교육학 석사 학위논문

p-row graphs and p-competition graphs

(p-행 그래프와 p-경쟁 그래프)

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서울대학교 대학원

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이 논문을 교육학 석사 학위논문으로 제출함

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p-row graphs and p-competition graphs

A dissertation
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by

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Abstract

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For a positive integer p, the p-competition graph of a digraph D is a graph which has the same vertex set as D and an edge between distinct vertices x and y if and only if x and y have at least p common out-neighbors in D. A graph is said to be a p-competition graph if it is the p-competition graph of a digraph. Given a graph G, we call the set of positive integers p such that G is a p-competition the competition-realizer of a graph G. We denote by $G/\sim$ the graph obtained from a graph G by identifying each pair of adjacent vertices which share the same closed neighborhood. In this paper, we introduce the notion of p-row graph of a matrix which generalizes the existing notion of row graph. Using the notions of p-row graph and $G/\sim$ for a graph G, we study competition-realizers for various graphs to extend results given by Kim *et al.* [p-competition graphs, *Linear Algebra Appl.* **217** (1995) 167–178]. Especially, we find all the elements in the competition-realizer for each caterpillar.

Key words: p-competition graph; p-edge clique cover; competition-realizer; p-row graph; $G/\sim$; caterpillar
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Chapter 1

Introduction

Given a digraph $D = (V, A)$, the *competition graph* of D has the same vertex set as D and has an edge uv if for some vertex $x \in V$, the arcs (u, x) and (v, x) are in D. The notion of competition graph is due to Cohen [1] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modelling of complex economic systems. (See [13] and [14] for a summary of these applications and [4] for a sample paper on the modelling application.) Since Cohen introduced the notion of competition graph, various variations have been defined and studied by many authors (see the survey articles by Kim [6] and Lundgren [11]). For recent work on this topic, see [3, 5, 9, 10, 15].

Kim *et al.* [7] introduced p-competition graphs as a variant of competition graph. For a positive integer p, the *p-competition graph* $C_p(D)$ corresponding to a digraph $D = (V, A)$ is defined to have vertex set V with an edge between two distinct vertices x and y if and only if, for some distinct $a_1, \ldots, a_p$ in V, the pairs $(x, a_1), (y, a_1), (x, a_2), (y, a_2), \ldots, (x, a_p), (y, a_p)$ are arcs. Note that $C_1(D)$ is the ordinary competition graph, which implies that the notion of p-competition graph generalizes that of competition graph. A graph G is called a *p-competition graph* if there exists a digraph D such that $G = C_p(D)$. By definition, it is obvious that if a nonempty graph G is a p-competition graph, then $p \leq |V(G)|$.

Competition graphs are closely related to edge clique covers and the edge clique cover numbers of graphs. A *clique* of a graph G is a subset of the vertex set of G such that its induced subgraph of G is a complete graph. We regard an empty set also as a clique of G for convenience. An *edge clique cover* of a graph G is a family of cliques of G such that the end vertices of each edge of G are contained in some clique in the family. The minimum size of an edge clique cover of G is called the *edge clique cover number* of the graph G, and is denoted by $\theta_e(G)$. Dutton and Brigham [2] characterized a competition graph in terms of its edge clique cover number.

Theorem 1.1 ([2]). *A graph G with n vertices is a competition graph if and only if $\theta_e(G) \leq n$.*

A p-competition graph G can be characterized in terms of the "p-edge clique cover number" of G. For a positive integer p, a *p-edge clique cover* (*p-ECC* for short) of a graph G is defined to be a multifamily $\mathcal{F} = \{S_1, \ldots, S_r\}$ of subsets of the vertex set of G satisfying the following:

- For any $J \in \binom{[r]}{p}$, the set $\bigcap_{j \in J} S_j$ is a clique of G;
- The collection $\left\{ \bigcap_{j \in J} S_j \mid J \in \binom{[r]}{p} \right\}$ covers all the edges of G,

where $\binom{[r]}{p}$ denotes the set of p-element subsets of the set $\{1, \ldots, r\}$. The minimum size r of a p-edge clique cover of G is called the *p-edge clique cover number* of G, and is denoted by $\theta_e^p(G)$. The following theorem characterizes p-competition graphs and so generalizes Theorem 1.1.

Theorem 1.2 ([7]). *A graph G with n vertices is a p-competition graph if and only if $\theta_e^p(G) \leq n$.*

In chapter 2, we introduce the notion of p-row graph of a matrix which generalizes the existing notion of row graph, and the notion of competition-realizer. Using the notions of p-row graph and $G/\!\sim$ for a graph G, we study competition-realizers for various graphs to extend results given by Kim *et al.* [7]. In chapter 3, we study the competition-realizers for trees. Especially, we find all the elements in the competition-realizer for each caterpillar.

Chapter 2

p-row graphs and competition-realizers

In this chapter, we introduce the notion of p-row graph of a matrix which generalize the notion of row graph of a matrix and the notion of competition-realizer for a graph. Then we study competition-realizers for various graphs in terms of p-row graphs and $G/\sim$ for a graph G. Particularly, we identify the graphs with n vertices the competition-realizers for which contain n and $n-1$, respectively.

Definition 2.1. Given a positive integer p and a $(0,1)$-matrix A, a graph G is called the *p-row graph* of A if the vertices of G are the rows of A, and two vertices are adjacent in G if and only if their corresponding rows have common nonzero entries in at least p columns of A.

If $p = 1$, then G is called the *row graph* of A, which was introduced by Greenberg *et al.* [4].

Suppose that a graph G is a p-competition graph with the vertex set $\{v_1, \ldots, v_n\}$. Then there exists a p-ECC $\mathcal{F} = \{F_1, \ldots, F_m\}$ of G for a non-negative integer $m \leq n$ by Theorem 1.2. Now we define a square matrix

$A = (a_{ij})$ of order n by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \in F_j; \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

By the definition of p-ECC, it is easy to see that G is isomorphic to the p-row graph of A. Conversely, suppose that a graph G with n vertices is isomorphic to the p-row graph of a square $(0,1)$-matrix A of order n. Let

$$F_j = \{v_i \mid a_{ij} = 1\}.$$

and let $\mathcal{F} = \{F_1, \ldots, F_n\}$. By the definition of p-row graph, a vertex v_s and a vertex v_t are adjacent if and only if the sth row and the tth row of A have common nonzero entries in at least p columns, which is equivalent to the statement that v_s and v_t are contained in the sets in $\mathcal{F}$ corresponding to those columns. Then, by Theorem 1.2, G is a p-competition graph.

Now we have shown the following statement:

Theorem 2.2. *A graph G with n vertices is a p-competition graph if and only if G is isomorphic to the p-row graph of a square $(0,1)$-matrix of order n.*

For simplicity's sake, we denote $J_{m,n}$ for the $(0,1)$-matrix of size m by n such that every entry is 1, I_n for the identity matrix of order n, and $O_{m,n}$ for the zero matrix of size m by n.

For a graph G with n vertices, we denote the set

$$\{p \in [n] \mid G \text{ is a } p\text{-competition graph}\}$$

by $\Upsilon(G)$ and call it the *competition-realizer* for G.

We make the following simple but useful observations.

