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Algebraic Geometry / *Géométrie algébrique*

# Note on the monodromy conjecture for a space monomial curve with a plane semigroup

*Note sur la conjecture de la monodromie pour une courbe d'espace monomiale dont le semi-groupe est celui d'une branche plane*

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**Abstract.** Roughly speaking, the monodromy conjecture for a singularity states that every pole of its motivic Igusa zeta function induces an eigenvalue of its monodromy. In this note, we determine both the motivic Igusa zeta function and the eigenvalues of monodromy for a space monomial curve that appears as the special fiber of an equisingular family whose generic fiber is a plane branch. In particular, this yields a proof of the monodromy conjecture for such a curve.

**Résumé.** En gros, la conjecture de la monodromie pour une singularité dit que chaque pôle de sa fonction zêta d'Igusa motivique induit une valeur propre de sa monodromie. Dans cette note, nous déterminons la fonction zêta d'Igusa motivique ainsi que les valeurs propres de la monodromie pour une courbe d'espace monomiale qui apparaît comme fibre spéciale d'une famille équisingulière dont la fibre générique est une branche plane. En particulier, il en résulte une démonstration de la conjecture de la monodromie pour une telle courbe.

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## Introduction

The *monodromy conjecture* for ideals predicts a relation between two invariants associated with an ideal, one originating from number theory and the other from differential topology. More precisely, it states that every pole  $\mathbb{L}^{-s_0}$  of the motivic Igusa zeta function of an ideal  $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$  induces an eigenvalue of monodromy  $e^{2\pi i s_0}$  associated with  $\mathcal{I}$ . To date, this conjecture has only been proven in full generality for ideals in two variables [13]. In higher dimension, there exist various partial results for one polynomial (see for instance the introduction of [2] for a list of references), while for multiple polynomials, the most general result so far is a proof for monomial ideals [4].

The *motivic Igusa zeta function* for a polynomial  $f \in \mathbb{C}[x_0, \dots, x_n]$  was introduced by Denef and Loeser [3] in analogy with the *p-adic Igusa zeta function*. Roughly speaking, it counts the  $\mathbb{C}[t]/(t^{m+1})$ -points on the hypersurface  $X = \{f = 0\}$  and is defined in terms of the *jet schemes* of  $X$ . In a straightforward way, this definition can be generalized to any ideal  $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$  with associated subscheme  $X$  in  $\mathbb{C}^{n+1}$ . Furthermore, the motivic Igusa zeta function is a rational function with poles of the form  $\mathbb{L}^{-s_0}$ , where  $\mathbb{L}$  denotes the class of the affine line in the *Grothendieck ring* of complex varieties and  $s_0 \in \mathbb{Q}$ .

The *eigenvalues of monodromy* were originally introduced for one polynomial  $f \in \mathbb{C}[x_0, \dots, x_n]$  as follows. Assume that  $f(0) = 0$  and consider  $f$  as a function  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ . Take a point  $x \in X = f^{-1}(0)$  and a small ball  $B$  in  $\mathbb{C}^{n+1}$  with center  $x$ . Milnor [7] showed that the restriction  $f|_B$  is a smooth locally trivial fibration over a small enough pointed disc  $D^* = D \setminus \{0\}$  in  $\mathbb{C}$  with center 0. The corresponding fiber is called the (*local*) *Milnor fiber*  $F_x$  of  $f$  at  $x$ . The lifting of a loop in  $D^*$  going once around the origin counterclockwise induces well-defined automorphisms on the cohomologies  $H^m(F_x, \mathbb{C})$ , which are called the (*local*) *monodromy transformations* of  $f$  at  $x$ . An *eigenvalue of monodromy* or a *monodromy eigenvalue* of  $f$  is an eigenvalue of such a monodromy action at some point  $x \in X$ . Although the notion of local Milnor fiber is not well-defined for a general ideal  $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ , there is an abstract construction of Verdier [14] in this setting, yielding *Verdier monodromy eigenvalues*. Furthermore, A'Campo [1] showed an expression for the monodromy eigenvalues of a polynomial  $f$  in terms of an *embedded resolution of singularities* of  $f$ , which can be generalized to an ideal  $\mathcal{I}$  by using a *principalization* of  $\mathcal{I}$ , see [13]. It is also well-known that all eigenvalues of monodromy are roots of unity, or thus, of the form  $e^{2\pi i s_0}$  for some  $s_0 \in \mathbb{Q}$ .

This note investigates the monodromy conjecture for the class of binomial ideals in arbitrary dimension defining so-called *space monomial curves*  $Y \subset \mathbb{C}^{g+1}$  with  $g \geq 1$ . These curves arise as the special fibers of equisingular families of curves whose generic fibers are isomorphic to some plane branch. After introducing these curves in more detail, we show the results, without proofs, obtained in [10]; by studying the jet schemes of a space monomial curve  $Y \subset \mathbb{C}^{g+1}$ , we are able to compute the motivic Igusa zeta function and to determine its poles, see Theorem 3 and Theorem 4, respectively. Then, we explain the approach of [6], again without proofs, to reduce the problem of studying the monodromy eigenvalues associated with  $Y \subset \mathbb{C}^{g+1}$  by considering  $Y$  as a Cartier divisor on a *generic embedding surface*  $S$ . This way, we can use an A'Campo formula in terms of an *embedded  $\mathbb{Q}$ -resolution* of  $Y \subset S$  to compute the monodromy zeta function of  $Y$ , see Theorem 8. Finally, we combine all results to conclude the monodromy conjecture for a space monomial curve  $Y \subset \mathbb{C}^{g+1}$  in Theorem 10.

## 1. Space monomial curves with a plane semigroup

We start this note with introducing the type of singularities we are interested in. They are defined by binomial equations and arise as (equisingular) deformations of irreducible plane branches.

