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Estimation of Implied Volatilities and Interest Rate Derivative Prices

(내재변동성과 이자율 파생상품 가격 추정)

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Estimation of Implied Volatilities and Interest Rate Derivative Prices

A dissertation
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by

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Abstract

The first purpose of this thesis is to derive the implied volatility of call options asymptotically under stochastic volatility models including the Heston model and the SABR model and under local volatility models including the CEV model and polynomial models. By minimizing the mean square error between the derived asymptotic implied volatility and the market implied volatility data, parameters can be estimated. Also, under the Black-Scholes model with the assumption that the short rate is zero, the asymptotic implied volatility of Asian call options can be derived such that the Asian call options can be priced without monte carlo simulations.

The second purpose of this thesis is to price interest rate derivatives under the Hull-White model. From the treasury yield data, the yield curve can be derived by the cubic spline curve method. With this curve, the formula for $\theta(t)$ in the Hull-White model is derived. By setting $b = 0.5$ and estimating σ from the historical volatility, the Hull-White model can be calibrated. Now, interest rate derivatives can be calculated with closed form formula except for swaptions. Since there is no closed form solution for swaptions, they are priced approximately.

Keywords : Heston model, SABR model, local volatility model, Black-Scholes model, Hull-White model, cubic spline curve
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Contents

Chapter 1

Introduction

As is described in the abstract, the first purpose of this paper is to derive the asymptotic implied volatility of call options under stochastic volatility models including the Heston model and the SABR model and under local volatility models including the CEV model and polynomial models. By comparing the implied volatility with the poly-fit of the market implied volatility data, we can decide parameters. This method is valid since there is a one to one correspondence between the option price and the implied volatility due to the positivity of the vega value of the call option under the Black-Scholes model. This means the option price increases as the implied volatility increases. Also, estimating parameters is meaningful since the future behavior of the stock can be simulated by monte carlo simulations.

The strategy for deriving implied volatility asymptotically is as follow. Firstly, derive the partial differential equation of the implied volatility. Secondly, expand the implied volatility as a polynomial series with the variables $x_t = \log(\frac{K}{F_t})$ and $\tau = T - t$. Finally, compare this with the poly-fit of the market implied volatility. By this process, we can estimate parameters.

In the case of Asian options, firstly assume that the risk-neutral interest rate is zero. This assumption is required since deriving the partial differential equation of the implied volatility without this assumption is too complex to handle. Even though the assumption seems not realistic, it turns out that this doesn't affect the implied volatility much. Secondly, derive the partial differential equation of the implied volatility. Finally, derive the asymptotic implied volatility curve.

For the second purpose, we calculate the prices of interest rate derivatives such as bond options, caplets, caps, swaps, and swaptions under the Hull-White model. By deriving the cubic spline curve that fits the market yield data, setting $b = 0.5$, and estimating σ as historical volatility, the Hull-White model is calibrated. Then, the bond dynamics is derived under which interest rate derivative prices are calculated.

Chapter 2

Main Ideas on the Implied Volatility

Even though there is a closed form call option price formula in a model, sometimes it is hard to estimate the market-fitting parameters. However, it is easy to estimate parameters when the asymptotic implied volatility curve is derived.

Now, let's change the stock price S_t to the log-money $x_t = \log\left(\frac{K}{F_t}\right)$ as a variable of the implied volatility where $F_t = S_t e^{r(T-t)}$ is the forward price of the stock at time t and K is the strike. This change is due to the fact that x_t is closer to 0 than S_t is such that the implied volatility expressed as a polynomial series in terms of x_t rather than S_t is more accurate.

Let's think of the call option with strike K and expiration time T and introduce the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with $\mathcal{F}_t = \mathcal{F}_t^W$ which means that the completed filtration is generated by the standard Brownian motion W_t. At time $t < T$, the call price is given by

$$e^{-r(T-t)} E((S_T - K)_+ | \mathcal{F}_t).$$

Since $S_T = F_T$ holds, this can also be written as

$$e^{-r(T-t)} E((F_T - K)_+ | \mathcal{F}_t).$$

Under the Black-Scholes model, the call price is of the form

$$S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

where N is the cumulative normal distribution and d_1, d_2 are defined as

$$d_1 = \frac{\log(\frac{F_t}{K})}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

with the volatility σ. Since $S_t = F_t e^{-r(T-t)}$, this can also be written as

$$e^{-r(T-t)}(F_t N(d_1) - K N(d_2)).$$

Since the implied volatility is the volatility of the Black-Scholes model that makes these two call prices equal, the equation

$$E((F_T - K)_+ | \mathcal{F}_t) = F_t N(d_1) - K N(d_2)$$

is satisfied under the implied volatility. By dividing both sides by K and expressing F_t in terms of $x_t = \log(K/F_t)$, we get

$$E((e^{-x_T} - 1)_+|\mathcal{F}_t) = e^{-x_t}N(d_1) - N(d_2).$$

Now, define C as

$$C(t, x_t, \nu_t) = E((e^{-x_T} - 1)_+|\mathcal{F}_t)$$

for stochastic volatility models and

$$C(t, x_t) = E((e^{-x_T} - 1)_+|\mathcal{F}_t)$$

for local volatility models. Define C^{BS} as

$$C^{BS}(x_t, w) = e^{-x_t}N(d_1) - N(d_2).$$

In the above equation, d_1 and d_2 are defined as $d_1 = -\frac{x_t}{w} + \frac{1}{2}w$, $d_2 = -\frac{x_t}{w} - \frac{1}{2}w$ where $x_t = \log(\frac{K}{F_t})$ and $w = \sigma(t, x, \nu)\sqrt{T-t}$ for stochastic volatility models and $w = \sigma(t, x)\sqrt{T-t}$ for local volatility models. Note that σ is the implied volatility and ν is the stochastic volatility. For $t_2 > t_1$, since

$$E(C(t_2, x_{t_2}, \nu_{t_2})|\mathcal{F}_{t_1}) = E(E((e^{-x_T} - 1)_+|\mathcal{F}_{t_2})|\mathcal{F}_{t_1}) = E((e^{-x_T} - 1)_+|\mathcal{F}_{t_1}) = C(t_1, x_{t_1}, \nu_{t_1})$$

holds, the process C is a martingale. Therefore, the dt term of dC must be zero. In stochastic volatility models,

$$dC(t, x, \nu) = C_t dt + C_x dx + \frac{1}{2}C_{xx}(dx)^2 + C_\nu d\nu + \frac{1}{2}C_{\nu\nu}(d\nu)^2 + C_{x\nu}(dxd\nu).$$

In local volatility models,

$$dC(t, x) = C_t dt + C_x dx + \frac{1}{2}C_{xx}(dx)^2.$$

With these, we can derive the partial differential equation of C. Since $C = C^{BS}$, we can express the derivatives of C in terms of the derivatives of C^{BS}. Now, we need to calculate the values of $C_\tau, C_x, C_{xx}, C_\nu, C_{\nu\nu}, C_{x\nu}$ where $\tau = T - t$. Before calculating these values, we need to calculate $C_x^{BS}, C_w^{BS}, C_{xx}^{BS}, C_{xw}^{BS}, C_{ww}^{BS}$.

$$C_x^{BS} = -e^{-x}N(d_1), \quad C_w^{BS} = e^{-x}N'(d_1)$$

$$C_{xx}^{BS} = -C_x^{BS} + C_w^{BS}\frac{1}{w}, \quad C_{xw}^{BS} = C_w^{BS}(-\frac{x}{w^2} - \frac{1}{2}), \quad C_{ww}^{BS} = C_w^{BS}(\frac{x^2}{w^3} - \frac{w}{4}).$$

Now, let's calculate $C_\tau, C_x, C_{xx}, C_\nu, C_{\nu\nu}, C_{x\nu}$.

$$C_\tau = C_w^{BS}w_\tau, \quad C_x = C_x^{BS} + C_w^{BS}w_x, \quad C_{xx} = C_{xx}^{BS} + 2C_{xw}^{BS}w_x + C_{ww}^{BS}w_x^2 + C_w^{BS}w_{xx}$$

$$C_\nu = C_w^{BS}w_\nu, \quad C_{\nu\nu} = C_{ww}^{BS}w_\nu^2 + C_w^{BS}w_{\nu\nu}, \quad C_{x\nu} = C_{wx}^{BS}w_\nu + C_{ww}^{BS}w_\nu w_x + C_w^{BS}w_{\nu x}.$$

By inserting $C_{xx}^{BS}, C_{xw}^{BS}, C_{ww}^{BS}$ into these equations, we can express $C_\tau, C_x, C_{xx}, C_\nu, C_{\nu\nu}, C_{x\nu}$ in terms of C_x^{BS} and C_x^{BS} as below.

$$C_\tau = C_w^{BS}w_\tau \tag{2.1}$$

$$C_x = C_x^{BS} + C_w^{BS} w_x \tag{2.2}$$

$$C_{xx} = -C_x^{BS} + C_w^{BS}(\frac{1}{w} + (-\frac{2x}{w^2} - 1)w_x + (\frac{x^2}{w^3} - \frac{w}{4})w_x^2 + w_{xx}) \tag{2.3}$$

$$C_\nu = C_w^{BS} w_\nu \tag{2.4}$$

$$C_{\nu\nu} = C_w^{BS}((\frac{x^2}{w^3} - \frac{w}{4})w_\nu^2 + w_{\nu\nu}) \tag{2.5}$$

$$C_{x\nu} = C_w^{BS}((-\frac{x}{w^2} - \frac{1}{2})w_\nu + (\frac{x^2}{w^3} - \frac{w}{4})w_\nu w_x + w_{\nu x}). \tag{2.6}$$

By inserting these to the partial differential equation of C, we can get the partial differential equation of the implied volatility σ which is also shown in p.5 of [7]. However, it is hard to solve the PDE. Therefore, the asymptotic method is required. In stochastic volatility models, let's expand the series as

$$\sigma(\tau, x, \nu) = a(x, \nu) + b(x, \nu)\tau + c(x, \nu)\tau^2 + \cdots$$

where

$$a(x, \nu) = a_0(\nu) + a_1(\nu)x + a_2(\nu)x^2 + a_3(\nu)x^3 + \cdots$$
$$b(x, \nu) = b_0(\nu) + b_1(\nu)x + b_2(\nu)x^2 + b_3(\nu)x^3 + \cdots$$
$$c(x, \nu) = c_0(\nu) + c_1(\nu)x + c_2(\nu)x^2 + c_3(\nu)x^3 + \cdots .$$

In local volatility models, there is no ν. For both models, the partial differential equations of a, b, and c are derived by comparing τ^i terms in the PDE of σ. Likewise, a_n, b_n, and c_n are derived by comparing x^i terms in the PDE of a, b, and c, respectively.

Let the market implied volatility data be $X_1, X_2, \cdots, X_n$ and the corresponding market log-money data and the maturity term data be $x_1, x_2, \cdots, x_n$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively. Then, the mean square error is

$$MSE = \sum_{i=1}^{n}(\sigma(\tau_i, x_i, \nu) - X_i)^2$$

with no ν in local volatility models. By minimizing this, all the parameters can be estimated.

Chapter 3

Main Results on the Implied Volatility

3.1 The Heston Model

The Heston model follows the stochastic volatility model

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_{1t}$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dW_{2t}$$

$$dW_{1t}dW_{2t} = \rho dt.$$

This model was introduced in [6]. Now, let $x_t = \log\left(\frac{K}{F_t}\right)$ where $F_t = S_t e^{r(T-t)}$ is the forward price of a stock at time t. Then,

$$dF_t = d(S_t e^{r(T-t)}) = e^{r(T-t)}dS_t - re^{r(T-t)}S_t dt = \sqrt{\nu_t}S_t e^{r(T-t)}dW_{1t} = \sqrt{\nu_t}F_t dW_{1t}$$

$$dx_t = -d\log F_t = -\frac{dF_t}{F_t} + \frac{1}{2}\Big(\frac{dF_t}{F_t}\Big)^2 = \frac{1}{2}\nu_t dt - \sqrt{\nu_t}dW_{1t}$$

such that the Heston model can be transformed to

$$dx_t = \frac{1}{2}\nu_t dt - \sqrt{\nu_t}dW_{1t}$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dW_{2t}$$

$$dW_{1t}dW_{2t} = \rho dt.$$

Now, let's calculate the differential of C,

$$\begin{aligned} dC(t,x,\nu) =& C_t dt + C_x dx + \frac{1}{2}C_{xx}(dx)^2 + C_\nu d\nu + \frac{1}{2}C_{\nu\nu}(d\nu)^2 + C_{x\nu}(dxd\nu) \\ =& \Big(C_t + \frac{1}{2}\nu(C_x + C_{xx}) + \kappa(\theta-\nu)C_\nu + \frac{1}{2}\xi^2\nu C_{\nu\nu} - \rho\xi\nu C_{x\nu}\Big)dt \\ &- C_x\sqrt{\nu}dW_{1t} + C_\nu\xi\sqrt{\nu}dW_{2t}. \end{aligned}$$

Since the dt term is zero due to the martingale property of C, we get

$$C_t + \frac{1}{2}\nu(C_x + C_{xx}) + \kappa(\theta-\nu)C_\nu + \frac{1}{2}\xi^2\nu C_{\nu\nu} - \rho\xi\nu C_{x\nu} = 0.$$

By changing the variable system from (t, x, ν) to (τ, x, ν) where $\tau = T - t$,

$$C_\tau = \frac{1}{2}\nu(C_x + C_{xx}) + \kappa(\theta - \nu)C_\nu + \frac{1}{2}\xi^2\nu C_{\nu\nu} - \rho\xi\nu C_{x\nu}.$$

By substituting (2.1)-(2.6) to the above equation, we get

$$\begin{aligned} w_\tau =& \frac{1}{2}\nu(w_x + \frac{1}{w} - (\frac{2x}{w^2} + 1)w_x + (\frac{x^2}{w^3} - \frac{w}{4})w_x^2 + w_{xx}) + \kappa(\theta - \nu)w_\nu \\ &+ \frac{1}{2}\xi^2\nu((\frac{x^2}{w^3} - \frac{w}{4})w_\nu^2 + w_{\nu\nu}) - \rho\xi\nu(-(\frac{x}{w^2} + \frac{1}{2})w_\nu + (\frac{x^2}{w^3} - \frac{w}{4})w_\nu w_x + w_{\nu x}). \end{aligned}$$

By inserting $w = \sigma(\tau, x, \nu)\sqrt{\tau}$ and multiplying both sides by $\sigma^3\sqrt{\tau}$, we get

$$\begin{aligned} \sigma^3\sigma_\tau\tau + \frac{1}{2}\sigma^4 =& \frac{1}{2}\nu(\sigma^2 - 2x\sigma\sigma_x + x^2\sigma_x^2 - \frac{1}{4}\sigma^4\sigma_x^2\tau^2 + \sigma^3\sigma_{xx}\tau) + \kappa(\theta - \nu)\sigma^3\sigma_\nu\tau \\ &+ \frac{1}{2}\xi^2\nu(x^2\sigma_\nu^2 - \frac{1}{4}\sigma^4\sigma_\nu^2\tau^2 + \sigma^3\sigma_{\nu\nu}\tau) - \rho\xi\nu(-x\sigma\sigma_\nu - \frac{1}{2}\sigma^3\sigma_\nu\tau \\ &+ x^2\sigma_\nu\sigma_x - \frac{1}{4}\sigma^4\sigma_\nu\sigma_x\tau^2 + \sigma^3\sigma_{\nu x}\tau). \end{aligned}$$

Now, by letting $\sigma(\tau, x, \nu)$ as

$$\sigma(\tau, x, \nu) = a(x, \nu) + b(x, \nu)\tau + c(x, \nu)\tau^2 + \cdots$$

and comparing the τ^0 coefficients, we get the PDE of $a(x, \nu)$,

$$\frac{1}{2}a^4 = \frac{1}{2}\nu(a^2 - 2xaa_x + x^2a_x^2) + \frac{1}{2}\xi^2\nu x^2a_\nu^2 + \rho\xi\nu xaa_\nu - \rho\xi\nu x^2a_\nu a_x.$$

By letting $a(x, \nu) = a_0(\nu) + a_1(\nu)x + a_2(\nu)x^2 + a_3(\nu)x^3 + \cdots$ and comparing x^i coefficients, we get

$$\begin{aligned} a_0(\nu) &= \nu^{1/2} \\ a_1(\nu) &= \frac{1}{4}\rho\xi\nu^{-1/2} \\ a_2(\nu) &= \frac{1}{48}(2 - 5\rho^2)\xi^2\nu^{-3/2} \\ a_3(\nu) &= -\frac{1}{96}\rho(5 - 8\rho^2)\xi^3\nu^{-5/2}. \end{aligned}$$

Note that in these equations, κ and θ don't appear and they cannot be estimated. Therefore, τ^1 asymptotic implied volatility terms are necessary. This means $b(x, \nu)$ that includes κ and θ must be calculated for some extent from the PDE of a and b,

$$\begin{aligned} 3a^3b =& \frac{1}{2}\nu(2ab - 2x(ab_x + a_xb) + 2x^2a_xb_x + a^3a_{xx}) + \kappa(\theta - \nu)a^3a_\nu + \frac{1}{2}\xi^2\nu(2x^2a_\nu b_\nu \\ &+ a^3a_{\nu\nu}) - \rho\xi\nu(-x(ab_\nu + a_\nu b) - \frac{1}{2}a^3a_\nu + x^2(a_\nu b_x + a_xb_\nu) + a^3a_{x\nu}). \end{aligned}$$

By letting $b(x, \nu) = b_0(\nu) + b_1(\nu)x + \cdots$ and comparing x^i coefficients, we get

$$b_0(\nu) = \frac{1}{8}\rho\xi\nu^{1/2} + \frac{1}{4}\kappa(\theta - \nu)\nu^{-1/2} - \frac{1}{96}(4 - \rho^2)\xi^2\nu^{-1/2}$$

$$
\begin{aligned}
b_1(\nu) =& -\frac{1}{96}\rho^2\xi^2\nu^{-1/2} - \frac{5}{48}\rho\xi\kappa(\theta-\nu)\nu^{-3/2} - \frac{1}{12}\rho\xi\kappa\nu^{-1/2} \\
& - \frac{1}{96}\rho(5-8\rho^2)\xi^2\nu^{-3/2} + \frac{1}{384}\rho(16-13\rho^2)\xi^3\nu^{-3/2}.
\end{aligned}
$$

