

ON DISCRETE-TIME SEMI-MARKOV PROCESSES

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ABSTRACT. In the last years, several authors studied a class of continuous-time semi-Markov processes obtained by time-changing Markov processes by hitting times of independent subordinators. Such processes are governed by integro-differential convolution equations of generalized fractional type. The aim of this paper is to develop a discrete-time counterpart of such a theory and to show relationships and differences with respect to the continuous time case. We present a class of discrete-time semi-Markov chains which can be constructed as time-changed Markov chains and we obtain the related governing convolution type equations. Such processes converge weakly to those in continuous time under suitable scaling limits.

1. Introduction. It is well-known that the memoryless property of homogeneous Markov processes imposes restrictions on the waiting time spent in a state, which must be either exponentially distributed (in the continuous time case) or geometrically distributed (in the discrete time case). Indeed, the exponential and the geometric distributions are the only to enjoy the lack of memory property. However, in many applied models it is useful to relax the Markov assumption in order to allow arbitrarily distributed waiting times in any state. This leads to semi-Markov processes. The theory of semi-Markov processes was introduced by Lévy [35] and Smith [62] and was developed in many subsequent works, such as [13, 17, 22, 27, 30, 32, 53, 54].

Since their introduction, semi-Markov processes have been mostly studied in the continuous time case, while discrete time processes are rarer in the literature (see e.g. [2] and the references therein). Time is usually assumed to be continuous, even if some physical theories claim that it could be discrete; however, it is true that we observe nature at discrete time instants. Moreover, in many applications the time scale is intrinsically discrete. For instance, in DNA analysis, any approach is based on discrete time because one deals with (discrete) sequences of four bases: A, T, C, G. Also, the techniques of text recognition are based on discrete-time models. In reliability theory, one could be interested in the number of times (e.g. days, hours) that a specific event occurs. Thus, discrete-time semi-Markov processes undoubtedly deserve a more in-depth analysis and the aim of the present paper is to give a contribution in this direction.

2010 *Mathematics Subject Classification.* 60K15, 60J10, 60G50, 60G51.

Key words and phrases. Semi-Markov processes, discrete-time chains, discrete fractional operators, time change, fractional Bernoulli process, sibuya counting process.

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In this paper we study the large class of semi-Markov chains which is generated by time-changing a discrete-time Markov chain by discrete-time renewal counting processes, that is with compositional inverses of increasing random walks with i.i.d. positive integer jumps. A little variant of such processes is proved to be governed by equations of the form

$$\sum_{\tau=0}^{\infty} (p(x, t) - p(x, t - \tau)) \mu(\tau) = \mathcal{G}_x p(x, t), \quad t \in \mathbb{N}, \quad (1)$$

where $p(x, t)$ is the probability that the process is in state x at time t , μ is a probability mass function supported on the positive integers and \mathcal{G}_x is related to the generator of the original Markov chain. In the case in which the waiting times follow a discrete Mittag-Leffler distribution (see Section 4 for its definition), equation (1) reduces to

$$(I - \mathcal{B})^\alpha p(x, t) = \mathcal{G}_x p(x, t) \quad t \in \mathbb{N}, \quad \alpha \in (0, 1), \quad (2)$$

where \mathcal{B} is the backward shift operator in the time-variable, that is $\mathcal{B}p(x, t) = p(x, t - 1)$, while $I - \mathcal{B}$ is the discrete-time derivative and $(I - \mathcal{B})^\alpha$ is its fractional power.

By this construction, a class of discrete-time semi-Markov processes arises, which enjoys interesting statistical properties and is suitable for some types of applications in which time is intrinsically discrete. For instance, in [2], the authors construct nonparametric estimators for the main characteristics of discrete-time semi-Markov systems and study their asymptotic behavior; they also show some applications in reliability theory and biology. Note that the mathematical techniques in [2] are specific to discrete time and do not fit well with the continuous time case.

Recently, the theory of semi-Markov processes in continuous time has regained much interest. The literature concerning this topic is vast: see for example [3, 28, 29, 37, 38, 41, 42, 64]. Consult also [18, 49, 56], where the theory has been extended to models of motions in heterogeneous media. Following the theory, a semi-Markov process of this kind can be obtained by time-changing a Markov process by the inverse hitting time of a subordinator and can be seen as scaling limits of continuous-time random walks (CTRWs). Such processes are known to be governed (see the review in Section 2.2) by integro-differential equations, which, in the most general form, can be written as

$$\int_0^\infty (p(x, t) - p(x, t - \tau)) \nu(d\tau) = G_x p(x, t), \quad t \in \mathbb{R}^+, \quad (3)$$

where G_x is the generator of the original Markov process, ν is the Lévy measure of the underlying subordinator, while the operator on the left-hand side is usually called generalized fractional derivative. The reason of this name is that in the case where the random time is an inverse β -stable subordinator, equation (3) reduces to

$$\frac{\partial^\beta}{\partial t^\beta} p(x, t) = G_x p(x, t), \quad t \in \mathbb{R}^+, \quad (4)$$

where $\frac{d^\beta}{dt^\beta}$ is the Caputo fractional derivative of order $\beta \in (0, 1)$.

Note that, formally, (3) is the continuous version of (1), in the same way as (4) is the continuous version of (2). Here we prove that a similar but non equivalent scheme holds in discrete time. In fact our treatise on the discrete-time case goes actually beyond the classical scheme of the standard CTRW-theory, essentially for three reasons. First, the class of discrete-time processes studied in this paper is

not trivially given by sampling continuous-time semi Markov processes at integer times (see, for example, the discussion in remark 13). Second, this paper includes the case where the original process $X(n)$ is a generic discrete-time Markov chain on a general state space, thus without restricting us to random walks with i.i.d. jumps in \mathbb{R}^d . Third, we show that a connection with (generalized) time-fractional equations also exists in discrete time, and not only after the continuous-time limit.

The structure of the paper is the following. In Section 2, besides a brief literature overview, we introduce the time change of discrete-time Markov chains and we present renewal chains and their inverse counting processes; special attention is devoted to the Bernoulli process (related to identical sequential trials) and to the Sibuya process (related to sequential trials with memory), which are discrete-time approximations of the Poisson process and the inverse stable subordinator, respectively. Section 3 contains our main results on discrete-time semi-Markov chains and generalized fractional difference equations, together with some results on path space convergence in the continuous time limit. Finally, Section 4 is devoted to the so-called fractional Bernoulli counting processes, which are discrete-time approximations of the fractional Poisson process studied in several papers such as [5, 6, 31, 36, 39].

2. Literature overview and preliminary results.

2.1. Time-change of discrete-time processes. For the sake of clarity, we recall two important definitions in probability theory: n -divisibility and infinite divisibility. A random variable X is said to be n -divisible if there exist i.i.d. random variables Y_1, Y_2, \dots, Y_n , such that

$$X \stackrel{d}{=} Y_1 + Y_2 + \dots + Y_n.$$

Instead, a random variable X is said to be infinitely divisible if for each $n \in \mathbb{N}$ there exist i.i.d. random variables $Y_1^n, Y_2^n, \dots, Y_n^n$ such that

$$X \stackrel{d}{=} Y_1^n + Y_2^n + \dots + Y_n^n.$$

Both definitions are actually connected to stochastic processes with stationary and independent increments. We briefly recall some related facts, for a more in-depth discussion consult, for example, [[63], Chapter 1]. On one hand, n -divisibility is related to random walks defined by the partial sums

$$X(n) = \sum_{j=1}^n \Delta_j \quad n \in \mathbb{N}, \quad (5)$$

where the Δ_j are i.i.d. random variables.

On the other hand, the notion of infinite divisibility is intimately related to Lévy processes. The definition of Lévy process makes sense only in continuous time case. Indeed, a process is called Lévy if, besides having independent and stationary increments, it is continuous in probability, i.e. $X(s) \xrightarrow{P} X(t)$ if $s \rightarrow t$ (see, for example, [1] for basic notions).

Lévy processes with non decreasing sample paths are called subordinators (for an overview see [8]). They are often used as models of random time for the construction of time-changed Markov processes (on this point consult, for example, [1], [58], [59]

and [65]). A subordinator $\sigma(t)$ has Laplace transform

$$\mathbb{E}e^{-\eta\sigma(t)} = e^{-tf(\eta)}, \quad \text{with } f(\eta) = b\eta + \int_0^\infty (1 - e^{-\eta x})\nu(dx), \quad \eta > 0,$$

where $b > 0$ and the Lévy measure ν is such that $\int_{(0,\infty)}(x \wedge 1)\nu(dx) < \infty$. We let $b = 0$ and only consider strictly increasing subordinators, i.e. those such that $\int_0^\infty \nu(dx) = \infty$. The inverse hitting time process

$$L(t) = \inf\{x \geq 0 : \sigma(x) > t\} = \sup\{x \geq 0 : \sigma(x) \leq t\}$$

is non decreasing, and thus it can be used as a random time to construct time-changed processes.

In particular, if $\{M(t)\}_{t \in \mathbb{R}^+}$ is a Markov process, the composition $\{M(L(t))\}_{t \in \mathbb{R}^+}$ is a semi-Markov process, which has been deeply studied, having an interesting connection to many different topics, such as anomalous diffusion, continuous time random walk limits and integro-differential and fractional equations (see [40], chapter 8 of [29] and the references therein).

One of the main goals of this paper is to develop some aspects of the theory of time-change for discrete time Markov processes. The discrete-time analogue of a subordinator is a random walk of type (5) with positive integer jumps

$$\sigma_d(n) = \sum_{j=1}^n Z_j, \quad n \in \mathbb{N}, \quad Z_j \in \mathbb{N}, \quad \sigma_d(0) = 0. \quad (6)$$

Its inverse process is given by

$$L_d(t) = \max\{n \in \mathbb{N}_0 : \sigma_d(n) \leq t\}, \quad t \in \mathbb{N}_0, \quad (7)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Indeed $L_d(t) = 0$ for $0 \leq t < Z_1$, $L_d(t) = 1$ for $Z_1 \leq t < Z_1 + Z_2$, and so forth.

So, given a discrete-time Markov chain $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$ and an independent process $\{L_d(t)\}_{t \in \mathbb{N}_0}$ of type (7), we here study the time-changed process $\{\mathcal{X}(L_d(t))\}_{t \in \mathbb{N}_0}$. We refer to [63] for some definitions and techniques on the time change of discrete-time processes.

2.2. Brief review on continuous-time semi-Markov chains. We here summarize some facts known in the literature concerning the continuous-time case. Let us consider a continuous-time Markov chain $\{X(t)\}_{t \in \mathbb{R}^+}$ on the discrete space \mathcal{S}

$$X(t) = X_n, \quad V_n \leq t < V_{n+1}, \quad \text{with } V_0 = 0, \quad V_n = \sum_{k=0}^{n-1} E_k, \quad (8)$$

where $\{X_n\}_{n=0,1,\dots}$ is a discrete-time Markov chain in \mathcal{S} , whose stochastic matrix is defined as

$$H_{ij} = P(X_{n+1} = j | X_n = i)$$

and the waiting times E_k are exponentially distributed:

$$P(E_k > t | X_k = i) = e^{-\lambda_i t} \quad t \geq 0. \quad (9)$$

The transition probabilities

$$p_{ij}(t) = P(X(t) = j | X(0) = i), \quad i, j \in \mathcal{S},$$

are known to solve (see for example [44]) the Kolmogorov backward equations

$$\frac{d}{dt}p_{ij}(t) = \sum_{l \in S} \lambda_i(H_{il} - \delta_{il})p_{lj}(t), \quad p_{ij}(0) = \delta_{ij}, \quad (10)$$

as well as the related forward equations.

We now consider a semi-Markov process $\{Y(t)\}_{t \in \mathbb{R}^+}$ which is constructed in the same way of (8) except for the distribution of the waiting times, which are no longer exponentially distributed:

$$Y(t) = X_n \quad T_n \leq t < T_{n+1} \quad \text{where} \quad T_0 = 0, \quad T_n = \sum_{k=0}^{n-1} J_k, \quad (11)$$

where the waiting times follow an arbitrary continuous distribution

$$P(J_k > t | X_k = i) = \bar{F}_i(t).$$

We are interested in a particular subclass of (11): taking any strictly increasing subordinator $\{\sigma(t)\}_{t \in \mathbb{R}^+}$, independent of $\{X(t)\}_{t \in \mathbb{R}^+}$, we assume that the waiting times J_0, J_1, \dots are such that

$$P(J_k > t | X_k = i) = P(\sigma(E_k) > t | X_k = i), \quad (12)$$

where E_k are distributed as in (9). The main result concerning such a class of semi-Markov processes (which also justifies the choice of the compound exponential law $J_k = \sigma(E_k)$ for the waiting times) is the following: let $L(t)$ be the right continuous inverse process of $\sigma(t)$, the following time-change relation holds:

$$\{Y(t)\}_{t \in \mathbb{R}^+} = \{X(L(t))\}_{t \in \mathbb{R}^+},$$

where X and L are independent. A heuristic proof of this fact is the following: by using the definition (8), we have

$$X(L(t)) = X_n \quad V_n \leq L(t) < V_{n+1} \quad (13)$$

namely

$$X(L(t)) = X_n \quad \sigma(V_n^-) \leq t < \sigma(V_{n+1})$$

and thus the waiting times are such that $J_n = \sigma(V_{n+1}) - \sigma(V_n) \stackrel{d}{=} \sigma(V_{n+1} - V_n) = \sigma(E_n)$ (where we used the fact that σ has independent and stationary increments).