Proposition 2.3. *Let G be a graph with n vertices. If G is empty or complete, then $\Upsilon(G) = [n]$.*

Proof. If G is empty, then G is a p-row graph of $O_{n,n}$ and so, by Theorem 2.2, is a p-competition graph for any $p \in [n]$. If G is complete, then G is a p-row graph of $J_{n,n}$ and so, by the same theorem, is a p-competition graph for any $p \in [n]$. □

Proposition 2.4. *Given a graph G with n vertices, suppose that G is a p-row graph of a matrix of size n by m for positive integers p and $m \leq n$. Then $\Upsilon(G) \supset \{p + i \mid i \in [n-m] \cup \{0\}\}$.*

Proof. Let M be an $n \times m$ matrix whose p-row graph is G. For each $i \in [n-m] \cup \{0\}$, we add i all-one columns and $n-m-i$ all-zero columns to M to obtain a square matrix of order n whose $(p+i)$-row graph is G. □

Proposition 2.5. *Let G be a graph and G' be a graph obtained from G by adding k isolated vertices. Then $\Upsilon(G') \supset \{p + i \mid p \in \Upsilon(G),\ i \in [k] \cup \{0\}\}$.*

Proof. Let $|V(G)| = n$ and take $p \in \Upsilon(G)$. By Theorem 2.2, G is a p-row graph of a square $(0,1)$-matrix M of order n. Fix $i \in [k] \cup \{0\}$. We define the square $(0,1)$-matrix M_i of order $n+k$ as follows:

$$M_i = \left[\begin{array}{c:c:c} M & J_{n,i} & O_{n,k-i} \\ \hdashline \multicolumn{3}{c}{O_{k,n+k}} \end{array}\right]$$

Obviously, the $(p+i)$-row graph of M_i is G together with k isolated vertices. □

For a p-row graph G of a matrix M and a vertex u of G, we let

$$\Lambda_M(u) = \{i \mid \text{the } i\text{th component of the row corresponding to } u \text{ in } M \text{ is } 1\}.$$

Proposition 2.6. *Let G be a p-row graph of a matrix M. Then, for a non-isolated non-simplicial vertex u, $|\Lambda_M(u)| \geq p + 1$.*

Proof. By the condition on u, u is adjacent to two nonadjacent vertices v and w. Suppose that $|\Lambda_M(u)| \leq p$. Then

$$p \leq |\Lambda_M(u) \cap \Lambda_M(v)| \leq p \quad \text{and} \quad p \leq |\Lambda_M(u) \cap \Lambda_M(w)| \leq p.$$

Thus $|\Lambda_M(u) \cap \Lambda_M(v)| = |\Lambda_M(u) \cap \Lambda_M(w)| = p$ and so $\Lambda_M(u) \cap \Lambda_M(v) = \Lambda_M(u) \cap \Lambda_M(w) = \Lambda_M(u)$. Hence $\Lambda_M(u) \subset \Lambda_M(v) \cap \Lambda_M(w)$ and so $|\Lambda_M(v) \cap \Lambda_M(w)| \geq p$, which is a contradiction. □

The following proposition characterizes a graph G with n vertices and $n \in \Upsilon(G)$.

Proposition 2.7. *Let G be a graph with n vertices. Then G is an n-competition graph if and only if $G \cong K_m \cup \mathcal{I}_{n-m}$ for some m, $0 \leq m \leq n$.*

Proof. By definition, G is an n-competition graph if and only if $n \in \Upsilon(G)$. To show the "if" part, suppose that $G \cong K_m \cup \mathcal{I}_{n-m}$ for some m, $0 \leq m \leq n$. By Proposition 2.3, $m \in \Upsilon(K_m)$. By Proposition 2.5, $m + (n-m) \in \Upsilon(G)$. To show the "only if" part, suppose that G is an n-competition graph. Then G is isomorphic to the n-row graph of a matrix M by Theorem 2.2. Take a non-isolated vertex u in G. Then u is adjacent to a vertex v in G, so $|\Lambda_M(u) \cap \Lambda_M(v)| = n$. Thus we may conclude that each row of M corresponding to a non-isolated vertex is the all-one vector in $\mathbb{R}^n$. Thus the subgraph of G induced by non-isolated vertices is a clique. Hence $G = K_m \cup \mathcal{I}_{n-m}$ where m is the number of non-isolated vertices in G. □

Corollary 2.8. *Suppose that a graph G with n vertices has no isolated vertices. Then G is an n-competition graph if and only if $G = K_n$.*

Corollary 2.9. *Let G be a graph with n vertices. Then $\Upsilon(G) = [n]$ if and only if $G \cong K_m \cup \mathcal{I}_{n-m}$ for some m, $0 \leq m \leq n$.*

Proof. The 'only if' part immediately follows by Proposition 2.7. To show the 'if' part, suppose that $G \cong K_m \cup \mathcal{I}_{n-m}$ for some m, $0 \leq m \leq n$. Let M

be a square $(0,1)$-matrix of order n such that the first m rows are all-one vector and the other $n-m$ rows are all-zero vector. Then it is easy to check that the p-row graph of M is isomorphic to G for each $p \in [n]$. $\square$

The competition-realizer may be empty for some graph. For example, for the complete bipartite graph $K_{3,3}$, $\Upsilon(K_{3,3}) = \emptyset$. To see why, we note that the number of vertices of $K_{3,3}$ is 6 and $\theta_e(K_{3,3}) = 9$. Therefore $1 \notin \Upsilon(K_{3,3})$ by Theorem 1.2. By Proposition 2.6, $5 \notin \Upsilon(K_{3,3})$ and $6 \notin \Upsilon(K_{3,3})$. Suppose that $K_{3,3}$ is a p-competition graph for some $p \in \{2,3,4\}$. Then G is isomorphic to the p-row graph of a square $(0,1)$-matrix M_p by Theorem 2.2.

Consider the case $p = 4$. Then each row of M_4 contains at least five 1s by Proposition 2.6. This implies that any two rows of M_4 have at least four common 1s and so G is isomorphic to K_6, which is a contradiction. Thus $4 \notin \Upsilon(K_{3,3})$.

Now consider the case $p = 3$. Then each row of M_3 contains at least four 1s by Proposition 2.6. This implies that any two rows of M_3 have at least two common 1s. If there is a row containing at least five 1s, then it shares at least three common 1s with each of the other vertices, which is impossible. Thus each row of M_3 contains exactly four 1s. Since $K_{3,3}$ has a partite set of size 3, we may assume that M_3 contains the following submatrix by permuting columns, if necessary:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Now we take a vertex u in the other partite set. Then u is adjacent to the vertex corresponding to each row of the above submatrix. To have u and the vertex corresponding to the first row of the above submatrix be adjacent, $\Lambda_{M_3}(u) \cap \{1,2,3,4\}$ is one of $\{1,2,3\}, \{2,3,4\}, \{1,2,4\}$, and $\{1,3,4\}$. Then, in case of $\{1,2,3\}$ or $\{1,2,4\}$, u is not adjacent to the vertex corresponding to the second row, while, in case of $\{1,3,4\}$ or $\{2,3,4\}$, u is not adjacent to the vertex corresponding to the third row. Therefore we reach a contradiction.

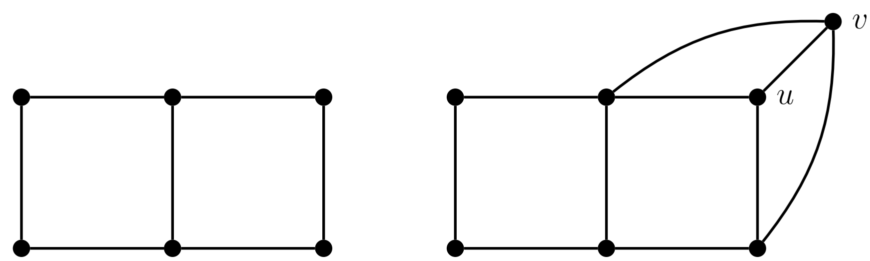
Thus $3 \notin \Upsilon(K_{3,3})$.

Now consider the case $p = 2$. Then each row of M_2 contains at least three 1s. Suppose that there is a row $\mathbf{r}_1$ containing at least four 1s. We may assume that $\mathbf{r}_1$ has 1s in the first component through the fourth component. Let v_1 be the vertex corresponding to $\mathbf{r}_1$ and v_2 and v_3 be the other vertices in the partite set to which v_1 belongs. Since v_1 and v_2 are not adjacent, $\mathbf{r}_1$ has exactly four 1s and the row $\mathbf{r}_2$ corresponding to v_2 has 1 in the fourth component through the sixth component. Then the row corresponding to v_3 must share at least two 1s with $\mathbf{r}_1$ or $\mathbf{r}_2$ and we reach a contradiction. Thus each row of M_2 contains exactly three 1s. If there are two vertices w_1 and w_2 in a partite set W such that their corresponding rows do not share 1s, then the row corresponding to the remaining vertex in W must share at least two 1s with one of the rows corresponding to w_1 and w_2, and we reach a contradiction. Therefore the rows corresponding to two vertices in the same partite set share exactly one 1. Thus we may assume that M_2 contains the following submatrix by permuting columns, if necessary:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Now we take a vertex x in the other partite set X. Then x is adjacent to each of the vertices corresponding to the rows of the above submatrix. Therefore $\Lambda_{M_2}(x) = \{1, 3, 5\}$. Since x is arbitrarily chosen, the rows corresponding to the other two vertices in X also have the first, the third, and the fifth component equal 1, which is impossible. Hence we have shown that $\Upsilon(K_{3,3}) = \emptyset$.

