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Intensity and spin anisotropy of three-dimensional polarization states

José J. Gil,[1,*] Andreas Norrman,[2] Ari T. Friberg,[3] and Tero Setälä[3]

[1]Department of Applied Physics, University of Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain
[2]Max Planck Institute for the Science of Light, Staudtstraße 2, D-91058 Erlangen, Germany
[3]Institute of Photonics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland
*Corresponding author: ppgil@unizar.es

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Anisotropy is a natural feature of polarization states, and only fully random three-dimensional (3D) states exhibit complete isotropy. In general, differences between the strengths of the electric field components along the three orthogonal directions give rise to intensity anisotropy. Moreover, polarization states involve an average spin whose inherent vectorial nature constitutes a source of spin anisotropy. In this work, appropriate descriptors are identified to characterize quantitatively the levels of intensity anisotropy and spin anisotropy of a general 3D polarization state, leading to a novel interpretation for the degree of polarimetric purity as a measure describing the overall polarimetric anisotropy of a 3D optical field. The mathematical representation, as well as the physical features of completely intensity-isotropic 3D polarization states with a maximum spin anisotropy, are also examined. The results provide new insights into the polarimetric field structure of random 3D electromagnetic light states.

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From the points of view of physical optics and electromagnetic radiation, it is essential to have an unambiguous mathematical description and understanding of the basic notion of polarization and of its true three-dimensional (3D) features [1–6]. In addition, the rapid advances of nano-optics and near-field phenomena [7], together with their applications, call for the analysis of the physical properties of 3D polarization. Important characteristics of 3D random electromagnetic fields are the sources of polarimetric anisotropy and their relations to the components of polarimetric purity [1,8] (viz., the degrees of linear polarization, circular polarization, and directionality), the polarimetric dimension [9], and the overall degree of polarimetric purity [1,10–12]. For example, while a linearly polarized state exhibits maximum intensity anisotropy and zero spin anisotropy, it was proven recently that, conversely, there exist states possessing full intensity isotropy but nonzero spin [13]. The aim of this work is to identify, characterize, and classify the 3D polarization states with different types of polarimetric anisotropy, namely (1) *intensity anisotropy* and (2) *spin anisotropy*, as well as to analyze their relation to the overall polarimetric anisotropy.

We start by recalling the concepts that are necessary for the developments in this work. The temporal polarization properties of a random, stationary 3D electromagnetic field at a fixed point in space are fully determined by the corresponding 3×3 polarization matrix $\mathbf{R}$, whose elements are $r_{ij} = \langle \varepsilon_i(t)\varepsilon_j^*(t)\rangle$ $(i,j = x,y,z)$ with $\varepsilon_i(t)$ being the zero-mean Cartesian electric field components. The angle brackets and the asterisk denote the time average and the complex conjugate, respectively. The matrix $\mathbf{R}$ can generally be expressed in terms of the spectral decomposition as [3]

$$\mathbf{R} = I\mathbf{U}\,\mathrm{diag}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3)\mathbf{U}^\dagger = I\sum_{i=1}^{3}\hat{\lambda}_i(\hat{\mathbf{u}}_i\otimes\hat{\mathbf{u}}_i^\dagger), \quad (1)$$

where diag refers to a diagonal matrix, $I = \mathrm{tr}\,\mathbf{R}$ is the intensity of the state, $\mathbf{U}$ is the unitary matrix that diagonalizes the Hermitian $\mathbf{R}$, and $\hat{\mathbf{u}}_i$ are the unit eigenvectors of the polarization density matrix $\hat{\mathbf{R}} = \mathbf{R}/I$, i.e., the normalized 3D Jones vectors of the polarization eigenstates. In addition, $\hat{\lambda}_i$ are the (nonnegative) eigenvalues of $\hat{\mathbf{R}}$, with the properties $\hat{\lambda}_1+\hat{\lambda}_2+\hat{\lambda}_3 = 1$ and $\hat{\lambda}_1\geq\hat{\lambda}_2\geq\hat{\lambda}_3\geq 0$, the superscript $\dagger$ indicates the conjugate transpose, and $\otimes$ stands for the Kronecker product.