More precisely, let  $\mathcal{C} := \{f = 0\} \subset (\mathbb{C}^2, 0)$  be the germ at the origin of a complex plane curve defined by an irreducible series  $f \in \mathbb{C}[[x_0, x_1]]$  satisfying  $f(0) = 0$ , and let

$$v_{\mathcal{C}} : \frac{\mathbb{C}[[x_0, x_1]]}{(f)} \setminus \{0\} \longrightarrow \mathbb{N} : h \mapsto \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_0, x_1]]}{(f, h)}$$

be its associated valuation. The *semigroup*  $\Gamma(\mathcal{C})$  of  $\mathcal{C}$  is the image of this valuation and can be generated by a unique minimal set of generators  $(\bar{\beta}_0, \dots, \bar{\beta}_g)$  with  $\bar{\beta}_0 < \dots < \bar{\beta}_g$  and  $\text{gcd}(\bar{\beta}_0, \dots, \bar{\beta}_g) = 1$  ( $\text{gcd}$  being the greatest common divisor), see for instance [15]. In terms of these generators, we define  $(Y, 0) \subset (\mathbb{C}^{g+1}, 0)$  as the image of the monomial map

$$M : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^{g+1}, 0) : t \mapsto (t^{\bar{\beta}_0}, t^{\bar{\beta}_1}, \dots, t^{\bar{\beta}_g}).$$

This is an irreducible (germ of a) curve which is smooth outside the origin and which has the “plane” semigroup  $\Gamma(\mathcal{C})$  as semigroup [12]. We call  $Y$  the *monomial curve associated with*  $\mathcal{C}$ .

The curve  $Y$  can be seen as a deformation of  $\mathcal{C}$  as follows. Firstly, we put  $e_i := \text{gcd}(\bar{\beta}_0, \dots, \bar{\beta}_i)$  for  $i = 0, \dots, g$ , and  $n_i := e_{i-1}/e_i$  for  $i = 1, \dots, g$ . These are positive integers satisfying  $\bar{\beta}_0 = e_0 > e_1 > \dots > e_g = 1$  and  $n_i \geq 2$ . Secondly, every  $n_i \bar{\beta}_i$  for  $i = 1, \dots, g$  belongs to the semigroup generated by  $\bar{\beta}_0, \dots, \bar{\beta}_{i-1}$ , or thus, there exist non-negative integers  $b_{ij}$  for  $0 \leq j < i$  such that  $n_i \bar{\beta}_i = b_{i0} \bar{\beta}_0 + \dots + b_{i(i-1)} \bar{\beta}_{i-1}$ . These integers are unique if we require that  $b_{ij} < n_j$  for  $j \neq 0$ . For notational reasons, we put  $n_0 := b_{10}$ . Thirdly, we can identify a *minimal generating sequence*  $(x_0, \dots, x_g)$  of the valuation  $v_{\mathcal{C}}$ , consisting of elements  $x_i \in \mathbb{C}[[x_0, x_1]]$  for  $i = 0, \dots, g$  such that  $v_{\mathcal{C}}(x_i) = \bar{\beta}_i$  and which satisfy equations of the form

$$x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \dots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0, \dots, \gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \dots x_i^{\gamma_i}, \quad i = 0, \dots, g,$$

where  $x_{g+1} = 0, c_i \in \mathbb{C} \setminus \{0\}, c_{i,\gamma} \in \mathbb{C}, 0 \leq \gamma_j < n_j$  for  $1 \leq j \leq i$ , and  $\sum_{j=0}^i \gamma_j \bar{\beta}_j > n_i \bar{\beta}_i$ . We refer to [9], [11] and [12] for more details. Finally, we can modify these equations a bit by involving an extra variable  $v \in \mathbb{C}$ :

$$v x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \dots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0, \dots, \gamma_i)} c_{i,\gamma} v x_0^{\gamma_0} \dots x_i^{\gamma_i}, \quad i = 0, \dots, g.$$

For every  $v$  in  $(\mathbb{C}, 0)$ , these new equations define a germ of a curve in  $(\mathbb{C}^{g+1}, 0)$  with semigroup  $\Gamma(\mathcal{C})$ . Even more, for  $v \neq 0$ , they are all isomorphic to  $\mathcal{C}$ , while for  $v = 0$ , we get the curve defined by  $x_i^{n_i} - c_i x_0^{b_{i0}} \dots x_{i-1}^{b_{i(i-1)}} = 0$  for  $i = 1, \dots, g$ . Hence, all these branches together define an equisingular family  $\eta : (\chi, 0) \subset (\mathbb{C}^{g+1} \times \mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  with generic fiber isomorphic to  $\mathcal{C}$  and special fiber given by the latter equations, in which the coefficients  $c_i$  are needed to see that any irreducible plane branch is a (equisingular) deformation of a such a curve. By changing the coordinates  $x_0, \dots, x_g$  if necessary, we can always assume that every  $c_i = 1$ ; these are the binomial equations defining the monomial curve  $(Y, 0) \subset (\mathbb{C}^{g+1}, 0)$  associated with  $\mathcal{C}$ .

Clearly, the equations defining  $(Y, 0)$  in  $(\mathbb{C}^{g+1}, 0)$  can also be considered in  $\mathbb{C}^{g+1}$ ; from now on, we define a (*space*) *monomial curve*  $Y \subset \mathbb{C}^{g+1}$  as the global curve defined by

$$\begin{cases} f_1 := x_1^{n_1} - x_0^{n_0} & = 0 \\ f_2 := x_2^{n_2} - x_0^{b_{20}} x_1^{b_{21}} & = 0 \\ \vdots & \\ f_g := x_g^{n_g} - x_0^{b_{g0}} x_1^{b_{g1}} \dots x_{g-1}^{b_{g(g-1)}} & = 0. \end{cases} \tag{1}$$

This is still an irreducible curve which is smooth outside the origin. Furthermore, it is a complete intersection curve given by the regular sequence  $f_1, \dots, f_g \in \mathbb{C}[x_0, \dots, x_g]$ .