Let G be a graph with n vertices. Two vertices u and v of G are said to be *homogeneous*, denoted by $u \sim v$, if they have the same closed neighborhood. Clearly $\sim$ is an equivalence relation on $V(G)$. We denote the equivalence class containing a vertex u of G by $[u]$. Then we define a new graph $G/\sim$ for



G_1 G_2

Figure 2.1: $G_1 \cong G_2/\sim$ (in G_2, $u \sim v$)

G by

$$V(G/\sim) = \{[u] \mid u \in V(G)\} \quad \text{and} \quad E(G/\sim) = \{[u][v] \mid u, v \in V(G) \text{ and } uv \in E(G)\}.$$

See Figure 2.1 for an illustration.

We note the following: Two vertices u and v are adjacent in $G \Leftrightarrow v \in N_G[u]$
$\Leftrightarrow v \in N_G[u']$ for any $u' \in [u] \Leftrightarrow u' \in N_G[v]$ for any $u' \in [u]$
$\Leftrightarrow u' \in N_G[v']$ for any $u' \in [u]$ and $v' \in [v]$
$\Leftrightarrow u'$ and v' are adjacent in G for any $u' \in [u]$ and $v' \in [v]$.
Therefore $G/\sim$ is well-defined.

It is obvious that

($\star$) for each isolated vertex in G, its equivalence class is isolated in $G/\sim$.

The notion of row graph provides a way of getting information on the competition-realizer for a graph G from the competition-realizer for a simpler graph $G/\sim$ as seen in the following results.

Proposition 2.10. *A connected non-complete graph G and $G/\sim$ have the same diameter.*

Proof. By the definition of $G/\sim$, there exists an induced (x, y)-path of length

l in G if and only if there exists an induced $([x],[y])$-path of length l in $G/\sim$ for some vertices x and y in G and an integer $l \geq 2$.

Let m be the diameter of G. Since G is not complete, $m \geq 2$. Then there exists an induced (u,v)-path of length m for some vertices u and v. By the above observation, there exists an induced $([u],[v])$-path of length m and there is no induced $([u],[v])$-path of length l for any $2 \leq l < m$ in $G/\sim$. If there exists a $([u],[v])$-path of length 1, then u and v are adjacent, which contradicts the choice of u and v so that $d_G(u,v) = m \geq 2$. Therefore the diameter of $G/\sim$ is greater than equal to m. By the symmetry of the above observation, it is also true that the diameter of $G/\sim$ is less than equal to m. □

Proposition 2.11. *A graph G is a p-competition graph if and only if there exists a square matrix M such that G is a p-row graph of M and the rows corresponding to two homogeneous vertices are identical.*

Proof. The "if" part is obvious. To show the "only if" part, suppose that a graph G is a p-competition graph for some positive integer p. Then, by Theorem 2.2, there exists a square matrix M' such that G is a p-row graph of M'. If there are at least two rows corresponding to homogeneous vertices, then we fix one row among them and replace the remaining rows with the fixed row. We denote by M the matrix obtained by applying the above procedure. It is easy to see that G is a p-row graph of M. □

Proposition 2.12. *Given a graph G with n vertices, suppose that $G/\sim$ is a p-row graph of a matrix M satisfying the property that M has m columns for a positive integer $m \leq n$ and every row of M has at least p 1s. Then $\Upsilon(G) \supset \{p+i \mid i \in [n-m] \cup \{0\}\}$.*

Proof. Let n_j be the size of equivalence class under $\sim$ corresponding to the jth row of M. We replace the jth row of M with n_j copies of it to obtain the matrix M^* which contains M as a submatrix. We note that the size of M^* is $n \times m$.

Take two vertices u and v in G and let $\mathbf{r}_u$ and $\mathbf{r}_v$ be the rows of M^* corresponding to u and v, respectively. If u and v are not homogenous, then they belong to distinct equivalence classes under $\sim$ and the following are true:

Two vertices u and v are adjacent in G

$\Leftrightarrow$ $[u]$ and $[v]$ are adjacent in $G/\sim$

$\Leftrightarrow$ the row corresponding to $[u]$ and the row corresponding to $[v]$ have at least p common 1s in M

$\Leftrightarrow$ u and v are adjacent in the p-row graph of M^*.

Suppose that u and v are homogenous. Then the rows $\mathbf{r}_u$ and $\mathbf{r}_v$ are identical. By the hypothesis, every row of M has at least p 1s. Thus $\mathbf{r}_u$ and $\mathbf{r}_v$ have at least p common 1s and so u and v are adjacent in the p-row graph of M^*. Hence G is p-row graph of M^* and so, by Proposition 2.4, G is $(p+i)$-competition graph, that is, $p+i \in \Upsilon(G)$ for any $i \in [n-m] \cup \{0\}$. □

For a positive integer p and the p-row graph G of a matrix M, each non-isolated vertex in G has at least p 1s in the row of M corresponding to it and so the following corollary is immediately true by the above theorem.

Corollary 2.13. *Given a graph G with n vertices, suppose that $G/\sim$ has no isolated vertices and is a p-row graph of a matrix M having m columns for a positive integer $m \le n$. Then $\Upsilon(G) \supset \{p+i \mid i \in [n-m] \cup \{0\}\}$.*

Corollary 2.14. *Given a graph G with n vertices, suppose that $G/\sim$ has m vertices none of which is isolated for a positive integer $m \le n$. Then*

$$\Upsilon(G) \supset \{p+i \mid p \in \Upsilon(G/\sim),\ i \in [n-m] \cup \{0\}\}.$$

Proof. By Theorem 2.2, $G/\sim$ is a p-row graph of a matrix M having m columns. Thus by Corollary 2.13, $p+i \in \Upsilon(G)$ for any $i \in [n-m] \cup \{0\}$. □

Remark 2.15. Even if G is a p-competition graph, $G/{\sim}$ may not be $(p-i)$-competition graph for some $i \in [n-m] \cup \{0\}$ where $n = |V(G)|$ and $m = |V(G/{\sim})|$. For example, the graph G_2 in Figure 2.1 is a 2-competition graph. For, G_1 is a 2-competition graph by Theorem 2.2 since G_1 is the 2-row graph of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Since G_1 is isomorphic to $G_2/{\sim}$, G_2 is a 2-competition graph. Yet, G_1, which is isomorphic to $G_2/{\sim}$, is not a 1-competition graph by Theorem 1.1 since $|V(G_1)| = 6 < 7 = |E(G_1)| = \theta_e(G_1)$.

We denote a set of m isolated vertices by $\mathcal{I}_m$. Technically, we let $\mathcal{I}_0 = \emptyset$ and $K_0 = \emptyset$.

A *union* $G \cup H$ of two graphs G and H is the graph having its vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. In this paper, the union of G and H means their disjoint union which has an additional condition $V(G) \cap V(H) = \emptyset$. A *join* $G \vee H$ of two graphs G and H is the graph having its vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.