The transitions functions solve the following equation:

$$\mathcal{D}_t p_{ij}(t) - \bar{\nu}(t)p_{ij}(0) = \sum_{l \in S} \lambda_i(H_{il} - \delta_{il})p_{lj}(t), \quad p_{ij}(0) = \delta_{ij}, \quad (14)$$

where λ_i are the rates of the E_n , while ν is the Lévy measure of the subordinator σ , $\bar{\nu}(t) = \int_t^\infty \nu(dy)$ and

$$\mathcal{D}_t p_{ij}(t) = \int_0^\infty (p_{ij}(t) - p_{ij}(t - \tau))\nu(d\tau) = \frac{d}{dt} \int_0^t p_{ij}(t - \tau)\bar{\nu}(\tau)d\tau.$$

Equation (14) is analogous to (10), but the time derivative $\frac{d}{dt}$ on the left side is replaced by the operator \mathcal{D}_t , which is sometimes called generalized fractional derivative. The reason of this name is that in the case where σ is an α -stable subordinator, we have $\nu(dx) = \alpha x^{-\alpha-1} dx / \Gamma(1 - \alpha)$ and (14) reduces to

$$\frac{d^\alpha}{dt^\alpha} p_{ij}(t) - \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} p_{ij}(0) = \sum_{l \in S} \lambda_i(H_{il} - \delta_{il})p_{lj}(t), \quad p_{ij}(0) = \delta_{ij}, \quad (15)$$

where

$$\frac{d^\alpha}{dt^\alpha} p_{ij}(t) = \frac{d}{dt} \int_0^t p_{ij}(t-\tau) \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} d\tau$$

is the Riemann–Liouville fractional derivative. In such a case, the waiting times have distribution

$$P(J_n > t | X_n = i) = \mathcal{E}(-\lambda_i t^\alpha) \quad (16)$$

where $\mathcal{E}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)}$ is the Mittag–Leffler function; indeed, by a simple conditioning argument, it is easy to check that the composition $J_n = \sigma(E_n)$ has Laplace transform

$$\mathbb{E}(e^{-sJ_n} | X_n = i) = \frac{\lambda_i}{\lambda_i + s^\alpha} \quad s \geq 0, \quad (17)$$

which is also the Laplace transform of $-\frac{d}{dt}\mathcal{E}(-\lambda_i t^\alpha)$. For analytical properties of the Mittag–Leffler function and its role in fractional calculus consult [57]; see also [4] and [19] for some applications on relaxation phenomena.

There is an extensive literature concerning the topics recalled in this section: see for example [3, 26, 28, 29, 40, 37, 41, 38, 42]. See also [20] for some applications and [25] for recent analytic results on fractional differential equations. Moreover, semi-Markov models of motion in heterogeneous media are studied in [18, 49, 56], where fractional equations of type (15) of state dependent order $\alpha = \alpha(x)$ arise. Consult also [48] where the authors study Markov processes time-changed by independent inverses of additive subordinators.

Remark 1. An important case is that of processes making jumps of height 1, i.e. $H_{ij} = 1$ if $j = i + 1$ almost surely. If besides $\lambda_i = \lambda$ for all $i \in \mathcal{S}$, they can be constructed as Poisson processes time-changed by inverses of subordinators (this is a class of renewal counting processes including the so-called fractional Poisson process studied e.g. in [5, 6, 31, 36, 39]). Other models of fractional point processes are studied in [45, 46, 47]. Thus, our investigation in the discrete time starts from the discrete analogue of renewal processes, known as renewal chains, which are treated in the next subsection.

2.3. Renewal chains. In the following we will make extensive use of generating functions. We recall that the generating function of a real sequence $\{a_t\}_{t \in \mathbb{N}_0}$ is defined by the power series

$$\mathcal{G}_a(u) = \sum_{t=0}^{\infty} u^t a_t$$

for all $u \in \mathbb{R}$ such that $|u| \leq R$, where $R \geq 0$ is the radius of convergence. Since \mathcal{G}_a can be differentiated term by term at all u inside the radius of convergence, the sequence $\{a_t\}_{t \in \mathbb{N}_0}$ can be uniquely reconstructed from the generating function by setting $a_t = \mathcal{G}_a^{(t)}(0)/t!$, where $\mathcal{G}_a^{(t)}(\cdot)$ denotes the t^{th} derivative of $\mathcal{G}_a(\cdot)$. A useful property is that the convolution of two sequences $\{a_t\}_{t \in \mathbb{N}_0}$ and $\{b_t\}_{t \in \mathbb{N}_0}$, which is defined as $\{a*b\}_{t \in \mathbb{N}_0} = \sum_{k=0}^t a_k b_{t-k}$, has generating function $\mathcal{G}_{a*b}(u) = \mathcal{G}_a(u)\mathcal{G}_b(u)$.

We now recall the notion of renewal chain; for a deeper insight consult [[2], Chapter 2]. Let W be a positive, integer valued random variable. Let W_0, W_1, \dots , be i.i.d. copies of W . Consider the increasing random walk

$$T_n = W_0 + W_1 + \dots + W_{n-1} \quad n \in \mathbb{N} \quad W_j \in \mathbb{N} \quad T_0 = 0. \quad (18)$$

Obviously (18) has the same form of (6), but here, intuitively, the T_n should be seen as the successive instants when a specific event occurs (and we call them renewal times) while the W_n represent the waiting times. The process $\{T_n\}_{n \in \mathbb{N}_0}$ is called renewal chain or discrete-time renewal process.

Its inverse

$$C(t) = \max\{n \in \mathbb{N}_0 : T_n \leq t\}, \quad t \in \mathbb{N}_0, \quad (19)$$

is a counting process representing the number of renewals up to time t . We now recall a useful result.

Proposition 1. *Let $\mathcal{G}_C(m, u) = \sum_{t=0}^{\infty} u^t P(C(t) = m)$, $|u| < 1$, be the generating function of the sequence $\{P(C(t) = m)\}_{t \in \mathbb{N}_0}$. Then*

$$\mathcal{G}_C(m, u) = \frac{1}{1-u} (\mathbb{E}u^W)^m (1 - \mathbb{E}u^W). \quad (20)$$

Proof. Observe that $\{C(t) \geq m\}$ if and only if $\{T_m \leq t\}$. Then $P(C(t) = m) = P(T_m \leq t, T_{m+1} > t)$. Furthermore, since $\{T_{m+1} \leq t\}$ implies $\{T_m \leq t\}$ we have $P(C(t) = m) = P(T_m \leq t) - P(T_{m+1} \leq t)$, and computing the generating function of both members, (20) is obtained. \square

We now focus on two types of renewal chains together with their related counting processes, which are called Bernoulli and Sibuya counting processes, respectively. In the continuous-time limit, they converge to a Poisson process and to an inverse stable subordinator, respectively.

2.3.1. The Bernoulli counting process and the Poisson process. Consider a sequence of Bernoulli trials, i.e. independent trials such that at each time step you can record the occurrence of either 1 event (with probability p) or of 0 events (with probability $q = 1 - p$). Let $N(t)$ be the number of events up to time t , which follows a binomial distribution $P(N(t) = k) = \binom{t}{k} p^k q^{t-k}$, $k = 0, \dots, t$. The waiting times between successive events are independent geometric random variables.

Definition 2.1. A Bernoulli counting process, denoted by $\{N(t)\}_{t \in \mathbb{N}_0}$, is a counting process of the type (19) such that the waiting times M_i , $i = 0, 1, \dots$, have common geometric distribution $P(M_i = k) = pq^{k-1}$, $k \in \mathbb{N}$.

By using (20) and the fact that $\mathbb{E}u^{M_i} = \frac{pu}{1-qu}$, the generating function of $\{P(N(t) = m)\}_{t \in \mathbb{N}_0}$ reads

$$\mathcal{G}_N(m, u) = \frac{(pu)^m}{(1-qu)^{m+1}}. \quad (21)$$

Note that $\{N(t)\}_{t \in \mathbb{N}_0}$ is a Markov process due to the independence of trials and to the lack of memory property of geometric distribution.

Let now $p_k(t) = P(N(t) = k)$. A simple conditioning argument gives

$$p_k(t) = qp_k(t-1) + pp_{k-1}(t-1) \quad t \in \mathbb{N}. \quad (22)$$

Re-writing the first member as $p_k(t) = qp_k(t) + pp_k(t)$ and then dividing by q , one obtains a finite difference equation governing the Bernoulli counting process:

$$(I - \mathcal{B})p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t-1), \quad t \in \mathbb{N}, \quad (23)$$

where $\lambda = p/q$, \mathcal{B} is the shift operator acting on the time variable such that $\mathcal{B}p(t) = p(t-1)$ and $I - \mathcal{B}$ is the discrete derivative acting on the time variable.

It is well-known that the Bernoulli process converges to the Poisson process under a suitable scaling limit. To see this intuitively (rigorous results on convergence will be given later), let the time steps have size $1/n$ (and thus the geometric waiting times scale as $M_i \rightarrow M_i/n$) and let the parameter p of the geometric distribution be substituted by λ/n . In the limit $n \rightarrow \infty$ each M_i converges to an exponential random variable of parameter λ .

Moreover, one could formally observe that, under the above scaling, equation (23) reads

$$p_k(t) - p_k(t - \frac{1}{n}) = -\frac{\lambda}{n}p_k(t) + \frac{\lambda}{n}p_{k-1}(t - \frac{1}{n}), \quad t \in \left\{ \frac{1}{n}, \frac{2}{n}, \dots \right\}. \quad (24)$$

Dividing by $1/n$ and letting $n \rightarrow \infty$, we formally get

$$\frac{d}{dt}p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t), \quad t \in \mathbb{R}^+,$$

which is the forward equation governing the Poisson process.

2.3.2. The Sibuya counting process and the inverse stable subordinator. Consider now a sequence of trials, each having two possible outcomes, such that the probability of success is not constant in time. If a success has just occurred, we assume that the probability of success in the r -th successive trial is equal to α/r , where $\alpha \in (0, 1)$. Unlike the Bernoulli process, which is Markovian, such a process has a memory, since it remembers the time elapsed from the previous success. In place of the geometric distribution of the Bernoulli process, the time Z in which the first success occurs here follows the so-called *Sibuya*(α) distribution (consult [15, 52])

$$\begin{aligned} P(Z = k) &= (1 - \alpha) \left(1 - \frac{\alpha}{2}\right) \dots \left(1 - \frac{\alpha}{k-1}\right) \frac{\alpha}{k} \\ &= (-1)^{k-1} \binom{\alpha}{k} \quad k = 1, 2, \dots \end{aligned} \quad (25)$$

having generating function

$$\mathbb{E}u^Z = 1 - (1 - u)^\alpha. \quad (26)$$

On this point, it is interesting to note that any discrete density $\{p_k\}_{k \in \mathbb{N}}$ can be expressed as

$$p_k = (1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_{k-1})\alpha_k \quad k \in \mathbb{N},$$

for $\alpha_k = p_k / \sum_{r=k}^{\infty} p_r$, and thus an interpretation within the sequential trial scheme makes sense. The Sibuya case is obtained by assuming $\alpha_k = \frac{\alpha}{k}$.

Unlike the geometric distribution, which has an exponential decay, the Sibuya distribution has power-law decay, as (see formula 2.5 in [40])

$$P(Z = k) = (-1)^{k-1} \binom{\alpha}{k} \sim \frac{\alpha}{\Gamma(1 - \alpha)} k^{-1-\alpha}, \quad \text{as } k \rightarrow \infty. \quad (27)$$

Now, we consider a random walk of type (6) defined by the partial sums of i.i.d. Sibuya random variables

$$\sigma_\alpha(n) = \sum_{j=1}^n Z_j \quad n \in \mathbb{N}, \quad \sigma_\alpha(0) = 0. \quad (28)$$

We define the process $\{L_\alpha(t)\}_{t \in \mathbb{N}_0}$ as the inverse of $\{\sigma_\alpha(t)\}_{t \in \mathbb{N}_0}$:

$$L_\alpha(t) = \max\{n \in \mathbb{N}_0 : \sigma_\alpha(n) \leq t\} \quad t \in \mathbb{N}_0. \quad (29)$$

We call $\{L_\alpha(t)\}_{t \in \mathbb{N}_0}$ Sibuya counting process, as it counts the number of successes up to time t in the case of Sibuya trials. By (20) and (26), the sequence $\{P(L_\alpha(t) = m)\}_{t \in \mathbb{N}_0}$ has generating function

$$\mathcal{G}_{L_\alpha}(m, u) = (1 - u)^{\alpha-1} [1 - (1 - u)^\alpha]^m. \quad (30)$$

Note that (30) can be expanded as

$$\begin{aligned} \mathcal{G}_{L_\alpha}(m, u) &= \sum_{t=0}^{\infty} u^t \left(\sum_{r=0}^m \binom{m}{r} (-1)^r (-1)^t \binom{\alpha r + \alpha - 1}{t} \right) \\ &= \sum_{t=0}^{\infty} u^t \left(\sum_{r=0}^m (-1)^r \binom{m}{r} \binom{t - \alpha r - \alpha}{t} \right), \end{aligned}$$

where in the last step we used that $(-1)^w \binom{a-1}{w} = \binom{w-a}{w}$ for $a > 0$. Hence $L_\alpha(t)$ has discrete density

$$P(L_\alpha(t) = m) = \sum_{r=0}^m (-1)^r \binom{m}{r} \binom{t - \alpha r - \alpha}{t}, \quad m = 0, 1, \dots \quad (31)$$

A crucial point is that the Sibuya counting process $\{L_\alpha(t)\}_{t \in \mathbb{N}_0}$ is a discrete-time approximation of the inverse stable subordinator. We remind that a stable subordinator $\{\sigma_*(t)\}_{t \in \mathbb{R}^+}$ is an increasing Lévy process with Laplace transform $\mathbb{E}e^{-\xi \sigma_*(t)} = e^{-t \xi^\alpha}$, $\alpha \in (0, 1)$, while its inverse hitting time process is defined as

$$L_*(t) = \inf\{x : \sigma_*(x) > t\} = \sup\{x : \sigma_*(x) \leq t\}. \quad (32)$$

The following Proposition, regarding convergence of the one dimensional distribution, is an anticipation of a more general result that will be treated in Proposition 5 where we will state that a suitable scaling of $\{\sigma_\alpha(\lfloor t \rfloor)\}_{t \in \mathbb{R}^+}$ and $\{L_\alpha(\lfloor t \rfloor)\}_{t \in \mathbb{R}^+}$ (where $\lfloor t \rfloor$ denotes the biggest integer less than or equal to t) respectively converge to a stable subordinator and its inverse in the Skorokhod J_1 sense and also in the sense of finite dimensional distributions.