## 2. The motivic Igusa zeta function of a space monomial curve

In this section, we explain the main results of [10], in which the motivic Igusa zeta function of a space monomial curve  $Y \subset \mathbb{C}^{g+1}$  is calculated by determining the structure of the jet schemes of  $Y$ . The *motivic Igusa zeta function* of  $Y \subset \mathbb{C}^{g+1}$  can be written as

$$Z_Y^{mot}(T) = 1 - \frac{1-T}{T} J_Y(T),$$

where  $J_Y(T)$  is the Poincaré series

$$J_Y(T) := \sum_{m \geq 0} [Y_m](\mathbb{L}^{-(g+1)} T)^{m+1} \in \mathcal{M}_{\mathbb{C}}[[T]].$$

Here,  $Y_m \subset \mathbb{C}^{(g+1)(m+1)}$  is the  $m$ -th jet scheme of  $Y$ , which is defined as the  $\mathbb{C}$ -scheme such that

$$\{\text{points of } Y_m \text{ with coordinates in } \mathbb{C}\} = \{\text{points of } Y \text{ with coordinates in } \mathbb{C}[t]/(t^{m+1})\},$$

and  $[Y_m]$  is its class in the Grothendieck ring  $K_0(\text{Var}_{\mathbb{C}})$  of complex varieties. By  $\mathbb{L}$  we denote the class of the affine line in  $K_0(\text{Var}_{\mathbb{C}})$  and by  $\mathcal{M}_{\mathbb{C}} := K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$  the localization with respect to  $\mathbb{L}$ . We further let  $\pi_{m,p} : Y_m \rightarrow Y_p$  for all  $m \geq p$  be the natural morphism induced by the truncation morphism  $\mathbb{C}[t]/(t^{m+1}) \rightarrow \mathbb{C}[t]/(t^{p+1})$  and we put  $\pi_m := \pi_{m,0} : Y_m \rightarrow Y_0 = Y$ . More details can be found for instance in [10, Section 2].

### 2.1. Structure of the jet schemes

Because  $Y_0 = Y$  and  $\pi_m$  induces a trivial fibration over  $Y \setminus \{0\}$  with fiber isomorphic to  $\mathbb{C}^m$ , it remains to investigate the fibers  $\pi_m^{-1}(0)$  for  $m \geq 1$ . Because these fibers for  $g = 1$  have already been determined in [8, Corollary 4.4], we assume that  $g \geq 2$ . For this purpose, we stratify each fiber  $\pi_m^{-1}(0)_{red}$  with its reduced structure as follows. Let  $l \in \mathbb{N}$  be such that  $ln_0 n_1 < m \leq (l+1)n_0 n_1$ . Then

$$\pi_m^{-1}(0)_{red} = \left( \bigsqcup_{k=1}^l D_{m,k} \right) \sqcup B_m, \tag{2}$$

where

$$\begin{aligned} D_{m,k} &:= \pi_{m, kn_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \dots = x_0^{(kn_1-1)} = 0\} \cap \{x_0^{(kn_1)} \neq 0\})_{red}, \\ B_m &:= \pi_{m, ln_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \dots = x_0^{(ln_1)} = 0\})_{red}. \end{aligned}$$

To explain the structure of these strata, we introduce some notation. First, we denote by  $[a/b]$  the integer part of a rational number  $a/b$ . Second, for  $k \geq 1$ , let  $j(k) \in \mathbb{N}$  be defined by

$$j(k) := \begin{cases} 2 & \text{if } n_2 \nmid k \\ \max_{l \in \mathbb{N}} \{n_2 \dots n_{l-1} \mid k\} & \text{otherwise.} \end{cases}$$

Note that  $2 \leq j(k) \leq g + 1$ . Third, for  $1 \leq i < j(k)$  and for  $m \in \mathbb{N}$  satisfying

$$\frac{kn_i \bar{\beta}_i}{e_1} \leq m < \frac{kn_{i+1} \bar{\beta}_{i+1}}{e_1},$$

where  $\bar{\beta}_{g+1} := +\infty$  by convention, we define

$$c_{i,k}(m) := k(n_0 + n_1) + \sum_{l=2}^i \frac{k \bar{\beta}_l}{e_1} + \sum_{l=1}^i \left( m - \frac{kn_l \bar{\beta}_l}{e_1} + 1 \right) + \sum_{l=i+1}^g \left( \left\lfloor \frac{m}{n_l} \right\rfloor + 1 \right).$$

Finally, let  $Y^i$  for  $i = 1, \dots, g$  be the complete intersection curve defined in  $\mathbb{C}^{i+1}$  by the first  $i$  equations  $f_1, \dots, f_i$  of the defining equations (1) of  $Y$ .

**Proposition 1.** Consider  $m \geq 1$  and let  $l \in \mathbb{N}$  be such that  $ln_0n_1 < m \leq (l + 1)n_0n_1$ . We have the following.

- (1) Let  $1 \leq k \leq l$ . If there exists some  $i \in \mathbb{N}$  such that  $1 \leq i < j(k)$  and  $\frac{kn_i\bar{\beta}_i}{e_1} \leq m < \frac{kn_{i+1}\bar{\beta}_{i+1}}{e_1}$ , then the stratum  $D_{m,k}$  is isomorphic to

$$(Y^i \setminus \{0\}) \times \mathbb{C}^{(g+1)(m+1)-c_{i,k}(m)-1} \simeq (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{(g+1)(m+1)-c_{i,k}(m)-1}.$$

In particular,  $\mathcal{C}_{m,k} := \overline{D_{m,k}}$  is irreducible and its codimension in  $\mathbb{C}^{(g+1)(m+1)}$  is equal to  $c_{i,k}(m)$ . If  $m \geq \frac{kn_{j(k)}\bar{\beta}_{j(k)}}{e_1}$ , then  $D_{m,k} = \emptyset$ .