Now, let's estimate ν, ρ, ξ. Let the market log-money be x_i and the market implied volatility be X_i for $i = 1, 2, \cdots, n$. Since the zero order approximation of $a(x, \nu)$ is

$$a(x,\nu) = \nu^{1/2} + \cdots,$$

the corresponding mean square error is given by

$$MSE(\nu) = \sum_{i=1}^{n}(\nu^{1/2} - X_i)^2.$$

By differentiating with respect to ν, we get the estimator of ν,

$$\hat{\nu} = \Big(\frac{\sum_{i=1}^{n} X_i}{n}\Big)^2.$$

The first order approximation of $a(x, \nu)$ is

$$a(x,\nu) = \nu^{1/2} + \frac{1}{4}\rho\xi\nu^{-1/2}x + \cdots$$

such that

$$MSE(\rho,\xi) = \sum_{i=1}^{n}(\frac{1}{4}\rho\xi\hat{\nu}^{-1/2}x_i + (\hat{\nu}^{1/2} - X_i))^2.$$

From this, we get the estimator of $\rho\xi$

$$\hat{\rho\xi} = -4\frac{\sum_{i=1}^{n}\hat{\nu}^{1/2}(\hat{\nu}^{1/2} - X_i)x_i}{\sum_{i=1}^{n} x_i^2}.$$

The second order approximation of $a(x, \nu)$ is

$$a(x,\nu) = \nu^{1/2} + \frac{1}{4}\rho\xi\nu^{-1/2}x + \frac{1}{48}(2-5\rho^2)\xi^2\nu^{-3/2}x^2 + \cdots$$

such that

$$
\begin{aligned}
MSE(\rho,\xi) =& \sum_{i=1}^{n}(\frac{1}{4}\rho\xi\hat{\nu}^{-1/2}x_i + \frac{1}{48}(2-5\rho^2)\xi^2\hat{\nu}^{-3/2}x_i^2 + (\hat{\nu}^{1/2} - X_i))^2 \\
=& \frac{1}{16}\rho^2\xi^2\hat{\nu}^{-1}\sum_{i=1}^{n} x_i^2 + \frac{1}{2304}(4\xi^4 - 20\rho^2\xi^4 + 25\rho^4\xi^4)\hat{\nu}^{-3}\sum_{i=1}^{n} x_i^4 \\
& + \sum_{i=1}^{n}(\hat{\nu}^{1/2} - X_i)^2 + \frac{1}{96}(2\rho\xi^3 - 5\rho^3\xi^3)\hat{\nu}^{-2}\sum_{i=1}^{n} x_i^3 \\
& + \frac{1}{2}\rho\xi\hat{\nu}^{-1/2}\sum_{i=1}^{n} x_i(\hat{\nu}^{1/2} - X_i) + \frac{1}{24}(2\xi^2 - 5\rho^2\xi^2)\hat{\nu}^{-3/2}\sum_{i=1}^{n} x_i^2(\hat{\nu}^{1/2} - X_i).
\end{aligned}
$$

Under the condition $\rho\xi = \widehat{\rho\xi}$, we can rewrite the MSE as

$$\begin{aligned}MSE(\xi) =& \frac{1}{16}(\widehat{\rho\xi})^2\hat{\nu}^{-1}\sum_{i=1}^n x_i^2 + \frac{1}{2304}(4\xi^4 - 20(\widehat{\rho\xi})^2\xi^2 + 25(\widehat{\rho\xi})^4)\hat{\nu}^{-3}\sum_{i=1}^n x_i^4 \\ &+ \sum_{i=1}^n(\hat{\nu}^{1/2} - X_i)^2 + \frac{1}{96}(2(\widehat{\rho\xi})\xi^2 - 5(\widehat{\rho\xi})^3)\hat{\nu}^{-2}\sum_{i=1}^n x_i^3 \\ &+ \frac{1}{2}(\widehat{\rho\xi})\hat{\nu}^{-1/2}\sum_{i=1}^n x_i(\hat{\nu}^{1/2} - X_i) + \frac{1}{24}(2\xi^2 - 5(\widehat{\rho\xi})^2)\hat{\nu}^{-3/2}\sum_{i=1}^n x_i^2(\hat{\nu}^{1/2} - X_i).\end{aligned}$$

Note that there is no ρ in the above equation. By differentiating this with respect to ξ, we get

$$\hat{\xi} = \sqrt{\frac{\frac{10}{3}(\widehat{\rho\xi})^2\sum_{i=1}^n x_i^4 - 8(\widehat{\rho\xi})\hat{\nu}\sum_{i=1}^n x_i^3 - 32\hat{\nu}^{3/2}\sum_{i=1}^n x_i^2(\hat{\nu}^{1/2} - X_i)}{\sum_{i=1}^n x_i^4}}.$$

Since $\hat{\rho}\hat{\xi} = \widehat{\rho\xi}$,

$$\hat{\rho} = \frac{\widehat{\rho\xi}}{\hat{\xi}}.$$

Now, Let's consider the τ^1 approximation of the implied volatility. In this case, let the maturity data be $\tau_1, \tau_2, \cdots, \tau_n$. Since the approximation of implied volatility is

$$\begin{aligned}a(x,\nu) + b(x,\nu)\tau =& \nu^{1/2} + \frac{1}{4}\rho\xi\nu^{-1/2}x + \frac{1}{48}(2 - 5\rho^2)\xi^2\nu^{-3/2}x^2 \\ &+ (\frac{1}{8}\rho\xi\nu^{1/2} + \frac{1}{4}\kappa(\theta - \nu)\nu^{-1/2} - \frac{1}{96}(4 - \rho^2)\xi^2\nu^{-1/2})\tau + \cdots,\end{aligned}$$

we get

$$\begin{aligned}MSE(\kappa,\theta) =& \sum_{i=1}^n\Big(\frac{1}{4}\kappa(\theta - \hat{\nu})\hat{\nu}^{-1/2}\tau_i + \Big(\hat{\nu}^{1/2} + \frac{1}{4}\hat{\rho}\hat{\xi}\hat{\nu}^{-1/2}x_i + \frac{1}{48}(2 - 5\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2}x_i^2 \\ &+ (\frac{1}{8}\hat{\rho}\hat{\xi}\hat{\nu}^{1/2} - \frac{1}{96}(4 - \hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-1/2})\tau_i - X_i\Big)\Big)^2.\end{aligned}$$

By differentiation, we get

$$\begin{aligned}\widehat{\kappa(\theta - \nu)} = \hat{\kappa}(\hat{\theta} - \hat{\nu}) =& -4\Big(\sum_{i=1}^n\hat{\nu}^{1/2}\tau_i\Big(\hat{\nu}^{1/2} + \frac{1}{4}\hat{\rho}\hat{\xi}\hat{\nu}^{-1/2}x_i + \frac{1}{48}(2 - 5\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2}x_i^2 \\ &+ (\frac{1}{8}\hat{\rho}\hat{\xi}\hat{\nu}^{1/2} - \frac{1}{96}(4 - \hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-1/2})\tau_i - X_i\Big)\Big)\Big/\Big(\sum_{i=1}^n\tau_i^2\Big).\end{aligned}$$

Since we have already estimated $\kappa(\theta - \nu)$ as $\widehat{\kappa(\theta - \nu)} = \hat{\kappa}(\hat{\theta} - \hat{\nu})$ and $b_1(\nu)$ has κ and θ terms as a form of $\kappa(\theta - \nu)$ and κ, the mean square error of $a_0(\nu) + a_1(\nu)x + a_2(\nu)x^2 + b_0(\nu) + b_1(\nu)x\tau$ to the market implied volatility data is actually a function of κ only.

By differentiating the mean square error of κ with respect to κ, $\hat{\kappa}$ is derived. The mean square error of κ is

$$
\begin{aligned}
MSE(\kappa) =& \sum_{i=1}^{n} \Big(-\frac{1}{12}\kappa\hat{\rho}\hat{\xi}\hat{\nu}^{-1/2}x_i\tau_i + \Big(\hat{\nu}^{1/2} + \frac{1}{4}\hat{\rho}\hat{\xi}\hat{\nu}^{-1/2}x_i + \frac{1}{48}(2-5\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2}x_i^2 + \Big(\frac{1}{8}\hat{\rho}\hat{\xi}\hat{\nu}^{1/2} \\
&+ \frac{1}{4}\widehat{\kappa(\theta-\nu)}\hat{\nu}^{-1/2} - \frac{1}{96}(4-\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-1/2}\Big)\tau_i + \Big(-\frac{1}{96}\hat{\rho}^2\hat{\xi}^2\hat{\nu}^{-1/2} - \frac{5}{48}\hat{\rho}\hat{\xi}\widehat{\kappa(\theta-\nu)}\hat{\nu}^{-3/2} \\
&- \frac{1}{96}\hat{\rho}(5-8\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2} + \frac{1}{384}\hat{\rho}(16-13\hat{\rho}^2)\hat{\xi}^3\hat{\nu}^{-3/2}\Big)x_i\tau_i - X_i\Big)\Big)^2
\end{aligned}
$$

such that

$$
\begin{aligned}
\hat{\kappa} =& 12\hat{\rho}^{-1}\hat{\xi}^{-1}\hat{\nu}^{1/2}\Big(\sum_{i=1}^{n} x_i\tau_i\Big(\hat{\nu}^{1/2} + \frac{1}{4}\hat{\rho}\hat{\xi}\hat{\nu}^{-1/2}x_i + \frac{1}{48}(2-5\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2}x_i^2 + \Big(\frac{1}{8}\hat{\rho}\hat{\xi}\hat{\nu}^{1/2} \\
&+ \frac{1}{4}\widehat{\kappa(\theta-\nu)}\hat{\nu}^{-1/2} - \frac{1}{96}(4-\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-1/2}\Big)\tau_i + \Big(-\frac{1}{96}\hat{\rho}^2\hat{\xi}^2\hat{\nu}^{-1/2} - \frac{5}{48}\hat{\rho}\hat{\xi}\widehat{\kappa(\theta-\nu)}\hat{\nu}^{-3/2} \\
&- \frac{1}{96}\hat{\rho}(5-8\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2} + \frac{1}{384}\hat{\rho}(16-13\hat{\rho}^2)\hat{\xi}^3\hat{\nu}^{-3/2}\Big)x_i\tau_i - X_i\Big)\Big)\Big/\Big(\sum_{i=1}^{n} x_i^2\tau_i^2\Big).
\end{aligned}
$$

Then, $\hat{\theta}$ is of the form

$$
\hat{\theta} = \hat{\nu} + \frac{\widehat{\kappa(\theta-\nu)}}{\hat{\kappa}}
$$

Another way is to derive the poly-fit of the implied volatility with

$$
MSE(m_0, m_1, m_2, m_3, m_4) = \sum_{i=1}^{n} \Big(m_0 + m_1x_i + m_2x_i^2 + m_3\tau_i + m_4\tau_i x_i - X_i\Big)^2.
$$

Let $\hat{m}_0, \hat{m}_1, \hat{m}_2, \hat{m}_3, \hat{m}_4$ be the corresponding values that minimize the MSE. Then $\hat{\nu}, \hat{\rho}, \hat{\xi}, \hat{\kappa}, \hat{\theta}$ is

$$
\begin{aligned}
\hat{\nu} &= \hat{m}_0^2 \\
\hat{\rho} &= \frac{2\hat{m}_0\hat{m}_1}{\sqrt{6\hat{m}_0^3\hat{m}_2 + 10\hat{m}_0^2\hat{m}_1^2}} \\
\hat{\xi} &= 2\sqrt{6\hat{m}_0^3\hat{m}_2 + 10\hat{m}_0^2\hat{m}_1^2} \\
\hat{\kappa} =& \frac{1}{\hat{m}_1}\Big(-5\frac{\hat{m}_1\hat{m}_3}{\hat{m}_0} + 2\hat{m}_0\hat{m}_1^2 - 5\hat{m}_0\hat{m}_1\hat{m}_2 - \frac{15}{2}\hat{m}_1^3 - 3\hat{m}_4 \\
&- \frac{1}{32}\hat{\rho}(5-8\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2} + \frac{1}{128}\hat{\rho}(16-13\hat{\rho}^2)\hat{\xi}^2\hat{\nu}^{-3/2}\Big) \\
\hat{\theta} &= \hat{\nu} + \frac{1}{\hat{\kappa}}(4\hat{m}_0\hat{m}_3 - 2\hat{m}_0^3\hat{m}_1 + 4\hat{m}_0^3\hat{m}_2 + 6\hat{m}_0^2\hat{m}_1^2).
\end{aligned}
$$

3.2 The SABR Model

The SABR model follows the stochastic volatility model

$$dF_t = \nu_t F_t^{\beta} dW_{1t}$$
$$d\nu_t = \alpha \nu_t dW_{2t}$$
$$dW_{1t} dW_{2t} = \rho dt.$$

This model was introduced in [4]. The estimation of parameters α, β, ρ, ν is required. The dynamics of $x_t = \log(\frac{K}{F_t})$ is

$$dx_t = -\frac{dF_t}{F_t} + \frac{1}{2}\Big(\frac{dF_t}{F_t}\Big)^2 = \frac{1}{2}\nu_t^2 F_t^{2\beta-2} dt - \nu_t F_t^{\beta-1} dW_{1t}.$$

Since $F_t = Ke^{-x_t}$, we get

$$dx_t = \frac{1}{2}\nu_t^2 K^{2\beta-2} e^{(2-2\beta)x_t} dt - \nu_t K^{\beta-1} e^{(1-\beta)x_t} dW_{1t}$$
$$d\nu_t = \alpha \nu_t dW_{2t}$$
$$dW_{1t} dW_{2t} = \rho dt$$

such that

$$\begin{aligned} dC(t,x,\nu) =& C_t dt + C_x dx + \frac{1}{2} C_{xx} (dx)^2 + C_\nu d\nu + \frac{1}{2} C_{\nu\nu} (d\nu)^2 + C_{x\nu}(dx d\nu) \\ =& \Big(C_t + \frac{1}{2}\nu^2 K^{2\beta-2} e^{(2-2\beta)x}(C_x + C_{xx}) + \frac{1}{2}\alpha^2\nu^2 C_{\nu\nu} - \rho\alpha\nu^2 K^{\beta-1} e^{(1-\beta)x} C_{x\nu}\Big) dt \\ & - \nu K^{\beta-1} e^{(1-\beta)x} C_x dW_{1t} + \alpha\nu C_\nu dW_{2t}. \end{aligned}$$

Since the dt term is zero due to the martingale property of C, we get

$$C_t + \frac{1}{2}\nu^2 K^{2\beta-2} e^{(2-2\beta)x}(C_x + C_{xx}) + \frac{1}{2}\alpha^2\nu^2 C_{\nu\nu} - \rho\alpha\nu^2 K^{\beta-1} e^{(1-\beta)x} C_{x\nu} = 0.$$

By changing the variables from (t, x, ν) to (τ, x, ν) where $\tau = T - t$, we get

$$C_\tau = \frac{1}{2}\nu^2 K^{2\beta-2} e^{(2-2\beta)x}(C_x + C_{xx}) + \frac{1}{2}\alpha^2\nu^2 C_{\nu\nu} - \rho\alpha\nu^2 K^{\beta-1} e^{(1-\beta)x} C_{x\nu}.$$

By substituting (2.1)-(2.6) to the above equation, we get

$$\begin{aligned} w_\tau =& \frac{1}{2}\nu^2 K^{2\beta-2} e^{(2-2\beta)x}(\frac{1}{w} - 2x\frac{w_x}{w^2} + (\frac{x^2}{w^3} - \frac{w}{4})w_x^2 + w_{xx}) + \frac{1}{2}\alpha^2\nu^2((\frac{x^2}{w^3} - \frac{w}{4})w_\nu^2 + w_{\nu\nu}) \\ & - \rho\alpha\nu^2 K^{\beta-1} e^{(1-\beta)x}((-\frac{x}{w^2} - \frac{1}{2})w_\nu + (\frac{x^2}{w^3} - \frac{w}{4})w_\nu w_x + w_{\nu x}). \end{aligned}$$

By inserting $w = \sigma(\tau, x, v)\sqrt{\tau}$ and multiplying by $\sigma^3\sqrt{\tau}$, we get

$$\begin{aligned} \sigma^3\sigma_\tau\tau + \frac{1}{2}\sigma^4 =& \frac{1}{2}\nu^2 K^{2\beta-2} e^{(2-2\beta)x}(\sigma^2 - 2x\sigma\sigma_x + x^2\sigma_x^2 - \frac{1}{4}\sigma^4\sigma_x^2\tau^2 + \sigma^3\sigma_{xx}\tau) \\ & + \frac{1}{2}\alpha^2\nu^2(x^2\sigma_\nu^2 - \frac{1}{4}\sigma^4\sigma_\nu^2\tau^2 + \sigma^3\sigma_{\nu\nu}\tau) - \rho\alpha\nu^2 K^{\beta-1} e^{(1-\beta)x}(-x\sigma\sigma_\nu \\ & - \frac{1}{2}\sigma^3\sigma_\nu\tau + x^2\sigma_\nu\sigma_x - \frac{1}{4}\sigma^4\sigma_\nu\sigma_x\tau^2 + \sigma^3\sigma_{\nu x}\tau). \end{aligned}$$

From $\sigma(\tau,x,\nu)=a(x,\nu)+b(x,\nu)\tau+c(x,\nu)\tau^2+\cdots$, we get the PDE of $a(x,\nu)$,

$$\frac{1}{2}a^4=\frac{1}{2}\nu^2K^{2\beta-2}e^{(2-2\beta)x}(a^2-2xaa_x+x^2a_x^2)+\frac{1}{2}\alpha^2\nu^2x^2a_\nu^2+\rho\alpha\nu^2K^{\beta-1}e^{(1-\beta)x}(xaa_\nu-x^2a_\nu a_x).$$