For a positive integer n, a nonnegative integer $k \leq n$, and the power set $\mathcal{P}([n])$ of $[n]$, we denote by $\Psi_{n,k}$ the graph with the vertex set $\mathcal{P}([n])$ and the edge set

$$\{ST \mid S, T \subseteq [n], |S \cup T| \leq k\}.$$

Theorem 2.16. *Let G be a connected graph with n vertices and k be a nonnegative integer less than or equal to n. Then G is an $(n-k)$-competition graph if and only if $G/{\sim}$ is isomorphic to an induced subgraph of $\Psi_{n,k}$.*

Proof. To show the "only if" part, suppose that G is an $(n-k)$-competition graph. Then, by Proposition 2.11, there exists a matrix M such that G is an $(n-k)$-row graph of M and the rows corresponding to two homogeneous vertices are identical. Let M' be a submatrix of M obtained by taking all the distinct rows of M. Then obviously $G/\!\sim$ is an $(n-k)$-row graph of M'. Therefore $[x]$ and $[y]$ are adjacent in $G/\!\sim$ if and only if $|\Lambda_{M'}([x]) \cap \Lambda_{M'}([y])| \geq n-k$ if and only if $|([n] \setminus \Lambda_{M'}([x])) \cup ([n] \setminus \Lambda_{M'}([y]))| \leq k$ if and only if $[n] \setminus \Lambda_{M'}([x])$ and $[n] \setminus \Lambda_{M'}([y])$ are adjacent in $\Psi_{n,k}$. Hence we have shown that $G/\!\sim$ is isomorphic to an induced subgraph of $\Psi_{n,k}$.

To show the "if" part, suppose that $G/\!\sim$ is isomorphic to an induced subgraph of $\Psi_{n,k}$. Then each vertex $[v]$ of $G/\!\sim$ is assigned a subset S_v so that $[v]$ and $[w]$ are adjacent in $G/\!\sim$ if and only if $|S_v \cup S_w| \leq k$. If $G/\!\sim$ is empty, then $\Upsilon(G) = [n]$ by Proposition 2.3 and Corollary 2.14. Since G is connected, $G/\!\sim$ is connected and so $|S_v| \leq k$. We denote by M' the matrix with each row corresponds to a vertex of $G/\!\sim$ in such a way that $[n] \setminus \Lambda_{M'}([v]) = S_v$. Then it is easy to see that $G/\!\sim$ is an $(n-k)$-row graph of M'. By Corollary 2.14, we can conclude that $n-k \in \Upsilon(G)$. □

Lemma 2.17. *The star graph $K_{1,n}$ is an n-competition graph.*

Proof. It is obvious that $K_{1,n}$ is the n-row graph of the following square matrix of order $n+1$:

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \end{bmatrix}.$$

□

The following proposition characterizes a graph G with n vertices and $n-1 \in \Upsilon(G)$.

Proposition 2.18. *Let G be a graph with n vertices. Then G is an $(n-1)$-competition graph if and only if $G \cong (K_{n_0} \vee (K_{n_1} \cup \cdots \cup K_{n_k})) \cup \mathcal{I}_m$ for some nonnegative integers $k, n_0, n_1, \ldots, n_k$, and m satisfying $m + \sum_{i=0}^{k} n_i = n$.*

Proof. We show the "if" part. If G is empty, then, by Proposition 2.3, G is a $(n-1)$-competition graph. Now suppose that G is a nonempty graph. Let G' be the subgraph of G resulting from deleting all the isolated vertices in G. It is easy to check that $G'/\sim$ is an empty graph or a star graph. Thus, by Proposition 2.3 and Lemma 2.17, $|V(G'/\sim)| - 1 \in \Upsilon(G'/\sim)$. Then $|V(G')| - 1 \in \Upsilon(G')$ by Corollary 2.14. By Proposition 2.5, G is an $(n-1)$-competition graph.

Now we show the "only if" part. Suppose that G is an $(n-1)$-competition graph. Then G is isomorphic to the $(n-1)$-row graph of a matrix M. Suppose that there exists a row, say $\mathbf{r}$, of M such that $\mathbf{r}$ contains at most $n-2$ 1s. Then the vertex in G corresponding to $\mathbf{r}$ is an isolated vertex, so G is still the $(n-1)$-row graph of the matrix resulting from replacing $\mathbf{r}$ with all-zero row. Therefore we may assume that M contains the rows of exactly three types:

1. the row with n 1s;
2. the row with $n-1$ 1s;
3. the row with 0 1s.

Obviously, the vertex corresponding to a row of Type 1 is a vertex which is adjacent to each of other non-isolated vertices and the vertex corresponding to a row of Type 3 is an isolated vertex. We note that two rows of Type 2 are identical if their corresponding vertices are adjacent in G. Therefore the vertex corresponding to a row of Type 2 is a simplicial vertex, that is, a vertex whose neighbors form a clique. Now the vertex set V of G can be partitioned into three subsets V_1, V_2, and V_3 such that V_i is the set of vertices corresponding to rows of Type i for $i = 1, 2, 3$. Let $n_0 = |V_1|$ and $m = |V_3|$. If $V_2 = \emptyset$, then $G \cong K_{n_0} \cup \mathcal{I}_m$ by Proposition 2.7. Now suppose that $V_2 \neq \emptyset$.

Then the subgraph of G induced by V_2 is a disjoint union of cliques. Let $W_1, W_2, \ldots, W_k$ be the vertex sets of those cliques and let $|W_i| = n_i$ for $1 \leq i \leq k$. Then $m + \sum_{i=0}^{k} n_i = n$. Since every vertex in V_1 is adjacent to each of other non-isolated vertices and every vertex in V_3 is isolated in G, $G \cong (K_{n_0} \vee (K_{n_1} \cup \cdots \cup K_{n_k})) \cup \mathcal{I}_m$. □

Theorem 2.19. *Let G be a graph with n vertices. Then $\Upsilon(G) = [n-1]$ if and only if $G \cong H \cup \mathcal{I}_m$ for some integer m, $0 \leq m \leq n$ and some graph H for which $H/\!\sim$ is an induced subgraph of a star graph with more than one vertex.*

Proof. To show the "if" part, suppose that $G \cong H \cup \mathcal{I}_m$ for some integer m, $0 \leq m \leq n$ and some graph H for which $H/\!\sim$ is an induced subgraph of a star graph Q with more than one vertex. We denote the number of vertices in $H/\!\sim$ by t. Take $p \in [t-1]$. We construct a square $(0,1)$-matrix M of order t in the following way. If $H/\!\sim$ contains a center of Q, then the row of M corresponding to it is the all-one vector. The rows of M corresponding to the vertices in $H/\!\sim$ which are not a center of Q are mutually distinct, and the number of 1s in each of them is p. Such a matrix M exists since $\binom{t}{p} \geq t$. It is easy to check that $H/\!\sim$ is isomorphic to the p-row graph of M. Thus $[t-1] \subset \Upsilon(H/\!\sim)$. By Proposition 2.12, $[n-m-1] \subset \Upsilon(H)$. Now, by Proposition 2.5, $[n-1] \subset \Upsilon(G)$. By Proposition 2.7, $n \notin \Upsilon(G)$ and so $\Upsilon(G) = [n-1]$.

To show the "only if" part, suppose that $\Upsilon(G) = [n-1]$. Then, by Proposition 2.18, $G \cong (K_{n_0} \vee (K_{n_1} \cup \cdots \cup K_{n_k})) \cup \mathcal{I}_m$ for some nonnegative integers $k, n_0, n_1, \ldots, n_k$, and m satisfying $m + \sum_{i=0}^{k} n_i = n$. If there is at most one nonzero integer among $n_1, n_2, \ldots, n_k$, then $G \cong K_l \cup \mathcal{I}_{n-l}$ for some l, $0 \leq l \leq n$ and so, by Corollary 2.9, $\Upsilon(G) = [n]$, which is a contradiction. Therefore there are at least two nonzero integers among $n_1, n_2, \ldots, n_k$ and so $H := K_{n_0} \vee (K_{n_1} \cup \cdots \cup K_{n_k})$ is an induced subgraph of G. It is easy to check that $H/\!\sim$ is an induced subgraph of a star graph with more than one vertex. □

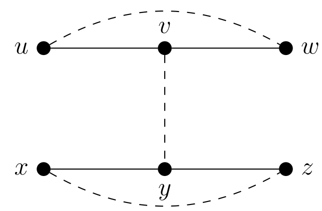


Figure 2.2: Paths uvw and xyz given in the proof of Proposition 2.20. The dotted line between two vertices means that they are not adjacent.