- (2) If  $m \neq (l + 1)n_0n_1$ , then

$$B_m \simeq \mathbb{C}^{(g+1)(m+1)-\sum_{i=0}^g (\lfloor \frac{m}{n_i} \rfloor + 1)}$$

is irreducible with codimension  $g + 1 + \sum_{i=0}^g \lfloor m/n_i \rfloor$  in  $\mathbb{C}^{(g+1)(m+1)}$ . If  $m = (l + 1)n_0n_1$ , then

$$B_m \simeq \{x_1^{((l+1)n_0)n_1} - x_0^{((l+1)n_1)n_0} = 0\} \times \mathbb{C}^{(g+1)(m+1)-(l+1)(n_0+n_1)-\sum_{i=2}^g (\lfloor \frac{(l+1)n_0n_1}{n_i} \rfloor + 1)}$$

is irreducible with codimension  $g + (l + 1)(n_0 + n_1) + \sum_{i=2}^g \lfloor (l + 1)n_0n_1/n_i \rfloor$  in  $\mathbb{C}^{(g+1)(m+1)}$ .

This result implies that the stratification (2) induces a decomposition

$$\pi_m^{-1}(0)_{red} = \left( \bigcup_{k=1}^l C_{m,k} \right) \cup B_m$$

into irreducible closed subvarieties with  $C_{m,k} = \emptyset$  if  $m \geq \frac{kn_{j(k)}\bar{\beta}_{j(k)}}{e_1}$ . The next theorem tells us that this is a decomposition into irreducible components.

**Theorem 2.** Consider  $m \geq 1$  and let  $l \in \mathbb{N}$  be such that  $ln_0n_1 < m \leq (l + 1)n_0n_1$ . The irreducible components of  $\pi_m^{-1}(0)_{red}$  are  $\mathcal{C}_{m,k} = \overline{D_{m,k}}$  for  $k = 1, \dots, l$  such that  $m < \frac{kn_{j(k)}\bar{\beta}_{j(k)}}{e_1}$  and  $B_m$ . Furthermore,  $B_m$  is a component of maximal dimension.

### 2.2. Formula for the motivic Igusa zeta function and its poles

With the structure of the jet schemes following from Proposition 1 for  $g \geq 2$  and from [8, Corollary 4.4] for  $g = 1$ , we can compute the motivic Igusa zeta function of  $Y \subset \mathbb{C}^{g+1}$ .

**Theorem 3.** Consider a space monomial curve  $Y \subset \mathbb{C}^{g+1}$  defined by the equations (1). Let  $N_i$  and  $v_i$  for  $i = 1, \dots, g$  be the positive integers defined as

$$N_i := \text{lcm} \left( \frac{\bar{\beta}_i}{e_i}, n_i, \dots, n_g \right), \quad v_i := N_i \left( \frac{1}{n_i\bar{\beta}_i} \left( \sum_{l=0}^i \bar{\beta}_l - \sum_{l=1}^{i-1} n_l\bar{\beta}_l \right) + (i-1) + \sum_{l=i+1}^g \frac{1}{n_l} \right),$$

where lcm denotes the least common multiple. The motivic Igusa zeta function associated with  $Y \subset \mathbb{C}^{g+1}$  is given by

$$Z_Y^{mot}(T) = 1 - (1 - T) \left( \frac{(\mathbb{L} - 1)\mathbb{L}^{-(g+1)}}{1 - \mathbb{L}^{-g}T} + \frac{\mathbb{L}^{-(g+1)}}{1 - \mathbb{L}^{-v_1}T^{N_1}} \sum_{r=0}^{N_1-1} \mathbb{L}^{-\sum_{i=0}^g \lfloor \frac{r}{n_i} \rfloor} T^r + \sum_{i=1}^{g-1} \frac{(\mathbb{L} - 1)Z_i(T)}{(1 - \mathbb{L}^{-v_i}T^{N_i})(1 - \mathbb{L}^{-v_{i+1}}T^{N_{i+1}})} + \frac{(\mathbb{L} - 1)\mathbb{L}^{-(v_g+g+1)}T^{N_g}}{(1 - \mathbb{L}^{-g}T)(1 - \mathbb{L}^{-v_g}T^{N_g})} \right).$$

Here,  $Z_i(T)$  for  $i = 1, \dots, g - 1$  are polynomials with coefficients in  $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$ . More precisely,

$$Z_1(T) := \sum_{r=0}^{\frac{N_1}{n_0 n_1} - 1} \sum_{r'=1}^{\frac{e_2 N_2}{\beta_2}} \sum_{m \in I_{1,r'+r}^{(r')}} \mathbb{L}^{-(c_{1,r'+r}(m)+1)} T^m,$$

$$Z_i(T) := \sum_{r=0}^{\frac{e_i N_i}{n_i \beta_i} - 1} \sum_{r'=1}^{\frac{e_{i+1} N_{i+1}}{\beta_{i+1}}} \sum_{m \in I_{i,(r'+r)n_2 \dots n_i}^{(r')}} \mathbb{L}^{-(c_{i,(r'+r)n_2 \dots n_i}(m)+1)} T^m, \quad i = 2, \dots, g - 1,$$

where  $I_{i,k}^{(p)}$  for  $i = 1, \dots, g - 1$  and  $k, p \in \mathbb{N}$  is the interval

$$I_{i,k}^{(p)} := \left[ \frac{kn_i \bar{\beta}_i}{e_1} + (p - 1) \left( \frac{n_{i+1} \bar{\beta}_{i+1}}{e_i} - \frac{n_i \bar{\beta}_i}{e_i} \right), \frac{kn_i \bar{\beta}_i}{e_1} + p \left( \frac{n_{i+1} \bar{\beta}_{i+1}}{e_i} - \frac{n_i \bar{\beta}_i}{e_i} \right) \right] \cap \mathbb{N}.$$

This expression immediately yields a complete list of  $g + 1$  candidate poles of  $Z_Y^{mot}(T)$ :

$$\mathbb{L}^g, \quad \mathbb{L}^{\frac{v_i}{N_i}}, \quad i = 1, \dots, g.$$

Using residues and the related *topological Igusa zeta function*, we are able to show that all these candidates are actual poles.