Note that the above equation includes α,β,ρ,ν all such that there is no need to calculate the b terms. By $a(x,\nu)=a_0(\nu)+a_1(\nu)x+a_2(\nu)x^2+a_3(\nu)x^3+\cdots$, we get

$$a_0(\nu)=\nu K^{\beta-1}$$

$$a_1(\nu)=\frac{1}{2}(1-\beta)a_0(\nu)+\frac{1}{2}\alpha\rho=\frac{1}{2}(1-\beta)\nu K^{\beta-1}+\frac{1}{2}\alpha\rho$$

$$a_2(\nu)=\frac{1}{12}(1-\beta)^2\nu K^{\beta-1}+\frac{1}{12}\alpha^2(2-3\rho^2)\nu^{-1}K^{1-\beta}$$

$$a_3(\nu)=-\frac{1}{24}\alpha^2(1-\beta)(2-3\rho^2)\nu^{-1}K^{1-\beta}-\frac{1}{24}\alpha^3\rho(5-6\rho^2)\nu^{-2}K^{2-2\beta}.$$

The log version MSE on $a_0(\nu)$ is

$$MSE(\nu,\beta)=\sum_{i=1}^{n}(\log\nu+(\beta-1)\log K_i-\log X_i)^2.$$

By differentiation, we get

$$\hat{\nu}=\exp\left(\frac{(\sum_{i=1}^n\frac{\log K_i}{n})(\sum_{i=1}^n\frac{\log K_i\log X_i}{n})-(\sum_{i=1}^n\frac{\log X_i}{n})(\sum_{i=1}^n\frac{(\log K_i)^2}{n})}{(\sum_{i=1}^n\frac{\log K_i}{n})^2-\sum_{i=1}^n\frac{(\log K_i)^2}{n}}\right)$$

$$\hat{\beta}=1+\frac{(\sum_{i=1}^n\frac{\log K_i}{n})(\sum_{i=1}^n\frac{\log X_i}{n})-\sum_{i=1}^n\frac{\log K_i\log X_i}{n}}{(\sum_{i=1}^n\frac{\log K_i}{n})^2-\sum_{i=1}^n\frac{(\log K_i)^2}{n}}.$$

An alternative way is to consider

$$MSE(m_0,m_1)=\sum_{i=1}^{n}(m_0+m_1\log K_i-\log X_i)^2.$$

We can get $\hat{m}_0,\hat{m}_1$ that minimize the mean square error such that

$$\hat{\nu}=e^{\hat{m}_0}$$

$$\hat{\beta}=\hat{m}_1+1.$$

For $a(x,\nu)$ with

$$a(x,\nu)=\nu K^{\beta-1}+(\frac{1}{2}(1-\beta)\nu K^{\beta-1}+\frac{1}{2}\alpha\rho)x+\cdots,$$

the mean square error is

$$MSE(\alpha,\rho)=\sum_{i=1}^{n}\left(\frac{1}{2}\alpha\rho x_i+\left(\hat{\nu}K_i^{\hat{\beta}-1}+\frac{1}{2}(1-\hat{\beta})\hat{\nu}K_i^{\hat{\beta}-1}x_i-X_i\right)\right)^2.$$

From this, we get

$$\hat{\alpha\rho} = -2\frac{\sum_{i=1}^{n}\left(\hat{\nu}K_i^{\hat{\beta}-1} + \frac{1}{2}(1-\hat{\beta})\hat{\nu}K_i^{\hat{\beta}-1}x_i - X_i\right)x_i}{\sum_{i=1}^{n} x_i^2}.$$

For $a(x,\nu)$ with

$$\begin{aligned} a(x,\nu) =& \nu K^{\beta-1} + (\frac{1}{2}(1-\beta)\nu K^{\beta-1} + \frac{1}{2}\alpha\rho)x \\ &+ (\frac{1}{12}(1-\beta)^2\nu K^{\beta-1} + \frac{1}{12}\alpha^2(2-3\rho^2)\nu^{-1}K^{1-\beta})x^2 + \cdots, \end{aligned}$$

the mean square error is

$$\begin{aligned} MSE(\alpha,\rho) =& \sum_{i=1}^{n}\Big(\frac{1}{6}\alpha^2\hat{\nu}^{-1}K_i^{1-\hat{\beta}}x_i^2 + \Big(\hat{\nu}K_i^{\hat{\beta}-1} + (\frac{1}{2}(1-\hat{\beta})\hat{\nu}K_i^{\hat{\beta}-1} + \frac{1}{2}(\hat{\alpha\rho}))x_i \\ &+ (\frac{1}{12}(1-\hat{\beta})^2\hat{\nu}K_i^{\hat{\beta}-1} - \frac{1}{4}(\hat{\alpha\rho})^2\hat{\nu}^{-1}K_i^{1-\hat{\beta}})x_i^2 - X_i\Big)\Big)^2. \end{aligned}$$

This equation is earned with the condition that $\alpha\rho = \hat{\alpha\rho}$. Since $\hat{\alpha\rho}$ is not a variable, this is actually a function of α only. Thus, by differentiating this with respect to α, we get

$$\begin{aligned} \hat{\alpha} =& \Big(-6\Big(\sum_{i=1}^{n}\hat{\nu}K_i^{\hat{\beta}-1}x_i^2\Big(\hat{\nu}K_i^{\hat{\beta}-1} + (\frac{1}{2}(1-\hat{\beta})\hat{\nu}K_i^{\hat{\beta}-1} + \frac{1}{2}(\hat{\alpha\rho}))x_i \\ &+ (\frac{1}{12}(1-\hat{\beta})^2\hat{\nu}K_i^{\hat{\beta}-1} - \frac{1}{4}(\hat{\alpha\rho})^2\hat{\nu}^{-1}K_i^{1-\hat{\beta}})x_i^2 - X_i\Big)\Big)\Big/\Big(\sum_{i=1}^{n}x_i^4\Big)\Big)^{\frac{1}{2}}. \end{aligned}$$

Since $\hat{\alpha\rho} = \hat{\alpha}\hat{\rho}$,

$$\hat{\rho} = \frac{\hat{\alpha\rho}}{\hat{\alpha}}.$$

By considering

$$\begin{aligned} MSE(m_2,m_3) =& \sum_{i=1}^{n}\Big(m_2x_i + m_3\hat{\nu}^{-1}K_i^{1-\hat{\beta}}x_i^2 + \Big(\hat{\nu}K_i^{\hat{\beta}-1} + \frac{1}{2}(1-\hat{\beta})\hat{\nu}K_i^{\hat{\beta}-1}x_i \\ &+ \frac{1}{12}(1-\hat{\beta})^2\hat{\nu}K_i^{\hat{\beta}-1}x_i^2 - X_i\Big)\Big)^2, \end{aligned}$$

we can get $\hat{m}_2, \hat{m}_3$ that minimize the mean square error such that

$$\hat{\alpha} = \sqrt{6\hat{m}_2^2 + 6\hat{m}_3}$$

$$\hat{\rho} = \frac{2\hat{m}_2}{\sqrt{6\hat{m}_2^2 + 6\hat{m}_3}}.$$

3.3 Local Volatility Models

Local volatility model follows the stochastic process

$$dS_t = rS_t + \xi(t, S_t)S_t dW_t.$$

The dynamics of $x_t = \log(\frac{K}{F_t})$ is

$$dx_t = \frac{1}{2}\xi(t, x_t)^2 dt - \xi(t, x_t)dW_t$$

such that

$$\begin{aligned} dC(t,x) &= C_t dt + C_x dx + \frac{1}{2}C_{xx}(dx)^2 \\ &= (C_t + \frac{1}{2}\xi(t,x)^2(C_x + C_{xx}))dt - \xi(t,x)C_x dW_t. \end{aligned}$$

Since dt term is zero due to the martingale property of C, we get

$$C_t + \frac{1}{2}\xi(t,x)^2(C_x + C_{xx}) = 0$$

such that

$$C_\tau = \frac{1}{2}\xi(\tau,x)^2(C_x + C_{xx})$$

with $\tau = T - t$. By substituting (2.1)-(2.3) to the above equation, we get

$$w_\tau = \frac{1}{2}\xi(\tau,x)^2(\frac{1}{w} - \frac{2x}{w^2}w_x + (\frac{x^2}{w^3} - \frac{w}{4})w_x^2 + w_{xx}).$$

Since $w(\tau, x) = \sigma(\tau, x)\sqrt{\tau}$,

$$2\sigma^3\sigma_\tau\tau + \sigma^4 = \xi(\tau,x)^2(\sigma^2 - 2x\sigma\sigma_x + x^2\sigma_x^2 - \frac{1}{4}\sigma^4\sigma_x^2\tau^2 + \sigma^3\sigma_{xx}\tau).$$

The squared local volatility $\xi(\tau, x)^2$ can be expressed as

$$\begin{aligned} \xi(\tau,x)^2 &= \frac{2\sigma^3\sigma_\tau\tau + \sigma^4}{\sigma^2 - 2x\sigma\sigma_x + x^2\sigma_x^2 - \frac{1}{4}\sigma^4\sigma_x^2\tau^2 + \sigma^3\sigma_{xx}\tau} \\ &= \frac{2\sigma^3\sigma_\tau\tau + \sigma^4}{(\sigma - x\sigma_x)^2 - \frac{1}{4}\sigma^4\sigma_x^2\tau^2 + \sigma^3\sigma_{xx}\tau} \\ &= \frac{2\sigma\sigma_\tau\tau + \sigma^2}{(1 - x\frac{\sigma_x}{\sigma})^2 - \frac{1}{4}\sigma^2\sigma_x^2\tau^2 + \sigma\sigma_{xx}\tau} \\ &= \frac{2\sigma\sigma_\tau\tau + \sigma^2}{(1 - x(\log\sigma)_x)^2 - \frac{1}{4}\sigma^2\sigma_x^2\tau^2 + \sigma\sigma_{xx}\tau}. \end{aligned}$$

Let the local volatility $\xi(\tau, x)$ as

$$\xi(\tau,x) = p(x) + q(x)\tau + r(x)\tau^2 + \cdots$$

where each $p(x), q(x), r(x)$ is of the form

$$p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots$$

$$q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots$$

$$r(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \cdots .$$

Let $\sigma(\tau, x)$ be

$$\sigma(\tau, x) = a(x) + b(x)\tau + c(x)\tau^2 + \cdots$$

and compare the coefficients of τ^i. Then, the PDE of $a(x)$ is

$$a^4 = p^2(a^2 - 2xaa_x + x^2 a_x^2) = p^2(a - xa_x)^2$$

such that

$$a^2 = p(a - xa_x).$$

By dividing both sides by a^2, this can be transformed to

$$1 = p(x)\left(\frac{x}{a(x)}\right)'$$

such that

$$a(x) = \frac{x}{\int_0^x \frac{1}{p(x)} dx}.$$

Even though this is a closed form formula, sometimes the integral of $\frac{1}{p(x)}$ is hard. Even in the case that the integral is easy, it might be hard to apply the MSE method. Therefore, considering another way is meaningful. One way is to regard $a(x)$ as $p(\frac{x}{2})$ since $\int_0^x \frac{1}{p(x)} dx \simeq \frac{x}{p(\frac{x}{2})}$. Another way is as usually letting $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and comparing the x^i coefficients. Then,

$$a_0 = p_0$$

$$a_1 = \frac{1}{2} p_1$$

$$a_2 = \frac{1}{3} p_2 - \frac{1}{12} \frac{p_1^2}{p_0}$$

$$a_3 = \frac{1}{4} p_3 - \frac{1}{6} \frac{p_1 p_2}{p_0} + \frac{1}{24} \frac{p_1^3}{p_0^2}$$

$$a_4 = \frac{1}{5} p_4 - \frac{3}{20} \frac{p_1 p_3}{p_0} - \frac{4}{45} \frac{p_2^2}{p_0} + \frac{23}{180} \frac{p_1^2 p_2}{p_0^2} - \frac{19}{720} \frac{p_1^4}{p_0^3}$$

$$a_5 = \frac{1}{6} p_5 - \frac{2}{15} \frac{p_1 p_4}{p_0} - \frac{1}{6} \frac{p_2 p_3}{p_0} + \frac{29}{240} \frac{p_1^2 p_3}{p_0^2} + \frac{133}{810} \frac{p_1 p_2^2}{p_0^2} - \frac{401}{3240} \frac{p_1^3 p_2}{p_0^3} + \frac{67}{3240} \frac{p_1^5}{p_0^4}.$$

Equivalently, there is a relation between the derivatives of a and p

$$a(0) = p(0)$$

$$a^{(1)}(0) = \frac{1}{2} p^{(1)}(0)$$

$$a^{(2)}(0) = \frac{1}{3}p^{(2)}(0) - \frac{1}{6}\frac{p^{(1)}(0)^2}{p(0)}$$

$$a^{(3)}(0) = \frac{1}{4}p^{(3)}(0) - \frac{1}{2}\frac{p^{(1)}(0)p^{(2)}(0)}{p(0)} + \frac{1}{4}\frac{p^{(1)}(0)^3}{p(0)^2}$$

$$a^{(4)}(0) = \frac{1}{5}p^{(4)}(0) - \frac{3}{5}\frac{p^{(1)}(0)p^{(3)}(0)}{p(0)} - \frac{8}{15}\frac{p^{(2)}(0)^2}{p(0)} + \frac{23}{15}\frac{p^{(1)}(0)^2p^{(2)}(0)}{p(0)^2} - \frac{19}{30}\frac{p^{(1)}(0)^4}{p(0)^3}$$

$$\begin{aligned} a^{(5)}(0) =& \frac{1}{6}p^{(5)}(0) - \frac{2}{3}\frac{p^{(1)}(0)p^{(4)}(0)}{p(0)} - \frac{5}{3}\frac{p^{(2)}(0)p^{(3)}(0)}{p(0)} + \frac{29}{12}\frac{p^{(1)}(0)^2p^{(3)}(0)}{p(0)^2} \\ &+ \frac{133}{27}\frac{p^{(1)}(0)p^{(2)}(0)^2}{p(0)^2} - \frac{401}{54}\frac{p^{(1)}(0)^3p^{(2)}(0)}{p(0)^3} + \frac{67}{27}\frac{p^{(1)}(0)^5}{p(0)^4}. \end{aligned}$$

For b terms, we get the PDE of the form

$$6a^3b = p^2(2ab - 2x(ab_x + a_xb) + 2x^2a_xb_x + a^3a_{xx}) + 2pq(a^2 - 2xaa_x + x^2a_x^2).$$

With $\phi(x) = \frac{b(x)}{a(x)}$, another equation can be derived. Let's go back to the equation

$$2\sigma\sigma_\tau\tau + \sigma^2 = \xi(\tau,x)^2\Big((1 - x(\log\sigma)_x)^2 - \frac{1}{4}\sigma^2\sigma_x^2\tau^2 + \sigma\sigma_{xx}\tau\Big).$$

Since

$$\log\sigma = \log(a + b\tau + \cdots) = \log a + \log(1 + \phi\tau + \cdots) = \log a + \phi\tau + \cdots$$

$$x(\log\sigma)_x = x(\frac{a_x}{a} + \phi_x\tau + \cdots)$$

holds, $(1 - x(\log\sigma)_x)^2$ is

$$\begin{aligned} (1 - x(\log\sigma)_x)^2 &= (1 - x\frac{a_x}{a} - x\phi_x\tau + \cdots)^2 \\ &= (1 - x\frac{a_x}{a})^2 - 2x\phi_x(1 - x\frac{a_x}{a})\tau + \cdots \\ &= \frac{a^2}{p^2} - 2x\phi_x\frac{a}{p}\tau + \cdots . \end{aligned}$$

The last equality is from the equation $1 - x\frac{a_x}{a} = \frac{a}{p}$. Now, the comparison of coefficients of τ^1 induces

$$\begin{aligned} 4a^2\phi &= p^2(-2x\phi_x\frac{a}{p} + aa_{xx}) + 2pq\frac{a^2}{p^2} \\ &= -2x\phi_xpa + p^2aa_{xx} + 2q\frac{a^2}{p}. \end{aligned}$$

Therefore, we get

$$4ap\phi + 2xp^2\phi_x = p^3a_{xx} + 2qa.$$

Equivalently,

$$x\phi_x + 2\frac{a}{p}\phi = \frac{1}{2}pa_{xx} + \frac{q}{p^2}a.$$

According to p.144 of [5], this equation has a solution of the form

$$\phi(x) = -\frac{1}{2}\Big(\frac{a(x)}{x}\Big)^2\Big(\log\big(\frac{a(x)^2}{p(x)p(0)}\big) - \int_0^x \frac{q(y)}{p(y)}\big(\frac{y^2}{a(y)^2}\big)' dy\Big).$$

Even though $\phi(x)$ has a closed form formula, it is useful to derive a polynomial series of $\phi(x)$,

$$\phi(x) = \phi_0 + \phi_1 x + \phi_2 x^2 + \phi_3 x^3 + \cdots ,$$

with

$$\phi_0 = \frac{1}{6}p_0p_2 - \frac{1}{24}p_1^2 + \frac{1}{2}\frac{q_0}{p_0}$$
$$\phi_1 = \frac{1}{4}p_0p_3 - \frac{1}{3}\frac{p_1q_0}{p_0^2} + \frac{1}{3}\frac{q_1}{p_0}.$$

When it is the case that the local volatility does not depend on τ, the local volatility is $\sigma(\tau, x) = p(x)$ and the calculation of ϕ_0, ϕ_1, ϕ_2, and ϕ_3 is easier. In this case, $q = 0$ such that the partial differential equation is

$$4ap\phi + 2xp^2\phi_x = p^3 a_{xx}$$

with

$$\phi_0 = \frac{1}{6}p_0p_2 - \frac{1}{24}p_1^2$$
$$\phi_1 = \frac{1}{4}p_0p_3$$
$$\phi_2 = \frac{3}{10}p_0p_4 + \frac{1}{40}p_1p_3 + \frac{1}{180}p_2^2 - \frac{1}{360}\frac{p_1^2p_2}{p_0} + \frac{23}{576}\frac{p_1^4}{p_0^2}$$
$$\phi_3 = \frac{1}{3}p_0p_5 + \frac{1}{30}p_1p_4 - \frac{1}{240}\frac{p_1^2p_3}{p_0} + \frac{217}{180}\frac{p_1p_2^2}{p_0} - \frac{91}{3240}\frac{p_1^3p_2}{p_0^2} - \frac{2623}{129600}\frac{p_1^5}{p_0^3}.$$