Proposition 2. *Let $n \in \mathbb{N}$, $t \in \mathbb{R}^+$. For any t , the random variable $n^{-\alpha} L_\alpha(\lfloor nt \rfloor)$ converges in distribution to $L_*(t)$ as $n \rightarrow \infty$.*

Proof. We remind (see e.g. [16], formula 1.11) that $L_*(t)$ has a density:

$$P(L_*(t) \in dx) = \frac{1}{t^\alpha} W_{-\alpha, 1-\alpha}(-x/t^\alpha) dx,$$

where $W_{\eta, \gamma}(z) = \sum_{r=0}^{\infty} \frac{z^r}{r! \Gamma(\eta r + \gamma)}$ is the Wright function. Then, for every $a < b$, with $a, b \in \mathbb{R}^+$, we have to show that

$$\lim_{n \rightarrow \infty} P\{a \leq n^{-\alpha} L_\alpha(\lfloor nt \rfloor) \leq b\} = \int_a^b \frac{W_{-\alpha, 1-\alpha}(-x/t^\alpha)}{t^\alpha} dx. \quad (33)$$

By (31), we have

$$\begin{aligned} P(L_\alpha(\lfloor nt \rfloor) = s) &= \sum_{r=0}^s \binom{s}{r} (-1)^r \binom{\lfloor nt \rfloor - \alpha(r+1)}{\lfloor nt \rfloor} \\ &= \sum_{r=0}^s \frac{\Gamma(s+1)}{\Gamma(s-r+1)r!} (-1)^r \frac{\Gamma(\lfloor nt \rfloor - \alpha(r+1) + 1)}{\Gamma(1 - \alpha(r+1))\Gamma(\lfloor nt \rfloor + 1)}. \end{aligned}$$

Taking s of the form $s = \lfloor n^\alpha h \rfloor$, with $h \in \mathbb{R}^+$, and using Tricomi formula

$$\frac{\Gamma(z+c)}{\Gamma(z+d)} = z^{c-d}(1 + O(1/z)) \quad c, d \in \mathbb{R}, \quad (34)$$

we write

$$\begin{aligned} P(L_\alpha(\lfloor nt \rfloor) = \lfloor n^\alpha h \rfloor) &\sim \frac{1}{(nt)^\alpha} \sum_{r=0}^{\lfloor n^\alpha h \rfloor} \left(\frac{-h}{t^\alpha}\right)^r \frac{1}{r! \Gamma(1-\alpha(r+1))} \left(1 + O\left(\frac{1}{n^\alpha}\right)\right) \\ &\sim \frac{1}{(nt)^\alpha} \sum_{r=0}^{\infty} \left(\frac{-h}{t^\alpha}\right)^r \frac{1}{r! \Gamma(1-\alpha(r+1))} \\ &\sim \frac{1}{(nt)^\alpha} W_{-\alpha, 1-\alpha}(-s/(nt)^\alpha). \end{aligned} \quad (35)$$

We use a regular partition of an interval $(a, b]$. Let $x_i = (i + \lfloor an^\alpha \rfloor)/n^\alpha$, $0 \leq i \leq \ell$, and $\ell = \max\{i : x_i \leq b\}$. Note that $x_{i+1} - x_i = 1/n^\alpha$, and since $L_\alpha(\lfloor nt \rfloor)$ is a discrete random variable,

$$P\{an^\alpha \leq L_\alpha(\lfloor nt \rfloor) \leq bn^\alpha\} = \sum_{s=\lfloor an^\alpha \rfloor}^{\lfloor bn^\alpha \rfloor} P[L_\alpha(\lfloor nt \rfloor) = s].$$

Substituting, in (35), each term with its asymptotic value, we obtain

$$\begin{aligned} P\{an^\alpha \leq L_\alpha(\lfloor nt \rfloor) \leq bn^\alpha\} &\sim \frac{1}{t^\alpha} \sum_{s=\lfloor an^\alpha \rfloor}^{\lfloor bn^\alpha \rfloor} \frac{1}{n^\alpha} W_{-\alpha, 1-\alpha}\left(\frac{-s}{(nt)^\alpha}\right) \\ &\sim \frac{1}{t^\alpha} \sum_{s=\lfloor an^\alpha \rfloor}^{\lfloor bn^\alpha \rfloor} \frac{1}{n^\alpha} W_{-\alpha, 1-\alpha}\left(\frac{-s}{(nt)^\alpha}\right). \end{aligned} \quad (36)$$

The correspondence between i and x_i is one to one and the interval $[x_0, x_{\ell+1}]$ contains the given $[a, b]$. In addition observe that $x_0 \leq a < x_1 < x_2 < \dots < x_\ell \leq b < x_{\ell+1}$, then (36) may be written as follows:

$$P\{an^\alpha \leq L_\alpha(\lfloor nt \rfloor) \leq bn^\alpha\} \sim \frac{1}{t^\alpha} \sum_{i=0}^{\ell-1} (x_{i+1} - x_i) W_{-\alpha, 1-\alpha}\left(\frac{-x_i}{t^\alpha}\right), \quad (37)$$

which is the Riemann sum converging to the integral on the right-hand side of (33). \square

We now compute the auto-correlation function of the process $\{L_\alpha(t)\}_{t \in \mathbb{N}_0}$.

Proposition 3. *The Sibuya counting process $\{L_\alpha(t)\}_{t \in \mathbb{N}_0}$ is such that*

$$\mathbb{E}[L_\alpha(t)] = \binom{t+\alpha}{t} - 1, \quad (38)$$

$$\mathbb{E}[L_\alpha(t)^2] = 2 \binom{t+2\alpha}{t} - 3 \binom{t+\alpha}{t} + 1, \quad (39)$$

$$\mathbb{E}[L_\alpha(t_1)L_\alpha(t_2)] = \sum_{\ell=1}^{\min(t_1, t_2)} \binom{\ell+\alpha-1}{\ell} \left[\binom{t_1-\ell+\alpha}{t_1-\ell} + \binom{t_2-\ell+\alpha}{t_2-\ell} - 1 \right]. \quad (40)$$

Moreover, as $t_2 \rightarrow \infty$,

$$\text{Corr}[L_\alpha(t_1), L_\alpha(t_2)] = \frac{\text{Cov}[L_\alpha(t_1), L_\alpha(t_2)]}{\sqrt{\mathbb{V}[L_\alpha(t_1)]\mathbb{V}[L_\alpha(t_2)]}} \sim C t_2^{-\alpha}, \quad (41)$$

where $C = C(t_1, \alpha)$.

Proof. In the following, we will use that

$$(-1)^t \binom{-a}{t} = \binom{a+t-1}{t}, \quad a > 0, \quad (42)$$

and

$$\sum_{\tau=0}^t (-1)^\tau \binom{-a}{\tau} = (-1)^t \binom{-a-1}{t} = \binom{a+t}{t}. \quad (43)$$

Note that (42) can be easily proved by expanding the binomial coefficients, while (43) can be checked by computing the generating function of both members. We now need to derive the mean time spent by the Sibuya random walk (28) at the location t :

$$\sum_{x=0}^{\infty} P(\sigma_\alpha(x) = t) = (-1)^t \binom{-\alpha}{t} = \binom{\alpha+t-1}{t}. \quad (44)$$

Formula (44) can be easily proved by computing (using (26)) the generating function of the left-hand side

$$\sum_{x=0}^{\infty} \sum_{t=0}^{\infty} u^t P(\sigma_\alpha(x) = t) = \sum_{x=0}^{\infty} \mathbb{E}u^{\sigma_\alpha(x)} = \mathbb{E}u^{\sigma_\alpha(0)} + \sum_{x=1}^{\infty} (\mathbb{E}u^Z)^x = (1-u)^{-\alpha},$$

which is also the generating function of $(-1)^t \binom{-\alpha}{t}$. By using (42), (43) and (44) we obtain

$$\begin{aligned} \mathbb{E}L_\alpha(t) &= \sum_{x=1}^{\infty} P(L_\alpha(t) \geq x) = \sum_{x=1}^{\infty} P(\sigma_\alpha(x) \leq t) = \sum_{\tau=0}^t \sum_{x=1}^{\infty} P(\sigma_\alpha(x) = \tau) \\ &= \sum_{\tau=0}^t \left((-1)^\tau \binom{-\alpha}{\tau} - \delta_0(\tau) \right) = \binom{t+\alpha}{t} - 1, \end{aligned}$$

and, by means of (34), we have

$$\mathbb{E}L_\alpha(t) \sim C_3 t^\alpha \quad \text{as } t \rightarrow \infty, \quad (45)$$

where $C_3 = C_3(\alpha)$. For $t_1 \leq t_2$ we have

$$\begin{aligned} \mathbb{E}[L_\alpha(t_1)L_\alpha(t_2)] &= \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} P(L_\alpha(t_1) \geq x, L_\alpha(t_2) \geq y) \\ &= \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} P(\sigma_\alpha(x) \leq t_1, \sigma_\alpha(y) \leq t_2) \\ &= \sum_{\tau_1=0}^{t_1} \sum_{\tau_2=0}^{t_2} \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} P(\sigma_\alpha(x) = \tau_1, \sigma_\alpha(y) = \tau_2) \\ &= \sum_{\tau_1=0}^{t_1} \sum_{\tau_2=\tau_1}^{t_2} \sum_{x=1}^{\infty} \sum_{y=x}^{\infty} P(\sigma_\alpha(x) = \tau_1) P(\sigma_\alpha(y-x) = \tau_2 - \tau_1) \end{aligned}$$

$$+ \sum_{\tau_2=0}^{t_1-1} \sum_{\tau_1=\tau_2+1}^{t_1} \sum_{y=1}^{\infty} \sum_{x=y+1}^{\infty} P(\sigma_\alpha(y) = \tau_2) P(\sigma_\alpha(x-y) = \tau_1 - \tau_2)$$

where in the last step we used that σ_α has independent and stationary increments. By using (42), (43), (44), we have

$$\begin{aligned} & \mathbb{E}[L_\alpha(t_1)L_\alpha(t_2)] \\ &= \sum_{\tau_1=0}^{t_1} \sum_{\tau_2=\tau_1}^{t_2} \binom{\alpha + \tau_2 - \tau_1 - 1}{\tau_2 - \tau_1} \left(\binom{\alpha + \tau_1 - 1}{\tau_1} - \delta_0(\tau_1) \right) \\ & \quad + \sum_{\tau_2=0}^{t_1-1} \sum_{\tau_1=\tau_2+1}^{t_1} \binom{\alpha + \tau_1 - \tau_2 - 1}{\tau_1 - \tau_2} \left(\binom{\alpha + \tau_2 - 1}{\tau_2} - \delta_0(\tau_2) \right) \\ &= \sum_{\tau_1=1}^{t_1} \binom{\alpha + t_2 - \tau_1}{t_2 - \tau_1} \binom{\alpha + \tau_1 - 1}{\tau_1} + \sum_{\tau_2=1}^{t_1-1} \left[\binom{\alpha + t_1 - \tau_2}{t_1 - \tau_2} - 1 \right] \binom{\alpha + \tau_2 - 1}{\tau_2} \\ &= \sum_{\ell=1}^{t_1} \binom{\ell + \alpha - 1}{\ell} \left[\binom{t_1 - \ell + \alpha}{t_1 - \ell} + \binom{t_2 - \ell + \alpha}{t_2 - \ell} - 1 \right] \end{aligned}$$

and this proves (40). In the end, (39) can be obtained by putting $t_1 = t_2$ and doing straightforward calculations (to this scope it is useful to recall formula 1.53 in [57], which states that $\sum_{j=0}^k \binom{\alpha}{j} \binom{\beta}{k-j} = \binom{\alpha+\beta}{k}$). By assuming $t_1 \leq t_2$, we now investigate the asymptotic behaviour of the correlation function for $t_2 \rightarrow \infty$ and t_1 fixed. First, by using (42) and (43), we have that

$$\begin{aligned} & \text{cov}(L_\alpha(t_1), L_\alpha(t_2)) \\ &= \sum_{\ell=1}^{t_1} \binom{\ell + \alpha - 1}{\ell} \left[\binom{t_1 - \ell + \alpha}{t_1 - \ell} + \binom{t_2 - \ell + \alpha}{t_2 - \ell} - 1 \right] \\ & \quad - \left(\binom{t_1 + \alpha}{t_1} - 1 \right) \left(\binom{t_2 + \alpha}{t_2} - 1 \right) \\ &= \sum_{\ell=1}^{t_1} (-1)^\ell \binom{-\alpha}{\ell} \left[\binom{t_1 - \ell + \alpha}{t_1 - \ell} + \binom{t_2 - \ell + \alpha}{t_2 - \ell} - 1 \right] \\ & \quad - \sum_{\ell=1}^{t_1} (-1)^\ell \binom{-\alpha}{\ell} \left(\binom{t_2 + \alpha}{t_2} - 1 \right) \\ &= \sum_{\ell=1}^{t_1} (-1)^\ell \binom{-\alpha}{\ell} \left[\binom{t_1 - \ell + \alpha}{t_1 - \ell} + \binom{t_2 - \ell + \alpha}{t_2 - \ell} - \binom{t_2 + \alpha}{t_2} \right] \\ &= \sum_{\ell=1}^{t_1} (-1)^\ell \binom{-\alpha}{\ell} \left[\binom{t_1 - \ell + \alpha}{t_1 - \ell} + \frac{\Gamma(t_2 - \ell + \alpha + 1)}{\Gamma(t_2 - \ell + 1)\Gamma(\alpha + 1)} - \frac{\Gamma(t_2 + \alpha + 1)}{\Gamma(t_2 + 1)\Gamma(\alpha + 1)} \right]. \end{aligned}$$