**Theorem 4.** *A complete list of the poles of the motivic Igusa zeta function associated with a space monomial curve  $Y \subset \mathbb{C}^{g+1}$  is given by*

$$\mathbb{L}^g, \quad \mathbb{L}^{\frac{v_i}{N_i}}, \quad i = 1, \dots, g,$$

and all these poles have order 1.

**Examples 5.**

- (1) The irreducible plane curve defined by  $(x_1^2 - x_0^3)^2 - x_0^5 x_1 = 0$  has  $(4, 6, 13)$  as minimal generating set of its semigroup and induces the space monomial curve  $Y_1 \subset \mathbb{C}^3$  given by

$$\begin{cases} x_1^2 - x_0^3 = 0 \\ x_2^2 - x_0^5 x_1 = 0. \end{cases}$$

From Theorem 3, one can compute that

$$Z_{Y_1}^{mot}(T) = \frac{(\mathbb{L} - 1)P_1(T)}{\mathbb{L}^{47}(1 - \mathbb{L}^{-2}T)(1 - \mathbb{L}^{-8}T^6)(1 - \mathbb{L}^{-37}T^{26})}$$

where  $P_1(T)$  is a concrete polynomial in  $T$  of degree 31 with coefficients in  $\mathbb{Z}[\mathbb{L}]$ , see [6, Example 4.1]. We see that all three candidate poles,  $\mathbb{L}^g = \mathbb{L}^2, \mathbb{L}^{\frac{v_1}{N_1}} = \mathbb{L}^{\frac{8}{6}}$  and  $\mathbb{L}^{\frac{v_2}{N_2}} = \mathbb{L}^{\frac{37}{26}}$ , are actual poles of order 1.

- (2) Consider the space monomial curve  $Y_2 \subseteq \mathbb{C}^4$  associated with the polynomial  $((x_1^2 - x_0^3)^2 - x_0^5 x_1)^2 - x_0^{10}(x_1^2 - x_0^3)$ . Its semigroup is minimally generated by  $(8, 12, 26, 53)$  and its defining equations are

$$\begin{cases} x_1^2 - x_0^3 = 0 \\ x_2^2 - x_0^5 x_1 = 0 \\ x_3^2 - x_0^{10} x_2 = 0. \end{cases}$$

For this curve, Theorem 3 gives

$$Z_{Y_2}^{mot}(T) = \frac{(\mathbb{L} - 1)P_2(T)}{\mathbb{L}^{299}(1 - \mathbb{L}^{-3}T)(1 - \mathbb{L}^{-11}T^6)(1 - \mathbb{L}^{-50}T^{26})(1 - \mathbb{L}^{-235}T^{106})},$$

for a concrete polynomial  $P_2(T)$  of degree 137 with coefficients in  $\mathbb{Z}[\mathbb{L}]$ . Again, we see that  $Z_{Y_2}^{mot}(T)$  has indeed poles  $\mathbb{L}^g = \mathbb{L}^3, \mathbb{L}^{\frac{v_1}{N_1}} = \mathbb{L}^{\frac{11}{6}}, \mathbb{L}^{\frac{v_2}{N_2}} = \mathbb{L}^{\frac{50}{26}}$ , and  $\mathbb{L}^{\frac{v_3}{N_3}} = \mathbb{L}^{\frac{235}{106}}$ .

### 3. The monodromy zeta function of a space monomial curve

This section provides an overview of the strategy in [6] to study the monodromy eigenvalues of a space monomial curve  $Y \subset \mathbb{C}^{g+1}$  by considering  $Y$  as a Cartier divisor on a generic embedding surface and computing an embedded  $\mathbb{Q}$ -resolution of this situation. Because the monodromy conjecture is already well-known in the case where  $g = 1$  (a monomial curve  $Y \subset \mathbb{C}^2$  of the above type is just a cusp), we assume that  $g \geq 2$ . The starting point is the following generalization of A'Campo's formula to express the *monodromy eigenvalues* associated with  $Y$  in terms of a *principalization*  $\varphi : \tilde{X} \rightarrow \mathbb{C}^{g+1}$  of its defining ideal  $\mathcal{I} = (f_1, \dots, f_g)$ . Denote by  $E_j$  for  $j \in J$  the irreducible components of  $\varphi^{-1}(Y)$  and put  $E_j^o := E_j \setminus \cup_{i \neq j} (E_i \cap E_j)$  for every  $j \in J$ . Let  $N_j$  and  $\nu_j - 1$  be the multiplicity of  $E_j$  in the divisor on  $\tilde{X}$  of  $\varphi^* \mathcal{I}$  and  $\varphi^*(dx_1 \wedge \dots \wedge dx_n)$ , respectively. Consider the blow-up  $\sigma : X' \rightarrow \mathbb{C}^{g+1}$  of  $\mathbb{C}^{g+1}$  along  $Y$  with exceptional divisor  $E' := \sigma^{-1}(Y)$ . By the universal property of the blow-up, there exists a unique morphism  $\psi : \tilde{X} \rightarrow X'$  such that  $\sigma \circ \psi = \varphi$ . It was proven in [13] that a complex number is a monodromy eigenvalue associated with  $Y$  if and only if it is a zero or pole of the *monodromy zeta function* at a point  $e \in E'$  given by

$$Z_{Y,e}^{mon}(t) = \prod_{j \in J} (1 - t^{N_j})^{\chi(E_j^o \cap \psi^{-1}(e))}, \tag{3}$$

where  $\chi$  is the topological Euler characteristic. For a more elaborate introduction to eigenvalues of monodromy and this formula, see for instance [6, Section 2]. In other words, to know the monodromy eigenvalues of  $Y$ , we need to compute these zeta functions. For example, it is not hard to show that the formula (3) for an arbitrary point  $e \in E' \setminus \sigma^{-1}(0)$  gives  $Z_{Y,e}^{mon}(t) = 1 - t$ , yielding the trivial monodromy eigenvalue 1. It thus remains to study the monodromy zeta functions at points in  $\sigma^{-1}(0)$ .