Now, under the assumption that the local volatility does not depend on τ, let's calculate τ^2 asymptotic implied volatility terms. With $\psi(x) = \frac{c(x)}{a(x)}$,

$$\begin{aligned}\log\sigma &= \log(a + b\tau + c\tau^2 + \cdots) \\ &= \log a + \log(1 + \phi\tau + \psi\tau^2 + \cdots) \\ &= \log a + (\phi\tau + \psi\tau^2 + \cdots) - \frac{1}{2}(\phi\tau + \psi\tau^2 + \cdots)^2 + \cdots \\ &= \log a + \phi\tau + (\psi - \frac{1}{2}\phi^2)\tau^2 + \cdots\end{aligned}$$

and $x(\log\sigma)_x$ is

$$x(\log\sigma)_x = x(\frac{a_x}{a} + \phi_x\tau + (\psi_x - \phi\phi_x)\tau^2 + \cdots).$$

Therefore, $(1 - x(\log\sigma)_x)^2$ is

$$\begin{aligned}&(1 - x(\log\sigma)_x)^2 = (1 - x\frac{a_x}{a} - x\phi_x\tau - x(\psi_x - \phi\phi_x)\tau^2 + \cdots)^2 \\ &= (1 - x\frac{a_x}{a})^2 - 2x\phi_x(1 - x\frac{a_x}{a})\tau + \Big(-2x(\psi_x - \phi\phi_x)(1 - x\frac{a_x}{a}) + x^2\phi_x^2\Big)\tau^2 + \cdots \\ &= \frac{a^2}{p^2} - 2x\phi_x\frac{a}{p}\tau + \Big(-2x(\psi_x - \phi\phi_x)\frac{a}{p} + x^2\phi_x^2\Big)\tau^2 + \cdots\end{aligned}$$

such that

$$6a^2\psi + 3a^2\phi^2 = p^2(-2x(\psi_x - \phi\phi_x)\frac{a}{p} + x^2\phi_x^2 - \frac{1}{4}a^2a_x^2 + aa_{xx}\phi + a(a\phi)_{xx}).$$

For $\psi(x)$ with

$$\psi(x) = \psi_0 + \psi_1 x + \cdots ,$$

we get

$$\psi_0 = -\frac{1}{96}p_0^2p_1^2 + \frac{1}{40}p_0^2p_2^2 - \frac{1}{80}p_0p_1^2p_2 + \frac{17}{1152}p_1^4 + \frac{1}{10}p_0^3p_4 + \frac{1}{20}p_0^2p_1p_3$$

$$\psi_1 = -\frac{1}{96}p_0p_1^3 - \frac{1}{48}p_0^2p_1p_2 + \frac{109}{120}p_0p_1p_2^2 - \frac{5}{216}p_1^3p_2 + \frac{319}{21600}\frac{p_1^5}{p_0} + \frac{1}{4}p_0^2p_1p_4 + \frac{7}{192}p_0p_1^2p_3 + \frac{5}{48}p_0^2p_2p_3 + \frac{1}{4}p_0^3p_5 + \frac{1}{96}p_0p_1p_2p_3 - \frac{1}{384}p_1^3p_3.$$

3.3.1 The CEV Model

Under the CEV model, a stock price follows the stochastic process

$$dS_t = rS_tdt + S_t^\gamma dW_t.$$

This model has a local volatility $\xi(t, S_t) = S_t^{\gamma-1}$. With $x_t = \log(\frac{K}{F_t})$, we get the following local volatility

$$\xi(t, x_t) = K^{\gamma-1}e^{(1-\gamma)x_t}e^{(1-\gamma)r(T-t)}.$$

With $\tau = T - t$, we get

$$\xi(\tau, x_t) = K^{\gamma-1}e^{r(1-\gamma)\tau}e^{(1-\gamma)x_t}.$$

By expressing the CEV model local volatility as a polynomial series,

$$\xi(\tau, x) = K^{\gamma-1}e^{(1-\gamma)x}\Big(1 + r(1-\gamma)\tau + \frac{1}{2}r^2(1-\gamma)^2\tau^2 + \cdots\Big)$$

with

$$p(x) = K^{\gamma-1}e^{(1-\gamma)x}$$

$$q(x) = r(1-\gamma)K^{\gamma-1}e^{(1-\gamma)x}$$

$$r(x) = \frac{1}{2}r^2(1-\gamma)^2K^{\gamma-1}e^{(1-\gamma)x}.$$

By inserting $p(x)$ to the closed form formula of $a(x)$, we get

$$a(x) = \frac{(1-\gamma)K^{\gamma-1}x}{(1-e^{-(1-\gamma)x})}.$$

By the formula of $\phi(x)$,

$$\phi(x) = -\frac{1}{2}\Big(\frac{(1-\gamma)K^{\gamma-1}}{1-e^{-(1-\gamma)x}}\Big)^2\Big(\log\Big(\frac{(1-\gamma)^2x^2}{e^{(1-\gamma)x}(1-e^{-(1-\gamma)x})^2}\Big) - \frac{r}{(1-\gamma)K^{2\gamma-2}}(1-e^{-(1-\gamma)x})^2\Big).$$

Since $b(x) = a(x)\phi(x)$, we get

$$b(x) = -\frac{1}{2}x\Big(\frac{(1-\gamma)K^{\gamma-1}}{1-e^{-(1-\gamma)x}}\Big)^3\Big(\log\Big(\frac{(1-\gamma)^2x^2}{e^{(1-\gamma)x}(1-e^{-(1-\gamma)x})^2}\Big)-\frac{r}{(1-\gamma)K^{2\gamma-2}}(1-e^{-(1-\gamma)x})^2\Big).$$

Note that $r = 0$ in the CEV model implies that $\xi(\tau, x) = K^{\gamma-1}e^{(1-\gamma)x}$ is not dependent on τ. In this case,

$$p(x) = K^{\gamma-1}e^{(1-\gamma)x}$$

with

$$p_0 = K^{\gamma-1}, \qquad p_1 = (1-\gamma)K^{\gamma-1}, \qquad p_2 = \frac{1}{2}(1-\gamma)^2K^{\gamma-1}$$

$$p_3 = \frac{1}{6}(1-\gamma)^3K^{\gamma-1}, \qquad p_4 = \frac{1}{24}(1-\gamma)^4K^{\gamma-1}, \qquad p_5 = \frac{1}{120}(1-\gamma)^5K^{\gamma-1}.$$

With these, we get

$$\psi_0 = K^{4\gamma-4}\Big(-\frac{1}{96}(1-\gamma)^2 + \frac{157}{5760}(1-\gamma)^4\Big)$$

$$\psi_1 = K^{4\gamma-4}\Big(-\frac{1}{48}(1-\gamma)^3 + \frac{7417}{28800}(1-\gamma)^5 + \frac{1}{2304}(1-\gamma)^6\Big)$$

and

$$c_0 = \frac{K^{5\gamma-5}x}{(1-e^{-(1-\gamma)x})}\Big(-\frac{1}{96}(1-\gamma)^3 + \frac{157}{5760}(1-\gamma)^5\Big)$$

$$c_1 = \frac{K^{5\gamma-5}x}{(1-e^{-(1-\gamma)x})}\Big(-\frac{1}{48}(1-\gamma)^4 + \frac{7417}{28800}(1-\gamma)^6 + \frac{1}{2304}(1-\gamma)^7\Big).$$

The log version MSE on a_0 is

$$MSE(\gamma) = \sum_{i=1}^{n}((\gamma-1)\log K_i - \log X_i)^2.$$

By differentiating with respect to γ, we get

$$\hat{\gamma} = 1 + \frac{\sum_{i=1}^{n}(\log K_i \log X_i)}{\sum_{i=1}^{n}(\log K_i)^2}.$$

In the case of $\xi(t, S_t) = \xi S_t^{\gamma-1}$, the MSE is

$$MSE(\xi, \gamma) = \sum_{i=1}^{n}(\log\xi + (\gamma-1)\log K_i - \log X_i)^2$$

such that

$$\hat{\xi} = \exp\Big(\frac{(\sum_{i=1}^{n}\frac{\log K_i}{n})(\sum_{i=1}^{n}\frac{\log K_i\log X_i}{n}) - (\sum_{i=1}^{n}\frac{\log X_i}{n})(\sum_{i=1}^{n}\frac{(\log K_i)^2}{n})}{(\sum_{i=1}^{n}\frac{\log K_i}{n})^2 - \sum_{i=1}^{n}\frac{(\log K_i)^2}{n}}\Big)$$

$$\hat{\gamma} = 1 + \frac{(\sum_{i=1}^{n}\frac{\log K_i}{n})(\sum_{i=1}^{n}\frac{\log X_i}{n}) - \sum_{i=1}^{n}\frac{\log K_i\log X_i}{n}}{(\sum_{i=1}^{n}\frac{\log K_i}{n})^2 - \sum_{i=1}^{n}\frac{(\log K_i)^2}{n}}.$$

3.3.2 Polynomial Model I

a polynomial model has a polynomial local volatility

$$\xi(\tau, x) = \alpha + \beta x + \gamma x^2 + \delta\tau + \theta x\tau.$$

In this model, $p(x)$ and $q(x)$ are

$$p(x) = \alpha + \beta x + \gamma x^2$$

$$q(x) = \delta + \theta x$$

such that $p_0 = \alpha, p_1 = \beta, p_2 = \gamma$ and $q_0 = \delta, q_1 = \theta$. Therefore, we get

$$a_0 = \alpha$$

$$a_1 = \frac{1}{2}\beta$$

$$a_2 = \frac{1}{3}\gamma - \frac{1}{12}\frac{\beta^2}{\alpha}$$

$$a_3 = -\frac{1}{6}\frac{\beta\gamma}{\alpha} + \frac{1}{24}\frac{\beta^3}{\alpha^2}$$

$$a_4 = -\frac{4}{45}\frac{\gamma^2}{\alpha} + \frac{23}{180}\frac{\beta^2\gamma}{\alpha^2} - \frac{19}{720}\frac{\beta^4}{\alpha^3}$$

$$a_5 = \frac{133}{810}\frac{\beta\gamma^2}{\alpha^2} - \frac{401}{3240}\frac{\beta^3\gamma}{\alpha^3} + \frac{67}{3240}\frac{\beta^5}{\alpha^4}.$$

MSE on a_0 is

$$MSE(\alpha) = \sum_{i=1}^{n}(\alpha - X_i)^2$$

such that

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The first order approximation of $a(x)$ is

$$a(x) = \alpha + \frac{1}{2}\beta x + \cdots$$

such that

$$MSE(\beta) = \sum_{i=1}^{n}(\hat{\alpha} + \frac{1}{2}\beta x_i - X_i)^2 = \sum_{\imath=1}^{n}(\frac{1}{2}\beta x_i + (\hat{\alpha} - X_i))^2.$$

By differentiating this with respect to β, we get

$$\hat{\beta} = -2\frac{\sum_{i=1}^{n} x_i(\hat{\alpha} - X_i)}{\sum_{i=1}^{n} x_i^2}.$$

The second order approximation of $a(x)$ is

$$a(x) = \alpha + \frac{1}{2}\beta x + (\frac{1}{3}\gamma - \frac{1}{12}\frac{\beta^2}{\alpha})x^2 + \cdots$$

such that

$$MSE(\gamma) = \sum_{i=1}^{n}(\hat{\alpha} + \frac{1}{2}\hat{\beta}x_i + (\frac{1}{3}\gamma - \frac{1}{12}\frac{\hat{\beta}^2}{\hat{\alpha}})x_i^2 - X_i)^2$$
$$= \sum_{i=1}^{n}(\frac{1}{3}\gamma x_i^2 + (\hat{\alpha} + \frac{1}{2}\hat{\beta}x_i - \frac{1}{12}\frac{\hat{\beta}^2}{\hat{\alpha}}x_i^2 - X_i))^2.$$

By differentiating this with respect to γ, we get

$$\hat{\gamma} = -3\frac{\sum_{i=1}^{n}(\hat{\alpha} + \frac{1}{2}\hat{\beta}x_i - \frac{1}{12}\frac{\hat{\beta}^2}{\hat{\alpha}}x_i^2 - X_i)x_i^2}{\sum_{i=1}^{n}x_i^4}.$$

Now, let's go to the case of $b(x)$. Since $b = a\phi$,

$$b_0 = a_0\phi_0 = \frac{1}{2}q_0 + \frac{1}{6}p_0^2p_2 - \frac{1}{24}p_0p_1^2$$
$$b_1 = a_0\phi_1 + a_1\phi_0 = \frac{1}{3}q_1 - \frac{1}{12}q_0\frac{p_1}{p_0} + \frac{1}{4}p_0^2p_3 + \frac{1}{12}p_0p_1p_2 - \frac{1}{48}p_1^3.$$

Therefore, we get

$$b_0 = \frac{1}{2}\delta + \frac{1}{6}\alpha^2\gamma - \frac{1}{24}\alpha\beta^2$$
$$b_1 = \frac{1}{3}\theta - \frac{1}{12}\delta\frac{\beta}{\alpha} + \frac{1}{12}\alpha\beta\gamma - \frac{1}{48}\beta^3.$$

The zero order approximation of the implied volatility including τ terms is

$$a(x) + b(x)\tau = \alpha + \frac{1}{2}\beta x + (\frac{1}{3}\gamma - \frac{1}{12}\frac{\beta^2}{\alpha})x^2 + (\frac{1}{2}\delta + \frac{1}{6}\alpha^2\gamma - \frac{1}{24}\alpha\beta^2)\tau + \cdots$$

such that

$$MSE(\delta) = \sum_{i=1}^{n}\Big(\frac{1}{2}\delta\tau_i + (\hat{\alpha} + \frac{1}{2}\hat{\beta}x_i + (\frac{1}{3}\hat{\gamma} - \frac{1}{12}\frac{\hat{\beta}^2}{\hat{\alpha}})x_i^2 + (\frac{1}{6}\hat{\alpha}^2\hat{\gamma} - \frac{1}{24}\hat{\alpha}\hat{\beta}^2)\tau_i - X_i)\Big)^2.$$

By differentiating this with respect to δ, we get

$$\hat{\delta} = -2\frac{\sum_{i=1}^{n}\tau_i\Big(\hat{\alpha} + \frac{1}{2}\hat{\beta}x_i + (\frac{1}{3}\hat{\gamma} - \frac{1}{12}\frac{\hat{\beta}^2}{\hat{\alpha}})x_i^2 + (\frac{1}{6}\hat{\alpha}^2\hat{\gamma} - \frac{1}{24}\hat{\alpha}\hat{\beta}^2)\tau_i - X_i\Big)}{\sum_{i=1}^{n}\tau_i^2}.$$

Now, the first order approximation of the implied volatility including τ terms is

$$a(x) + b(x)\tau = \alpha + \frac{1}{2}\beta x + (\frac{1}{3}\gamma - \frac{1}{12}\frac{\beta^2}{\alpha})x^2 + (\frac{1}{2}\delta + \frac{1}{6}\alpha^2\gamma - \frac{1}{24}\alpha\beta^2)\tau$$
$$+ (\frac{1}{3}\theta - \frac{1}{12}\delta\frac{\beta}{\alpha} + \frac{1}{12}\alpha\beta\gamma - \frac{1}{48}\beta^3)x\tau + \cdots$$

such that

$$MSE(\theta) = \sum_{i=1}^{n}\Big(\frac{1}{3}\theta x_i\tau_i + \Big(\hat{\alpha} + \frac{1}{2}\hat{\beta}x_i + (\frac{1}{3}\hat{\gamma} - \frac{1}{12}\frac{\hat{\beta}^2}{\hat{\alpha}})x_i^2 + (\frac{1}{2}\hat{\delta} + \frac{1}{6}\hat{\alpha}^2\hat{\gamma} - \frac{1}{24}\hat{\alpha}\hat{\beta}^2)\tau_i$$
$$+ (-\frac{1}{12}\hat{\delta}\frac{\hat{\beta}}{\hat{\alpha}} + \frac{1}{12}\hat{\alpha}\hat{\beta}\hat{\gamma} - \frac{1}{48}\hat{\beta}^3)x_i\tau_i - X_i\Big)\Big)^2.$$

Therefore, we can estimate θ as

$$\hat{\theta} = -3\sum_{i=1}^{n}\Big(\hat{\alpha} + \frac{1}{2}\hat{\beta}x_i + (\frac{1}{3}\hat{\gamma} - \frac{1}{12}\frac{\hat{\beta}^2}{\hat{\alpha}})x_i^2 + (\frac{1}{2}\hat{\delta} + \frac{1}{6}\hat{\alpha}^2\hat{\gamma} - \frac{1}{24}\hat{\alpha}\hat{\beta}^2)\tau_i$$
$$+ (-\frac{1}{12}\hat{\delta}\frac{\hat{\beta}}{\hat{\alpha}} + \frac{1}{12}\hat{\alpha}\hat{\beta}\hat{\gamma} - \frac{1}{48}\hat{\beta}^3)x_i\tau_i - X_i\Big)x_i\tau_i\Big/\Big(\sum_{i=1}^{n}x_i^2\tau_i^2\Big).$$

Equivalently, consider

$$m_0 + m_1 x + m_2 x^2 + m_3\tau + m_4 x\tau$$

such that

$$MSE(m_0, m_1, m_2, m_3, m_4) = \sum_{i=1}^{n}\Big(m_0 + m_1 x_i + m_2 x_i^2 + m_3\tau_i + m_4 x_i\tau_i - X_i\Big)^2.$$

By differentiating this with respect to each of m_i, $\hat{m}_0, \hat{m}_1, \hat{m}_2, \hat{m}_3$, and $\hat{m}_4$ are earned such that

$$\hat{\alpha} = \hat{m}_0$$
$$\hat{\beta} = 2\hat{m}_1$$
$$\hat{\gamma} = 3\hat{m}_2 + \frac{\hat{m}_1^2}{\hat{m}_0}$$
$$\hat{\delta} = 2\hat{m}_3 - \hat{m}_0^2\hat{m}_2$$
$$\hat{\theta} = 3\hat{m}_4 + \frac{\hat{m}_1\hat{m}_3}{\hat{m}_0} - 2\hat{m}_0\hat{m}_1\hat{m}_2.$$

