

$$\bar\theta_1 = \min\left\{\Big(\frac{\delta}{4c_*}\Big)^2, \Big[\mu^{-1}\Big(\frac{\delta}{4c_*}\Big)\Big]^2\right\}, \quad \theta_2 = \frac{\delta}{4c_*} \quad \text{and} \quad \theta_3 = \left(\frac{\delta}{4c_*}\right)^2,$$

we complete the proof. □

**Lemma 4.4.3.** *Assume* (4.32) *and* (4.38). *For any* $\tau \in (0,1/2]$ *there exist* $\theta_1 = \theta_1(\mathtt{data}, \mu(\cdot), \tau) \leq \tau^{n+2\alpha}$ *and* $\theta_2, \theta_3$ *depending on* $\mathtt{data}$ *and* $\tau$ *such that*

$$\big[E(Du,B_{\tau r})\big]^2 \leq c_3\tau^{2\alpha}\big[E(Du,B_r)\big]^2, \tag{4.49}$$

*where the constant* $c_3$ *depends only on* $\mathtt{data}$.

*Proof.* Recall (4.19) and (4.40). We are going to use $\mathcal{A}$-harmonic approximation lemma with an $N$-function

$$\Psi(t) = \frac{1}{[E(Du,B_r)]^2}\psi_{|\xi_1|}\left(\frac{E(Du,B_r)}{|\xi_1|^{\frac{p-2}{2}}}t\right), \qquad \forall t \geq 0.$$

Then we have

$$\Psi(|Dv|) = \frac{\psi_{|\xi_1|}(|Du - \xi_1|)}{[E(Du, B_r)]^2}.$$

It then follows from Lemma 4.3.2, Lemma 4.3.1 and Lemma 4.4.1 that

$$\begin{aligned}
&[E(Du, B_r)]^2 \left( \fint_{B_{r/2}} \Psi(|Dv|)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \\
&= \left( \fint_{B_{r/2}} \left[ \psi_{|\xi_1|}(|Du - \xi_1|) \right]^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \\
&\le c[E(Du, B_r)]^2 + c|\xi_1|^{2-p} \left( r^q \fint_{B_r} |f|^q \, dx \right)^{\frac{2}{q}} + c\omega(r)^2 |\xi_1|^p \\
&\quad + c \left( r^q \fint_{B_r} |f|^q \, dx \right)^{\frac{p}{q(p-1)}},
\end{aligned}$$

where $\sigma > 0$ is the number given in Lemma 4.3.2. Dividing this estimate by $[E(Du, B_r)]^2$ and using (4.32), (4.38) and (4.39), we discover

$$\left( \fint_{B_{r/2}} \Psi(|Dv|)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \le c \left( 1 + \theta_2^2 + \theta_3 + \theta_1^{\frac{2-p}{2(p-1)}} \theta_2^{\frac{p}{p-1}} \right) \le c. \tag{4.50}$$

Apparently, we also have

$$\fint_{B_r} \Psi(|Dv|) \, dx \le c. \tag{4.51}$$

In light of Lemma 4.4.2, one can apply Lemma 4.2.2 to see that for any $\varepsilon > 0$ there exists $\mathcal{A}$-harmonic function $\bar{h} \in v + W^{1,\Psi}(B_{r/2})$ such that

$$\fint_{B_{r/2}} \Psi(|Dv - D\bar{h}|) \, dx \le \varepsilon, \tag{4.52}$$

where we also have used (4.50) and (4.51). Using the notation

$$h(x) := \frac{E(Du, B_r)}{\sqrt{|\xi_1|^{p-2}}} \bar{h}(x),$$

(4.52) can be written as follows:

$$\fint_{B_{r/2}} \psi_{|\xi_1|}(|Du - Dh|)\, dx \le \varepsilon [E(Du, B_r)]^2. \tag{4.53}$$

So far we have shown comparison estimates between $u$ and $\mathcal{A}$-harmonic function $h$. To proceed further, let $\tau > 0$, and set $\xi_2 = (Du)_{B_{\tau r}}$. Then we have

$$\begin{aligned} |\xi_2 - \xi_1| &\le c \left( \fint_{B_{\tau r}} |Du - \xi_1|^p \, dx \right)^{\frac{1}{p}} \\ &\le \frac{c}{\tau^{\frac{n}{p}}} \left( \fint_{B_r} |V(Du) - V(\xi_1)|^p (|Du| + |\xi_1|)^{\frac{(2-p)p}{2}} \, dx \right)^{\frac{1}{p}} \\ &\overset{(4.32)}{\le} \frac{\tilde{c}_*}{\tau^{\frac{n}{p}}} \theta_1^{\frac{1}{2}} |\xi_1|, \end{aligned}$$

where $\tilde{c}_*$ depends only on `data`. Taking

$$\theta_1 = \min \left\{ \tau^{n+2\alpha}, \frac{\tau^{\frac{2n}{p}}}{4\tilde{c}_*^2}, \bar{\theta}_1 \right\},$$

we see $\frac{1}{2}|\xi_2| \le |\xi_1| \le 2|\xi_2|$, and (4.20) follows.

Using Jensen's inequality, we have

$$\begin{aligned} &\fint_{B_{r/2}} \psi_{|\xi_1|}(|Dh - (Dh)_{B_{r/2}}|)\, dx \\ &\le c \fint_{B_{r/2}} \psi_{|\xi_1|}(|Dh - Du|)\, dx + c \fint_{B_{r/2}} \psi_{|\xi_1|}(|Du - \xi_1|)\, dx \\ &\quad + c\, \psi_{|\xi_1|}(|\xi_1 - (Du)_{B_{r/2}}|) + c\, \psi_{|\xi_1|}(|(Du)_{B_{r/2}} - (Dh)_{B_{r/2}}|) \\ &\le c \fint_{B_{r/2}} \psi_{|\xi_1|}(|Dh - Du|)\, dx + c \fint_{B_{r/2}} \psi_{|\xi_1|}(|Du - \xi_1|)\, dx. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} &\left[E(Du, B_{\tau r})\right]^2 \\ &\overset{(4.21)}{\le} c \fint_{B_{\tau r}} \psi_{|\xi_2|}(|Du - Dh|)\, dx + c \fint_{B_{\tau r}} \psi_{|\xi_2|}(|Dh - (Dh)_{B_{\tau r}}|)\, dx. \end{aligned}$$

It then follows from (4.20) and (4.23) that

$$\begin{aligned}
&\big[E(Du, B_{\tau r})\big]^2 \\
&\le c\tau^{2\alpha} \fint_{B_{r/2}} \psi_{|\xi_1|}(|Dh - (Dh)_{B_{r/2}}|)\,dx + \frac{c}{\tau^n} \fint_{B_{r/2}} \psi_{|\xi_1|}(|Du - Dh|)\,dx \\
&\le c\tau^{2\alpha} \fint_{B_r} \psi_{|\xi_1|}(|Du - \xi_1|)\,dx + c\left(\tau^{2\alpha} + \frac{1}{\tau^n}\right) \fint_{B_{r/2}} \psi_{|\xi_1|}(|Du - Dh|)\,dx \\
&\overset{(4.53)}{\le} c\left(\tau^{2\alpha} + \frac{\varepsilon}{\tau^n}\right)[E(Du, B_r)]^2.
\end{aligned}$$

Taking $\varepsilon \le \tau^{n+2}$ in the above estimate yields the desired result. □

### 4.4.4 The singular case

In this section, we consider the case complementary to (4.32), that is

$$\big|(Du)_{B_r}\big|^p \le \frac{1}{\theta_1}\big[E(Du, B_r)\big]^2 \tag{4.54}$$

The following lemma is the singular counterpart of Lemma 4.4.1.

**Lemma 4.4.4.** *Assume that* (4.54) *holds. There exists a constant* $c = c(p) \ge 1$ *such that*

$$\fint_{B_r} |Du|^p\,dx \le \frac{c}{\theta_1}\big[E(Du, B_r)\big]^2.$$

*Proof.* This lemma can be proved by applying (4.54) instead of (4.32) in the very last part of the proof of Lemma 4.4.1. □

We are now able to obtain excess decay estimates for the singular case (4.54). We refer to [80, Proposition 4.1] for analogous estimates in case of $p \ge 2$.

**Lemma 4.4.5.** *Assume* (4.54)*. For any* $\tau \in (0,1)$*, there exists* $\varepsilon_1$ *depending only on* $\boldsymbol{data}, \theta_1, \eta(\cdot)$ *and* $\tau$ *such that if*

$$\big[E(Du, B_r)\big]^2 \le \varepsilon_1, \tag{4.55}$$

*then we have*

$$\big[E(Du, B_{\tau r})\big]^2 \le c_3 \tau^{2\alpha} \big[E(Du, B_r)\big]^2 + c_4 \bigg( r^q \fint_{B_r} |f|^q \, dx \bigg)^{\frac{p}{q(p-1)}},$$

*where* $c_3 = c_3(\boldsymbol{data})$ *and* $c_4 = c_4(\boldsymbol{data}, \theta_1, \tau)$ *are greater than or equal to 1, and* $\alpha \in (0,1]$ *is the constant determined in* (4.24).

*Proof.* Fix any $\varphi \in C_0^1(B_r)$, then by (4.1) we have

$$\begin{aligned} &\left| \fint_{B_r} a(x)|Du|^{p-2} \langle Du, D\varphi \rangle \, dx \right| \\ &\le \left| \fint_{B_r} \langle A(x,Du) - a(x)|Du|^{p-2} Du, D\varphi \rangle \, dx \right| + \left| \fint_{B_r} \varphi f \, dx \right| \\ &=: I_1 + I_2. \end{aligned} \tag{4.56}$$

We now use (4.2), (4.5) and Lemma 4.4.4 to estimate $I_1$. For any $s > 0$, it holds

$$\begin{aligned} I_1 \le{}& \frac{L}{|B_r|} \int_{B_r \cap \{|Du| > \eta(s)\}} \frac{|Du|^p}{\eta(s)} \, dx \|D\varphi\|_{L^\infty(B_r)} \\ &+ \frac{Ls}{|B_r|} \int_{B_r \cap \{|Du| \le \eta(s)\}} |Du|^{p-1} \, dx \|D\varphi\|_{L^\infty(B_r)} \\ \le{}& \left[ \frac{c}{\theta_1 \eta(s)} [E(Du, B_r)]^2 + \frac{cs}{\theta_1^{\frac{p-1}{p}}} [E(Du, B_r)]^{\frac{2(p-1)}{p}} \right] \|D\varphi\|_{L^\infty(B_r)}. \end{aligned} \tag{4.57}$$

On the other hand, Hölder's and Sobolev's inequalities yield

$$I_2 \le c \|D\varphi\|_{L^\infty(B_r)} \bigg( r^q \fint_{B_r} |f|^q \, dx \bigg)^{\frac{1}{q}}. \tag{4.58}$$

Define

$$w(x) := \frac{u(x)}{\lambda} \quad \text{with} \quad \lambda := \frac{[E(Du, B_r)]^{\frac{2}{p}}}{\theta_1^{\frac{1}{p}}} + \bigg( \frac{r^q}{\kappa_1} \fint_{B_r} |f|^q \, dx \bigg)^{\frac{1}{q(p-1)}}$$

for $\kappa_1 > 0$ to be chosen shortly. According to Lemma 4.4.4

$$\fint_{B_r} |Dw|^p \, dx \leq c.$$

We now combine (4.56)-(4.58) and use (4.55) to obtain

$$\fint_{B_r} |Dw|^{p-2}\langle Dw, D\varphi\rangle \, dx \leq \bar{c}_* \left( \frac{\varepsilon_1^{\frac{1}{p}}}{\theta_1^{\frac{1}{p}}\eta(s)} + s + \kappa_1^{\frac{1}{q}} \right) \|D\varphi\|_{L^\infty(B_r)},$$

where $\bar{c}_* \geq 1$ depends only on $\texttt{data}$. For any $\delta \in (0,1)$, we derive

$$\fint_{B_r} |Dw|^{p-2} \langle Dw, D\varphi\rangle \, dx \leq \delta \|D\varphi\|_{L^\infty(B_r)},$$

by taking $s \leq \frac{\delta}{3\bar{c}_*}$, thereby $\eta(s)$, and then

$$\varepsilon_1 \leq \left( \frac{\delta\theta_1^{\frac{1}{p}}\eta(s)}{3\bar{c}_*} \right)^{\frac{p}{2}} \quad \text{and} \quad \kappa_1 \leq \left( \frac{\delta}{3\bar{c}_*} \right)^q.$$

Set $\kappa_2 = \theta_1 \tau^{2n+4\alpha} > 0$ and $p_1 := p_2' := (2+2\sigma_g)'$ for the constant $\sigma_g > 0$ in Lemma 4.3.1. Taking $\delta = \delta(\texttt{data}, p_1, \kappa_2) = \delta(\texttt{data}, \theta_1, \tau) > 0$ sufficiently small, one can apply Lemma 4.2.3, so that there exists $p$-harmonic map $\bar{h} \in W^{1,p}(B_r;\mathbb{R}^N)$ with $\bar{h} = w$ on $\partial B_r$ such that

$$\left( \fint_{B_r} |V(Dw) - V(D\bar{h})|^{p_1} \, dx \right)^{\frac{2}{p_1}} \leq \kappa_2.$$

From the choice of $\delta$, we note that $\varepsilon_1$ depends only on $\texttt{data}, \theta_1, \tau$ and $\eta(\cdot)$. Scaling back the last estimate with $h(\xi) = \lambda\bar{h}(\xi)$, we find

$$\left( \fint_{B_r} |V(Du) - V(Dh)|^{p_1} \, dx \right)^{\frac{2}{p_1}} \leq c\frac{\kappa_2}{\theta_1} \big[E(Du, B_r)\big]^2 + c\kappa_2 \left( \frac{r^q}{\kappa_1} \fint_{B_r} |f|^q \, dx \right)^{\frac{p}{q(p-1)}}. \tag{4.59}$$

Higher integrability (see for instance [63, Section 6]) and energy minimizing property of $p$-harmonic map yield

$$\left(\fint_{B_{r/2}} |D\bar{h}|^{p(1+\sigma_g)}\,dx\right)^{\frac{1}{1+\sigma_g}} \le c\fint_{B_r} |D\bar{h}|^p\,dx \le c\fint_{B_r} |Dw|^p\,dx.$$

It then follows from the equalities $|Dh|^{p(1+\sigma_g)} = |V(Dh)|^{2(1+\sigma_g)} = |V(Dh)|^{p_2}$ and Lemma 4.4.4 that

$$\left(\fint_{B_{r/2}} |V(Dh)|^{p_2}\,dx\right)^{\frac{2}{p_2}} \le c\fint_{B_r} |Du|^p\,dx \le c\big[E(Du,B_r)\big]^2. \tag{4.60}$$

By Lemma 4.3.1 and Lemma 4.4.4, we further have

$$\left(\fint_{B_{r/2}} |V(Du)|^{p_2}\,dx\right)^{\frac{2}{p_2}} \le c\big[E(Du,B_r)\big]^2 + c\left(r^q\fint_{B_r} |f|^q\,dx\right)^{\frac{p}{q(p-1)}}. \tag{4.61}$$

We combine (4.59), (4.60) and (4.61) to discover

$$\begin{aligned}
&\fint_{B_{r/2}} |V(Du)-V(Dh)|^2\,dx \\
&\le \left(\fint_{B_{r/2}} |V(Du)-V(Dh)|^{p_1}\,dx\right)^{\frac{1}{p_1}} \left(\fint_{B_{r/2}} |V(Du)-V(Dh)|^{p_2}\,dx\right)^{\frac{1}{p_2}} \\
&\le c\tau^{n+2\alpha}\big[E(Du,B_r)\big]^2 + c(\kappa_1)\left(r^q\fint_{B_r} |f|^q\,dx\right)^{\frac{p}{q(p-1)}}.
\end{aligned} \tag{4.62}$$

In the last line, we have used $\kappa_2 = \theta_1\tau^{2n+4\alpha}$.

We now estimate the excess of $Du$ by using (4.17) and (4.24) as follows:

$$\begin{aligned}
\big[E(Du,B_{\tau r})\big]^2 &\le c\fint_{B_{\tau r}} \big|V(Du)-\big(V(Du)\big)_{B_{\tau r}}\big|^2\,dx \\
&\le c\fint_{B_{\tau r}} \big|V(Du)-\big(V(Dh)\big)_{B_{\tau r}}\big|^2\,dx \\
&\le c\big[E(Dh,B_{\tau r})\big]^2 + c\fint_{B_{\tau r}} \big|V(Du)-V(Dh)\big|^2\,dx \\
&\le c\,\tau^{2\alpha}\big[E(Dh,B_{r/2})\big]^2 + c\,\tau^{-n}\fint_{B_{r/2}} \big|V(Du)-V(Dh)\big|^2\,dx
\end{aligned}$$

$$\leq c\,\tau^{2\alpha}\big[E(Du,B_{r/2})\big]^2 + c\,\tau^{-n}\fint_{B_{r/2}}\big|V(Du)-V(Dh)\big|^2\,dx.$$

Consequently, (4.62) gives the desired estimate. □

We recall Lemma 4.4.3 to see that $\theta_1$ depends on $\texttt{data}, \mu(\cdot)$ and $\tau$. Therefore, $\varepsilon_1$ depends on $\texttt{data}, \mu(\cdot), \eta(\cdot)$ and $\tau$, while $c_4$ depends only on $\texttt{data}, \mu(\cdot)$ and $\tau$.

# 4.5 Proof of Theorem 4.1.1

We divide the proof of Theorem 4.1.1 into three steps. In step 1, we construct a sequence of concentric balls and revisit some well known properties of Riesz potentials. In step 2, we prove some iterative lemmas including Lemma 4.5.1, which insures that at least one of the estimates in Section 4.4 is still valid for each concentric balls. Finally in step 3, we complete the proof of Theorem 4.1.1.

## 4.5.1 Basic settings

We fix constants $0 < \rho \leq R \leq R_*$, where $R_*$ will be chosen shortly. Let us take a small constant $\tau \in (0, 1/2]$ satisfying

$$c_1^2 c_3 \tau^{2\alpha} \leq \frac{1}{16}, \tag{4.63}$$

where $c_1$ is determined in (4.17), and $c_3$ is determined in Lemma 4.4.3 and Lemma 4.4.5. Recall the constants $\theta_1, \theta_2, \theta_3, \varepsilon_1, c_4$ and $c_2$ given in Section 4.4 and their dependence. Notice that we have chosen $\theta_1 \in (0,1)$ in Lemma 4.4.3 to satisfy

$$\theta_1 \leq \tau^{n+2\alpha}. \tag{4.64}$$

For every $i \in \{0, 1, 2, \cdots\}$, we set a sequence of concentric balls

$$B_i := B_{r_i} = B_{r_i}(x_0) \quad \text{with} \quad r_i = \tau^{i+1}R,$$

and write

$$k_i = |(Du)_{B_i}|^{\frac{p}{2}} \quad \text{and} \quad E_i = E(Du, B_i).$$

Recall (4.8) and the basic property of $I^f_{1,q}$ from [80, (6.6)]

$$\sum_{i=0}^{\infty}\left(r_i^q \fint_{B_i} |f|^q\,dx\right)^{\frac{1}{q}} \le \frac{I^f_{1,q}(x_0,R)}{\tau^{2n}}. \tag{4.65}$$

Since $p/2(p-1) \ge 1$, we have

$$\sum_{i=0}^{\infty}\left(r_i^q \fint_{B_i} |f|^q\,dx\right)^{\frac{p}{2q(p-1)}} \le \left(\frac{I^f_{1,q}(x_0,R)}{\tau^{2n}}\right)^{\frac{p}{2(p-1)}}. \tag{4.66}$$

Moreover, (4.65) readily implies

$$\sup_{0<\rho\le\tau R}\left(\rho^q \fint_{B_\rho} |f|^q\,dx\right)^{\frac{1}{q}} \le \frac{I^f_{1,q}(x_0,R)}{\tau^{2n}} \text{ and } \lim_{\rho\to 0}\rho^q \fint_{B_\rho} |f|^q\,dx = 0. \tag{4.67}$$

Dini continuity of coefficient, (4.4) allow us to take $R_*(\mathtt{data}, d(\cdot)) > 0$ satisfying

$$d(R_*) = \int_0^{R_*}\omega(\rho)\frac{d\rho}{\rho} \le \frac{\theta_1^{\frac{1}{2}}\theta_3^{\frac{1}{2}}\tau^{\frac{n}{2}}}{100c_1c_2^{\frac{1}{2}}}.$$

Similarly to (4.65), we discover

$$\begin{aligned}\sum_{i=0}^{\infty}\omega(r_i) &\le \frac{1}{-\log\tau}\left(\int_{r_0}^{R}\omega(\rho)\frac{d\rho}{\rho} + \sum_{i=0}^{\infty}\int_{r_{i+1}}^{r_i}\omega(\rho)\frac{d\rho}{\rho}\right)\\ &\le d(R) \le \frac{\theta_1^{\frac{1}{2}}\theta_3^{\frac{1}{2}}\tau^{\frac{n}{2}}}{100c_1c_2^{\frac{1}{2}}}.\end{aligned} \tag{4.68}$$

### 4.5.2 Iterative lemmas

To begin with, let us remark about counterpart of Lemma 4.5.1 for the case $p \ge 2$ shown by Kuusi and Mingione in [80, Section 5]. They showed that $E_i$ is sufficiently small for every $i \ge 0$ when $E_0$ is small enough. However, this fails when $p < 2$, because of lack of degeneracy. More precisely,

$$|\xi_1 - \xi_2|^p \le c|V(\xi_1) - V(\xi_2)|^2 \quad (\xi_1, \xi_2 \in \mathbb{R}^{Nn})$$

does not holds in general, for instance, see [53, Lemma 4.2] and [80]. Therefore, it is not clear that Lemma 4.4.5 is applicable to the singular case (4.54). Nevertheless, we can show that if (4.54) holds at some $i \geq 0$, then $E_i$ is small enough, as we now state and prove.