#### 3.1. Monodromy via a generic embedding surface

To reduce the problem of investigating the monodromy zeta functions at points in  $\sigma^{-1}(0)$ , we introduce a generic embedding surface of  $Y$  in terms of its defining equations  $f_1, \dots, f_g$ . For every set  $(\lambda_2, \dots, \lambda_g)$  of  $g - 1$  non-zero complex numbers, we define an affine scheme  $S(\lambda_2, \dots, \lambda_g)$  in  $\mathbb{C}^{g+1}$  given by the equations

$$\begin{cases} f_1 + \lambda_2 f_2 = 0 \\ f_2 + \lambda_3 f_3 = 0 \\ \vdots \\ f_{g-1} + \lambda_g f_g = 0. \end{cases} \tag{4}$$

The curve  $Y$  is contained in every such scheme as a Cartier divisor defined by a single equation  $f_i = 0$  for some  $i \in \{1, \dots, g\}$ . For *generic* coefficients  $(\lambda_2, \dots, \lambda_g)$  (i.e., the point  $(\lambda_2, \dots, \lambda_g)$  is contained in the non-empty complement of a specific closed subset of  $(\mathbb{C} \setminus \{0\})^{g-1}$ ), one can show that  $S(\lambda_2, \dots, \lambda_g)$  is a normal surface which is smooth outside the origin [6, Proposition 4.1]. We call such  $S(\lambda_2, \dots, \lambda_g)$  a *generic (embedding) surface* of  $Y$  and, to simplify the notation, we put  $S := S(\lambda_2, \dots, \lambda_g)$ .

Because  $\text{Sing}(S) = \text{Sing}(Y) = \{0\}$ , we can apply the original formula of A'Campo [1] to compute the monodromy zeta function  $Z_{Y,0}^{mon}(t)$  of  $Y \subset S$  at the origin in terms of an *embedded resolution of singularities* of  $Y = \{f_1 = 0\}$  on  $S$ . To see the link between the latter formula and the eigenvalues associated with  $Y \subset \mathbb{C}^{g+1}$ , we consider the strict transform  $S' = \sigma^{-1}(S \setminus Y)$  of  $S$  under the blow-up  $\sigma$ . By the properties of the blow-up, we know that  $S'$  is isomorphic to  $S$  with  $Y' := S' \cap E'$  isomorphic to  $Y$ . In particular, the intersection  $S' \cap \sigma^{-1}(0)$  consists of a single point. We denote this point by  $p$  and refer to it as the *generic point* associated with  $S$ . It turns out that the monodromy zeta function  $Z_{Y,p}^{mon}(T)$  of  $Y \subset \mathbb{C}^{g+1}$  at the generic point  $p$  is equal to the monodromy



zeta function  $Z_{Y,0}^{mon}(t)$  of  $Y \subset S$  at the origin. As the next theorems shows, this result is true for a larger class of ideals. Hence, it could be possibly useful for studying the monodromy eigenvalues of other ideals in this class.

**Theorem 6.** *Consider a complete intersection curve  $Y = V(\mathcal{I}) \subset \mathbb{C}^{g+1}$  whose ideal  $\mathcal{I} = (f_1, \dots, f_g)$  is generated by a regular sequence  $f_1, \dots, f_g \in \mathbb{C}[x_0, \dots, x_g]$ , and whose singular set is  $\text{Sing}(Y) = \{0\}$ . Let  $S = S(\lambda_2, \dots, \lambda_g)$  be a generic embedding surface of  $Y$  defined by the equations (4). Denote by  $\sigma : X' \rightarrow \mathbb{C}^{g+1}$  the blow-up of  $\mathbb{C}^{g+1}$  with center  $Y$  and by  $S'$  the strict transform of  $S$  under  $\sigma$ . Then, the monodromy zeta function  $Z_{Y,p}^{mon}(t)$  of  $Y$  considered in  $\mathbb{C}^{g+1}$  at the generic point  $p = S' \cap \sigma^{-1}(0)$  is equal to the monodromy zeta function  $Z_{Y,0}^{mon}(t)$  of  $Y$  considered as a Cartier divisor on  $S$  at the origin.*

We will see later on that the monodromy zeta function  $Z_{Y,p}^{mon}(t)$  of  $Y \subset \mathbb{C}^{g+1}$  at the generic point  $p$  suffices to recover all non-trivial candidate monodromy eigenvalues coming from the poles of the motivic Igusa zeta function of  $Y$ . We will refer to  $Z_{Y,p}^{mon}(t)$  as *the monodromy zeta function* of  $Y$ . By the previous theorem, we can find this zeta function by considering  $Y$  on a generic surface  $S$ .

### 3.2. Monodromy via an embedded $\mathbb{Q}$ -resolution

To compute the monodromy zeta function at the origin of  $Y$  considered as a Cartier divisor on a generic embedding surface  $S$ , we make use of another generalization of A'Campo's formula using an embedded  $\mathbb{Q}$ -resolution of  $Y \subset S$ . Roughly speaking, an *embedded  $\mathbb{Q}$ -resolution* is a resolution in which we allow the final ambient space to have abelian quotient singularities and can be constructed as a sequence of *weighted blow-ups*. With the formula shown in [5], the monodromy zeta function of  $Y \subset S$  at the origin can be written in terms of an embedded  $\mathbb{Q}$ -resolution  $\varphi : \widehat{S} \rightarrow S$  as

$$Z_{Y,0}^{mon}(t) = \prod_{\substack{1 \leq j \leq r \\ t \in T}} (1 - t^{m_{j,t}})^{\chi(E_{j,t}^{\circ})},$$