By (34) we have

$$\text{cov}(L_\alpha(t_1), L_\alpha(t_2)) \sim C_1 \quad \text{as } t_2 \rightarrow \infty, \quad \text{where } C_1 = C_1(t_1, \alpha). \quad (46)$$

We further have

$$\text{Var}(L_\alpha(t_2)) = 2 \binom{t_2 + 2\alpha}{t_2} - \binom{t_2 + \alpha}{t_2}^2 - \binom{t_2 + \alpha}{t_2} \sim C_2(\alpha) t_2^{2\alpha} \quad \text{as } t_2 \rightarrow \infty, \quad (47)$$

where in the last step we expanded the binomial coefficients and used again (34). Putting all together, we have

$$\text{corr}(L_\alpha(t_1), L_\alpha(t_2)) = \frac{\text{cov}(L_\alpha(t_1), L_\alpha(t_2))}{\sqrt{\text{Var}(L_\alpha(t_1))}\sqrt{\text{Var}(L_\alpha(t_2))}} \sim C t_2^{-\alpha} \quad t_2 \rightarrow \infty.$$

where $C = C(t_1, \alpha)$. \square

3. Discrete-time semi-Markov chains. Consider a discrete-time homogeneous Markov chain $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$ on the discrete space \mathcal{S} with transition matrix

$$A_{ij} = P(\mathcal{X}(t+1) = j | \mathcal{X}(t) = i), \quad i, j \in \mathcal{S}, \quad \forall t \in \mathbb{N}_0.$$

The set of functions

$$P_{ij}(t) = P(\mathcal{X}(t) = j | \mathcal{X}(0) = i)$$

satisfy the backward equation

$$P_{ij}(t) = \sum_{k \in \mathcal{S}} A_{ik} P_{kj}(t-1) \quad (48)$$

by virtue of the Chapman-Kolmogorov equality. If the process is in the state $i \in \mathcal{S}$ at time t , then at time $t+1$ it remains in the same state with probability $q_i = A_{ii}$, while it makes a jump to a different state with probability $p_i = 1 - A_{ii}$. Thus, if $A_{ii} \in (0, 1)$, the waiting time in the state i has geometric distribution of parameter p_i , if $A_{ii} = 0$ then the waiting time is equal to 1 almost surely (which can be considered a degenerate geometric law), while $A_{ii} = 1$ implies that i is an absorbing state and the waiting time is infinity. Under the assumption $A_{ii} < 1$ for each $i \in \mathcal{S}$, equation (48) can be re-written as

$$P_{ij}(t) = q_i P_{ij}(t-1) + p_i \sum_{l \in \mathcal{S}} H_{il} P_{lj}(t-1) \quad (49)$$

where H_{ij} is the probability of a jump from i to j conditioned to the fact that $i \neq j$. The matrix

$$H_{ij} = \begin{cases} 0 & j = i \\ \frac{A_{ij}}{1 - A_{ii}} & j \neq i \end{cases}$$

defines a new Markov chain $\{X_n\}_{n \in \mathbb{N}_0}$ in \mathcal{S} :

$$H_{ij} = P(X_{n+1} = j | X_n = i). \quad (50)$$

By construction, all the waiting times of $\{X_n\}_{n \in \mathbb{N}_0}$ are equal to 1 almost surely. Now, the original Markov chain $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$ can also be re-written in the alternative form of a discrete-time jump process having geometric (possibly degenerate) waiting times M_k

$$\mathcal{X}(t) = X_n \quad V_n \leq t < V_{n+1} \quad \text{with } V_0 = 0, \quad V_n = \sum_{k=0}^{n-1} M_k \quad (51)$$

with

$$P(M_k = r | X_k = i) = q_i^{r-1} p_i \quad p_i \in (0, 1] \quad r = 1, 2, \dots \quad k = 0, 1, 2, \dots$$

In some sense, Markovianity of $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$ is a consequence of both markovianity of $\{X_n\}_{n \in \mathbb{N}_0}$ and the lack of memory of the geometric distribution.

From now on, we will consider discrete-time homogeneous semi-Markov chains, i.e. processes constructed in the same way of (51) except for the law of the waiting times, which are now no-longer geometrically distributed:

$$\mathcal{Y}(t) = X_n \quad T_n \leq t < T_{n+1} \quad \text{where} \quad T_0 = 0, \quad T_n = \sum_{k=0}^{n-1} J_k \quad (52)$$

where H_{ij} is the same matrix defined in (50), while the waiting times are such that

$$P(J_k = r | X_k = i) = f(r, i)$$

for an arbitrary discrete density $f(r, i)$ on $r = 1, 2, \dots$. Note that the conditional distribution of the waiting time J_k is time-homogeneous, i.e. depends only on the position of the process and not on the number k .

We are actually interested in two particular subclasses of (52). Such subclasses consist of semi-Markov chains that are constructed as particular modifications of the Markov chain (51), in the sense that the new waiting times J_k are given by particular functions of the geometrically distributed waiting times M_k . In the following definition we propose these two models.

Definition 3.1. Let $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$ be a Markov chain of type (51), having waiting times M_k with geometric law $P(M_k = r | X_k = i) = p_i q_i^{r-1}$, $r = 1, 2, \dots$, $p_i \in (0, 1]$. Let $\{\sigma_d(t)\}_{t \in \mathbb{N}_0}$ be a random walk of type (6), independent of $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$, whose jumps Z_1, Z_2, \dots are i.i.d. copies of a positive integer-valued random variable Z .

i) We say that (52) is a semi-Markov chain of type A if the waiting times J_k have the compound geometric form

$$J_k = \sigma_d(M_k) = \sum_{i=1}^{M_k} Z_i \quad (53)$$

and thus having generating function

$$\mathbb{E}(u^{J_k} | X_k = i) = \frac{p_i \mathbb{E}u^Z}{1 - q_i \mathbb{E}u^Z}. \quad (54)$$

ii) We say that (52) is a semi-Markov chain of type B if the waiting times J_k have the compound shifted geometric form

$$J_k = 1 + \sigma_d(M_k - 1) = 1 + \sum_{i=1}^{M_k-1} Z_i \quad (55)$$

and thus having generating function

$$\mathbb{E}(u^{J_k} | X_k = i) = \frac{p_i u}{1 - q_i \mathbb{E}u^Z}. \quad (56)$$

Observe that if $p_i \in (0, 1)$, then the J_k follow a standard compound geometric law, while in the degenerate case $p_i = 1$ we have $M_k = 1$ and $J_k = Z_k$ almost surely. Note that, for both semi-Markov chains of type A and B, the waiting times are delayed with respect to those of the original Markov ones, i.e. $J_k \geq M_k$ almost surely, being the Z_j strictly positive.

In the special case in which $Z \sim Sibuya(\alpha)$, the random variables J_k follow the discrete Mittag-Leffler distributions of type A and B, which will be respectively defined in (75) and (78).

Remark 2. The reason why we are interested in waiting times having compound geometric type distributions is due to the analogy with the related continuous time processes treated in the literature. Indeed, as said in Section 2.2, we focused on a subclass of continuous time semi-Markov processes with waiting times following a compound exponential distribution $\sigma(E_k)$, where E_k is the exponential waiting time of the original Markov process and σ is an independent subordinator (for example, in the case of fractional processes, σ is a stable subordinator and $\sigma(E_k)$ follows the Mittag-Leffler distribution). In discrete time, the subordinator σ is replaced by the increasing random walk σ_d (see (6)) and the exponential distribution is replaced by the (possibly degenerate) geometric one. With such a choice of compound geometric waiting times, we will prove that these discrete-time semi-Markov chains retain important features of the related continuous time processes (and converge to them under suitable scaling limits). In particular, our type A chains exhibits the property of time-change construction, while type B chains are governed by convolution equations of generalized fractional type.

3.1. Semi-Markov chains of type A: The time change construction.

Theorem 3.2. *Let $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$ be a Markov chain of type (51) having no absorbing states and let $\{\sigma_d(t)\}_{t \in \mathbb{N}_0}$ be an independent random walk of type (6), whose inverse is $\{L_d(t)\}_{t \in \mathbb{N}_0}$ defined in (7). Then the time changed process $\{\mathcal{Y}(t)\}_{t \in \mathbb{N}_0} = \{\mathcal{X}(L_d(t))\}_{t \in \mathbb{N}_0}$ is a semi-Markov chain of type A (according to Definition 3.1).*

Proof. Since the jumps of L_d are at most of size 1, both the jumps of \mathcal{X} and \mathcal{Y} are described by the chain X_n . By using definition (51) we have

$$\mathcal{X}(L_d(t)) = X_n \quad V_n \leq L_d(t) < V_{n+1} \quad \text{with } V_0 = 0, \quad V_n = \sum_{k=0}^{n-1} M_k \quad (57)$$

where each M_k is finite since there are no absorbing states. Definition (7) implies that

$$\mathcal{X}(L_d(t)) = X_n \quad \sigma_d(V_n) \leq t < \sigma_d(V_{n+1}).$$

Thus the waiting times are given by

$$J_n = \sigma_d(V_{n+1}) - \sigma_d(V_n) \stackrel{d}{=} \sigma_d(V_{n+1} - V_n) = \sigma_d(M_n) = \sum_{i=1}^{M_n} Z_i$$

as σ_d has independent and stationary increments. \square

3.1.1. Time-changing Markov chains with i.i.d. jumps. We now gain more insights on a particular subclass of semi-Markov chains of type A. We consider Markov chains of type (51) with values in a discrete state space $\mathcal{S} \subseteq \mathbb{R}$, having independent and stationary increments, i.e. the transition matrix elements A_{ij} only depend on the jump $j - i$. Such processes can be equivalently written as the random walk

$$X(t) = \sum_{j=1}^t X_j \quad t \in \mathbb{N} \quad X(0) = 0, \quad (58)$$

where X_1, X_2, \dots are i.i.d. random variables. Let σ_d be an increasing random walk of type (6), independent of (58), with inverse L_d defined in (7). We are interested

in the time-changed process

$$Y(t) = X(L_d(t)) = \sum_{j=1}^{L_d(t)} X_j \quad t \in \mathbb{N}, \quad Y(0) = 0. \quad (59)$$

By Theorem 3.2, we have that (59) is a semi-Markov chain of type A.

The time-change (59) introduces a memory tail effect, which is evident by investigating the behavior of the auto-correlation function

$$\rho(s, t) = \frac{\text{cov}(Y(t); Y(s))}{\sqrt{\text{Var}Y(t)}\sqrt{\text{Var}Y(s)}} \quad s \leq t \quad s \in \mathbb{N} \quad t \in \mathbb{N}.$$

A remarkable example of this fact is analyzed in the following proposition, where we prove that in the case where L_d is the Sibuya counting process (defined in (29)), then, for fixed s and large t , the autocorrelation function of $Y_\alpha(t) = X(L_\alpha(t))$ exhibits a different decay with respect to the $t^{-\frac{1}{2}}$ decay characterizing the original Markov chain X . This seems to be useful in many applications, such as the problem of modeling memory effects in evolving graphs (see [50], [51]). The following Proposition is the discrete-time counterpart of the analogous result holding in continuous time, when considering Lévy processes time-changed by inverse α -stable subordinators (see [33] for the computation of the auto-correlation function). This is consistent with the fact that the Sibuya counting process is just a discrete time approximation of the inverse stable subordinator.

Proposition 4. *Let $\{X(t)\}_{t \in \mathbb{N}_0}$ be a process of type (58), such that X_1 has finite mean and variance, and let $\{L_d(t)\}_{t \in \mathbb{N}_0}$ be a counting process of type (7), independent of $\{X(t)\}_{t \in \mathbb{N}_0}$. Let $\{Y(t)\}_{t \in \mathbb{N}_0}$ be the process defined in (59). Then*

a) $\{Y(t)\}_{t \in \mathbb{N}_0}$ has auto-correlation function

$$\begin{aligned} \rho(s, t) &= \frac{\text{cov}(L_d(t); L_d(s))(\mathbb{E}X_1)^2 + \mathbb{E}L_d(s)\text{Var}X_1}{\sqrt{\text{Var}[L_d(t)](\mathbb{E}X_1)^2 + \mathbb{E}L_d(t)\text{Var}X_1}\sqrt{\text{Var}[L_d(s)](\mathbb{E}X_1)^2 + \mathbb{E}L_d(s)\text{Var}X_1}}, \end{aligned}$$

with $s \leq t$.

b) Consider the Sibuya counting process $\{L_\alpha(t)\}_{t \in \mathbb{N}_0}$. Then, for fixed s and large t , $\{Y_\alpha(t)\}_{t \in \mathbb{N}_0} = \{X(L_\alpha(t))\}_{t \in \mathbb{N}_0}$ has auto-correlation function

$$\rho_\alpha(s, t) \sim \frac{k_1}{t^\alpha} \quad \text{if} \quad \mathbb{E}X_1 \neq 0$$

and

$$\rho_\alpha(s, t) \sim \frac{k_2}{t^{\alpha/2}} \quad \text{if} \quad \mathbb{E}X_1 = 0,$$

where $k_1 = k_1(s)$ and $k_2 = k_2(s)$.