**Lemma 4.5.1.** *There exists a constant $H_1 = H_1(\boldsymbol{data}, \mu(\cdot), \tau) \geq 1$ such that for the constant $\varepsilon_1$ given in Lemma 4.4.5, if*

$$E_0^2 + H_1 \sup_{j\geq 0} \left( r_j^q \fint_{B_j} |f|^q \, dx \right)^{\frac{p}{q(p-1)}} \leq \varepsilon_1, \tag{4.69}$$

*then*

$$E_i^2 \leq \varepsilon_1 \quad \text{or} \quad \frac{1}{\theta_1} E_i^2 < k_i^2 \tag{4.70}$$

*holds for every $i \geq 0$.*

*Proof.* We prove by induction. For $i = 0$, $(4.70)_1$ holds true by the assumption (4.69). We now assume that (4.70) holds for some $i \geq 0$. We write

$$H_1 = \min \left\{ 4c_4, 4\left( \frac{16c_2}{\theta_2^2 \tau^n} \right)^{\frac{2p-1}{p-1}} \right\}. \tag{4.71}$$

***Case 1)*** $(4.70)_2$ *fails.*

In this case, we assume not only $(4.70)_1$, but also $\frac{1}{\theta_1} E_i^2 \geq k_i^2$. It then follows from Lemma 4.4.5, (4.63) and (4.71) that

$$E_{i+1}^2 \leq 2c_3 \tau^{2\alpha} E_i^2 + 2c_4 \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{q(p-1)}} \leq \varepsilon_1.$$

***Case 2)*** $(4.70)_2$ *holds.*

We now suppose that $(4.70)_2$ holds. Taking (4.18) and (4.70) into account, we see

$$k_i \leq k_{i+1} + 3\tau^{-\frac{n}{2}} E_i \leq k_{i+1} + 3c_1 \tau^{-\frac{n}{2}} \theta_1^{\frac{1}{2}} k_i.$$

Accordingly, (4.63) and (4.64) yield

$$k_i \leq 2k_{i+1}. \tag{4.72}$$

Next, we consider the following alternatives: Either

$$k_i^2 > \frac{1}{\theta_*}\left(r_i^q \fint_{B_i} |f|^q\, dx\right)^{\frac{p}{q(p-1)}} \tag{4.73}$$

or

$$k_i^2 \leq \frac{1}{\theta_*}\left(r_i^q \fint_{B_i} |f|^q\, dx\right)^{\frac{p}{q(p-1)}} \tag{4.74}$$

for some $\theta_* \in (0,1)$ to be chosen shortly. For the case (4.73), we combine (4.35), (4.37) and (4.49) to discover

$$\begin{aligned} E_{i+1}^2 &\leq \frac{c_2}{\theta_2 \tau^n} k_i^{\frac{2}{p}} \left(r_i^q \fint_{B_{r_i}} |f|^q\, dx\right)^{\frac{1}{q}} + \frac{c_2}{\theta_3 \tau^n}\omega(r_i)^2 k_i^2 + c_3 \tau^{2\alpha} E_i^2 \\ &\leq \left[\frac{c_2 \theta_*^{\frac{p-1}{p}}}{\theta_2 \tau^n} + \frac{c_2}{\theta_3 \tau^n}\omega(r_i)^2 + c_3 \tau^{2\alpha}\theta_1\right] k_i^2. \end{aligned}$$

Taking $\theta_* = \left(\frac{\theta_1 \theta_2 \tau^n}{16 c_2}\right)^{\frac{p}{p-1}}$, the estimate

$$\frac{1}{\theta_1} E_{i+1}^2 \leq k_{i+1}^2$$

follows from (4.63), (4.68) and (4.72).

The only case left to be considered is that $(4.70)_2$ holds with (4.74). If we assume

$$\left(r_i^q \fint_{B_i} |f|^q\, dx\right)^{\frac{1}{q}} \leq \theta_2 E_i k_i^{\frac{p-2}{p}} \quad \text{and} \quad \omega(r_i)^2 k_i^2 \leq \theta_3 E_i^2,$$

then Lemma 4.4.3 and (4.72) yield

$$\frac{1}{\theta_1} E_{i+1}^2 \leq \frac{c_3 \tau^{2\alpha}}{\theta_1} E_i^2 \overset{(4.63)}{\leq} \frac{1}{16} k_i^2 \leq k_{i+1}^2.$$

On the contrary, when

$$\left(r_i^q \fint_{B_i} |f|^q\, dx\right)^{\frac{1}{q}} > \theta_2 E_i k_i^{\frac{p-2}{p}} \quad \text{or} \quad \omega(r_i)^2 k_i^2 > \theta_3 E_i^2$$

holds, then (4.35), (4.37), (4.68), (4.74) and the choice of $\theta_*$ yield

$$\begin{aligned} E_{i+1}^2 &\le \frac{c_2}{\theta_2\tau^n} k_i^{\frac{2}{p}} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{1}{q}} + \frac{c_2}{\theta_3\tau^n}\omega(r_i)^2 k_i^2 \\ &\le \frac{c_2}{\theta_2\theta_*\tau^n} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{q(p-1)}} + \frac{c_2}{\theta_3\theta_*\tau^n}\omega(r_i)^2 \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{q(p-1)}} \\ &\le H_1 \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{q(p-1)}}. \end{aligned}$$

Recalling (4.69), we obtain $E_{i+1}^2 \le \varepsilon_1$. This finishes the proof. □

**Remark 4.5.2.** Assume (4.69). Consequently, in light of Lemma 4.5.1, we can apply Lemma 4.4.5, Lemma 4.4.3, (4.36) and (4.37) to obtain

$$\begin{aligned} E_{i+1} \le \frac{1}{4} E_i + c_4^{\frac{1}{2}} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{2q(p-1)}} \\ + \frac{c_2^{\frac{1}{2}}}{\theta_2 \tau^{\frac{n}{2}}} \Lambda^{\frac{2-p}{2}} \left( R^q \fint_{B_i} |f|^q \, dx \right)^{\frac{1}{q}} + \frac{c_2^{\frac{1}{2}}\omega(r_i)}{\theta_3^{\frac{1}{2}}\tau^{\frac{n}{2}}} \Lambda^{\frac{p}{2}}, \end{aligned} \tag{4.75}$$

whenever $k_i^2 \le \Lambda^p$ for any $i \ge 0$. Here, we also have used (4.63), (4.65) and (4.66).

**Lemma 4.5.3.** *Under the assumption* (4.69), *if*

$$k_l + \tau^{-\frac{n}{2}} E_l \le \frac{\Lambda^{\frac{p}{2}}}{100} \tag{4.76}$$

*and*

$$H_2 I_{1,q}^f(x_0, r_l) \le \Lambda^{p-1} \quad \textit{with} \quad H_2 = \frac{100 c_1 c_2^{\frac{1}{2}} c_4^{\frac{1}{2}}}{\theta_2 \tau^{3n}} \tag{4.77}$$

*holds for some* $l \ge 0$*, then for every* $0 < \rho \le r_l$ *we have*

$$|(Du)_{B_\rho}| \le \Lambda. \tag{4.78}$$

*Proof.* Firstly, we prove

$$k_i^2 \le \frac{\Lambda^p}{2} \tag{4.79}$$

for every $i \geq l$ by induction and by using the estimate (4.75). The case $i = l$ follows directly from (4.76).

We now assume that (4.79) holds for every $i \in \{l, l+1, \ldots, m\}$. Using (4.75) repeatedly, we have

$$\begin{aligned}
\sum_{i=l}^{m} E_i &\leq E_l + \frac{1}{4}\sum_{i=l}^{m-1} E_i + c_4^{\frac{1}{2}} \sum_{i=l}^{m-1} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{2q(p-1)}} \\
&\qquad + \frac{c_2^{\frac{1}{2}}}{\theta_2 \tau^{\frac{n}{2}}} \Lambda^{\frac{2-p}{2}} \sum_{i=l}^{m-1} \left( R^q \fint_{B_i} |f|^q \, dx \right)^{\frac{1}{q}} + \frac{c_2^{\frac{1}{2}}}{\theta_3^{\frac{1}{2}} \tau^{\frac{n}{2}}} \Lambda^{\frac{p}{2}} \sum_{i=l}^{m-1} \omega(r_i) \\
&\leq 2E_l + 2c_4^{\frac{1}{2}} \sum_{i=l}^{m-1} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{2q(p-1)}} \\
&\qquad + \frac{2c_2^{\frac{1}{2}}}{\theta_2 \tau^{\frac{n}{2}}} \Lambda^{\frac{2-p}{2}} \sum_{i=l}^{m-1} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{1}{q}} + \frac{2c_2^{\frac{1}{2}}}{\theta_3^{\frac{1}{2}} \tau^{\frac{n}{2}}} \Lambda^{\frac{p}{2}} \sum_{i=l}^{m-1} \omega(r_i).
\end{aligned}$$

It then follows from (4.65), (4.66), (4.68) and (4.79) that

$$\begin{aligned}
\sum_{i=l}^{m} E_i &\leq 2E_l + 2c_4^{\frac{1}{2}} \left( \frac{I_{1,q}^f(x_0, r_l)}{\tau^{2n}} \right)^{\frac{p}{2(p-1)}} \\
&\qquad + \frac{2c_2^{\frac{1}{2}}}{\theta_2 \tau^{\frac{n}{2}}} \Lambda^{\frac{2-p}{2}} \frac{I_{1,q}^f(x_0, r_l)}{\tau^{2n}} + \frac{2c_2^{\frac{1}{2}}}{\theta_3^{\frac{1}{2}} \tau^{\frac{n}{2}}} \Lambda^{\frac{p}{2}} \sum_{i=l}^{m-1} \omega(r_i) \leq \frac{\tau^{\frac{n}{2}}}{10} \Lambda^{\frac{p}{2}}.
\end{aligned} \tag{4.80}$$

For the last inequality, we have used the definition of $H_2$ given in (4.77) and the fact that $p/2(p-1) \geq 1$. It then follows from (4.18) that

$$k_{m+1} \leq k_l + 2\tau^{-\frac{n}{2}} \sum_{i=l}^{l} E_i \leq \frac{\Lambda^{\frac{p}{2}}}{2}.$$

This shows (4.79).

We next prove (4.78). For any $0 < \rho \leq r_l$, there exists an integer $m \geq l$ such that $\rho \in (r_{m+1}, r_m]$. Using again (4.18), we discover

$$|(Du)_{B_\rho}|^{\frac{p}{2}} \leq 2\tau^{-\frac{n}{2}} E_m + k_m.$$

Now, (4.78) is a direct consequence of (4.79) and (4.80). □

For each $x_0 \in \Omega$, we write

$$\begin{aligned}\Lambda^{\frac{p}{2}} &:= \Lambda(x_0)^{\frac{p}{2}} \\ &:= 200\tau^{-\frac{n}{2}}\left[\left(\fint_{B_{\tau R}(x_0)} |Du|^p\,dx\right)^{\frac{1}{2}} + \left(H_2 I^f_{1,q}(x_0,\tau R)\right)^{\frac{p}{2(p-1)}}\right]. \quad (4.81)\end{aligned}$$

The assumptions, (4.76) and (4.77) for $l = 0$, immediately hold with above $\Lambda$.

**Remark 4.5.4.** A consequence of Lemma 4.5.1 and Lemma 4.5.3 is pointwise BMO-regularity for $Du$ under (4.69). In light of Lemma 4.5.1, for every $i \geq 0$, we have

$$E_i^2 \leq \varepsilon_1 + \theta_1 k_i^2.$$

For any $\rho \in (0, r_0]$, there exists $j \geq 0$ such that $\rho \in (r_{j+1}, r_j]$. Using (4.17), we obtain

$$E(Du, B_\rho) \leq c_1 \widetilde{E}(Du, B_\rho) \leq c_1 \tau^{-\frac{n}{2}} E_j \leq c_1 \tau^{-\frac{n}{2}}(1 + k_j). \qquad (4.82)$$

On the other hand, Lemma 4.5.3 implies that $k_j \leq \Lambda^{\frac{p}{2}}$. Consequently, $E(Du, B_\rho)$ is uniformly bounded with respect to $\rho \in (0, r_0]$ with $r_0 = \tau R$.

We are now ready to prove Theorem 4.1.2.

*Proof of Theorem 4.1.2.* Set $\varepsilon_* = \tau^n \varepsilon_1 / H_1$, where $\tau, H_1, \varepsilon_1$ are the given constants in (4.63), Lemma 4.5.1 and Lemma 4.4.5. Then (4.69) holds. Therefore, Lemma 4.5.1 holds, and also Lemma 4.5.3 holds for the constant $\Lambda$ given in (4.81) and $l = 0$ .

Our first goal is to prove that for any small constant $s > 0$, there exists $i_s \in \mathbb{N}$ such that

$$\sup_{0<\rho\leq i_s} E(Du, B_\rho(x_0)) \leq s, \qquad (4.83)$$

which is equivalent to (4.13). We now fix a small constant $s > 0$. Taking (4.67) and (4.68) into account, there exists $i_0$ such that

$$H_2(1 + \Lambda^{\frac{2-p}{2}}) \sup_{i\geq i_0}\left(r_m^q \fint_{B_m} |f|^q\,dx\right)^{\frac{p}{2q(p-1)}} \leq \frac{\tau^{\frac{n}{2}} s}{8} \qquad (4.84)$$

and

$$H_2 \omega(r_{i_0}) \Lambda^{\frac{p}{2}} \leq \frac{\tau^{\frac{n}{2}} s}{4}. \qquad (4.85)$$

Fix any $\rho \in (0, r_{i_0+1}]$, then there exists $j \geq i_0 + 1$ such that $\rho \in (r_{j+1}, r_j]$, and so

$$E(Du, B_\rho(x_0)) \leq \tau^{-\frac{n}{2}} E_j. \tag{4.86}$$

Applying (4.75) iteratively and using (4.84) and (4.85), we find

$$\begin{aligned} E_j &\leq \frac{1}{4^{j-i_0}} E_{i_0} + H_2 \sup_{i \geq i_0} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{p}{2q(p-1)}} \\ &\quad + H_2 \Lambda^{\frac{2-p}{2}} \sup_{i \geq i_0} \left( r_i^q \fint_{B_i} |f|^q \, dx \right)^{\frac{1}{q}} + H_2 \omega(r_{i_0}) \Lambda^{\frac{p}{2}} \\ &\leq \frac{1}{4^{j-i_0}} E_{i_0} + \frac{\tau^{\frac{n}{2}} s}{2}, \end{aligned}$$

where we also have used the fact $p/2(p-1) \geq 1$. We now combine (4.86) and (4.82) to discover

$$E(Du, B_\rho(x_0)) \leq \frac{c_1(1+\Lambda)}{\tau^n 4^{j-i_0}} + \frac{s}{2}.$$

Consequently, there exists $i_s = i_s(\tau, \Lambda) > i_0$ such that (4.83) holds. This shows (4.13).

We now turn our attention to local VMO-regularity. Assume (4.14) and (4.15), and take a point $x_0 \in \Omega_u$. Note that (4.14) and continuity property of integral imply the uniform boundedness of map $x \mapsto \Lambda(x)$, where $\Lambda(\cdot)$ is defined in (4.81).

For a fixed constant $s > 0$, (4.15) and uniform boundedness of $\Lambda(\cdot)$ allow us to take a neighborhood of $x_0$(denote by $\mathcal{O}$), such that there exists $i_0$ satisfying (4.84) for every points in $\mathcal{O}$. In light of (4.68), we can further assume that (4.85) holds for every $x \in \mathcal{O}$. Then the same proof as above gives

$$E(Du, B_\rho(x)) \leq \frac{c_1(1+\Lambda(x))}{\tau^n 4^{j-i_0}} + \frac{s}{2}$$

for every $x \in \mathcal{O}$. Consequently, uniform boundedness of $x \to \Lambda(x)$ in $\mathcal{O}$ yields that $E(Du, B_\rho(x)) \to 0$ uniformly in $\mathcal{O}$. This completes the proof. □

### 4.5.3 Proof of Theorem 4.1.1

We first prove that every point $x_0 \in \Omega$ satisfying (4.9) is a Lebesgue point of $Du$. Recall that the choice of $\varepsilon_*$ given in the proof of Theorem 4.1.2 implies (4.69), and so Lemma 4.5.1 is available.

*Proof of* (4.10) *and* (4.11). For any $i \geq 0$, (4.18) implies

$$|V\big((Du)_{B_{i+1}}\big) - V\big((Du)_{B_i}\big)| \leq 3\tau^{-\frac{n}{2}} E_i.$$

In light of (4.9), we can apply (4.80) for any $1 \leq m < l$ to discover

$$\begin{aligned} &|V\big((Du)_{B_l}\big) - V\big((Du)_{B_m}\big)| \\ &\leq 3\tau^{-\frac{n}{2}} \sum_{m \leq i \leq l-1} E_i \\ &\leq 6\tau^{-\frac{n}{2}} E_m + H_2 \Big[ I^f_{1,q}(x_0, r_{m-1}) \Big]^{\frac{p}{2(p-1)}} \\ &\quad + H_2 \Lambda^{\frac{2-p}{2}} I^f_{1,q}(x_0, r_{m-1}) + H_2 \Lambda^{\frac{p}{2}} d(r_{m-1}). \end{aligned} \tag{4.87}$$

Recall (4.4) and (4.8) to observe that $I^f_{1,q}(x_0, r_{m-1})$ and $d(r_{m-1})$ converges to 0, as $m \to \infty$. In addition, the right-hand side of (4.87) does not depend on $l$, and it converges to 0 as $m \to \infty$ by pointwise VMO regularity, (4.13).

On the other hand, using Lemma 4.5.3, we have

$$|(Du)_{B_l} - (Du)_{B_m}| \leq c\Lambda^{\frac{2-p}{2}} |V\big((Du)_{B_l}\big) - V\big((Du)_{B_m}\big)|.$$

Hence, $\{(Du)_{B_i}\}_{i \geq 1}$ is a Cauchy sequence, and denote the limit of the sequence by $\mathcal{L} \in \mathbb{R}^{Nn}$, i.e.,

$$\lim_{i \to \infty} (Du)_{B_i} = \mathcal{L}.$$

For any $\rho \in (0, \tau^2 r]$, there exists $m \geq 1$ such that $\rho \in (r_{m+1}, r_m]$. Again (4.18) implies

$$|V\big((Du)_{B_\rho(x_0)}\big) - V\big((Du)_{B_m}\big)| \leq 3\tau^{-\frac{n}{2}} E_m.$$

Therefore, we obtain

$$\begin{aligned} \lim_{\rho \to 0} |\mathcal{L} - (Du)_{B_\rho(x_0)}| \leq \lim_{m \to \infty} &|\mathcal{L} - (Du)_{B_m}| \\ &+ c\Lambda^{(2-p)/2} \lim_{\rho \to 0} \big|V\big((Du)_{B_m}\big) - V\big((Du)_{B_\rho(x_0)}\big)\big| \end{aligned}$$

$$\leq \lim_{m\to\infty} |\mathcal{L} - (Du)_{B_m}| + 3c\tau^{-\frac{n}{2}}\Lambda^{\frac{2-p}{2}} \lim_{m\to\infty} E_m = 0.$$

This completes the proof of (4.10).

A slight modification of the proof of $(4.10)_1$ yields $(4.10)_2$. Hence we omit the proof of $(4.10)_2$. For more details, we refer to [80, Lemma 6.4].

At this stage, the proof of (4.11) follows from (4.10) and the following estimate

$$\left|\big(V(Du)\big)_{B_\rho(x_0)} - V\big((Du)_{B_\rho(x_0)}\big)\right| \leq E(Du, B_\rho(x_0))$$

for every $\rho \in (0, \tau^2 R]$. Specifically, we have

$$\begin{aligned}|V(Du)(x_0) - V(Du(x_0))| \leq{}& |V(Du)(x_0) - \big(V(Du)\big)_{B_\rho(x_0)}| \\ &+ |\big(V(Du)\big)_{B_\rho(x_0)} - V\big((Du)_{B_\rho(x_0)}\big)| \\ &+ |V\big((Du)_{B_\rho(x_0)}\big) - V(Du(x_0))|.\end{aligned}$$

Taking (4.10) and (4.13) into account, the right-hand side converges to 0 as $\rho$ goes to 0.

*Proof of* (4.12). First, we assume $\rho \in (0, \tau R]$. There exists $m_\rho \in \{0, 1, 2, \cdots\}$ such that $\rho \in (r_{m_\rho+1}, r_{m_\rho}]$. In this proof, we set

$$\Lambda^{\frac{p}{2}} = 200\tau^{-n}\left[\left(\fint_{B_\rho(x_0)} |Du|^p\, dx\right)^{\frac{1}{2}} + \left(H_2 I^f_{1,q}(x_0, \rho)\right)^{\frac{p}{2(p-1)}}\right].$$

A straightforward calculation shows

$$k_{m_\rho+1} + \tau^{-\frac{n}{2}} E_{m_\rho+1} \leq \tau^{-\frac{n}{2}} k_\rho + \tau^{-n} E(Du, B_\rho) \leq \frac{\Lambda^{\frac{p}{2}}}{100}$$

and

$$H_2 I^f_{1,q}(x_0, r_l) \leq \Lambda^{p-1}.$$

Therefore, Lemma 4.5.3 gives

$$|(Du)_{B_\varrho}| \leq \Lambda$$

for every $\varrho \in (0, m_\rho + 1]$. Moreover, similarly to (4.87), we have

$$|V\big((Du)_{B_l}\big) - V\big((Du)_{B_{m_\rho+2}}\big)|$$

$$\begin{aligned}
&\le 6\tau^{-\frac{n}{2}} E_{m_\rho+2} + H_2 \Big[ I^f_{1,q}(x_0, r_{m_\rho+1}) \Big]^{\frac{p}{2(p-1)}} \\
&\quad + H_2 \Lambda^{\frac{2-p}{2}} I^f_{1,q}(x_0, r_{m_\rho+1}) + H_2 \Lambda^{\frac{p}{2}} d(r_{m_\rho+1})
\end{aligned}$$

for any $l \ge m_\rho + 2$.

Letting $l \to \infty$ in the previous estimates, we discover

$$\begin{aligned}
&|V(Du)(x_0) - V\big((Du)_{B_{m_\rho+2}}\big)| \\
&\le 6\tau^{-\frac{n}{2}} E_{m_\rho+2} + H_2 \Big[ I^f_{1,q}(x_0, r_{m_\rho+1}) \Big]^{\frac{p}{2(p-1)}} \\
&\quad + H_2 \Lambda_2^{\frac{2-p}{2}} I^f_{1,q}(x_0, r_{m_\rho+1}) + H_2 \Lambda_2^{\frac{p}{2}} d(r_{m_\rho+1}) \\
&\le 6\tau^{-\frac{3n}{2}} E(Du, B_\rho) + H_2 \Big[ I^f_{1,q}(x_0, \rho) \Big]^{\frac{p}{2(p-1)}} \\
&\quad + H_2 \Lambda_2^{\frac{2-p}{2}} I^f_{1,q}(x_0, \rho) + H_2 \Lambda_2^{\frac{p}{2}} d(\rho),
\end{aligned} \tag{4.88}$$

where we have used (4.10).