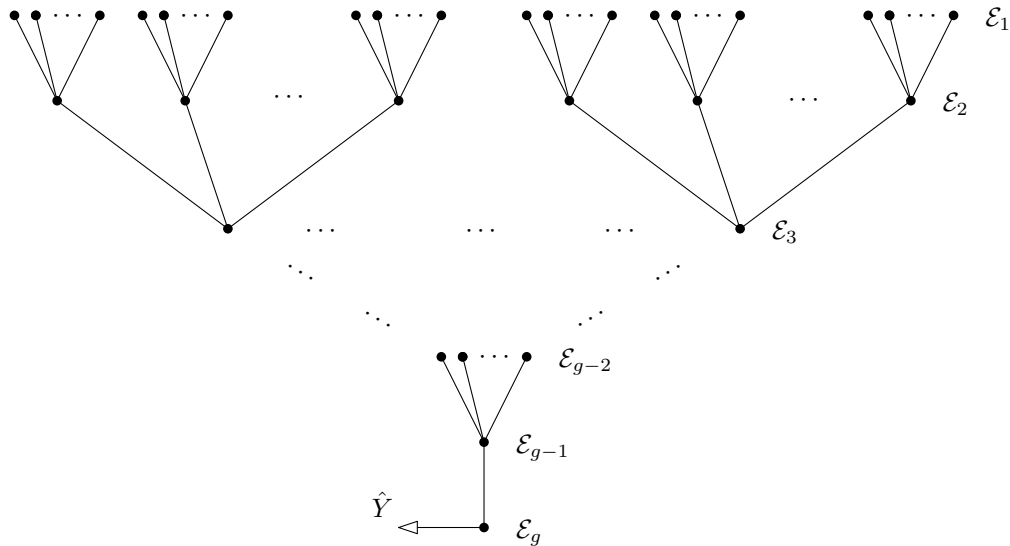
where  $\{E_{j,t}\}_{j=1,\dots,r,t \in T}$  is a finite stratification of the exceptional varieties  $E_1, \dots, E_r$  of  $\varphi$  such that the *multiplicity*  $m_{j,t}$  of  $E_j$  along each  $E_{j,t}$  is constant. Here, the multiplicity  $m_{j,t}$  is defined as  $m/d$  where  $d$  is the order of the small group  $\mu_d$  acting at a general point  $q$  in  $E_{j,t}$  and  $x^m : \mathbb{C}^2 / \mu_d \rightarrow \mathbb{C}$  is the local equation of  $E_j$  at  $q$ . More information on abelian quotient singularities, embedded  $\mathbb{Q}$ -resolutions, weighted blow-ups and this A'Campo formula can be found for instance in [6, Section 3].

In [6, Section 5], the computation of  $g$  weighted blow-ups in higher dimension yields an embedded  $\mathbb{Q}$ -resolution of  $Y \subset S$  with dual graph a tree as in Figure 1. After each blow-up, one variable can be eliminated so that the new situation is very similar to the one before the blow-up, but with one equation in  $Y$  and  $S$  less. Therefore, the last step coincides with the resolution of a cusp in a Hirzebruch–Jung singularity of type  $\frac{1}{d}(1, q)$  by one weighted blow-up. It is worth mentioning that this process is similar to the resolution of an irreducible plane curve with  $g$  Puiseux pairs; after each weighted blow-up, the number of Puiseux pairs is lowered by one so that the last step coincides with the resolution of an irreducible plane curve with one Puiseux pair. However, our resolution is more complicated as the strict transform of  $Y$  intersects in general the singular locus of the ambient space. For instance, the computations of the numerical data of the resolution, such as the multiplicity of the exceptional divisor in each step, its number of irreducible components and its Euler characteristic, give rise to interesting arithmetic challenges.

**Theorem 7.** *Let  $Y \subset \mathbb{C}^{g+1}$  be a space monomial curve defined by the equations (1) with  $g \geq 2$  and consider  $Y$  as a Cartier divisor on a generic surface  $S = S(\lambda_2, \dots, \lambda_g) \subset \mathbb{C}^{g+1}$  given by (4). There exists an embedded  $\mathbb{Q}$ -resolution  $\varphi = \varphi_1 \circ \dots \circ \varphi_g : \widehat{S} \rightarrow S$  of  $Y \subset S$  which is a composition of  $g$  weighted blow-ups  $\varphi_k$  with exceptional divisor  $\mathcal{E}_k$  such that the pull-back of  $Y$  is given by*

$$\varphi^* Y = \widehat{Y} + \sum_{\substack{1 \leq k \leq g \\ 1 \leq j \leq r_k}} N_k \mathcal{E}_{kj},$$

where  $\mathcal{E}_k = \mathcal{E}_{k1} + \dots + \mathcal{E}_{kr_k}$  is the decomposition of  $\mathcal{E}_k$  into  $r_k = e_k / \text{lcm}(n_{k+1}, \dots, n_g)$  if  $k = 1, \dots, g-2$  and  $r_{g-1} = r_g = 1$  irreducible components, and  $N_k = \text{lcm}(\overline{\beta}_k / e_k, n_k, \dots, n_g)$  is the multiplicity of  $\mathcal{E}_k$ . Furthermore, each divisor  $\mathcal{E}_k$  for  $k = 2, \dots, g-1$  only intersects  $\mathcal{E}_{k-1}$  and  $\mathcal{E}_{k+1}$ , and  $\mathcal{E}_g$  only intersects  $\mathcal{E}_{g-1}$ . Finally, for every  $k = 2, \dots, g$ , the intersections of  $\mathcal{E}_{k-1}$  and  $\mathcal{E}_k$  are equally distributed; each of the components  $\mathcal{E}_{kj}$  of  $\mathcal{E}_k$  intersects precisely  $r_{k-1} / r_k$  components of  $\mathcal{E}_{k-1}$ , each component  $\mathcal{E}_{(k-1)j'}$  of  $\mathcal{E}_{k-1}$  is intersected by only one of the components of  $\mathcal{E}_k$ , and each non-empty intersection between two components  $\mathcal{E}_{kj}$  and  $\mathcal{E}_{(k-1)j'}$  consists of a single point. In particular, the dual graph of the resolution is a tree as in Figure 1.