3.3.3 Polynomial Model II

Another polynomial model has a polynomial local volatility

$$\xi(\tau, x) = \alpha + \beta x + \gamma x^2 + \delta x^3 + \theta x^4.$$

In this case, $p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3 + \theta x^4$ such that $p_0 = \alpha, p_1 = \beta, p_2 = \gamma, p_3 = \delta, p_4 = \theta$. Therefore, we get

$$a_0 = \alpha$$
$$a_1 = \frac{1}{2}\beta$$
$$a_2 = \frac{1}{3}\gamma - \frac{1}{12}\frac{\beta^2}{\alpha}$$
$$a_3 = \frac{1}{4}\delta - \frac{1}{6}\frac{\beta\gamma}{\alpha} + \frac{1}{24}\frac{\beta^3}{\alpha^2}$$
$$a_4 = \frac{1}{5}\theta - \frac{3}{20}\frac{\beta\delta}{\alpha} - \frac{4}{45}\frac{\gamma^2}{\alpha} + \frac{23}{180}\frac{\beta^2\gamma}{\alpha^2} - \frac{19}{720}\frac{\beta^4}{\alpha^3}.$$

Now, let's find the 4th order polynomial that fits the data most. The MSE is

$$MSE(m_0, m_1, m_2, m_3, m_4) = \sum_{i=1}^{n}\Big(m_0 + m_1 x_i + m_2 x_i^2 + m_3 x_i^3 + m_4 x_i^4 - X_i\Big)^2.$$

By differentiation, we get $\hat{m}_0, \hat{m}_1, \hat{m}_2, \hat{m}_3$, and $\hat{m}_4$ such that

$$\hat{\alpha} = \hat{m}_0$$

$$\hat{\beta} = 2\hat{m}_1$$

$$\hat{\gamma} = 3\hat{m}_2 + \frac{\hat{m}_1^2}{\hat{m}_0}$$

$$\hat{\delta} = 4\hat{m}_3 + 4\frac{\hat{m}_1\hat{m}_2}{\hat{m}_0}$$

$$\hat{\theta} = 5\hat{m}_4 + 6\frac{\hat{m}_1\hat{m}_3}{\hat{m}_0} + 4\frac{\hat{m}_2^2}{\hat{m}_0} + \frac{\hat{m}_1^2\hat{m}_2}{\hat{m}_0^2}.$$

3.4 Asian Options under the Black-Scholes Model

The Black-Scholes model is

$$dS_t = rS_t dt + \xi S_t dW_t.$$

Under this model, the implied volatility of Asian options can be derived when r is zero. Note that the Asian call option price with zero short rate, strike K, and maturity T is given by

$$E\Big(\Big(\frac{1}{T}\int_0^T S_t dt - K\Big)_+\Big).$$

According to [1] and [2] that are also mentioned in p.2 of [8], the process $\int_0^T S_t dt$ has the same distribution with $S_0 Y_T$ where the dynamics of Y_T is

$$dY_T = dT + \xi Y_T dW_T$$

such that the Asian call option price can also be written as

$$E\Big(\Big(\frac{S_0 Y_T}{T} - K\Big)_+\Big).$$

Under the implied volatility, this equals to

$$S_0 N(d_1) - KN(d_2)$$

since

$$E\Big(\frac{1}{T}\int_0^T S_t dt\Big) = \frac{1}{T}\int_0^T E(S_t)dt = S_0.$$

By equating both and dividing by S_0,

$$E\Big(\Big(\frac{Y_T}{T} - \frac{K}{S_0}\Big)_+\Big) = N(d_1) - \frac{K}{S_0}N(d_2).$$

With $X = \log(\frac{K}{S_0})$,

$$E\Big(\Big(\frac{Y_T}{T} - e^X\Big)_+\Big) = N(d_1) - e^X N(d_2).$$

Now, define C as

$$C(T,X) = E\Big(\Big(\frac{Y_T}{T} - e^X\Big)_+\Big)$$

and define C^{BS} as

$$C^{BS}(X,W) = N(d_1) - e^X N(d_2)$$

where d_1 and d_2 are

$$d_1 = -\frac{X}{W} + \frac{1}{2}W$$
$$d_2 = -\frac{X}{W} - \frac{1}{2}W$$

with $X = \log(\frac{K}{S_0})$ and $W = \sigma(T,X)\sqrt{T}$. Let's define a function F as

$$F(T,Y) = \Big(\frac{Y}{T} - e^X\Big)_+$$

with X fixed. Then, the dynamics is

$$dF = F_T dT + F_Y dY + \frac{1}{2}F_{YY}(dY)^2$$

where F_T, F_Y, F_{YY} are given by

$$F_T = -\frac{Y}{T^2}\theta\Big(\frac{Y}{T} - e^X\Big)$$
$$F_Y = \frac{1}{T}\theta\Big(\frac{Y}{T} - e^X\Big)$$
$$F_{YY} = \frac{1}{T^2}\delta\Big(\frac{Y}{T} - e^X\Big).$$

Note that θ is the Heaviside step function and δ is the Dirac delta function. This technique is introduced in pp.13-14 of [3]. By inserting these values to dF and deleting the dW_t term,

$$\begin{aligned} d\Big(\frac{Y}{T} - e^X\Big)_+ =& \Big(-\frac{1}{T}\Big(\frac{Y}{T} - e^X\Big)\theta\Big(\frac{Y}{T} - e^X\Big) - \frac{e^X}{T}\theta\Big(\frac{Y}{T} - e^X\Big) \\ &+ \frac{1}{T}\theta\Big(\frac{Y}{T} - e^X\Big) + \frac{\xi^2}{2}\frac{Y^2}{T^2}\delta\Big(\frac{Y}{T} - e^X\Big)\Big)dT \end{aligned}$$

holds. The dW_t term is deleted since it is zero under the expectation that is to happen in the next step. By taking E on both sides of dF and applying the rule $E(dF) = d(E(F))$, we get

$$\frac{dC}{dT} = -\frac{1}{T}E\Big(\Big(\frac{Y}{T} - e^X\Big)\theta\Big(\frac{Y}{T} - e^X\Big)\Big) + \frac{1-e^X}{T}E\Big(\theta\Big(\frac{Y}{T} - e^X\Big)\Big) + \frac{\xi^2}{2T^2}E\Big(Y^2\delta\Big(\frac{Y}{T} - e^X\Big)\Big).$$

Since

$$E\Big(\Big(\frac{Y}{T} - e^X\Big)\theta\Big(\frac{Y}{T} - e^X\Big)\Big) = E\Big(\Big(\frac{Y}{T} - e^X\Big)_+\Big) = C$$
$$E\Big(\theta\Big(\frac{Y}{T} - e^X\Big)\Big) = -C_X e^{-X}$$

$$E\Big(\delta\Big(\frac{Y}{T}-e^X\Big)\Big)=C_{XX}e^{-2X}-C_Xe^{-2X}$$

holds, the PDE of C can be earned

$$C_T+\frac{1}{T}C=\frac{1}{T}C_X-\frac{1}{T}e^{-X}C_X+\frac{1}{2}\xi^2(C_{XX}-C_X).$$

Let's multiply both sides by T and differentiate both sides by X. Then, we get

$$TC_{TX}=(1-e^{-X})(C_{XX}-C_X)+\frac{1}{2}\xi^2T(C_{XXX}-C_{XX}). \tag{3.1}$$

Now, it is necessary to calculate the derivatives of C in terms of C^{BS} as

$$C_X=C^{BS}_X+C^{BS}_WW_X$$

$$C_{XX}=C^{BS}_{XX}+2C^{BS}_{XW}W_X+C^{BS}_{WW}(W_X)^2+C^{BS}_WW_{XX}$$

$$\begin{aligned}C_{XXX}=&C^{BS}_{XXX}+3C^{BS}_{XXW}W_X+3C^{BS}_{XWW}(W_X)^2+C^{BS}_{WWW}(W_X)^3\\&+3C^{BS}_{XW}W_{XX}+3C^{BS}_{WW}W_XW_{XX}+C^{BS}_WW_{XXX}\end{aligned}$$

$$C_{TX}=C^{BS}_WW_{TX}+C^{BS}_{WX}W_T+C^{BS}_{WW}W_XW_T.$$

To calculate above values, we need to know the values C^{BS}_X, C^{BS}_W, C^{BS}_{XX}, C^{BS}_{XW}, C^{BS}_{WW}, C^{BS}_{XXX}, C^{BS}_{XXW}, C^{BS}_{XWW}, C^{BS}_{WWW}

$$C^{BS}_W=e^XN'(d_2),\quad C^{BS}_X=-e^XN(d_2)$$

$$C^{BS}_{WW}=C^{BS}_W(X^2W^{-3}-\frac{1}{4}W),\quad C^{BS}_{WX}=C^{BS}_W(\frac{1}{2}-XW^{-2}),\quad C^{BS}_{XX}=C^{BS}_X+C^{BS}_WW^{-1}$$

$$C^{BS}_{WWW}=C^{BS}_W(X^4W^{-6}-\frac{1}{2}X^2W^{-2}+\frac{1}{16}W^2-3X^2W^{-4}-\frac{1}{4})$$

$$C^{BS}_{WWX}=C^{BS}_W(\frac{1}{2}X^2W^{-3}-X^3W^{-5}-\frac{1}{8}W+\frac{1}{4}XW^{-1}+2XW^{-3})$$

$$C^{BS}_{WXX}=C^{BS}_W(-W^{-2}+\frac{1}{4}-XW^{-2}+X^2W^{-4})$$

$$C^{BS}_{XXX}=C^{BS}_X+C^{BS}_W(\frac{3}{2}W^{-1}-XW^{-3}).$$

Now, let's get the values of $C_X, C_{XX}, C_{XXX}, C_{TX}$ in terms of C^{BS}_X and C^{BS}_W

$$C_X=C^{BS}_X+C^{BS}_WW_X$$

$$C_{XX}=C^{BS}_X+C^{BS}_W(W^{-1}+(1-2XW^{-2})W_X+(X^2W^{-3}-\frac{1}{4}W)(W_X)^2+W_{XX})$$

$$\begin{aligned}C_{XXX}=&C^{BS}_X+C^{BS}_W((\frac{3}{2}W^{-1}-XW^{-3})+(-3W^{-2}+\frac{3}{4}-3XW^{-2}+3X^2W^{-4})W_X\\&+(\frac{3}{2}X^2W^{-3}-3X^3W^{-5}-\frac{3}{8}W+\frac{3}{4}XW^{-1}+6XW^{-3})(W_X)^2\\&+(\frac{3}{2}-3XW^{-2})W_{XX}+(X^4W^{-6}-\frac{1}{2}X^2W^{-2}+\frac{1}{16}W^2-3X^2W^{-4}\\&-\frac{1}{4})(W_X)^3+(3X^2W^{-3}-\frac{3}{4}W)W_XW_{XX}+W_{XXX})\end{aligned}$$

$$C_{TX} = C_W^{BS}((\frac{1}{2} - XW^{-2})W_T + (X^2W^{-3} - \frac{1}{4}W)W_XW_T + W_{TX}).$$

By substituting the above C_X, C_{XX}, C_{XXX}, and C_{TX} formulas to (3.1), we get

$$\begin{aligned}
&T((\frac{1}{2} - XW^{-2})W_T + (X^2W^{-3} - \frac{1}{4}W)W_XW_T + W_{TX}) = (1-e^{-X})(W^{-1} - 2XW^{-2}W_X \\
&+ (X^2W^{-3} - \frac{1}{4}W)(W_X)^2 + W_{XX}) + \frac{1}{2}\xi^2 T\Big(- W^{-1} - (1-2XW^{-2})W_X - (X^2W^{-3} \\
&- \frac{1}{4}W)(W_X)^2 - W_{XX} + (\frac{3}{2}W^{-1} - XW^{-3}) + (-3W^{-2} + \frac{3}{4} - 3XW^{-2} + 3X^2W^{-4})W_X \\
&+ (\frac{3}{2}X^2W^{-3} - 3X^3W^{-5} - \frac{3}{8}W + \frac{3}{4}XW^{-1} + 6XW^{-3})(W_X)^2 + (\frac{3}{2} - 3XW^{-2})W_{XX} \\
&+ (X^4W^{-6} - \frac{1}{2}X^2W^{-2} + \frac{1}{16}W^2 - 3X^2W^{-4} - \frac{1}{4})(W_X)^3 + (3X^2W^{-3} \\
&- \frac{3}{4}W)W_XW_{XX} + W_{XXX}\Big).
\end{aligned}$$

With $W = \sigma(X,T)\sqrt{T}$ and multiplying both sides by $\sigma(X,T)^6\sqrt{T}$, we get

$$\begin{aligned}
&\frac{1}{4}\sigma^7 T - \frac{1}{2}x\sigma^5 + \frac{1}{2}\sigma^6\sigma_T T^2 - x\sigma^4\sigma_T T + \frac{1}{2}x^2\sigma^4\sigma_X - \frac{1}{8}\sigma^8\sigma_X T^2 + x^2\sigma^3\sigma_X\sigma_T T - \frac{1}{4}\sigma^7\sigma_X\sigma_T T^3 \\
&+ \frac{1}{2}\sigma^6\sigma_X T + \sigma^6\sigma_{XT}T^2 = (1-e^{-X})(\sigma^5 - 2x\sigma^4\sigma_X + x^2\sigma^3\sigma_X^2 - \frac{1}{4}\sigma^7\sigma_X^2 T^2 + \sigma^6\sigma_{XX}T) \\
&+ \frac{1}{2}\xi^2\Big(- \sigma^5 T - \sigma^6\sigma_X T^2 + 2x\sigma^4\sigma_X T - x^2\sigma^3\sigma_X^2 T + \frac{1}{4}\sigma^7\sigma_X^2 T^3 - \sigma^6\sigma_{XX}T^2 + \frac{3}{2}\sigma^5 T - x\sigma^3 \\
&- 3\sigma^4\sigma_X T + \frac{3}{4}\sigma^6\sigma_X T^2 - 3x\sigma^4\sigma_X T + 3x^2\sigma^2\sigma_X + \frac{3}{2}x^2\sigma^3\sigma_X^2 T - 3x^3\sigma\sigma_X^2 - \frac{3}{8}\sigma^7\sigma_X^2 T^3 \\
&+ \frac{3}{4}x\sigma^5\sigma_X^2 T^2 + 6x\sigma^3\sigma_X^2 T + \frac{3}{2}\sigma^6\sigma_{XX}T^2 - 3x\sigma^4\sigma_{XX}T + x^4\sigma_X^3 - \frac{1}{2}x^2\sigma^4\sigma_X^3 T^2 + \frac{1}{16}\sigma^8\sigma_X^3 T^4 \\
&- 3x^2\sigma^2\sigma_X^3 T - \frac{1}{4}\sigma^6\sigma_X^3 T^3 + 3x^2\sigma^3\sigma_X\sigma_{XX}T - \frac{3}{4}\sigma^7\sigma_X\sigma_{XX}T^3 + \sigma^6\sigma_{XXX}T^2\Big).
\end{aligned}$$

Let $\sigma(X,T)$ be

$$\sigma(X,T) = a(X) + b(X)T + c(X)T^2 + \cdots .$$

By comparing the T^0 coefficients, we get

$$\begin{aligned}
-\frac{1}{2}Xa^5 + \frac{1}{2}X^2a^4a_X =&(1-e^{-X})(a^5 - 2Xa^4a_X + X^2a^3a_X^2) \\
&+ \frac{1}{2}\xi^2(-Xa^3 + 3X^2a^2a_X - 3X^3aa_X^2 + X^4a_X^3)
\end{aligned}$$

with

$$\begin{aligned}
a_0 &= \frac{1}{\sqrt{3}}\xi \\
a_1 &= \frac{1}{10}a_0 \\
a_2 &= -\frac{23}{2100}a_0 \\
a_3 &= \frac{1}{3500}a_0.
\end{aligned}$$

This result is the same as the equation (55) in p.11 of [8]. By comparing the T^1 coefficients, we get

$$\frac{1}{4}a^7-\frac{7}{2}Xa^4b+\frac{1}{2}X^2a^4b_X+3X^2a^3a_Xb+\frac{1}{2}a^6a_X=(1-e^{-X})(5a^4b-2X(a^4b_X+4a^3a_Xb)$$
$$+X^2(2a^3a_Xb_X+3a^2a_X^2b)+a^6a_{XX})+\frac{1}{2}\xi^2\Big(-a^5+2Xa^4a_X-X^2a^3a_X^2+\frac{3}{2}a^5-3Xa^2b$$
$$-3a^4a_X-3Xa^4a_X+3X^2(2aa_Xb+a^2b_X)+\frac{3}{2}X^2a^3a_X^2-3X^3(2aa_Xb_X+a_X^2b)$$
$$+6Xa^3a_X^2-3Xa^4a_{XX}+3X^4a_X^2b_X-3X^2a^2a_X^3+3X^2a^3a_Xa_{XX}\Big)$$

with

$$b_0=-\frac{61}{2100}a_0^3$$
$$b_1=-\frac{33}{1400}a_0^3.$$

Here, we can price Asian options with the asymptotic implied volatility

$$a(x)=\frac{1}{\sqrt{3}}\xi(1+\frac{1}{10}x-\frac{23}{2100}x^2+\frac{1}{3500}x^3+\cdots).$$

The price formula is

$$AN(d_1)-Ke^{-rT}N(d_2)$$

where A, d_1, d_2 are defined as

$$A=e^{-rT}\frac{1}{T}E(\int_0^T S_tdt)=e^{-rT}\frac{1}{T}\int_0^T S_0e^{rt}dt=\frac{S_0}{rT}(1-e^{-rT})$$
$$d_1=\frac{\log(\frac{A}{K})+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2=d_1-\sigma\sqrt{T}.$$

Note that the assumption that the short rate is zero is only used in the process of deriving the implied volatility.

3.5 Simulation Results

Now, let's check if the formulas for parameters $\alpha, \beta, \nu, \cdots$ are appropriate. Focusing on the Amazon call options with stock price at May 29 being 1604.01 USD and maturity being July 6, we can figure out if every formula is sketched well by comparing the price in the market with the price derived from monte carlo simulations. Further, Asian option price can be derived.