Again we have

$$\Big| V\big((Du)_{B_{m_\rho+2}}\big) - V\big((Du)_{B_\rho}\big) \Big| \le 3\tau^{-n} E(Du, B_\rho). \tag{4.89}$$

Combining (4.88) and (4.89), we conclude that (4.12) holds for every $\rho \le \tau R$.

We use (4.12) for $\tau\rho > 0$ to obtain (4.12) for $\rho \in (\tau R, R]$. This completes the proof.

# Chapter 5

# Measure data problems with general growth

In this chapter, we study Calderón-Zygmund type estimates for nonlinear elliptic measure data problems in terms of the fractional maximal function of order 1 and later, we investigate similar estimates for integral functionals with $p(x)$-growth. Indeed, for the second problems, we take quasi-minimizers into account and present only the proof of a comparison estimate which essentially deals with controlling quasi-minimality, since the remaining parts of the proofs are quite similar to the ones for former problems.

We remark that these researches delivered from the process of developing a unified method in the gradient potential theory.

## 5.1 Main result

Let us consider the following measure data problem with general growth:

$$\begin{cases} -\mathrm{div}(A(x, Du)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$ and $\mu$ is a Radon measure with finite mass. The given vector field $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be $C^1$-regular in the second variable, with $\partial A(\cdot) = \partial_\xi A(\cdot)$ being Carathéodory regular, and satisfy the following growth and ellipticity assumptions:

$$\begin{cases} |A(x,\xi)| + |\partial A(x,\xi)||\xi| \le Lg(|\xi|), \\ \nu \frac{g(|\xi|)}{|\xi|}|\zeta|^2 \le \langle \partial A(x,\xi)\zeta, \zeta \rangle \end{cases} \tag{5.2}$$

for every $\xi, \zeta \in \mathbb{R}^n$ and $x \in \Omega$, where $0 < \nu \le L < \infty$. Here, $g$ is the derivative of an $N$-function $G \in C^2(0,\infty) \cap C^1[0,\infty)$ satisfying

$$0 < \gamma_1 - 1 \le \frac{tg'(t)}{g(t)} \le \gamma_2 - 1 < \infty \tag{5.3}$$

for some constants $\gamma_1, \gamma_2 > 1$. Recall the definition of $N$-functions and corresponding function spaces given in Section 2.2.

For (5.1), we are going to prove the existence of a SOLA and the global Calderón-Zygmund estimates for (5.1) in terms of $M_1(\mu)$, under possibly the weakest assumptions both on $A(\cdot)$ and $\Omega$.

So far, there have been only a few regularity results for SOLAs to (5.1) with general $g(t)$ satisfying (5.3), while there are many research papers when $g(t) = t^{p-1}$ in (5.2). We refer to the very fine paper [10] which obtained Riesz potential estimates for (5.1) with general growth. To avoid the difficulties that arise from the lack of monotonicity of the map $t \mapsto g'(t)$, it is assumed in [10] that

$$2 \le \gamma_1, \tag{5.4}$$

in order to obtain gradient potential estimates. On the other hand, in the spirit of Calderón-Zygmund estimates, (5.4) can be relaxed as

$$2 - 1/n < \gamma_1 \le \gamma_2 < \infty, \tag{5.5}$$

which covers the whole range of $p \in (2-1/n, \infty)$ for $p$-Laplacian type equations, see [94].

We would like to emphasize that comparison estimates between $p$-Laplacian type measure data problem and the corresponding homogeneous problem have different forms, stemming from (5.14), according to the range of $p$. Roughly speaking, constructing some auxiliary functions, we can obtain desired comparison estimates in the sense of $L^1$ without distinguishing $p \ge 2$ and $p \in (2-1/n, 2)$, see Lemma 5.3.5.

We now turn our attention to our assumptions on the couple $(A(\cdot), \Omega)$.

**Definition 5.1.1.** For some $R > 0$ and $\delta \in (0, 1/8)$, we say that $(A(\cdot), \Omega)$ is $(\delta, R)$-vanishing, whenever the followings hold:

1. Denoting
$$\theta(U)(x) := \frac{1}{g(|\xi|)} \sup_{\xi \in \mathbb{R}^n} \left| A(x, \xi) - \fint_U A(z, \xi)\, dz \right| \quad (\leq 2L)$$
for any measurable set $U \subset \mathbb{R}^n$ and $x \in U$, we have
$$\sup_{0<r<R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} \theta(B_r(y))(x)\, dx \leq \delta.$$

2. For each $y \in \partial\Omega$ and $r \in (0, R]$, there exists a coordinate system $\{\tilde{y}_1, \cdots, \tilde{y}_n\}$ with the origin at $y$ satisfying
$$B_r(0) \cap \{\tilde{y}_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{\tilde{y}_n > -\delta r\}.$$
We call such $\Omega$ as a $(\delta, R)$-Riefenberg flat domain.

We remark that a Riefenberg flat domain has its boundary trapped between two hyperplanes. Moreover, $(\delta, R)$-Riefenberg flatness guarantees the measure density condition
$$\sup_{\substack{0<r\leq R \\ x \in \Omega}} \frac{|B_r(x)|}{|\Omega \cap B_r(x)|} \geq \left(\frac{2}{1-\delta}\right)^n \geq \left(\frac{16}{7}\right)^n$$
and
$$\inf_{\substack{0<r\leq R \\ x \in \partial\Omega}} \frac{|\Omega^c \cap B_r(x)|}{|B_r(x)|} \geq \left(\frac{1-\delta}{2}\right)^n \geq \left(\frac{7}{16}\right)^n.$$
This condition will be used several times later, without referring to it. For a further discussion on Riefenberg domains, we refer to [28, 82, 96].

We now state our main result.

**Theorem 5.1.2.** *Under the assumptions* (5.2) *and* (5.3), *let* $u \in W_0^{1,1}(\Omega)$ *be a SOLA to the problem* (5.1). *Suppose that* $g^{-1}(M_1(\mu)) \in L^H(\Omega)$ *for some* $N$*-function* $H \in C^2(0, \infty) \cap C^1[0, \infty)$ *with its derivative* $h$ *satisfying*
$$0 < \gamma_3 - 1 = \inf_{t>0} \frac{th'(t)}{h(t)} \leq \frac{th'(t)}{h(t)} \leq \sup_{t>0} \frac{th'(t)}{h(t)} = \gamma_4 - 1. \tag{5.6}$$

*Then there exists a small constant* $\delta = \delta(n, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \nu, L) > 0$ *such that if* $(A(\cdot), \Omega)$ *is* $(\delta, R)$*-vanishing, then* $Du \in L^H(\Omega)$ *with the estimate*

$$\int_\Omega H(|Du|)\,dx \le c \int_\Omega H \circ g^{-1}(M_1(\mu))\,dx, \tag{5.7}$$

*where* $c$ *depends only on* $n, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \nu, L$ *and* $\operatorname{diam}(\Omega)/R$.

As a consequence of Theorem 5.1.2, we find a sharp gradient estimate in the frame of Lorentz spaces, see Remark 5.4.2.

Our main approach to the proof of Theorem 5.1.2 is based on the so-called maximal function free technique which is introduced in [5] and revisited by [27, 32], along with the following observation

$$\begin{aligned} g^{-1}\left(\frac{r|\mu|(\Omega_r(x_0))}{|B_r(x_0)|}\right) &\le c \fint_{\Omega_r(x_0)} g^{-1}\left(\frac{2r|\mu|(\Omega_{2r}(x))}{|B_{2r}(x)|}\right)dx \\ &\le c \fint_{\Omega_r(x_0)} g^{-1}(M_1(\mu))\,dx, \end{aligned} \tag{5.8}$$

for any $x_0 \in \Omega$ and some constant $c = c(n, \gamma_1, \gamma_2)$. The main feature in this chapter is that we are able to find a unified way working both the degenerate and singular cases to prove (5.7).

## 5.2 Existence of SOLA

Until Section 5.5, we use the abbreviation

$$\texttt{data} := \{n, \gamma_1, \gamma_2, \nu, L\}$$

and set the auxiliary vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ by

$$V(\xi) := \left(\frac{g(|\xi|)}{|\xi|}\right)^{\frac{1}{2}} \xi$$

for each $\xi \in \mathbb{R}^n$. The monotonicity of $A$ can be written simply in terms of $V$ as in (2.18).

This section is devoted to introducing the so-called SOLAs to (5.1) and proving its existence. Recall that the right-hand side datum $\mu$ given in (5.1) is a bounded Radon measure. As mentioned in Section 1.1, several notions of solutions were mentioned to deal with measure data problems. Before introducing SOLAs, we first recall approximable solutions with the following notation:

$$\Lambda^{1,G} := \{u \text{ is measurable in } \Omega : T_t(u) \in W^{1,G}(\Omega) \text{ for all } t > 0\}.$$

**Definition 5.2.1.** We say that $u \in \Lambda^{1,G}$ is called an approximable solution to (5.1), if $u$ solves (5.1) in the distributional sense which means

$$\int_\Omega \langle A(x, Du), D\phi\rangle \, dx = \int_\Omega \phi \, d\mu, \quad {}^\forall \phi \in W_0^{1,\infty}(\Omega),$$

and the following statement holds: There exists a sequence of weak solutions $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,G}(\Omega)$ to

$$\begin{cases} -\mathrm{div}(A(x, Du_k)) = \mu_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.9}$$

where $\mu_k \in L^\infty(\Omega)$ converges to $\mu$ weakly in $\mathcal{M}_b(\Omega)$, such that $u_k \to u$ almost everywhere.

We now introduce SOLAs.

**Definition 5.2.2.** We say that $u \in W_0^{1,1}(\Omega)$ is a SOLA to (5.1), if $u$ is an approximable solution with a sequence of functions $\{u_k\}_{k\geq 0}$, then $u_k$ converges to $u$ strongly in $W^{1,1}(\Omega)$ up to a subsequence.

A main difference between the notion of SOLA and approximable solution is that an approximable solution $u$ only requires $Du_k \to Du$ almost everywhere, while a SOLA requires $Du_k \to Du$ strongly in $L^1$. We point out that the almost everywhere convergence is not enough in proving the desired Calderón-Zygmund estimate (5.7), as far as we are concerned. In this regard, we need the strong convergence in $L^1$ for which we are dealing with SOLA instead of approximable solution.

Except for this section where the existence of a SOLA is proved under (5.12), we always assume $\gamma_1 > 2 - \frac{1}{n}$ but do not assume $\gamma_2 \leq n$. Instead, we consider the following slow growth conditions

$$\int^\infty \left(\frac{t}{G(t)}\right)^{\frac{1}{n-1}} dt = \infty \quad \text{and} \quad \int_0 \left(\frac{t}{G(t)}\right)^{\frac{1}{n-1}} dt < \infty. \tag{5.10}$$

Note that for the case $G(t) = t^p$, $(5.10)_1$ implies $p \le n$. If $(5.10)_1$ does not hold, that is

$$\int^{\infty} \left( \frac{t}{G(t)} \right)^{\frac{1}{n-1}} dt < \infty,$$

then $u \in W_0^{1,G}(\Omega)$ is continuous as follows from [33, Theorem 1a], which ensures $\mu \in (W_0^{1,G}(\Omega))^*$ and the existence and uniqueness of a weak solution to (5.1) follows from the monotone operator theory. Therefore $(5.10)_1$ can be regarded to be a natural generalization of $p \le n$ for the case of $G(t) = t^p$. On the other hand, assumption $(5.10)_2$ is given for a technical reason. If $(5.10)_2$ does not hold, then we define $\bar{G}$, a modification of $G$ near 0, so that $\bar{G}$ satisfies (5.10) and $L^G = L^{\bar{G}}$, see [10, Section 5] and [37, Section 3].

Our proof of the existence of a SOLA to (5.1) is motivated from the previous paper [37]. Let us start with the introduction of the following functions from [37]:

$$\phi_n(t) := \int_0^t \left( \frac{s}{G(s)} \right)^{\frac{1}{n-1}} ds, \quad H_n(t) := \phi_n(t)^{\frac{1}{n'}}, \quad \Psi_n(t) := \frac{G(t)}{\phi_n(t)}$$

and

$$G_n(t) := G(H_n^{-1}(t)). \tag{5.11}$$

Let us present important lemmas from [37]. Throughout this section we always assume that $G$ satisfies (5.3).

**Lemma 5.2.3.** *[37, Lemma 4.1] Let $\Omega$ be a domain in $R^n$ such that $|\Omega| < \infty$ and $u \in W_0^{1,G}(\Omega)$. We assume that there exists $M > 0$ and $t_0 \ge 0$ such that*

$$\int_{\{|u| \le t\}} G(|Du|)\, dx \le Mt \quad \text{for } t \ge t_0.$$

*Then there exists $c = c(n)$ such that*

$$|\{|u| > t\}| \le \frac{Mt}{G_n(ct^{\frac{1}{n'}}/M^{\frac{1}{n}})} \quad \text{for } t \ge t_0,$$

*where $n' = \frac{n}{n-1}$.*

By using the standard mollification, there exist a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ with $\mu_k \rightharpoonup \mu$ in measure, and a sequence of weak solutions $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,G}(\Omega)$ to (5.9). In light of (5.3), $u_m$ and $Du_k$ converge to $u$ and $Du$ in measure, respectively, see [37, Section 5]. As mentioned earlier, we need $L^1$ convergence of $Du_k$ to $Du$ in Definition 5.2.2. To show the existence of a SOLA, we assume that

$$\int^{\infty} \frac{dt}{\Psi_n(t)} < \infty. \tag{5.12}$$

When $G(t) = t^p$, $\Psi_n(t) = t^{\frac{n}{n-1}(p-1)}$ and (5.12) is equivalent to $p > 2 - \frac{1}{n}$. In this way we regard (5.12) to be a necessary assumption to the existence of a SOLA to (5.1).

**Lemma 5.2.4.** *Let $u$ be a weak solution to*

$$\begin{cases} -\mathrm{div} A(x, Du) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.13}$$

*where $f \in \left(W_0^{1,G}(\Omega)\right)^* \cap L^1(\Omega)$. If $||f||_{L^1} \leq M$ for some positive constant, then there exists a constant $c = c(n, \nu, L, M) > 0$ such that*

$$|\{|Du| > t\}| \Psi_n(t) \leq c \quad \text{for all } t > 0.$$

*Proof.* By taking a test function $\psi = T_l(u)$ with $l > 0$ in (5.13) and using (5.2), we find

$$\begin{aligned} \frac{1}{c} \int_{\{|u| \leq l\}} G(|Du|)\, dx &\leq \int_{\{|u| \leq l\}} A(x, Du) Du\, dx \\ &= \int_{\Omega} f \psi\, dx \leq l ||f||_{L^1(\Omega)} \leq Ml, \end{aligned}$$

where $c = c(\nu, L)$. Then according to Lemma 5.2.3, there exists $c(n)$ such that

$$|\{|u| > l\}| \leq \frac{cMl}{G_n(c(n) l^{\frac{1}{n'}} / M^{\frac{1}{n}})}$$

for all $t > 0$. Also, the following inequality holds:

$$|\{G(|Du|)) > s, |u| \leq t\}| \leq \frac{1}{s} \int_{\{G(|Du|) > s, |u| \leq t\}} G(|Du|)\, dx \leq \frac{cMl}{s},$$

for all $s > 0$. Consequently, we have

$$|\{G(|Du|) > s\}| \leq |\{G(|Du|) > s, |u| \leq l\}| + |\{|u| > l\}|$$

$$\leq \frac{cMl}{s} + \frac{cMl}{G_n(cl^{\frac{1}{n'}}/M^{\frac{1}{n}})}.$$

Then taking $l = (\frac{1}{c(n)}M^{\frac{1}{n}}G_n^{-1}(s))^{n'}$, we find

$$|\{G(|Du|) > s\}| \leq c\frac{M^{n'}G_n^{-1}(s)^{n'}}{s}$$

where $c = c(c, \nu, L)$. We then select $s = G(t)$ and recall (5.11) to have

$$|\{|Du| > t\}| \leq c\frac{H_n(t)^{n'}}{G(t)} = \frac{c}{\Psi_n(t)},$$

as required. □

We now prove $L^1$ convergence of $Du_k$, where $u_k$ is a weak solution to (5.9), using Lemma 5.2.4.

**Theorem 5.2.5.** *Let $G$ satisfy* (5.3) *and* (5.12). *If $u_k$ is a weak solution to* (5.9)*, then $Du_k$ converges to $Du$ strongly in $L^1$ up to a subsequence, where $u$ is an approximable solution to* (5.1). *Therefore, any approximable solutions to* (5.1) *are SOLAs.*

*Proof.* Out first step is to show that $\{Du_k\}_{k\geq 1}$ is a Cauchy sequence in $L^1$. To do this, we split domain of the following integral into three parts. For some small $\varepsilon > 0$ and large $M_0 > 0$, we have

$$\begin{aligned}\int_\Omega |Du_k - Du_m|\,dx &= \int_{\{|Du_k - Du_m| \leq \varepsilon\}} |Du_k - Du_m|\,dx \\ &\quad + \int_{\{\varepsilon < |Du_k - Du_m| \leq M_0\}} |Du_k - Du_m|\,dx \\ &\quad + \int_{\{|Du_k - Du_m| > M_0\}} |Du_k - Du_m|\,dx \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.\end{aligned}$$

Clearly, we have

$$\mathcal{I}_1 \leq \varepsilon|\Omega| \quad \text{and} \quad \mathcal{I}_2 \leq M_0|\{|Du_k - Du_m| > \varepsilon\}|.$$

From Lemma 5.2.4, we have

$$|\{|Du_k - Du_m| > t\}| \le |\{|Du_k| + |Du_m| > t\}| \le |\{|Du_k| > \frac{t}{2}\}| + |\{|Du_m| > \frac{t}{2}\}| \le \frac{c}{\Psi_n(t/2)}.$$

Then we have

$$\begin{aligned}\mathcal{I}_3 \le & \int_{M_0}^{\infty} |\{|Du_k - Du_m| > t\}| dt + M_0 |\{|Du_k - Du_m| > M_0\}| \\ \le & c^* \int_{M_0}^{\infty} \frac{1}{\Psi_n(\frac{t}{2})}\, dt + M_0 |\{|Du_k - Du_m| > \varepsilon\}|,\end{aligned}$$

where $c^* = c^*(n, \nu, L, |\mu|(\Omega))$. Choose $M_0$ large enough so that

$$c^* \int_{M_0}^{\infty} \frac{1}{\Psi_n(\frac{t}{2})}\, dt \le \frac{\varepsilon}{3},$$

which is possible by (5.12). Since $Du_m$ is a Cauchy sequence in measure, see the proof of [37, Theorem 3.8], there exists a positive integer $N_0$ such that $k, m > N_0$ implies

$$M_0 |\{|Du_k - Du_m| > \varepsilon\}| \le \frac{\varepsilon}{3}.$$

Consequently

$$\int_\Omega |Du_k - Du_m|\, dx \le \varepsilon(|\Omega| + 1),$$

whenever $k, m \ge N_0$, which proves that $\{Du_k\}$ is a Cauchy sequence in $L^1$.

Let $Z = (Z_1, \cdots, Z_n)$ be a vector-valued function satisfying $Du_k \to Z$ strongly in $L^1$. Then $u_k$ converges to $u$ almost everywhere, as shown in [37]. According to Definition 5.2.1, we are left to show that $u \in W^{1,1}(\Omega)$ and $Du = Z$. By Rellich-Kondrachov compactness theorem, we have $u_k \to u$ strongly in $L^s(\Omega)$ for all $1 \le s < \frac{n}{n-1}$. For any $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega u\, D_i\phi\, dx &= \lim_{k\to\infty} \int_\Omega u_k\, D_i\phi\, dx \\ &= -\lim_{k\to\infty} \int_\Omega D_i u_k\, \phi\, dx\end{aligned}$$

$$= -\int_{\Omega} Z_i \phi \, dx,$$

which implies that $Du = Z$. This completes the proof. □

## 5.3 Comparison estimates

This section is devoted to deriving the desired comparison estimates. To this end, we divide it into three parts. In the first part, we define some $N$-functions related to $G$. Boundary estimates are in the second part, while interior case is in the last part. Throughout this section, we assume that $u \in W_0^{1,G}(\Omega)$ is the weak solution to (5.1) with $\mu \in L^\infty(\Omega)$. In addition, without lose of generality, we assume (5.3), (5.5) and $\gamma_1 < n$, here and in the sequel.

### 5.3.1 Technical estimates

We want to remark that for $G_1(t) = t^p$ and any $\varepsilon > 0$, we have

$$G_1(|\xi_1 - \xi_2|) \le \begin{cases} |V_p(\xi_1) - V_p(\xi_2)|^2 & \text{if} \quad p \ge 2, \\ \varepsilon^{\frac{p-2}{p}} |V_p(\xi_1) - V_p(\xi_2)|^2 + \varepsilon(|\xi_1| + |\xi_2|)^p & \text{if} \quad p < 2, \end{cases} \tag{5.14}$$

where $V_p(\xi) = |\xi|^{(p-2)/2}\xi$ for every $\xi \in \mathbb{R}^n$. As far as we know, there is not a proper analogy of (5.14) for general $N$-functions, which makes it hard to apply the argument given in [24, Lemma 3.4] to the measure data problem (5.1). To overcome this difficulty, we construct some auxiliary functions in the followings. In turn, we obtain $L^1$-comparison estimates for $Du$ in order to prove Theorem 5.1.2.

Let us define an auxiliary function

$$\Psi_g^{-1}(t) := \int_0^t \frac{\mathring{\Psi}_g^{-1}(s)}{s} \, ds, \quad \text{where} \quad \mathring{\Psi}_g^{-1}(t) := \left( \frac{t^{\gamma_2+1}}{g(t)} \right)^{\frac{1}{\gamma_2+2}}. \tag{5.15}$$

Then we have the following properties.