For $p = n$ or $n-1$, a p-competition graph is a chordal graph by Propositions 2.7 and 2.18. As a matter of fact, an $(n-2)$-competition graph is also chordal.

An *induced path* of a graph means a path as an induced subgraph of the graph.

Proposition 2.20. *If a graph G with n vertices contains two internally disjoint induced paths of length* 2 *whose internal vertices are nonadjacent, then* $\Upsilon(G) \subset [n-3]$

Proof. Let G be a graph with n vertices containing two internally disjoint induced paths uvw and xyz of length 2 with v and y nonadjacent (see Figure 2.2). Then, by Propositions 2.7 and 2.18, $\Upsilon(G) \subset [n-2]$. Suppose, to the contrary, that $n-2 \in \Upsilon(G)$. By Theorem 2.2, G is isomorphic to $(n-2)$-row graph of a square matrix M of order n. If $|\Lambda_M(v)| \geq n-1$ and $|\Lambda_M(y)| \geq n-1$, then $|\Lambda_M(v) \cap \Lambda_M(y)| \geq n-2$ and so v and y are adjacent, which is impossible. Thus $|\Lambda_M(v)| \leq n-2$ or $|\Lambda_M(y)| \leq n-2$. Without loss of generality, we may assume that $|\Lambda_M(v)| \leq n-2$. Since v is non-isolated, $|\Lambda_M(v)| = n-2$. Since u and v (resp. w and v) are adjacent, $|\Lambda_M(u) \cap \Lambda_M(v)| \geq n-2$ (resp. $|\Lambda_M(w) \cap \Lambda_M(v)| \geq n-2$). Since $|\Lambda_M(v)| = n-2$, $\Lambda_M(v) \subset \Lambda_M(u)$ and $\Lambda_M(v) \subset \Lambda_M(w)$. Therefore $\Lambda_M(v) \subset \Lambda_M(u) \cap \Lambda_M(w)$ and so $|\Lambda_M(u) \cap \Lambda_M(w)| \geq n-2$. Then u and v are adjacent in G and we reach a contradiction. □

A *hole* of a graph is a cycle of length greater than or equal to 4 which is an induced subgraph of the graph. A graph without holes is said to be *chordal*.

Corollary 2.21. *If a graph G with $n \geq 4$ vertices is non-chordal or has an induced path of length 4, then $\Upsilon(G) \subset [n-3]$*

Proof. If a p-competition graph with n vertices is non-chordal or has an induced path of length 4 for integers $n \geq 4$ and $p \in [n]$, then it contains two internally disjoint induced paths of length 2 whose internal vertices are nonadjacent and the statement is true by Proposition 2.20. □

By Corollary 2.8, it is trivially true that if a connected graph G with n vertices is an n-competition graph, then the diameter of G is 1. By Proposition 2.18, the diameter of a connected $(n-1)$-competition graph which has n vertices is at most 2. The diameter of a connected $(n-2)$-competition graph which has n vertices is at most 3 by Corollary 2.21. However, interestingly, the diameter of a connected $(n-3)$-competition graph with n vertices can be arbitrarily large, which will be shown by Lemma 3.4.

Proposition 2.22. *For a graph G with $\theta_e(G) \leq |V(G)|$, $[|V(G)| - \theta_e(G) + 1] \subset \Upsilon(G)$.*

Proof. Let $|V(G)| = n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. There is an edge clique cover $\mathcal{C} := \{C_1, C_2, \ldots, C_{\theta_e(G)}\}$ of G as $\theta_e(G)$ is the edge clique number of G. We define an $n \times \theta_e(G)$ matrix $M = (m_{ij})$ as follows:

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \in C_j \\ 0 & \text{if } v_i \notin C_j \end{cases}$$

Then G is isomorphic to the 1-row graph of M. Therefore the statement is true by Proposition 2.4. □

Given a graph G, it is known that it can be made into the competition graph of an acyclic digraph as long as it is allowed to add new isolated vertices. The smallest among such numbers is called the *competition number* of G and denoted by $k(G)$. Opsut [12] showed that

$$k(G) \geq \theta_e(G) - |V(G)| + 2. \tag{2.2}$$

Corollary 2.23. *Let G be a graph with ω components. If each component of G has competition number one, then $[\omega + 1] \subset \Upsilon(G)$.*

Proof. Let $G_1, G_2, \ldots, G_\omega$ be the components of G. Then, by (2.2), $|V(G_i)| - \theta_e(G_i) \geq 2 - k(G_i) = 1$ for any $1 \leq i \leq \omega$. Since $|V(G)| = \sum_{i=1}^{\omega} |V(G_i)|$ and $\theta_e(G) = \sum_{i=1}^{\omega} \theta_e(G_i)$, $|V(G)| - \theta_e(G) + 1 \geq \omega + 1$. Thus, by Proposition 2.22, the corollary is true. □

Since it is known that the competition numbers of a chordal graph and a forest are at most 1, the following corollaries immediately follow from Corollary 2.23.

Corollary 2.24. *For a chordal graph G having ω components, $[\omega + 1] \subset \Upsilon(G)$.*

Corollary 2.25. *For a forest G having ω components, $[\omega + 1] \subset \Upsilon(G)$.*

Chapter 3

Competition-realizers for trees

In this chapter, we study p-competition trees. Especially, we completely characterize the competition-realizers for caterpillars.

Let G be a p-competition graph. Then G is isomorphic to the p-row graph of a matrix $M = (m_{ij})$. If $|\Lambda_M(v)| \le p-1$, then v is an isolated vertex in G, and so G is still the p-row graph of the resulting matrix even if the row corresponding to v is replaced by the row with $p-1$ 1s. Thus we may conclude as follows:

(§) If a p-competition graph G is isomorphic to the p-row graph of a matrix M, then we may assume that $|\Lambda_M(v)| \ge p-1$ for each vertex v in G.

Adding a pendant vertex v to a graph G means obtaining a graph G' such that $v \notin V(G)$, $V(G') = V(G) \cup \{v\}$, and $E(G') = E(G) \cup \{vu\}$ for a vertex u in G.

Theorem 3.1. *Suppose that G is a p-competition graph. Then $p \in \Upsilon(G')$ if G' is obtained from G by adding a pendant vertex.*

Proof. Suppose that G has n vertices and let G' be a graph obtained from G by adding a pendant vertex u at vertex v in G. Since G is a p-competition graph, G is isomorphic to the p-row graph of a matrix $M = (m_{ij})$. By (§), we may assume that $|\Lambda_M(v)| \geq p-1$. Without loss of generality, we may assume that the row corresponding to v is located at the bottom of M and $\Lambda_M(v) = \{1, 2, \ldots, |\Lambda_M(v)|\}$.

Now we define a matrix $M' = (m'_{ij})$ of order $n+1$ by

$$M' = \left[\begin{array}{c:c} & 0 \\ & \vdots \\ M & 0 \\ & 1 \\ \hdashline \underbrace{1\cdots1}_{p-1}\ \underbrace{0\cdots0}_{n-p+1} & \underbrace{1}_{1} \end{array}\right]\begin{array}{l} \\ \\ \\ v \\ \\ u \end{array}$$

It is easy to check that G' is the p-row graph of M'. By the Theorem 2.2, G' is a p-competition graph. $\square$

Kim *et al.* [8] specified the length of a cycle which is a p-competition graph in terms of p.