The detailed quantification of the polarimetric randomness (or conversely, the polarimetric purity) of $\mathbf{R}$ is achieved through its characteristic decomposition [3,14,15]

$$\mathbf{R} = I[P_1\hat{\mathbf{R}}_p + (P_2-P_1)\hat{\mathbf{R}}_m + (1-P_2)\hat{\mathbf{R}}_{u\text{-}3D}],$$
$$\hat{\mathbf{R}}_p = \mathbf{U}\,\mathrm{diag}(1,0,0)\mathbf{U}^\dagger,$$
$$\hat{\mathbf{R}}_m = \mathbf{U}\,\mathrm{diag}(1,1,0)\mathbf{U}^\dagger/2, \qquad \hat{\mathbf{R}}_{u\text{-}3D} = \mathbf{I}/3, \quad (2)$$

where $\mathbf{I}$ is the 3×3 identity matrix. Whereas $\hat{\mathbf{R}}_p$ and $\hat{\mathbf{R}}_{u\text{-}3D}$, respectively, represent a pure state and a fully random state (a 3D unpolarized state), the interpretation of $\hat{\mathbf{R}}_m$ is more involved and leads to the concept of *regular* polarization states [$\hat{\mathbf{R}}_m$ represents 2D (two-dimensional) unpolarized light] and *nonregular* polarization states ($\hat{\mathbf{R}}_m$ represents genuine 3D light) [9,13,16]. Further, P_1, P_2 in Eq. (2) are the so-called *indices of polarimetric purity* (IPP) [3,10,11], defined as $P_1 = \hat{\lambda}_1-\hat{\lambda}_2$,

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$P_2 = 1 - 3\hat{\lambda}_3$. The IPP of a given state $\mathbf{R}$ are limited by $0 \le P_1 \le P_2 \le 1$,
and their physical interpretations can be found in [3,11,15]. The overall *degree of polarimetric purity* [3] (also called the *3D degree of polarization* [17–19]) can be expressed as $P_{3D} = \sqrt{3P_1^2 + P_2^2}/2$ in terms of the IPP [10].

While the IPP provide information on the quantitative structure of polarimetric purity [15], the *components of purity* (CP) of $\mathbf{R}$, namely the degrees of linear polarization P_l, circular polarization P_c, and directionality P_d, constitute another set of meaningful parameters that give qualitative information on the polarimetric purity structure of the state $\mathbf{R}$. The CP are introduced in terms of the *intrinsic polarization matrix* $\mathbf{R}_O$ associated with $\mathbf{R}$. The matrix $\mathbf{R}_O$ is given by [20,21]

$$\mathbf{R}_O = I\mathbf{Q}_O\hat{\mathbf{R}}\mathbf{Q}_O^{\mathrm{T}} = I\begin{pmatrix} \hat{a}_1 & -i\hat{n}_{O3}/2 & i\hat{n}_{O2}/2 \\ i\hat{n}_{O3}/2 & \hat{a}_2 & -i\hat{n}_{O1}/2 \\ -i\hat{n}_{O2}/2 & i\hat{n}_{O1}/2 & \hat{a}_3 \end{pmatrix}, \tag{3}$$

where $\mathbf{Q}_O$ is the orthogonal matrix that diagonalizes the (symmetric) real part of $\mathbf{R}$, and T denotes the transpose. Thus, $\mathbf{R}_O$ represents the same state of polarization as $\mathbf{R}$ but expressed with respect to the corresponding *intrinsic reference frame* $X_O Y_O Z_O$ [21]. The (nonnegative) eigenvalues $\hat{a}_i$ of $\mathrm{Re}(\hat{\mathbf{R}})$ are called the intensity-normalized *principal intensities*, $\mathbf{Q}_O$ is defined such that $\hat{a}_1 \ge \hat{a}_2 \ge \hat{a}_3$, with $\hat{a}_1 + \hat{a}_2 + \hat{a}_3 = 1$, while $\hat{n}_{Oi}$ are the components of the intensity-normalized spin vector $\hat{\mathbf{n}}_O \equiv (\hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})^{\mathrm{T}}$ of the state with respect to $X_O Y_O Z_O$.