Proof. a) First observe that

$$\begin{aligned} \mathbb{E}[X(t)X(s)] &= \mathbb{E}[(X(t) - X(s))X(s)] + \mathbb{E}[X(s)^2] \\ &= \mathbb{E}[X(t) - X(s)]\mathbb{E}[X(s)] + \text{Var}[X(s)] + (\mathbb{E}X(s))^2 \\ &= (t - s)s(\mathbb{E}X_1)^2 + s\text{Var}X_1 + s^2(\mathbb{E}X_1)^2 \\ &= ts(\mathbb{E}X_1)^2 + s\text{Var}X_1 \end{aligned}$$

where we have used independence and stationarity of the increments and the fact that $\mathbb{E}X(t) = t\mathbb{E}X_1$ and $\text{Var}X(t) = t\text{Var}X_1$. Then, a standard conditioning argument yields

$$\begin{aligned} & \mathbb{E}[X(L_d(t))X(L_d(s))] \\ &= \sum_{w=0}^{\infty} \sum_{v=0}^{\infty} \mathbb{E}[X(w)X(v)]P(L_d(t) = w, L_d(s) = v) \\ &= \sum_{w=0}^{\infty} \sum_{v=0}^{\infty} (wv(\mathbb{E}X_1)^2 + v\text{Var}X_1)P(L_d(t) = w, L_d(s) = v) \\ &= \mathbb{E}(L_d(t)L_d(s))(\mathbb{E}X_1)^2 + \mathbb{E}L_d(s)\text{Var}X_1. \end{aligned}$$

Taking into account that

$$\mathbb{E}X(L_d(t)) = \mathbb{E}X_1\mathbb{E}L_d(t)$$

by Wald formula, we have

$$\text{cov}(X(L_d(t)); X(L_d(s))) = \text{cov}(L_d(t); L_d(s))(\mathbb{E}X_1)^2 + \mathbb{E}L_d(s)\text{Var}X_1$$

whence, in the special case $s = t$ we have

$$\text{Var}[X(L_d(t))] = \text{Var}[L_d(t)](\mathbb{E}X_1)^2 + \mathbb{E}L_d(t)\text{Var}X_1.$$

Then the auto-correlation function reads

$$\begin{aligned} \rho(s, t) &= \frac{\text{cov}(X(L_d(t)); X(L_d(s)))}{\sqrt{\text{Var}[X(L_d(t))]} \sqrt{\text{Var}[X(L_d(s))]}} \\ &= \frac{\text{cov}(L_d(t); L_d(s))(\mathbb{E}X_1)^2 + \mathbb{E}L_d(s)\text{Var}X_1}{h(t)h(s)}, \end{aligned}$$

where $h(t) = \sqrt{\text{Var}[L_d(t)](\mathbb{E}X_1)^2 + \mathbb{E}L_d(t)\text{Var}X_1}$.

- b) Since (45), (47) and (46) state that for fixed s and large t the following relations hold:

$$\text{cov}(L_\alpha(t); L_\alpha(s)) \sim C_1 \quad \text{Var}[L_\alpha(t)] \sim C_2 t^{2\alpha} \quad \mathbb{E}L_\alpha(t) \sim C_3 t^\alpha,$$

then by straightforward calculations we obtain the result. \square

3.1.2. Continuous-time limits of discrete-time random walks. As often mentioned in previous sections, many of the processes studied in this paper are discrete approximation of continuous-time processes: the increasing random walks of type (6) and their inverses of type (7) respectively converge to subordinators and their inverses, discrete-time Markov chains converge to continuous-time Markov processes, and so forth. In this section we prove rigorous results on continuous-time limits. For the notion of Skorokhod J_1 and M_1 topology, and an exhaustive treaty on path space convergence, consult, for example, [9], [60] and [67]. In the following, we denote by $D[0, \infty)$ the space of càdlàg functions $x : [0, \infty) \rightarrow \mathbb{R}$.

As a first result (that we have already anticipated in section 2.3.2), we state that the Sibuya random walk (28) and its inverse (29) are discrete-time approximations of the stable subordinator and its inverse respectively. The following proposition indeed gives convergence of finite dimensional distributions and also convergence in J_1 sense. The proof is not reported since it follows as a special case of the theory given in [40], [37] and all the references therein. The key point of the proof is that is that the Sibuya distribution lies in the domain of attraction of a stable law. For convenience of the reader, we recall that a random variable Z lies in the domain of

attraction of a stable law if, given Z_1, Z_2, \dots, Z_n independent copies of Z , it holds that

$$c_n(Z_1 + Z_2 + \dots + Z_n) \xrightarrow{d} S, \quad (60)$$

where S is stable, for some $c_n \rightarrow 0$.

Proposition 5. *Under the following scaling limit, the Sibuya random walk (28) and its inverse (29) respectively converge to a α -stable subordinator and to its inverse in the sense of finite dimensional distributions:*

$$\begin{aligned} \{n^{-\frac{1}{\alpha}}\sigma_\alpha(\lfloor nt \rfloor)\}_{t \in \mathbb{R}^+} &\xrightarrow{fdd} \{\sigma_*(t)\}_{t \in \mathbb{R}^+}, & n \rightarrow \infty, \\ \{n^{-1}L_\alpha(\lfloor n^{\frac{1}{\alpha}} t \rfloor)\}_{t \in \mathbb{R}^+} &\xrightarrow{fdd} \{L_*(t)\}_{t \in \mathbb{R}^+}, & n \rightarrow \infty, \end{aligned} \quad (61)$$

where $\lfloor a \rfloor$ denotes the largest integer less than a (or equal to a). The convergence also holds in weak sense under the J_1 topology on $D[0, \infty)$.

We now study the continuous-time limit of the same subclass of semi-Markov chains of type A which has been considered in section 3.1.1, namely those processes obtained by time changing Markov chains having i.i.d. jumps.

We firstly construct a rescaled version of such processes. Consider a sequence of discrete-time Markov chains with i.i.d. jumps, indexed by the parameter n :

$$X^{(n)}(t) = \sum_{j=1}^t X_j^{(n)} \quad t \in \mathbb{N} \quad X^{(n)}(0) = 0. \quad (62)$$

Furthermore we consider a sequence of rescaled random walks with positive jumps of type (6), indexed by n :

$$\sigma_d^{(n)}(t) = \sum_{j=1}^t Z_j^{(n)}, \quad t \in \mathbb{N}, \quad (63)$$

such that $Z_j^{(n)}$ may now have, in general, real values. Its inverse counting process

$$L_d^{(n)}(t) = \max\{k \in \mathbb{N}_0 : \sigma_d^{(n)}(k) \leq t\} \quad t \in \mathbb{R}^+ \quad (64)$$

allows us to define the time-changed process

$$Y^{(n)}(t) = X^{(n)}(L_d^{(n)}(t)), \quad t \in \mathbb{R}^+.$$

We also consider

$$X^{(n)}(\lfloor t \rfloor) = \sum_{j=1}^{\lfloor t \rfloor} X_j^{(n)}, \quad \sigma_d^{(n)}(\lfloor t \rfloor) = \sum_{j=1}^{\lfloor t \rfloor} Z_j^{(n)}, \quad t \in \mathbb{R}^+.$$

In the following we denote by $\mathcal{D}(W)$ the set of points of discontinuity of the process W , namely $\mathcal{D}(W) = \{t > 0 : W(t^-) \neq W(t)\}$.

Theorem 3.3. *If, for $n \rightarrow \infty$, the following three conditions hold:*

- i) $X^{(n)}(\lfloor nt \rfloor)$ converges to a Lévy Process $A(t)$ in J_1 sense,*
- ii) $\sigma_d^{(n)}(\lfloor nt \rfloor)$ converges to a subordinator $\sigma(t)$ in J_1 sense,*
- iii) the limit processes $A(t)$ and $\sigma(t)$ are such that $\mathcal{D}(A) \cap \mathcal{D}(\sigma) = \emptyset$ almost surely,*

then $Y^{(n)}(t) = X^{(n)}(L_d^{(n)}(t))$ converges in M_1 sense to the time changed process $A(L(t))$, where L is the inverse of σ .

Proof. We follow the same steps as in [[3], Theorem 3.1] and [[38], Theorem 2.1]. We also make use of the continuous mapping theorem (see [[67], Theorem 3.4.3] and [66]). By ii), we have that $n^{-1}L_d^{(n)}(t)$ converges to $L(t)$ in M_1 topology by a continuous mapping argument (a random function is mapped into its inverse). Then, by another continuous mapping argument, where the couple $(X^{(n)}(\lfloor nt \rfloor), n^{-1}L_d^{(n)}(t))$ is mapped into the composition, the proof is completed. \square

Remark 3. Theorem 3.3 applies, for instance, to random walks whose jumps lie in the domain of attraction of a stable law (indeed, following the same steps as in proof of Prop. 5, we actually get the hypotheses i) and ii) on J_1 convergence). But, to be able to understand the importance of Theorem 3.3, it is natural to wonder which other random walks converge in J_1 sense, that is, if there exists some simple criterion to characterize random walks converging in J_1 sense under a suitable scaling limit. One answer is given by Skorokhod in Theorem 2.7 of [61]: a sequence of processes $\xi_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \xi_k^n$, such that the ξ_k^n are i.i.d. for each n , converges weakly in J_1 topology to the process $\xi(t)$ if for each t the random variable $\xi_n(t)$ converges in distribution to $\xi(t)$. So, if the addends $X_j^{(n)}$ in (62) are such that $X^{(n)}(\lfloor nt \rfloor) \xrightarrow{d} A(t)$ for any $t \in \mathbb{R}^+$, then $X^{(n)}(\lfloor nt \rfloor)$ converges weakly to the Lévy process $A(t)$ under the J_1 topology. For suitable sequences $P(X_j^{(n)} = 0)$, the limit Lévy process $A(t)$ will have finite activity, that is it will be a continuous-time Markov chain with i.i.d. jumps. In the same way, if the positive addends $Z_j^{(n)}$ in (63) are such that $\sigma_d^{(n)}(\lfloor nt \rfloor) \xrightarrow{d} \sigma(t)$ for any $t \in \mathbb{R}^+$, then $\sigma_d^{(n)}(\lfloor nt \rfloor)$ converges weakly to a subordinator $\sigma(t)$ in J_1 sense.

Remark 4. Conversely, one could wonder if, given a continuous-time semi-Markov process $A(L(t))$, there exists a discrete-time semi-Markov chain converging to it under a suitable scaling limit. The answer is positive. The first step is to observe that, given the limit processes $A(t)$ and $\sigma(t)$, there exist approximating discrete-time random walks converging to them. This is due to a well-known result on triangular array convergence (see, for example, [[24], page 442] and also [21] for a complete discussion): since, for each t , the random variables $A(t)$ and $\sigma(t)$ are infinitely divisible, there exist i.i.d. random variables $X_k^{(n)}$ and i.i.d. random variables $Z_k^{(n)}$ such that $X^{(n)}(\lfloor nt \rfloor) = \sum_{k=1}^{\lfloor nt \rfloor} X_k^{(n)}$ converges in distribution to $A(t)$ and $\sigma_d^{(n)}(\lfloor nt \rfloor) = \sum_{k=1}^{\lfloor nt \rfloor} Z_k^{(n)}$ converges in distribution to $\sigma(t)$ (furthermore Theorem 2.7 in [61] guarantees also J_1 weak convergence). Then, once identified $\sigma_d^{(n)}$, one uses its inverse $L_d^{(n)}$ to construct the semi-Markov process $Y^{(n)}(t) = X^{(n)}(L_d^{(n)}(t))$.

3.2. Semi-Markov chains of type B: Generalized fractional finite-difference equations. We note that, by some algebraic manipulations, equation (49) governing the Markov chain $\{\mathcal{X}(t)\}_{t \in \mathbb{N}_0}$ can be re-written in the form of a finite difference equation:

$$(\mathcal{I} - \mathcal{B})P_{ij}(t) = \sum_{l \in S} \lambda_i (H_{il} \mathcal{B} - \delta_{il}) P_{lj}(t), \quad P_{ij}(0) = \delta_{ij}, \quad (65)$$

where $\lambda_i = \frac{p_i}{q_i}$, δ_{ij} denotes the Kronecker delta, $\mathcal{B}p(t) = p(t-1)$ is the shift operator acting on the time variable, and hence $\mathcal{I} - \mathcal{B}$ represents the discrete-time derivative.

Equation (65) is a discrete-time version of equation (10) governing continuous time Markov chains. This fact can be heuristically seen by making use of a suitable

scaling limit. Indeed, assume that the time steps have size $1/n$, so that the shift operator acts as $\mathcal{B}_{1/n}p(t) = p(t - 1/n)$. Then scale $\lambda_i \rightarrow \lambda_i/n$ and divide both members of (65) by $1/n$. The equation reads

$$\frac{\mathcal{I} - \mathcal{B}_{1/n}}{1/n} P_{ij}(t) = \sum_{l \in S} \lambda_i (H_{il} \mathcal{B}_{1/n} - \delta_{il}) P_{lj}(t), \quad t \in \left\{ \frac{1}{n}, \frac{2}{n}, \dots \right\}, \quad P_{ij}(0) = \delta_{ij}, \quad (66)$$

and the continuous time limit $n \rightarrow \infty$ gives

$$\frac{d}{dt} P_{ij}(t) = \sum_{l \in S} \lambda_i (H_{il} - \delta_{il}) P_{lj}(t), \quad P_{ij}(0) = \delta_{ij} \quad t \in \mathbb{R}^+, \quad (67)$$

which is the Kolmogorov backward equation (10).

We now consider a semi-Markov chain of type B (according to Definition 3.1) and we denote by $\gamma(t)$ the (discrete) time spent by the process in the current position:

$$\gamma(t) = \inf\{k \in \mathbb{N} : \mathcal{Y}(t - k) \neq \mathcal{Y}(t)\} \quad \gamma(0) = 1.$$

Starting from a generic renewal time τ (i.e. such that $\gamma(\tau) = 1$), which is a regeneration time for the process, we derive a system of backward equations for the transition functions

$$\begin{aligned} p_{ij}(t) &= P(\mathcal{Y}(t + \tau) = j | \mathcal{Y}(\tau) = i, \gamma(\tau) = 1) \\ &= P(\mathcal{Y}(t) = j | \mathcal{Y}(0) = i, \gamma(0) = 1) \quad i, j \in S \quad t \in \mathbb{N}_0, \end{aligned}$$

where the last equality follows by time homogeneity. Such a system is given by (68) of the following theorem.