**Lemma 5.3.1.** *The functions given in* (5.15) *are equivalent:*

$$\Psi_g^{-1}(t) \approx \mathring{\Psi}_g^{-1}(t).$$

*Moreover, $\Psi_g^{-1}$ has the inverse function $\Psi_g$ that is an $N$-function.*

*Proof.* Note that

$$\mathring{\Psi}_g^{-1}(t) > 0 \quad \text{and} \quad \frac{d}{dt}\mathring{\Psi}_g^{-1}(t) = \frac{g(t)^{\frac{-1}{\gamma_2+2}}}{(\gamma_2+2)t^{\frac{1}{\gamma_2+2}}}\left(\gamma_2 + 1 - \frac{tg'(t)}{g(t)}\right) > 0 \quad (5.16)$$

for $t > 0$, thus, $\mathring{\Psi}_g^{-1}$ and $\Psi_g^{-1}$ are strictly increasing $C^1$-functions on $[0, \infty)$. Moreover, by a direct calculation, we discover that $\Psi_g^{-1}$ has its continuous second derivative on $(0, \infty)$ as follows:

$$\frac{d^2}{dt^2}\Psi_g^{-1}(t) = \frac{t\frac{d}{dt}\mathring{\Psi}_g^{-1}(t) - \mathring{\Psi}_g^{-1}(t)}{t^2} = \frac{-g(t)^{\frac{-1}{\gamma_2+2}}}{(\gamma_2+2)t^{\frac{\gamma_2+3}{\gamma_2+2}}}\left(1 + \frac{tg'(t)}{g(t)}\right) < 0.$$

It also turns out that $\Psi_g^{-1}$ is concave. We apply the inverse function theorem to find $\Psi_g \in C^2(0, \infty)$ which is convex and satisfies $\Psi_g \circ \Psi_g^{-1}(t) = t$.

On the other hand, the first part of the lemma can be shown by using (5.15), (5.16) and (5.3) as follows:

$$c(\gamma_2)\mathring{\Psi}_g^{-1}(t) \le \mathring{\Psi}_g^{-1}\left(\frac{t}{2}\right) \le \Psi_g^{-1}(t) \quad (5.17)$$

and

$$\Psi_g^{-1}(t) \le \sum_{i=1}^{\infty} \mathring{\Psi}_g^{-1}\left(\frac{t}{2^{i-1}}\right) \le c(\gamma_1, \gamma_2)\mathring{\Psi}_g^{-1}(t).$$

To complete the proof, it remains to show that

$$\lim_{t\to 0} \frac{\Psi_g(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Psi_g(t)}{t} = \infty. \quad (5.18)$$

Indeed, we use (5.3) to see

$$\lim_{t\to 0} \frac{\mathring{\Psi}_g^{-1}(t)}{t} = \lim_{t\to 0}\left(\frac{1}{tg(t)}\right)^{\frac{1}{\gamma_2+2}} \ge \lim_{t\to 0}\left(\frac{1}{\gamma_2 G(t)}\right)^{\frac{1}{\gamma_2+2}} = \infty.$$

It then follows from (5.17) that

$$\lim_{t\to 0} \frac{\Psi_g^{-1}(t)}{t} = \infty,$$

or equivalently

$$\lim_{t\to 0}\frac{\Psi_g(t)}{t}=0.$$

Similarly, we can show $(5.18)_2$. This completes the proof. □

**Remark 5.3.2.** Note that whenever $t \mapsto \mathring{\Psi}_g^{-1}(t)/t$ is decreasing and $t \mapsto \mathring{\Psi}_g^{-1}(t)$ is increasing, there are a few indirect ways to construct concave functions like $\Psi_g^{-1}$, which is neither a differentiable function nor an $N$-function. See for instance [93, Lemma 2.7] and [67, Lemma 2.2].

For $\widetilde{\Psi}_g$, the complementary $N$-function of $\Psi_g$, we now claim that

$$\widetilde{\Psi}_g\Big(G(t)^{\frac{1}{\gamma_2+2}}\Big)\approx t. \tag{5.19}$$

Recalling Lemma 5.3.1 and (5.3), we have

$$\Psi_g^{-1}(t)\approx \mathring{\Psi}_g^{-1}(t)\approx \frac{t}{[G(t)]^{\frac{1}{\gamma_2+2}}} \tag{5.20}$$

and

$$-1<-\frac{\gamma_2}{\gamma_2+2}\le \frac{t[\Psi_g^{-1}]''(t)}{[\Psi_g^{-1}]'(t)}=\frac{-1}{\gamma_2+2}\left(1+\frac{tg'(t)}{g(t)}\right)\le -\frac{\gamma_1}{\gamma_2+2}<0.$$

This estimate implies

$$0<\frac{2}{\gamma_2+2}\le\frac{t[\Psi_g^{-1}]'(t)}{\Psi_g^{-1}(t)}\le\frac{\gamma_2-\gamma_1+2}{\gamma_2+2}<1,$$

and so we obtain

$$1<\frac{\gamma_2+2}{\gamma_2-\gamma_1+2}\le\frac{t\Psi_g'(t)}{\Psi_g(t)}\le\frac{\gamma_2+2}{2} \tag{5.21}$$

and

$$1<\frac{\gamma_2+2}{\gamma_2}\le\frac{t\widetilde{\Psi}_g'(t)}{\widetilde{\Psi}_g(t)}\le\frac{\gamma_2+2}{\gamma_1}.$$

Here, we have used (2.4) and the fact that

$$[\Psi_g^{-1}]'\circ\Psi_g(t)=1/\Psi_g'(t).$$

It then follows from (5.20) that

$$t \approx \Psi_g\left(\frac{t}{[G(t)]^{\frac{1}{\gamma_2+2}}}\right) \approx \frac{t}{[G(t)]^{\frac{1}{\gamma_2+2}}}[\widetilde{\Psi}'_g]^{-1}\left(\frac{t}{[G(t)]^{\frac{1}{\gamma_2+2}}}\right).$$

This estimate can be written as

$$\widetilde{\Psi}'_g\left([G(t)]^{\frac{1}{\gamma_2+2}}\right) \approx \frac{t}{[G(t)]^{\frac{1}{\gamma_2+2}}}.$$

At this stage the claim (5.19) is a direct consequence of (5.21).

We further note that from (5.21) there exists $\gamma = \gamma(\gamma_1, \gamma_2) > 1$ such that for any $\varepsilon \in (0, 1]$ and $s, t \geq 0$

$$st \leq \varepsilon\Psi(s) + \varepsilon^{-\gamma}\widetilde{\Psi}(t). \tag{5.22}$$

**Remark 5.3.3.** We define another auxiliary function

$$\mathring{G}(t) = \int_0^t \frac{[G(s)]^{\frac{1}{\gamma_1}}}{s}\,ds \approx [G(t)]^{\frac{1}{\gamma_1}} \tag{5.23}$$

for every $t \geq 0$. As in the proof of Lemma 5.3.1, we can show that for $t > 0$, $t \mapsto [G(t)]^{1/\gamma_1}/t$ has non-negative derivative, and so $\mathring{G}$ is convex.

### 5.3.2 Boundary comparison estimates

We recall first our assumption that $(\Omega, A(\cdot))$ is $(\delta, R)$-vanishing. Take $x_0 \in \Omega$ and $r \in (0, R/5]$ such that

$$B_{5r}^+ \subset \Omega_{5r} \subset B_{5r} \cap \{x_n > -10\delta r\}, \tag{5.24}$$

and consider the homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x, Dw)) = 0 & \text{in } \Omega_{5r}, \\ w = u & \text{on } \partial\Omega_{5r}. \end{cases} \tag{5.25}$$

One can find a higher integrability result for (5.25) in [44, Theorem 9], which we state as follows in view of [63, Remark 6.12].

**Lemma 5.3.4.** *Let $w \in W^{1,G}(B_{5r})$ be the weak solution to* (5.25)*. Then for every $q \geq 1$, there exists $\sigma = \sigma(\boldsymbol{data}) \in (0,1)$ such that*

$$\left( \fint_{\Omega_{4r}} G(|Dw|)^{1+\sigma}\, dx \right)^{\frac{1}{1+\sigma}} \leq c \left( \fint_{\Omega_{5r}} G(|Dw|)^{\frac{1}{q}}\, dx \right)^{q},$$

*where $c = c(\boldsymbol{data}, q)$.*

We can see that $t \mapsto \big[G(t)\big]^{1/2\gamma_2}/t$ is decreasing, by differentiating it. Similarly to Lemma 5.3.1, the map $t \mapsto \big[G(t)\big]^{1/2\gamma_2}$ is comparable to some concave function. Therefore, in light of Lemma 5.3.4 with $q = 2\gamma_2$ and Jensen's inequality, we have

$$\left( \fint_{\Omega_{4r}} G(|Dw|)^{1+\sigma}\, dx \right)^{\frac{1}{1+\sigma}} \leq c\, G\left( \fint_{\Omega_{5r}} |Dw|\, dx \right), \tag{5.26}$$

where $c$ depends only on $\mathtt{data}$.

We now move on to a $L^1$-comparison estimate between (5.1) and (5.25). The functions $\Psi_g$, $\mathring{\Psi}_g$ and $\mathring{G}$ investigated in Subsection 5.3.1 are useful in the following lemma.

**Lemma 5.3.5.** *Let $w \in W^{1,G}(\Omega_{5r})$ be the weak solution to* (5.25)*. For any $\varepsilon \in (0,1]$, there exists $\delta = \delta(\boldsymbol{data}, \varepsilon) > 0$ such that if*

$$\fint_{\Omega_{5r}} |Du|\, dx + \frac{1}{\delta} \fint_{\Omega_{5r}} g^{-1}(M_1(\mu))\, dx \leq \lambda \tag{5.27}$$

*for some $\lambda > 0$, then*

$$\fint_{\Omega_{5r}} |Du - Dw|\, dx \leq \varepsilon\lambda.$$

*Proof.* We start with scaling and normalization arguments. For some constants $\kappa \in (0,1]$ and $\theta \geq 1$ to be chosen later, set

$$M = \kappa \fint_{\Omega_{5r}} |Du|\, dx + \frac{1}{\kappa^{\theta}} g^{-1}\left( \frac{|\mu|(\Omega_{5r})}{r^{n-1}} \right) \geq 0.$$

If $|\mu|(\Omega_{5r}) = 0$, there is nothing to prove. So, without loss of generality, assume $|\mu|(\Omega_{5r}) > 0$, which implies $M > 0$. We now set

$$\begin{aligned}
&\hat{A}(y,\xi) = \frac{A(x_0+ry, M\xi)}{g(M)}, \quad \hat{g}(t) = \frac{g(Mt)}{g(M)}, \quad \hat{\Psi}(t) = \Psi_{\hat{g}}(t),\\
&\hat{G}(t) = \frac{G(Mt)}{G(M)}, \quad \hat{u}(y) = \frac{u(x_0+ry)}{Mr}, \quad \hat{w}(y) = \frac{w(x_0+ry)}{Mr}\\
&\text{and} \quad \hat{\mu}(y) = \frac{r\mu(x_0+ry)}{g(M)}
\end{aligned} \tag{5.28}$$

for $y \in \hat{\Omega}_5 := \{y \in \mathbb{R}^n : x_0 + ry \in \Omega_{5r}\}$. It is readily seen that

$$|\hat{\mu}|(\hat{\Omega}_5) \le \kappa^{(\gamma_1-1)\theta}, \qquad \fint_{\hat{\Omega}_5} |D\hat{u}|\,dx \le \kappa^{-1}, \qquad \hat{G}(1) = 1 \tag{5.29}$$

and

$$\langle \partial\hat{A}(y,\xi)\eta, \eta\rangle \ge \nu\frac{g(M|\xi|)}{g(M)|\xi|}|\eta|^2 = \nu\frac{\hat{g}(|\xi|)}{|\xi|}|\eta|^2. \tag{5.30}$$

Accordingly, once we have

$$\int_{\hat{\Omega}_5} |D\hat{u} - D\hat{w}|\,dx \le c \tag{5.31}$$

for some $c \ge 1$, then

$$\fint_{\Omega_{5r}} |Du - Dv|\,dx \le c\,\kappa\bigg(\fint_{\Omega_{5r}} |Du|\,dx + \frac{1}{\kappa^{1+\theta}}\fint_{\Omega_{5r}} g^{-1}(M_1(\mu))\,dx\bigg),$$

where we also have used (5.8). Taking $\kappa = \varepsilon/c$ and $\delta = \kappa^{1+\theta}$, we obtain the desired estimate. Therefore, it is enough to verify (5.31).

From now on, for simplicity of notation, we omit $\hat{}$ over characters. As mentioned at the beginning of Subsection 5.3.1, there is no analogy of (5.14) for general $N$-function. Instead, for any $\xi_1, \xi_2 \in \mathbb{R}^n$, Lemma 5.3.1 and (5.22) yield

$$|\xi_1 - \xi_2| \le c\left(\frac{g(|\xi_1| + |\xi_2|)}{|\xi_1| + |\xi_2|}|\xi_1 - \xi_2|^2\right)^{\frac{1}{\gamma_2+2}}\Psi^{-1}(|\xi_1| + |\xi_2|)$$

$$\leq c\left(\frac{g(|\xi_1|+|\xi_2|)}{|\xi_1|+|\xi_2|}|\xi_1-\xi_2|^2\right)^{\frac{1}{\gamma_2+2}}\left[\Psi^{-1}(|\xi_1-\xi_2|)+\Psi^{-1}(|\xi_1|)\right]$$
$$\leq \frac{1}{2}|\xi_1-\xi_2|+\kappa|\xi_1|+c\,\kappa^{-\gamma}\widetilde{\Psi}\left(\left(\frac{g(|\xi_1|+|\xi_2|)}{|\xi_1|+|\xi_2|}|\xi_1-\xi_2|^2\right)^{\frac{1}{\gamma_2+2}}\right),$$

where $\gamma > 1$ is the constant determined in (5.22). Here, we also have used the concavity of $\Psi^{-1}$. Taking (5.19) and (5.23) into account, we discover

$$\widetilde{\Psi}\left(t^{\frac{1}{\gamma_2+2}}\right) \approx G^{-1}(t),$$

and we find

$$|\xi_1-\xi_2| \leq 2\kappa|\xi_1|+c\,\kappa^{-\gamma}\mathring{G}^{-1}\left(\left[\frac{g(|\xi_1|+|\xi_2|)}{|\xi_1|+|\xi_2|}|\xi_1-\xi_2|^2\right]^{\frac{1}{\gamma_1}}\right). \tag{5.32}$$

Recall the truncation operators given in Section 2.1, and denote

$$C_s := \{x \in \Omega_5 : |u(x)-w(x)| \leq s\}$$

and

$$D_s := \{x \in \Omega_5 : s < |u(x)-w(x)| \leq s+1\}$$

for every $s \in \mathbb{N}$. Testing $T_s(u-w)$ and $\mathfrak{T}_s(u-w)$ to both (5.1) and (5.25) and using (5.30), we have

$$\int_{C_s}\frac{g(|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2\,dx \leq c\,s|\mu|(\Omega_5)$$

and

$$\int_{D_s}\frac{g(|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2\,dx \leq c\,|\mu|(\Omega_5).$$

Then applying Hölder's inequality, we find

$$\int_{C_s}\left(\frac{g(|Du|+|Dw|)}{|Du|+|Dw|}|Du-Dw|^2\right)^{\frac{1}{\gamma_1}}dx \leq c\,s^{\frac{1}{\gamma_1}}|\mu|(\Omega_5)^{\frac{1}{\gamma_1}}$$

and

$$\int_{D_s} \left( \frac{g(|Du|+|Dw|)}{|Du|+|Dw|} |Du-Dw|^2 \right)^{\frac{1}{\gamma_1}} dx$$
$$\leq c\,|D_s|^{1-\frac{1}{\gamma_1}} |\mu|(\Omega_5)^{\frac{1}{\gamma_1}} \leq c \left( \int_{D_s} \left( \frac{|u-w|}{s} \right)^{\frac{n}{n-1}} dx \right)^{1-\frac{1}{\gamma_1}} |\mu|(\Omega_5)^{\frac{1}{\gamma_1}}.$$

It then follows from Sobolev-Poincaré's inequality that for any $s_0 \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\Omega_5} \left( \frac{g(|Du|+|Dw|)}{|Du|+|Dw|} |Du-Dw|^2 \right)^{\frac{1}{\gamma_1}} dx \\
&\leq \int_{C_{s_0}} \left( \frac{g(|Du|+|Dw|)}{|Du|+|Dw|} |Du-Dw|^2 \right)^{\frac{1}{\gamma_1}} dx \\
&\quad + \sum_{s \geq s_0} \int_{D_s} \left( \frac{g(|Du|+|Dw|)}{|Du|+|Dw|} |Du-Dw|^2 \right)^{\frac{1}{\gamma_1}} dx \\
&\leq c\, s_0^{\frac{1}{\gamma_1}} |\mu|(\Omega_5)^{\frac{1}{\gamma_1}} + c \sum_{s \geq s_0} \left( \frac{|\mu|(\Omega_5)}{s^{\frac{n(\gamma_1-1)}{(n-1)}}} \right)^{\frac{1}{\gamma_1}} \left( \int_{D_s} |u-w|^{\frac{n}{n-1}} dx \right)^{1-\frac{1}{\gamma_1}} \\
&\leq c\, s_0^{\frac{1}{\gamma_1}} |\mu|(\Omega_5)^{\frac{1}{\gamma_1}} + c \left( \sum_{s \geq s_0} \frac{|\mu|(\Omega_5)}{s^{\frac{n(\gamma_1-1)}{(n-1)}}} \right)^{\frac{1}{\gamma_1}} \left( \int_{\Omega_5} |u-w|^{\frac{n}{n-1}} dx \right)^{1-\frac{1}{\gamma_1}} \\
&\leq c\, s_0^{\frac{1}{\gamma_1}} |\mu|(\Omega_5)^{\frac{1}{\gamma_1}} + c(s_0) |\mu|(\Omega_5)^{\frac{1}{\gamma_1}} \left( \int_{\Omega_5} |Du-Dw|\, dx \right)^{\frac{n(\gamma_1-1)}{(n-1)\gamma_1}}. \qquad (5.33)
\end{aligned}$$

Note that $c(s_0) < \infty$ and $c(s_0) \to 0$ as $s_0$ goes to $\infty$, since $\gamma_1 > 2 - \frac{1}{n}$.

By (5.32), we have

$$\begin{aligned}
\int_{\Omega_5} |Du-Dw|\, dx &\leq 2\kappa \int_{\Omega_5} |Du|\, dx \\
&\quad + c\kappa^{-\gamma} \int_{\Omega_5} \mathring{G}^{-1} \left( \left[ \frac{g(|Du|+|Dw|)}{|Du|+|Dw|} |Du-Dw|^2 \right]^{\frac{1}{\gamma_1}} \right) dx \\
&\leq I + II. \qquad (5.34)
\end{aligned}$$

It is readily checked that $I \leq 2$ in (5.29).

We now recall Remark 5.3.3 to see that $\mathring{G}^{-1}$ is concave. Then Jensen's inequality and (5.33) yield

$$\begin{aligned}\kappa^\gamma II &\le c\,\mathring{G}^{-1}\left(\fint_{\Omega_5}\left[\frac{g(|\xi_1|+|\xi_2|)}{|\xi_1|+|\xi_2|}|\xi_1-\xi_2|^2\right]^{\frac{1}{\gamma_1}}dx\right)\\ &\le c\,G^{-1}\big(s_0|\mu|(\Omega_5)\big)+c\,G^{-1}\left(c(s_0)^{\gamma_1}|\mu|(\Omega_5)\left[\fint_{\Omega_5}|Du-Dw|\,dx\right]^{\frac{n(\gamma_1-1)}{n-1}}\right).\end{aligned}$$

Note that $\frac{n(\gamma_1-1)}{(n-1)\gamma_2}\le\frac{n(\gamma_1-1)}{(n-1)\gamma_1}<1$, since $\gamma_1<n$. In light of (5.3) and $(5.29)_3$, we discover that for any $t\ge 0$, $\alpha\in(0,1)$ and any small $\bar\kappa>0$,

$$\begin{aligned}G^{-1}\left(\alpha t^{\frac{n(\gamma_1-1)}{n-1}}\right)&\le G^{-1}(\alpha)\max\left\{t^{\frac{n(\gamma_1-1)}{(n-1)\gamma_1}},t^{\frac{n(\gamma_1-1)}{(n-1)\gamma_2}}\right\}\\ &\le c(n,\gamma_1,\gamma_2)\bar\kappa^{-q_1}\alpha^{q_2}+\bar\kappa t,\end{aligned}$$

where $q_1$ and $q_2$ depend only on `data`, which are introduced for simplicity of notation. It then follows from (5.29) that

$$\kappa^\gamma II\le c\,G^{-1}(s_0)\kappa^{\frac{(\gamma_1-1)\theta}{\gamma_2}}+c(s_0)\kappa^{(\gamma_1-1)\theta q_2}\bar\kappa^{-q_1}+\bar\kappa\fint_{\Omega_5}|Du-Dw|\,dx, \quad (5.35)$$

where $c(s_0)\to 0$ as $s_0$ goes to $\infty$.

Combining (5.34) and (5.35), we have

$$\begin{aligned}\fint_{\Omega_5}|Du-Dw|\,dx &\le 2+G^{-1}(s_0)\kappa^{\frac{(\gamma_1-1)\theta}{\gamma_2}-\gamma}\\ &\quad +c(s_0)\kappa^{(\gamma_1-1)\theta q_2-\gamma}\bar\kappa^{-q_1}+c_*\kappa^{-\gamma}\bar\kappa\fint_{\Omega_5}|Du-Dw|\,dx,\end{aligned}$$

where we temporarily fix a constant $c_*$ depending only on `data`. We now take $\bar\kappa=\frac{\kappa^\gamma}{2c_*}$, and then take $\theta$ large enough to satisfy

$$(\gamma_1-1)\theta-\gamma\gamma_2>0 \quad\text{and}\quad (\gamma_1-1)\theta q_2-\gamma-\gamma q_1>0.$$

Consequently, we have (5.31) and this completes the proof. □

**Remark 5.3.6.** We note that a suitable modification of the proof of Lemma

5.3.5 gives

$$\fint_{\Omega} |Du| \, dx \le c \fint_{\Omega} g^{-1}(M_1(\mu)) \, dx, \tag{5.36}$$

where $c$ depends only on $\texttt{data}$. We will use this estimate later in the proof of Theorem 5.1.2.