The CP are now defined as [21]

$$P_l = \hat{a}_1 - \hat{a}_2, \qquad P_d = 1 - 3\hat{a}_3, \qquad P_c = |\hat{\mathbf{n}}_O|, \tag{4}$$

with physical meanings closely related to the intrinsic Stokes parameters [22], and they are connected to the overall degree of polarimetric purity according to [8]

$$P_{3D} = \sqrt{\frac{3}{4}(P_l^2 + P_c^2) + \frac{1}{4}P_d^2}, \tag{5}$$

justifying the name components of purity for the set of parameters (P_l, P_c, P_d). The maximum $P_{3D} = 1$ is saturated exclusively when $\sqrt{P_l^2 + P_c^2} = P_d = 1$, while the minimum $P_{3D} = 0$ is associated uniquely with $P_l = P_c = P_d = 0$.

The states obeying $\hat{a}_1 = \hat{a}_2 = \hat{a}_3 = 1/3$ are fully intensity isotropic [9,13], i.e., the strengths of the three electric field components along the respective intrinsic reference axes $X_O Y_O Z_O$ are equal. For such states, $\mathrm{Re}(\hat{\mathbf{R}}) = \mathbf{I}/3$, and therefore $\mathrm{Re}(\hat{\mathbf{R}})$ is invariant under any orthogonal transformation (rotation of the Cartesian reference frame). Consequently, for a maximally intensity-isotropic state, all Cartesian reference systems XYZ may be considered intrinsic. Indeed, full intensity isotropy is characterized completely by this genuine invariance of $\mathrm{Re}(\hat{\mathbf{R}})$. A specific kind of a fully intensity-isotropic state is the unpolarized 3D state $\hat{\mathbf{R}}_{u\text{-}3D} = \mathbf{I}/3$. In general, however, maximum intensity isotropy ($\hat{a}_1 = \hat{a}_2 = \hat{a}_3$) does not imply that the state is fully 3D unpolarized ($\hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3$), as shown below. On the other hand, a state with complete lack of intensity isotropy, i.e., one with maximum intensity anisotropy, is characterized by $\hat{a}_1 = 1$, $\hat{a}_2 = \hat{a}_3 = 0$ [9,13], which according to Eq. (4) corresponds to a linearly polarized pure state ($P_l = 1$).

Let us now analyze the *dimensionality index*

$$d = \sqrt{[(\hat{a}_1 - \hat{a}_2)^2 + (\hat{a}_1 - \hat{a}_3)^2 + (\hat{a}_2 - \hat{a}_3)^2]/2}, \tag{6}$$

specifying the *polarimetric dimension* $D = 3 - 2d$ of a light field [9]. The dimensionality index varies continuously and monotonically in the interval $0 \le d \le 1$ as a function of the principal intensities. The lower limit $d = 0$ is met only for fully intensity-isotropic states ($\hat{a}_1 = \hat{a}_2 = \hat{a}_3 = 1/3$), whereas the upper bound $d = 1$ corresponds uniquely to linearly polarized pure states ($\hat{a}_1 = 1$, $\hat{a}_2 = \hat{a}_3 = 0$) of maximum intensity anisotropy. Thus, in addition of determining the polarimetric dimension of a light field, the parameter d constitutes a proper measure that quantifies the *degree of intensity anisotropy* of the polarization state $\mathbf{R}$.

From Eqs. (4) and (6), we especially find that

$$d = \sqrt{(3P_l^2 + P_d^2)/2}, \tag{7}$$

showing that the degree of intensity anisotropy d is a weighted quadratic average of the degree of linear polarization P_l and the degree of directionality P_d. Equation (7) thus implies that the parameters P_l and P_d do not alone give complete information on the intensity anisotropy of the state $\mathbf{R}$. For example, the degree of directionality P_d, describing the stability of the polarization-ellipse plane for time intervals much larger than the polarization time [23–25], attains its maximum value $P_d = 1$ whenever $\hat{a}_3 = 0$, irrespective of the values of $\hat{a}_1$ and $\hat{a}_2$. Therefore, maximum intensity anisotropy ($d = 1$) necessitates that both $P_l = 1$ and $P_d = 1$ are satisfied. The lowest value $P_d = 0$, implying $P_l = 0$ as $P_l \le P_d$ according to Eq. (4), is an exception that corresponds uniquely to the case $\hat{a}_1 = \hat{a}_2 = \hat{a}_3$ of minimum intensity anisotropy ($d = 0$).