Theorem 3.4. *The set of functions $\{p_{ij}(t), i, j \in S, t \in \mathbb{N}_0\}$, under the initial condition $p_{ij}(0) = \delta_{ij}$, solve the following system of equations:*

$$\tilde{\mathcal{D}}_t p_{ij}(t) - P(Z > t) p_{ij}(0) = \sum_{l \in S} \lambda_i (H_{il} \mathcal{B} - \delta_{il}) p_{lj}(t) + \lambda_i p_{ij}(0) \delta_{0t}, \quad (68)$$

where

$$\tilde{\mathcal{D}}_t p_{ij}(t) = \sum_{\tau=0}^{\infty} (p_{ij}(t) - p_{ij}(t - \tau)) P(Z = \tau), \quad t \in \mathbb{N}_0, \quad (69)$$

while \mathcal{B} is the shift operator such that $\mathcal{B}p(t) = p(t - 1)$ and $\lambda_i = p_i/q_i$.

Remark 5. It is remarkable to note that (68) (governing discrete-time semi-Markov chains) can be obtained by (65) (governing the corresponding Markov ones), by substituting the discrete-time derivative on the left-hand side with the convolution operator $\tilde{\mathcal{D}}_t$. Conversely, $\tilde{\mathcal{D}}_t$ reduces to the discrete derivative $\mathcal{I} - \mathcal{B}$ in the trivial case where $Z = 1$ almost surely.

This is analogous to what happens in continuous time, where equation (14) (governing semi-Markov processes) is obtained from (10) (governing Markov processes) by changing the time derivative with the generalized fractional derivative

$$\mathcal{D}_t p_{ij}(t) = \int_0^{\infty} (p_{ij}(t) - p_{ij}(t - \tau)) \nu(d\tau) \quad t \in \mathbb{R}^+.$$

Equation (68) can be interpreted as a discrete-time version of (14), where the integral in the time variable is replaced by a series, the Lévy measure ν is replaced by the discrete density of Z and the tail of the Lévy measure $\bar{\nu}$ is replaced by the survival function of Z .

Remark 6. We call the operator (69) generalized fractional discrete derivative. The reason of this name is that in the case where Z follows the Sibuya distribution (25), the operator (69) reduces to the fractional power of the discrete derivative. Indeed, by simple calculations, we have

$$\begin{aligned}\tilde{\mathcal{D}}_t p_{ij}(t) &= \sum_{\tau=0}^{\infty} (p_{ij}(t) - p_{ij}(t - \tau)) P(Z = \tau) \\ &= \sum_{\tau=1}^{\infty} (p_{ij}(t) - p_{ij}(t - \tau)) (-1)^{\tau-1} \binom{\alpha}{\tau} \\ &= \sum_{\tau=0}^{\infty} \binom{\alpha}{\tau} (-1)^{\tau} p_{ij}(t - \tau) \\ &= (\mathcal{I} - \mathcal{B})^{\alpha} p_{ij}(t),\end{aligned}$$

where \mathcal{B} is the shift operator in the time variable, such that $\mathcal{B}p(t) = p(t - 1)$. We observe that such operator also appears in ARFIMA models (see [23]). The interested reader can find a pioneering study of the operator (69) in [43].

Proof of Theorem 3.4. The discrete-time renewal equation reads

$$p_{ij}(t) = \sum_{\tau=0}^t \sum_{l \in S} H_{il} P(J_0 = \tau | X_0 = i) p_{lj}(t - \tau) + P(J_0 > t | X_0 = i) \delta_{ij}.$$

By applying the generating function to both members we have

$$\tilde{p}_{ij}(u) = \sum_{l \in S} H_{il} \mathbb{E}(u^{J_0} | X_0 = i) \tilde{p}_{lj}(u) + \sum_{t=0}^{\infty} u^t P(J_0 > t | X_0 = i) \delta_{ij}. \quad (70)$$

Note that for any positive and integer valued random variable Y we have

$$\begin{aligned}\sum_{t=0}^{\infty} u^t P(Y > t) &= \sum_{t=0}^{\infty} u^t \sum_{k=t+1}^{\infty} P(Y = k) = \sum_{k=1}^{\infty} \sum_{t=0}^{k-1} u^t P(Y = k) \\ &= \sum_{k=1}^{\infty} \frac{1 - u^k}{1 - u} P(Y = k) = \frac{1 - \mathbb{E}u^Y}{1 - u}.\end{aligned} \quad (71)$$

Hence, using (56) we have

$$\begin{aligned}\sum_{t=0}^{\infty} u^t P(J_0 > t | X_0 = i) &= \frac{1 - \mathbb{E}(u^{J_0} | X_0 = i)}{1 - u} \\ &= \frac{1}{1 - u} \frac{1 - q_i \mathbb{E}u^Z - p_i u}{1 - q_i \mathbb{E}u^Z} = \frac{p_i + q_i \frac{1 - \mathbb{E}u^Z}{1 - u}}{1 - q_i \mathbb{E}u^Z}.\end{aligned}$$

Thus, (70) becomes

$$\tilde{p}_{ij}(u) = \sum_{l \in S} H_{il} \frac{p_i u}{1 - q_i \mathbb{E}u^Z} \tilde{p}_{lj}(u) + \frac{p_i + q_i \frac{1 - \mathbb{E}u^Z}{1 - u}}{1 - q_i \mathbb{E}u^Z} \delta_{ij}.$$

and can be re-written as

$$\tilde{p}_{ij}(u) - q_i \mathbb{E}u^Z \tilde{p}_{ij}(u) = \sum_{l \in S} H_{il} p_i u \tilde{p}_{lj}(u) + \left(p_i + q_i \frac{1 - \mathbb{E}u^Z}{1 - u} \right) \delta_{ij}.$$

Taking the inverse transform (by using (71) for the variable Z), the previous equation becomes

$$p_{ij}(t) - q_i \sum_{\tau=0}^{\infty} p_{ij}(t-\tau)P(Z=\tau) = \sum_{l \in S} H_{il} p_i p_{lj}(t-1) + (p_i \delta_{0t} + q_i P(Z > t)) \delta_{ij}$$

where the summation is extended to infinity because $p_{ij}(t) = 0$ for $t < 0$. Writing $p_{ij}(t) = p_i p_{ij}(t) + q_i p_{ij}(t)$ on the left-hand side, and dividing both sides by q_i (with the position $\lambda_i = p_i/q_i$) we obtain the desired equation. \square

Remark 7. We can retrace the same steps as in the proof of Theorem 3.4 in order to find a fractional-type equation governing the Sibuya counting process (29). Let $J_0 \sim \text{Sibuya}(\alpha)$ and $H_{ij} = 1$ if $j = i + 1$. We have

$$(I - \mathcal{B})^\alpha p_{(i+1)j}(t) - (-1)^t \binom{\alpha-1}{t} \delta_{ij} = p_{(i+1)j}(t) - p_{ij}(t),$$

which is a discrete-time version of equation $\partial_t^\alpha l(x, t) = -\partial_x l(x, t) + \bar{\nu}(t) \delta_{ij}$ governing the inverse stable subordinator.

Remark 8. To be exhaustive, we specify that also semi-Markov chains of type A have governing equations involving the operator (69). However, such equations have a cumbersome form, which is certainly not the fractional counterpart of the Markovian equation (65). The interested reader can find them by writing the related Markov renewal equation and following the same steps as in the proof of Theorem 3.4.

3.2.1. *Continuous-time limit of the governing equation.* Convergence of (68) to (14) can be easily obtained in the special case in which Z is regularly varying of order $\alpha \in (0, 1)$. For convenience of the reader, we recall below the notion of regular variation (for further details consult [11] and [12]):

Definition 3.5. A non negative random variable Z is said to be regularly varying of order α if its survival function satisfies one of the following equivalent conditions:

- a) $P(Z > t) = t^{-\alpha} L(t)$, with L a slowly varying function (i.e. $\lim_{n \rightarrow \infty} \frac{L(nt)}{L(n)} = 1$).
- b) $\lim_{n \rightarrow \infty} \frac{P(Z > nt)}{P(Z > n)} = t^{-\alpha}$.

Moreover, observe that the class of regularly varying distributions includes the Sibuya distribution (25). If Z is regularly varying, then, under a suitable scaling limit, equation (68) converges, for $t > 0$, to equation (15) governing fractional processes, i.e. Markov processes time changed by an independent inverse stable subordinator. Indeed, by letting the time steps have size $1/n$ and by the scaling

$$Z \rightarrow \frac{Z}{n} \quad \lambda_i \rightarrow \lambda_i P(Z > n) \Gamma(1 - \alpha) \quad \forall i \in S \quad \alpha \in (0, 1),$$

then, for $t \in \{\frac{1}{n}, \frac{2}{n}, \dots\}$, equation (68) reads

$$\begin{aligned} & \sum_{\tau=\frac{1}{n}}^{\infty} (p_{ij}(t) - p_{ij}(t-\tau)) \frac{1}{\Gamma(1-\alpha)} \frac{P(Z=n\tau)}{P(Z>n)} - \frac{1}{\Gamma(1-\alpha)} \frac{P(Z>nt)}{P(Z>n)} p_{ij}(0) \\ & = \sum_{l \in S} \lambda_i (H_{il} \mathcal{B}_{\frac{1}{n}} - \delta_{il}) p_{lj}(t), \end{aligned} \tag{72}$$

where $\mathcal{B}_{\frac{1}{n}}p(t) = p(t - \frac{1}{n})$. Taking into account that ¹

$$\frac{P(Z = nt)}{P(Z > n)} \sim \alpha t^{-\alpha-1} \frac{1}{n} \quad \text{for large } n, \quad (73)$$

in the limit $n \rightarrow \infty$ equation (72) reduces to

$$\begin{aligned} & \int_0^\infty (p_{ij}(t) - p_{ij}(t - \tau)) \frac{\alpha \tau^{-\alpha-1}}{\Gamma(1-\alpha)} d\tau - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \delta_{ij} \\ &= \sum_{l \in S} \lambda_i (H_{il} - \delta_{il}) p_{lj}(t) \quad t \in \mathbb{R}^+ \end{aligned}$$

which coincides with (15).

4. A remarkable special case: The fractional Bernoulli processes. We here analyze a special case of the theory expounded in the previous section, by constructing two processes which are discrete-time versions of the fractional Poisson process. We recall that the fractional Poisson process is a continuous time counting process whose i.i.d. waiting times J_0, J_1, \dots , have common law $P(J_k > t) = \mathcal{E}(-\lambda t^\alpha)$, $\alpha \in (0, 1)$, where $\mathcal{E}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1+\alpha k)}$ is the Mittag–Leffler function. As recalled in Section 2.2, it is obtained by the composition of a Poisson process with an independent inverse stable subordinator. For some references, consult [5, 6, 31, 36, 39]; see also [7] and [34] for time-inhomogeneous extensions of the model. The fractional Poisson process is intimately connected to fractional calculus as its state probabilities solve the forward equation

$$\frac{d^\alpha}{dt^\alpha} p_k(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} p_k(0) = -\lambda p_k(t) + \lambda p_{k-1}(t) \quad (74)$$

where $\frac{d^\alpha}{dt^\alpha} p_k(t)$ denotes the Riemann–Liouville fractional derivative. We propose two discrete-time approximations of such a process, respectively called fractional Bernoulli of types A and B (in the sense of the classification of semi-Markov chains given in the previous section). We have been inspired by [52], where the author defines a so-called Discrete Mittag–Leffler distribution. The reason of this name is that such a distribution converges to the classical continuous Mittag–Leffler distribution (16) under a suitable scaling limit. We here define two distributions, named discrete Mittag–Leffler distributions of type A and B, which are similar to that studied in [52], on which we base the definition of the related Bernoulli processes.

4.1. Fractional Bernoulli process of type A.

Definition 4.1. Let M be a geometric random variable with law $P(M = k) = pq^{k-1}$, $k \in \mathbb{N}$, and let the Z_j be i.i.d. Sibuya random variables as in (25). A

¹ Since Z is slowly varying, we have

$$\begin{aligned} \frac{P(Z = nt)}{P(Z > n)} &= \frac{P(Z > nt - 1) - P(Z > nt)}{P(Z > n)} = \frac{(nt - 1)^{-\alpha} L(nt - 1) - (nt)^{-\alpha} L(nt)}{n^{-\alpha} L(n)} \\ &= \frac{(nt)^{-\alpha} L(nt) \left[\left(1 - \frac{1}{nt}\right)^{-\alpha} \frac{L(nt-1)}{L(nt)} - 1 \right]}{n^{-\alpha} L(n)}. \end{aligned}$$

Thus, using the properties of the slowly varying function L and the expansion $(1 - \frac{1}{nt})^{-\alpha} \sim 1 + \frac{\alpha}{nt}$ for large n , the proof is complete.

random variable J^A is said to follow a DML_A (i.e. discrete Mittag–Leffler of type A) distribution if it can be expressed as a compound geometric sum

$$J^A = \sum_{k=1}^M Z_k$$

and thus has generating function

$$\mathbb{E}u^{J^A} = \frac{1 - (1 - u)^\alpha}{1 + \frac{q}{p}(1 - u)^\alpha}, \quad \alpha \in (0, 1). \quad (75)$$

Remark 9. DML_A is a discrete approximation of the Mittag–Leffler distribution (16). Indeed by rescaling

$$Z_k \rightarrow \frac{Z_k}{n} \quad J^A \rightarrow \frac{J^A}{n} \quad \frac{p}{q} = \lambda \rightarrow \frac{\lambda}{n^\alpha}, \quad (76)$$

we obtain the rescaled random variable

$$J^{A(n)} = \frac{1}{n} \sum_{j=1}^{M^{(n)}} Z_j \quad (77)$$

such that

$$\lim_{n \rightarrow \infty} \mathbb{E}e^{-sJ^{A(n)}} = \frac{\lambda}{\lambda + s^\alpha} \quad s \in \mathbb{R}^+,$$

where $\lambda/(\lambda + s^\alpha)$ is the Laplace transform of the Mittag–Leffler distribution (see (17)).