3.5.1 European Call Price

Amazon option at May 29 with maturity July 6 and stock price 1604.01					
Strike	Market	Heston	SABR	CEV	Poly I
1,580.00	67.76	60.645444	62.013403	61.032228	64.252564
1,585.00	62.93	56.480284	58.660591	59.730168	61.604195
1,590.00	60.75	53.23883	57.386392	56.830303	59.248404
1,595.00	54.00	50.430531	52.002971	52.659427	54.10636
1,600.00	52.65	49.743746	50.306041	50.877847	53.304915
1,605.00	49.77	46.732249	47.595102	48.054903	50.419906
1,607.50	46.75	45.715583	46.773268	45.260608	48.754384
1,610.00	47.64	43.549376	46.178112	45.183742	46.684461
1,612.50	46.50	42.894484	44.649311	44.410964	45.229907
1,615.00	44.65	40.632874	43.385943	43.340332	44.176436
1,617.50	45.81	39.554687	40.865873	41.766556	42.619492
1,620.00	41.83	37.094714	40.373545	40.426768	40.603717
1,625.00	38.62	36.445526	38.245524	39.694598	38.905353
1,630.00	36.45	34.915661	35.761398	36.552927	37.149369
1,635.00	35.09	33.879022	34.668515	35.334606	35.689422
1,645.00	30.05	28.711506	31.135566	30.345976	30.914223
1,650.00	26.55	27.451116	27.253689	28.82947	29.661548
1,655.00	24.86	25.39695	26.576591	26.872059	26.576432
1,660.00	23.20	24.181304	23.687733	25.131676	24.582443
1,665.00	25.05	22.242366	23.535113	24.618654	24.216653
1,670.00	22.62	21.125216	21.335455	22.136501	21.439892
1,675.00	21.65	19.653131	19.624961	21.105888	21.109992
1,685.00	16.77	17.209686	16.976963	18.407306	17.465296
1,690.00	15.67	16.823898	16.577191	17.036417	16.878361
1,695.00	13.71	14.734694	14.257787	15.918647	15.498633
1,700.00	13.07	14.467532	13.34931	14.636945	13.914245
1,705.00	11.40	13.3584127	11.690837	13.308013	12.6997
1,710.00	12.20	11.87443	11.375263	12.610207	11.331498
1,715.00	11.09	11.641844	10.632232	11.690617	10.484525
1,720.00	10.13	10.290186	10.206354	11.057457	9.718831
1,730.00	8.00	8.944739	8.550755	9.795459	7.537434
1,735.00	7.63	8.178623	7.625618	9.0609	6.832857
1,745.00	6.33	6.368693	6.449534	7.552583	5.894523
1,750.00	5.52	6.254262	5.878925	6.948637	5.260258
1,755.00	5.33	5.62798	5.487726	5.921315	4.544115

The Heston model parameters are $\hat{\nu} = 0.050190081, \hat{\xi} = 0.39701068, \hat{\rho} = -0.365239988, \hat{\kappa} = 594.4852068$, and $\hat{\theta} = 0.047395476$. The SABR model parameters are $\hat{\nu} = 2.163752732, \hat{\beta} = 0.69346191, \hat{\alpha} = 0.679324515$, and $\hat{\rho} = -0.426107241$. The CEV model parameter is $\hat{\gamma} = 0.798033$. The polynomial model I parameters are $\hat{\alpha} = 0.233796, \hat{\beta} = -0.541884, \hat{\gamma} = -1.648903, \hat{\delta} = 0.035764$, and $\hat{\theta} = -0.082893$. Next, let's check the polynomial model II.

Amazon option at May 29 with maturity July 6 and stock price 1604.01					
Strike	Market	1st Order	2nd Order	3rd Order	4th Order
1580.00	67.76	62.328396	66.276659	70.942473	35.665267
1,585.00	62.93	57.521639	61.212552	66.233476	35.257526
1,590.00	60.75	55.385036	57.887647	60.657215	34.846038
1,595.00	54.00	53.100918	54.434931	54.907698	32.060563
1,600.00	52.65	50.346866	52.17523	53.392602	31.268189
1,605.00	49.77	46.300455	48.696107	50.612378	29.804021
1,607.50	46.75	45.780669	46.840566	49.182237	29.635226
1,610.00	47.64	45.568062	46.503396	47.709509	29.331383
1,612.50	46.50	43.685226	46.021092	46.990064	28.739526
1,615.00	44.65	42.501372	45.015383	45.487439	28.451286
1,617.50	45.81	42.248048	43.11987	44.804047	27.312222
1,620.00	41.83	40.341442	41.049844	43.57711	27.230459
1,625.00	38.62	39.395962	38.216222	41.027755	25.924875
1,630.00	36.45	36.495998	36.276953	38.551065	25.038825
1,635.00	35.09	35.270748	34.538835	35.322679	24.176529
1,645.00	30.05	30.597934	32.286329	31.464469	21.449801
1,650.00	26.55	27.951415	29.669126	28.865567	21.091353
1,655.00	24.86	26.410051	27.708456	27.423512	19.339344
1,660.00	23.20	25.636072	24.828057	25.750921	18.925484
1,665.00	25.05	23.485878	24.437263	25.066976	18.158616
1,670.00	22.62	23.103578	22.763239	22.261224	16.47043
1,675.00	21.65	21.09127	19.911892	20.053077	16.32382
1,685.00	16.77	18.134223	16.478888	18.234761	14.592728
1,690.00	15.67	16.720867	15.291114	16.69328	13.594538
1,695.00	13.71	16.222728	14.944222	14.755005	13.218692
1,700.00	13.07	14.939491	14.382047	14.221829	11.500508
1,705.00	11.40	13.547906	12.057629	13.766016	10.929602
1,710.00	12.20	12.902524	11.443239	12.941215	10.508059
1,715.00	11.09	12.351622	10.216018	11.418627	9.623293
1,720.00	10.13	10.752734	9.056485	10.996763	9.003824
1,730.00	8.00	9.257586	7.822373	9.112074	7.271972
1,735.00	7.63	8.692838	7.011516	8.797915	6.803217
1,745.00	6.33	7.443033	5.892997	7.683662	6.554882
1,750.00	5.52	6.337309	5.200923	7.197661	6.126521
1,755.00	5.33	5.951882	4.503373	6.90293	5.618845

In this case, the 1st order local volatility is

$$\hat{\alpha} + \hat{\beta}x_t = 0.225394 - 0.13804x_t.$$

The 2nd order local volatility is

$$\hat{\alpha} + \hat{\beta}x_t + \hat{\gamma}x_t^2 = 0.233796 - 0.541884x_t - 1.648903x_t^2.$$

The 3rd order local volatility is

$$\hat{\alpha} + \hat{\beta}x_t + \hat{\gamma}x_t^2 + \hat{\delta}x_t^3 = 0.235593 - 1.266821x_t + 12.09113x_t^2 + 11.799255x_t^3.$$

The 4th order local volatility is

$\hat{\alpha}+\hat{\beta}x_t+\hat{\gamma}x_t^2+\hat{\delta}x_t^3+\hat{\theta}x_t^4 = 0.235429-1.321967x_t+15.916928x_t^2-37.821392x_t^3+204.75312x_t^4.$

Note that the 1st, 2nd, and 3rd order models explain market fairly well even though the 4th order model doesn't.

3.5.2 Asian Call Price

First, let's price Asian call options under the Black-Scholes model.

Amazon Asian Call Option with Maturity 1 Year					
Strike	Formula	Euler	Milstein	Bridge	Stock Control
1,580.00	101.951521	101.661401	101.264012	100.209363	100.864552
1,585.00	99.297037	99.607137	98.635993	98.264957	98.265233
1,590.00	96.690221	97.638489	97.209778	96.932233	96.249238
1,595.00	93.130987	93.947572	94.241585	92.610073	94.484778
1,600.00	91.619219	91.089917	91.142138	91.889317	92.01557
1,605.00	89.154776	89.74931	89.123067	88.706974	88.387647
1,607.50	87.94025	88.307991	87.456185	88.018193	86.876253
1,610.00	86.73749	87.447825	86.429189	86.9784	86.001714
1,612.50	85.5467	85.57048	85.359617	86.527994	85.759805
1,615.00	84.367164	85.038648	84.070083	85.466623	84.379683
1,617.50	83.199544	83.010626	83.91466	84.494741	83.56915
1,620.00	82.043579	82.02161	82.098848	80.41922	81.164784
1,625.00	79.76649	81.1463	79.047355	79.449461	78.767065
1,630.00	77.535627	78.170877	78.309989	76.110906	77.578718
1,635.00	75.350697	75.3226	74.477888	74.804431	74.901031
1,645.00	71.117355	70.893669	72.22147	71.009718	71.583787
1,650.00	69.068247	68.355917	68.612929	69.734688	68.52528
1,655.00	67.063682	66.669746	67.428251	67.860845	66.291325
1,660.00	65.103263	65.683964	65.431809	64.98461	65.662776
1,665.00	63.186572	64.110556	63.539897	63.641629	63.187965
1,670.00	61.313174	62.076701	61.378468	62.483906	62.670316
1,675.00	59.482618	58.058394	59.786357	58.652946	59.392709
1,685.00	55.948145	56.025288	55.7248	56.11776	56.048974
1,690.00	54.243246	54.285876	53.618738	53.422115	55.279546
1,695.00	52.57923	51.577684	51.738471	52.275648	52.190972
1,700.00	50.955571	50.466192	51.334716	51.521421	50.773997
1,705.00	49.371735	49.700448	49.938619	48.917785	48.632252
1,710.00	47.827174	48.156503	47.498294	48.230225	47.440606
1,715.00	46.321333	46.500226	46.325398	46.423396	46.268999
1,720.00	44.853645	44.235724	44.654502	44.619907	44.960818
1,730.00	42.030421	42.628294	42.261414	41.555439	41.90753
1,735.00	40.673714	40.779332	40.092479	40.493431	40.766003
1,745.00	38.067137	37.37563	38.676672	38.854242	37.581538
1,750.00	36.816061	36.594717	36.661995	37.6115599	36.876322
1,755.00	35.598983	35.971209	35.38337	36.67878	35.869562

The formula price is based on the implied volatility formula and other columns are based on diverse monte carlo simulations based on the Black-Scholes model. Next, under other models, the monte carlo simulation results are

Amazon Asian Call Option with Maturity 1 Year					
Strike	Heston	SABR	CEV	1st Order	2nd Order
1,580.00	102.85886	104.28139	101.319368	102.804388	102.695124
1,585.00	98.719744	102.273437	99.286485	98.645186	101.023366
1,590.00	95.001414	98.347106	98.570933	98.088922	99.922473
1,595.00	92.698616	94.22033	95.782532	95.951691	97.626
1,600.00	90.531723	91.880014	93.614542	93.244371	95.18548
1,605.00	87.944065	90.61483	91.193755	89.575208	91.347945
1,607.50	87.287801	89.48443	87.840067	87.649145	89.088982
1,610.00	86.49296	87.8165	86.012761	87.050365	88.252163
1,612.50	85.378265	86.595407	85.672479	86.213429	87.629311
1,615.00	85.771944	83.907949	85.056746	84.509309	85.868848
1,617.50	81.757993	82.312999	82.659401	83.707642	85.233741
1,620.00	80.355757	82.185273	82.281425	83.223167	84.224041
1,625.00	78.927099	80.272812	79.618891	80.354138	79.384088
1,630.00	78.553445	79.631274	78.441506	76.967679	79.094563
1,635.00	73.71579	74.522618	76.418274	75.039239	76.021561
1,645.00	68.25551	71.450994	69.915669	71.65051	71.394887
1,650.00	66.98601	68.604666	67.776137	70.289003	70.972442
1,655.00	64.642901	67.573066	66.778664	68.530379	65.503976
1,660.00	62.212031	63.856481	65.297168	66.587879	64.269332
1,665.00	60.282579	61.938023	63.803681	63.376417	63.200619
1,670.00	59.965112	60.057196	62.496415	60.524482	60.740511
1,675.00	59.412316	59.026992	59.890025	59.043799	58.449542
1,685.00	53.199375	54.119771	57.347846	55.700794	55.096271
1,690.00	51.670541	53.200522	54.678129	54.49611	54.79586
1,695.00	49.24666	50.982459	53.521944	52.94002	52.876272
1,700.00	49.462499	48.872987	51.018748	50.898062	50.831587
1,705.00	48.288415	46.475375	48.597325	49.442755	48.734044
1,710.00	45.584649	44.921446	47.736674	48.239677	47.38926
1,715.00	44.527962	44.073906	46.715422	45.853458	45.086832
1,720.00	42.263989	41.84129	45.478643	43.95536	43.482088
1,730.00	39.966029	37.925755	41.862285	40.348447	40.427321
1,735.00	38.731563	37.628129	40.23742	38.885514	38.708159
1,745.00	36.607961	34.291962	38.643142	37.316314	33.934781
1,750.00	35.37858	33.955819	35.673521	36.492449	31.371797
1,755.00	35.153902	32.45489	34.867192	35.399209	30.550834

Note that there is no result for the polynomial model I and the 3rd and 4th order polynomial model II. I excluded these since the results are far from the other models. This means that models with more number of parameters don't guarantee better accuracy than models with few parameters.

Chapter 4

Interest Rate Derivatives

Define the measure P_i as the measure derived by the numeraire $P(t, T_i)$. Then, let the dynamics of the bond price $P(t, T_i)$ be

$$\frac{dP(t, T_i)}{P(t, T_i)} = r_t dt + \zeta_i(t) dB_t.$$

Then,

$$B_t^i = B_t - \int_0^t \zeta_i(s) ds$$

is the standard Brownian motion under P_i. Also note that

$$\begin{aligned}
d\Big(\frac{P(t, T_j)}{P(t, T_i)}\Big) &= \frac{dP(t, T_j)}{P(t, T_i)} - \frac{P(t, T_j)}{P(t, T_i)^2} dP(t, T_i) + \frac{P(t, T_j)}{P(t, T_i)^3} dP(t, T_i)^2 - \frac{1}{P(t, T_i)^2} dP(t, T_j) dP(t, T_i) \\
&= \frac{P(t, T_j)}{P(t, T_i)} (r_t dt + \zeta_j(t) dB_t - (r_t dt + \zeta_i(t) dB_t) + \zeta_i(t)^2 dt - \zeta_j(t)\zeta_i(t) dt) \\
&= \frac{P(t, T_j)}{P(t, T_i)} ((\zeta_j(t) - \zeta_i(t)) dB_t - \zeta_i(t)(\zeta_j(t) - \zeta_i(t)) dt) \\
&= \frac{P(t, T_j)}{P(t, T_i)} (\zeta_j(t) - \zeta_i(t))(dB_t - \zeta_i(t) dt) \\
&= \frac{P(t, T_j)}{P(t, T_i)} (\zeta_j(t) - \zeta_i(t)) dB_t^i,
\end{aligned}$$

which is also shown in the theorem 9.2.2 of [11]. With this dynamics, we get

$$P(T_i, T_j) = \frac{P(T_i, T_j)}{P(T_i, T_i)} = \frac{P(t, T_j)}{P(t, T_i)} e^{\int_t^{T_i} (\zeta_j(s) - \zeta_i(s)) dB_s^i - \frac{1}{2} \int_t^{T_i} (\zeta_j(s) - \zeta_i(s))^2 ds}.$$

4.1 Bond Option

The price of bond call options at time t with the option maturity T and the bond maturity S is given by

$$\begin{aligned}
C(t) &= E(e^{-\int_t^T r_s ds}(P(T,S)-K)^+|\mathcal{F}_t) \\
&= E(e^{-\int_t^T r_s ds}(\frac{P(t,S)}{P(t,T)}e^{\int_t^T(\zeta_S(s)-\zeta_T(s))dB_s^T-\frac{1}{2}\int_t^T(\zeta_S(s)-\zeta_T(s))^2ds}-K)^+|\mathcal{F}_t) \\
&= P(t,T)E^T((\frac{P(t,S)}{P(t,T)}e^{\int_t^T(\zeta_S(s)-\zeta_T(s))dB_s^T-\frac{1}{2}\int_t^T(\zeta_S(s)-\zeta_T(s))^2ds}-K)^+|\mathcal{F}_t) \\
&= P(t,S)N(d_1)-KP(t,T)N(d_2)
\end{aligned}$$

where d_1 and d_2 are given by

$$d_1 = \frac{\log(\frac{P(t,S)}{KP(t,T)})}{v}+\frac{1}{2}v$$

$$d_2 = \frac{\log(\frac{P(t,S)}{KP(t,T)})}{v}-\frac{1}{2}v$$

with

$$v^2 = \int_t^T(\zeta_S(s)-\zeta_T(s))^2ds.$$

4.2 Caplet

The price of caplets at time t with the option maturity T is given by

$$\begin{aligned}
C(t) &= E(e^{-\int_t^S r_s ds}(L(T,T,S)-\kappa)^+|\mathcal{F}_t) \\
&= P(t,S)E^S((L(T,T,S)-\kappa)^+|\mathcal{F}_t) \\
&= P(t,S)E^S(\frac{1}{P(T,S)}-(1+\kappa))^+|\mathcal{F}_t) \\
&= P(t,S)E^S((\frac{P(t,T)}{P(t,S)}e^{\int_t^T(\zeta_T(s)-\zeta_S(s))dB_s^S-\frac{1}{2}\int_t^T(\zeta_T(s)-\zeta_S(s))^2ds}-(1+\kappa))^+|\mathcal{F}_t) \\
&= P(t,T)N(d_1)-(1+\kappa)P(t,S)N(d_2)
\end{aligned}$$

where d_1 and d_2 are given by

$$d_1 = \frac{\log(\frac{P(t,T)}{(1+\kappa)P(t,S)})}{v}+\frac{1}{2}v$$

$$d_2 = \frac{\log(\frac{P(t,T)}{(1+\kappa)P(t,S)})}{v}-\frac{1}{2}v$$

with

$$v^2 = \int_t^T(\zeta_T(s)-\zeta_S(s))^2ds.$$

Note that $L(t,T,S)$ is the Libor rate that is defined by

$$L(t,T,S) = \frac{1}{S-T}(\frac{P(t,T)}{P(t,S)} - 1).$$

Another approach is by the dynamics of $L(t,T,S)$,

$$\begin{aligned} dL(t,T,S) &= \frac{1}{S-T} d\Big(\frac{P(t,T)}{P(t,S)}\Big) \\ &= \frac{1}{S-T}\frac{P(t,T)}{P(t,S)}(\zeta_T(t) - \zeta_S(t))dB_t^S \\ &= (L(t,T,S) + \frac{1}{S-T})(\zeta_T(t) - \zeta_S(t))dB_t^S \end{aligned}$$

such that

$$d(L(t,T,S) + \frac{1}{S-T}) = (L(t,T,S) + \frac{1}{S-T})(\zeta_T(t) - \zeta_S(t))dB_t^S.$$