Theorem 3.2 ([8])**.** *Let C_n be a cycle with n vertices for a positive integer $n \geq 4$. Then $\Upsilon(C_n) = [n-3]$.*

In the proof of Theorem 3.2, $\mathcal{F} := \{S_0, \ldots, S_{n-1}\}$, where, for each $i = 0, \ldots, n-1$, $S_i := \{v_i, v_{i+1}, \ldots, v_{i+p}\}$, is given as a p-edge clique cover of $C_n = v_0 v_1 \ldots v_{n-1} v_0$ for which all the subscripts are reduced modulo n. The following square matrix of order n is obtained from $\mathcal{F}$ by (2.1):

$$M_{p,n} := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}, \tag{3.1}$$

where ith row (resp. column) is corresponding to v_{i-1} (resp. S_{i-1}) and the $(i+1)$st row is obtained by cyclically shifting the ith row by 1 to the right for each i, $1 \le i \le n$. Therefore the p-row graph of $M_{p,n}$ is isomorphic to C_n and the following proposition is immediately true.

Proposition 3.3. *Let n be an integer greater than or equal to 4 and p be a positive integer less than or equal to $n-3$. Then, for the matrix $M_{p,n}$ given in* (3.1)*, the following are true:*

(1) the kth row contains exactly $p+1$ 1s for each integer k , $1 \le k \le n$;

(2) the kth row and the $(k+1)$st row have common 1s in exactly p columns for each integer k , $1 \le k \le n$ (we identify the $n+1$st row with the first row);

(3) the kth row and the lth row have common 1s in at most $p-1$ columns for integers k, l satisfying $1 \le k, l \le n$ and $2 \le |k-l|$.

We denote the path graph with n vertices by P_n.

Lemma 3.4. *For an integer $n \ge 3$,*

$$\Upsilon(P_n) = \begin{cases} \{1,2\} & \text{if } n \in \{3,4\}; \\ [n-3] & \text{if } n \ge 5. \end{cases}$$

Proof. For $n \geq 5$, then $\Upsilon(P_n) \subset [n-3]$ by Corollary 2.21. If $n = 3$, then $\Upsilon(P_3) \subset \{1, 2\}$ by Corollary 2.8. If $n = 4$, then $\Upsilon(P_4) \subset \{1, 2\}$ by Corollary 2.8 and Proposition 2.18.

Now we show the other direction containment. If $n = 3$ and $p \leq 2$, then $\{1, 2\} \subset \Upsilon(P_n)$ by Corollary 2.25. It is easy to check that P_4 is isomorphic to the 2-row graph of the following matrix:

$$M^*_{2,4} := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Thus $2 \in \Upsilon(P_4)$. Now suppose that $n \geq 4$ and $p \leq n-3$. In $M_{p,n}$ given in (3.1), we replace 1 in the $(1, n-p+1)$-entry with 0 to obtain a square matrix $M^*_{p,n}$ of order n. Let G' be the p-row graph of $M^*_{p,n}$. Then the first row and the second row of $M^*_{p,n}$ still share p 1s. Yet the first row and the nth row of $M^*_{p,n}$ share only $p-1$ 1s. Thus, by (2) and (3) of Proposition 3.3, G' is isomorphic to a path graph with n vertices. Hence $[n-3] \subset \Upsilon(P_n)$ and this completes the proof. □

Proposition 3.5. *Let T be a tree with the diameter m. Then $\Upsilon(T) \supset [m-2]$.*

Proof. Take $p \in [m-2]$. Since the diameter of T is m, there exists an induced path of length m. Since $m \geq p+2$, we may take a section P of this path which has length $p+2$. Then P is a p-competition graph by Lemma 3.4. Since T can be obtained from P by adding pendant vertices sequentially, G is a p-competition graph by Theorem 3.1. □

Corollary 3.6. *Given a graph G with n vertices and diameter m, if $G/\sim$ is a tree with n' vertices, then $\Upsilon(G) \supset [n - n' + m - 2]$.*

Proof. Suppose $G/\sim$ is a tree with n' vertices. Then, by Proposition 2.10, $G/\sim$ has diameter m. Thus, by Proposition 3.5, $\Upsilon(G/\sim) \supset [m-2]$. By

Corollary 2.14,

$$\Upsilon(G) \supset \{p + i \mid p \in [m-2] \text{ and } i \in [n-n'] \cup \{0\}\} = [n - n' + m - 2].$$

□

A *caterpillar* is a tree with at least 3 vertices the removal of whose pendant vertices produces a path called a *central path*. A *spine* of a caterpillar is the longest path of the caterpillar. In the following, for a caterpillar T with n vertices, we shall find all the positive integers p such that T is a p-competition graph in terms of n. To do so, we need the following lemma.

Lemma 3.7. *Let n and p be positive integers such that either $(n,p) = (4,2)$ or $n \geq 4$ and $p \leq n-3$. Then, for any nonnegative integer k and a path graph P of length $n-1$, a caterpillar T obtained by adding k new vertices to P in such a way that the added vertices are pendent vertices of T adjacent to interior vertices of P is a $(p+k)$-competition graph.*

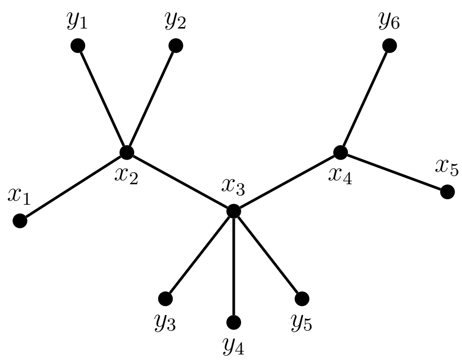
Proof. Let $P = x_1x_2 \cdots x_n$ and $y_1, \ldots, y_k$ be the vertices added to P as described in the theorem statement. Since either $(n,p) = (4,2)$ or $n \geq 4$ and $p \leq n-3$, P is a p-competition graph by Lemma 3.4. By the way, P is the p-row graph of $M^*_{p,n}$ where $M^*_{p,n}$ is the matrix defined in the proof of the same lemma.

There exists a map $\phi : [k] \to [n]$ such that $x_{\phi(i)}$ is a vertex on P adjacent to y_i. By the way that $y_1, \ldots, y_k$ were added, ϕ is well-defined. We define a $k \times n$ $(0,1)$-matrix A so that the ith row of A is the same as the row of $M^*_{p,n}$ corresponding to $x_{\phi(i)}$. Then, for $1 \leq i < j \leq k$,

(M-1) if y_i and y_j are adjacent to the same vertex on P, then the ith row and the jth row of A are identical and have exactly $p+1$ common 1s; otherwise, the ith row and the jth row of A have at most p common 1s;

$$
M := \left[\begin{array}{c:c}
\overbrace{\qquad M^*_{p,n} \qquad}^{n} & \overbrace{\qquad J_{n,k} \qquad}^{k} \\
\hdashline
A & J_{k,k} - I_k
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l} x_1 \\ \vdots \\ x_n \end{array}\right\} \\
\left.\begin{array}{l} y_1 \\ \vdots \\ y_k \end{array}\right\}
\end{array}.
$$

Figure 3.1: A matrix whose p-row graph is a caterpillar



$$
\begin{array}{c}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6
\end{array}
\left[\begin{array}{ccccc:cccccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hdashline
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right]
$$

Figure 3.2: A caterpillar T and a matrix whose 8-row graph is isomorphic to G where the row labeled with w corresponds to the vertex w in T.

(M-2) if y_i and x_l are adjacent in T, then the lth row of the $(1,1)$-block of M and the ith row of the $(2,1)$-block of M have exactly $p+1$ common 1s; otherwise, ith row and the lth row of the $(1,1)$-block of M and the ith row of the $(2,1)$-block of M have at most p common 1s.

Now we consider the matrix M in Figure 3.1. For an example of M, see Figure 3.2.

Let G be the $(p+k)$-row graph of M. We denote the row of M containing the row of $M^*_{p,n}$ corresponding to x_i by $\mathbf{x}_i$ for each $i = 1, \ldots, n$, the row of M containing the ith row of A by $\mathbf{y}_i$ for each $i = 1, \ldots, k$.

By the definition of $M^*_{p,n}$, the row of $M^*_{p,n}$ corresponding to x_i and the row of $M^*_{p,n}$ corresponding to x_j have at most $p-1$ common 1s if and only if $|j-i| \geq 2$. Thus the rows $\mathbf{x}_i$ and $\mathbf{x}_j$ have at most $p+k-1$ common 1s if and only if $|j-i| \geq 2$ and therefore P is an induced subgraph of G.