We are now ready to define a discrete-time approximation of the fractional Poisson process.

Definition 4.2. Let $T_n^A = J_0^A + J_1^A + \dots + J_{n-1}^A$ be a renewal chain of type (18), with waiting times J_0^A, J_1^A, \dots , having common DML_A distribution. The related counting process

$$\{N_A(t)\}_{t \in \mathbb{N}_0} = \max\{n \in \mathbb{N}_0 : T_n^A \leq t\}$$

is called fractional Bernoulli counting process of type A.

In the case $\alpha = 1$, we have $Z_k = 1$ for each k almost surely, and J^A defined in (75) reduces to a geometric random variable $J^A \stackrel{d}{=} M$ and thus the counting process N_A reduces to the Bernoulli counting process N defined in section 2.3.1.

We now give an important time-change relation regarding $\{N_A(t)\}_{t \in \mathbb{N}_0}$.

Proposition 6. Let $\{N(t)\}_{t \in \mathbb{N}_0}$ be a Bernoulli counting process and $\{L_\alpha(t)\}_{t \in \mathbb{N}_0}$ be an independent Sibuya counting process defined in (29). For each $t \in \mathbb{N}_0$, the following equality holds in distribution

$$N_A(t) \stackrel{d}{=} N(L_\alpha(t)).$$

Proof. By using (20) with waiting times (75), the generating function of $\{N_A(t)\}_{t \in \mathbb{N}}$ reads

$$\mathcal{G}_{N_A}(m, u) = (1 - u)^{\alpha-1} \frac{(p - p(1 - u)^\alpha)^m}{(p + q(1 - u)^\alpha)^{m+1}}.$$

The same form can be obtained by computing the generating function of $N(L_\alpha(t))$ by a simple conditioning argument

$$\begin{aligned} \mathcal{G}_{N_A}(m, u) &= \sum_{t=0}^{\infty} u^t P(N(L_\alpha(t)) = m) = \sum_{t=0}^{\infty} \sum_{j=0}^{\infty} u^t P(N(j) = m) P(L_\alpha(t) = j) \\ &= \sum_{j=0}^{\infty} P(N(j) = m) \mathcal{G}_{L_\alpha}(j, u) = (1-u)^{\alpha-1} \frac{(p-p(1-u)^\alpha)^m}{(p+q(1-u)^\alpha)^{m+1}} \end{aligned}$$

where we used (30) and (21). \square

The above proposition shows that N_A exhibits a time-change construction similar to that of the fractional Poisson process. Indeed, while N_A is given by the composition of a Bernoulli with a Sibuya process, the fractional Poisson process is given by the composition of a Poisson process with an inverse stable subordinator. The meaning of this construction is clear if we recall that the Bernoulli and Sibuya processes converge the Poisson process and to the inverse stable subordinator respectively.

4.2. Fractional Bernoulli process of type B.

Definition 4.3. Let M be a geometric random variable with law $P(M = k) = pq^{k-1}$, $k \in \mathbb{N}$, and let the Z_j be i.i.d. Sibuya random variables as in (25). A random variable J_B is said to follow a DML_B (e.g. discrete Mittag–Leffler of type B) distribution if it can be expressed as a compound shifted geometric sum

$$J^B = 1 + \sum_{k=1}^{M-1} Z_k$$

and thus has generating function

$$\mathbb{E}u^{J^B} = \frac{u}{1 + \frac{q}{p}(1-u)^\alpha}, \quad \alpha \in (0, 1). \quad (78)$$

Remark 10. DML_B is a discrete approximation of the Mittag–Leffler distribution (16). Indeed by rescaling

$$Z_k \rightarrow \frac{Z_k}{n} \quad J^B \rightarrow \frac{J^B}{n} \quad \frac{p}{q} = \lambda \rightarrow \frac{\lambda}{n^\alpha}, \quad (79)$$

we obtain the rescaled random variable

$$J^{B(n)} = \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{M^{(n)}-1} Z_j, \quad (80)$$

such that

$$\lim_{n \rightarrow \infty} \mathbb{E}e^{-sJ^{B(n)}} = \frac{\lambda}{\lambda + s^\alpha}, \quad s \in \mathbb{R}^+,$$

where $\lambda/(\lambda + s^\alpha)$ is the Laplace transform of the Mittag–Leffler distribution (see (17)).

We now define another discrete-time approximation of the fractional Poisson process.

Definition 4.4. Let $T_n^B = J_0^B + J_1^B + \dots + J_{n-1}^B$ be a renewal chain of type (18), with waiting times J_0^B, J_1^B, \dots , having common DML_B distribution. The related counting process

$$\{N_B(t)\}_{t \in \mathbb{N}_0} = \max\{n \in \mathbb{N}_0 : T_n^B \leq t\}$$

is called fractional Bernoulli counting process of type B.

In the case $\alpha = 1$, we have $Z_k = 1$ for each k almost surely, and J^B defined in (78) reduces to a geometric random variable $J^B \stackrel{d}{=} M$ and thus the counting process N_B reduces to the Bernoulli counting process N .

We observe that the process N_B has an interesting connection to fractional calculus, as $p_k(t) = P(N_B(t) = k)$ solves a forward equation which is analogous to (23) governing N , but where the discrete-time derivative $I - \mathcal{B}$ is replaced by its fractional power $(I - \mathcal{B})^\alpha$:

$$(I - \mathcal{B})^\alpha p(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k p(t - k).$$

Proposition 7. For $t \in \mathbb{N}_0$, the state probabilities $p_k(t) = P(N_B(t) = k)$ solve the following system

$$(I - \mathcal{B})^\alpha p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t-1), \quad k \geq 1, \quad (81)$$

$$(I - \mathcal{B})^\alpha p_0(t) - (-1)^t \binom{\alpha-1}{t} = -\lambda p_0(t) + \lambda \delta_{0t}. \quad (82)$$

under the initial condition $p_k(0) = \delta_{0k}$.

Proof. By computing the generating function of both members of (82) one has

$$(1-u)^\alpha \tilde{p}_0(u) - (1-u)^{\alpha-1} = -\lambda \tilde{p}_0(u) + \lambda,$$

which gives

$$\tilde{p}_0(u) = \frac{\lambda + (1-u)^{\alpha-1}}{\lambda + (1-u)^\alpha}.$$

By further computing the generating function of both members of (81) one has

$$(1-u)^\alpha \tilde{p}_k(u) = -\lambda \tilde{p}_k(u) + \lambda u \tilde{p}_{k-1}(u).$$

Solving the last equation by iteration, we have

$$\tilde{p}_k(u) = \frac{\lambda u}{\lambda + (1-u)^\alpha} \tilde{p}_{k-1}(u) = \frac{(\lambda u)^k}{[\lambda + (1-u)^\alpha]^k} \tilde{p}_0(u) = \frac{(\lambda u)^k [\lambda + (1-u)^{\alpha-1}]}{[\lambda + (1-u)^\alpha]^{k+1}},$$

which coincides with the generating function obtained by (20) with waiting times (78), and this concludes the proof. \square

Remark 11. Note that for $t \neq 0$ equations (81) and (82) can be written in compact form as

$$(I - \mathcal{B})^\alpha p_k(t) - P(Z > t) p_k(0) = -\lambda p_k(t) + \lambda p_{k-1}(t-1), \quad t \in \mathbb{N}, \quad (83)$$

where Z is the Sibuya random variable defined in (25).

We let the time steps have size $1/n$ and, following (79), we scale $\lambda \rightarrow \lambda/n^\alpha$ and $Z \rightarrow Z/n$. Thus (83) becomes

$$(I - \mathcal{B}_{\frac{1}{n}})^\alpha p_k(t) - P\left(\frac{Z}{n} > t\right) p_k(0)$$

$$= -\frac{\lambda}{n^\alpha} p_k(t) + \frac{\lambda}{n^\alpha} p_{k-1}\left(t - \frac{1}{n}\right), \quad t \in \left\{\frac{1}{n}, \frac{2}{n}, \dots\right\},$$

where $\mathcal{B}_{\frac{1}{n}} p_k(t) = p_k\left(t - \frac{1}{n}\right)$, namely

$$\begin{aligned} & \left(\frac{I - \mathcal{B}_{\frac{1}{n}}}{\frac{1}{n}}\right)^\alpha p_k(t) - n^\alpha P(Z > nt) p_k(0) \\ &= -\lambda p_k(t) + \lambda p_{k-1}\left(t - \frac{1}{n}\right), \quad t \in \left\{\frac{1}{n}, \frac{2}{n}, \dots\right\}. \end{aligned}$$

We let $n \rightarrow \infty$ and recall that the operator on the left-hand side

$$\lim_{n \rightarrow \infty} \left(\frac{I - \mathcal{B}_{\frac{1}{n}}}{\frac{1}{n}}\right)^\alpha$$

is known as Grünwald–Letnikov derivative (see [[57] Chapter 4, sect. 20]), and coincides with the fractional Riemann–Liouville derivative. Moreover, by using (42) and (34), we have

$$P(Z > nt) = (-1)^{nt} \binom{\alpha - 1}{nt} \sim \frac{(nt)^{-\alpha}}{\Gamma(1 - \alpha)}, \quad \text{as } n \rightarrow \infty.$$

Thus, formally, the limiting equation coincides with the forward equation (74) governing the fractional Poisson process.

Remark 12. We note that (7) is in the forward form. By adapting (68) we can also write the backward equation. Indeed, for $\lambda_i = \lambda \forall i \in \mathcal{S}$, $H_{ij} = 1$ if $j = i + 1$ and Z following the Sibuya distribution, then the waiting times follow a *DMLB* distribution (see (78)) and (68) reduces to

$$(I - \mathcal{B})^\alpha p_{ij}(t) - (-1)^t \binom{\alpha - 1}{t} \delta_{ij} = \lambda p_{(i+1),j}(t - 1) - \lambda p_{ij}(t) + \lambda \delta_{ij} \delta_0 t.$$

4.3. Convergence to the fractional Poisson process. We finally prove that, under a suitable limit, both the fractional Bernoulli counting processes $\{N_A(t)\}_{t \in \mathbb{N}_0}$ and $\{N_B(t)\}_{t \in \mathbb{N}_0}$ defined in this section converge to a fractional Poisson process. The convergence holds in the sense of finite dimensional distributions and also in J_1 Skorokhod sense.

Let $J_0^{A(n)}, J_1^{A(n)}, \dots$ be i.i.d. copies of $J^{A(n)}$, defined in (77), and let $T_k^{A(n)} = J_0^{A(n)} + J_1^{A(n)} + \dots + J_{k-1}^{A(n)}$ be the renewal chain associated with the counting process

$$N_A^{(n)}(t) = \max\{y \in \mathbb{N}_0 : T_y^{A(n)} \leq t\} \quad t \in \mathbb{R}^+.$$

Moreover, let $J_0^{B(n)}, J_1^{B(n)}, \dots$ be i.i.d. copies of $J^{B(n)}$ defined in (80) and let $T_k^{B(n)} = J_0^{B(n)} + J_1^{B(n)} + \dots + J_{k-1}^{B(n)}$ be the renewal chain associated with the counting process

$$N_B^{(n)}(t) = \max\{y \in \mathbb{N}_0 : T_y^{B(n)} \leq t\} \quad t \in \mathbb{R}^+.$$

Finally, let W_0, W_1, \dots be i.i.d. random variables with common Mittag–Leffler law (16) and let $T_k = W_0 + W_1 + \dots + W_{k-1}$ be the renewal process whose counting process is the fractional Poisson process

$$\Pi(t) = \max\{y \in \mathbb{N}_0 : T_y \leq t\}, \quad t \in \mathbb{R}^+.$$

Proposition 8. *Under the scaling limit (79) and (76), we have that:*

a) for $n \rightarrow \infty$,

$$\{N_A^{(n)}(t)\}_{t \in \mathbb{R}^+} \xrightarrow{fdd} \{\Pi(t)\}_{t \in \mathbb{R}^+}, \quad \{N_B^{(n)}(t)\}_{t \in \mathbb{R}^+} \xrightarrow{fdd} \{\Pi(t)\}_{t \in \mathbb{R}^+}.$$

where \xrightarrow{fdd} denotes convergence of finite dimensional distributions.

b) for $n \rightarrow \infty$,

$$\{N_A^{(n)}(t)\}_{t \in \mathbb{R}^+} \xrightarrow{J_1} \{\Pi(t)\}_{t \in \mathbb{R}^+}, \quad \{N_B^{(n)}(t)\}_{t \in \mathbb{R}^+} \xrightarrow{J_1} \{\Pi(t)\}_{t \in \mathbb{R}^+}, \quad \text{on } D[0, \infty),$$

where $\xrightarrow{J_1}$ denotes convergence in weak sense under the J_1 Skorokhod topology on the space of càdlàg functions $D[0, \infty)$.