**Figure 1.** Dual graph of the resolution of  $Y \subset S$ . Graphe dual de la résolution de  $Y \subset S$ .

### 3.3. Formula for the monodromy zeta function

By stratifying the exceptional divisor of the embedded  $\mathbb{Q}$ -resolution of  $Y \subset S$  from Theorem 7 and computing the Euler characteristic of these strata, we can compute the monodromy zeta function  $Z_{Y,0}^{mon}(t)$  of  $Y \subset S$  at the origin, which is equal to the monodromy zeta function of  $Y$  by Theorem 6.

**Theorem 8.** *Let  $Y \subset \mathbb{C}^{g+1}$  be a space monomial curve defined by the equations (1) with  $g \geq 2$  and consider a generic embedding surface  $S = S(\lambda_2, \dots, \lambda_g) \subset \mathbb{C}^{g+1}$  given by (4). Denote by  $\sigma : X' \rightarrow \mathbb{C}^{g+1}$  the blow-up of  $\mathbb{C}^{g+1}$  with center  $Y$  and by  $S'$  the strict transform of  $S$  under  $\sigma$ . Then, the monodromy zeta function of  $Y$  considered in  $\mathbb{C}^{g+1}$  at the generic point  $p = S' \cap \sigma^{-1}(0)$  is given by*

$$Z_{Y,p}^{mon}(t) = \frac{\prod_{k=0}^g (1 - t^{M_k})^{\frac{\beta_k}{M_k}}}{\prod_{k=1}^g (1 - t^{N_k})^{\frac{n_k \beta_k}{N_k}}},$$

where  $M_k := \text{lcm}(\bar{\beta}_k/e_k, n_{k+1}, \dots, n_g)$  for  $k = 0, \dots, g$ , and  $N_k := \text{lcm}(\bar{\beta}_k/e_k, n_k, \dots, n_g)$  for  $k = 1, \dots, g$ .

**Example 9.** Recall the two space monomial curves from Example 5.

(1) For the space monomial curve  $Y_1 \subset \mathbb{C}^3$ , we find with Theorem 8 that

$$Z_{Y_1,p_1}^{mon}(t) = \frac{(1 - t^2)^2(1 - t^6)(1 - t^{13})}{(1 - t^6)^2(1 - t^{26})} = \frac{(1 - t^2)^2(1 - t^{13})}{(1 - t^6)(1 - t^{26})}.$$

Every pole  $\mathbb{L}^{-s_0}$  of the motivic Igusa zeta function of  $Y_1$  induces a monodromy eigenvalue  $e^{2\pi i s_0}$ :  $e^{-4\pi i}$  is a zero of  $Z_{Y_1,p_1}^{mon}(t)$ , while  $e^{-8\pi i/3}$  and  $e^{-37\pi i/13}$  are poles of  $Z_{Y_1,p_1}^{mon}(t)$ .

(2) A simple computation for the space monomial curve  $Y_2 \subseteq \mathbb{C}^4$  gives

$$Z_{Y_2,p_2}^{mon}(T) = \frac{(1 - t^2)^4(1 - t^6)^2(1 - t^{26})(1 - t^{53})}{(1 - t^6)^4(1 - t^{26})^2(1 - t^{106})} = \frac{(1 - t^2)^4(1 - t^{53})}{(1 - t^6)^2(1 - t^{26})(1 - t^{106})},$$

from which it is easy to check that all four poles of the motivic zeta function induce a monodromy eigenvalue.

#### 4. The monodromy conjecture for a space monomial curve

By combining the results of the previous two sections, we can show the monodromy conjecture for a space monomial curve  $Y \subset \mathbb{C}^{g+1}$  with  $g \geq 2$ .

**Theorem 10.** *Let  $Y \subset \mathbb{C}^{g+1}$  be a space monomial curve defined by the equations (1) with  $g \geq 2$  and denote by  $\sigma : X' \rightarrow \mathbb{C}^{g+1}$  the blow-up of  $\mathbb{C}^{g+1}$  with center  $Y$ . Every pole  $\mathbb{L}^{-s_0}$  of the motivic Igusa zeta function associated with  $Y$  induces a monodromy eigenvalue  $e^{2\pi i s_0}$  of  $Y$  at a point in  $\sigma^{-1}(Y)$ .*

In [6, Section 7], it is shown that the pole  $\mathbb{L}^g$  and poles  $\mathbb{L}^{\frac{v_k}{N_k}}$  for  $k \in \{1, \dots, g\}$  with  $v_k/N_k \in \mathbb{N}$  induce the trivial monodromy eigenvalue 1, while every other candidate monodromy eigenvalue  $e^{-2\pi i v_k/N_k}$  is a pole of the monodromy zeta function  $Z_{Y,p}^{mon}(t)$  of  $Y$ .

**Remark 11.** The structure of the jet schemes can also be used to compute the *local motivic Igusa zeta function* of a space monomial curve  $Y \subset \mathbb{C}^{g+1}$ , see [10, Section 4]. In particular, this local version has the same poles as the global one. Hence, the monodromy conjecture also holds for the local motivic zeta function: every pole  $\mathbb{L}^{-s_0}$  of the local motivic Igusa zeta function associated with a space monomial curve  $Y \subset \mathbb{C}^{g+1}$  induces a monodromy eigenvalue  $e^{2\pi i s_0}$  of  $Y$  at a point in  $\sigma^{-1}(B \cap Y)$  for  $B$  a small ball around 0. Likewise, one can state and conclude the monodromy conjecture for the related global/local *topological and p-adic Igusa zeta function*, which are specializations of the global/local motivic Igusa zeta function, see [6, Section 7].

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