We note that the ith row and the jth row of the $(2,2)$-block of M have exactly $k-2$ common 1s for $1 \leq i < j \leq k$. Thus, by (M-1), $\mathbf{y}_i$ and $\mathbf{y}_j$ have at most $p+k-1$ common 1s for $1 \leq i < j \leq k$. Thus y_i and y_j are not adjacent in G for $1 \leq i < j \leq k$.

We note that the lth row of the $(1,2)$-block of M and the ith row of the $(2,2)$-block of M have at least $k-1$ common 1s if and only if y_i and x_l are adjacent. Thus, by (M-2), y_i and x_l are adjacent in T if and only if $\mathbf{y}_i$ and $\mathbf{x}_l$ have at least $p+k$ common 1s. Hence we have shown that T is isomorphic to G. □

Lemma 3.8. *Given an integer $n \geq 2$ and the star graph $K_{1,n}$, $\Upsilon(K_{1,n}) = [n]$.*

Proof. By Corollary 2.8, $n+1 \notin \Upsilon(K_{1,n})$, so $\Upsilon(K_{1,n}) \subset [n]$.

Now we show the converse containment. By Corollary 2.25, $\{1,2\} \subset \Upsilon(K_{1,n})$ for $n \geq 2$. By Lemma 2.17 and Corollary 2.25, $\{1,2,n\} \subset \Upsilon(K_{1,n})$ for $n \geq 3$. Therefore $[n] \subset \Upsilon(K_{1,n})$ for $n = 2, 3$ and

$$\{1,2,n\} \subset \Upsilon(K_{1,n}) \tag{3.2}$$

for $n \geq 3$. Now suppose $n \geq 4$. By (3.2), it is sufficient to show that $p \in \Upsilon(K_{1,n})$ for $3 \leq p \leq n-1$. Now we consider the following matrix M:

$$M = \left[\begin{array}{ccc|c} & & & 1 \\ & M_{p-2,n} & & \vdots \\ & & & 1 \\ \hline 1 & \cdots & 1 & 1 \end{array}\right]$$

where $M_{p-2,n}$ is the matrix defined in (3.1). Let G be the p-row graph of M. In addition, let $\mathbf{r}_i$ denote the ith row of M and v_i be the vertex of G

corresponding to $\mathbf{r}_i$. For $i = 1, \ldots, n$, the ith row of $M_{p-2,n}$ contains exactly $p-1$ 1s by Proposition 3.3(1). Thus $\mathbf{r}_i$ contains exactly p 1s in M and so v_i and v_{n+1} are adjacent in G.

By (2) and (3) of Proposition 3.3, the ith row and the jth row of $M_{p-2,n}$ have at most $p-2$ common 1s for distinct i and j in $[n]$. Thus $\mathbf{r}_i$ and $\mathbf{r}_j$ have at most $p-1$ common 1s and so v_i and v_j are not adjacent in G for distinct i and j in $[n]$. Hence G is isomorphic to $K_{1,n}$. □

Theorem 3.9. *For a caterpillar T with n vertices,*

$$\Upsilon(T) = \begin{cases} [n-1] & \text{if } d(T) = 2; \\ [n-2] & \text{if } d(T) = 3; \\ [n-3] & \text{if } d(T) \geq 4 \end{cases}$$

where $d(T)$ denotes the diameter of T.

Proof. If $d(T) = 2$, then $T \cong K_{1,n-1}$ and, by Lemma 3.8, $\Upsilon(T) = [n-1]$.

Suppose $d(T) \geq 3$. If $d(T) = 3$, $\Upsilon(T) \subset [n-2]$ by Corollary 2.8 and Proposition 2.18. If $d(T) \geq 4$, $\Upsilon(T) \subset [n-3]$ by Proposition 2.20.

To show the converse containment, let $k(T)$ denotes the number of vertices which are attached to the spine of T. Now take a positive integer $p \in [n-t]$ where $t = 2$ if $d(T) = 3$ and $t = 3$ if $d(T) \geq 4$.

Since $d(T)$ is the length of the spine of T, $n = d(T) + 1 + k(T)$. Thus $p \leq (d(T) + 1 + k(T)) - t$ or

$$p - d(T) + t - 1 \leq k(T). \tag{3.3}$$

If either $d(T) = 3$ and $p \leq 2$ or $d(T) \geq 4$ and $p \leq d(T) - 2$, then the spine of T is a p-competition graph by Lemma 3.4 and so T is a p-competition graph by Theorem 3.1.

Now assume that either $d(T) = 3$ and $p > 2$ hold or $d(T) \geq 4$ and

$p > d(T) - 2$. Let

$$\alpha = \begin{cases} p-2 & \text{if } d(T) = 3 \text{ and } p > 2, \\ p - d(T) + 2 & \text{if } d(T) \geq 4 \text{ and } p > d(T) - 2. \end{cases}$$

By (3.3), we have $\alpha \leq k(T)$. Let T' be a caterpillar obtained from T by deleting some pendent vertices of T so that $d(T') = d(T)$ and $\alpha = k(T')$. Then, by Lemma 3.7, T' is a p-competition graph and so, by Theorem 3.1, T is a p-competition graph. □

Corollary 3.10. *Let G be a graph with n vertices such that $G/\sim$ is a caterpillar. Then*

$$\Upsilon(G) = \begin{cases} [n-1] & \textit{if } d(G) = 2; \\ [n-2] & \textit{if } d(G) = 3; \\ [n-3] & \textit{if } d(G) \geq 4 \end{cases}$$

where $d(G)$ denotes the diameter of G.

Proof. By Theorem 3.9,

$$\Upsilon(G/\sim) = \begin{cases} [m-1] & \text{if } d(G/\sim) = 2; \\ [m-2] & \text{if } d(G/\sim) = 3; \\ [m-3] & \text{if } d(G/\sim) \geq 4, \end{cases}$$

where $m = |V(G/\sim)|$ and $d(G/\sim)$ denotes the diameter of $G/\sim$. Since $G/\sim$ has no isolated vertices,

$$\Upsilon(G) \supset \begin{cases} [n-1] & \text{if } d(G) = 2; \\ [n-2] & \text{if } d(G) = 3; \\ [n-3] & \text{if } d(G) \geq 4, \end{cases}$$

by Proposition 2.10 and Corollary 2.14. By Corollary 2.8, $n \notin \Upsilon(G)$. Therefore $\Upsilon(G) = [n-1]$ if $d(G) = 2$. By Proposition 2.18, $n-1 \notin \Upsilon(G)$ if $d(G) \geq 3$. Thus $\Upsilon(G) = [n-2]$ if $d(G) = 3$. By Proposition 2.20, $n-2 \notin \Upsilon(G)$ if $d(G) \geq 4$. Hence $\Upsilon(G) = [n-3]$ if $d(G) \geq 4$. $\square$

Lemma 3.11. *Given a p-competition graph G with n vertices, suppose that 2^r+1 neighbors of a vertex v of G form an independent set for some positive integer r. Then, if $p \geq n-r$, then there are two nonadjacent neighbors x and y of v with $|\Lambda_M(x)| < |\Lambda_M(v)|$ and $|\Lambda_M(y)| < |\Lambda_M(v)|$ for any p-row matrix M of G.*

Proof. Since v is not isolated, $p \leq |\Lambda_M(v)| \leq n$. For notational convenience, we let $\overline{\Lambda}_M(v) = [n] \setminus \Lambda_M(v)$ Now suppose $p \geq n-r$. Then $0 \leq |\overline{\Lambda}_M(v)| \leq n-p \leq r$. Thus the number of subsets of $\overline{\Lambda}_M(v)$ is less than 2^r+1. For each neighbor x of v, $\overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(x)$ is a subset of $\overline{\Lambda}_M(v)$. Since v has 2^r+1 neighbors which form an independent set by the hypothesis, there are two nonadjacent neighbors x and y of v such that $\overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(x) = \overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(y)$ by the Pigeonhole principle. Since $\overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(x)$ and $\overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(y)$ are subsets of $\overline{\Lambda}_M(x)$ and $\overline{\Lambda}_M(y)$, respectively, we have

$$\overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(x) = \overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(y) \subset \overline{\Lambda}_M(x) \cap \overline{\Lambda}_M(y). \tag{3.4}$$