Proof. a) The proof proceeds along the same lines for both processes A and B . For the sake of brevity, we only show the proof for case A . Since $J_k^{A(n)} \xrightarrow{d} W_k$ for each k , where W_k has a Mittag–Leffler distribution (16), then, fixed $r \in \mathbb{N}$, the vector $(J_0^{A(n)}, J_1^{A(n)}, \dots, J_{r-1}^{A(n)})$ converges in distribution to $(W_0, W_1, \dots, W_{r-1})$. Let $T_k = W_0 + W_1 + \dots + W_{k-1}$. By considering the function $h(x_0, x_1, \dots, x_{r-1}) = (x_0, x_0 + x_1, x_0 + x_1 + x_2, \dots, x_0 + x_1 + \dots + x_{r-1})$, a continuous mapping argument gives

$$\begin{aligned} (T_1^{A(n)}, \dots, T_r^{A(n)}) &= h(J_0^{A(n)}, J_1^{A(n)}, \dots, J_{r-1}^{A(n)}) \\ &\xrightarrow{d} h(W_0, W_1, \dots, W_{r-1}) = (T_1, \dots, T_r). \end{aligned}$$

Hence, by fixing r times t_1, \dots, t_r in \mathbb{R}^+ and r numbers m_1, \dots, m_r in \mathbb{N} , we have

$$\begin{aligned} P(N_A^{(n)}(t_1) \leq m_1, \dots, N_A^{(n)}(t_r) \leq m_r) &= P(T_{m_1}^{A(n)} \geq t_1, \dots, T_{m_r}^{A(n)} \geq t_r) \\ &\xrightarrow{n \rightarrow \infty} P(T_{m_1} \geq t_1, \dots, T_{m_r} \geq t_r) = P(\Pi(t_1) \leq m_1, \dots, \Pi(t_r) \leq m_r). \end{aligned}$$

b) It is sufficient to apply Theorem 3 in [10]. Indeed $N_A^{(n)}(t)$ has non decreasing sample paths and, according to the proof of statement a) above, it converges in the sense of finite dimensional distribution to a fractional Poisson process, the latter being continuous in probability. \square

Remark 13. There are many possible discrete-time approximations of the Fractional Poisson process (at least one for each different discrete version of the Mittag–Leffler distribution). We have only presented two of these possible approximating processes. None of these two is trivially given by sampling the fractional Poisson process at integer times (as well as the Bernoulli process is not given by the Poisson process sampled at integer times). Indeed, by sampling the fractional Poisson process at integer times, we obtain a process with jumps of size possibly greater than 1. In general, the discrete-time semi-Markov processes treated in this paper are not obtained by sampling the related continuous-time processes at discrete times.

Acknowledgments. F. Polito and C. Ricciuti have been partially supported by the project “Memory in Evolving Graphs” (Compagnia di San Paolo–Università di Torino). F. Polito has been also partially supported by INdAM/GNAMPA.

A. Pachon has been partially supported by the project “Complex Networks and Data Science” (CES RIS 2019, University of South Wales).

REFERENCES

- [1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics, 116, Cambridge University Press, Cambridge, 2009.
- [2] V. S. Barbu and N. Limnios, *Semi-Markov Chains and Hidden Semi-Markov Models Toward Applications*, Lecture Notes in Statistics, 191, Springer, New York, 2008.
- [3] P. Becker-Kern, M. M. Meerschaert and H.-P. Scheffler, [Limit theorems for coupled continuous-time random walks](#), *Ann. Probab.*, **32** (2004), 730–756.
- [4] L. Beghin, [Fractional relaxation equations and Brownian crossing probabilities of a random boundary](#), *Adv. in Appl. Probab.*, **44** (2012), 479–505.
- [5] L. Beghin and E. Orsingher, [Fractional Poisson processes and related planar random motions](#), *Electron. J. Probab.*, **14** (2009), 1790–1827.
- [6] L. Beghin and E. Orsingher, [Poisson-type processes governed by fractional and higher-order recursive differential equations](#), *Electron. J. Probab.*, **15** (2010), 684–709.
- [7] L. Beghin and C. Ricciuti, [Time-inhomogeneous fractional Poisson processes defined by the multistable subordinator](#), *Stoch. Anal. Appl.*, **37** (2019), 171–188.
- [8] J. Bertoin, [Subordinators: Examples and Applications](#), in *Lectures on Probability Theory and Statistics*, Lectures Notes in Math., 1717, Springer, Berlin, 1999, 1–91.
- [9] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, Inc., New York-London-Sydney, 1968.
- [10] N. H. Bingham, [Limit theorems for occupation times of Markov processes](#), *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **17** (1971), 1–22.
- [11] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, 27, Cambridge University Press, Cambridge, 1987.
- [12] D. Buraczewski, E. Damek and T. Mikosch, *Stochastic Models with Power-Law Tails*, Springer Series in Operations Research and Financial Engineering, Springer, Cham, 2016.
- [13] E. Çinlar, Markov additive processes and semi-regeneration, in *Proceedings of the Fifth Conference on Probability Theory*, Editura Acad. R. S. R., Bucharest, 1977, 33–49.
- [14] D. R. Cox, *Renewal Theory*, Methuen & Co. Ltd., London; John Wiley & Sons, Inc., New York, 1962.
- [15] L. Devroye, [A triptych of discrete distributions related to the stable law](#), *Statist. Probab. Lett.*, **18** (1993), 349–351.
- [16] M. D’Ovidio, [Wright functions governed by fractional directional derivatives and fractional advection diffusion equations](#), *Methods Appl. Anal.*, **22** (2015), 1–36.
- [17] W. Feller, [On semi-Markov processes](#), *Proc. Nat. Acad. Sci. U.S.A.*, **51** (1964), 653–659.
- [18] R. Garra, E. Orsingher and F. Polito, [State-dependent fractional point processes](#), *J. Appl. Probab.*, **52** (2015), 18–36.
- [19] R. Garra, A. Giusti, F. Mainardi and G. Pagnini, [Fractional relaxation with time-varying coefficient](#), *Fract. Calc. Appl. Anal.*, **17** (2014), 424–439.
- [20] N. Georgiou, I. Z. Kiss and E. Scalas, [Solvable non-Markovian dynamic network](#), *Phys. Rev. E* (3), **92** (2015), 9pp.
- [21] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Educational Publishers Inc., Cambridge, Mass., 1954.
- [22] I. I. Gihman and A. V. Skorohod, *The Theory of Stochastic Processes. II*, der Mathematischen Wissenschaften, Band 218, Springer-Verlag, New York-Heidelberg, 1975.
- [23] C. W. J. Granger and R. Joyeux, [An introduction to long-memory time series models and fractional differencing](#), *J. Time Ser. Anal.*, **1** (1980), 15–29.
- [24] A. Gut, *Probability: A Graduate Course*, Springer Texts in Statistics, 75, Springer, New York, 2013.
- [25] M. E. Hernández-Hernández and V. N. Kolokoltsov, [Probabilistic solutions to nonlinear fractional differential equations of generalized Caputo and Riemann–Liouville type](#), *Stochastics*, **90** (2018), 224–255.
- [26] M. E. Hernández-Hernández, V. N. Kolokoltsov and L. Toniazzi, [Generalized fractional evolutions equations of Caputo type](#), *Chaos Solitons Fractals*, **102** (2017), 184–196.
- [27] J. Jacod, [Systèmes régénératifs and processus semi-markoviens](#), *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **31** (1974/75), 1–23.
- [28] V. N. Kolokol’tsov, [Generalized continuous-time random walks, subordination by hitting times, and fractional dynamics](#), *Theory Probab. Appl.*, **53** (2009), 594–609.

- [29] V. N. Kolokoltsov, *Markov Processes, Semigroups and Generators*, De Gruyter Studies in Mathematics, 38, Walter de Gruyter & Co., Berlin, 2011.
- [30] V. Korolyuk and A. Swishchuk, *Semi-Markov Random Evolutions*, Mathematics and its Applications, 308, Kluwer Academic Publishers, Dordrecht, 1995.
- [31] A. Kumar, E. Nane and P. Vellaisamy, *Time-changed Poisson processes*, *Statist. Probab. Lett.*, **81** (2011), 1899–1910.
- [32] T. G. Kurtz, *Comparison of semi-Markov and Markov processes*, *Ann. Math. Statist.*, **42** (1971), 991–1002.
- [33] N. N. Leonenko, M. M. Meerschaert, R. L. Shilling and A. Sikorskii, *Correlation structure of time-changed Lévy processes*, *Commun. Appl. Ind. Math.*, **6** (2014), 22pp.
- [34] N. N. Leonenko, E. Scalas and M. Trinh, *The fractional non-homogeneous Poisson process*, *Statist. Probab. Lett.*, **120** (2017), 147–156.
- [35] P. Levy, *Processus semi-markoviens*, *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam*, Erven P. Noordhoff N. V., Groningen; North-Holland Publishing Co., Amsterdam, 1956, 416–426.
- [36] F. Mainardi, R. Gorenflo and E. Scalas, *A fractional generalization of the Poisson process*, *Vietnam J. Math.*, **32** (2007), 53–64.
- [37] M. M. Meerschaert and H.-P. Scheffler, *Limit theorems for continuous-time random walks with infinite mean waiting times*, *J. Appl. Probab.*, **41** (2004), 623–638.
- [38] M. M. Meerschaert and H.-P. Scheffler, *Triangular array limits for continuous time random walks*, *Stochastic Process. Appl.*, **118** (2008), 1606–1633.
- [39] M. M. Meerschaert, E. Nane and P. Vellaisamy, *The fractional Poisson process and the inverse stable subordinator*, *Electron. J. Probab.*, **16** (2011), 1600–1620.
- [40] M. M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus*, De Gruyter Studies in Mathematics, 43, Walter de Gruyter & Co., Berlin, 2012.
- [41] M. M. Meerschaert and P. Straka, *Semi-Markov approach to continuous time random walk limit processes*, *Ann. Probab.*, **42** (2014), 1699–1723.
- [42] M. M. Meerschaert and B. Toaldo, *Relaxation patterns and semi-Markov dynamics*, *Stochastic Process. Appl.*, **129** (2019), 2850–2879.
- [43] T. Michelitsch, G. Maugin, S. Derogar and A. Nowakowski, *Sur une généralisation de l’opérateur fractionnaire*, preprint, [arXiv:1111.1898v1](https://arxiv.org/abs/1111.1898v1).
- [44] J. R. Norris, *Markov Chains*, Cambridge Series in Statistical and Probabilistic Mathematics, 2, Cambridge University Press, Cambridge, 1998.
- [45] E. Orsingher and F. Polito, *Fractional pure birth processes*, *Bernoulli*, **16** (2010), 858–881.
- [46] E. Orsingher and F. Polito, *On a fractional linear birth-death process*, *Bernoulli*, **17** (2011), 114–137.
- [47] E. Orsingher, F. Polito and L. Sakhno, *Fractional non-linear, linear and sub-linear death processes*, *J. Stat. Phys.*, **141** (2010), 68–93.
- [48] E. Orsingher, C. Ricciuti and B. Toaldo, *Time-inhomogeneous jump processes and variable order operators*, *Potential Anal.*, **45** (2016), 435–461.
- [49] E. Orsingher, C. Ricciuti and B. Toaldo, *On semi-Markov processes and their Kolmogorov’s integro-differential equations*, *J. Funct. Anal.*, **275** (2018), 830–868.
- [50] A. Pachon, F. Polito and L. Sacerdote, *Random graphs associated to some discrete and continuous time preferential attachment models*, *J. Stat. Phys.*, **162** (2016), 1608–1638.
- [51] A. Pachon, L. Sacerdote and S. Yang, *Scale-free behaviour of networks with the copresence of preferential and uniform attachment rules*, *Phys. D*, **371** (2018), 1–12.
- [52] R. N. Pillai and K. Jayakumar, *Discrete Mittag-Leffler distributions*, *Statist. Probab. Lett.*, **23** (1995), 271–274.
- [53] R. Pyke, *Markov renewal processes with finitely many states*, *Ann. Math. Statist.*, **32** (1961), 1243–1259.
- [54] R. Pyke, *Markov renewal processes with infinitely many states*, *Ann. Math. Statist.*, **35** (1964), 1746–1764.
- [55] M. Raberto, F. Rapallo and E. Scalas, *Semi-Markov graph dynamics*, *PLoS ONE*, **6** (2011).
- [56] C. Ricciuti and B. Toaldo, *Semi-Markov models and motion in heterogeneous media*, *J. Stat. Phys.*, **169** (2017), 340–361.
- [57] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, 1993.
- [58] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, 68, Cambridge University Press, Cambridge, 1999.

- [59] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions. Theory and Applications*, De Gruyter Studies in Mathematics, 37, Walter de Gruyter & Co., Berlin, 2010.
- [60] A. V. Skorohod, [Limit theorems for stochastic processes](#), *Teor. Veroyatnost. i Primenen.*, **1** (1956), 289–319.
- [61] A. V. Skorohod, [Limit theorems for stochastic processes with independent increments](#), *Teor. Veroyatnost. i Primenen.*, **2** (1957), 145–177.
- [62] W. L. Smith, [Regenerative stochastic processes](#), *Proc. Roy. Soc. London Ser. A*, **232** (1955), 6–31.
- [63] F. W. Steutel and K. van Harn, *Infinite Divisibility of Probability Distributions on the Real Line*, Monographs and Textbooks in Pure and Applied Mathematics, 259, Marcel Dekker, Inc., New York, 2004.
- [64] P. Straka and B. I. Henry, [Lagging and leading coupled continuous time random walks, renewal times and their joint limits](#), *Stochastic Process. Appl.*, **121** (2011), 324–336.
- [65] B. Toaldo, [Lévy mixing related to distributed order calculus, subordinators and slow diffusions](#), *J. Math. Anal. Appl.*, **430** (2015), 1009–1036.
- [66] W. Whitt, [Some useful functions for functional limit theorems](#), *Math. Oper. Res.*, **5** (1980), 67–85.
- [67] W. Whitt, *Stochastic-Process Limits*, Springer Series in Operations Research, Springer-Verlag, New York, 2002.

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