Here, we give a sketch of the proof of (5.36). We denote $d = \operatorname{diam}(\Omega)$ and take any $\tilde{x} \in \Omega$ to see that $\Omega \subset B_d(\tilde{x})$. We now set

$$M = \kappa \fint_{\Omega} |Du| \, dx + \frac{1}{\kappa^{\theta}} g^{-1}\left( \frac{|\mu|(\Omega)}{d^{n-1}} \right) \ge 0$$

for some constants $\kappa \in (0,1]$ and $\theta \ge 1$. We use a scaling similar to (5.28) with $\tilde{x}, d$ replacing $x_0, r$, respectively. That is, for $y \in \hat{\Omega} := \{y \in \mathbb{R}^n : \tilde{x} + dy \in \Omega\}$,

$$\hat{u}(y) = \frac{u(\tilde{x}+dy)}{Md}, \quad \hat{\mu}(y) = \frac{d\mu(\tilde{x}+dy)}{g(M)}, \quad \hat{A}(y,\xi) = \frac{A(\tilde{x}+dy, M\xi)}{g(M)}$$

and so on. Testing $T_s(u), \mathfrak{T}_s(u) \in W_0^{1,G}(\Omega)$ to (5.1) as in the proof of Lemma 5.3.5, we discover

$$\fint_{\Omega} |Du| \, dx \le c\kappa \fint_{\Omega} |Du| \, dx + \frac{c}{\kappa^{\theta}} \fint_{\Omega} g^{-1}(M_1(\mu)) \, dx,$$

where $c$ depends only on $\texttt{data}$. Taking $\kappa > 0$ small enough, we obtain the desired estimate (5.36).

We now consider the weak solution $\bar{w} \in W^{1,G}(\Omega_{5r})$ to

$$\begin{cases} -\operatorname{div}(\bar{A}(D\bar{w})) = 0 & \text{in } \Omega_{4r}, \\ \bar{w} = w & \text{on } \partial\Omega_{4r}, \end{cases} \tag{5.37}$$

where $\bar{A} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\bar{A}(\xi) = \frac{1}{|B_{4r}^+|} \int_{B_{4r}^+} A(x,\xi) \, dx.$$

By (5.2), we have the following ellipticity and growth of $\bar{A}$:

$$\begin{cases} |\bar{A}(\xi)| + |\partial \bar{A}(\xi)||\xi| \leq Lg(|\xi|), \\ \nu \frac{g(|\xi|)}{|\xi|}|\zeta|^2 \leq \langle \partial \bar{A}(\zeta)\zeta, \zeta \rangle \end{cases} \tag{5.38}$$

for every $\xi, \zeta \in \mathbb{R}^n$, where $\nu, L$ are the constant given in (5.2). Apparently, there holds that for any $\varphi \in w + W_0^{1,G}(\Omega_{4r})$

$$\int_{\Omega_{4r}} G(|D\bar{w}|)\,dx \leq c \int_{\Omega_{4r}} G(|D\varphi|)\,dx. \tag{5.39}$$

In other words, $\bar{w}$ is a quasi-minimizer of the above functional.

**Lemma 5.3.7.** *Let $\bar{w} \in W^{1,G}(\Omega_{4r})$ is the weak solution to* (5.37)*. For any $\varepsilon \in (0,1]$, there exists $\delta = \delta(\boldsymbol{data}, \varepsilon) > 0$ such that if* (5.27) *holds for some $\lambda > 0$, then*

$$G^{-1}\left( \fint_{\Omega_{4r}} |V(Dw) - V(D\bar{w})|^2\,dx \right) \leq \varepsilon\lambda \tag{5.40}$$

*and*

$$G^{-1}\left( \fint_{\Omega_{4r}} G(|D\bar{w}|)\,dx \right) \leq c\lambda.$$

*Proof.* We give a brief sketch of the proof, since it is similar as in [32, Lemma 5.10]. By (5.38), we have

$$\begin{aligned} \frac{1}{c}\fint_{\Omega_{4r}} |V(Dw) - V(D\bar{w})|^2\,dx &\leq \fint_{\Omega_{4r}} \langle \bar{A}(Dw) - \bar{A}(D\bar{w}), Dw - D\bar{w} \rangle\,dx \\ &= \fint_{\Omega_{4r}} \langle \bar{A}(Dw) - A(x, Dw), Dw - D\bar{w} \rangle\,dx \\ &\leq \fint_{\Omega_{4r}} \theta(B_{4r}^+)(x) g(|Dw|)|Dw - D\bar{w}|\,dx \\ &=: I. \end{aligned}$$

Young's inequality for any $\bar{\varepsilon} > 0$ and Hölder's inequality yield

$$\begin{aligned} I &\leq c(\bar{\varepsilon}) \fint_{\Omega_{4r}} \theta(B_{4r}^+)(x) G(|Dw|)\,dx + \bar{\varepsilon} \fint_{\Omega_{4r}} \theta(B_{4r}^+)(x) G(|D\bar{w}|)\,dx \\ &\leq c(\bar{\varepsilon}) \left( \fint_{\Omega_{4r}} \left(\theta(B_{4r}^+)(x)\right)^{\frac{1+\sigma}{\sigma}}\,dx \right)^{\frac{\sigma}{1+\sigma}} \left( \fint_{\Omega_{4r}} G(|Dw|)^{1+\sigma}\,dx \right)^{\frac{1}{1+\sigma}} \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\bar{\varepsilon} L \fint_{\Omega_{4r}} G(|D\bar{w}|)\,dx \\
&\overset{(5.39)}{\leq} c(\bar{\varepsilon}) \left( L^{\frac{1}{\sigma}} \fint_{B_{4r}^{+}} \theta(B_{4r}^{+})(x)\,dx + L^{\frac{1+\sigma}{\sigma}} \delta \right)^{\frac{\sigma}{1+\sigma}} \left( \fint_{\Omega_{4r}} G(|Dw|)^{1+\sigma}\,dx \right)^{\frac{1}{1+\sigma}} \\
&\quad + c\bar{\varepsilon} \fint_{\Omega_{4r}} G(|Dw|)\,dx \\
&\leq \left( c(\bar{\varepsilon}) \delta^{\frac{\sigma}{1+\sigma}} + c\bar{\varepsilon} \right) \left( \fint_{\Omega_{4r}} G(|Dw|)^{1+\sigma}\,dx \right)^{\frac{1}{1+\sigma}} \\
&\overset{(5.26)}{\leq} \left( c(\bar{\varepsilon}) \delta^{\frac{\sigma}{1+\sigma}} + c\bar{\varepsilon} \right) G\left( \fint_{\Omega_{5r}} |Dw|\,dx \right),
\end{aligned}
$$

where $\sigma$ is the constant given in Lemma 5.3.4. Taking $\bar{\varepsilon}, \delta > 0$ small enough and employing Lemma 5.3.5, we have (5.40).

Moreover, we use (5.40), (5.26) and Lemma 5.3.5 to obtain

$$
\begin{aligned}
&G^{-1}\left( \fint_{\Omega_{4r}} G(D\bar{w})\,dx \right) \\
&\leq c\, G^{-1}\left( \fint_{\Omega_{4r}} |V(Dw) - V(D\bar{w})|^2\,dx \right) + c \fint_{\Omega_{5r}} |Dw|\,dx \\
&\leq c\lambda + c \fint_{\Omega_{5r}} |Du - Dw|\,dx + c \fint_{\Omega_{5r}} |Du|\,dx \leq c\lambda.
\end{aligned}
$$

This completes the proof. □

In the next lemma, we construct a function defined on a flat boundary, which is close enough to $\bar{w}$. Note that it is also possible to construct such a function by using compactness argument, see for instance [32, Lemma 5.8]. Here, we present a simple proof of the lemma by modifying the proof given in [73, Lemma 2.5].

**Lemma 5.3.8.** *For any $\varepsilon \in (0,1]$, there exists $\delta = \delta(\texttt{data}, \varepsilon) > 0$ such that the following statement holds: Let $\bar{w} \in W^{1,G}(\Omega_{4r})$ be the weak solution to* (5.37) *and $v \in W^{1,G}(B_{2r}^{+})$ be the weak solution to*

$$
\begin{cases} -\mathrm{div}(\bar{A}(Dv)) = 0 & \text{in } B_{2r}^{+}, \\ v = \eta\bar{w} & \text{on } \partial B_{2r}^{+}, \end{cases} \tag{5.41}
$$

*where* $\eta = \eta(x_n) \in C^\infty(\mathbb{R})$ *satisfies*

$$0 \le \eta \le 1, \quad \eta \equiv 1 \ \text{on } [\delta r, 2r], \quad \eta \equiv 0 \ \text{on } (-\infty, 0] \quad \text{and} \quad |D\eta| \le \frac{2}{\delta r}.$$

*We extend* $v$ *to* $\Omega_{2r}$ *by setting* $v = 0$ *in* $\Omega_{2r} \setminus B_{2r}^+$. *Then we have*

$$\fint_{\Omega_{2r}} |V(D\bar{w}) - V(Dv)|^2 \, dx \le \varepsilon \fint_{\Omega_{4r}} G(|D\bar{w}|) \, dx \tag{5.42}$$

*and*

$$\fint_{\Omega_{2r}} G(|Dv|) \, dx \le c \fint_{\Omega_{4r}} G(|D\bar{w}|) \, dx. \tag{5.43}$$

*Proof.* We apply Lemma 5.3.4 to $\bar{w}$ and Hölder's inequality to estimate

$$\begin{aligned} &\frac{1}{|B_{3r}|} \int_{\Omega_{3r} \cap \{x_n \le \delta r\}} G(|D\bar{w}|) \, dx \\ &\le c \left( \frac{1}{|B_{3r}|} \int_{\Omega_{3r} \cap \{x_n \le \delta r\}} G(|D\bar{w}|)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \left( \frac{|\Omega_{3r} \cap \{x_n \le \delta r\}|}{|B_{3r}|} \right)^{\frac{\sigma}{1+\sigma}} \\ &\overset{(5.24)}{\le} c \delta^{\frac{\sigma}{1+\sigma}} \left( \fint_{\Omega_{3r}} G(|D\bar{w}|)^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \\ &\le c \delta^{\frac{\sigma}{1+\sigma}} \fint_{\Omega_{4r}} G(|D\bar{w}|) \, dx, \end{aligned} \tag{5.44}$$

where $\sigma = \sigma(n, \nu, L, p)$ is the constant given in Lemma 5.3.4.

Moreover, we discover from the fact that $\{|x'| < 2r\} \times \{|x_n| < 2r\} \subset B_{3r}$ and $\bar{w} = 0$ in $B'_{2r} \times \{x_n \le -10\delta r\}$ that

$$\begin{aligned} &\int_{\Omega_{2r} \cap \{x_n \le \delta r\}} G(|\bar{w}||D\eta|) \, dx \\ &\le c \int_{\Omega_{2r} \cap \{x_n \le \delta r\}} G\left( \frac{1}{\delta r} \left| \int_{-10\delta r}^{x_n} \frac{\partial}{\partial y} \bar{w}(x', y) \, dy \right| \right) dx \\ &\le c \int_{\Omega_{2r} \cap \{x_n \le \delta r\}} G\left( \fint_{-10\delta r}^{\delta r} |D\bar{w}(x', y)| \, dy \right) dx \\ &\le \frac{c}{\delta r} \int_{\Omega_{2r} \cap \{x_n \le \delta r\}} \int_{-10\delta r}^{\delta r} G(|D\bar{w}(x', y)|) \, dy \, dx \end{aligned}$$

$$\leq c\int_{\Omega_{3r}\cap\{x_n\leq\delta r\}} G(|D\bar{w}|)\,dx. \tag{5.45}$$

Testing $v-\eta\bar{w}\in W_0^{1,G}(B_{2r}^+)$ to (5.41) and using (5.45), we obtain (5.43) as follows:

$$\begin{aligned}\int_{B_{2r}^+} G(|Dv|)\,dx &\leq c\int_{B_{2r}^+} G(|D(\eta\bar{w})|)\,dx\\ &\leq c\int_{B_{2r}^+\cap\{x_n\leq\delta r\}} G(|\bar{w}||D\eta|)\,dx + c\int_{B_{2r}^+} G(|D\bar{w}|)\,dx\\ &\leq c\int_{\Omega_{3r}} G(|D\bar{w}|)\,dx. \end{aligned}\tag{5.46}$$

To prove (5.42), we now test $v-\eta\bar{w}\in W_0^{1,G}(B_{2r}^+)$ for (5.37) and (5.41) to discover

$$\int_{B_{2r}^+} (\bar{A}(Dv)-\bar{A}(D\bar{w}))\cdot D(v-\eta\bar{w})\,dx = 0.$$

It then follows from Young's inequality that

$$\begin{aligned}&\fint_{B_{2r}^+} (\bar{A}(Dv)-\bar{A}(D\bar{w}))\cdot D(v-\bar{w})\,dx\\ &= \fint_{B_{2r}^+} (\bar{A}(Dv)-\bar{A}(D\bar{w}))\cdot D(\eta\bar{w}-\bar{w})\,dx\\ &= \frac{2}{|B_{2r}|}\int_{B_{2r}^+\cap\{x_n\leq\delta r\}} (\bar{A}(Dv)-\bar{A}(D\bar{w}))\cdot(\bar{w}D\eta+(\eta-1)D\bar{w})\,dx\\ &\leq \bar{\varepsilon}\fint_{B_{2r}^+} G(|Dv|)\,dx + \bar{\varepsilon}\fint_{B_{2r}^+} G(|D\bar{w}|)\,dx\\ &\quad + \frac{c(\bar{\varepsilon})}{|B_{2r}|}\int_{B_{2r}^+\cap\{x_n\leq\delta r\}} G(|\bar{w}||D\eta|+|D\bar{w}|)\,dx\end{aligned}$$

for any $\bar{\varepsilon}>0$. Using (2.18), (5.43), (5.44), (5.45) and (5.46), we discover

$$\begin{aligned}\fint_{\Omega_{2r}} |V(Dv)-V(D\bar{w})|^2\,dx &= \frac{1}{|\Omega_{2r}|}\int_{\Omega_{2r}\setminus B_{2r}^+} |V(D\bar{w})|^2\,dx\\ &\quad + \frac{1}{|\Omega_{2r}|}\int_{B_{2r}^+} |V(Dv)-V(D\bar{w})|^2\,dx\end{aligned}$$

$$\leq \left(c\bar{\varepsilon} + c(\bar{\varepsilon})\delta^{\frac{\sigma}{1+\sigma}}\right) \fint_{\Omega_{4r}} G(|D\bar{w}|)\, dx.$$

Taking $\bar{\varepsilon}, \delta$ small enough, we finally obtain the desired estimates. □

**Remark 5.3.9.** According to Lemma 5.3.8 and Lemma 5.3.7, we see that for any $\varepsilon > 0$ there exists a small $\delta = \delta(\texttt{data}, \varepsilon) > 0$ such that

$$\fint_{\Omega_{2r}} |V(D\bar{w}) - V(Dv)|^2\, dx \leq \varepsilon \fint_{\Omega_{4r}} G(|D\bar{w}|)\, dx \leq c\varepsilon G(\lambda),$$

whenever (5.27) holds for some $\lambda > 0$.

Therefore, we combine this estimate and Lemma 2.3.2 to obtain

$$\begin{aligned} \sup_{B_r^+} |Dv| &\leq G^{-1}\left(\fint_{\Omega_{2r}} |V(D\bar{w}) - V(Dv)|^2\, dx\right) + G^{-1}\left(\fint_{\Omega_{2r}} G(|D\bar{w}|)\, dx\right) \\ &\leq c\lambda. \end{aligned} \tag{5.47}$$

### 5.3.3 Interior comparison estimates

In this subsection, we study the interior counterparts of the comparison estimates given in Section 5.3.2. In what follows, we state lemmas without their proofs, as they are similar to those in Section 5.3.2.

Take any $B_{5r} \subset \Omega$. Let $w \in W^{1,G}(B_{5r})$ be the weak solution to

$$\begin{cases} -\mathrm{div}(A(x, Dw)) = 0 & \text{in } B_{5r}, \\ w = u & \text{on } \partial B_{5r}, \end{cases} \tag{5.48}$$

and $v \in W^{1,G}(\Omega_{2r})$ be the weak solution to

$$\begin{cases} -\operatorname{div}(\bar{A}(Dv)) = 0 & \text{in } B_{2r}, \\ v = w & \text{on } \partial B_{2r}, \end{cases} \tag{5.49}$$

where the vector field $\bar{A} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\bar{A}(\xi) = \frac{1}{|B_{2r}|} \int_{B_{2r}} A(x, \xi)\, dx \quad \xi \in \mathbb{R}^n.$$

Then the following lemma is an interior version of Lemma 5.3.5.

**Lemma 5.3.10.** *For any $\varepsilon \in (0,1]$, there exists $\delta = \delta(\texttt{data}, \varepsilon) > 0$ such that if*

$$\fint_{\Omega_{5r}} |Du|\,dx + \frac{1}{\delta}\fint_{\Omega_{5r}} M_1(\mu)\,dx \leq \lambda$$

*for some $\lambda > 0$, then*

$$\fint_{\Omega_{5r}} |Du - Dw|\,dx \leq \varepsilon\lambda.$$

For the interior case, the following comparison estimate and interior Lipschitz regularity are enough to prove Theorem 5.1.2 (cf. Lemma 5.3.8, Lemma 2.3.2 and Remark 5.3.9).

**Lemma 5.3.11.** *Under the same assumptions as in Lemma 5.3.10, we have*

$$G^{-1}\left(\fint_{\Omega_{2r}} |V(Dw) - V(Dv)|^2\,dx\right) \leq \varepsilon\lambda$$

*and*

$$\sup_{B_r} |Dv| \leq c_l\lambda, \tag{5.50}$$

*where $c_l$ is the constant given in Lemma 2.3.2.*

## 5.4 Proof of the main theorem

To prove regularity results for a SOLA given in Section 5.2, we consider a sequence of measurable functions $\{\mu_k\}_{k\in\mathbb{N}} \in L^\infty(\Omega)$ and a sequence of weak solutions $\{u_k\}_{k\geq 1}$ to

$$\begin{cases} -\mathrm{div}(A(x, Du_k)) = \mu_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.51}$$

where $\mu_k \to \mu$ weakly in measure and $u_k \to u$ in $W_0^{1,1}(\Omega)$. The convergence of $\{u_k\}$ implies that for every $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\int_{\Omega} |Du - Du_k|\,dx \leq \varepsilon\lambda_0 \tag{5.52}$$

for every $k \geq k_0$, where $\lambda_0 > 0$ will be determined later in (5.54).

We recall here a property of weak convergence in measure from the classical measure theory. Extending $\{\mu_k\}_{k\in\mathbb{N}} \in L^\infty(\Omega)$ by 0 in $\mathbb{R}^n \setminus \Omega$, the sequence can be regarded as a sequence of bounded Radon measures converging to $\mu$ weakly in measure. Therefore, in light of [56, Theorem 1.3.1], we have

$$\limsup_{k\to\infty} |\mu_k|(\bar{\mathcal{O}}) \le |\mu|(\bar{\mathcal{O}}),$$

for every measurable subset $\mathcal{O} \subset \Omega$. In other words, for each $\mathcal{O} \subset \Omega$, we can take large enough $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ the following inequality holds:

$$|\mu_k|(\bar{\mathcal{O}}) \le 2|\mu|(\bar{\mathcal{O}}). \tag{5.53}$$

Lemmas in Section 5.3 hold for $u_k \in W^{1,G}(\Omega)$, the weak solutions to (5.51), but do not hold for $u \in W^{1,1}(\Omega)$, a SOLA to (5.1). Accordingly, to estimate integral quantities of $Du$, (5.52) and (5.53) have an important role in our analysis.

We now denote super level sets of $Du$ by

$$E(\lambda) = \{x \in \Omega : |Du(x)| > \lambda\}, \quad \lambda > 0,$$

and write

$$\lambda_0 = \fint_\Omega |Du|\,dx + \frac{1}{\delta}\fint_\Omega g^{-1}\big(M_1(\mu)\big)\,dx \text{ and } A = \left(\frac{2000\cdot \operatorname{diam}(\Omega)}{R}\right)^n, \tag{5.54}$$

where $\delta$ is the constant given in Lemma 5.3.5 and Lemma 5.3.10.

The following is a Vitali type covering lemma, which can be obtained by a modification of [27, Lemma 4.1].

**Lemma 5.4.1.** *For any $\lambda \ge A\lambda_0$, there exists a negligible set $N$ and a disjoint covering $\{B_{r_i}(x^i)\}_{i\ge1}$ with center $x_i \in \Omega$ and radii $r_i \le R/500$ such that*

$$E(\lambda) \setminus N \subset \bigcup_{i\ge1} B_{5r_i}(x^i),$$

$$\fint_{\Omega_{r_i}(x^i)} |Du|\,dx + \frac{1}{\delta}\fint_{\Omega_{r_i}(x^i)} g^{-1}\big(M_1(\mu)\big)\,dx = \lambda \tag{5.55}$$

*and*

$$\fint_{\Omega_\rho(x^i)} |Du|\,dx + \frac{1}{\delta}\fint_{\Omega_\rho(x^i)} g^{-1}\big(M_1(\mu)\big)\,dx \le \lambda, \quad \rho \in (r_i, R].$$

We are now ready to prove our main result.

*Proof of Theorem 5.1.2.* Throughout this proof, we denote by $|Du|_l = \min\{|Du|, l\}$ for any $l \geq KA\lambda_0$, where $K = 2c_l$ with the constant $c_l$ given in (5.47) and (5.50). A straightforward calculation yields

$$\begin{aligned}\int_\Omega h(|Du|_l)|Du|\,dx &= K\int_0^{l/K}\int_{E(K\lambda)}|Du|\,dx\,h'(K\lambda)\,d\lambda\\ &= K\int_0^{A\lambda_0}\int_{E(K\lambda)}|Du|\,dx\,h'(K\lambda)\,d\lambda\\ &\quad + K\int_{A\lambda_0}^{l/K}\int_{E(K\lambda)}|Du|\,dx\,h'(K\lambda)\,d\lambda\\ &=: I + II, \end{aligned} \tag{5.56}$$

where $h$ is the derivative of $H$ given in Theorem 5.1.2.

We first estimate $I$ as

$$I \leq h(KA\lambda_0)\int_\Omega |Du|\,dx \leq \gamma_3|\Omega|(KA)^{\gamma_3-1}H(\lambda_0). \tag{5.57}$$

On the other hand, we use the covering given in Lemma 5.4.1 to see

$$II \leq K\sum_{i\geq 1}\int_{A\lambda_0}^{l/K}\int_{E(K\lambda)\cap B_{5r_i}(x^i)}|Du|\,dx\,h'(K\lambda)\,d\lambda. \tag{5.58}$$

We now distinguish two cases, $B_{25r_i}(x^i) \subset \Omega$ and $B_{25r_i}(x^i) \not\subset \Omega$.