Since v is adjacent x and y, $|\overline{\Lambda}_M(v) \cup (\overline{\Lambda}_M(x)| \leq n-p$ and $|\overline{\Lambda}_M(v) \cup \overline{\Lambda}_M(y)| \leq n-p$. Since x and y are not adjacent, $|\overline{\Lambda}_M(x) \cup \overline{\Lambda}_M(y)| > n-p$. Thus

$$\begin{aligned} |\overline{\Lambda}_M(v)| + |\overline{\Lambda}_M(x)| - |\overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(x)| &= |\overline{\Lambda}_M(v) \cup \overline{\Lambda}_M(x)| \\ < |\overline{\Lambda}_M(x) \cup \overline{\Lambda}_M(y)| &= |\overline{\Lambda}_M(x)| + |\overline{\Lambda}_M(y)| - |\overline{\Lambda}_M(x) \cap \overline{\Lambda}_M(y)| \end{aligned}$$

Then, by (3.4), $|\overline{\Lambda}_M(y)| > |\overline{\Lambda}_M(v)|$. By the same argument, one can show that $|\overline{\Lambda}_M(x)| > |\overline{\Lambda}_M(v)|$ and we complete the proof. $\square$

Let k be a positive integer. A *k-ary tree* is a rooted tree in which each vertex has no more than k children. A *full k-ary tree* is a rooted tree exactly

k children or no children. A *perfect k-ary tree* is a full k-ary tree in which all pendant vertices are at the same depth.

By Proposition 3.5, $\Upsilon(T) \neq \emptyset$ for a tree T, so $\max(\Upsilon(T))$ exists. We have shown that $|V(T)| - \max(\Upsilon(T)) \leq 3$ for a caterpillar T. One might think by this result that there exists a positive integer t such that $|V(T)| - \max(\Upsilon(T)) \leq t$ for any tree T, yet it is not true by the following theorem.

Theorem 3.12. *For any positive integer r, there is a tree T with $|V(T)| - \max(\Upsilon(T)) > r$.*

Proof. Let T be a perfect (2^r+1)-ary tree with height $r+1$ and a root x_0. Suppose that $|V(T)| - \max(\Upsilon(T)) \leq r$. Then by the definition of $\Upsilon(G)$ for a graph G, T is a $\max(\Upsilon(T))$-competition graph and $\max(\Upsilon(T)) \geq |V(T)| - r$. Then T is a $\max(\Upsilon(T))$-row graph of a matrix M. By Lemma 3.11, $|\Lambda_M(x_1)| < |\Lambda_M(x_0)| \leq |V(T)|$ for some children x_1 of x_0. Then, by the same lemma again, $|\Lambda_M(x_2)| < |\Lambda_M(x_1)| \leq |V(T)| - 1$ for some children x_2 of x_1. We apply the lemma repeatedly to have $|\Lambda_M(x_{r+1})| < |V(T)| - r$. We have reached a contradiction since x_{r+1} is non-isolated. Hence $|V(T)| - \max(\Upsilon) > r$. □

Chapter 4

Closing Remarks

We have shown that $\Upsilon(K_{3,3}) = \emptyset$. We would like to know $\Upsilon(K_{n,n}) = \emptyset$ for any $n \geq 4$. We have characterized the graphs with n vertices and the competition-realizer $[n]$ and $[n-1]$, respectively. It would be interesting to characterize the graphs with n vertices and competition-realizer $[n-2]$. Finally we suggest to find the realizer for a Lobster to extend our result which gives every element in the competition-realizer for a caterpillar.

Bibliography

[1] Joel E Cohen. Interval graphs and food webs: a finding and a problem. *RAND Corporation Document*, 17696, 1968.

[2] Ronald D Dutton and Robert C Brigham. A characterization of competition graphs. *Discrete Applied Mathematics*, 6(3):315–317, 1983.

[3] Kim A S Factor and Sarah K Merz. The $(1,2)$-step competition graph of a tournament. *Discrete Applied Mathematics*, 159(2):100–103, 2011.

[4] HJ Greenberg, JR Lundgren, and JS Maybee. Graph-theoretic foundations of computer-assisted analysis. *Computer-Assisted Analysis and Model Simplification, Academic Press, New York*, pages 481–495, 1981.

[5] Akira Kamibeppu. A sufficient condition for kim's conjecture on the competition numbers of graphs. *Discrete Mathematics*, 312(6):1123–1127, 2012.

[6] Suh-Ryung Kim. The competition number and its variants. *Annals of Discrete Mathematics*, 55:313–326, 1993.

[7] Suh-ryung Kim, Terry A McKee, FR McMorris, and Fred S Roberts. p-competition graphs. *Linear algebra and its applications*, 217:167–178, 1995.

[8] Suh-Ryung Kim, Boram Park, and Yoshio Sano. Cycles and p-competition graphs. *Congressus Numerantium*, pages 23–30, 2009.

[9] Jaromy Kuhl. Transversals and competition numbers of complete multipartite graphs. *Discrete Applied Mathematics*, 161(3):435–440, 2013.

[10] Bo-Jr Li and Gerard J Chang. Competition numbers of complete r-partite graphs. *Discrete Applied Mathematics*, 160(15):2271–2276, 2012.

[11] J Richard Lundgren. Food webs, competition graphs, competition-common enemy graphs, and niche graphs. *Applications of Combinatorics and Graph Theory in the Biological and Social Sciences, IMA Volumes in Mathematics and its Applications*, 17:221–243, 1989.

[12] Robert J Opsut. On the computation of the competition number of a graph. *SIAM Journal on Algebraic Discrete Methods*, 3(4):420–428, 1982.

[13] Arundhati Raychaudhuri and Fred S Roberts. Generalized competition graphs and their applications. *Methods of Operations Research*, 49:295–311, 1985.

[14] Fred S Roberts. Competition graphs and phylogeny graphs, in l. lovasz (ed.),. *Graph Theory and Combinatorial Biology (Bolyai Math. Stud.)*, 7:333–362, 1999.

[15] Xinhong Zhang and Ruijuan Li. The $(1, 2)$-step competition graph of a pure local tournament that is not round decomposable. *Discrete Applied Mathematics*, 205:180–190, 2016.

국문초록

양의 정수 p에 대해, 어떤 유향그래프 D의 p-경쟁그래프는 꼭짓점의 집합을 D와 같이 갖고 두 꼭짓점 x와 y가 변으로 이어질 필요충분조건을 x와 y가 최소한 p개 이상의 공통된 외이웃을 가지는 것으로 하는 그래프로 정의된다. 어떤 그래프 G에 대해 어떤 유향그래프 D가 존재해서 G가 D의 p-경쟁그래프가 될 때 G를 p-경쟁그래프라고 부른다. 그래프 G가 주어졌을 때, G가 p-경쟁그래프가 되는 양의 정수 p의 집합을 G의 경쟁실현집합이라고 정의한다. 그래프 G에 대해서 서로 같은 닫힌 외이웃을 갖는 인접한 꼭짓점을 동일시하여 얻은 그래프를 $G/\sim$으로 나타낸다. 본 연구에서는 기존의 행그래프에 대한 개념을 확장한 p-행그래프를 도입하였고, p-행그래프와 $G/\sim$을 이용하여 다양한 그래프의 경쟁실현집합을 찾아내었으며 기존의 연구에서 나왔던 결과들을 확장하였으며 수형도의 일종인 캐터필러의 경쟁실현집합의 모든 원소를 찾아내었다.

주요어휘: p-경쟁그래프; p-변 완전부분그래프 덮개; 경쟁실현집합; p-행그래프; $G/\sim$; 캐터필러
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