Next, we explore the physical interpretation of fully intensity-isotropic states and their mathematical representation by means of incoherent compositions of simple polarization states. The polarization density matrix $\hat{\mathbf{R}}_I$ of an intensity-isotropic state is characterized univocally by $d = 0$, and it has the general form

$$\hat{\mathbf{R}}_I = \begin{pmatrix} 1/3 & -i\hat{n}_3/2 & i\hat{n}_2/2 \\ i\hat{n}_3/2 & 1/3 & -i\hat{n}_1/2 \\ -i\hat{n}_2/2 & i\hat{n}_1/2 & 1/3 \end{pmatrix}. \tag{8}$$

We observe that, unlike $\mathrm{Re}(\hat{\mathbf{R}}_I)$, $\mathrm{Im}(\hat{\mathbf{R}}_I)$ is not invariant under orthogonal transformations, and therefore, as expected, the components of the intensity-normalized spin vector $\hat{\mathbf{n}} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)^{\mathrm{T}}$ depend on the particular Cartesian reference frame considered. The direction of the vector $\hat{\mathbf{n}}$, as well as the magnitude $\hat{n} \equiv |\hat{\mathbf{n}}| = P_c$, are nonetheless preserved.

The eigenvalues of $\hat{\mathbf{R}}_I$ in Eq. (8) are $\hat{\lambda}_1 = 1/3 + P_c/2$, $\hat{\lambda}_2 = 1/3$, $\hat{\lambda}_3 = 1/3 - P_c/2$, with the corresponding 3D Jones eigenstates

$$\hat{\mathbf{u}}_1 = \frac{1}{P_c}\begin{pmatrix}\sqrt{\frac{\hat{n}_2^2+\hat{n}_3^2}{2}}\\ \frac{-\hat{n}_1\hat{n}_2+i\hat{n}_3P_c}{\sqrt{2(\hat{n}_2^2+\hat{n}_3^2)}}\\ \frac{-\hat{n}_1\hat{n}_3-i\hat{n}_2P_c}{\sqrt{2(\hat{n}_2^2+\hat{n}_3^2)}}\end{pmatrix}, \qquad \hat{\mathbf{u}}_2 = \frac{1}{P_c}\begin{pmatrix}\hat{n}_1\\ \hat{n}_2\\ \hat{n}_3\end{pmatrix},$$

$$\hat{\mathbf{u}}_3 = \frac{1}{P_c}\begin{pmatrix}\sqrt{\frac{\hat{n}_2^2+\hat{n}_3^2}{2}}\\ \frac{-\hat{n}_1\hat{n}_2-i\hat{n}_3P_c}{\sqrt{2(\hat{n}_2^2+\hat{n}_3^2)}}\\ \frac{-\hat{n}_1\hat{n}_3+i\hat{n}_2P_c}{\sqrt{2(\hat{n}_2^2+\hat{n}_3^2)}}\end{pmatrix}. \tag{9}$$

The eigenvalues are seen to depend only on P_c, with $0 \le P_c \le 2/3$. The smaller P_c is, the closer the state is to the unpolarized 3D state $\hat{\mathbf{R}}_{u\text{-}3D}$, and vice versa. The minimum $P_c = 0$ corresponds strictly to $\hat{\mathbf{R}}_{u\text{-}3D}$, while the maximum $P_c = 2/3$ is met only if rank $\hat{\mathbf{R}}_I = 2$ ($\hat{\lambda}_3 = 0$), in which case the state corresponds to an incoherent composition of two pure states. Whenever $P_c < 2/3$, then $\hat{\lambda}_3 > 0$, which in turn implies rank $\hat{\mathbf{R}}_I = 3$ and, by virtue of the arbitrary decomposition of a polarization matrix [3], that the state $\hat{\mathbf{R}}_I$ is given by the incoherent composition of three independent pure states (in general not necessarily orthogonal), including the particular case of $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3$.

In addition, as $\hat{\mathbf{R}}_I$ is specified by $d = P_l = P_d = 0$ and $0 \le P_c \le 2/3$, we obtain from Eq. (5) that, for totally intensity-isotropic states, the overall degree of polarimetric purity is bounded as $0 \le P_{3D} \le 1/\sqrt{3}$. An intensity-isotropic 3D state is thereby usually not unpolarized, but may in fact possess a rather high degree of polarimetric purity, depending on the spin of the state.

A classification of intensity-isotropic polarization states in terms of the achievable values for P_c is summarized in Table 1.