**The first case** Using (5.52) and (5.53), for any $\varepsilon > 0$ and each $i \geq 1$, there exists $k \in \mathbb{N}$ such that

$$\fint_{B_{25r_i}(x^i)} |Du - Du_k| \leq \varepsilon\lambda \quad \text{and} \quad |\mu_k|(B_{25r_i}(x^i)) \leq 2|\mu|(B_{25r_i}(x^i)). \tag{5.59}$$

Let $w_{i,k} \in u_k + W^{1,G}(B_{25r_i})$ be the weak solution to (5.48) with $B_{5r} = B_{25r_i}(x^i)$ and $u = u_k$, and $v_{i,k} \in w_{i,k} + W^{1,G}(B_{10r_i})$ be the weak solution to (5.49) with $B_{2r} = B_{10r_i}(x^i)$ and $w = w_{i,k}$.

It then follows from (5.50) that in $E(K\lambda) \cap B_{5r_i}(x^i)$, there holds $|Dv_{i,k}| \leq$

$K\lambda/2$, and so

$$\begin{aligned}|Du| &\leq |Du - Du_k| + |Du_k - Dw_{i,k}| \\ &\quad + G^{-1}(|V(Dw_{i,k}) - V(Dv_{i,k})|^2) + |Dv_{i,k}| \\ &\leq |Du - Du_k| + |Du_k - Dw_{i,k}| \\ &\quad + G^{-1}(|V(Dw_{i,k}) - V(Dv_{i,k})|^2) + \frac{1}{2}|Du|.\end{aligned}$$

Hence, for any $\varepsilon > 0$, Jensen's inequality, Lemma 5.3.10, Lemma 5.3.11 and (5.59) imply

$$\begin{aligned}&\frac{1}{|B_{5r_i}|}\int_{E(K\lambda)\cap B_{5r_i}(x^i)} |Du|\,dx \\ &\leq \fint_{B_{5r_i}(x^i)} |Du - Du_k|\,dx + \fint_{B_{5r_i}(x^i)} |Du_k - Dw_{i,k}|\,dx \\ &\quad + c\,G^{-1}\left(\fint_{B_{5r_i}(x^i)} |V(Dw_{i,k}) - V(Dv_{i,k})|^2\,dx\right) \\ &\leq c\varepsilon\lambda. \end{aligned} \tag{5.60}$$

**The second case** In this case, we take a point $y^i \in B_{25r_i}(x^i)$ with a coordinate system $y = (y_1, y_2, \ldots, y_n)$ such that

$$B^+_{400r_i}(y^i) \subset \Omega_{400r_i}(y^i) \subset B_{400r_i}(y^i) \cap \{y_n > -400\delta r_i\}$$

and

$$|y^i - x^i| \leq 25r_i + 400\delta r_i \leq 75r_i.$$

Noting $B_{400r_i}(y^i) \subset B_{500r_i}(x^i)$, we discover

$$\fint_{\Omega_{400r_i}(y^i)} |Du|\,dx + \frac{1}{\delta}\fint_{B_{400r_i}(y^i)} g^{-1}\big(M_1(\mu)\big)\,dx \leq 5^n\lambda.$$

Moreover, similarly to **The first case**, using lemmas in Subsection 5.3.2 for $B_{5r} = B_{400r_i}(y^i)$, we discover

$$\sup_{\Omega_{5r_i}(x^i)} |Dv_{i,k}| \leq \sup_{\Omega_{80r_i}(y^i)} |Dv_{i,k}| \leq c_l\lambda$$

and

$$\int_{E(K\lambda)\cap B_{5r_i}(x^i)} |Du|\,dx \le c\varepsilon\lambda|\Omega_{5r_i}(x^i)|. \tag{5.61}$$

Here, we also have used the fact that $B_{5r_i}(x^i) \subset B_{80r_i}(y^i)$.

Combining (5.60) and (5.61), we have

$$\int_{E(K\lambda)\cap B_{5r_i}(x^i)} |Du|\,dx \le c\varepsilon\lambda|\Omega_{r_i}(x^i)|. \tag{5.62}$$

To estimate $\lambda|\Omega_{r_i}(x^i)|$ in the above inequality, we recall (5.55) to see either

$$\fint_{\Omega_{r_i}(x^i)} |Du|\,dx \ge \frac{\lambda}{2} \quad \text{or} \quad \frac{1}{\delta}\fint_{\Omega_{r_i}(x^i)} g^{-1}\big(M_1(\mu)\big)\,dx \ge \frac{\lambda}{2}. \tag{5.63}$$

In case of $(5.63)_1$, we estimate as follows:

$$\begin{aligned} \lambda|\Omega_{r_i}(x^i)| &\le 2\int_{\Omega_{r_i}(x^i)\cap\{|Du|>\frac{\lambda}{4}\}} |Du|\,dx + 2\int_{\Omega_{r_i}(x^i)\cap\{|Du|\le\frac{\lambda}{4}\}} |Du|\,dx \\ &\le 2\int_{\Omega_{r_i}(x^i)\cap\{|Du|>\frac{\lambda}{4}\}} |Du|\,dx + \frac{\lambda}{2}|\Omega_{r_i}(x^i)|. \end{aligned} \tag{5.64}$$

Similarly, we estimate $(5.63)_2$ as

$$\lambda|\Omega_{r_i}(x^i)| \le 2\int_{\Omega_{r_i}(x^i)\cap\{g^{-1}(M_1(\mu))>\frac{\lambda}{4}\}} g^{-1}\big(M_1(\mu)\big)\,dx + \frac{\lambda}{2}|\Omega_{r_i}(x^i)|. \tag{5.65}$$

Applying (5.64) and (5.65) to (5.62), we obtain

$$\begin{aligned} \int_{E(K\lambda)\cap B_{5r_i}(x^i)} |Du|\,dx &\le c\varepsilon\int_{\Omega_{r_i}(x^i)\cap\{|Du|>\frac{\lambda}{4}\}} |Du|\,dx \\ &\quad + c\varepsilon\int_{\Omega_{r_i}(x^i)\cap\{g^{-1}(M_1(\mu))>\frac{\lambda}{4}\}} g^{-1}\big(M_1(\mu)\big)\,dx. \end{aligned} \tag{5.66}$$

Since $\{B_{r_i}(x^i)\}_{i\in\mathbb{N}}$ are mutually disjoint, we combine (5.58) and (5.66) to discover

$$II \le c\varepsilon K\int_0^{l/K}\int_{\Omega\cap\{|Du|>\frac{\lambda}{4}\}} |Du|\,dx\,h'(K\lambda)\,d\lambda$$

$$
\begin{aligned}
&+ c\varepsilon K \int_0^{l/K} \int_{\Omega\cap\{g^{-1}(M_1(\mu))>\frac{\lambda}{4}\}} g^{-1}\big(M_1(\mu)\big)\,dx\,h'(K\lambda)\,d\lambda \\
&=: III + IV. \qquad (5.67)
\end{aligned}
$$

We estimate $III$ directly as

$$
\begin{aligned}
III &\le c\varepsilon K^{\gamma_4-1}\frac{\gamma_4-1}{\gamma_3-1}\int_0^l \int_{\Omega\cap\{|Du|>\lambda\}} |Du|\,dx\,h'(\lambda)\,d\lambda \\
&\le c(\gamma_3,\gamma_4)\varepsilon \int_\Omega h(|Du|_l)|Du|\,dx. \qquad (5.68)
\end{aligned}
$$

Likewise, $IV$ can be estimated as

$$
IV \le c(\gamma_3,\gamma_4)\varepsilon \int_\Omega H\circ g^{-1}\big(M_1(\mu)\big)\,dx. \qquad (5.69)
$$

Combining (5.56),(5.57),(5.67),(5.68) and (5.69) and then taking $\varepsilon$ small enough depending only on $\texttt{data}, \gamma_3, \gamma_4$ and $K$, we discover

$$
\int_\Omega h(|Du|_l)|Du|\,dx \le c\int_\Omega H\circ g^{-1}\big(M_1(\mu)\big)\,dx + \gamma_3|\Omega|(KA)^{\gamma_3-1}H(\lambda_0).
$$

Then we take limit as $l$ goes to $\infty$, use (5.54) and Jensen's inequality, to observe

$$
\int_\Omega H(|Du|)\,dx \le c\int_\Omega H\circ g^{-1}\big(M_1(\mu)\big)\,dx + \gamma_3|\Omega|(KA)^{\gamma_3-1}H\bigg(\fint_\Omega |Du|\,dx\bigg).
$$

Recalling (5.36) and using Jensen's inequality, we finally derive the conclusion. □

**Remark 5.4.2.** In this final remark, we present an mapping property of Riesz potential to derive a direct consequence of Theorem 5.1.2. Recall the mapping property

$$
I_1 : L(p,q) \to L(np/(n-p),q) \quad \text{for } 1<p<n \text{ and } q>0,
$$

where $L(p,q)$ for $p>1$ and $q>0$ is Lorentz space defined by

$$
\int_0^\infty \big(t^p|\{x\in\Omega : |\mu(x)|>t\}|\big)^{\frac{q}{p}}\frac{dt}{t} < \infty,
$$

see for instance [64, 77]. If $p = \frac{nq}{n+q}$ for some $q > \frac{n}{n-1}$, then the mapping property and (1.3) imply that

$$M_1(\mu) \in L(q,q) = L^q,$$

whenever $\mu \in L(\frac{nq}{n+q}, q)$.

Setting

$$H_1(t) = g(t)^q \quad \text{and} \quad H_2(t) = \int_0^t \frac{H_1(s)}{s}\, ds,$$

we see that $H_2$ is an $N$-function satisfying (5.6). Moreover, similar calculations as in (5.17) show that $H_1 \approx H_2$. Applying Theorem 5.1.2 with $H = H_2$, we conclude that

$$\int_\Omega \big[g(|Du|)\big]^q\, dx \le c \int_\Omega [M_1(\mu)]^q\, dx.$$

This claims that $\mu \in L(\frac{nq}{n+q}, q)$ implies $g(|Du|) \in L^q$ for any $q > \frac{n}{n-1}$. Recalling $L(s, s^*) \subset L(s,s) = L^s$, for every $s > 1$ we cover the fact which is given in [19, Theorem 3] that $\mu \in L^s$ implies $g(|Du|) \in L^{s^*}$.

## 5.5 Calderón-Zygmund theory for integral functionals with $p(x)$-growth

In the rest of this chapter, we study spherical quasi-minimizers (or $Q$-minimizers with $Q \ge 1$), along with $\omega$-minimizers, of integral functionals with $p(x)$ growth of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x, Du) - |F|^{p(x)-2} F \cdot Du\, dx \tag{5.70}$$

that is already introduces in (1.8), where $\Omega \subset \mathbb{R}^n$ ($n \ge 2$) is a bounded domain. We assume that $p(\cdot) : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$1 < \gamma_1 \le p(x) \le \gamma_2 < \infty \quad \text{for } x \in \mathbb{R}^n \tag{5.71}$$

for some constants $\gamma_1, \gamma_2$ and it is logarithmic Hölder continuous with $\omega_p(\cdot)$ a modulus of continuity of $p(\cdot)$, see 1.5.

We are given a function $F : \Omega \to \mathbb{R}^n$ with $|F|^{p(\cdot)} \in L^1$ and the integral function

$$f = f(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

satisfying the following growth and ellipticity conditions:

$$\begin{aligned} \nu(|\xi|^2+s^2)^{\frac{p(x)}{2}} &\le f(x,\xi) \le L(|\xi|^2+s^2)^{\frac{p(x)}{2}} \\ \nu(|\xi|^2+s^2)^{\frac{p(x)-2}{2}}|\eta|^2 &\le D^2 f(x,\xi)\eta\cdot\eta \le L(|\xi|^2+s^2)^{\frac{p(x)-2}{2}}|\eta|^2 \end{aligned} \tag{5.72}$$

for almost every $x \in \mathbb{R}^n$, every $\xi \in R^n$, any $s \in [0,1]$ and some $0 < \nu \le 1 \le L < \infty$ , where $D^2 f = D^2_\xi f$ (if $s = 0$ and $p(x) < 2$, then we do not consider $D^2 f(x,\xi)$ at $\xi = 0$).

Now let us introduce various weak type minimizers for the functional (5.70).

**Definition 5.5.1.** We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a quasi-minimizer(briefly $Q$-minimizer) with $Q \ge 1$ of $\mathcal{F}$ in (5.70), if for any ball $B_r(y)$ with $y \in \Omega$ and any $\varphi \in W_0^{1,p(\cdot)}(\Omega_r(y))$, we have

$$\mathcal{F}(u, \operatorname{supp}\varphi) \le Q\mathcal{F}(u+\varphi, \operatorname{supp}\varphi).$$

**Definition 5.5.2.** We say that $u \in W^{1,p(\cdot)}(\Omega)$ is an $\omega$-minimizer of the functional $\mathcal{F}$ in (5.70), if there exists a concave nonnegative function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ such that for any ball $B_r(y)$ with $y \in \Omega$ and any $\varphi \in W_0^{1,p(\cdot)}(\Omega_r(y))$, we have

$$\mathcal{F}(u, \Omega_r(y)) \le (1+\omega(r))\mathcal{F}(u+\varphi, \Omega_r(y)).$$

**Definition 5.5.3.** We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a spherical quasi-minimizer(briefly spherical $Q$-minimizer) with $Q \ge 1$ of the functional $\mathcal{F}$ in (5.70), if for any ball $B_r(y)$ with $y \in \Omega$ and any $\varphi \in W_0^{1,p(\cdot)}(\Omega_r(y))$, we have

$$\mathcal{F}(u, \Omega_r(y)) \le Q\mathcal{F}(u+\varphi, \Omega_r(y)). \tag{5.73}$$

We point out some comments regarding the mentioned quasi-minimizers: When $Q = 1$, a quasi-minimizer is the same as a minimizer. A $\omega$-minimizer is a spherical $Q$-minimizer with $Q = 1+\omega(\operatorname{diam}(\Omega))$. A quasi-minimizer is a spherical quasi-minimizer.

We now return to our main regularity assumptions and results.

**Definition 5.5.4.** For any $R > 0$ and $\delta \in (0, \frac{1}{8})$, we say that $(p(\cdot), f(\cdot), \Omega)$ is $(\delta, R)$-vanishing provided the following conditions hold:

1. For the modulus of continuity function $\omega_p(\cdot)$ of $p(\cdot)$, we have

$$\sup_{0<r\leq R} \omega_p(r) \log\left(\frac{1}{r}\right) \leq \delta.$$

2. For any measurable set $U \subset \mathbb{R}^n$ and any $x \in U$, we write

$$\theta(U)(x) := \sup_{\xi\in\mathbb{R}^n} \left| \frac{f(x,\xi)}{(|\xi|^2 + s^2)^{p(x)}} - \fint_U \frac{f(z,\xi)}{(|\xi|^2 + s^2)^{p(z)}}\, dz \right| \leq 2L. \quad (5.74)$$

   Then we have

$$\sup_{0<r<R} \sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \theta(B_r(y))(x)\, dx \leq \delta.$$

3. $\Omega$ is a $(\delta, R)$-Reifenberg flat domain. In other words, for each $y \in \partial\Omega$ and each $r \in (0, R]$, there exists a coordinate system $\{\tilde{y}_1, \cdots, \tilde{y}_n\}$ with the origin at $y$ such that

$$B_r(0) \cap \{\tilde{y}_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{\tilde{y}_n > -\delta r\}.$$

We now state the main results.

**Theorem 5.5.5.** *Assume $|F|^{p(\cdot)} \in L^q(\Omega)$ for $1 < q < \infty$. Then there exists $\delta = \delta(n, \nu, L, \gamma_1, \gamma_2, q) \in (0, \frac{1}{8})$ such that if $(p(\cdot), f(\cdot), \Omega)$ is $(\delta, R)$-vanishing for some $R \in (0, 1]$ and $1 \leq Q \leq 1 + \delta$, then any spherical $Q$-minimizer $u \in W_0^{1,p(\cdot)}(\Omega)$ of (5.70) satisfies $|Du|^{p(\cdot)} \in L^q(\Omega)$ with the estimate*

$$\int_\Omega |Du|^{p(x)q} dx \leq c \left(\frac{\operatorname{diam}(\Omega)}{R_0}\right)^{n(q-1)} \int_\Omega \left[|F|^{p(x)q} + 1\right]\, dx \quad (5.75)$$

*for some constants $c = c(n, \nu, L, \gamma_1, \gamma_2, q) \geq 1$ and $R_0 = R_0(n, \nu, L, \gamma_1, \gamma_2, q, \omega_p(\cdot), R, m_1) > 1$ is given in (5.94).*

With the help of the so called maximal function free method introduced in [5], our idea to the proof of Theorem 5.5.5 is based on a perturbation argument which was mainly developed in [4]. In particular, we regard the integrand $f$ as a perturbation of a function with standard growth in order to utilize the Lipschitz regularity for the associated integral functional. Knowing that we deal with quasi-minimizers of functionals, we need to make comparison estimates for integral functional instead of equations as studied in [4].

## 5.6 Proof of Theorem 5.5.5

As we mentioned, in the remaining part of this chapter, we only give the comparison estimates for spherical $Q$-minimizers, since the remaining part of the proofs are similar to the one in Section 5.4.

### 5.6.1 Auxiliary results for frozen functionals

Let $1 < p < \infty$ and $f_0 \in C^2(\mathbb{R}^n; \mathbb{R})$. We assume that

$$\begin{aligned} \nu(|\xi|^2 + s^2)^{\frac{p}{2}} \leq & f_0(\xi) \leq L(|\xi|^2 + s^2)^{\frac{p}{2}} \\ \nu(|\xi|^2 + s^2)^{\frac{p-2}{2}} |\eta|^2 \leq & D^2 f_0(\xi)\eta \cdot \eta \leq L(|\xi|^2 + s^2)^{\frac{p-2}{2}} |\eta|^2 \end{aligned}$$

for every $\xi, \eta \in \mathbb{R}^n$, for some $s \in [0, 1]$ and $0 < \nu \leq L < \infty$. Let $w \in W^{1,p}(\Omega)$ be a minimizer of the functional of

$$\mathcal{F}_0(w, \Omega) := \int_\Omega f_0(Dw)\, dx \tag{5.76}$$

and write $a(\xi) = D_\xi f_0$. Then it is readily checked that

$$\begin{cases} |a(\xi)| + |Da(\xi)| & \leq \quad L(|\xi|^2 + s^2)^{\frac{p-1}{2}} \\ (a(\xi_1) - a(\xi_2) \cdot (\xi_1 - \xi_2) & \geq \quad \nu(|\xi_1|^2 + |\xi_2|^2 + s^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \end{cases} \tag{5.77}$$

and that $w$ is a weak solution to

$$\operatorname{div}\, a(Dw) = 0 \quad \text{in} \quad \Omega.$$

Let us first state Lipschitz regularity for the frozen functional (5.76).

**Lemma 5.6.1.** *(Lipschitz regularity, see [41, 83])*

1. *Let $w \in W^{1,p}(B_r)$ be a minimizer of the functional* (5.76) *with $\Omega$ replaced by $B_r$. Then there holds*

$$\|Dw\|^p_{L^\infty(B_{\frac{r}{2}})} \le c\left(\fint_{B_r} |Dw|^p dx + 1\right)$$

*for some $c = c(n, \nu, L, p) \ge 1$.*

2. *(See for instance [83]) Let $v \in W^{1,p}(B_r^+)$ with $v = 0$ on $T_r$ be a minimizer of* (5.76) *with $\Omega$ replaced by $B_r^+$. Then we have*

$$\|Dw\|^p_{L^\infty(B^+_{\frac{r}{2}})} \le c\left(\fint_{B_r^+} |Dw|^p dx + 1\right)$$

*for some $c = c(n, \nu, L, p) \ge 1$.*

In the above lemma we have considered a weak solution with the zero value on the flat boundary. We next consider a weak solution with the zero value on the rough boundary. In this case we cannot obtain Lipschitz regularity. Instead we make comparison estimates as in the next lemma.

**Lemma 5.6.2.** *Let $0 < r, \epsilon < 1$. Then there exists $\delta = \delta(n, p, \nu, L, \epsilon) > 0$ such that if $w \in W^{1,p}(\Omega_{3r})$ is a weak solution to*

$$\begin{cases} -\operatorname{div}(a(Dw)) = 0 & \text{in } \Omega_{4r}, \\ w = 0 & \text{on } \partial_w \Omega_{4r} := B_{4r} \cap \partial\Omega \end{cases} \tag{5.78}$$

*with*

$$B_{5r}^+ \subset \Omega_{5r} \subset B_{5r} \cap \{x_n > -10\delta r\},$$

*and $v \in W^{1,p}(B_{2r}^+)$ is a weak solution to*

$$\begin{cases} -\operatorname{div}(a(Dv)) = 0 & \text{in } B_{2r}^+, \\ v = \eta w & \text{on } \partial B_{2r}^+ \end{cases} \tag{5.79}$$

*with $\eta = \eta(x_n) \in C^\infty(\mathbb{R})$ satisfying*

$$0 \le \eta \le 1, \ \eta \equiv 1 \text{ on } [\delta r, 2r], \ \eta \equiv 0 \text{ on } (-\infty, 0], \ |D\eta| \le 2/(\delta r),$$

*then there holds*

$$\fint_{\Omega_{2r}} |Dw - Dv|^p dx \le \epsilon \left( \fint_{\Omega_{4r}} |Dw|^p \, dx + 1 \right), \tag{5.80}$$

*where $v$ is extended to $B_{2r}$ by zero. Moreover, we also have*

$$\fint_{\Omega_{2r}} |Dv|^p dx \le c \left( \fint_{\Omega_{4r}} |Dw|^p \, dx + 1 \right). \tag{5.81}$$

*Proof.* We first observe from Hölder's inequality and the higher integrability of $Dw$ that

$$\begin{aligned}
&\int_{\Omega_{3r} \cap \{x_n \le \delta r\}} |Dw|^p \, dx \\
&\le c \left( \int_{\Omega_{3r} \cap \{x_n \le \delta r\}} |Dw|^{p(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} |\Omega_{3r} \cap \{x_n \le \delta r\}|^{\frac{\sigma}{1+\sigma}} \\
&\le c \delta^{\frac{\sigma}{1+\sigma}} |B_{3r}| \left( \fint_{\Omega_{3r}} |Dw|^{p(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} \\
&\le c \delta^{\frac{\sigma}{1+\sigma}} \left( \int_{\Omega_{4r}} |Dw|^p \, dx + 1 \right),
\end{aligned} \tag{5.82}$$

for some positive constant $\sigma = \sigma(n, \nu, L, p)$. From the fact that $\{|x'| < 2r\} \times \{|x_n| < 2r\} \subset B_{3r}$ and $w = 0$ in $B'_{2r} \times \{x_n < -10\delta r\}$ and Hölder's inequality, we find

$$\begin{aligned}
\int_{\Omega_{2r} \cap \{x_n \le \delta r\}} |D\eta|^p |w|^p \, dx &\le \frac{c}{(\delta r)^p} \int_{\Omega_{2r} \cap \{x_n \le \delta r\}} \left| \int_{-10\delta r}^{x_n} \frac{\partial}{\partial y} w(x', y) \, dy \right|^p \, dx \\
&\le \frac{c}{(\delta r)^p} \int_{\Omega_{2r} \cap \{x_n \le \delta r\}} \left( \int_{-10\delta r}^{\delta r} |D_n w(x', y)| \, dy \right)^p \, dx \\
&\le \frac{c}{\delta r} \int_{\Omega_{2r} \cap \{x_n \le \delta r\}} \int_{-10\delta r}^{\delta r} |D_n w(x', y)|^p \, dy \, dx \\
&\le c \int_{\Omega_{3r} \cap \{x_n \le \delta r\}} |Dw(x', y)|^p \, dx' \, dy.
\end{aligned} \tag{5.83}$$

By standard energy estimate for (5.79), it follows from (5.83) that

$$\begin{aligned}\int_{B_{2r}^+} |Dv|^p\,dx &\le c\left(\int_{B_{2r}^+} \left[|D(\eta w)|^p + 1\right]\,dx\right)\\ &\le c\left(\int_{B_{2r}^+\cap\{x_n\le\delta r\}} |D\eta|^p|w|^p\,dx + \int_{B_{2r}^+} |Dw|^p\,dx + |B_{2r}^+|\right)\\ &\le c\int_{\Omega_{3r}} \left[|Dw|^p + 1\right]\,dx, \qquad (5.84)\end{aligned}$$

and so (5.81) is proved.