Therefore, the price of caplets is

$$\begin{aligned} C(t) &= E(e^{-\int_t^S r_s ds}(L(T,T,S) - \kappa)^+|\mathcal{F}_t) \\ &= P(t,S)E^S((L(T,T,S) - \kappa)^+|\mathcal{F}_t) \\ &= P(t,S)E^S((L(T,T,S) + \frac{1}{S-T} - (\kappa + \frac{1}{S-T}))^+|\mathcal{F}_t) \\ &= P(t,S)(L(t,T,S) + \frac{1}{S-T})N(d_1) - P(t,S)(\kappa + \frac{1}{S-T})N(d_2) \end{aligned}$$

where d_1 and d_2 are given by

$$d_1 = \frac{\log(\frac{L(t,T,S)+\frac{1}{S-T}}{\kappa+\frac{1}{S-T}})}{v} + \frac{1}{2}v$$

$$d_2 = \frac{\log(\frac{L(t,T,S)+\frac{1}{S-T}}{\kappa+\frac{1}{S-T}})}{v} - \frac{1}{2}v.$$

4.3 Caps

The price of caps is given by

$$\begin{aligned} C(t) &= \sum_{k=i}^{j-1}(T_{k+1} - T_k)E(e^{-\int_t^{T_{k+1}} r_s ds}(L(T_k,T_k,T_{k+1}) - \kappa)^+|\mathcal{F}_t) \\ &= \sum_{k=i}^{j-1}(T_{k+1} - T_k)(P(t,T_k)N(d_{1k}) - (1+\kappa)P(t,T_{k+1})N(d_{2k})) \end{aligned}$$

where d_{1k} and d_{2k} are given by

$$d_{1k} = \frac{\log(\frac{P(t,T_k)}{(1+\kappa)P(t,T_{k+1})})}{v_k} + \frac{1}{2}v_k$$

$$d_{2k} = \frac{\log(\frac{P(t,T_k)}{(1+\kappa)P(t,T_{k+1})})}{v_k} - \frac{1}{2}v_k$$

with

$$v_k^2 = \int_t^{T_k} (\zeta_{T_k}(s) - \zeta_{T_{k+1}}(s))^2 ds.$$

4.4 Swap

The swap rate κ satisfies the following equation

$$\sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t, T_{k+1})(L(t, T_k, T_{k+1}) - \kappa) = 0$$

such that the swap rate $S(t, T, S)$ is given by

$$\begin{aligned} S(t, T, S) &= \frac{\sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t, T_{k+1})L(t, T_k, T_{k+1})}{\sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t, T_{k+1})} \\ &= \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)} \end{aligned}$$

where $P(t, T_i, T_j)$ is

$$P(t, T_i, T_j) = \sum_{k=i}^{j-1}(T_{k+1} - T_k)P(t, T_{k+1}).$$

4.5 Swaption

The swaption price is given by

$$C(t) = E(e^{-\int_t^{T_i} r_s ds}(\sum_{k=i}^{j-1}(T_{k+1} - T_k)e^{-\int_{T_i}^{T_{k+1}} r_s ds}(L(T_i, T_k, T_{k+1}) - \kappa))^+ |\mathcal{F}_t).$$

But this is hard to calculate. An alternative price is given by

$$\begin{aligned} C(t) &= E(e^{-\int_t^{T_i} r_s ds}(\sum_{k=i}^{j-1}(T_{k+1} - T_k)P(T_i, T_{k+1})(L(T_i, T_k, T_{k+1}) - \kappa))^+ |\mathcal{F}_t) \\ &= E(e^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j)(S(T_i, T_i, T_j) - \kappa)^+ |\mathcal{F}_t) \\ &= P(t, T_i, T_j)E_{i,j}((S(T_i, T_i, T_j) - \kappa)^+ |\mathcal{F}_t) \end{aligned}$$

where the measure $P_{i,j}$ is derived by setting $P(t, T_i, T_j)$ as a nemeraire. Now, let's introduce a new notation

$$v_k^{i,j}(t) = \frac{P(t, T_k)}{P(t, T_i, T_j)}.$$

Then, the swap rate is

$$
\begin{aligned}
S(t, T_i, T_j) &= \frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)} \\
&= v_i^{i,j}(t) - v_j^{i,j}(t).
\end{aligned}
$$

According to the chapter 10 of [9] and the chapter 12 of [10], the dynamics of $v_k^{i,j}(t)$ is

$$
dv_k^{i,j}(t) = v_k^{i,j}(t) \sum_{l=i}^{j-1} \delta_l v_{l+1}^{i,j}(t)(\zeta_k(t) - \zeta_{l+1}(t)) dB_t^{i,j}
$$

where

$$
B_t^{i,j} = B_t - \sum_{l=i}^{j-1} \int_0^t \delta_l v_{l+1}^{i,j}(s) \zeta_{l+1}(s) ds
$$

is the standard Brownian motion under the measure $P_{i,j}$ and $\delta_l = T_{l+1} - T_l$. With this, we can get the dynamics of the swap rate

$$
dS(t, T_i, T_j) = S(t, T_i, T_j) \sigma_{i,j}(t) dB_t^{i,j}
$$

where $\sigma_{i,j}(t)$ is given by the following.

$$
\begin{aligned}
dS(t, T_i, T_j) &= d\Big(\frac{P(t, T_i) - P(t, T_j)}{P(t, T_i, T_j)}\Big) \\
&= dv_i^{i,j}(t) - dv_j^{i,j}(t) \\
&= \sum_{l=i}^{j-1} \delta_l v_{l+1}^{i,j}(t)(v_i^{i,j}(t)(\zeta_i(t) - \zeta_{l+1}(t)) - v_j^{i,j}(t)(\zeta_j(t) - \zeta_{l+1}(t))) dB_t^{i,j} \\
&= \sum_{l=i}^{j-1} \zeta_{l+1}(t) \delta_l v_{l+1}^{i,j}(t)(v_j^{i,j}(t) - v_i^{i,j}(t)) dB_t^{i,j} + (v_i^{i,j}(t)\zeta_i(t) - v_j^{i,j}(t)\zeta_j(t)) dB_t^{i,j} \\
&= \sum_{l=i}^{j-1} (\zeta_{l+1}(t) - \zeta_i(t)) \delta_l v_{l+1}^{i,j}(t)(v_j^{i,j}(t) - v_i^{i,j}(t)) dB_t^{i,j} + v_j^{i,j}(t)(\zeta_i(t) - \zeta_j(t)) dB_t^{i,j} \\
&= S(t, T_i, T_j)(\sum_{l=i}^{j-1} \delta_l v_{l+1}^{i,j}(t)(\zeta_i(t) - \zeta_{l+1}(t)) + \frac{P(t, T_j)}{P(t, T_i) - P(t, T_j)}(\zeta_i(t) - \zeta_j(t))) dB_t^{i,j} \\
&= S(t, T_i, T_j) \sigma_{i,j}(t) dB_t^{i,j}
\end{aligned}
$$

such that

$$
\sigma_{i,j}(t) = \sum_{l=i}^{j-1} \delta_l v_{l+1}^{i,j}(t)(\zeta_i(t) - \zeta_{l+1}(t)) + \frac{P(t, T_j)}{P(t, T_i) - P(t, T_j)}(\zeta_i(t) - \zeta_j(t)).
$$

This is also shown in [9] and [10]. Then, v^2 is

$$
\begin{aligned}
v^2 &= \int_t^{T_i} \sigma_{i,j}^2(s)ds \\
&\approx \sum_{l,l'=i}^{j-1} \delta_l \delta_{l'} v_{l+1}^{i,j} v_{l'+1}^{i,j} \int_t^{T_i} (\zeta_i(s) - \zeta_{l+1}(s))(\zeta_i(s) - \zeta_{l'+1}(s))ds \\
&\quad + \frac{2P(t,T_j)}{P(t,T_i) - P(t,T_j)} \sum_{l=i}^{j-1} \delta_l v_{l+1}^{i,j}(t) \int_t^{T_i} (\zeta_i(s) - \zeta_{l+1}(s))(\zeta_i(s) - \zeta_j(s))ds \\
&\quad + \Big(\frac{P(t,T_j)}{P(t,T_i) - P(t,T_j)}\Big)^2 \int_t^{T_i} (\zeta_i(s) - \zeta_j(s))^2 ds.
\end{aligned}
$$

Another approach is based on the dynamics of the Libor rate. Firstly, let's check the dynamics of $L(t, T_k, T_{k+1})$.

$$
\begin{aligned}
dL(t, T_k, T_{k+1}) &= \frac{1}{T_{k+1} - T_k} d\Big(\frac{P(t,T_k)}{P(t,T_{k+1})}\Big) \\
&= \frac{1}{T_{k+1} - T_k} \frac{P(t,T_k)}{P(t,T_{k+1})} (\zeta_k(t) - \zeta_{k+1}(t)) dB_t^{k+1}
\end{aligned}
$$

such that the dynamics of the swap rate is given by

$$
\begin{aligned}
dS(t, T_i, T_j) &\approx \frac{1}{P(t, T_i, T_j)} \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) dL(t, T_k, T_{k+1}) \\
&= \frac{1}{P(t, T_i, T_j)} \sum_{k=i}^{j-1} P(t, T_k)(\zeta_k(t) - \zeta_{k+1}(t)) dB_t^{k+1} \\
&= \frac{S(t, T_i, T_j)}{P(t,T_i) - P(t,T_j)} \sum_{k=i}^{j-1} P(t, T_k)(\zeta_k(t) - \zeta_{k+1}(t)) dB_t^{k+1}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sigma_{i,j}^2(t)dt &= \Big(\frac{dS(t, T_i, T_j)}{S(t, T_i, T_j)}\Big)^2 \\
&\approx \sum_{k,k'=i}^{j-1} \frac{P(t,T_k)P(t,T_{k'})}{(P(t,T_i) - P(t,T_j))^2} (\zeta_k(t) - \zeta_{k+1}(t))(\zeta_{k'}(t) - \zeta_{k'+1})dt
\end{aligned}
$$

such that

$$
\begin{aligned}
v^2 &= \int_t^{T_i} \sigma_{i,j}^2(s)ds \\
&\approx \sum_{k,k'=i}^{j-1} \frac{P(t,T_k)P(t,T_{k'})}{(P(t,T_i) - P(t,T_j))^2} \int_t^{T_i} (\zeta_k(s) - \zeta_{k+1}(s))(\zeta_{k'}(s) - \zeta_{k'+1}(s))ds.
\end{aligned}
$$

The approximately calculated swaption price is

$$
\begin{aligned}
C(t) &= P(t,T_i,T_j)E_{i,j}((S(T_i,T_i,T_j)-\kappa)^+|\mathcal{F}_t) \\
&= P(t,T_i,T_j)E_{i,j}((S(t,T_i,T_j)e^{\int_t^{T_i}\sigma_{i,j}(s)dB_s^{i,j}-\frac{1}{2}\int_t^{T_i}\sigma_{i,j}^2(s)ds}-\kappa)^+|\mathcal{F}_t) \\
&\approx P(t,T_i,T_j)S(t,T_i,T_j)N(d_1)-\kappa P(t,T_i,T_j)N(d_2) \\
&= (P(t,T_i)-P(t,T_j))N(d_1)-\kappa P(t,T_i,T_j)N(d_2)
\end{aligned}
$$

where d_1 and d_2 are given by

$$
d_1 = \frac{\log(\frac{S(t,T_i,T_j)}{\kappa})}{v} + \frac{1}{2}v
$$

$$
d_2 = \frac{\log(\frac{S(t,T_i,T_j)}{\kappa})}{v} - \frac{1}{2}v.
$$

Chapter 5

The Hull-White Model

5.1 Basic Concept

The Hull-White model is given by

$$dr_t = (\theta(t) - br_t)dt + \sigma dB_t$$

where b and σ are constant. From the equation

$$P(t,T) = E(e^{-\int_t^T r_s ds}|\mathcal{F}_t),$$

the bond price $P(t,T)$ can be seen as a function of r_t and t. By the Feynman Kac formula, the bond price satisfies the partial differential equation

$$P_t + (\theta(t) - br)P_r + \frac{1}{2}\sigma^2 P_{rr} - rP = 0.$$

Let $P(t,T)$ be of the form

$$P(t,T) = e^{A(t,T) - rB(t,T)}$$

where $A(T,T) = B(T,T) = 0$ from the condition $P(T,T) = 1$. Then, we get the following two ordinary differential equations

$$A^{'}(t,T) - \theta(t)B(t,T) + \frac{1}{2}\sigma^2 B(t,T)^2 = 0$$

$$-B^{'}(t,T) + bB(t,T) - 1 = 0$$

such that we get

$$A(t,T) = \frac{\sigma^2}{2b^2}\Big((T-t) - \frac{2}{b}(1 - e^{-b(T-t)}) + \frac{1}{2b}(1 - e^{-2b(T-t)})\Big) - \frac{1}{b}\int_t^T \theta(s)(1 - e^{-b(T-s)})ds$$

$$B(t,T) = \frac{1}{b}(1 - e^{-b(T-t)}).$$

The bond price function can be rewritten as $P(t,r)$. By the Ito formula,

$$\begin{aligned} dP(t,r) =& P_t(t,r)dt + P_r(t,r)dr + \frac{1}{2}P_{rr}(t,r)(dr)^2 \\ =& P(t,r)\Big(A'(t,T) - rB'(t,T) - B(t,T)(\theta(t) - br) + \frac{1}{2}\sigma^2 B(t,T)^2\Big)dt \\ & - P(t,T)\sigma B(t,T)dB_t \\ =& rP(t,r)dt - \sigma B(t,T)P(t,r)dB_t. \end{aligned}$$

As a result,

$$dP(t,T) = r_t P(t,T)dt - \sigma B(t,T)P(t,T)dB_t.$$

Therefore, $\zeta_T(t)$ is

$$\begin{aligned}\zeta_T(t) &= -\sigma B(t,T)\\ &= -\frac{\sigma}{b}(1-e^{-b(T-t)}).\end{aligned}$$

For the case of bond options, caplets, and caps,

$$\begin{aligned}v^2 &= \int_t^T (\zeta_S(s)-\zeta_T(s))^2 ds\\ &= \frac{\sigma^2}{b^2}\int_t^T (e^{-b(T-s)}-e^{-b(S-s)})^2 ds\\ &= \frac{\sigma^2}{b^2}\int_t^T (e^{-2b(T-s)}-2e^{-b(T+S-2s)}+e^{-2b(S-s)})ds\\ &= \frac{\sigma^2}{2b^3}(1-e^{-2b(T-t)}-2(e^{-b(S-T)}-e^{-b(T+S-2t)})+e^{-2b(S-T)}-e^{-2b(S-t)})\\ &= \frac{\sigma^2}{2b^3}(1-e^{-b(S-T)})^2(1-e^{-2b(T-t)})\end{aligned}$$

such that

$$\begin{aligned}v &= \frac{\sigma}{b}(1-e^{-b(S-T)})\sqrt{\frac{1-e^{-2b(T-t)}}{2b}}\\ v_k &= \frac{\sigma}{b}(1-e^{-b(T_{k+1}-T_k)})\sqrt{\frac{1-e^{-2b(T_k-t)}}{2b}}.\end{aligned}$$

For the calculation of swaptions, the following are required

$$\int_t^{T_i} (\zeta_i(s)-\zeta_{l+1}(s))(\zeta_i(s)-\zeta_{l'+1}(s))ds = \frac{\sigma^2}{2b^3}(1-e^{-2b(T_i-t)})(1-e^{-b(T_{l+1}-T_i)})(1-e^{-b(T_{l'+1}-T_i)})$$

$$\int_t^{T_i} (\zeta_i(s)-\zeta_{l+1}(s))(\zeta_i(s)-\zeta_j(s))ds = \frac{\sigma^2}{2b^3}(1-e^{-2b(T_i-t)})(1-e^{-b(T_{l+1}-T_i)})(1-e^{-b(T_j-T_i)})$$

$$\int_t^{T_i} (\zeta_i(s)-\zeta_j(s))^2 ds = \frac{\sigma^2}{2b^3}(1-e^{-b(T_j-T_i)})^2(1-e^{-2b(T_i-t)})$$

such that

$$\begin{aligned}v^2 \approx& \sum_{l,l'=i}^{j-1} \delta_l\delta_{l'} v_{l+1}^{i,j} v_{l'+1}^{i,j} \frac{\sigma^2}{2b^3}(1-e^{-2b(T_i-t)})(1-e^{-b(T_{l+1}-T_i)})(1-e^{-b(T_{l'+1}-T_i)})\\ &+ \frac{2P(t,T_j)}{P(t,T_i)-P(t,T_j)}\sum_{l=i}^{j-1}\delta_l v_{l+1}^{i,j}(t)\frac{\sigma^2}{2b^3}(1-e^{-2b(T_i-t)})(1-e^{-b(T_{l+1}-T_i)})(1-e^{-b(T_j-T_i)})\\ &+ \Big(\frac{P(t,T_j)}{P(t,T_i)-P(t,T_j)}\Big)^2\frac{\sigma^2}{2b^3}(1-e^{-b(T_j-T_i)})^2(1-e^{-2b(T_i-t)}).\end{aligned}$$

For another approach, the following is required.