The above analysis shows that even a polarization state of zero intensity anisotropy may exhibit spatial spin asymmetry. In other words, $d = 0$ does not necessarily result in $P_c = |\hat{\mathbf{n}}| = 0$. Hence, and because $\hat{\mathbf{n}}$ has a vector nature, the spin of a state $\mathbf{R}$ entails another source of polarimetric anisotropy, which substantially differs from the concept of intensity anisotropy and which is not accounted for by the degree of intensity anisotropy d. The relation $P_c = |\hat{\mathbf{n}}|$, on the other hand, promotes the degree of circular polarization P_c as an appropriate measure that describes the *degree of spin anisotropy* of a polarization state $\mathbf{R}$.

Let us examine further intensity-isotropic states with a maximum degree of spin anisotropy, viz., $d = 0$, $P_c = 2/3$. In this case, $\hat{\lambda}_3 = 0$, and therefore only the first two eigenvectors $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2$ appear in the spectral decomposition (1), with corresponding weights $\hat{\lambda}_1 = 2/3$, $\hat{\lambda}_2 = 1/3$. By means of the method described in [26], the respective ellipticity angles are found to be $\chi_1 = \pm\pi/4$, $\chi_2 = 0$, showing that $\hat{\mathbf{u}}_1$ represents a circularly polarized state, while $\hat{\mathbf{u}}_2$ represents a linearly polarized state. Moreover, the electric field associated with $\hat{\mathbf{u}}_2$ vibrates along the direction normal to the polarization-circle plane of $\hat{\mathbf{u}}_1$. Thus, we find that, up to a rotation of the Cartesian reference frame, a completely intensity-isotropic state ($d = 0$) with maximum spin anisotropy ($P_c = 2/3$) can always be represented as the following composition of a circularly polarized state and a linearly polarized state:

Table 1. Classification of Intensity-Isotropic States ($d = 0$) According to P_c

$P_c = 0$	$0 < P_c < 2/3$	$P_c = 2/3$
rank $\hat{\mathbf{R}}_I = 3$	rank $\hat{\mathbf{R}}_I = 3$	rank $\hat{\mathbf{R}}_I = 2$
$P_{3D} = 0$	$0 < P_{3D} < 1/\sqrt{3}$	$P_{3D} = 1/\sqrt{3}$
$\hat{\mathbf{R}}_I = \hat{\mathbf{R}}_{u\text{-}3D}$	$\hat{\mathbf{R}}_I \neq \hat{\mathbf{R}}_{u\text{-}3D}$	$\hat{\mathbf{R}}_I \neq \hat{\mathbf{R}}_{u\text{-}3D}$

$$\hat{\mathbf{R}}_{IAz} = \frac{2}{3}\begin{pmatrix}1/2 & \mp i/2 & 0\\ \pm i/2 & 1/2 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}. \tag{10}$$

Consequently, any polarization density matrix $\hat{\mathbf{R}}_{IA}$ of an intensity-isotropic state with maximum spin anisotropy can, in general, be expressed through a rotation transformation of the form $\hat{\mathbf{R}}_{IA} = \mathbf{Q}\hat{\mathbf{R}}_{IAz}\mathbf{Q}^{\mathrm{T}}$, where $\mathbf{Q}$ is a proper orthogonal matrix. This result stresses the fact that all intensity-isotropic states ($d = 0$) exhibiting a maximum degree of spin anisotropy ($P_c = 2/3$) are polarimetrically equivalent since they have the same polarization density matrix up to a rotation transformation, and hence they have the same spectral structure.

It is remarkable that the same incoherent composition, but with equal coefficients $\hat{\lambda}_1 = \hat{\lambda}_2 = 1/2$, represents a *maximally (perfect) nonregular state* [13].