To prove (5.80), we first note that $v - \eta w \in W_0^{1,p}(B_{2r}^+)$. Define $\bar{v} \in W^{1,p}(B_{2r})$ by $\bar{v} = v$ in $B_{2r}^+$ and $\bar{v} = 0$ in $\Omega_{2r} \setminus B_{2r}^+$. Then $\bar{v} - \eta w \in W_0^{1,p}(\Omega_{2r})$, and by taking $\bar{v} - \eta w$ as a test function in both (5.78) and (5.79), we discover

$$\int_{\Omega_{2r}} (a(Dv) - a(Dw)) \cdot D(\bar{v} - \eta w)\,dx = 0.$$

To estimate the integral

$$I := \fint_{B_{2r}^+} (a(Dv) - a(Dw)) \cdot D(v - w)\,dx,$$

we use the previous identity, the fact that $\bar{v} - \eta w \equiv 0$ in $\Omega_{2r} \setminus B_{2r}^+$ and Hölder's inequality, to find

$$\begin{aligned}I &= \fint_{B_{2r}^+} (a(Dv) - a(Dw)) \cdot D(\eta w - w)\,dx\\ &= \frac{2}{|B_{2r}|}\int_{B_{2r}^+\cap\{x_n\le\delta r\}} (a(Dv) - a(Dw)) \cdot (wD\eta + (\eta - 1)Dw)\,dx\\ &\le \frac{c}{|B_{2r}|}\left(\int_{B_{2r}^+\cap\{x_n\le\delta r\}} \left[|Dv|^p + |Dw|^p + 1\right]\,dx\right)^{\frac{p-1}{p}}\\ &\quad \cdot \left(\int_{B_{2r}^+\cap\{x_n\le\delta r\}} \left[|w|^p|D\eta|^p + |Dw|^p\right]\,dx\right)^{\frac{1}{p}}.\end{aligned}$$

Thus (5.82), (5.83) and (5.84) imply

$$I \leq c\delta^{\frac{\sigma}{(1+\sigma)p}} \fint_{\Omega_{4r}} [|Dw|^p + 1] \, dx$$

for some constant $c = c(n, \nu, L, p) \geq 1$. Employing (5.77), we deduce

$$\fint_{\Omega_{2r}} (|Dw|^2 + |Dv|^2 + s^2)^{\frac{p-2}{2}} |Dw - Dv|^2 \, dx \leq c\delta^{\frac{\sigma}{(1+\sigma)p}} \fint_{\Omega_{4r}} [|Dw|^p + 1] \, dx.$$

On one hand, if $p \geq 2$, then we have

$$\fint_{\Omega_{2r}} |Dw - Dv|^p \leq c\delta^{\frac{\sigma}{(1+\sigma)p}} \fint_{\Omega_{4r}} [|Dw|^p + 1] \, dx,$$

which implies (5.80) by taking sufficiently small $\delta$ depending on $\epsilon$ and the other universal constants.

On the other hand, if $1 < p < 2$, then by Young's inequality and (5.84), we have that for any $\kappa \in (0, 1)$,

$$\begin{aligned}
\fint_{\Omega_{2r}} |Dw - Dv|^p &\leq \kappa \fint_{\Omega_{2r}} (|Dw|^2 + |Dv|^2 + s^2)^{\frac{p}{2}} \, dx \\
&\quad + c(\kappa) \fint_{\Omega_{2r}} (|Dw|^2 + |Dv|^2 + s^2)^{\frac{p-2}{2}} |Dw - Dv|^2 \, dx \\
&\leq c_1\kappa \fint_{\Omega_{2r}} [|Dw|^p + 1] \, dx + c_2(\kappa)\delta^{\frac{\sigma}{(1+\sigma)p}} \fint_{\Omega_{2r}} [|Dw|^p + 1] \, dx
\end{aligned}$$

for some universal constants $c_1, c_2$. We first select $\kappa$ so that $c_1\kappa \leq \epsilon/2$, and then $\delta$ so that $c_2(\kappa)\delta^{\frac{\delta}{(1+\delta)p}} \leq \epsilon/2$. This proves assertion (5.80). □

### 5.6.2 Comparison estimates

Throughout this subsection, we assume that $\delta \in (0, \frac{1}{8})$ is a sufficiently small number depending on a given parameter $\epsilon$ and structure numbers like $n, \nu, L, p$, while $R > 0$ is a given small number. We start with a self improving property of the gradient of spherical quasi-minimizers under consideration.

**Lemma 5.6.3.** *Let $f$ satisfy the first condition in* (5.72)*, and assume that $p(\cdot)$ and $\Omega$ satisfy* (1) *and* (3) *in Definition 5.5.4, respectively, and $F^{p(\cdot)} \in$*

*$L^q(\Omega)$ for some $q > 1$. Suppose $u \in W_0^{1,p(\cdot)}(\Omega)$ is a spherical $Q$-minimizer of (5.70) for any $Q \geq 1$ and*

$$\int_\Omega |Du|^{p(x)} dx + 1 \leq m$$

*for some $m \geq 1$. Assume finally that $y_0 \in \Omega$ and $R_0 > 0$ satisfy*

$$R_0 \leq \min\left\{\frac{R}{8}, \frac{1}{4m}\right\} \quad \text{and} \quad \omega_p(2R_0) \leq \sqrt{\frac{n+1}{n}} - 1. \tag{5.85}$$

*Then there exists positive constants $\sigma_1 = \sigma_1(n, \nu, L, \gamma_1, \gamma_2, Q, q) < q$ and $c = c(n, \nu, L, \gamma_1, \gamma_2, Q)$ such that*

$$\left(\fint_{\Omega_{\tilde r}(\tilde y)} |Du|^{p(x)(1+\sigma)}\, dx\right)^{\frac{1}{1+\sigma}} \leq c \fint_{\Omega_{2\tilde r}(\tilde y)} |Du|^{p(x)}\, dx + \left(\fint_{\Omega_{2\tilde r}(\tilde y)} \left[|F|^{p(x)(1+\sigma)} + 1\right]\, dx\right)^{\frac{1}{1+\sigma}},$$

*whenever $0 < \sigma \leq \sigma_1$ and $\Omega_{2\tilde r}(\tilde y) \subset \Omega_{R_0}(y)$ with $\tilde y \in \Omega_{R_0}(y)$ and $0 < \tilde r \leq \frac{R_0}{2}$.*

*Proof.* It suffices to consider the boundary case $B_{R_0}(y) \not\subset \Omega$, as the interior case $B_{R_0}(y) \subset \Omega$ can be handled in the same way. We now fix any $B_{8\tilde r}(\tilde y) \subset B_{R_0}(y)$. For simplicity, we will omit the center $\tilde y$ and denote $p_1 := \inf_{x\in\Omega_{8\tilde r}} p(x)$, $p_2 := \sup_{x\in\Omega_{8\tilde r}} p(x)$.

We first assume $B_{2\tilde r} \subset \Omega$. Let $\tilde r \leq \rho_1 < \rho_2 \leq 2\tilde r$ and $\eta \in C_c^\infty(B_{\rho_2})$ be a cutoff function satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\rho_1}$ and $|D\eta| \leq \frac{2}{\rho_2 - \rho_1}$. We substitute $\varphi = (u - (u)_{B_{2\tilde r}})\eta \in W_0^{1,p(\cdot)}(B_{2\tilde r})$ into the right-hand side of (5.73). By following the proof of [4, Theorem 5] under a suitable modification for quasi-minimizers with [63, Theorem 7.1], we have the following Caccioppoli type inequality

$$\fint_{B_{\tilde r}} |Du|^{p(x)}\, dx \leq c \fint_{B_{2\tilde r}} \left(\frac{|u - (u)_{B_{2\tilde r}}|}{\tilde r}\right)^{p_2} dx + c \fint_{\Omega_{2\tilde r}} \left[|F|^{p(x)} + 1\right]\, dx. \tag{5.86}$$

According to (5.85), we have $1 \leq \frac{p_2}{p_1} \leq \sqrt{\frac{n+1}{n}} =: t$ and $p_2 \leq \left(\frac{p_1}{t}\right)^*$, where

$\left(\frac{p_1}{t}\right)^*$ is the Sobolev conjugate of $\frac{p_1}{t}$. By Sobolev-Poincáre inequality, we have

$$\fint_{B_{2\tilde{r}}} \left(\frac{|u-(u)_{B_{2\tilde{r}}}|}{\tilde{r}}\right)^{p_2} dx \le c\left(\fint_{B_{2\tilde{r}}} |Du|^{\frac{p_1}{t}}\, dx\right)^{\frac{tp_2}{p_1}}. \tag{5.87}$$

In the light of (5.85), we find

$$\left(\fint_{B_{2\tilde{r}}} |Du|^{\frac{p_1}{t}}\, dx\right)^{\omega_p(16\tilde{r})} \le c\left(\frac{m}{\tilde{r}^n}\right)^{\frac{\omega_p(16\tilde{r})}{t}} \le c\left(\frac{1}{\tilde{r}}\right)^{\frac{(n+1)}{t}\omega_p(4\tilde{r})} \le c. \tag{5.88}$$

Combining (5.86), (5.87) and (5.88), we have

$$\begin{aligned}\fint_{B_{\tilde{r}}} |Du|^{p(x)}\, dx \le{}& c\left(\fint_{B_{2\tilde{r}}} (|Du|+1)^{\frac{p(x)}{t}}\, dx\right)^t \\ &+ c\fint_{B_{2\tilde{r}}} \left[|F|^{p(x)}+1\right] dx.\end{aligned} \tag{5.89}$$

On the other hand, if $B_{2\tilde{r}} \not\subset \Omega$ and $B_{2\tilde{r}} \cap \Omega \neq \emptyset$, then one can find $\tilde{y}' \in \partial\Omega$ such that $B_{\tilde{r}} \subset B_{3\tilde{r}}(\tilde{y}')$ and $B_{6\tilde{r}}(\tilde{y}') \subset B_{8\tilde{r}}$. Let $3\tilde{r} \le \rho_1 < \rho_2 \le 6\tilde{r}$ and $\eta \in C_c^\infty(B_{\rho_2})$ be a cutoff function satisfying $0 \le \eta \le 1$, $\eta = 1$ in $B_{\rho_1}$ and $|D\eta| \le \frac{2}{\rho_2-\rho_1}$. Taking $\varphi = u\eta \in W_0^{1,p(\cdot)}(\Omega_{\rho_2}(\tilde{y}'))$ in (5.73), we discover

$$\fint_{\Omega_{3\tilde{r}}(\tilde{y}')} |Du|^{p(x)}\, dx \le c\fint_{\Omega_{6\tilde{r}}(\tilde{y}')} \left(\frac{|u|}{\tilde{r}}\right)^{p_2} dx + c\fint_{\Omega_{6\tilde{r}}(\tilde{y}')} \left[|F|^{p(x)}+1\right] dx.$$

Using Sobolev-Poincáre inequality, we discover with the same spirit as in (5.88) and (5.89) that

$$\fint_{\Omega_{3\tilde{r}}(\tilde{y}')} |Du|^{p(x)}\, dx \le c\left(\fint_{\Omega_{6\tilde{r}}(\tilde{y}')} |Du|^{\frac{p(x)}{t}}\, dx\right)^t + c\fint_{\Omega_{6\tilde{r}}(\tilde{y}')} \left[|F|^{p(x)}+1\right] dx,$$

which yields

$$\fint_{B_{\tilde{r}}} |Du|^{p(x)}\, dx \le c\left(\fint_{B_{8\tilde{r}}} |Du|^{\frac{p(x)}{t}}\, dx\right)^t + c\fint_{B_{8\tilde{r}}} \left[|F|^{p(x)}+1\right] dx. \tag{5.90}$$

We see that (5.89) (5.90) holds for any $B_{8\tilde{r}} \subset B_{R_0}(y)$. Applying Gehring's lemma, see [63, Corollary 6.1], [4, Theorem 4], the conclusion follows. $\square$

**Remark 5.6.4.** If $R_0 > 0$ satisfies that

$$R_0 \leq \min\left\{\frac{R}{8}, \frac{1}{4m_0}\right\} \quad \text{and} \quad \omega_p(4R_0) \leq \sqrt{\frac{n+1}{n}} - 1, \tag{5.91}$$

where $m_0 \geq 1$ is denoted by

$$m_0 := \int_\Omega |Du|^{p(x)} dx + 1.$$

Then we see that $\Omega_{R_0}(y)$ satisfies the assumption in Lemma 5.6.3 for any $y \in \Omega$. Therefore we discover that $u \in W^{1,p(\cdot)(1+\sigma_0)}(B_{\tilde{r}}(\tilde{y}))$ for any $0 < \tilde{r} \leq \frac{R_0}{2}$ and $\tilde{y} \in \Omega$.

We also need the following self improving property.

**Lemma 5.6.5.** *Suppose $1 < p < \infty$, $s \in [0,1]$, and $f_0 : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\nu(|\xi|^2 + s^2)^{\frac{p}{2}} \leq f_0(\xi) \leq L(|\xi|^2 + s^2)^{\frac{p}{2}}.$$

*Let $w \in w_0 + W_0^{1,p}(\Omega_{2r}(y))$, $y \in \Omega$, be a minimizer of (5.76) with $w_0 \in W^{1,p(1+\sigma_1)}(\Omega_{2r}(y))$ for some $\sigma_1 > 0$. Then there exists a positive constant $\sigma_2 = \sigma_2(n, p, \nu, L, \sigma_1) < \sigma_1$ such that for any $\sigma \in (0, \sigma_2]$, we have*

$$\begin{aligned} \left(\fint_{\Omega_r(y)} |Dw|^{p(1+\sigma)}\, dx\right)^{\frac{1}{1+\sigma}} &\leq c \fint_{\Omega_{2r}(y)} |Dw|^p\, dx \\ &\quad + \left(\fint_{\Omega_{2r}(y)} \left[|Dw_0|^{p(1+\sigma)} + 1\right] dx\right)^{\frac{1}{1+\sigma}}, \end{aligned}$$

*where $c > 1$ depends only on $n, \nu, L, p$. In particular, if $\gamma_1 \leq p \leq \gamma_2$, then the dependence $p$ of $\sigma_2$ and $c$ can be replaced by $\gamma_1, \gamma_2$.*

*Proof.* The proof is similar to that of Lemma 5.6.3 and so we use the same notation. Let $B_{8\tilde{r}}(\tilde{y}) \subset B_{2r}(y)$. If $B_{2\tilde{r}}(\tilde{y}) \subset \Omega_{2r}(y)$, then by taking a test function $\varphi = (w - (w)_{B_{2\tilde{r}}})\eta \in W_0^{1,p(\cdot)}(B_{2\tilde{r}})$, we have

$$\fint_{B_{\tilde{r}}} |Dw|^p\, dx \leq c \fint_{B_{2\tilde{r}}} \left(\frac{|w - (w)_{B_{2\tilde{r}}}|}{\tilde{r}}\right)^p dx + c \leq c \left(\fint_{B_{2\tilde{r}}} |Dw|^{p_*}\, dx\right)^{\frac{p}{p_*}} + c,$$

where $p_* := \max\{1, \frac{np}{n+p}\}$.

If $B_{2\tilde{r}} \not\subset \Omega$ but $B_{2\tilde{r}} \cap \Omega \neq \emptyset$, then we take $\varphi = (w - w_0)\eta \in W_0^{1,p(\cdot)}(\Omega_{6\tilde{r}}(\tilde{y}'))$ in (5.73) to find

$$\begin{aligned}
\fint_{\Omega_{3\tilde{r}}(\tilde{y}')} |Du|^p \, dx &\le c \fint_{\Omega_{6\tilde{r}}(\tilde{y}')} \left( \frac{|w - w_0|}{\tilde{r}} \right)^p dx + c \\
&\le c \left( \fint_{\Omega_{6\tilde{r}}(\tilde{y}')} |Dw - Dw_0|^{p_*} \, dx \right)^{\frac{p}{p_*}} + c \\
&\le c \left( \fint_{\Omega_{6\tilde{r}}(\tilde{y}')} |Dw|^{p_*} \, dx \right)^{\frac{p}{p_*}} + c \fint_{\Omega_{6\tilde{r}}(\tilde{y}')} \left[ |Dw_0|^p + 1 \right] \, dx.
\end{aligned}$$

Therefore, we have

$$\fint_{B_{\tilde{r}}} |Du|^p \, dx \le c \left( \fint_{B_{8\tilde{r}}} |Dw|^{p_*} \, dx \right)^{\frac{p}{p_*}} + c \fint_{B_{8\tilde{r}}} \left[ |Dw_0|^p + 1 \right] \, dx$$

for any $B_{8\tilde{r}} \subset B_{2r}(y)$, by putting $Dw$ and $Dw_0$ by 0 in $B_{2r}(y) \setminus \Omega$. Now Gehring's lemma gives the desired estimate. □

**Remark 5.6.6.** We define $\sigma_0 = \sigma_0(n, \nu, L, \gamma_1, \gamma_2, Q, q) > 0$ by $\sigma_2$ in Lemma 5.6.5, where $\sigma_1$ is the one determined in Lemma 5.6.3 and $p \in [\gamma_1, \gamma_2]$. Of course, we have $\sigma_0 = \sigma_2 \le \sigma_1$.

From now on, we present boundary comparison estimates. Suppose $(p(\cdot), f(\cdot), \Omega)$ is $(\delta, R)$-vanishing for some $R \in (0, 1)$, and let $u \in W_0^{1,p(\cdot)}(\Omega)$ be a spherical $Q$-minimizer of (5.70) with

$$1 \le Q \le 1 + \delta. \tag{5.92}$$

We first denote $m_1 > 0$ by

$$m_1 := \int_\Omega |Du|^{p(x)} dx + \int_\Omega \left[ |F|^{p(x)(1+\sigma_0)} + 1 \right] \, dx + 1 \ge m_0, \tag{5.93}$$

where $\sigma_0 > 0$ is determined in Remark 5.6.6. Then $R_0 > 0$ is assumed to satisfy

$$R_0 \le \min \left\{ \frac{R}{8}, \frac{1}{4m_1} \right\} \tag{5.94}$$

and

$$\omega_p(2R_0) \leq \min\left\{\sqrt{\frac{n+1}{n}} - 1, \frac{\sigma_0}{4}, \frac{\nu}{8(L+\tilde{L})}\right\}.$$

Note that $R_0$ satisfies (5.91). Let us consider any boundary region $\Omega_{5r}(y)$ with $y \in \Omega$ and $0 < r \leq \frac{R_0}{5}$ satisfying

$$B_{5r}(y)^+ \subset \Omega_{5r}(y) \subset B_{5r}(y) \cap \{x_n \geq -10\delta r\}. \tag{5.95}$$

Finally, set

$$p_1 = \inf_{\Omega_{5r}(y)} p(x),\ p_2 = \sup_{\Omega_{5r}(y)} p(x),\ \text{and } h(x,\xi) = f(x,\xi)(|\xi|^2+s^2)^{\frac{p_2-p(x)}{2}}.$$

Then we have from (5.94) that

$$p_2 \leq p(x)\left(1 + \frac{\omega_p(10r)}{p_1}\right) \leq p(x)(1+\sigma_0) \tag{5.96}$$

and

$$(1+\omega_p(10r))(1+\frac{\sigma_0}{4}) \leq 1+\sigma_0. \tag{5.97}$$

We next construct a frozen functional relevant to $\mathcal{F}$. We shall omit the center $y$ when no confusion arises in the context. According to [41, Proposition 2.32], it follows from (5.72) that

$$|Df(x,\xi)| \leq \tilde{L}(|\xi|^2+s^2)^{\frac{p(x)-1}{2}} \tag{5.98}$$

for some $\tilde{L} = \tilde{L}(\gamma_1,\gamma_2,L) \geq 0$. Define function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{h} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$h(x,\xi) := f(x,\xi)(|\xi|^2+s^2)^{\frac{p_2-p(x)}{2}} \quad \text{and} \quad \bar{h}(\xi) := \fint_{B_{4r}^+} h(x,\xi)\,dx.$$

Then we have

$$\begin{cases} \nu(|\xi|^2+s^2)^{\frac{p_2}{2}} \leq \bar{h}(\xi) \leq L(|\xi|^2+s^2)^{\frac{p_2}{2}} \\ \frac{\nu}{8}(|\xi|^2+s^2)^{\frac{p_2-2}{2}}|\eta|^2 \leq D^2\bar{h}(\xi)\eta\cdot\eta \leq 2L(|\xi|^2+s^2)^{\frac{p_2-2}{2}}|\eta|^2. \end{cases} \tag{5.99}$$