$$
\begin{aligned}
&\int_t^{T_i} (\zeta_k(s) - \zeta_{k+1}(s))(\zeta_{k'}(s) - \zeta_{k'+1}(s))ds \\
&= \frac{\sigma^2}{b^2} \int_t^{T_i} (e^{-b(T_k - s)} - e^{-b(T_{k+1}-s)})(e^{-b(T_{k'}-s)} - e^{-b(T_{k'+1}-s)})ds \\
&= \frac{\sigma^2}{b^2} \int_t^{T_i} (e^{-b(T_k+T_{k'}-2s)} - e^{-b(T_{k+1}+T_{k'}-2s)} - e^{-b(T_k+T_{k'+1}-2s)} + e^{-b(T_{k+1}+T_{k'+1}-2s)})ds \\
&= \frac{\sigma^2}{2b^3}(e^{-b(T_k+T_{k'}-2T_i)} - e^{-b(T_{k+1}+T_{k'}-2T_i)} - e^{-b(T_k+T_{k'+1}-2T_i)} + e^{-b(T_{k+1}+T_{k'+1}-2T_i)} \\
&\quad - e^{-b(T_k+T_{k'}-2t)} + e^{-b(T_{k+1}+T_{k'}-2t)} + e^{-b(T_k+T_{k'+1}-2t)} - e^{-b(T_{k+1}+T_{k'+1}-2t)}) \\
&= \frac{\sigma^2}{2b^3} e^{-b(T_k+T_{k'}-2T_i)}(1 - e^{-b(T_{k+1}-T_k)} - e^{-b(T_{k'+1}-T_{k'})} + e^{-b(T_{k+1}-T_k+T_{k'+1}-T_{k'})}) \\
&\quad - \frac{\sigma^2}{2b^3} e^{-b(T_k+T_{k'}-2t)}(1 - e^{-b(T_{k+1}-T_k)} - e^{-b(T_{k'+1}-T_{k'})} + e^{-b(T_{k+1}-T_k+T_{k'+1}-T_{k'})}) \\
&= \frac{\sigma^2}{2b^3} e^{-b(T_k+T_{k'}-2T_i)}(1 - e^{-2b(T_i-t)})(1 - e^{-b(T_{k+1}-T_k)})(1 - e^{-b(T_{k'+1}-T_{k'})})
\end{aligned}
$$

such that

$$
v^2 \approx \sum_{k,k'=i}^{j-1} \frac{P(t,T_k)P(t,T_{k'})}{(P(t,T_i) - P(t,T_j))^2} \frac{\sigma^2}{2b^3} e^{-b(T_k+T_{k'}-2T_i)}(1-e^{-2b(T_i-t)})(1-e^{-b(T_{k+1}-T_k)})(1-e^{-b(T_{k'+1}-T_{k'})}).
$$

5.2 Cubic Spline

The yield curve is $Y(t,T)$ that satisfies

$$
P(t,T) = e^{-Y(t,T)(T-t)}
$$

such that the yield curve is of the form

$$
\begin{aligned}
Y(t,T) &= -\frac{\log P(t,T)}{T-t} \\
&= \frac{rB(t,T) - A(t,T)}{T-t} \\
&= \frac{r}{b(T-t)}(1 - e^{-b(T-t)}) - \frac{\sigma^2}{2b^2(T-t)}\Big((T-t) - \frac{2}{b}(1 - e^{-b(T-t)}) + \frac{1}{2b}(1 - e^{-2b(T-t)})\Big) \\
&\quad + \frac{1}{b(T-t)} \int_t^T \theta(s)(1 - e^{-b(T-s)})ds.
\end{aligned}
$$

From the current time to the maturity time t, the yield curve is

$$
Y(0,t) = \frac{r}{bt}(1-e^{-bt}) - \frac{\sigma^2}{2b^2t}\Big(t - \frac{2}{b}(1-e^{-bt}) + \frac{1}{2b}(1-e^{-2bt})\Big) + \frac{1}{bt}\int_0^t \theta(s)(1-e^{-b(t-s)})ds.
$$

Now, let's think of the spline curve of the daily treasury yield data. Following is the daily treasury yield data as of the date June 19th, 2018.

Treasury Yield	
Term	Treasury Yield
1/12	0.0185
1/4	0.0194
1/2	0.0213
1	0.0234
2	0.0254
3	0.0264
5	0.0277
7	0.0284
10	0.0289
20	0.0295
30	0.0302

Let $f(t)$ be the cubic spline curve of the daily treasury yield data. Then, we get

$$f(t) = \left\{ \begin{array}{ll} 0.02084t^3 - 0.00521t^2 + 0.00526t + 0.01809 & \text{for } 0 \le t \le \frac{1}{4} \\ -0.025t^3 + 0.02917t^2 - 0.00334t + 0.0188 & \text{for } \frac{1}{4} \le t \le \frac{1}{2} \\ 0.00514t^3 - 0.01605t^2 + 0.01927t + 0.01503 & \text{for } \frac{1}{2} \le t \le 1 \\ 0.0000109t^3 - 0.00064955t^2 + 0.003872t + 0.020166 & \text{for } 1 \le t \le 2 \\ 0.000179t^3 - 0.001658t^2 + 0.00589t + 0.01882 & \text{for } 2 \le t \le 3 \\ -0.00000759t^3 + 0.0000214t^2 + 0.00085t + 0.02386 & \text{for } 3 \le t \le 5 \\ 0.00000979t^3 - 0.000239t^2 + 0.00215367t + 0.0216888 & \text{for } 5 \le t \le 7 \\ 0.000002665t^3 - 0.0000896t^2 + 0.0011t + 0.024133 & \text{for } 7 \le t \le 10 \\ 0.00000042678t^3 - 0.000022446t^2 + 0.0004346t + 0.02637145 & \text{for } 10 \le t \le 20 \\ -0.000000105356t^3 + 0.000009482t^2 - 0.0002039t + 0.030628549 & \text{for } 20 \le t \le 30 \end{array} \right\}.$$

The purpose is to match $tf(t)$ to $tY(0,t)$ as close as possible such that

$$tY(0,t) = \frac{r}{b}(1-e^{-bt}) - \frac{\sigma^2}{2b^2}\Big(t - \frac{2}{b}(1-e^{-bt}) + \frac{1}{2b}(1-e^{-2bt})\Big) + \frac{1}{b}\int_0^t \theta(s)(1-e^{-b(t-s)})ds.$$

In this case, $r = f(0)$. By differentiating this,

$$\frac{d}{dt}(tY(0,t)) = re^{-bt} - \frac{\sigma^2}{2b^2}\Big(1 - 2e^{-bt} + e^{-2bt}\Big) + \int_0^t \theta(s)e^{-b(t-s)}ds$$

$$\frac{d^2}{dt^2}(tY(0,t)) = -rbe^{-bt} - \frac{\sigma^2}{b}(e^{-bt} - e^{-2bt}) + \theta(t) - b\int_0^t \theta(s)e^{-b(t-s)}ds$$

such that we can express $\theta(t)$ as

$$\theta(t) = \frac{d^2}{dt^2}(tY(0,t)) + rbe^{-bt} + \frac{\sigma^2}{b}(e^{-bt} - e^{-2bt}) + b\int_0^t \theta(s)e^{-b(t-s)}ds.$$

Since

$$b\int_0^t \theta(s)e^{-b(t-s)}ds = b\frac{d}{dt}(tY(0,t)) - rbe^{-bt} + \frac{\sigma^2}{2b}\Big(1 - 2e^{-bt} + e^{-2bt}\Big),$$

we get

$$\theta(t) = \frac{d^2}{dt^2}(tY(0,t)) + b\frac{d}{dt}(tY(0,t)) + \frac{\sigma^2}{2b}(1 - e^{-2bt}).$$

Since we want to match $Y(0,t)$ to $f(t)$, let $tY(0,t) = tf(t)$ such that

$$\theta(t) = \frac{d^2}{dt^2}(tf(t)) + b\frac{d}{dt}(tf(t)) + \frac{\sigma^2}{2b}(1 - e^{-2bt}).$$

This $\theta(t)$ makes the cubic spline curve be the yield curve. Now, let $b = 0.5$. $\sigma = 0.00465147$. The volatility σ is derived based on historical estimation. The monte carlo simulations are

Yield Monte Carlo Simulations		
Term	Treasury Yield	simulation
1/12	0.0185	0.018518
1/4	0.0194	0.0194
1/2	0.0213	0.021291
1	0.0234	0.023401
2	0.0254	0.025397
3	0.0264	0.026405
5	0.0277	0.027676
7	0.0284	0.028359
10	0.0289	0.028869
20	0.0295	0.029502
30	0.0302	0.030182

This result shows that the parameters are estimated accurately.

5.3 Pricing

Now, let's calculate the prices of interest rate derivatives. For bond options with $T = 1$,

Bond Option			
K	S=2	S=3	S=5
0.80	0.168971	0.142358	0.089166
0.81	0.159203	0.132589	0.079397
0.82	0.149434	0.122820	0.069628
0.83	0.139665	0.113052	0.059860
0.84	0.129897	0.103283	0.050091
0.85	0.120128	0.093514	0.040322
0.86	0.110359	0.083746	0.030554
0.87	0.100590	0.073977	0.020785
0.88	0.090822	0.064208	0.011064
0.89	0.081053	0.054439	0.002899
0.90	0.071284	0.044671	0.000156
0.91	0.061516	0.034902	0.000001
0.92	0.051747	0.025133	0.000000
0.93	0.041978	0.015365	0.000000
0.94	0.032209	0.005792	0.000000
0.95	0.022441	0.000386	0.000000
0.96	0.012672	0.000001	0.000000
0.97	0.003112	0.000000	0.000000
0.98	0.000006	0.000000	0.000000

For caplets with $T = 1$,

Caplet			
κ	S=2	S=3	S=5
0.010	0.016898	0.043778	0.097502
0.011	0.015948	0.042854	0.096631
0.012	0.014997	0.041930	0.095760
0.013	0.014047	0.041006	0.094890
0.014	0.013096	0.040083	0.094019
0.015	0.012146	0.039159	0.093148
0.016	0.011195	0.038235	0.092278
0.017	0.010245	0.037311	0.091407
0.018	0.009295	0.036387	0.090536
0.019	0.008345	0.035463	0.089666
0.020	0.007398	0.034539	0.088795
0.021	0.006454	0.033616	0.087924
0.022	0.005521	0.032692	0.087054
0.023	0.004608	0.031768	0.086183
0.024	0.003731	0.030844	0.085312
0.025	0.002911	0.029920	0.084442
0.026	0.002174	0.028996	0.083571
0.027	0.001542	0.028072	0.082700
0.028	0.001032	0.027149	0.081830
0.029	0.000648	0.026225	0.080959
0.030	0.000379	0.025301	0.080089

For caps with $T = 1$,

Caps						
S		3			5	
κ	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
0.0005	0.012306	0.025562	0.052074	0.024718	0.051283	0.104414
0.0010	0.011358	0.024618	0.051136	0.022877	0.049449	0.102593
0.0015	0.010411	0.023674	0.050199	0.021036	0.047614	0.100771
0.0020	0.009464	0.022730	0.049262	0.019195	0.045780	0.098950
0.0025	0.008517	0.021786	0.048325	0.017353	0.043945	0.097129
0.0030	0.007570	0.020842	0.047388	0.015512	0.042111	0.095308
0.0035	0.006623	0.019899	0.046451	0.013672	0.040277	0.093487
0.0040	0.005676	0.018955	0.045513	0.011831	0.038442	0.091666
0.0045	0.004732	0.018011	0.044576	0.009995	0.036608	0.089845
0.0050	0.003797	0.017067	0.043639	0.008172	0.034773	0.088024
0.0055	0.002889	0.016123	0.042702	0.006392	0.032939	0.086203
0.0060	0.002046	0.015179	0.041765	0.004710	0.031105	0.084382
0.0065	0.001320	0.014236	0.040828	0.003215	0.029270	0.082561
0.0070	0.000759	0.013292	0.039890	0.001998	0.027436	0.080739
0.0075	0.000382	0.012348	0.038953	0.001112	0.025602	0.078918
0.0080	0.000166	0.011405	0.038016	0.000548	0.023768	0.077097
0.0085	0.000061	0.010462	0.037079	0.000236	0.021935	0.075276
0.0090	0.000019	0.009520	0.036142	0.000088	0.020104	0.073455
0.0095	0.000005	0.008581	0.035205	0.000028	0.018277	0.071634
0.0100	0.000001	0.007647	0.034267	0.000008	0.016458	0.069813
0.0105	0.000000	0.006723	0.033330	0.000002	0.014654	0.067992
0.0110	0.000000	0.005816	0.032393	0.000000	0.012875	0.066171
0.0115	0.000000	0.004937	0.031456	0.000000	0.011139	0.064350
0.0120	0.000000	0.004098	0.030519	0.000000	0.009465	0.062529
0.0125	0.000000	0.003316	0.029582	0.000000	0.007879	0.060708

For swaps with $\delta = 0.25$, $\delta = 0.5$, and $\delta = 1$,

Interest Rate Swap							
Maturity	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	Maturity	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
1	0.023452	0.023524	0.023675	16	0.029277	0.029385	0.029602
2	0.025445	0.025528	0.025699	17	0.029318	0.029426	0.029644
3	0.026433	0.026522	0.026704	18	0.029356	0.029464	0.029683
4	0.027138	0.027232	0.027423	19	0.029395	0.029503	0.029723
5	0.027711	0.027808	0.028006	20	0.029436	0.029545	0.029765
6	0.028114	0.028214	0.028417	21	0.029481	0.029590	0.029810
7	0.028345	0.028446	0.028651	22	0.029528	0.029637	0.029859
8	0.028544	0.028647	0.028855	23	0.029578	0.029688	0.029910
9	0.028695	0.028798	0.029008	24	0.029631	0.029741	0.029964
10	0.028864	0.028969	0.029182	25	0.029685	0.029796	0.030019
11	0.028966	0.029071	0.029285	26	0.029741	0.029852	0.030076
12	0.029050	0.029156	0.029371	27	0.029798	0.029909	0.030134
13	0.029121	0.029228	0.029444	28	0.029855	0.029967	0.030193
14	0.029181	0.029288	0.029505	29	0.029912	0.030024	0.030251
15	0.029232	0.029340	0.029557	30	0.029969	0.030081	0.030309

For swaptions with $T = 1$ and $S = 3$,

Interest Rate Swaption							
κ	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	κ	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
0.0100	0.034068	0.034134	0.034267	0.0175	0.019861	0.019977	0.020210
0.0105	0.033121	0.033190	0.033330	0.0180	0.018914	0.019033	0.019273
0.0110	0.032174	0.032247	0.032393	0.0185	0.017966	0.018089	0.018336
0.0115	0.031226	0.031303	0.031456	0.0190	0.017019	0.017145	0.017398
0.0120	0.030279	0.030359	0.030519	0.0195	0.016072	0.016202	0.016461
0.0125	0.029332	0.029415	0.029582	0.0200	0.015125	0.015258	0.015524
0.0130	0.028385	0.028471	0.028644	0.0205	0.014178	0.014314	0.014587
0.0135	0.027438	0.027527	0.027707	0.0210	0.013231	0.013370	0.013650
0.0140	0.026491	0.026584	0.026770	0.0215	0.012284	0.012427	0.012713
0.0145	0.025544	0.025640	0.025833	0.0220	0.011339	0.011485	0.011777
0.0150	0.024596	0.024696	0.024896	0.0225	0.010395	0.010544	0.010843
0.0155	0.023649	0.023752	0.023959	0.0230	0.009456	0.009607	0.009911
0.0160	0.022702	0.022808	0.023021	0.0235	0.008524	0.008677	0.008986
0.0165	0.021755	0.021865	0.022084	0.0240	0.007604	0.007759	0.008070
0.0170	0.020808	0.020921	0.021147	0.0245	0.006704	0.006859	0.007170

For swaptions with $T = 1$ and $S = 5$,

Interest Rate Swaption							
κ	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$	κ	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
0.0100	0.069414	0.069547	0.069813	0.0175	0.041798	0.042031	0.042497
0.0105	0.067573	0.067712	0.067992	0.0180	0.039957	0.040196	0.040676
0.0110	0.065732	0.065878	0.066171	0.0185	0.038116	0.038362	0.038854
0.0115	0.063891	0.064044	0.064350	0.0190	0.036275	0.036528	0.037033
0.0120	0.062050	0.062209	0.062529	0.0195	0.034434	0.034693	0.035212
0.0125	0.060209	0.060375	0.060708	0.0200	0.032593	0.032859	0.033391
0.0130	0.058368	0.058540	0.058886	0.0205	0.030752	0.031024	0.031570
0.0135	0.056527	0.056706	0.057065	0.0210	0.028911	0.029190	0.029749
0.0140	0.054685	0.054872	0.055244	0.0215	0.027070	0.027356	0.027928
0.0145	0.052844	0.053037	0.053423	0.0220	0.025229	0.025521	0.026107
0.0150	0.051003	0.051203	0.051602	0.0225	0.023388	0.023687	0.024286
0.0155	0.049162	0.049368	0.049781	0.0230	0.021547	0.021853	0.022465
0.0160	0.047321	0.047534	0.047960	0.0235	0.019706	0.020018	0.020644
0.0165	0.045480	0.045700	0.046139	0.0240	0.017865	0.018184	0.018823
0.0170	0.043639	0.043865	0.044318	0.0245	0.016027	0.016352	0.017003

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국문초록

이 논문의 첫 목적은 헤스톤 모델과 SABR 모델을 포함한 확률 변동성 모델과 CEV 모델과 다항식 모델을 포함한 국소 변동성 모델 하에서 콜 옵션의 내재변동성을 점근적으로 도출하는 것이다. 시장의 내재변동성 데이터와 도출된 내재변동성간의 평균 제곱오차를 최소화함으로써 매개변수를 추정할 수 있다. 또한, 이자율이 0이라는 가정 하에 블랙-숄즈 모델에서 아시안 옵션의 내재변동성을 점근적으로 도출할 수 있어서 몬테 카를로 시뮬레이션 없이도 아시안 옵션의 가격을 계산할 수 있다.

두 번째 목적은 헐-화이트 모델에서 이자율 파생상품의 가격을 계산하는 것이다. 국채 수익률 데이터로부터 3차 스플라인 곡선 기법을 통해 국채 수익률 곡선을 도출할 수 있다. 이로부터 헐-화이트 모델의 $\theta(t)$를 도출할 수 있다. $b = 0.5$라 두고 σ를 경험적 방법으로 추정할 수 있다. 스왑션을 제외한 이자율 파생상품의 가격은 공식이 있어 계산할 수 있고 스왑션은 근사적으로 계산된다.

주요어 : 헤스톤 모델, SABR 모델, 국소 변동성 모델, 블랙-숄즈 모델, 헐-화이트 모델, 3차 스플라인 곡선
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