Note that the arbitrary decomposition [3,21] implies that $\hat{\mathbf{R}}_{IA}$ can also be synthesized by the incoherent composition of certain pairs of pure states that are not necessarily mutually orthogonal. In fact, a state $\hat{\mathbf{R}}_{IA}$ can be obtained by the incoherent combination of two elliptically polarized states whose ellipses lie in respective planes that are mutually orthogonal, their major semi-axes being equal (with value $\hat{a} = 1/3$) and mutually orthogonal, while their minor semi-axes lie in a common direction and their sum equals the value of the major semi-axes. An example of such combination is that generated by the 3D Jones vector pair $\mathbf{v}_1 = (1, i\sqrt{b}, 0)^{\mathrm{T}}/\sqrt{3}$, $\mathbf{v}_2 = (0, \sqrt{1-b}, i)^{\mathrm{T}}/\sqrt{3}$, with $0 \le b \le 1$, so that

$$\hat{\mathbf{R}}_{IA} = \frac{1}{3}\begin{pmatrix}1 & -i\sqrt{b} & 0\\ i\sqrt{b} & b & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0 & 0 & 0\\ 0 & 1-b & -i\sqrt{1-b}\\ 0 & i\sqrt{1-b} & 1\end{pmatrix}, \tag{11}$$

where

$$\hat{\mathbf{n}} = \frac{2}{3}\left(\sqrt{1-b}, 0, \sqrt{b}\right), \qquad |\hat{\mathbf{n}}| = \frac{2}{3}. \tag{12}$$

Concerning fully spin-isotropic polarization states, characterized by $P_c = 0$ and $\mathbf{R}_O = \mathrm{diag}(\hat{a}_1, \hat{a}_2, \hat{a}_3)$, they form a particular family of regular states [13,15] that can be considered either as an incoherent mixture of three linearly polarized states vibrating along mutually orthogonal axes [spectral representation (1)], or as an incoherent composition of a linearly polarized state, an unpolarized 2D state, and an unpolarized 3D state [characteristic representation (2)].

As indicated above, generally two different sources of anisotropy can affect a polarization state: (1) intensity anisotropy, measured by d; and (2) spin anisotropy, measured

Table 2. Classification of 3D Polarization States According to Their Anisotropy Descriptors, Namely the Degree of Intensity Anisotropy *d*, the Degree of Spin Anisotropy P_c, and the Degree of Polarimetric Anisotropy P_{3D}

Pure States			
Linearly Polarized	**Circularly Polarized**	**Elliptically Polarized**	
$d = 1$ $P_c = 0$ $P_{3D} = 1$	$d = 1/2$ $P_c = 1$ $P_{3D} = 1$	$1/2 < d < 1$ $0 < P_c < 1$ $P_{3D} = 1$	
2D Mixed States			
2D Partially Polarized	**2D Unpolarized**		
$1/2 < d < 1$ $0 < P_c < 1$ $1/2 < P_{3D} < 1$	$d = 1/2$ $P_c = 0$ $P_{3D} = 1/2$		
Genuine 3D States			
Intensity-Anisotropic	**Intensity-Isotropic with $P_c = 2/3$**	**Intensity-Isotropic with $0 < P_c < 2/3$**	**Isotropic**
$0 < d < 1$ $0 < P_c < 1$ $0 < P_{3D} < 1$	$d = 0$ $P_c = 2/3$ $P_{3D} = 1/\sqrt{3}$	$d = 0$ $0 < P_c < 2/3$ $0 < P_{3D} < 1/\sqrt{3}$	$d = 0$ $P_c = 0$ $P_{3D} = 0$

in terms of P_c. Interestingly, on combining Eqs. (5) and (7), we find that

$$P_{3D} = \sqrt{d^2 + \frac{3}{4}P_c^2}, \tag{13}$$

showing how both sources of polarimetric anisotropy constitute, in a complementary way, the overall degree of polarimetric purity. Likewise, Eq. (13) reveals that the overall degree of polarimetric purity (3D degree of polarization) is nothing more than a measure of the degree of polarimetric anisotropy of a 3D polarization state. Moreover, we recall from our earlier discussion that fully intensity-isotropic states ($d = 0$) satisfy $0 \leq P_c \leq 2/3$, whereby $0 \leq P_{3D} \leq 1/\sqrt{3}$ for such states. On the other hand, states having $P_c = 0$ possess no such restriction but can exhibit any value in the range $0 \leq P_{3D} \leq 1$: the upper limit is saturated for a state that approaches a linearly polarized state ($d = 1$) with zero spin, while the lower bound is encountered for the unpolarized 3D state $\hat{\mathbf{R}}_{u-3D}$ ($d = 0$). Consequently, zero intensity anisotropy places a stricter constraint on the degree of polarimetric purity than zero spin anisotropy. This and other peculiar features of P_{3D} can readily be seen by the inspection of Table 2, in which the values for d, P_c, and P_{3D} are shown for different types of polarization states.