Indeed, it suffices to show that $h$ satisfies (5.99) with $h$ replaced by $\bar{h}$. We first calculate $D^2h$:

$$\begin{aligned}
&D^2h(x,\xi)\\
&= D^2f(x,\xi)(|\xi|^2+s^2)^{\frac{p_2-p(x)}{2}} + (p_2-p(x))(|\xi|^2+s^2)^{\frac{p_2-p(x)}{2}-1} f(x,\xi)I\\
&\quad + (p_2-p(x))(|\xi|^2+s^2)^{\frac{p_2-p(x)}{2}-1}\left[Df(x,\xi)\otimes\xi+\xi\otimes Df(x,\xi)\right]\\
&\quad + (p_2-p(x))(p_2-p(x)-2)(|\xi|^2+s^2)^{\frac{p_2-p(x)}{2}-2} f(x,\xi)\xi\otimes\xi,
\end{aligned}$$

where $I$ is the $n\times n$ identity matrix. It then follows from (5.72), (5.98) and (5.94) that

$$\frac{\nu}{8}(|\xi|^2+s^2)^{\frac{p_2-2}{2}}|\eta|^2 \le D^2h(x,\xi)\eta\cdot\eta \le 2L(|\xi|^2+s^2)^{\frac{p_2-2}{2}}|\eta|^2.$$

Notice that as a direct consequence of (5.99), there exists a constant $c\ge 1$ such that

$$\bar h(\xi_2)-\bar h(\xi_1)-D\bar h(\xi_1)\cdot(\xi_2-\xi_1) \ge c^{-1}(|\xi_1|^2+|\xi_2|^2+s^2)^{\frac{p_2-2}{2}}|\xi_1-\xi_2|^2 \tag{5.100}$$

and

$$|D\bar h(\xi)| \le c(|\xi|^2+s^2)^{\frac{p_2-1}{2}}.$$

In addition, for any $x\in B_{4r}^+$, it follows from (5.74) that

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}^n}\frac{|h(x,\xi)-\tilde h(\xi)|}{(|\xi|^2+s^2)^{\frac{p_2}{2}}} &= \sup_{\xi\in\mathbb{R}^n}\left|\frac{h(x,\xi)}{(|\xi|^2+s^2)^{\frac{p_2}{2}}} - \fint_{B_{4r}^+}\frac{h(z,\xi)}{(|\xi|^2+s^2)^{\frac{p_2}{2}}}dz\right|\\
&= \sup_{\xi\in\mathbb{R}^n}\left|\frac{f(x,\xi)}{(|\xi|^2+s^2)^{\frac{p(x)}{2}}} - \fint_{B_{4r}^+}\frac{f(z,\xi)}{(|\xi|^2+s^2)^{\frac{p(z)}{2}}}dz\right|\\
&= \theta(B_{4r}^+)(x).
\end{aligned} \tag{5.101}$$

We now consider a minimizer $w\in u+W_0^{1,p_2}(\Omega_{4r})$ of the functional

$$\mathcal{F}_0(Dw) := \int_{\Omega_{4r}}\bar h(Dw)\,dx \le \int_{\Omega_{4r}}\bar h(Dw+D\varphi)\,dx \tag{5.102}$$

for every $\varphi \in W_0^{1,p_2}(\Omega_{4r})$. Then $w$ solves the following Dirichlet problem:

$$\begin{cases} -\operatorname{div}\big(D\bar{h}(Dw)\big) = 0 & \text{in } \Omega_{4r} \\ \qquad\qquad w = u & \text{on } \partial\Omega_{4r}. \end{cases} \tag{5.103}$$

It then follows from (5.96), (5.102) with $\varphi = w - u$ and Lemma 5.6.3 that

$$\begin{aligned} \fint_{\Omega_{4r}} |Dw|^{p_2} dx &\le \frac{1}{\nu} \fint_{\Omega_{4r}} \bar{h}(Dw)\, dx \le \frac{1}{\nu} \fint_{\Omega_{4r}} \bar{h}(Du)\, dx \\ &\le \frac{L}{\nu} \fint_{\Omega_{4r}} \left[|Du|^{p_2} + 1\right] dx \\ &\le c \fint_{\Omega_{4r}} \left[|Du|^{p(x)(1+\frac{\omega_p(10r)}{p_1})} + 1\right] dx \\ &\le c \left(\fint_{\Omega_{5r}} |Du|^{p(x)}\, dx\right)^{1+\frac{\omega_p(10r)}{p_1}} \\ &\quad + \left(\fint_{\Omega_{5r}} \left[|F|^{p(x)(1+\sigma_0)} + 1\right] dx\right)^{\frac{1}{1+\sigma_0} + \frac{\omega_p(10r)}{p_1(1+\sigma_0)}}. \end{aligned} \tag{5.104}$$

Using (5.94), we estimate

$$\left(\fint_{\Omega_{5r}} |Du|^{p(x)} dx\right)^{\omega_p(10r)} \le c \left(\frac{m_1}{r^n}\right)^{\omega_p(10r)} \le \left(\frac{1}{r}\right)^{(n+1)\omega_p(10r)} \tag{5.105}$$

and similarly,

$$\begin{aligned} \left(\fint_{\Omega_{5r}} \left[|F|^{p(x)(1+\sigma_0)} + 1\right] dx\right)^{\omega_p(10r)} &\le c \left(\frac{m_1}{r^n}\right)^{\omega_p(10r)} \\ &\le \left(\frac{1}{r}\right)^{(n+1)\omega_p(10r)}. \end{aligned} \tag{5.106}$$

Combining (5.104), (5.105) and (5.106), we have

$$\begin{aligned} \fint_{\Omega_{4r}} |Dw|^{p_2}\, dx &\le \frac{L}{\nu} \fint_{\Omega_{4r}} \left[|Du|^{p_2} + 1\right] dx \\ &\le c \fint_{\Omega_{5r}} |Du|^{p(x)}\, dx \end{aligned}$$

$$+ c\left(\fint_{\Omega_{5r}} \left[|F|^{p(x)(1+\sigma_0)} + 1\right] dx\right)^{\frac{1}{1+\sigma_0}}. \qquad (5.107)$$

Now we are ready to derive comparison estimates.

**Lemma 5.6.7.** *For any small $\epsilon > 0$, there exists $\delta = \delta(n, \nu, L, \gamma_1, \gamma_2, \epsilon) > 0$ such that the following statement holds: For any $\lambda \geq 1$, if*

$$\fint_{\Omega_{5r}} \left[|Du|^{p(x)} + 1\right] dx + \frac{1}{\delta}\left(\fint_{\Omega_{5r}} |F|^{p(x)(1+\sigma_0)} dx\right)^{\frac{1}{1+\sigma_0}} \leq \lambda, \qquad (5.108)$$

*then we have*

$$\fint_{\Omega_{3r}} |Dw|^{p_2}\, dx \leq c_1\lambda \quad \text{and} \quad \fint_{\Omega_{3r}} |Du - Dw|^{p_2}\, dx \leq \epsilon\lambda.$$

*Proof.* We first observe from (5.107) and (5.108) that

$$\fint_{\Omega_{3r}} |Dw|^{p_2}\, dx \leq c\fint_{\Omega_{4r}} |Dw|^{p_2}\, dx \leq c\lambda$$

Moreover, using Lemma 5.6.5, Lemma 5.6.3, (5.97), (5.105), (5.106) and (5.108), we discover

$$\begin{aligned}
&\left(\fint_{\Omega_{3r}} |Dw|^{p_2(1+\frac{\sigma_0}{4})} dx\right)^{\frac{4}{4+\sigma_0}} \\
&\leq c\fint_{\Omega_{4r}} |Dw|^{p_2} dx + c\left(\fint_{\Omega_{4r}} |Du|^{p_2(1+\frac{\sigma_0}{4})} dx\right)^{\frac{4}{4+\sigma_0}} \\
&\leq c\lambda + c\left(\fint_{\Omega_{4r}} \left[|Du|^{p(x)(1+\omega_p(8r)(1+\frac{\sigma_0}{4}))} + 1\right] dx\right)^{\frac{4}{4+\sigma_0}} \\
&\leq c\lambda + c\left(\fint_{\Omega_{5r}} |Du|^{p(x)} dx\right)^{1+\omega_p(8r)} + c\left(\fint_{\Omega_{5r}} \left[|F|^{p(x)(1+\sigma_0)} + 1\right] dx\right)^{\frac{1+\omega_p(8r)}{1+\sigma_0}} \\
&\leq c\lambda + c\fint_{\Omega_{5r}} |Du|^{p(x)} dx + c\left(\fint_{\Omega_{5r}} \left[|F|^{p(x)(1+\sigma_0)} + 1\right] dx\right)^{\frac{1}{1+\sigma_0}} \\
&\leq c\lambda. \qquad (5.109)
\end{aligned}$$

When $p_2 \geq 2$, we see

$$|Du - Dw|^{p_2} \leq (|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2-2}{2}} |Du - Dw|^2.$$

On the other hand, when $p_2 < 2$, we see

$$\begin{aligned} |Du - Dw|^{p_2} &= (|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2(2-p_2)+p_2(p_2-2)}{4}} |Du - Dw|^{p_2} \\ &\leq \epsilon_1 (|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2}{2}} \\ &\quad + c(\epsilon_1)(|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2-2}{2}} |Du - Dw|^2 \end{aligned} \tag{5.110}$$

for any $\epsilon_1 > 0$. In (5.110), $c(\epsilon_1) \geq 1$ depends only on $n, \gamma_1, \gamma_2, \nu, L$ and $\epsilon_1$, and it is stable as $p_2 \nearrow 2$ for each $\epsilon_1 > 0$. Therefore, regardless of whether $p_2 < 2$ or not, we have

$$\begin{aligned} &\fint_{\Omega_{3r}} |Du - Dw|^{p_2}\, dx \\ &\leq \epsilon_1 \fint_{\Omega_{3r}} (|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2}{2}}\, dx \\ &\quad + c(\epsilon_1) \fint_{\Omega_{3r}} (|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2-2}{2}} |Du - Dw|^2\, dx \\ &\leq c\epsilon_1 \lambda + c(\epsilon_1) \fint_{\Omega_{3r}} (|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2-2}{2}} |Du - Dw|^2\, dx. \end{aligned}$$

To estimate further, recall that $w$ solves (5.103). It then follows from (5.100) that

$$\begin{aligned} &\fint_{\Omega_{3r}} (|Du|^2 + |Dw|^2 + s^2)^{\frac{p_2-2}{2}} |Du - Dw|^2\, dx \\ &\leq c \fint_{\Omega_{3r}} \bar{h}(Du) - \bar{h}(Dw) - D\bar{h}(Dw) \cdot (Du - Dw)\, dx \\ &= \fint_{\Omega_{3r}} (\bar{h}(Du) - h(x, Du))\, dx + \fint_{\Omega_{3r}} (h(x, Du) - f(x, Du))\, dx \\ &\quad + \fint_{\Omega_{3r}} (f(x, Du) - f(x, Dw))\, dx + \fint_{\Omega_{3r}} (f(x, Dw) - h(x, Dw))\, dx \\ &\quad + \fint_{\Omega_{3r}} (h(x, Dw) - \bar{h}(Dw))\, dx =: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

*Estimates $I_1$ and $I_5$:* Applying (5.101), Hölder's inequality and (5.109), we estimate

$$
\begin{aligned}
|I_1| + |I_5| &\le c \fint_{\Omega_{3r}} \theta(B_{4r}^+) \left[ (|Du|^2 + s^2)^{\frac{p_2}{2}} + (|Dw|^2 + s^2)^{\frac{p_2}{2}} \right] dx \\
&\le c \left( \fint_{\Omega_{3r}} \theta(B_{4r}^+)^{\frac{4+\sigma_0}{\sigma_0}} \, dx \right)^{\frac{\sigma_0}{4+\sigma_0}} \lambda \\
&\le c \left( L^{\frac{4}{\sigma_0}} \fint_{B_{4r}^+} \theta(B_{4r}^+) \, dx + L^{\frac{4+\sigma_0}{\sigma_0}} \delta \right)^{\frac{\sigma_0}{4+\sigma_0}} \lambda \le c \delta^{\frac{\sigma_0}{4+\sigma_0}} \lambda
\end{aligned}
$$

Here, we have used (2) in Definition 5.5.4, so that

$$
\fint_{B_{4r}^+(y)} \theta(B_{4r}^+(y))(x) \, dx \le 4 \fint_{B_{4r}(y)} \theta(B_{4r}(y))(x) \, dx \le 4\delta,
$$

and (5.95).

*Estimates $I_2$ and $I_4$:* The following estimates can be obtained by a direct calculation.

$$
\begin{aligned}
|I_2| &\le L \fint_{\Omega_{3r}} \Bigg[ \int_0^1 \frac{p_2 - p(x)}{2} |\log(|Du|^2 + s^2)| \\
&\qquad\qquad \cdot (|Du|^2 + s^2)^{\frac{p_2 - p(x)}{2} t} dt \Bigg] (|Du|^2 + s^2)^{\frac{p(x)}{2}} \, dx \\
&\le c \omega_p(10r) \int_0^1 \fint_{\Omega_{3r}} |\log(|Du|^2 + s^2)| (|Du|^2 + s^2)^{\frac{p_2 - p(x)}{2} t + \frac{\gamma_1}{4}} \\
&\qquad\qquad \cdot (|Du|^2 + s^2)^{\frac{2p(x) - \gamma_1}{4}} \, dx dt.
\end{aligned}
$$

For any $\alpha > 0$ and $\beta > 1$ we see

$$
t^\alpha |\log t| \le \begin{cases} \dfrac{e^\alpha}{\alpha} & \text{if} \quad 0 < t \le e, \\ 2t^\alpha \log(e + t^{\frac{\beta}{2}}) & \text{if} \quad e < t, \end{cases}
$$

and for every $t_1, t_2 > 0$ we have $\log(e + t_1 t_2) \le \log(e + t_1) + \log(e + t_2)$. It then follows that

$$
|\log(|Du|^2 + s^2)| (|Du|^2 + s^2)^{\frac{p_2 - p(x)}{2} t + \frac{\gamma_1}{4}} (|Du|^2 + s^2)^{\frac{2p(x) - \gamma_1}{4}}
$$

$$
\begin{aligned}
&\leq 2|\log(e+(|Du|^2+1)^{\frac{p_2}{2}})|(|Du|^2+1)^{\frac{p_2}{2}} + \frac{4e^{\gamma_2}}{\gamma_1}(|Du|^2+s^2)^{\frac{2p(x)-\gamma_1}{4}} \\
&\leq 2\log\left(e+\frac{(|Du|^2+1)^{\frac{p_2}{2}}}{((|Du|^2+1)^{\frac{p_2}{2}})_{\Omega_{3r}}}\right)(|Du|^2+1)^{\frac{p_2}{2}} \\
&\quad + 2\log\left(e+\left((|Du|^2+1)^{\frac{p_2}{2}}\right)_{\Omega_{3r}}\right)(|Du|^2+1)^{\frac{p_2}{2}} \\
&\quad + c(\gamma_1,\gamma_2)(|Du|^2+1)^{\frac{p(x)}{2}}.
\end{aligned}
$$

Thus, (2.15), (5.94), (5.109) and (1) in Definition 5.5.4 yield

$$
\begin{aligned}
|I_2| &\leq c\omega_p(10r)\fint_{\Omega_{3r}}\log\left(e+\frac{(|Du|^2+1)^{\frac{p_2}{2}}}{(|Du|^2+1)^{\frac{p_2}{2}})_{\Omega_{3r}}}\right)(|Du|^2+1)^{\frac{p_2}{2}}\,dx \\
&\quad + c\omega_p(10r)\fint_{\Omega_{3r}}\log\left(e+\left((|Du|^2+1)^{\frac{p_2}{2}}\right)_{\Omega_{3r}}\right)(|Du|^2+1)^{\frac{p_2}{2}}\,dx \\
&\quad + c\omega_p(10r)\fint_{\Omega_{3r}}(|Du|^2+1)^{\frac{p(x)}{2}}\,dx \\
&\leq c\omega_p(10r)\left(\fint_{\Omega_{3r}}(|Du|+1)^{p_2\left(1+\frac{\sigma_0}{4}\right)}\,dx\right)^{\frac{4}{4+\sigma_0}} \\
&\quad + c\omega_p(10r)\log\left(\frac{1}{r}\right)\fint_{\Omega_{3r}}(|Du|+1)^{p_2}\,dx \\
&\quad + c\omega_p(10r)\fint_{\Omega_{3r}}(|Du|+1)^{p(x)}\,dx \\
&\leq c\omega_p(10r)\log\left(\frac{1}{r}\right)\left(\fint_{\Omega_{3r}}(|Du|+1)^{p_2\left(1+\frac{\sigma_0}{4}\right)}\,dx\right)^{\frac{4}{4+\sigma_0}} \\
&\leq c\delta\lambda.
\end{aligned}
$$

In the same spirit, we find

$$
|I_4| \leq c\omega_p(10r)\log\left(\frac{1}{10r}\right)\left(\fint_{\Omega_{3r}}(|Dw|+1)^{p_2\left(1+\frac{\sigma_0}{4}\right)}\,dx\right)^{\frac{4}{4+\sigma_0}} \leq c\delta\lambda.
$$

*Estimate* $I_3$. Using Young's inequality and that $u$ is quasi-minimizer of

(5.70) with (5.92), we have

$$
\begin{aligned}
|I_3| &\leq (Q-1) \fint_{\Omega_{3r}} f(x, Dw)\, dx + \fint_{\Omega_{3r}} |F|^{p(x)-2} F \cdot (Du - QDw)\, dx \\
&\leq L\delta \fint_{\Omega_{3r}} (|Dw|^2 + s^2)^{\frac{p(x)}{2}}\, dx + \fint_{\Omega_{3r}} |F|^{p(x)-1} |Du - Dw|\, dx \\
&\quad + \delta \fint_{\Omega_{3r}} |F|^{p(x)-1} |Dw|\, dx \\
&\leq c\delta \fint_{\Omega_{3r}} (|Dw| + 1)^{p_2}\, dx + \epsilon_2 \fint_{\Omega_{3r}} |Du - Dw|^{p_2}\, dx \\
&\quad + c(\epsilon_2) \fint_{\Omega_{3r}} (|F| + 1)^{p(x)} dx
\end{aligned}
$$

for any $\epsilon_2 > 0$. Taking (5.109) into account, we see

$$
|I_3| \leq c(\epsilon_2)\delta\lambda + \epsilon_2 \fint_{\Omega_{3r}} |Du - Dw|^{p_2}\, dx.
$$

Summing up the previous inequalities gives

$$
\begin{aligned}
&\fint_{\Omega_{3r}} |Du - Dw|^{p_2}\, dx \\
&\leq c\epsilon_1\lambda + c(\epsilon_1, \epsilon_2) \left( \delta^{\frac{\sigma_0}{4+\sigma_0}} + \omega_p(10r) \log\left(\frac{1}{10r}\right)\right) \lambda \\
&\quad + c(\epsilon_1)\epsilon_2 \fint_{\Omega_{3r}} |Du - Dw|^{p_2} dx.
\end{aligned}
$$

Taking $\epsilon_1, \epsilon_2$ and $\delta$ small enough, the desired results follows. □

Finally using the previous lemma, Lemma 5.6.2 and (2) in Lemma 5.6.1, we can deduce the following lemma.

**Lemma 5.6.8.** *For any small $\epsilon > 0$, there exists $\delta = \delta(n, \nu, L, \gamma_1, \gamma_2, \epsilon) > 0$ such that the following statement holds: For any $\lambda \geq 1$, if*

$$
\fint_{\Omega_{5r}} \left[|Du|^{p(x)} + 1\right]\, dx + \frac{1}{\delta} \left( \fint_{\Omega_{5r}} |F|^{p(x)(1+\sigma_0)}\, dx \right)^{\frac{1}{1+\sigma_0}} \leq \lambda,
$$

*then there exists* $v \in W^{1,p}(\Omega_{2r}) \cap W^{1,\infty}(\Omega_r)$ *with* $v = 0$ *on* $T_{2r}$ *such that*

$$\fint_{\Omega_{2r}} |Du - Dv|^{p_2}\, dx \le \epsilon\lambda$$

*and*

$$\sup_{\Omega_r} \|Dv\|^{p_2} = \sup_{B_r^+} \|Dv\|^{p_2} \le c \left( \fint_{\Omega_{2r}} |Dv|^{p_2}\, dx + 1 \right) \le c\lambda,$$

*for some* $c = c(n, \nu, L, \gamma_1, \gamma_2) \ge 1$.

As we mentioned, the remaining part of the proof of Theorem 5.5.5 is similar to the one for Theorem 5.1.2. Therefore, we end the proof here.

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#### 국문초록

이 학위 논문에서는 측도데이터를 갖는 비선형 편미분 방정식에 대하여 해의 그라디언트가 각 점 별로 주어진 데이터의 1-리즈 퍼텐셜 가늠을 갖는다는 것을 다양한 비표준 성장조건 하에서 증명하였다.

특히 1-리즈 퍼텐셜이 선형 연산자에 대응하는 퍼텐셜이라는 사실로 인하여 선형화 기법을 통하여 주어진 비선형 연산자를 선형 연산자로 근사하는 방법이 요구된다. 이러한 과정에서 현재까지의 접근법에는 주어진 편미분 방정식의 성장조건이 2차보다 높은 차수인 경우와 낮은 차수인 경우에 대해 근본적인 차이가 있었다.

위와 같은 차이는 이 논문의 4장의 접근법을 통하여 주어진 데이터가 약해의 존재성을 보장할 수 있는 경우에 대하여 극복된다. 그러나 측도데이터를 갖는 방정식에 대해서는 아직까지 위와 같은 통합적인 접근법이 제시되지 않았으며, 이러한 접근법을 제시하는 첫 걸음으로서 5장에서 측도데이터를 갖는 올리즈 성장조건 방정식에 대하여 칼데론 지그먼드 이론을 설명하고 있다.

**주요어휘:** 측도데이터, 퍼텐셜 이론, 비표준 성장, 선형화 기법, 조화적 근사
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바쁜 일정에도 시간을 할애하여 논문 심사를 해주시고 졸업논문에 진심어린 충고를 해주신 이기암 선생님, Lihe Wang 선생님, Giuseppe Mingione 선생님, Paolo Baroni 선생님께 감사드립니다. 특히, 2017년 이탈리아의 파르마에서 보낸 4달 동안 여러모로 도움을 주었고, 멘토이자 친구가 되어주었던 Paolo Baroni 선생님께 진심으로 감사드립니다. 또한, 실해석학 과목을 통해 강의자로서의 교수에 대한 표본이 되어주신 이우영 선생님, 조화해석학 과목을 통해 연구자가 가져야 할 마음가짐을 가르쳐주신 이상혁 선생님께 감사드립니다. 그리고 대학원 과정동안 수업과 세미나에서 양질의 지식을 탐구할 기회를 제공해주신 수리과학부의 모든 선생님들께 감사드립니다.

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