In summary, we conclude that the concept of anisotropy of 3D polarization states involves two combined types of separable and complementary sources: (1) the intensity anisotropy, and (2) the spin anisotropy. Whereas the former follows from the anisotropic distribution of the principal intensities, the latter arises from the directional nature of the averaged spin angular momentum vector. These sources thus carry different information on the polarimetric anisotropy, and they were shown to admit respective descriptors, namely the degree of intensity anisotropy d and the degree of spin anisotropy P_c, having natural and meaningful physical properties. In particular, the 3D degree of polarimetric purity (or degree of polarization) P_{3D} reveals its character as a weighted quadratic average of d and P_c, thereby leading to a novel interpretation for P_{3D} as a quantitative measure of the overall polarimetric anisotropy of a general 3D optical field. These results, together with the identification and characterization of totally intensity-isotropic 3D states with maximum spin isotropy, provide new tools and deeper insights for a rigorous assessment of the polarimetric structure of random 3D electromagnetic fields.

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REFERENCES

1. J. J. Gil and R. Ossikovski, *Polarized Light and the Mueller Matrix Approach* (CRC Press, 2016).
2. C. Brosseau and A. Dogariu, "Symmetry properties and polarization descriptors for an arbitrary electromagnetic wavefield," in *Progress in Optics*, E. Wolf, ed. (Elsevier, 2006), Vol. **49**, Chap. 4.
3. J. J. Gil, Eur. Phys. J. Appl. Phys. **40**, 1 (2007).
4. C. J. R. Sheppard, J. Opt. Soc. Am. A **28**, 2655 (2011).
5. J. M. Auñón and M. Nieto-Vesperinas, Opt. Lett. **38**, 58 (2013).
6. O. Gamel and D. F. V. James, J. Opt. Soc. Am. A **31**, 1620 (2014).
7. L. Novotny and B. Hecht, *Principles of Nano-Optics*, 2nd ed. (Cambridge University, 2012).
8. J. J. Gil, J. Opt. Soc. Am. A **33**, 40 (2016).
9. A. Norrman, A. T. Friberg, J. J. Gil, and T. Setälä, J. Eur. Opt. Soc. Rapid Publ. **13**, 36 (2017).
10. J. J. Gil, J. M. Correas, P. A. Melero, and C. Ferreira, Monog. Sem. Mat. Garcia Galdeano **31**, 161 (2004).
11. I. San José and J. J. Gil, Opt. Commun. **284**, 38 (2011).
12. J. J. Gil, A. Norrman, A. T. Friberg, and T. Setälä, Phys. Rev. A **97**, 023838 (2018).
13. J. J. Gil, A. Norrman, A. T. Friberg, and T. Setälä, Opt. Lett. **43**, 4611 (2018).
14. J. J. Gil and I. San José, Opt. Commun. **283**, 4430 (2010).
15. J. J. Gil, A. T. Friberg, T. Setälä, and I. San José, Phys. Rev. A **95**, 053856 (2017).
16. A. Norrman, J. J. Gil, A. T. Friberg, and T. Setälä, Opt. Lett. **44**, 215 (2019).
17. J. C. Samson, Geophys. J. R. Astron. Soc. **34**, 403 (1973).
18. T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, Phys. Rev. E **66**, 016615 (2002).
19. T. Setälä, M. Kaivola, and A. T. Friberg, Phys. Rev. Lett. **88**, 123902 (2002).
20. M. R. Dennis, J. Opt. A **6**, S26 (2004).
21. J. J. Gil, Phys. Rev. A **90**, 043858 (2014).
22. J. J. Gil, J. Eur. Opt. Soc. Rapid Publ. **10**, 15054 (2015).
23. T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, Phys. Rev. A **78**, 033817 (2008).
24. T. Voipio, T. Setälä, A. Shevchenko, and A. T. Friberg, Phys. Rev. A **82**, 063807 (2010).
25. A. Shevchenko, M. Roussey, A. T. Friberg, and T. Setälä, Optica **4**, 64 (2017).
26. J. J. Gil, I. San José, A. Norrman, A. T. Friberg, and T. Setälä, Sets of orthogonal three-dimensional polarization states and their physical interpretation (to be published).